MAT292 - Fall 2019

Term Test 1 - October 17, 2019

Time allotted: 100 minutes

Aids permitted: None

Total marks: 60

Full Name: ______________________________
Last First

Student Number: ______________________________

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- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers and brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 11–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

Question	Q1-Q6	Q7	Q8	Q9	Q10	Q11	**Total**
Marks	10	10	10	10	10	10	**60**

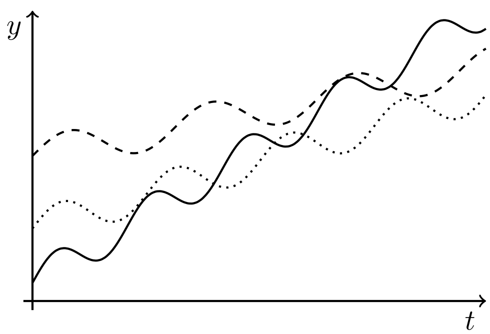
HAVE FUN!

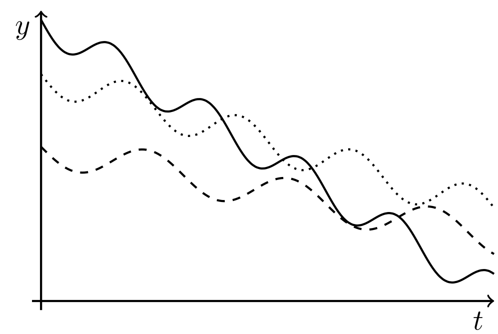
SECTION I Provide the final answer. Justify briefly when asked. **(10 marks)**

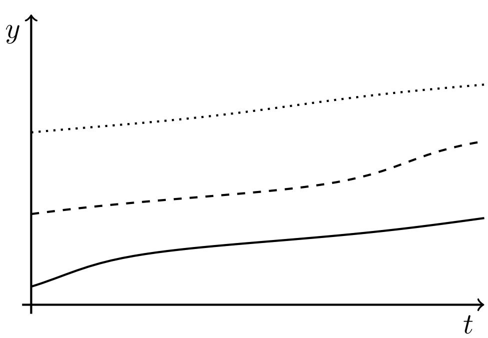
1. **(1 mark)** Give an example of an ODE that has *third* order, is *not* autonomous, and is *not* linear.

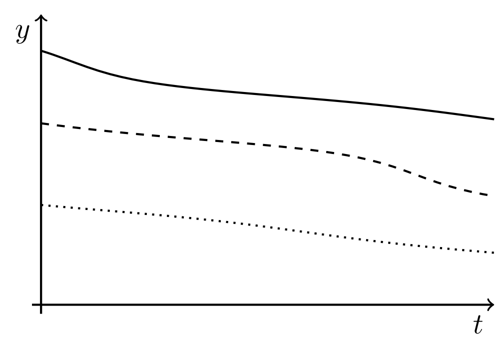
$y''' + y^2 + t = 0$

2. **(1 mark)** Consider the ODE $y' = \frac{1}{2}(\sin y)^2 + \frac{1}{2}e^{\sin y}$. Only one of the following four diagrams shows plots of three solutions to this ODE. Which one? Highlight one of the four circles.

○ 

○ 

⊗ 

○ 

3. **(2 marks)** **Briefly** justify your choice in the previous question (question 2).

y must be an increasing function. Also, the uniqueness theorem applies, so solutions can't cross.

4. **(2 marks)** Consider this statement: Every first-order linear ODE is separable.

Make a choice: ◯ The statement is TRUE. ⊗ The statement is FALSE.

Justify **briefly**:

The ODE $y' + y = t$ is linear but not separable.

5. **(2 marks)** Consider this statement: Every system of ODEs $\vec{x}\,' = A\vec{x}$ has exactly one equilibrium.

Make a choice: ◯ The statement is TRUE. ⊗ The statement is FALSE.

Justify **briefly**:

If $A = 0$, every point is an equilibrium. Even if just one eigenvalue is zero, there are many equilibria.

6. **(2 marks)** Consider the ODE $y' = f(y)$. You are given the following three facts:

(i) The function f is continuous everywhere.

(ii) $f(2) = 3$

(iii) $f(3) = -2$

Make a choice:

⊗ Based on these conditions, this ODE must have at least one stable equilibrium.

◯ Based on these conditions, this ODE must have at least one unstable equilibrium.

◯ We would need more information to decide this.

Justify **briefly**:

At some point, $f(y)$ must change sign from positive to negative, which is where a stable equilibrium is located.

SECTION II **Justify** your answers.

7. **(a)** **(3 marks)** Show that the differential equation $(y^2 - t^4)\, y' - 4t^3 y = 0$ is exact. Then find a one-parameter family of implicit solutions.

Solution: $\partial_y(-4t^3 y) = -4t^3 = \partial_t(y^2 - t^4)$

$\partial_y \phi = y^2 - t^4 \to \phi = y^3/3 - t^4 y + f(t) \to \partial_t \phi = -4t^3 y + f'(t) = -4t^3 y$

Therefore, a one-parameter family of implicit solutions are $y^3 - 3t^4 y = c$.

(b) **(2 marks)** Find all solutions passing through $(1, 1)$.

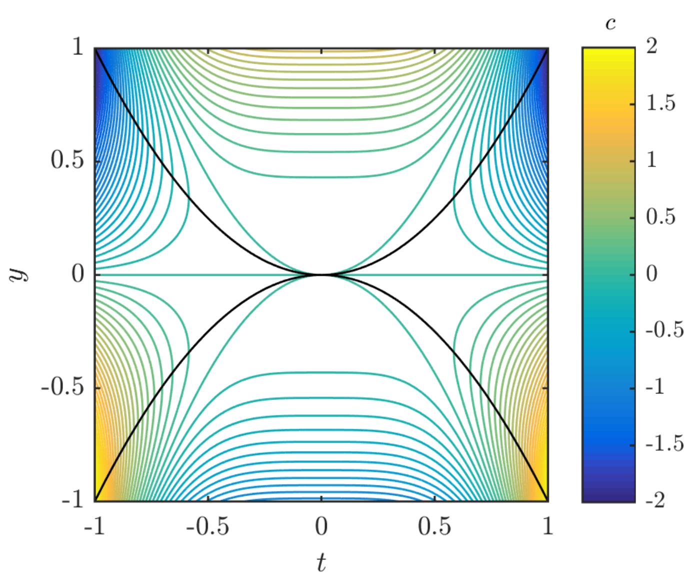
Solution: If $t = 1$ and $y = 1$ then $c = -2$ giving the implicit solution $y^3 - 3t^4 y + 2 = 0$. This is the only solution, see below.

(c) **(2 marks)** Find all solutions passing through $(0, 0)$.

Solution: If $t = 0$ and $y = 0$ then $c = 0$ giving the implicit solution $y^3 - 3t^4 y = 0$. There are three real roots to the cubic in y, $y = 0$ and $y = \pm\sqrt{3}t^2$.

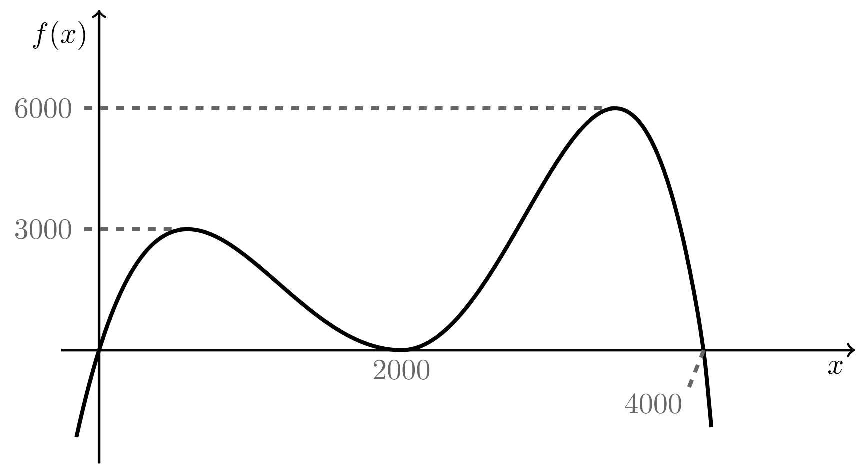
(d) **(3 marks)** Discuss your answers to parts **(b)** and **(c)** in terms of the Existence-Uniqueness Theorem.

Solution: Writing the ODE in the form $\frac{dy}{dt} = \frac{4t^3 y}{y^2 - t^4}$, we see that the right-hand side is not coninuous when $y = \pm t^2$, but is otherwise continuous in t and y and has a continuous derivative in y. The existence-uniqueness theorem says that there is a unique solution in some neighbourhood of $(1, 1)$, but is inconclusive about initial conditions on the curves $y = \pm t^2$. In fact, there are three solutions passing through $(0, 0)$ and two through (t_0, y_0) if $y_0 = \pm t_0^2$, but they are only defined for $t \geqslant t_0$ if $t_0 > 0$ and $t \leqslant t_0$ if $t_0 < 0$. See the contour plot of $y^3 - 3t^4 y$ below, which shows $y = \pm t^2$ in black.



8. Consider a species named *unicornobos* living on a remote island. **(10 marks)**

The population of unicornobos on the island over time is described by $U(t)$ which is governed by the differential equation $U' = f(U)$. The diagram below shows the graph of $f(x)$.



(a) **(4 marks)** The following facts are known about unicornobos:

(i) Unicornobos reproduce at a constant rate, unless they are inhibited by external factors.

(ii) Unicornobos can only survive in the forest. There are two forests on the island. Due to limited resources, each forest can support up to C unicornobos. Whenever there are less than C unicornobos on the island, they all live in one forest together. Unicornobos never move from one forest to the other.

Explain in your own words how these effects can be seen in the plot that is provided and deduce from the plot what the value of C is.

Solution: Between populations of 0 and 4000, U' is always positive. This reflects the fact that the capacity of both forests has not been reached. So there is positive growth from regular reproduction. If there are more than 4000 unicornobos, the population is in decline since the forests are overpopulated.

If however, there are less than 2000 unicornobos, they all live in a single forest and their growth is logistic in nature, since the carrying capacity is given by 2000. It follows that $C = 2000$. On the other hand, if there are more than 2000 unicornobos, they live in two forests, so their growth is logistic in nature with a carrying capacity of 4000.

Note that all of this is justified by the numbers 2000 and 4000 on the x-axis, NOT by the numbers of the $f(x)$-axis.

(b) **(4 marks)** Find all equilibria of the unicornobo population and classify each of them.

Solution: The equilibria are: $U = 0$ unstable, $U = 2000$ semistable, $U = 4000$ stable.

(c) **(2 marks)** Given that $U(0) = 1500$, find $\lim_{t\to\infty} U(t)$. Justify your result.

Solution: In this case, the population grows to the semistable equilibrium, i.e. $\lim_{t\to\infty} U(t) = 2000$.

9. Solve the following initial value problem using the eigenvalue method. **(10 marks)**

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Also sketch the solution in the phase plane.

You do not need to draw to scale, a qualitative plot is enough.

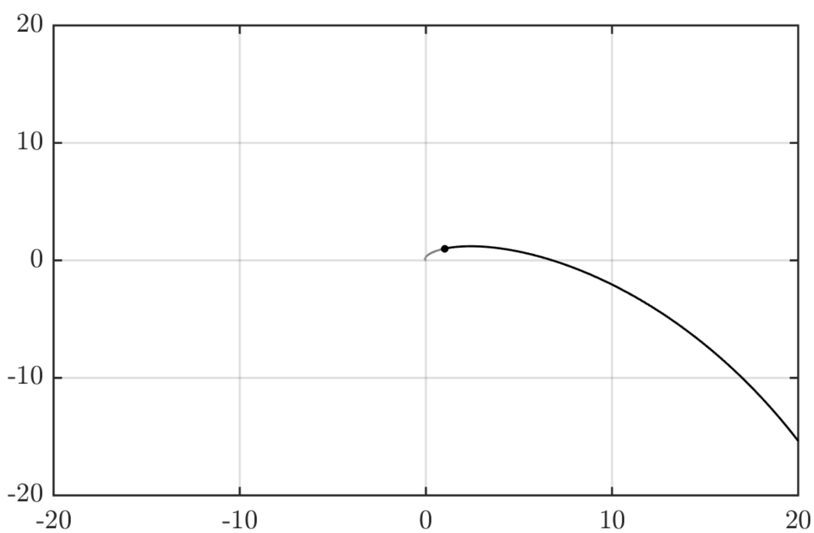
Solution: The characteristic equation is $(2-\lambda)^2 - 1(-1) = \lambda^2 - 4\lambda + 5 = 0$ with roots $\lambda = 2 \pm i$. An eigenvector for $\lambda = 2+i$ is $\begin{pmatrix} 1 \\ i \end{pmatrix}$. The general solution is

$$\vec{x}(t) = C_1 e^{(2+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{(2-i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

The real general solution is

$$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

To solve the initial value problem, we need to find $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ satisfying $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The solution is $\vec{x}(t) = e^{2t} \begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}$.



10. Consider the following linear, homogeneous, constant coefficient system of differential equations that has a parameter $a \in \mathbb{R}$, **(10 marks)**

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad A = \begin{pmatrix} 1+a & 1-a \\ 1-a & 1+a \end{pmatrix}.$$

(a) **(5 marks)** Find the general solution using the eigenvalue method.

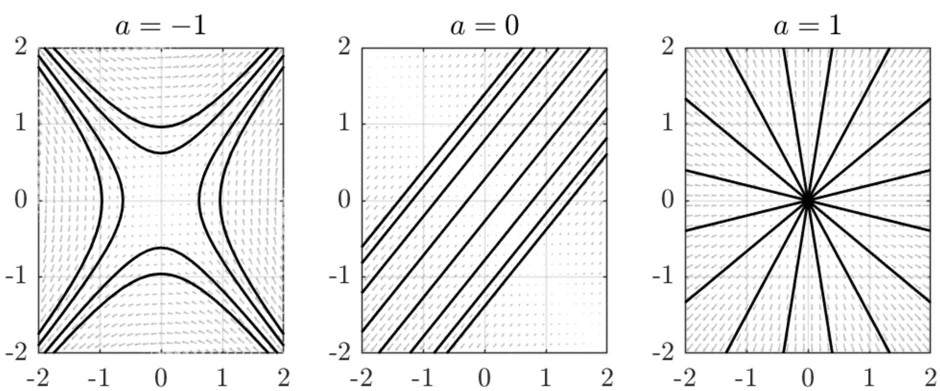
Solution: The characteristic equation is $(1+a-\lambda)^2 - (1-a)^2 = (2a-\lambda)(2-\lambda) = 0$, so the eigenvalues are $\lambda = 2a,\ \lambda = 2$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The general solution is

$$\vec{x}(t) = c_1 e^{2at}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The eigenvalues are linearly independent for all values of a, so the general solution above constains all possible solutions of this linear differential equation. When $a = 0$, all of the points on $x_1 = -x_2$, along the first eigenvectors are equilibrium points. When $a = 1$, the eigenvalues are repeated, but we still have a full set of eigenvectors, which is a basis for $\mathbb{R}^2$.

(b) **(3 marks)** Sketch the phase plane for the following three values of a:

Solution:



(c) **(2 marks)** For which values of a is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a stable equilibrium? Justify your answer.

Solution: The origin is unstable for every $a \in \mathbb{R}$ since (at least) one of the eigenvalues is positive.

11. You are attempting to land a probe on Mars with mass $m = 500$kg. **(10 marks)**

The probe is initially at altitude $h_0 = 2000$m and has an initial downward speed of $v_0 = 500$m/s.

It experiences a drag force $F = \mu v$ proportional to its downward speed. Assume $\mu = 10$Ns/m.

The probe has an engine that provides a constant upward thrust of force u to slow down the descent.

To simplify calculations, assume that the gravitational acceleration on Mars is $g = 4\text{m/s}^2$.

(a) **(2 marks)** Write down a differential equation for the probe's downward speed $v(t)$.

Note: You are being asked to write a differential equation for the speed, NOT for the height.

Solution: $m\frac{dv}{dt} = mg - \mu v - u$

(b) **(1 mark)** What is the probe's terminal velocity if the engine is not used, i.e. if $u = 0$?

Solution: $v_{\text{terminal}} = \frac{mg}{\mu} = 200$m/s.

(c) **(1 mark)** Express the probe's altitude $h(t)$ in terms of its downward speed $v(t)$.

Solution: $\frac{dh}{dt} = -v \to h(t) = h_0 - \int_0^t v(\tau)\, d\tau$.

(d) (2 marks) Solve for $v(t)$.

Solution: Using an integrating factor,

$$\begin{aligned}
\frac{dv}{dt} &= g - \frac{\mu}{m}v - \frac{u}{m} \\
\frac{dv}{dt} + \frac{\mu}{m}v &= g - \frac{u}{m} \\
\frac{d}{dt}\left(e^{\mu t/m}v\right) &= \left(g - \frac{u}{m}\right)e^{\mu t/m} \\
v(t) &= \left(\frac{mg-u}{\mu}\right) + Ce^{-\mu t/m}.
\end{aligned}$$

Imposing $v(0) = v_0$, the solution is

$$v(t) = \left(\frac{mg-u}{\mu}\right) + \left(v_0 - \frac{mg-u}{\mu}\right)e^{-\mu t/m}.$$

(e) (1 mark) Find u so that $\lim_{t\to\infty} v(t) = 0$.

Solution: Either from imposing a terminal speed of zero in the differential equation or directly from the solution, $u = mg$.

(f) (1 mark) Using the value of u found in part **(e)**, find $h(t)$.

Solution: Integrating and applying the initial condition $h(0) = h_0$ gives

$$\begin{aligned}
h(t) &= -\left(\frac{mg-u}{\mu}\right)t - \left(v_0 - \frac{mg-u}{\mu}\right)\left(-\frac{m}{\mu}\right)e^{-\mu t/m} + K \\
&= \left(\frac{u-mg}{\mu}\right)t + \left(v_0 + \frac{u-mg}{\mu}\right)\frac{m}{\mu}e^{-\mu t/m} + h_0 - \left(v_0 + \frac{u-mg}{\mu}\right)\frac{m}{\mu} \\
&= \frac{v_0 m}{\mu}e^{-\mu t/m} + h_0 - \frac{v_0 m}{\mu} \quad \text{using } u = mg.
\end{aligned}$$

(g) (2 marks) Using the value of u found in part **(e)**, find the landing time T so that $h(T) = 0$. What is $v(T)$?

Solution: At $t = T$,

$$h(T) = \frac{v_0 m}{\mu}\left(e^{-\mu T/m} - 1\right) + h_0 = 0.$$

Solving gives

$$T = \frac{m}{\mu}\log\left(\frac{v_0 m}{v_0 m - h_0\mu}\right) = 50\log(50/46),$$

or 4.2 seconds. Note that $v_0 m \geqslant h_0\mu$ so that the probe reaches the ground. The speed at landing is $v(T) = v_0 e^{-\mu T/m} = v_0 - \frac{h_0\mu}{m} = 460\text{m/s}$, which is way too fast.

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