MAT292 - Fall 2018

Term Test 1 - October 25, 2018

Time allotted: 100 minutes   Aids permitted: None

Total marks: 65

Full Name: ____
Last   First

Student Number: ____

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test contains 10 pages (including this title page). Make sure you have all of them.
- You can use pages 9–10 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 9–10.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

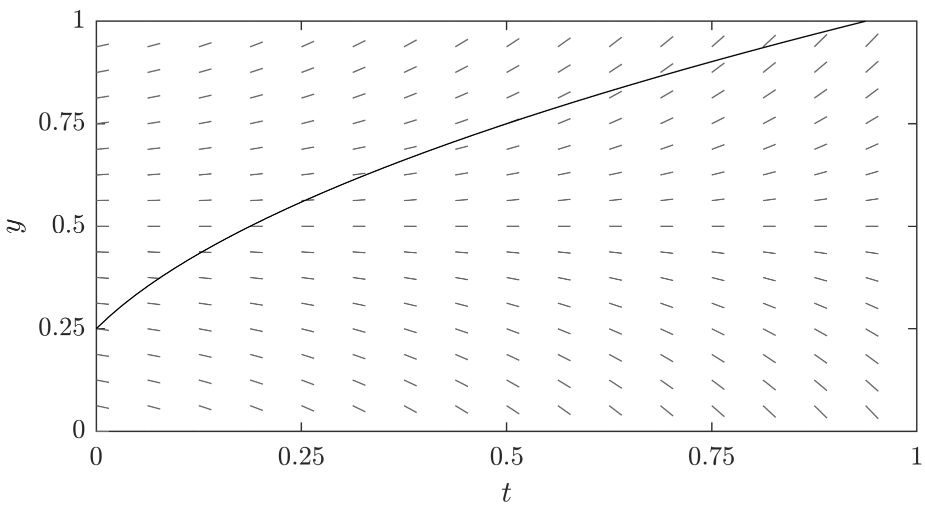
HAVE FUN!

SECTION I No explanation is necessary. (10 marks)

1. **(2 marks)** Give the general solution to differential equation $\frac{dy}{dt} = y$: $y(t) =$ ________________

2. **(1 mark)** Is the function $y(t) = e^t - 1$, a solution to the differential equation $\frac{dy}{dt} = 1 + \max(-2y, y)$? Answer 'yes' or 'no'. ______________

3. **(2 marks)** The direction field for a differential equation is given below. What are the equilibrium solution(s) to this differential equation? ____________________________



4. **(1 mark)** Is the black curve a solution to the differential equation? Answer 'yes' or 'no'. __________

5. **(1 mark)** What is the order of the differential equation $y^4 + (1 - y^2)y''' + \sin(y'') + 1 = 0$? ______________

6. **(1 mark)** Is $e^y y' + e^{-t+y} y = 0$, a linear or a non-linear differential equation? ______________

7. **(1 mark)** Does the initial value problem $y' = \sin(\pi e^{t+y})$, $y(0) = 0$ have a unique solution on $t \in (-\infty, \infty)$? Answer 'yes' or 'no'. ______________

8. **(1 mark)** Give an example of a first order, non-linear, autonomous ordinary differential equation.

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SECTION II **Justify** your answers. **(55 marks)**

9. The solution to the initial value problem $dy/dt = 4/(1+t^2)$, $y(0) = 0$ has the value $y(1) = \pi$. **(5 marks)**

(a) **(2 marks)** Use Euler's method to estimate the value of π with $\Delta t = 1$ and $\Delta t = 0.5$.

(b) **(2 marks)** How small would you have to make Δt to expect to get two correct digits to $\pi \approx 3.14159265358979$?

(c) **(1 mark)** Let π_N be the approximation using $N = 1/\Delta t$ steps of Euler's method. We can extrapolate a more accurate value by combining π_N and π_{2N} as $\hat{\pi}_N := 2\pi_{2N} - \pi_N$. What is $\hat{\pi}_1$?

10. Solve the following initial value problem using the integrating factor method: **(10 marks)**

$$\sin(t)\frac{dy}{dt} + \cos(t)y = \sin(t)\cos(t), \quad y\left(\frac{\pi}{2}\right) = 0.$$

11. Consider the differential equation $2ty + (t^2 - y^2)\frac{dy}{dt} = 0$. **(10 marks)**

(a) **(4 marks)** Find value(s) of c so that $y = ct$ is a solution to the differential equation. Does this contradict *the* existence/uniqueness theorem for the initial condition $y(0) = 0$?

(b) **(6 marks)** Find an implicit solution to the exact differential equation.

12. Consider the differential equation $y' = y^2(y^2 - 1)$. **(10 marks)**

(a) **(1 mark)** Find all equilibrium solutions.

(b) **(3 marks)** Determine which of the equilibrium solutions are stable, unstable, or semi-stable.

(c) **(2 marks)** Let $y_1(t), y_2(t)$ be the solutions with initial conditions $y_1(0) = -1/2,\ y_2(0) = 1/2$. Without writing formulas for these solutions, find the following limits:

i) $\lim_{t\to-\infty} y_1(t)$ ii) $\lim_{t\to\infty} y_1(t)$

iii) $\lim_{t\to-\infty} y_2(t)$ iv) $\lim_{t\to\infty} y_2(t)$

(d) **(2 marks)** Let $y_3(t)$ be the solution with initial condition $y_3(0) = 2$. Explain why the solution $y_3(t)$ is only defined on $-\infty < t < T$ for $T = \int_2^\infty y^{-2}(y^2-1)^{-1}\,dy$.

(e) **(1 mark)** Find the following limits:

i) $\lim_{t\to-\infty} y_3(t)$ ii) $\lim_{t\to T} y_3(t)$

(f) **(1 mark)** For $y > 2$, the function $f(y) = 1/(y^4 - y^2) = y^{-4} \cdot 1/(1 - y^{-2})$ satisfies $y^{-4} < f(y) < \frac{4}{3}y^{-4}$. Show that the number T satisfies $1/24 < T < 1/18$.

13. Consider the differential equation $\frac{dy}{dt} = 3y^{2/3}$. **(10 marks)**

(a) **(3 marks)** Find the general solution for $y > 0$ and for $y < 0$. Show that $y \equiv 0$ is a solution.

(b) **(1 mark)** Find the solution $y_0^+(t)$ in the domain $y > 0$ that satisfies $\lim_{t \to 0^+} y_0^+(t) = 0$.

(c) **(2 marks)** Let $y_a^+(t) = y_0^+(t-a)$ for $t > a$, where y_0^+ is the function defined in **(b)**. Show that $y_a^+(t)$ is the solution for the domain $y > 0$ and that $\lim_{t \to a^+} y_a^+(t) = 0$.

(d) **(4 marks)** Let $y_b^-(t)$ be the solution in the domain $y < 0$ for $t < b$ satisfying $\lim_{t \to b^-} y_b^-(t) = 0$. With reference to $y_a^+(t)$ and $y_b^-(t)$, show that there are an infinite number of solutions to the differential equation satisfying $y(1) = 1$. Why does this not contradict *the* existence/uniqueness theorem?

14. A *well-mixed* tank of constant volume contains bacteria in water. **(10 marks)**

Water is pumped into and out of the tank at equal rates. The concentration $C(t)$ of bacteria in the tank is modelled by the differential equation $C' = \alpha C(C_* - C) + rC_0 - rC$, with non-negative parameters α, C_0, C_*, r.

(a) **(3 marks)** Explain the meaning of each term and parameter in the differential equation.

(b) **(3 marks)** Find the equilibrium solutions and state their stability.

(c) **(4 marks)** Suppose that $C_0 = 0$ and that the system is used to produce the bacteria. Consider α and C_* to be fixed, but r can be varied. What value of r maximizes the equilibrium rate that bacteria is removed from the tank?

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