**MAT292 - Fall 2020**

**Term Test 1 - October 1, 2020**

Time allotted: 90 minutes
Total marks: 59

Aids permitted: see "OK list"

**Full Name:** SOLUTIONS
Last First

**Student Number:**

**Email:** @mail.utoronto.ca

Do not forget to fill in the integrity statement on the last page!

- In the first section, only answers and sometimes brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 11 pages (including this title page). Make sure you have all of them.
- You can use pages 9–10 for rough work or to complete a question (**Mark clearly**).
- Make sure to follow this timeline:
  - 9:10 am – Start test.
  - 10:40 am – Stop writing test, fill in integrity page (last page).
    You MUST stop writing the test at 10:40 am. The last 15 minutes are for submission, not for test writing.
  - 10:55 am – Upload deadline.
    No extensions will be given.

| **Question** | Q1-5 | Q6 | Q7 | Q8 | Q9 | **Total** |
|---|---|---|---|---|---|---|
| **Marks** | 11 | 12 | 12 | 12 | 12 | **59** |

HAVE FUN!

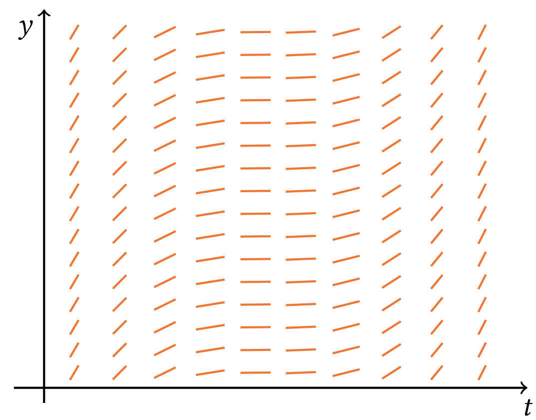
## SECTION I    Provide the final answer. Justify briefly when asked.    (11 marks)

1. **(2 marks)** Does the following problem have a solution? Explain briefly.

$$\begin{cases} y' = \sin(y^2) \\ y(0) = 1 \\ y(1) = \pi \end{cases}$$

**Solution:** No, there is an equilibrium solution $y = \sqrt{\pi}$. Note that $1 < \sqrt{\pi} < \pi$. The solution to the above problem would have to "cross" that equilibrium.

2. **(2 marks)** Consider this direction field of an ODE. You can assume there are "no surprises" outside the region that is plotted.




Make a choice:

⊗ The ODE must be separable.
○ The ODE must not be separable.
○ More information is necessary to decide.

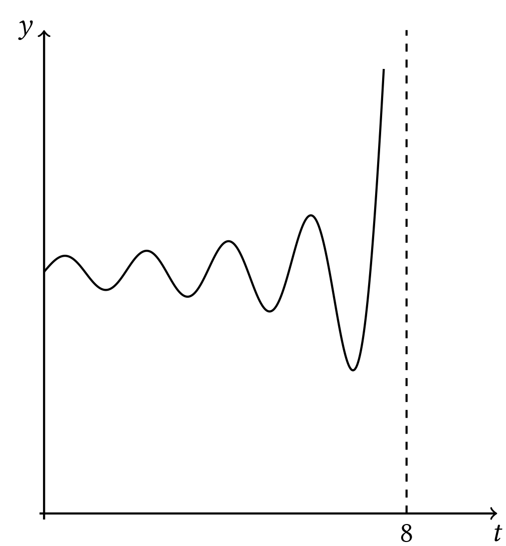
Briefly justify your choice:
**Solution:** The slope $y'$ only depends on $t$ and not on $y$. Therefore, the ODE can be written as $y' = f(t)$, which already separated and so the ODE is separable.

3. **(2 marks)** Write down the most general autonomous, linear, second order ODE for which $y = 0$ is *not* a solution.

**Solution:** The most general autonomous, linear, second order ODE is $a_0 y(t) + a_1 y'(t) + a_2 y''(t) = c$. Now note that if $c = 0$, $y = 0$ is a solution, but that $c \neq 0$ guarantees that $y = 0$ is not a solution.

4. **(2 marks)** The interval of definition of the solution to $y' = y^3,\ y(0) = 1$ is $\boxed{(-\infty, \frac{1}{2})}$.

5. **(3 marks)** Consider the function $y(t)$ graphed to the right with a vertical asymptote at $t = 8$.




Is it possible for $y(t)$ to be a solution to an ODE of each of the following forms?

**Choose 'YES' or 'NO'. No explanation necessary.**

(a) $y' + p(t)y = g(t)$    for polynomial $p(t)$ and $g(t)$

○ YES    ⊗ NO

(b) $y' = f(y)$

○ YES    ⊗ NO

(c) $y' = f(y)g(t)$

⊗ YES    ○ NO

## SECTION II **Justify** your answers.

**6.** Imagine that you are a swordsmith with extensive knowledge of ODEs. **(12 marks)**

You are forging swords. After being forged, every sword goes through two cooling steps: First cooling in a water bath, then cooling in air in the backyard.

You just finished forging a sword. Let $t = 0$ be the time the sword goes in the water bath. Let $t = T$ be the time it is moved to the backyard.

You know the following facts. We will use degrees Celsius for all temperatures and minutes for time.

- You are late with the order and the buyer will pick up the sword in 10 min.
- The sword needs to be cooled down to at most $50°C$ before it can be picked up by the buyer.
- At the end of forging, the sword has a temperature of $1500°C$.
- The water in the bath is constantly being replenished during the cooling process, so you can assume the water has constant temperature $20°C$.
- The temperature in your backyard is $30°C$ (it's a hot summer day).
- You can assume that the cooling constant for steel-in-water is $0.5\ \frac{1}{\text{min}}$.
- You can assume that the cooling constant for steel-in-air is $0.1\ \frac{1}{\text{min}}$.

Your water bath has space for only one sword at a time but you have lots of space in the backyard. This means you want to get this sword out of the water bath as early as possible so you have space for the next.

You want to find the earliest time $T$ you can take the sword out of the water bath.

**(a)** **(3 marks)** Set up an IVP for the sword in the water bath. Explain the IVP in detail.

**Solution:** This is Newton's Law of Cooling, which says that the rate of change of temperature $y'$ is proportional (with cooling constant 0.5) to the difference between the ambient temperature $20°C$ and the temperature $y$. The sword's initial temperature is $1500°C$.

IVP: $y' = -0.5(y - 20), \quad y(0) = 1500$

**(b)** **(0 marks)** State the solution of the IVP in part **(a)** without explanation.
*You may want to use WolframAlpha.*

Solution to the IVP: $y(t) = 1480e^{-0.5t} + 20$

**(c)** **(1 mark)** Give an expression for the temperature of the sword when taken out of the water bath.

Temperature when taken out of water bath: $y(T) = 1480e^{-0.5T} + 20$

**(d)** **(3 marks)** Set up an IVP for the sword in the backyard. Explain the IVP in detail.

**Solution:** This is again Newton's Law of Cooling, but with an ambient temperature of $30°C$ and a cooling constant of $0.1$. The initial value is the final value from the cooling in the water, so $y(T)$ is solution in part **(c)** evaluated at $t = T$.

IVP: $y' = -0.1(y - 30), \quad y(T) = 1480e^{-0.5T} + 20$

**(e)** **(0 marks)** State the general solution (before using the initial value) of the ODE in part **(d)** without explanation. *You may want to use WolframAlpha.*

General solution of the ODE: $y(t) = ce^{-0.1t} + 30$

**(f)** **(2 marks)** Use the initial value to arrive at the solution of the IVP in part **(d)**.
Solve $ce^{-0.1T} + 30 = 1480e^{-0.5T} + 20$ for $c$ to obtain the solution to the IVP.

Solution of the IVP: $y(t) = (1480e^{-0.5T} - 10)e^{-0.1(t-T)} + 30$

**(g)** **(1 mark)** Give an expression for the temperature of the sword when the buyer picks it up.

Temperature when buyer picks up the sword: $y(10) = (1480e^{-0.5T} - 10)e^{-0.1(10-T)} + 30$

**(h)** **(2 mark)** Finally, find the earliest time $T$ you can take the sword out of the water bath given the situation. *You may want to use WolframAlpha, but make sure to explain what you are looking for.*

**Solution:** We want $y(10) = (1480e^{-0.5T} - 10)e^{-0.1(10-T)} + 30 \leq 50$. Since the left-hand side of this inequality is monotonically decreasing in $T$, we can solve $(1480e^{-0.5T} - 10)e^{-0.1(10-T)} + 30 = 50$ for $T$ to obtain the earliest time that the sword can be removed from the water bath. We can see that $y(10)$ is monotonic in $T$ from the Desmos plot `https://www.desmos.com/calculator/ynshybj7kl` or from a physical argument: the more time in the water with the larger cooling constant, the cooler the sword will be at time 10. WolframAlpha gives $T \approx 7.4\, min$. An acceptable estimate can also be obtained by eye from the Desmos plot. See page 9 for another way to solve for $T$ using just a calculator. For interest, if we left the sword in the water for the whole 10 minutes, it would be at approximately $30°\,C$ after 10 minutes.

$T = 7.4\, min$

**7.** A generalization of the logistic growth model is **(12 marks)**

$$P' = r\left[1 - \left(\frac{P}{K}\right)^s\right]P,$$

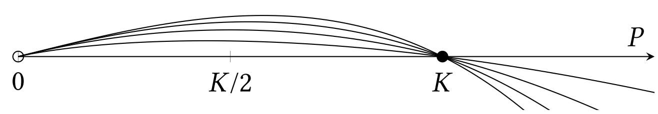
with parameters $r > 0$, $K > 0$, and $s > 0$.

**(a)** **(4 marks)** Find the equilibria and their stability and list them here:

| **Equilibria** | $P = 0$ | $P = K$ |
|---|---|---|
| **Stability** | unstable | stable |

Explain the equilibria and classification here:

**Solution:** The roots of $f(P) = r\left[1 - \left(\frac{P}{K}\right)^s\right]P = 0$ are $P = 0$ and $P = K$. Phase-line analysis (for $s = 0.5, 1.0, 1.5, 2.0$ shows that $P = 0$ is unstable and $P = K$ is stable.




The stability can also be seen with $f'(0) = r > 0$ and $f'(K) = -pr < 0$.

**(b)** **(4 marks)** Explain why $K$ is referred to as the *carrying capacity*.

**Solution:** In the long-run, $P \longrightarrow K$. When $P = K$, the births and deaths are balanced. If $P > K$, the habitat cannot support, or *carry*, that many animals and the population size will decrease. If $P < K$, the habitat can support more animals and the population size will increase.

**(c)** **(4 marks)** One effect of climate change is that carrying capacities of many habitats are decreasing over time. Suppose that $K(t) = 1000e^{-t}$.

Find $P_0$ and $b$ so that $P(t) = P_0e^{-bt}$ is a solution.

**Solution:** Substituting $P_0e^{-bt}$ into the ODE with $K = 1000e^{-t}$ gives

$$P_0e^{-bt}(-b) = r\left[1 - \left(\frac{P_0e^{-bt}}{1000e^{-t}}\right)^s\right]P_0e^{-bt},$$

$$-b/r = 1 - \left(\frac{P_0}{1000}\right)^s e^{s(1-b)t}.$$

In order to remove the time-dependence, we can take $b = 1$. Solving

$$-1/r = 1 - \left(\frac{P_0}{1000}\right)^s,$$

for $P_0$ gives

$$P_0 = 1000\sqrt[s]{1 + \frac{1}{r}}.$$

$P_0 = 1000\sqrt[s]{1 + \frac{1}{r}}$ $\qquad$ $b = 1$

**8.** The velocity $v(t)$ of a sky diver falling under gravity experiencing laminar drag might be modelled as **(12 marks)**

$$v' + \frac{q'(t)}{q(t)} v = 1.$$

The initial velocity is zero, $v(0) = 0$. The function $q(t)$ is under the control of the sky diver and it can be selected to change the drag coefficient. To be a physical model, $q(t) > 0$ and $q'(t) \geq 0$ for every $t \geq 0$.

**(a)** **(4 marks)** *For this part only*, assume $q(t) = 1 + e^t$.
Solve for $v(t)$ using the method of integrating factors.
**Solution:** The ODE is

$$\frac{q'}{q} = \frac{e^t}{1+e^t} \longrightarrow v' + \frac{e^t}{1+e^t} v = 1.$$

The integrating factor, using the $u$-substitution $u = 1 + e^t$, is

$$\mu(t) = \exp\left(\int \frac{e^t}{1+e^t}\, dt\right) = \exp\left(\int \frac{du}{u}\right) = \exp\left(\ln(1+e^t)\right) = 1 + e^t.$$

Solving

$$\begin{aligned} \left((1+e^t)v\right)' &= 1 + e^t \\ v(t) &= \frac{t + e^t + C}{1+e^t} = \frac{t + e^t - 1}{1+e^t} \end{aligned}$$

by choosing $C = -1$ to satisfy the initial condition.

$$\boxed{v(t) = \frac{t+e^t+C}{1+e^t} = \frac{t+e^t-1}{1+e^t}}$$

**(b)** **(4 marks)** Show that $q(t)$ can be used as an integrating factor for this ODE (no matter the exact $q(t)$).
**Solution:** An integrating factor is given by

$$\mu(t) = \exp\left(\int \frac{q'(t)}{q(t)}\, dt\right) = \exp\left(\int \frac{d}{dt}\ln(q(t))\, dt\right) = \exp\left(\ln(q(t))\right) = q(t).$$

Alternatively, write the ODE as

$$\begin{aligned} q(t)v' + q'(t)v &= q(t) \\ (q(t)v(t))' &= q(t) \end{aligned}$$

so that $q(t)$ is an integrating factor.

**(c)** **(4 marks)** Write the solution $v(t)$ in terms of $q(t)$ and a definite integral of $q(t)$.
**Solution:**

$$\begin{aligned} q(t)v' + q'(t)v &= q(t) \\ (q(t)v(t))' &= q(t) \end{aligned}$$

Therefore,

$$v(t) = \frac{1}{q(t)} \int_0^t q(\tau)\, d\tau,$$

which has the correct initial condition, $v(0) = 0$.

$$\boxed{v(t) = \frac{1}{q(t)} \int_0^t q(\tau)\, d\tau}$$

**9.** One case of Clairant's equation is **(12 marks)**

$$(y'(x))^2 + xy'(x) - y(x) = 0, \quad x \geq 0, \quad y' \geq 0.$$

**(a)** **(4 marks)** Write the differential equation in the standard form $y'(x) = f(x, y)$.
*Hint: The equation is quadratic in $y'$.*
**Solution:** Solving the quadratic equation $y'^2 + xy' - y = 0$ gives

$$y'(x) = -\frac{x}{2} \pm \frac{1}{2}\sqrt{x^2 + 4y}.$$

Since the quadratic for $y'$ has two roots, Clairant's equation is a pair of ODEs in disguise. The restrictions $x \geq 0$, $y' \geq 0$ (which implies $y \geq 0$ from the original equation) mean that we are interested in the '+' root so

$$f(x, y) = -\frac{x}{2} + \frac{1}{2}\sqrt{x^2 + 4y}.$$

$$\boxed{y'(x) = -\tfrac{x}{2} + \tfrac{1}{2}\sqrt{x^2 + 4y}}$$

**(b)** **(4 marks)** Consider the ODE from part **(a)** and find all solutions of the form $y = ax + b$ and $y = ax^2 + bx + c$.
**Solution:** Substituting $y = ax + b$ gives

$$a^2 + xa - ax - b = 0,$$

which shows that $b = a^2$. Therefore, any line of the form $y = ax + a^2$ is a solution. So that $y' \geq 0$, we need $a \geq 0$.
Substituting $y = ax^2 + bx + c$ gives

$$(2ax + b)^2 + x(2ax + b) - (ax^2 + bx + c) = 0$$
$$(4a^2 + 2a - a)x^2 + (4ab + b - b)x + (b^2 - c) = 0.$$

We see that $4a^2 + a = 0$, so either $a = 0$, which can be ignored since lines were already considered, or $a = -\frac{1}{4}$. Consequently, $b = 0$ and $c = 0$. Therefore, the only solution of the form $y = ax^2 + bx + c$ is $y = -\frac{1}{4}x^2$. This solution does not meet the condition $y' \geq 0$, but is nonetheless important for the subsequent questions.

$$\boxed{\text{Solutions: } y = ax + a^2,\ a \geq 0 \text{ and } y = -\tfrac{1}{4}x^2 \text{ (ignoring } y' \geq 0)}$$

**(c)** **(2 marks)** $y = x+1$ solves the ODE of part **(a)** together with the initial value $y(0) = 1$. Is this solution unique? Explain.

**Solution:** Since $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x^2+4y}}$, we can see that $f$ and $\frac{\partial f}{\partial y}$ are continuous near $(0,1)$. In particular, $f$ and $\frac{\partial f}{\partial y}$ are continuous in the rectangle $(-\infty, \infty) \times (0, \infty)$, which contains $(0,1)$. According to the existence/uniqueness theorem, the IVP has a unique solution. That solution is $y = x + 1$.

**(d)** **(2 marks)** Both $y = 0$ and $y = -\frac{1}{4}x^2$ solve the ODE of part **(a)** together with the initial value $y(0) = 0$. Why does this not contradict the existence-uniqueness theorem? Explain.

**Solution:** The fact that $y = 0$ and $y = -\frac{1}{4}x^2$ both solve the ODE and satisfy $y(0) = 0$ does not contradict the existence-uniqueness theorem. This is because $\frac{\partial f}{\partial y}$ does not exist at $y = 0, x = 0$ and therefore is not continuous at $x = 0, y = 0$. The existence-uniqueness theorem does not apply to this initial value problem.

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We can solve the equation

$$(1480e^{-0.5T} - 10)e^{-0.1(10-T)} + 30 = 50$$

with only a calculator as follows. The variable $T$ appears in two locations. Re-arrange the equation so that the first $T$ appears on the left-hand side:

$$T = -2\log\left(\frac{20e^{1-0.1T} + 10}{1480}\right).$$

We solve for $T$ using an iteration:

$$T_n = -2\log\left(\frac{20e^{1-0.1T_{n-1}} + 10}{1480}\right).$$

Starting with $T_0 = 0$, we obtain a sequence of estimates:

$$T_1 = 6.2704, T_2 = 7.2704, T_3 = 7.4172, T_4 = 7.4385, T_5 = 7.4415, T_6 = 7.4420, \ldots$$

Interestingly, if you create an iteration based on re-arranging for the second $T$ appearing in the orignal equation, then the iteration does not find the solution. The solution $T \approx 7.44$ is a fixed point of the iteration, but it is not *stable*.

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