MAT292 - Calculus III - Fall 2016

Term Test 2 - November 17, 2016

Time allotted: 70 minutes | Aids permitted: None

Total marks: 50

Full Name: Solution

Last | First

Student Number:

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 11–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

GOOD LUCK!

PART I No explanation is necessary. **(10 marks)**

1. **(1 mark)** If the origin of the system $\vec{x}'(t) = \begin{pmatrix} a & 2 \\ -2 & 1 \end{pmatrix} \vec{x}(t)$ is a centre, then $a =$ $\underline{-1}$.

$(a-\lambda)(1-\lambda)+4=0 \iff \lambda^2-(1+a)\lambda+a+4=0$

$\iff \lambda = \dfrac{1+a \pm \sqrt{(1+a)^2-4(a+4)}}{2}$

Centre if λ is complex with real part $=0$:

$1+a=0 \iff a=-1$

2. **(2 marks)** If y_1 and y_2 form a fundamental set of solutions to $y''+t^2y'+7y=0$, and $W[y_1,y_2](0)=2$, then

$W[y_1,y_2](t) =$ $\underline{2e^{-t^3/3}}$.

Abel's Thm. $W[y_1,y_2](t) = Ce^{-\int p(t)dt} = Ce^{-\int t^2 dt}$

$= Ce^{-t^3/3}$

$W[y_1,y_2](0) = C = 2$

3. **(2 marks)** If $y(t)=t^r$ solves $t^2y''(t)-4ty'(t)+4y(t)=0$ for $t>0$, then $r =$ $\underline{1 \text{ or } r=4}$.

$y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$

$r(r-1)t^r - 4rt^r + 4t^r = 0 \iff r^2-r-4r+4=0$

$\iff r^2-5r+4=0 \iff r = \dfrac{5 \pm \sqrt{5^2-4^2}}{2}$

$\iff r = \dfrac{5 \pm 3}{2}$

Continued...

4. (3 marks) Consider the ODE

$$y^{(6)} - 5y^{(5)} + 11y^{(4)} - 37y^{(3)} + 32y'' + 10y' = te^t + e^t\cos(2t) + t\sin(3t) + t^2.$$

Hint. $r^6 - 5r^5 + 11r^4 - 37r^3 + 32r^2 + 10r = r(r-1)^3\big((r-1)^2+9\big)$.

Then its particular solution will have the following terms: **(Circle all correct options)**

(a) A	**(e)** Et^2e^t	**(i)** $Ie^t\cos(2t)$	**(m)** $M\cos(3t)$	**(q)** $Qe^t\cos(3t)$
(b) Bt	**(f)** Ft^3e^t	**(j)** $Je^t\sin(2t)$	**(n)** $N\sin(3t)$	**(r)** $Re^t\sin(3t)$
(c) Ct^2	**(g)** Gt^4e^t	**(k)** $Kte^t\cos(2t)$	**(o)** $Ot\cos(3t)$	**(s)** $Ste^t\cos(3t)$
(d) Dt^3	**(h)** Ht^5e^t	**(l)** $Lte^t\sin(2t)$	**(p)** $Pt\sin(3t)$	**(t)** $Tte^t\sin(3t)$

where $A, B, \ldots, T \neq 0$.

5. (1 mark) Consider the system of two first-order linear differential equations

$$\vec{x}' = A\vec{x},$$

where the matrix A has the eigenvalues $\lambda = \alpha \pm i\beta$ with $\alpha \leqslant 0$ and $\beta > 0$.

(Circle all possible options)

(a) $\lim_{t\to+\infty} \vec{x}(t) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$.

(b) $\lim_{t\to+\infty} \|\vec{x}(t)\| = +\infty$.

(c) $\vec{x}(t)$ keeps orbiting $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ forever, so the limit as $t \to +\infty$ doesn't exist.

6. (1 mark) Consider the second-order differential equation $y'' + y = g(t)$. Give an example of a **bounded function $g(t)$** such that $\big|y(t)\big|$ **is not bounded**.

$g(t) =$ sin(t) or cos(t) or 3 sin(t) - 5cos(t).

Continued...

PART II **Justify** your answers.

7. Consider the following system of differential equations. **(13 marks)**

$$\vec{x}\,'(t) = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \vec{x}(t)$$

(a) **(6 marks)** Find the solution to the system that satisfies the initial condition

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues: $(7-r)(3-r)+4=0 \iff r^2-10r+25=0$

$\iff r = 5 \pm \sqrt{5^2-25} = 5$

Eigenvectors: $\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \xi_2 = -2\xi_1 \iff \vec{\xi} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Generalized Eigenvector: $\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \iff v_2 = 1-2v_1$

$\iff \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General Solution: $\boxed{\vec{x}(t) = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{5t} + c_2\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}t + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]e^{5t}}$

Initial Condition: $\begin{cases} c_1 + c_2 = 1 \\ -2c_1 - c_2 = 0 \end{cases} \iff \begin{cases} c_1 - 2c_1 = 1 \\ c_2 = -2c_1 \end{cases} \iff \begin{cases} c_1 = -1 \\ c_2 = 2 \end{cases}$

Solution: $\boxed{\vec{x}(t) = \begin{pmatrix} 1+2t \\ -4t \end{pmatrix}e^{5t}}$

Continued...

(b) **(3 marks)** Find the solution to the system that satisfies the initial condition

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We only need to find the constants c_1, c_2 again:

$$\begin{cases} c_1 + c_2 = 0 \\ -2c_1 - c_2 = 1 \end{cases} \iff \begin{cases} c_2 = -c_1 \\ -2c_1 + c_1 = 1 \end{cases} \iff \begin{cases} c_2 = 1 \\ c_1 = -1 \end{cases}$$

Solution

$$\boxed{\vec{x}(t) = \begin{pmatrix} t \\ 1-2t \end{pmatrix} e^{5t}}$$

(c) **(4 marks)** Find the special fundamental matrix.

The special fundamental matrix is constructed by using the solutions of (a) and (b) as columns:

$$\boxed{\Phi(t) = \begin{pmatrix} 1+2t & t \\ -4t & 1-2t \end{pmatrix} e^{5t}}$$

8. Consider the following initial-value problem. **(13 marks)**

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

where

$$p(t) = \begin{cases} 2 & \text{if } t < 1 \\ 0 & \text{if } t \geqslant 1 \end{cases}, \quad q(t) = \begin{cases} 1 & \text{if } t < 1 \\ 0 & \text{if } t \geqslant 1 \end{cases}, \quad g(t) = \begin{cases} 9e^{2t} & \text{if } t < 1 \\ 0 & \text{if } t \geqslant 1 \end{cases}$$

Find the solution $y(t)$ for all $t \geqslant 0$.

We need to solve the problem separately for $t<1$ and $t\geq 1$.

$\boxed{t<1}$
$$\begin{cases} y'' + 2y' + y = 9e^{2t} \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

step 1. Characteristic Equation is $r^2+2r+1=0 \iff (r+1)^2=0$

So $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$.

step 2. The particular solution is $y_p(t) = Ae^{2t}$, so

$$4Ae^{2t} + 4Ae^{2t} + Ae^{2t} = 9e^{2t} \iff 9A = 9 \iff A = 1$$

So $y_p(t) = e^{2t}$

step 3. We now have $y(t) = c_1 e^{-t} + c_2 t e^{-t} + e^{2t}$ for $t<1$.

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To be used for the answer to question **8.**

Step 4. Using the initial conditions :

$$y(0) = c_1 + 1 = 1 \iff c_1 = 0$$
$$y'(0) = c_2 + 2 = 2 \iff c_2 = 0$$

So $\boxed{y(t) = e^{2t} \quad \text{for } t<1}$

$\boxed{t \geq 1}$ $\begin{cases} y'' = 0 \\ y(1) = e^2 \\ y'(1) = 2e^2 \end{cases}$ > from the solution for $t<1$, because the solution and its derivative have to be continuous.

$$\Rightarrow y(t) = c_3 + c_4 t$$

Using the initial conditions : $\begin{cases} y(1) = c_3 + c_4 = e^2 \\ y'(1) = c_4 = 2e^2 \end{cases} \Rightarrow c_3 = -e^2$

So $\boxed{y(t) = e^2(2t-1) \quad \text{for } t \geq 1}$

The solution is

$$\boxed{y(t) = \begin{cases} e^{2t} & \text{if } t<1 \\ e^2(2t-1) & \text{if } t \geq 1 \end{cases}}$$

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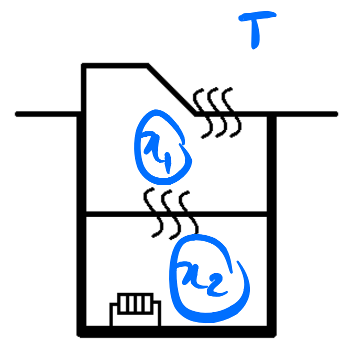
9. Consider an underground facility consisting of two floors one on top of the other with temperatures $x_1(t)$ and $x_2(t)$. **(14 marks)**



1
- Heat is transferred between the floors at a rate equal to the difference of the temperatures.

2
- Heat is transferred from the outside to the upper room at a rate proportional to the difference between the outside temperature T and the temperature of the upper floor, with proportionality constant $P > 0$.

3
- Finally, there is an adjustable temperature regulator heating (or cooling) the lower floor at a constant rate H.

Hint. You can answer all the following questions without finding the solution of the system.

(a) **(3 marks)** What system of first order linear differential equations does the vector $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ satisfy?

$$\begin{cases} x_1' = \overbrace{\boxed{x_2 - x_1}}^{1} + \overbrace{\boxed{P(T - x_1)}}^{2} \\ x_2' = \boxed{x_1 - x_2} + \boxed{H} \ 3 \end{cases}$$

So $$\vec{x}' = \begin{pmatrix} -1-P & 1 \\ 1 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} PT \\ H \end{pmatrix}$$

Continued...

(b) **(3 marks)** Find the equilibrium solution $\vec{x}_e = \begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix}$ of the system from **(a)** .

The equilibrium solution satisfies $\vec{x_e}' = \vec{0}$:

$$\begin{cases} (-1-P)x_e^1 + x_e^2 + PT = 0 \\ x_e^1 - x_e^2 + H = 0 \end{cases} \iff \begin{cases} (\cancel{-1}-P+\cancel{1})x_e^1 = -PT-H \\ x_e^2 = x_e^1 + H \end{cases}$$

$$\iff \begin{cases} x_1^e = \dfrac{PT+H}{P} = T + \dfrac{H}{P} \\ x_2^e = T + H + \dfrac{H}{P} \end{cases} \iff \boxed{\vec{x_e} = \begin{cases} T + H/P \\ T + H/P + H \end{cases}}$$

(c) **(3 marks)** Let $u_1(t) = x_1(t) - x_1^e$ and $u_2(t) = x_2(t) - x_2^e$.

What first order linear system of differential equations does $\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ solve? Justify.

First observe that $A\vec{x_e} + \vec{b} = \vec{0}$ where $A = \begin{pmatrix} -1-P & 1 \\ 1 & -1 \end{pmatrix}$

$\vec{b} = \begin{pmatrix} PT \\ H \end{pmatrix}$

So

$$\vec{u}' = \vec{x}' = A\vec{x} + \vec{b} = A(\vec{u} + \vec{x_e}) + \vec{b}$$

$$= A\vec{u} + \underbrace{A\vec{x_e} + \vec{b}}_{=\vec{0}} = A\vec{u}$$

So $\boxed{\vec{u}' = \begin{pmatrix} -1-P & 1 \\ 1 & -1 \end{pmatrix} \vec{u}}$

Continued...

(d) **(2 marks)** For what values of $P > 0$ is $\binom{0}{0}$ a stable critical point of the system from **(c)** ? Justify.

Hint: $(a+b)^{1/2} < a^{1/2} + b^{1/2}$ for $a, b > 0$.

The equilibrium is stable if the real part of all eigenvalues is negative or zero.

Eigenvalues: $(1+P+r)(1+r) - 1 = 0 \iff r^2 + (2+P)r + P = 0$

$$\iff r = \frac{-(2+P) \pm \sqrt{(2+P)^2 - 4P}}{2} = \frac{-(2+P) \pm \sqrt{4+P^2}}{2}$$

$4+P^2 > 0$

The eigenvalues are real numbers, so they both must be negative for a stable equil.:

Hint

$$\lambda_- < \lambda_+ = \frac{-(2+P) + \sqrt{4+P^2}}{2} < \frac{-2-P+2+P}{2} = 0$$

Both eigenvalues are always negative, so $\binom{0}{0}$ is stable for all $P > 0$.

(e) **(3 marks)** If $H = 0$ (i.e. the temperature regulator is off) and for the values of $P > 0$ found in **(d)** , what is $\lim_{t\to\infty} \vec{x}(t)$? Does this make physical sense? Explain why or why not.

Since the equilibrium is asymptotically stable, then

$$\lim_{t\to\infty} \vec{x}(t) = \vec{x}_e = \begin{pmatrix} T \\ T \end{pmatrix} \quad (\text{for } H=0)$$

This makes physical sense. There is no heater on the bottom floor, so the temperature exchange with the exterior will slowly (or quickly) bring the temperatures inside to match the exterior temperature T.

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