Laplace Transform Lab: Solving ODEs using Laplace Transform in MATLAB

This lab will teach you to solve ODEs using a built in MATLAB Laplace transform function `laplace`.

There are five (5) exercises in this lab that are to be handed in. Write your solutions in a separate file, including appropriate descriptions in each step.

Include your name and student number in the submitted file.

Student Information

```
Student Name: Robert Purcaru

Student Number: 1007019842
```

Using symbolic variables to define functions

In this exercise we will use symbolic variables and functions.

```
syms t s x y

f = cos(t)
```

f = $\cos(t)$

```
h = exp(2*x)
```

h = e^{2x}

Laplace transform and its inverse

```
% The routine |laplace| computes the Laplace transform of a function

F=laplace(f)
```

F =

$$\frac{s}{s^2+1}$$

By default it uses the variable `s` for the Laplace transform But we can specify which variable we want:

```
H=laplace(h)
```

H =

$$\frac{1}{s-2}$$

```
laplace(h,y)
```

ans =

$$\frac{1}{y-2}$$

```
% Observe that the results are identical: one in the variable |s| and the
% other in the variable |y|
```

We can also specify which variable to use to compute the Laplace transform:

```
j = exp(x*t)
```

j = $e^{t\,x}$

```
laplace(j)
```

ans =

$$\frac{1}{s-x}$$

```
laplace(j,x,s)
```

ans =

$$\frac{1}{s-t}$$

```
% By default, MATLAB assumes that the Laplace transform is to be computed
% using the variable |t|, unless we specify that we should use the variable
% |x|
```

We can also use inline functions with `laplace`. When using inline functions, we always have to specify the variable of the function.

```
l = @(t) t^2+t+1
```

```
l = function_handle with value:
    @(t)t^2+t+1
```

```
laplace(l(t))
```

ans =

$$\frac{s+1}{s^2}+\frac{2}{s^3}$$

MATLAB also has the routine `ilaplace` to compute the inverse Laplace transform

```
ilaplace(F)
```

ans = $\cos(t)$

```
ilaplace(H)
```

ans = e^{2t}

```
ilaplace(laplace(f))
```

ans = $\cos(t)$

If `laplace` cannot compute the Laplace transform, it returns an unevaluated call.

```
g = 1/sqrt(t^2+1)
```

g =

$$\frac{1}{\sqrt{t^2+1}}$$

```
G = laplace(g)
```

G =

$$\mathrm{laplace}\left(\frac{1}{\sqrt{t^2+1}}, t, s\right)$$

But MATLAB "knows" that it is supposed to be a Laplace transform of a function. So if we compute the inverse Laplace transform, we obtain the original function

```
ilaplace(G)
```

ans =

$$\frac{1}{\sqrt{t^2+1}}$$

The Laplace transform of a function is related to the Laplace transform of its derivative:

```
syms g(t)
laplace(diff(g,t),t,s)
```

ans = $s\,\mathrm{laplace}(g(t), t, s) - g(0)$

Exercise 1

Objective: Compute the Laplace transform and use it to show that MATLAB 'knows' some of its properties.

Details:

(a) Define the function `f(t)=exp(2t)*t^3`, and compute its Laplace transform `F(s)`. (b) Find a function `f(t)` such that its Laplace transform is `(s - 1)*(s - 2))/(s*(s + 2)*(s - 3)` (c) Show that MATLAB 'knows' that if `F(s)` is the Laplace transform of `f(t)`, then the Laplace transform of `exp(at)f(t)` is `F(s-a)`

(in your answer, explain part (c) using comments).

Observe that MATLAB splits the rational function automatically when solving the inverse Laplace transform.

(a)

```
clc;
close all;
clear;

syms f t;
f = exp(2*t) * t^3;

F = laplace(f)
```

F =

$$\frac{6}{(s-2)^4}$$

(b)

```
syms g G s;
G = ((s-1)*(s-2))/(s*(s+2)*(s-3));
g = ilaplace(G)
```

g =

$$\frac{6\,e^{-2\,t}}{5}+\frac{2\,e^{3\,t}}{15}-\frac{1}{3}$$

(c)

```
syms F(t) f(t) a t
g(t) = exp(a*t)*f(t);
F=laplace(f(t))
```

F = $\mathrm{laplace}(f(t), t, s)$

```
G=laplace(g(t))
```

G = $\mathrm{laplace}(f(t), t, s-a)$

```
% f(t) is a non specific function with laplace transfrom F(s) as a fucntion
% of (t,s, f(t)). Since g is a function of (a, t f(t)), MATLAB shows that G
% is a function of (f(t), t, s - a) implying that G is a function of
% F(s-a), since the (s) term in the G arguments is replaced by (s - a)
```

Heaviside and Dirac functions

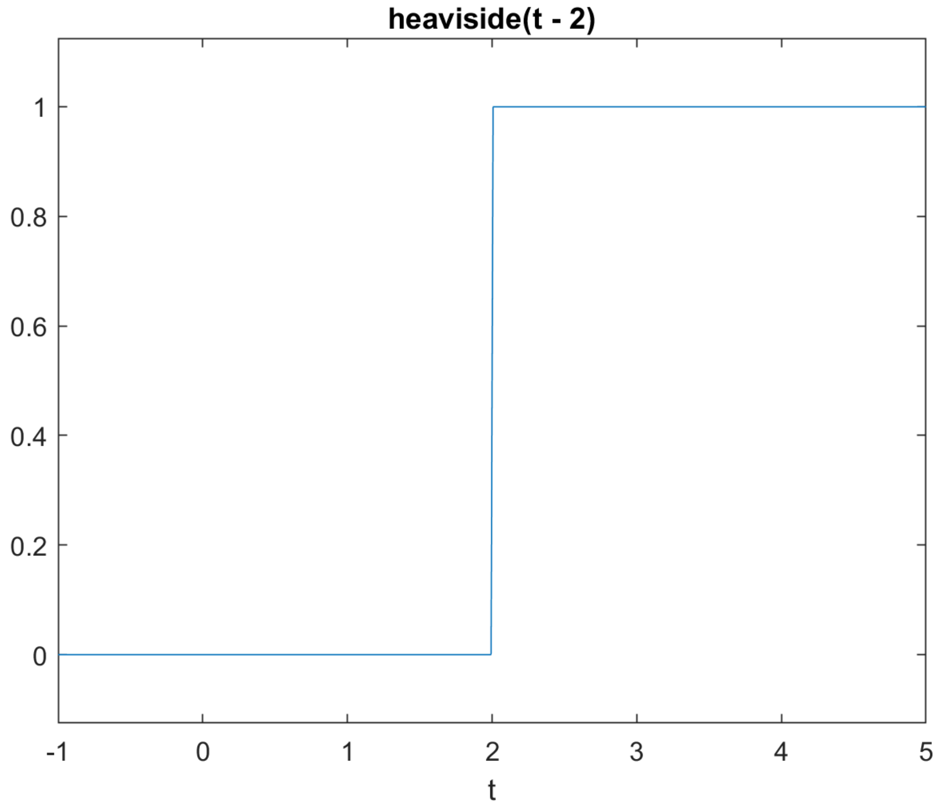
These two functions are builtin to MATLAB: `heaviside` is the Heaviside function `u_0(t)` at `0`

To define `u_2(t)`, we need to write

```
f=heaviside(t-2)
```

f = $\mathrm{heaviside}(t-2)$

```
ezplot(f,[-1,5])
```




```
% The Dirac delta function (at |0|) is also defined with the routine |dirac|

g = dirac(t-3)
```

g = $\delta(t-3)$

```
% MATLAB "knows" how to compute the Laplace transform of these functions

laplace(f)
```

ans =

$$\frac{e^{-2\,s}}{s}$$

```
laplace(g)
```

ans = $e^{-3\,s}$

Exercise 2

Objective: Find a formula comparing the Laplace transform of a translation of `f(t)` by `t-a` with the Laplace transform of `f(t)`

Details:

- Give a value to a
- Let `G(s)` be the Laplace transform of `g(t)=u_a(t)f(t-a)` and `F(s)` is the Laplace transform of `f(t)`, then find a formula relating `G(s)` and `F(s)`

In your answer, explain the 'proof' using comments.

```
clc;
close all;
clear;

syms a s t;

a = 10;

ua(t) = heaviside(t-a);

f(t) = t*exp(t);
F = laplace(f(t))
```

F =

$$\frac{1}{(s-1)^2}$$

```
g(t) = ua(t)*f(t-a);
G = laplace(g(t))
```

G =

$$\frac{e^{-10\,s}}{(s-1)^2}$$

```
% The laplace transform of a function u_a(t)*f(t-a) = exp(-c*s) * F(s)
% where F(s) is the laplace transform of f(t). This is confirmed here as
% the laplace transform of f(t) is F(s) = 1/(s-1)^2 and G(s) is e^(-a*s) * F(s)
```

Solving IVPs using Laplace transforms

Consider the following IVP, `y''-3y = 5t` with the initial conditions `y(0)=1` and `y'(0)=2`. We can use MATLAB to solve this problem using Laplace transforms:

```
% First we define the unknown function and its variable and the Laplace
% tranform of the unknown

syms y(t) t Y s

% Then we define the ODE
```

```
ODE=diff(y(t),t,2)-3*y(t)-5*t == 0
```

ODE =

$$\frac{\partial^2}{\partial t^2}\; y(t) - 5\,t - 3\,y(t) = 0$$

```
% Now we compute the Laplace transform of the ODE.

L_ODE = laplace(ODE)
```

L_ODE =

$$s^2\,\mathrm{laplace}(y(t),t,s) - s\,y(0) - \left(\left(\frac{\partial}{\partial t}\;y(t)\right)\Big|_{t=0}\right) - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t),t,s) = 0$$

```
% Use the initial conditions

L_ODE=subs(L_ODE,y(0),1)
```

L_ODE =

$$s^2\,\mathrm{laplace}(y(t),t,s) - s - \left(\left(\frac{\partial}{\partial t}\;y(t)\right)\Big|_{t=0}\right) - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t),t,s) = 0$$

```
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),2)
```

L_ODE =

$$s^2\,\mathrm{laplace}(y(t),t,s) - s - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t),t,s) - 2 = 0$$

```
% We then need to factor out the Laplace transform of |y(t)|

L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

L_ODE =

$$Y\,s^2 - s - 3\,Y - \frac{5}{s^2} - 2 = 0$$

```
Y=solve(L_ODE,Y)
```

Y =

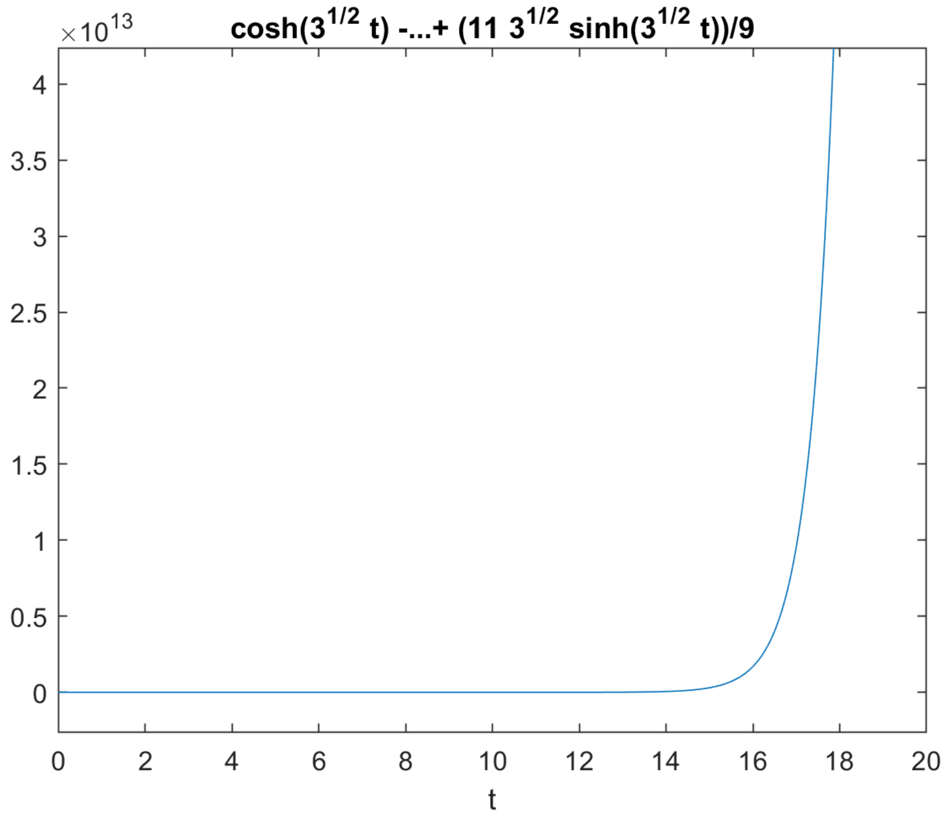
$$\frac{s + \frac{5}{s^2} + 2}{s^2 - 3}$$

```
% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP

y = ilaplace(Y)
```

y =

$\cosh(\sqrt{3}\ t) - \frac{5\,t}{3} + \frac{11\ \sqrt{3}\ \sinh(\sqrt{3}\ t)}{9}$

```
% We can plot the solution

ezplot(y,[0,20])
```




```
% We can check that this is indeed the solution

diff(y,t,2)-3*y
```

`ans =` $5\,t$

Exercise 3

Objective: Solve an IVP using the Laplace transform

Details: Explain your steps using comments

- Solve the IVP
- `y'''+2y''+y'+2*y=-cos(t)`
- `y(0)=0`, `y'(0)=0`, and `y''(0)=0`
- for `t` in `[0,10*pi]`
- Is there an initial condition for which `y` remains bounded as `t` goes to infinity? If so, find it.

```
clc;
close all;
clear;

% Declaring variables to be used
syms y(t) t Y s

% define the ODE

ODE=diff(y(t),t,3) + 2*diff(y(t),t,2) + diff(y(t),t,1) + 2*y(t) + cos(t) == 0
```

ODE =

$$\frac{\partial^3}{\partial t^3}\ y(t) + 2\frac{\partial^2}{\partial t^2}\ y(t) + \frac{\partial}{\partial t}\ y(t) + \cos(t) + 2\ y(t) = 0$$

```
% laplace transform of the ODE.

L_ODE = laplace(ODE)
```

L_ODE =

$$s\,\sigma_1 - y(0) - 2\,s\,y(0) - s\left(\left(\frac{\partial}{\partial t}\ y(t)\right)\Big|_{t=0}\right) + \frac{s}{s^2+1} + 2\,s^2\,\sigma_1 + s^3\,\sigma_1 - 2\left(\left(\frac{\partial}{\partial t}\ y(t)\right)\Big|_{t=0}\right) - s^2\,y(0) - \Bigg(\Bigg($$

where

$$\sigma_1 = \mathrm{laplace}(y(t), t, s)$$

```
% initial conditions

L_ODE=subs(L_ODE,y(0),0)
```

L_ODE =

$$s\,\sigma_1 - s\left(\left(\frac{\partial}{\partial t}\ y(t)\right)\Big|_{t=0}\right) + \frac{s}{s^2+1} + 2\,s^2\,\sigma_1 + s^3\,\sigma_1 - 2\left(\left(\frac{\partial}{\partial t}\ y(t)\right)\Big|_{t=0}\right) - \left(\left(\frac{\partial^2}{\partial t^2}\ y(t)\right)\Big|_{t=0}\right) + 2\,\sigma_1 = 0$$

where

$$\sigma_1 = \mathrm{laplace}(y(t), t, s)$$

```
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),0)
```

L_ODE =

$$s\,\mathrm{laplace}(y(t), t, s) + \frac{s}{s^2+1} + 2\,s^2\,\mathrm{laplace}(y(t), t, s) + s^3\,\mathrm{laplace}(y(t), t, s) - \left(\left(\frac{\partial^2}{\partial t^2}\ y(t)\right)\Big|_{t=0}\right) + 2\,\mathrm{laplace}($$

```
L_ODE=subs(L_ODE,subs(diff(y(t), t, 2), t, 0),0)
```

L_ODE =

$$s\,\mathrm{laplace}(y(t),t,s)+\frac{s}{s^2+1}+2\,s^2\,\mathrm{laplace}(y(t),t,s)+s^3\,\mathrm{laplace}(y(t),t,s)+2\,\mathrm{laplace}(y(t),t,s)=0$$

```
% factor out the Laplace transform of |y(t)|

L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

L_ODE =

$$2\,Y+Y\,s+\frac{s}{s^2+1}+2\,Y\,s^2+Y\,s^3=0$$

```
Y=solve(L_ODE,Y)
```

Y =

$$-\frac{s}{(s^2+1)\,(s^3+2\,s^2+s+2)}$$

```
% Assign y the solution to the inverse laplace transform

y = ilaplace(Y)
```

y =

$$\frac{2\,e^{-2\,t}}{25}-\frac{2\cos(t)}{25}+\frac{3\sin(t)}{50}+\frac{t\cos(t)}{10}-\frac{t\sin(t)}{5}$$

```
% plot the solution

ezplot(y,[0,10*pi])
title("y'''+2y''+y'+2*y=-cos(t), y(0) = 0, y'(0)=0, y''(0)=0");
ylabel('y');
xlabel('t');

% there is no solution that bounds y as t goes to infinty, the function
% oscillates with increasing magnitude regardless of where it starts.
```

Exercise 4

Objective: Solve an IVP using the Laplace transform

Details:

- Define
- `g(t) = 3 if 0 < t < 2`
- `g(t) = t+1 if 2 < t < 5`
- `g(t) = 5 if t > 5`
- Solve the IVP
- `y''+2y'+5y=g(t)`
- `y(0)=2 and y'(0)=1`

- Plot the solution for `t` in `[0,12]` and y in `[0,2.25]`.

In your answer, explain your steps using comments.

```
clc;
close all;
clear;

% Declaring variables to be used
syms y(t) t Y s

% defining g(t) using heaviside functions, each subsequent
% function needs to account for thevalue set by the previous;
% the last term is the sum of previous terms that gives a constant 5
u0(t) = heaviside(t);
u2(t) = heaviside(t-2);
u5(t) = heaviside(t-5);

g(t) = 3*u0(t)+(t-2)*u2(t)+(-t+4)*u5(t);

% define the ODE

ODE = diff(y(t),t,2) + 2*diff(y(t),t) + 5*y(t) - g(t) == 0
```

ODE =

$$\frac{\partial^2}{\partial t^2}\ y(t) + 2\frac{\partial}{\partial t}\ y(t) - 3\,\mathrm{heaviside}(t) + 5\ y(t) - \mathrm{heaviside}(t-2)\ (t-2) + \mathrm{heaviside}(t-5)\ (t-4) = 0$$

```

% laplace transform of the ODE.

L_ODE = laplace(ODE)
```

L_ODE =

$$2\,s\,\sigma_1 - 2\ y(0) - s\ y(0) - \frac{e^{-2\,s}}{s^2} + s^2\,\sigma_1 - \left(\left(\frac{\partial}{\partial t}\ y(t)\right)\Big|_{t=0}\right) - \frac{3}{s} + \frac{e^{-5\,s}\ (s+1)}{s^2} + 5\,\sigma_1 = 0$$

where

$$\sigma_1 = \mathrm{laplace}(y(t), t, s)$$

```

% initial conditions

L_ODE=subs(L_ODE,y(0),2)
```

L_ODE =

$$2\,s\ \mathrm{laplace}(y(t), t, s) - 2\,s - \frac{e^{-2\,s}}{s^2} + s^2\ \mathrm{laplace}(y(t), t, s) - \left(\left(\frac{\partial}{\partial t}\ y(t)\right)\Big|_{t=0}\right) - \frac{3}{s} + \frac{e^{-5\,s}\ (s+1)}{s^2} + 5\ \mathrm{laplace}(y$$

```
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),1)
```

```
L_ODE =
```

$$2\,s\,\mathrm{laplace}(y(t),t,s)-2\,s-\frac{e^{-2\,s}}{s^2}+s^2\,\mathrm{laplace}(y(t),t,s)-\frac{3}{s}+\frac{e^{-5\,s}\,(s+1)}{s^2}+5\,\mathrm{laplace}(y(t),t,s)-5=0$$

```
% factor out the Laplace transform of |y(t)|

L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

```
L_ODE =
```

$$5\,Y-2\,s+2\,Y\,s-\frac{e^{-2\,s}}{s^2}+Y\,s^2-\frac{3}{s}+\frac{e^{-5\,s}\,(s+1)}{s^2}-5=0$$

```
Y=solve(L_ODE,Y)
```

```
Y =
```

$$\frac{2\,s+\frac{e^{-2\,s}}{s^2}+\frac{3}{s}-\frac{e^{-5\,s}\,(s+1)}{s^2}+5}{s^2+2\,s+5}$$

```
% Assign y the solution to the inverse laplace transform

y = ilaplace(Y)
```

```
y =
```

$$\mathrm{heaviside}(t-2)\,\left(\frac{t}{5}+\frac{2\,e^{2-t}\,\left(\cos(2\,t-4)-\frac{3\,\sin(2\,t-4)}{4}\right)}{25}-\frac{12}{25}\right)-\mathrm{heaviside}(t-5)\,\left(\frac{t}{5}+\frac{2\,e^{5-t}\,\left(\sigma_3-\right.}{25}\right.$$

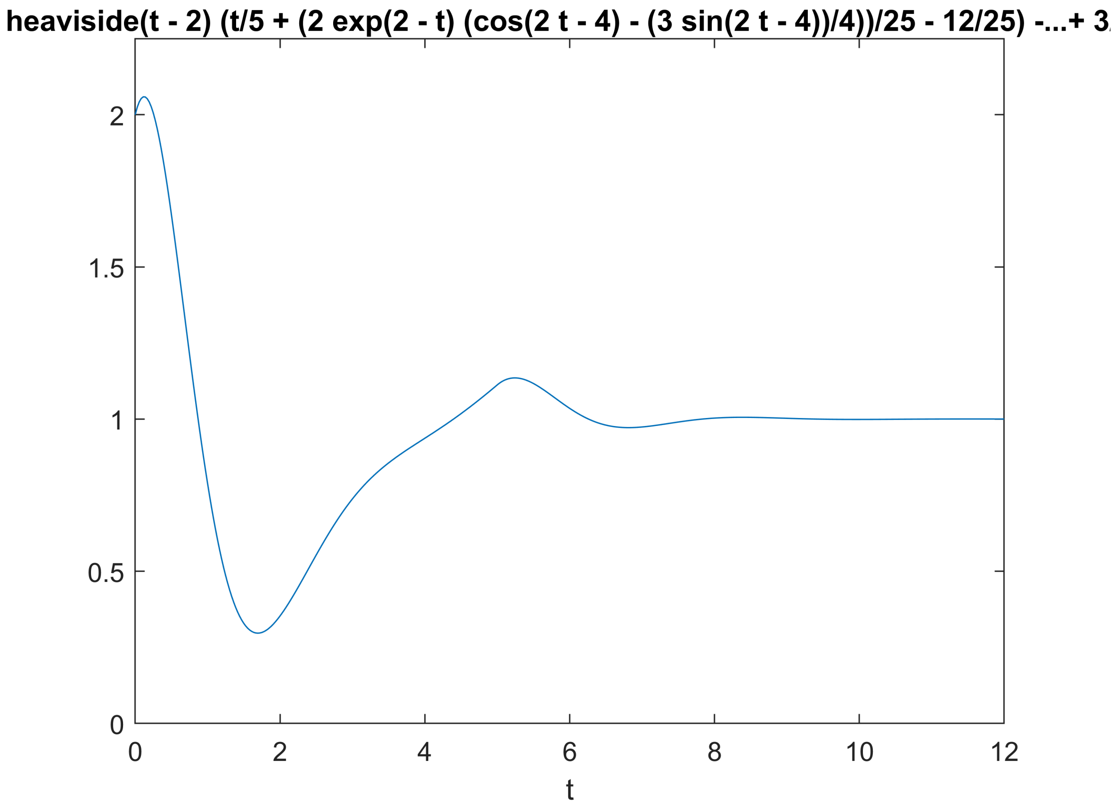
where

$$\sigma_1=\frac{\sin(2\,t)}{2}$$

$$\sigma_2=\sin(2\,t-10)$$

$$\sigma_3=\cos(2\,t-10)$$

```
% plot the solution

ezplot(y, [0,12, 0, 2.25])
```




```
% there is no solution that bounds y as t goes to infinty, the function
% oscillates with increasing magnitude regardless of where it starts.
```

Exercise 5

Objective: Use the Laplace transform to solve an integral equation

Verify that MATLAB knowns about the convolution theorem by explaining why the following transform is computed correctly.

```
clc;
close all;
clear;

syms t tau y(tau) s
I=int(exp(-2*(t-tau))*y(tau),tau,0,t)
```

I =

$$\int_0^t e^{2\,\tau-2\,t}\,y(\tau)\mathrm{d}\tau$$

```
laplace(I,t,s)
```

ans =

$$\frac{\mathrm{laplace}(y(t), t, s)}{s+2}$$

```
% By definition, I is the convolution of e^(2\tau - 2t) and y(\tau)
% so the laplace transfrom of I is the product of the laplace transform
% of y(\tau) and e^(2\tau - 2t). The laplace transfrom of y is given on the
% numerator and the laplace transform of e^(2\tau - 2t) is 1/(s+2).
```