MAT292 - Fall 2020

Final Exam - December 14, 2020

Time allotted: 90 minutes

Total marks: 48

Aids permitted: see "OK list"

Full Name: SOLUTIONS
Last First

Student Number:

Email: @mail.utoronto.ca

Do not forget to fill in the integrity statement on the second page!

- In the first section, only answers and sometimes brief justifications are required.
- In the second section, justify your answers fully.
- This exam contains 9 pages (including this title page). Make sure you have all of them.
- You can use pages 8–9 for rough work or to complete a question (**Mark clearly**).
- Make sure to follow this timeline:
  - 7:05 pm – Start writing the exam.
  - 8:35 pm – Stop writing the exam; fill in the integrity page (second page).
    You MUST stop writing at 8:35 pm.
    The last 25 minutes are for submission, not for writing.
  - 9:00 pm – Upload deadline.
    No extensions will be given.

Question	Q1-4	Q5	Q6	Q7	Q8	**Total**
Marks	8	10	10	10	10	**48**

HAVE FUN!

This is the ONLY page you can fill in *after* 8:35 pm.

If you don't complete and sign this page, you will receive a grade of zero for the entire test.

We at U of T want you to feel proud of what you accomplish as a student. Please respect all of the hard work you're doing this term by making sure that the work you do is your own.

We don't expect you to score perfectly on the assessments and there will be some things that you may not know. Using an unauthorized resource or asking someone else for the answer robs you of the chance later to feel proud of how well you did because you'll know that it wasn't really your work that got you there.

Success in university isn't about getting a certain mark, it's about becoming the very best person you can by enriching yourself with knowledge, strengthening yourself with skills, and building a healthy self-esteem based on how much you've grown and achieved. No one assessment captures that but your conscience will stay with you forever.

Make yourself and your loved ones proud of the student that you are by conducting yourself honestly at all times. Hold each other accountable to these standards.

In submitting this assessment ...	**Short sentences**
... I confirm that my conduct regarding this test adheres to the Code of Behaviour on Academic Matters.	I know the Code.
... I confirm that I have not acted in such a way that would constitute cheating, misrepresentation, or unfairness, including but not limited to, using unauthorized aids and assistance, impersonating another person, and committing plagiarism.	I didn't cheat.
... I confirm that the work I am submitting in my name is the work of no one but myself.	This is only my work.
... I confirm that all pages have been handwritten by myself.	I wrote all pages.
... I confirm that I have not received help from others, whether directly or indirectly.	I didn't receive help.
... I confirm that I have not provided help to others, whether directly or indirectly.	I didn't provide help.
... I confirm that I have only used the aids marked as "OK" on the list.	I only used "OK" aids.
... I am aware that not disclosing another student's misconduct despite my knowledge is an academic offence.	I know I must report cheating.

In this box, handwrite the sequence of short sentences (starting with "I know the Code. I didn't cheat...").

Your student number

Your signature

Submission date

SECTION I Provide the final answer. **Justification required only in questions 4 and 5. (18 marks)**

1. **(2 marks)** State the inverse Laplace transform of $\dfrac{1}{s(s+1)(s^2+1)}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s^2+1)}\right\} = 1 - e^{-t}/2 - 1/2(\cos t + \sin t)$$

2. **(2 marks)** The Fourier series for $f(x) = |x|, \ -\pi \le x \le \pi$ is given by: $A_0 - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}$

First, determine the value of the constant term: $A_0 =$

Then use the result to determine a value for this sum: $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots =$

Solution: $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{2}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$. Evaluating the Fourier series at $x = 0$ and setting it equal to $f(0) = 0$ gives $S = \frac{\pi^2}{8}$.

3. **(2 marks)** Consider the following IVP

$$y' = \begin{cases} -y & y \ge 0 \\ 0 & y < 0 \end{cases} \qquad y(0) = 1$$

Euler's method gives approximate values $y_n \approx y(n \cdot \Delta t)$ to this IVP. Consider the following two cases.

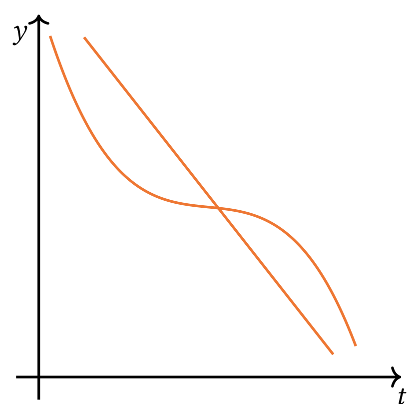
If $\Delta t < 1$, then $\lim\limits_{n\to\infty} y_n =$ If $\Delta t > 1$, then $\lim\limits_{n\to\infty} y_n =$

Solution: For $y_n > 0$, the Euler step is $y_{n+1} = (1-\Delta t)y_n$. The limit is zero if $\Delta t < 1$. If $\Delta t > 1$ then y_1 will be negative, where it will remain. Therefore $\lim_{n\to\infty} y_n = \begin{cases} 0 & \Delta t < 1 \\ 1-\Delta t & \Delta t > 0 \end{cases}$.

4. **(2 marks)** Why can the plot on the right NOT be a plot of solutions to this ODE?

$$y' + e^{t^2}y = 0$$

Reason: This equation is of the form $y' + p(t)y = 0$ for continuous $p(t)$. Therefore the E-U theorem for linear ODEs applies and solutions must be unique given an initial value. The point of intersection in the plot contradicts that fact.



5. (10 marks) For each of the following statements, decide if it is true or false. Then justify your choice.

Remember: A statement is only true if it is always true. If a statement only works in special cases/under certain circumstances, it is false.

(a) If $f'(3) = 1$ and $f(3) = 0$, then $y' = f(y)$ has an unstable equilibrium at $y = 3$. ⊗ TRUE ○ FALSE

Solution: Comparing this with a phase plot yields the answer. An alternative, more detailed argument is this: Due to $f'(3) > 0$ and $f(3) = 0$, we get $y' = f(y) < 0$ for values slightly smaller than 3. Likewise the function is increasing for values slightly above 3. These behaviours characterize an unstable equilibrium.

(b) Consider an IVP $y' = f(y), y(0) = 0$ and two numerical methods A and B. ○ TRUE ⊗ FALSE

If method A has first order and method B has second order, then – given the same step size – method B will produce a better approximation of $y(1)$.

Solution: The order of a method only gives a bound for the error, not an exact number. We might get (un)lucky in which case a lower-order method gives a better approximation in a special case.

(c) Assume $f(t)$ is continuous for all t. If a solution exists for ○ TRUE ⊗ FALSE

$(t^2 + 1)y'' + f(t)y' + y = 0, y(0) = 3$, then it must be unique.

Solution: This is a second order ODE. Uniqueness requires to also have an initial value $y'(0)$ given.

(d) If A is a square, nilpotent matrix then e^{At} is NOT invertible. ○ TRUE ⊗ FALSE

Solution: e^{At} is always invertible with inverse e^{-At}. Also, plugging in the zero matrix for A, which is certainly nilpotent, is a counterexample.

(e) If $f(t)$ is continuous for all $t \in \mathbb{R}$ then $\mathcal{L}\{f\}(s)$ is continous for all $s \in \mathbb{R}$. ○ TRUE ⊗ FALSE

Solution: A counterexample is given by $f(t) = 1$ and $F(s) = 1/s$.

SECTION II **Justify** your answers. **(30 marks)**

6. Consider the IVP

$$y' + y = \delta(t-7) + g(t) \qquad y(0) = 0 \qquad \text{where} \qquad g(t) = u_3(t)\int_0^{t-3} e^{\tau} \cdot (t-3-\tau)\,d\tau$$

(a) **(4 marks)** Find $G(s) = \mathcal{L}\{g(t)\}$. Simplify as much as possible.

Solution: $g(t)$ is a time-shift by 3 of the convolution

$$\int_0^t e^{\tau} \cdot (t-\tau)\,d\tau = (h * k)(t)$$

for $h(t) = e^t$ and $k(t) = t$.

Using the convolution theorem, we get $\mathcal{L}\{h * k\} = H(s)K(s) = \frac{1}{s-1}\frac{1}{s^2}$.

Using the theorem for time shift, we get $G(s) = e^{-3s}H(s) = e^{-3s}\frac{1}{s-1}\frac{1}{s^2}$.

$G(s) = \mathcal{L}\{g(t)\} = e^{-3s}\frac{1}{s-1}\frac{1}{s^2}$

If you were unable to find an expression for $G(s)$ in part (a), you can just write "$G(s)$" in part (b).

(b) **(4 marks)** Find the Laplace transform $Y(s)$ of the solution to the IVP. Simplify as much as possible. *Note that you are NOT asked to find an expression for $y(t)$.*

Solution: Using the formula for the Laplace transform of derivatives and the Laplace transform of the unit impulse we get

$$sY + Y = e^{-7s} + e^{-3s}\frac{1}{s-1}\frac{1}{s^2}$$

Rearranging this, we get

$$Y = \frac{e^{-7s}}{s+1} + e^{-3s}\frac{1}{s-1}\frac{1}{s^2}\frac{1}{s+1}$$

$Y(s) = \frac{e^{-7s}}{s+1} + e^{-3s}\frac{1}{s-1}\frac{1}{s^2}\frac{1}{s+1}$

(c) **(2 marks)** What is $y(2)$? Explain.

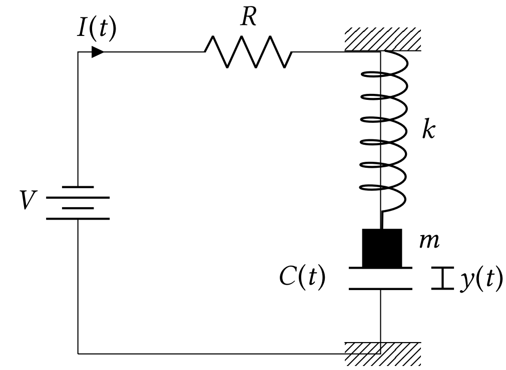
Solution: The solution has initial value $y(0) = 0$. The impulse has no effect before $t = 7$ and $g(t)$, as a time-shifted function, has no effect before $t = 3$. So for $t < 3$ the IVP is equivalent to $y' + y = 0$, $y(0) = 0$. The solution to this IVP is $y = 0$ and therefore $y(2) = 0$.

$y(2) = 0$

7. An electro-mechanical system, like the one found in smartphone accelerometers, is diagrammed below.

Let's first have a look at the *mechanical* part of the system. The mass m is attached to a spring with spring constant k and is able to move relative to the other plate.

The plates are separated by a distance $y(t)$. The equilibrium position of the mass is $y = d$. The mass experiences and external force $f(t)$.



(a) **(3 marks)** Assume the spring satisfies Hooke's law. Use Newton's second law to obtain a second-order ODE for the plate separation $y(t)$.
Solution:

$$my'' + k(y - d) = f(t)$$

Now let's have a look at the *electrical* part of the system. A voltage source V (constant), a resistor R (constant), and a parallel-plate capacitor $C(t)$ are in series.

An electrically conductive mass forms one plate of the parallel-plate capacitor. The capacitance of the parallel-plate capacitor is given as $C(t) = \frac{\kappa}{y(t)}$, for some constant $\kappa > 0$.

(b) **(3 marks)** Use Kirchhoff's voltage law to obtain a first-order ODE for the capacitor voltage, $V_C(t)$.
Solution:

$$V = V_R(t) + V_C(t) = I(t)R + V_C, \quad I(t) = C(t)\frac{dV_C}{dt} \quad \longrightarrow \quad \frac{dV_C}{dt} = \frac{V - V_C}{RC(t)}$$

(c) **(2 marks)** Express $V_C(t)$ in terms of an integral of $y(t)$ assuming the initial condition $V_C(0) = 0$.
Solution: Separating variables

$$\frac{dV_C}{V - V_C} = \frac{y}{R\kappa}dt$$

$$\longrightarrow \quad V_C(t) = V\left(1 - \exp\left(\frac{1}{R\kappa}\int_0^t y(\tau)\, d\tau\right)\right)$$

Finally, consider how the mechanical and the electrical parts of the system *interact*.

(d) **(2 marks)** Explain how this system could be used to electronically detect acceleration.
Solution: As the device is accelerated in the vertical direction, the mass will experience a force $f(t)$. If the parameters V, R, κ, m, k, d are precisely known then the measured voltage V_C, or more easily V_R, is a reflection of the acceleration. To measure an arbitrary acceleration, three devices aligned in independent (mutually perpendicular is best) direction could be used.

8. Inside a thermally conducting rod with an insulated surface and length $L = 1$m, a chemical reaction is occuring. This reaction is exothermic (heat-releasing) at a rate that increases with temperature – an explosive combination. We attempt to cool the rod by placing its ends in ice water at 0°C.

This leads to the model for the rod's temperature $u(x,t)$ in °C at position x along the rod and time t,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku, \quad u(0,t) = 0 = u(1,t), \quad u(x,0) = 20°\text{C}.$$

(a) **(2 marks)** Substitute $u(x,t) = X(x)T(t)$ into the PDE and use $-\lambda$ as the separation constant. After separation, we arrive at the following equation (fill in the blanks).

$$XT' = X''T + kXT \quad \longrightarrow \quad \frac{T'}{T} = \frac{X''}{X} + k = -\lambda$$

(b) **(2 marks)** What ODE does $T(t)$ satisfy? What BVP does $X(x)$ satisfy?

ODE for $T(t)$:	$T' = -\lambda T$
BVP for $X(x)$:	$X'' + (k+\lambda)X = 0, \quad X(0) = 0 = X(1)$

(c) **(2 marks)** For which values of λ does i) $T(t) \longrightarrow \infty$ and ii) $T(t) \longrightarrow 0$ as $t \longrightarrow \infty$?
Solution: From the ODE for $T(t)$, $T' = -\lambda T$, $T \longrightarrow \infty$ as $t \longrightarrow \infty$ if $\lambda < 0$ and $T \longrightarrow 0$ as $t \longrightarrow \infty$ if $\lambda > 0$.

(d) **(2 marks)** Determine the values of λ that give a non-trival solution to the BVP for $X(x)$.
Solution: Let $\mu = k + \lambda$ then the general solution is

$$X(x) = c_1 \sin\left(\sqrt{\mu}x\right) + c_2 \cos\left(\sqrt{\mu}x\right).$$

Imposing $X(0) = 0$ implies $c_2 = 0$. Imposing $X(1) = 0$, a non-trivial solution exists for $\sqrt{\mu} = n\pi, n = 1, 2, 3, \cdots$. Therefore, the values of λ that give a non-trivial solution to the BVP are $\lambda = n^2\pi^2 - k, n = 1, 2, 3, \cdots$.

(e) **(2 marks)** For which values of $k > 0$ does the temperature in the rod go to zero, i.e. $\lim\limits_{t\to\infty} u(x,t) = 0°\text{C}$?
Solution: So long as all of the λ that give non-trivial solutions to the BVP have $\lambda > 0$ then $\lim_{t\to\infty} u(x,t) = 0$. Therefore, we need $1^2\pi^2 - k > 0$ or $k < \pi^2$. The reaction strength k needs to small compared to the diffusivity with value 1, so that the heat generated inside the rod has time to diffuse to the ends and be carried away into the ice water before the reaction runs away.

Page for scratch work or for clearly-labelled overflow from previous pages

Page for scratch work or for clearly-labelled overflow from previous pages

Do not submit the blue pages.

Delete them/do not scan them when preparing for submission.

MAT292 Ordinary Differential Equations – Fall 2020

Final Exam – Dec 14, 7:05 pm - 8:35 pm (+25 min for submission)

Do **NOT** upload the instruction pages with a blue background.

Option 1: Print, Write, Scan

- Print on A4 or Letter paper.
- Handwrite on the printout. Neatness counts.
- You must *not* use additional pages.
- Scan the pages. Do NOT use raw photos. Use a "scanner app".
- It is totally fine if your document has a "watermark" because you used a free app.

Option 2: Write Digitally, Create PDF

- Open this PDF file in a notetaking app on your iPad/Surface/...
- Use a "stylus" (digital pen) to handwrite your answers. Neatness counts.
- You must *not* use additional pages.
- Export your notes as a PDF file.
- It is totally fine if your document has a "watermark" because you used a free app.

Examples Of What Is *Not* Accepted

- Uploading raw photos instead of using a scanner app. If you do not use a scanner app and upload a photo instead, a penalty will be applied. Make sure to use the app correctly so that it actually exports a cropped, high contrast version. Some apps allow you to export the raw photo as well. Do not upload that.
- Typed solutions, e.g. using MS Word, LaTeX, OpenOffice, ... will receive a **mark of zero**.
- Wrongly ordered pages or missing pages. A penalty will be applied if pages are not correctly ordered.
- Using empty sheets of paper instead of the printed template. Any solutions on empty sheets of paper will be ignored and given a **mark of zero**.
- Submission via email. Any email submissions will be ignored.
- Submissions after the deadline. You must stop writing at 8:35 pm and submit the test on gradscope.ca within 25 minutes. Late submissions will not be accepted.

If you have neither a printer nor a tablet for digital writing, you must email fparsch@math.toronto.edu at least three days before the test explaining your situation and asking for instructions. If you receive such instructions directly from him, they supersede the rules on this page.

Asking questions during the test

- Go to the MAT292 Piazza Page
- Make a private (!) post to all instructors. Be brief and concise.
- Continue with your test until you receive a reply.

Just like in a regular term test, it is quite likely that our reply will have to be “sorry we can’t answer that”. Please don’t ask us things like “is it enough if I write...”. It is part of the test to assess your judgement on what constitutes a sufficient solution.

The OK List

This OK list is a closed list, not just a list of examples. If you are unsure of what you are allowed to use, do not hesitate to ask on Piazza.

- OK: Anything that can be found on the MAT292 Quercus page
- OK: Your own (!) previous MAT292 work, including any assignments
- OK: Your own (!) MAT292 lecture notes
- OK: Any Calculus/ODE textbook that you have access to, digitally or printed
- OK: Online learning videos (e.g. the ones published by Khan Academy)
- OK: wolframalpha.com, desmos.com, symbolab.com, any calculator
- OK: Anything declared as OK in a written announcement by the instructors

Anything else is NOT OK

Examples Of Not OK Things

Here are some examples of things that are not OK. These are just *examples*. Unless something is on the OK list, it is not OK.

- NO: Communicating with a MAT292 student between 7:00 pm and 9:00 pm. It is irrelevant what you are communicating about, you **must not** interact in any way, even if you just want to talk about the weather. This obviously includes any kind of group chats. The mere fact that you were seen to be online in any specific messaging app during the test window already indicates an academic offence. To further clarify: if you choose to hand in early, you must wait until 9:00 pm to talk to other students.
- NO: Getting any outside help from anyone, no matter if they are a U of T student or not.
- NO: Posting on or accessing so-called "tutoring websites" like chegg.com.
- NO: Posting in online forums like stackexchange.
- NO: Entering the question text into a search engine.

If you are in any online chat that involves MAT292 students and – after the exam – you observe that someone communicated in that chat between 7:00 pm and 9:00 pm, you must inform us of the fact immediately. Otherwise you – the person who doesn’t tell us – are party to the academic offence according to section B.II.1.(a) of the Code. You will therefore be equally prosecuted should the matter come to our attention.