MAT292 - Fall 2019

Term Test 2 - November 14, 2019

Time allotted: 105 minutes | Aids permitted: None

Total marks: 66

Full Name: ______________________________
Last | First

Student Number: ______________________________

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- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers and brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 10–12 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 10–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

Question	Q1-Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	**Total**
Marks	10	5	5	10	8	10	8	10	**66**

HAVE FUN!

SECTION I No explanation is necessary. **(10 marks)**

1. **(2 marks)** Use Euler's method with step size $h = 0.5$ for the IVP $y' = t + y$, $y(0) = 1$.

Final answer, *no justification required*: $y(1) \approx$ ______________

2. **(2 marks)** Select all properties that apply to the solution of $y'' - y' = 0$, $y(0) = 1$, $y'(0) = 1$.

○ bounded ○ always positive ○ periodic ○ increasing

3. **(2 marks)** Consider the IVP $y' = f(t, y)$, $y(0) = y_0$. You estimate $y(1)$ two ways:

- Using Euler's Method with two steps ($h = 0.5$). You get $y(1) \approx y_{\text{Euler}}$.
- Using the Runge-Kutta Method with two steps ($h = 0.5$). You get $y(1) \approx y_{\text{RK}}$.

Make a choice:

○ $|y(1) - y_{\text{RK}}| \leqslant |y(1) - y_{\text{Euler}}|$ ○ $|y(1) - y_{\text{RK}}| \geqslant |y(1) - y_{\text{Euler}}|$ ○ This can not be decided.

Justify **briefly**:

4. **(2 marks)** Consider this statement: For a homogeneous system of three linear first-order ODEs, if we found three distinct solutions $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$, then any other solution can be written as $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$ for some coefficients c_1, c_2, c_3.

Make a choice: ○ The statement is TRUE. ○ The statement is FALSE.

Justify **briefly**:

5. **(2 marks)** Consider this statement: **There is NO** second order homogeneous linear ODE with constant coefficients such that $y(t) = 3e^{2t} + 5\cos(2t)$ is a solution.

Make a choice: ○ The statement is TRUE. ○ The statement is FALSE.

Justify **briefly**:

SECTION II **Justify** your answers. **(56 marks)**

6. **(5 marks)** Show the following: If $Y_1(t)$ is a particular solution of $y'' + p(t)y' + q(t)y = g_1(t)$, and $Y_2(t)$ is a particular solution of $y'' + p(t)y' + q(t)y = g_2(t)$, then $Y(t) = Y_1(t) + Y_2(t)$ is a particular solution of $y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$

7. **(5 marks)** Consider an initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ such that $y'' > 0$ at all times.

When using Euler's Method to numerically approximate the solution to this IVP, explain why, at every step, you would underestimate $y(t)$.

8. Solve the initial value problem

$$y'' + y' - 2y = -10\sin(t), \quad y(0) = 1, \ \ y'(0) = 0,$$

using the following three steps.

(a) **(3 marks)** Use the characteristic polynomial to solve the complementary equation.

(b) **(4 marks)** Use the method of undetermined coefficients to find a particular solution.

(c) **(3 marks)** Now solve the initial value problem.

9. Consider the ODE problem

$$x_1'(t) = x_2(t), \quad x_2'(t) = -\frac{\pi^2}{4}x_1(t), \quad x_1(0) = 0, \quad x_2(1) = 0.$$

Note in the conditions that x_1 is evaluated at $t = 0$ and x_2 is evaluated at $t = 1$.

(a) (4 marks) Verify that it has solutions $x_1(t) = A\sin\left(\frac{\pi}{2}t\right), x_2(t) = A\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)$ for any $A \in \mathbb{R}$.

(b) (4 marks) Why does this result not violate the existence-uniqueness theorem? Explain.

10. A MAT292 student implemented the improved Euler method for an initial value problem $x'(t) = f(t,x),\ x(0) = 0,\ t \in [0,1]$. The student's MATLAB code below produced the results in the table for various values of the stepsize h and number of steps N

N	h	estimated $x(1)$	estimated error
10	0.10000	-0.07394	0.00666
20	0.05000	-0.07791	0.00269
40	0.02500	-0.07942	0.00118
80	0.01250	-0.08005	0.00055
160	0.00625	-0.08034	0.00026
320	0.00313	-0.08047	0.00013

```
t = 0; x = 0; h = 1/N;
for n=1:N
   k1 = f(t,x);
   k2 = f(t,x+h*k1);
   x = x + h*(k1+k2)/2;
   t = t + h;
end
```

(a) **(4 marks)** Considering the table on the left, for what order k do you estimate that the error can be bounded in terms of $C \cdot h^k$, for some constant C? Explain.

(b) **(2 marks)** What order k did you expect?

(c) **(4 marks)** Find the bug in the code that lead to the discrepancy between parts (a) and (b).

11. Let $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$.

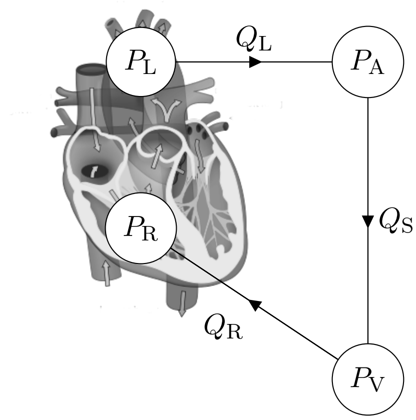
(a) **(5 marks)** Use the definition of the matrix exponential to compute e^{At}.

(b) **(3 marks)** Substitute $-t$ for t to explicitly verify for this matrix A that $e^{At}e^{-At} = I$.

Continued...

12. The human systemic circulatory system can be modelled with a system of differential equations based on the following assumptions. Time t is given in minutes.

There are *four compartments*: **L**eft heart, **A**rteries, **V**eins, and **R**ight heart. They are connected by *three vessel networks*. The flow through each network, given in L/min, is described by Q_{L}, Q_{S} and Q_{R}.



(a) **(1 mark)** The flow out of the heart is given as

$$Q_{\mathrm{L}}(t) = 5 + 5\sin\left(\frac{2\pi t}{T}\right).$$

What do you think is a realistic value for the parameter T?

(b) **(2 marks)** The pressure in the compartments A and V, measured in mmHg, is $P_{\mathrm{A}}(t)$ and $P_{\mathrm{V}}(t)$. The volume of blood, measured in litres, is $V_{\mathrm{A}}(t)$ and $V_{\mathrm{V}}(t)$. We assume that in each case, volume is proportional to pressure. Express this in two formulas introducing constants.

(c) **(2 marks)** Blood flows between connected compartments at a rate proportional to their difference in pressures, from the high pressure to the low pressure compartment. Use this fact to express Q_{S} in terms of P_{A} and P_{V}, also introducing a constant.

(d) **(1 mark)** Using the same assumptions, now also express Q_{R} in terms of P_{V} and P_{R}.

Continued...

For the remainder of the question, consider that the net flow of blood into a compartment changes its volume. For example, $\frac{d}{dt}V_{\mathrm{A}} = Q_{\mathrm{L}} - Q_{\mathrm{S}}$.

Also note that the pressure in compartment R can be assumed to be constant at $P_{\mathrm{R}} = 5\mathrm{mmHg}$.

(e) **(3 marks)** Use the assumptions and diagram above to write down a system of differential equations for $P_{\mathrm{V}}(t)$ and $P_{\mathrm{A}}(t)$.

(f) **(1 mark)** If a person, due to injury, were to lose a significant amount of blood, how would you account for this within the model?

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