MAT292 - Fall 2019

Term Test 1 - October 17, 2019

Time allotted: 100 minutes

Aids permitted: None

Total marks: 60

Full Name: ______________________________
Last First

Student Number: ______________________________

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- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers and brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 11–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

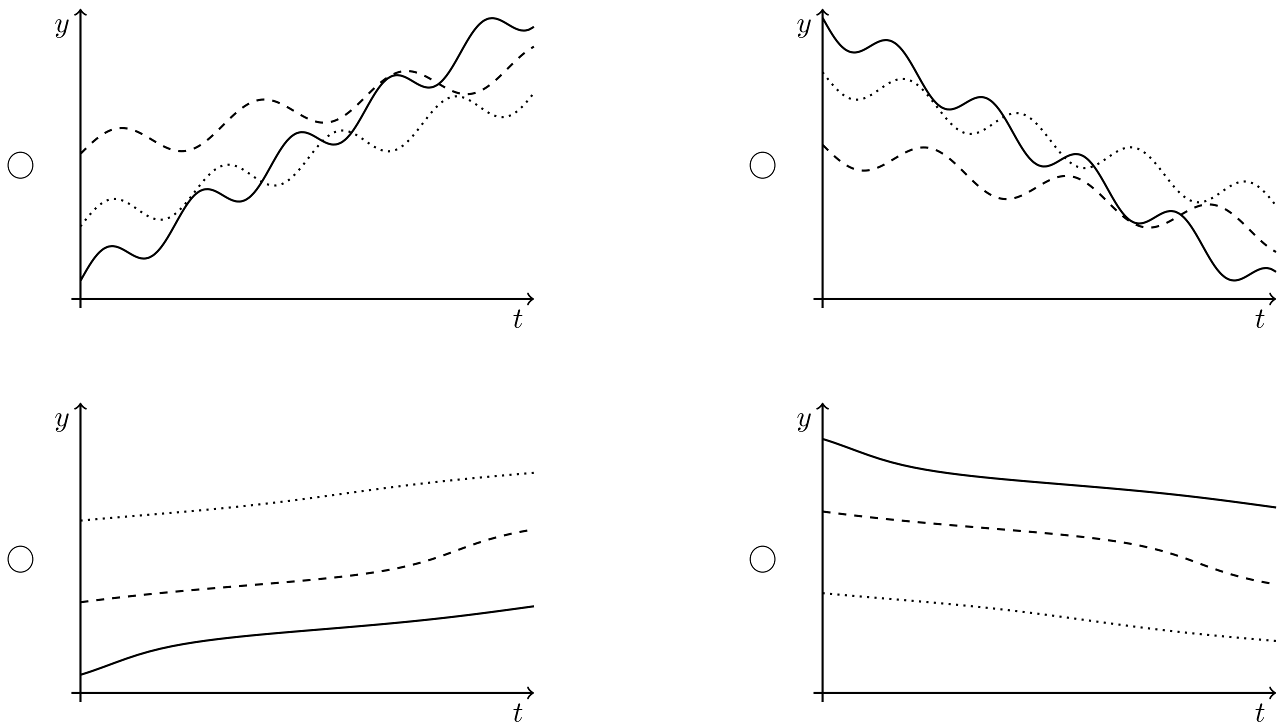
Question	Q1-Q6	Q7	Q8	Q9	Q10	Q11	**Total**
Marks	10	10	10	10	10	10	**60**

HAVE FUN!

SECTION I Provide the final answer. Justify briefly when asked. **(10 marks)**

1. **(1 mark)** Give an example of an ODE that has *third* order, is *not* autonomous, and is *not* linear.

2. **(1 mark)** Consider the ODE $y' = \frac{1}{2}(\sin y)^2 + \frac{1}{2}e^{\sin y}$. Only one of the following four diagrams shows plots of three solutions to this ODE. Which one? Highlight one of the four circles.



3. **(2 marks) Briefly** justify your choice in the previous question (question 2).

4. **(2 marks)** Consider this statement: Every first-order linear ODE is separable.

Make a choice: ◯ The statement is TRUE. ◯ The statement is FALSE.

Justify **briefly**:

5. **(2 marks)** Consider this statement: Every system of ODEs $\vec{x}\,' = A\vec{x}$ has exactly one equilibrium.

Make a choice: ◯ The statement is TRUE. ◯ The statement is FALSE.

Justify **briefly**:

6. **(2 marks)** Consider the ODE $y' = f(y)$. You are given the following three facts:

(i) The function f is continuous everywhere.

(ii) $f(2) = 3$

(iii) $f(3) = -2$

Make a choice:

◯ Based on these conditions, this ODE must have at least one stable equilibrium.

◯ Based on these conditions, this ODE must have at least one unstable equilibrium.

◯ We would need more information to decide this.

Justify **briefly**:

SECTION II **Justify** your answers.

7. **(a)** **(3 marks)** Show that the differential equation $(y^2 - t^4)\, y' - 4t^3 y = 0$ is exact. Then find a one-parameter family of implicit solutions.

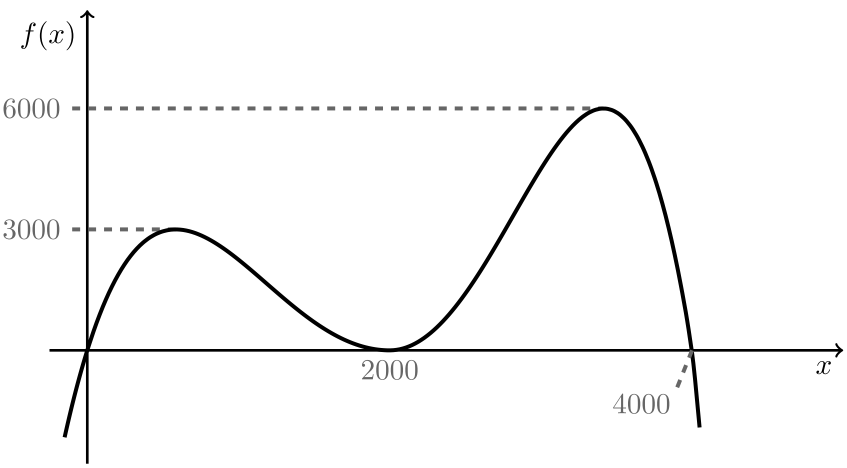
(b) **(2 marks)** Find all solutions passing through $(1, 1)$.

(c) **(2 marks)** Find all solutions passing through $(0, 0)$.

(d) **(3 marks)** Discuss your answers to parts **(b)** and **(c)** in terms of the Existence-Uniqueness Theorem.

8. Consider a species named *unicornobos* living on a remote island. **(10 marks)**

The population of unicornobos on the island over time is described by $U(t)$ which is governed by the differential equation $U' = f(U)$. The diagram below shows the graph of $f(x)$.



(a) **(4 marks)** The following facts are known about unicornobos:

(i) Unicornobos reproduce at a constant rate, unless they are inhibited by external factors.

(ii) Unicornobos can only survive in the forest. There are two forests on the island. Due to limited resources, each forest can support up to C unicornobos. Whenever there are less than C unicornobos on the island, they all live in one forest together. Unicornobos never move from one forest to the other.

Explain in your own words how these effects can be seen in the plot that is provided and deduce from the plot what the value of C is.

(b) **(4 marks)** Find all equilibria of the unicornobo population and classify each of them.

(c) **(2 marks)** Given that $U(0) = 1500$, find $\lim_{t \to \infty} U(t)$. Justify your result.

9. Solve the following initial value problem using the eigenvalue method. **(10 marks)**

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Also sketch the solution in the phase plane.

You do not need to draw to scale, a qualitative plot is enough.

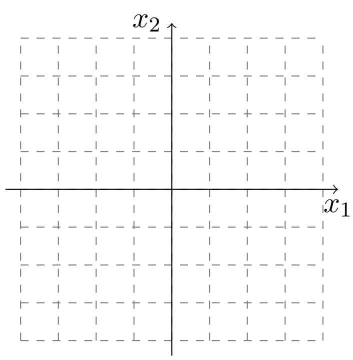
10. Consider the following linear, homogeneous, constant coefficient system of differential equations that has a parameter $a \in \mathbb{R}$, **(10 marks)**

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad A = \begin{pmatrix} 1+a & 1-a \\ 1-a & 1+a \end{pmatrix}.$$

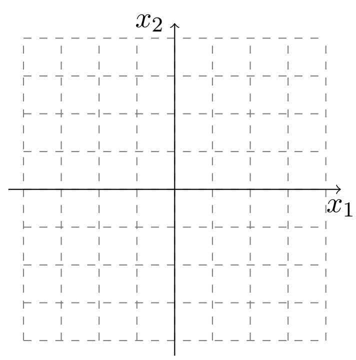
(a) **(5 marks)** Find the general solution using the eigenvalue method.

(b) **(3 marks)** Sketch the phase plane for the following three values of a:

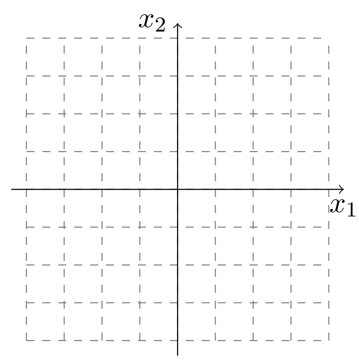
$a = -1$



$a = 0$



$a = 1$



(c) **(2 marks)** For which values of a is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a stable equilibrium? Justify your answer.

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11. You are attempting to land a probe on Mars with mass $m = 500$kg. **(10 marks)**

The probe is initially at altitude $h_0 = 2000$m and has an initial downward speed of $v_0 = 500$m/s.

It experiences a drag force $F = \mu v$ proportional to its downward speed. Assume $\mu = 10$Ns/m.

The probe has an engine that provides a constant upward thrust of force u to slow down the descent.

To simplify calculations, assume that the gravitational acceleration on Mars is $g = 4\text{m/s}^2$.

(a) **(2 marks)** Write down a differential equation for the probe's downward speed $v(t)$.

Note: You are being asked to write a differential equation for the speed, NOT for the height.

(b) **(1 mark)** What is the probe's terminal velocity if the engine is not used, i.e. if $u = 0$?

(c) **(1 mark)** Express the probe's altitude $h(t)$ in terms of its downward speed $v(t)$.

(d) **(2 marks)** Solve for $v(t)$.

(e) **(1 mark)** Find u so that $\lim_{t\to\infty} v(t) = 0$.

(f) **(1 mark)** Using the value of u found in part **(e)**, find $h(t)$.

(g) **(2 marks)** Using the value of u found in part **(e)**, find the landing time T so that $h(T) = 0$. What is $v(T)$?

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