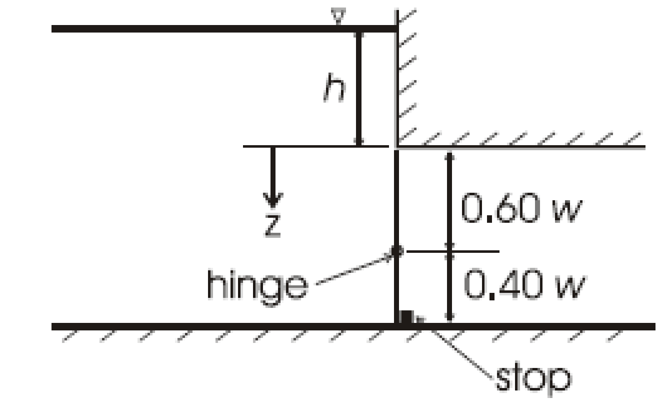
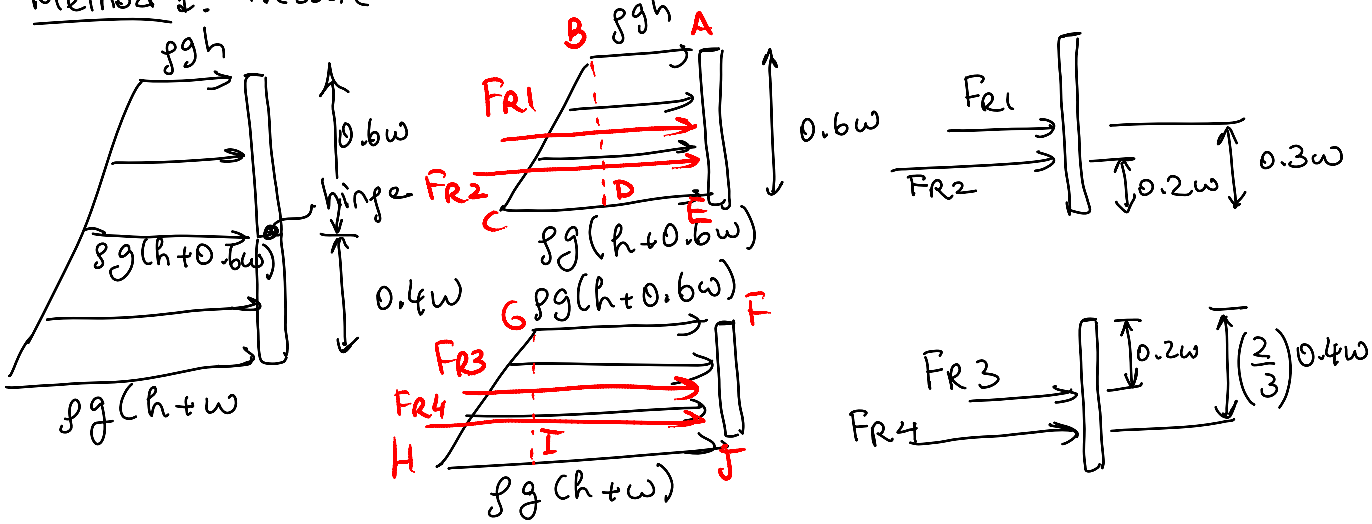
The square gate (with a length w and width w) is eccentrically pivoted as shown below so that it automatically opens at a certain value of h. What is that h value in terms of w?

Hint: Note that the gate is free to rotate in the clockwise direction around the pivot (without the effect of the stopper).



Method 1: Pressure Prism



$$F_{R1} = (\text{Volume})_{ABDE} = 0.6\,\rho g h w^2$$

$$F_{R2} = (\text{Volume})_{BCD} = \rho g \frac{(0.6w)(0.6w)}{2} \cdot w = (0.18)\,\rho g w^3$$

$$F_{R3} = (\text{Volume})_{FGIJ} = \rho g (h+0.6w)(0.4w)w = \rho g w \left(0.4hw + 0.24w^2\right)$$

$$F_{R4} = (\text{Volume})_{GHI} = \rho g (h+w-h-0.6w)(0.4w)\frac{1}{2}w = (0.08)\,\rho g w^3$$

Name: ____________________

EXTRA PAGE

$\Sigma M_{hinge} = 0$

$$\Rightarrow (F_{R1})(0.3w) + (F_{R2})(0.2w) = (F_{R3})(0.2w) + (F_{R4})\left(\frac{2}{3}\right)(0.4w)$$

$$(0.6\rho g h w^2)(0.3w) + (0.18)\rho g w^3 (0.2w) = 0.4\rho g h w^2 (0.2w) + 0.24\rho g w^3 (0.2w) + (0.08)\rho g w^3 \frac{2}{3}(0.4w)$$

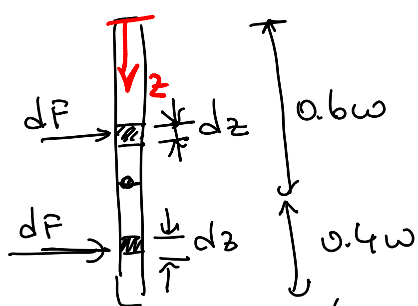
$$0.18\cancel{\rho}\cancel{g}h\cancel{w^3} + 0.036\cancel{\rho}\cancel{g}\cancel{w^4}^{\,w} = 0.08\cancel{\rho}\cancel{g}h\cancel{w^3} + 0.048\cancel{\rho}\cancel{g}\cancel{w^4}^{\,w} + \frac{0.064}{3}\cancel{\rho}\cancel{g}\cancel{w^4}^{\,w}$$

$$0.18h + 0.036w = 0.08h + 0.048w + \frac{0.064}{3}w$$

$$(0.18 - 0.08)h = \left(0.048 - 0.036 + \frac{0.064}{3}\right)w$$

$$0.1h = \frac{0.1}{3}w \Rightarrow \boxed{h = \frac{w}{3}} \quad *\,*\,*$$

Method 2: (Integration):



Above the pivot

$$dM_{above} = (dF).(0.6w - z)$$

$$dM_{below} = (dF)(z - 0.6w)$$

$$dF = p\,dA_{0.6w} = \rho g(h+z)\,w\,dz$$

$$M_{above} = \iint_{area} dM_{above} = \iint_{area} dF\,(0.6w - z) = \int_{z=0} \rho g\,(h+z)\,w\,(0.6w - z)\,dz$$

$$= \rho g w \int_0^{0.6w} \left(0.6wh - hz + 0.6wz - z^2\right) dz$$

$$= \rho g w \left[0.6wh z - \frac{h z^2}{2} + 0.6w \frac{z^2}{2} - \frac{z^3}{3} \right]_0^{0.6w}$$

$$= \rho g w \left[0.6^2 w^2 h - \frac{h(0.6)^2 w^2}{2} - \frac{0.6w(0.6)^2 w^2}{2} - \frac{(0.6)^3 w^3}{3} \right]$$

$$= \rho g w^3 \left[0.18h - 0.18w \right]$$

$$\boxed{M_{above} = 0.18 \rho g w^3 (h - w)}$$

$*\,*$
$*$

$$M_{below} = \iint_{area} dM_{below} = \iint_{area} dF\,(z - 0.6w) = \int_{z=0.6w}^{w} \rho g (h+z) w (z - 0.6w)\, dz$$

$$= \rho g w \int_{0.6w}^{w} \left(hz - 0.6hw + z^2 - 0.6wz \right) dz$$

$$= \rho g w \left[\frac{hz^2}{2} - 0.6hwz + \frac{z^3}{3} - 0.6w\frac{z^2}{2} \right]_{0.6w}^{w}$$

$$= \rho g w \left[\frac{hw^2}{2} - 0.6hw^2 + \frac{w^3}{3} - \frac{0.6}{2} w^3 - \frac{h(0.6)^2 w^2}{2} - (0.6)^2 hw^2 - \frac{(0.6)(0.6)^2 w^3}{3} - \frac{(0.6)^3}{2} w^3 \right]$$

$$= \rho g w \left[-0.64\,hw^2 - 0.0267\,w^3 \right]$$

$$\boxed{M_{below} = \rho g w^3 \left[-0.64h - 0.0267w \right]}$$

$*\,*$
$*$

$$M_{above} = M_{below}$$

$$\rho g w^3 \left[0.18h - 0.18w \right] = \rho g w^3 \left[-0.64h - 0.0267w \right]$$

$$-0.46h = -0.1533w$$

$$h = \frac{0.1533}{0.46} w = 0.333w \Rightarrow \boxed{h = \frac{1}{3} w}$$