MAT292 - Fall 2017

Term Test 2 - November 16, 2017

Time allotted: 100 minutes

Aids permitted: None

Total marks: 60

Full Name: ______________________________
Last First

Student Number: ______________________________

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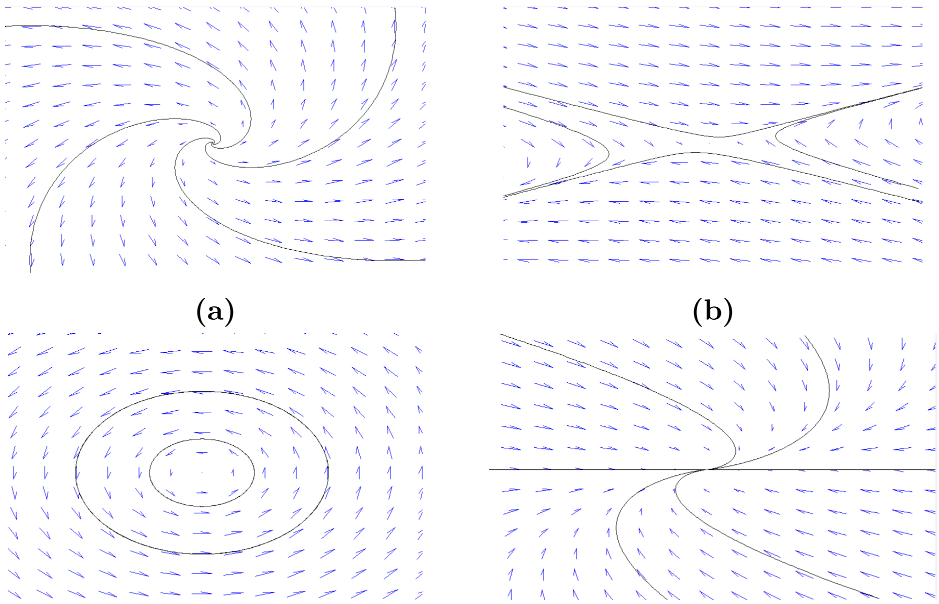
<u>Instructions</u>

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test is **double-sided**. Make sure you don't skip any problems.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 11–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

HAVE FUN!

SECTION I No explanation is necessary. **(15 marks)**

For questions 1–4, please match the differential equations with the phase portraits and circle the correct option about stability.



(a) **(b)** **(c)** **(d)**

1. **(2 marks)** $\mathbf{x}' = \begin{pmatrix} -1 & 6 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ **(a)** **(b)** **(c)** **(d)**

The equilibrium solution is **stable** / **unstable** .

2. **(2 marks)** $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x}$ **(a)** **(b)** **(c)** **(d)**

The equilibrium solution is **stable** / **unstable** .

3. **(2 marks)** $\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \mathbf{x}$ **(a)** **(b)** **(c)** **(d)**

The equilibrium solution is **stable** / **unstable** .

4. **(2 marks)** $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x}$ **(a)** **(b)** **(c)** **(d)**

The equilibrium solution is **stable** / **unstable** .

For questions 5–8, please fill in the blanks.

5. **(2 marks)** Suppose y_1 and y_2 are two solutions to $y''(t) - \frac{1}{t}y'(t) = 0$ for $t > 0$, and the general solution is given by $y(t) = c_1 + c_2 t^2$.
If $W[y_1, y_2](1) = 1$, then $W[y_1, y_2](t) =$ ____.

6. **(1 mark)** For which value(s) of γ are solutions to $3y''(t) + y(t) = \cos(\gamma t)$ unbounded?
$\gamma =$ __________________________.

7. **(2 marks)** For which value(s) of r is x^r a solution to $x^2 y''(x) + 5xy'(x) + 4y(x) = 0$?
$r =$ __________________________.

8. **(2 marks)** If $\mathbf{x}' = \begin{pmatrix} t & 1 \\ 2 & t \end{pmatrix} \mathbf{x}$, $\mathbf{x}_0 = \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\mathbf{x}_1$ is the result of applying Euler's method to numerically approximate this system with step size $h = 1$, then

$\mathbf{x}_1 =$ __________________________.

SECTION II **Justify** your answers. **(45 marks)**

9. Consider the system $\mathbf{x}'(t) = \begin{pmatrix} 1 & 4 \\ \alpha & 1 \end{pmatrix} \mathbf{x}(t)$ for $\alpha \in \mathbb{R}$. **(5 marks)**

a) **(1 mark)** For which value(s) of α is $\mathbf{0}$ an unstable spiral point?

b) **(4 marks)** Find the general solution to the system for $\alpha = -1/4$, and write it in terms of real valued functions.

10. Use the method of undetermined coefficient to solve the initial value problem **(10 marks)**

$$\begin{cases} y''(t) + y'(t) - 2y(t) = 5\sin(t) + 2 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

11. **(10 marks)**

Suppose $y_1(x)$ is a solution to the differential equation $x^2y''(x) = 2xy'(x) - 2y(x)$, for $x > 0$.

a) **(2 marks)** Let $z(x) = \dfrac{y_1(x)}{x}$. Using $y_1(x) = x \cdot z(x)$, find $y_1'(x)$ and $y_1''(x)$ in terms of x, $z'(x)$, and $z''(x)$.

b) **(2 marks)** Substitute the results of **a)** into the differential equation for y_1 to get a differential equation for $z(x)$.

c) **(2 marks)** Find $z(x)$.

d) **(2 marks)** If $z(1) = 1$, and $z'(57) = -1$, find $y_1(x)$.

e) **(2 marks)** Check that $y_1(x)$ satisfies the differential equation.

12. For a 2×2 real valued constant matrix A, consider the initial value problem, **(10 marks)**

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{$\star$}$$

a) (2 marks) Write the solution to the initial value problem $(\star)$ in terms of e^{At} if $t_0 = 0$.

b) (2 marks) Recall that $(e^{At})^{-1} = e^{-At}$. Write the solution to the initial value problem $(\star)$ using matrix exponentials if $t_0 = 6$.

c) (3 marks) If $A = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$, then the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t}$ for $c_1, c_2 \in \mathbb{R}$. Find e^{At}.

Hint: $e^{At}\big|_{t=0} = I$.

d) (3 marks) Use the method of integrating factors to find the vector function $\mathbf{v}(t)$ such that the general solution to the system of differential equations,

$$\mathbf{x}'(t) + \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} \mathbf{x}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

can be written as $\mathbf{x}(t) = e^{-At}\mathbf{v}(t) + e^{-At}\mathbf{c}$ where $\mathbf{c} \in \mathbb{R}^2$.

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13. **(10 marks)**

Lidocaine is a drug used for treating ventricular arrhythmia. When lidocaine is given to a patient, it first passes into body tissue before moving into the bloodstream where it is effective. Additionally, over time the drug is filtered out of body tissue and the bloodstream. Let

- $x(t)$ = the amount of lidocaine in the bloodstream,
- $y(t)$ = the amount of lidocaine in body tissue,

where t represents time in hours. A system of differential equations that models $x(t)$ and $y(t)$ is

$$\begin{cases} x'(t) &= -9x(t) + \alpha \cdot y(t) \\ y'(t) &= \ \ 6x(t) - \beta \cdot y(t) \end{cases} \tag{\#}$$

where α and β are strictly positive real numbers.

a) **(1 mark)** What does the initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$ represent?

b) **(2 marks)** If all of the lidocaine that leaves body tissue must enter the bloodstream (it can't go anywhere else), what can you say about α and β? Justify in one or two sentences.

c) **(3 marks)** If $\alpha = \beta$, classify the critical point $\mathbf{0}$ of the system (#).

d) **(1 mark)** What is the physical interpretation of your classification of the critical point from part **c**? Does it make sense?

e) **(3 marks)** Ideally, we would have a treatment plan where $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for some optimized constant lidocaine levels x_1, and y_1. A researcher has developed a new treatment that they claim will keep $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ close to $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for all $t > 0$. The new treatment, with a sinusodially driven bloodstream lidocaine, is modelled by the differential equations

$$\begin{cases} (x(t) - x_1)' &= -(y(t) - y_1) + \cos(t) \\ (y(t) - y_1)' &= \quad (x(t) - x_1) \end{cases}$$

Will $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ remain close to $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for all $t > 0$ under this treatment plan?
Hint: Let $\tilde{x}(t) = x(t) - x_1$ and $\tilde{y}(t) = y(t) - y_1$, then find a second order differential equation for $\tilde{y}(t)$.

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