MAT292 - Fall 2018

Term Test 2 - November 12, 2018

Time allotted: 100 minutes

Aids permitted: None

Total marks: 65

Full Name: ______________________________
Last   First

Student Number: ______________________________

Email: ______________________________ @mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test contains 10 pages (including this title page). Make sure you have all of them.
- You can use pages 9–10 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 9–10.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

HAVE FUN!

SECTION I No explanation is necessary. **(10 marks)**

1. **(4 marks)** Each curve in the phase plane is a solution to which differential equation $\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}$?

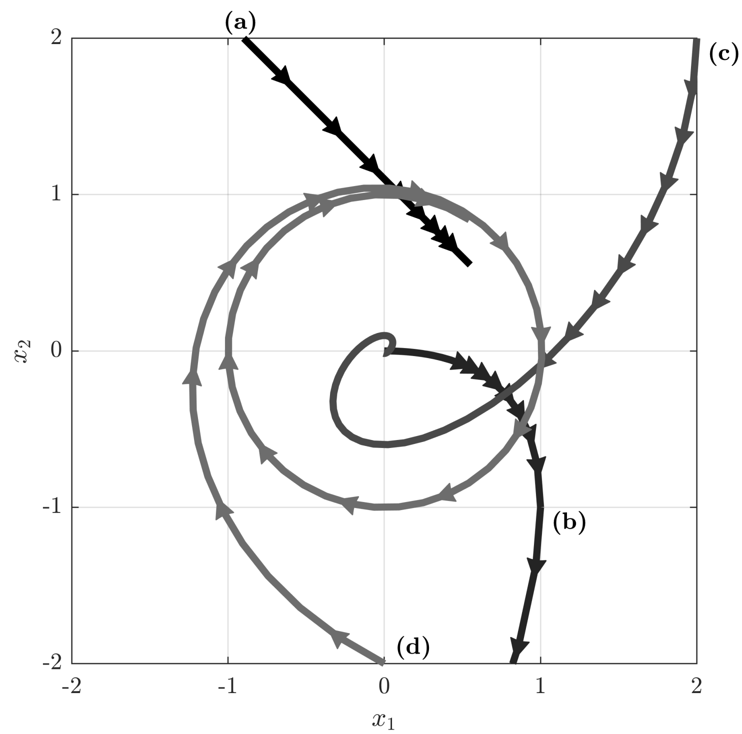
i) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

ii) $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

iii) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

iv) $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

v) none of the above



(a) ________

(b) ________

(c) ________

(d) ________

2. **(2 marks)** Let y_1 and y_2 be solutions to $y''(t) + 7y = 0$ with initial values $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$. Compute the Wronskian of $y_1(t)$ and $y_2(t)$: $W[y_1, y_2](t) =$ ________________

3. **(1 mark)** Find γ so that $y = cte^{2t}$ (for some constant c) is a solution to $y'' - 3y' + \gamma y = e^{2t}$.
$\gamma =$ ________________

4. **(2 marks)** Find a and b so that $\boldsymbol{x}(t) = e^{at}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution of the system $\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 1 & 2b \\ 2 & 3b \end{pmatrix}\boldsymbol{x}$.

$a =$ ____________ $b =$ ____________

5. **(1 mark)** For $\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\boldsymbol{x}$, $\boldsymbol{x}_0 = \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find $\boldsymbol{x}_1$, the result of applying Euler's method with step size $h = 1$.

$\boldsymbol{x}_1 =$ ________________

SECTION II **Justify** your answers. **(55 marks)**

6. Solve the following initial value problem with two different methods, **(10 marks)**

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \frac{1}{3}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}, \quad x_1(0) = 1,\ x_2(0) = 2.$$

(a) **(5 marks)** The eigenvalue method.

(b) **(5 marks)** Let $z_1 = x_1 + x_2$ and $z_2 = x_1 - 2x_2$. Solve separate differential equations for z_1 and z_2 and then determine $x_1 = (2z_1 + z_2)/3$ and $x_2 = (z_1 - z_2)/3$.

7. Show that for two solutions $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ to the system of differential equations $\frac{d\boldsymbol{x}}{dt} = P(t)\boldsymbol{x}$, that $\boldsymbol{x}_1(t) + \boldsymbol{x}_2(t)$ is a solution. Is it necessary that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be linearly independent? **(5 marks)**

8. Consider the differential equation $y'' - 2\alpha y' + (\alpha^2 - \alpha + 1)y = 0$ with parameter $\alpha \in \mathbb{R}$. **(10 marks)**

(a) **(2 marks)** For which values of α are solutions (except $y \equiv 0$) i) growing amplitude oscillations, and ii) decaying amplitude oscillations.

(b) **(3 marks)** For all α, find the general real solution. Consider the cases of distinct real, repeated real, and complex conjugate pairs of eigenvalues separately.

(c) **(5 marks)** Using the method of undetermined coefficients **for all** α find a particular solution of $y'' - 2\alpha y' + (\alpha^2 - \alpha + 1)y = e^t$.

9. If a solution y_1 is known for the differential equation $y'' + p(t)y' + q(t)y = 0$, **(10 marks)**
then one can find the general solution using the Wronskian $W[y_1, y_2] = y_1 y_2' - y_2 y_1'$ as follows.

(a) **(2 marks)** Show that $\left(\frac{y_2}{y_1}\right)' = \frac{W[y_1,y_2]}{y_1^2}$.

(b) **(2 marks)** Show that $W[y_1, y_2]$ satisfies $W' + p(t)W = 0$ and therefore $W[y_1, y_2] = c_1 \exp\left(-\int_0^t p(\tau)d\tau\right)$.

(c) **(6 marks)** Check that $y_1 = t^2$ is a solution of $t^2 y'' - 2y = 0$. Use the Wronskian to find a second linearly independent solution. What is the general solution?

10. Consider the system of equations $\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. **(10 marks)**

(a) **(4 marks)** Find the eigenvalues of A. What are the equilibrium solution(s)? Are they stable?

(b) **(4 marks)** Find solutions $\boldsymbol{x}_1(t) = \begin{pmatrix} a_{1,1}(t) \\ a_{2,1}(t) \end{pmatrix}$ and $\boldsymbol{x}_2(t) = \begin{pmatrix} a_{1,2}(t) \\ a_{2,2}(t) \end{pmatrix}$ with initial conditions $\boldsymbol{x}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{x}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) **(2 marks)** Explain why the solution $\boldsymbol{x}(t)$ with the initial data $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is equal $B(t)\boldsymbol{x}_0$, where $B(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$. (Note that the matrix $B(t)$ is called $\exp(At)$).

11. Aerial refueling is a dangerous procedure in which the receiver aircraft **(10 marks)** approaches a tanker aircraft from below. The receiver aircraft has altitude $h(t)$ beginning at $h(0) = h_0$ and no vertical speed $h'(0) = 0$. The time until 'docking' is T. The altitude $h(t)$ is modelled with the differential equation

$$mh'' = U\left(1 - \frac{t}{T}\right) - \gamma h' - mg.$$

(a) **(4 marks)** Explain the meaning of each term in the differential equation.

(b) **(6 marks)** For a 'soft-landing', $h'(T) = 0$. In some units, $m = 1$, $\gamma = 2$, $g = 1$, and $T = 5$. Find U so that there is a soft-landing. How far apart were the aircraft initially?

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