MAT292 - Fall 2020

Final Exam - December 14, 2020

Time allotted: 90 minutes
Total marks: 48

Aids permitted: see "OK list"

Full Name: ______________________________
Last First

Student Number: ______________________________

Email: ______________________________ @mail.utoronto.ca

Do not forget to fill in the integrity statement on the second page!

- In the first section, only answers and sometimes brief justifications are required.
- In the second section, justify your answers fully.
- This exam contains 9 pages (including this title page). Make sure you have all of them.
- You can use pages 8–9 for rough work or to complete a question (**Mark clearly**).
- Make sure to follow this timeline:
  - 7:05 pm – Start writing the exam.
  - 8:35 pm – Stop writing the exam; fill in the integrity page (second page).
    You MUST stop writing at 8:35 pm.
    The last 25 minutes are for submission, not for writing.
  - 9:00 pm – Upload deadline.
    No extensions will be given.

Question	Q1-4	Q5	Q6	Q7	Q8	**Total**
Marks	8	10	10	10	10	**48**

HAVE FUN!

This is the ONLY page you can fill in *after* 8:35 pm.

If you don't complete and sign this page, you will receive a grade of zero for the entire test.

We at U of T want you to feel proud of what you accomplish as a student. Please respect all of the hard work you're doing this term by making sure that the work you do is your own.

We don't expect you to score perfectly on the assessments and there will be some things that you may not know. Using an unauthorized resource or asking someone else for the answer robs you of the chance later to feel proud of how well you did because you'll know that it wasn't really your work that got you there.

Success in university isn't about getting a certain mark, it's about becoming the very best person you can by enriching yourself with knowledge, strengthening yourself with skills, and building a healthy self-esteem based on how much you've grown and achieved. No one assessment captures that but your conscience will stay with you forever.

Make yourself and your loved ones proud of the student that you are by conducting yourself honestly at all times. Hold each other accountable to these standards.

In submitting this assessment ...	**Short sentences**
... I confirm that my conduct regarding this test adheres to the Code of Behaviour on Academic Matters.	I know the Code.
... I confirm that I have not acted in such a way that would constitute cheating, misrepresentation, or unfairness, including but not limited to, using unauthorized aids and assistance, impersonating another person, and committing plagiarism.	I didn't cheat.
... I confirm that the work I am submitting in my name is the work of no one but myself.	This is only my work.
... I confirm that all pages have been handwritten by myself.	I wrote all pages.
... I confirm that I have not received help from others, whether directly or indirectly.	I didn't receive help.
... I confirm that I have not provided help to others, whether directly or indirectly.	I didn't provide help.
... I confirm that I have only used the aids marked as "OK" on the list.	I only used "OK" aids.
... I am aware that not disclosing another student's misconduct despite my knowledge is an academic offence.	I know I must report cheating.

In this box, handwrite the sequence of short sentences (starting with "I know the Code. I didn't cheat...").

Your student number

Your signature

Submission date

SECTION I Provide the final answer. **Justification required only in questions 4 and 5. (18 marks)**

1. **(2 marks)** State the inverse Laplace transform of $\dfrac{1}{s(s+1)(s^2+1)}$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s^2+1)}\right\} =$

2. **(2 marks)** The Fourier series for $f(x) = |x|, \ -\pi \le x \le \pi$ is given by: $A_0 - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}$

First, determine the value of the constant term: $A_0 =$

Then use the result to determine a value for this sum: $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots =$

3. **(2 marks)** Consider the following IVP

$$y' = \begin{cases} -y & y \ge 0 \\ 0 & y < 0 \end{cases} \qquad y(0) = 1$$

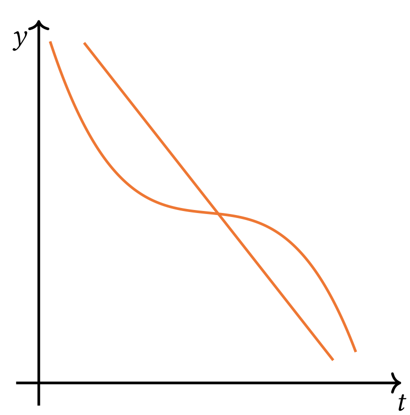
Euler's method gives approximate values $y_n \approx y(n \cdot \Delta t)$ to this IVP. Consider the following two cases.

If $\Delta t < 1$, then $\lim\limits_{n\to\infty} y_n =$

If $\Delta t > 1$, then $\lim\limits_{n\to\infty} y_n =$

4. **(2 marks)** Why can the plot on the right NOT be a plot of solutions to this ODE?

$$y' + e^{t^2} y = 0$$



Reason:

5. **(10 marks)** For each of the following statements, decide if it is true or false. Then justify your choice.

Remember: A statement is only true if it is always true. If a statement only works in special cases/under certain circumstances, it is false.

(a) If $f'(3) = 1$ and $f(3) = 0$, then $y' = f(y)$ has an unstable equilibrium at $y = 3$. ○ TRUE ○ FALSE

(b) Consider an IVP $y' = f(y), y(0) = 0$ and two numerical methods A and B. ○ TRUE ○ FALSE
If method A has first order and method B has second order, then – given the same step size – method B will produce a better approximation of $y(1)$.

(c) Assume $f(t)$ is continuous for all t. If a solution exists for $(t^2 + 1)y'' + f(t)y' + y = 0, y(0) = 3$, then it must be unique. ○ TRUE ○ FALSE

(d) If A is a square, nilpotent matrix then e^{At} is NOT invertible. ○ TRUE ○ FALSE

(e) If $f(t)$ is continuous for all $t \in \mathbb{R}$ then $\mathcal{L}\{f\}(s)$ is continous for all $s \in \mathbb{R}$. ○ TRUE ○ FALSE

SECTION II **Justify** your answers. **(30 marks)**

6. Consider the IVP

$$y' + y = \delta(t-7) + g(t) \qquad y(0) = 0 \qquad \text{where} \qquad g(t) = u_3(t)\int_0^{t-3} e^{\tau} \cdot (t-3-\tau)\, d\tau$$

(a) **(4 marks)** Find $G(s) = \mathcal{L}\{g(t)\}$. Simplify as much as possible.

$G(s) = \mathcal{L}\{g(t)\} =$

If you were unable to find an expression for $G(s)$ in part (a), you can just write "$G(s)$" in part (b).

(b) **(4 marks)** Find the Laplace transform $Y(s)$ of the solution to the IVP. Simplify as much as possible. *Note that you are NOT asked to find an expression for $y(t)$.*

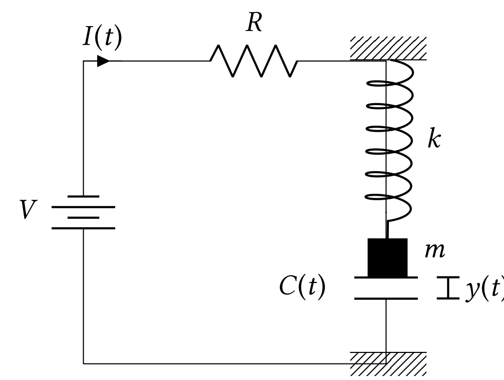
$Y(s) =$

(c) **(2 marks)** What is $y(2)$? Explain.

$y(2) =$

7. An electro-mechanical system, like the one found in smartphone accelerometers, is diagrammed below.

Let's first have a look at the *mechanical* part of the system. The mass m is attached to a spring with spring constant k and is able to move relative to the other plate.



The plates are separated by a distance $y(t)$. The equilibrium position of the mass is $y = d$. The mass experiences and external force $f(t)$.

(a) **(3 marks)** Assume the spring satisfies Hooke's law. Use Newton's second law to obtain a second-order ODE for the plate separation $y(t)$.

Now let's have a look at the *electrical* part of the system. A voltage source V (constant), a resistor R (constant), and a parallel-plate capacitor $C(t)$ are in series.

An electrically conductive mass forms one plate of the parallel-plate capacitor. The capacitance of the parallel-plate capacitor is given as $C(t) = \frac{\kappa}{y(t)}$, for some constant $\kappa > 0$.

(b) **(3 marks)** Use Kirchhoff's voltage law to obtain a first-order ODE for the capacitor voltage, $V_C(t)$.

(c) **(2 marks)** Express $V_C(t)$ in terms of an integral of $y(t)$ assuming the initial condition $V_C(0) = 0$.

Finally, consider how the mechanical and the electrical parts of the system *interact*.

(d) **(2 marks)** Explain how this system could be used to electronically detect acceleration.

8. Inside a thermally conducting rod with an insulated surface and length $L = 1$m, a chemical reaction is occuring. This reaction is exothermic (heat-releasing) at a rate that increases with temperature – an explosive combination. We attempt to cool the rod by placing its ends in ice water at 0°C.

This leads to the model for the rod's temperature $u(x,t)$ in °C at position x along the rod and time t,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku, \quad u(0,t) = 0 = u(1,t), \quad u(x,0) = 20°\text{C}.$$

(a) **(2 marks)** Substitute $u(x,t) = X(x)T(t)$ into the PDE and use $-\lambda$ as the separation constant. After separation, we arrive at the following equation (fill in the blanks).

$$\frac{\qquad}{T(t)} = \frac{\qquad}{\qquad} + \qquad = -\lambda$$

(b) **(2 marks)** What ODE does $T(t)$ satisfy? What BVP does $X(x)$ satisfy?

ODE for $T(t)$: $\boxed{\qquad\qquad\qquad}$

BVP for $X(x)$: $\boxed{\qquad\qquad\qquad}$

(c) **(2 marks)** For which values of λ does i) $T(t) \longrightarrow \infty$ and ii) $T(t) \longrightarrow 0$ as $t \longrightarrow \infty$?

(d) **(2 marks)** Determine the values of λ that give a non-trival solution to the BVP for $X(x)$.

(e) **(2 marks)** For which values of $k > 0$ does the temperature in the rod go to zero, i.e. $\lim_{t\to\infty} u(x,t) = 0$°C?

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Page for scratch work or for clearly-labelled overflow from previous pages