MAT292 - Fall 2020

Term Test 3 - November 26, 2020

Time allotted: 90 minutes
Total marks: 47

Aids permitted: see "OK list"

Full Name: SOLUTIONS
Last First

Student Number:

Email: @mail.utoronto.ca

Do not forget to fill in the integrity statement on the second page!

- In the first section, only answers and sometimes brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 10 pages (including this title page). Make sure you have all of them.
- You can use pages 9–10 for rough work or to complete a question (**Mark clearly**).
- Make sure to follow this timeline:
  - 9:05 am – Start test.
  - 10:35 am – Stop writing test, fill in integrity page (second page).
    You MUST stop writing the test at 10:35 am.
    The last 25 minutes are for submission, not for test writing.
  - 11:00 am – Upload deadline.
    No extensions will be given.

Question	Q1-4	Q5	Q6	Q7	Q8	**Total**
Marks	9	10	8	8	12	**47**

HAVE FUN!

This is the ONLY page you can fill in *after* 10:35 am.

If you don't complete and sign this page, you will receive a grade of zero for the entire test.

We at U of T want you to feel proud of what you accomplish as a student. Please respect all of the hard work you're doing this term by making sure that the work you do is your own.

We don't expect you to score perfectly on the assessments and there will be some things that you may not know. Using an unauthorized resource or asking someone else for the answer robs you of the chance later to feel proud of how well you did because you'll know that it wasn't really your work that got you there.

Success in university isn't about getting a certain mark, it's about becoming the very best person you can by enriching yourself with knowledge, strengthening yourself with skills, and building a healthy self-esteem based on how much you've grown and achieved. No one assessment captures that but your conscience will stay with you forever.

Make yourself and your loved ones proud of the student that you are by conducting yourself honestly at all times. Hold each other accountable to these standards.

In submitting this assessment ...	Short sentences
... I confirm that my conduct regarding this test adheres to the Code of Behaviour on Academic Matters.	I know the Code.
... I confirm that I have not acted in such a way that would constitute cheating, misrepresentation, or unfairness, including but not limited to, using unauthorized aids and assistance, impersonating another person, and committing plagiarism.	I didn't cheat.
... I confirm that the work I am submitting in my name is the work of no one but myself.	This is only my work.
... I confirm that all pages have been handwritten by myself.	I wrote all pages.
... I confirm that I have not received help from others, whether directly or indirectly.	I didn't receive help.
... I confirm that I have not provided help to others, whether directly or indirectly.	I didn't provide help.
... I confirm that I have only used the aids marked as "OK" on the list.	I only used "OK" aids.
... I am aware that not disclosing another student's misconduct despite my knowledge is an academic offence.	I know I must report cheating.

In this box, handwrite the sequence of short sentences (starting with "I know the Code. I didn't cheat...").

Your student number

Your signature

Submission date

SECTION I Provide the final answer. **Justification required only in question 5.** **(19 marks)**

1. **(2 marks)** Find e^{At} if A is a 2×2 matrix with the following eigenvalue-eigenvector pairs

$$\lambda_1 = 2, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda_2 = -1, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} e^{2t} & e^{2t} - e^{-t} \\ 0 & e^{-t} \end{bmatrix}}$$

2. **(2 marks)** Find two values for a such that $y(t) = t^a$ is a solution to $t^2 y'' + 2ty' - 2y = 0$.

smaller value: $\boxed{a = -2}$ larger value: $\boxed{a = 1}$

3. **(2 marks)** Using the method of undermined coefficients, what is the simplest guess for a particular solution to $y'' + ry' = te^{-rt}$ that will be successful? The parameter r is real and positive.
You do not need to find the coefficients. Stating the guess is enough.

$$\boxed{Y = Ate^{-rt} + Bt^2 e^{-rt}}$$

4. **(3 marks)** For each value of k below, solve the initial value problem

$$y'' + 2y' + ky = 0, \; y(0) = 1, \; y'(0) = 0$$

and then determine the **smallest** non-negative integer n for which $|y(t)| < \frac{1}{10}$ for all $t \geq n$.

Note: As for all the other questions on this page, only a final answer is required.
You may use WolframAlpha or Symbolab.

(a) If $k = \frac{8}{9}$, then $\boxed{y(t) = 2e^{-2t/3} - e^{-4t/3}}$ $\boxed{n = 5}$

(b) If $k = 1$, then $\boxed{y(t) = e^{-t}(t + 1)}$ $\boxed{n = 4}$

(c) If $k = 2$, then $\boxed{y(t) = e^{-t}(\cos t + \sin t)}$ $\boxed{n = 2}$

5. **(10 marks)** For each of the following statements, decide if it is true or false. Then justify your choice.

Remember: A statement is only true if it is always true. If a statement only works in special cases/under certain circumstances, it is false.

(a) Consider a three dimensional linear system of ODEs. ○ TRUE ⊗ FALSE
If $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are solutions to this system, then the matrix $\begin{bmatrix} | & | & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ | & | & | \end{bmatrix}$ is invertible.

Solution: The solutions $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ may be linearly dependent, which would make the matrix $\begin{bmatrix} | & | & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ | & | & | \end{bmatrix}$ not invertible. For example, if $\vec{x}_2 = 2\vec{x}_2$ and $\vec{x}_3 = 3\vec{x}_1$ for a solution $\vec{x}_1$, then $\vec{x}_2$ and $\vec{x}_3$ would be solutions due to linearity, but the matrix would not be invertible.

(b) If $X(t)$ is a fundamental matrix for an n-dimensional system of ODEs $\vec{x}\,' = A\vec{x}$ ⊗ TRUE ○ FALSE
and B is an invertible $n \times n$ matrix, then $X(t) \cdot B$ is also a fundamental matrix for the same system.
Solution: $X(t) \cdot B$ is invertible since the produce of invertible matrices are invertible: $(X(t) \cdot B)^{-1} = B^{-1} \cdot X(t)^{-1}$. The columns of $X(t) \cdot B$ are solutions: $(X(t) \cdot B)' = X(t)' \cdot B = (A \cdot X(t)) \cdot B = A(X(t) \cdot B)$. In other words, the columns of $X(t) \cdot B$ are linear combinations of solutions (the columns of $X(t)$) and hence are solutions.

(c) If $y_1(t)$ and $y_2(t)$ are linearly dependent solutions to $y'' + ty' + \cos(t^2)y = 0$ ⊗ TRUE ○ FALSE
then $y_1(t) + y_2(t)$ is also a solution.
Solution: The ODE is linear, so a sum of solutions is a solution. It does not matter that the solutions are linearly dependent.

(d) The solution to the following IVP has the largest maximum amplitude if $\omega = 1$. ○ TRUE ⊗ FALSE
"Maximum amplitude" refers to $\max_{t \geq 0} |y(t)|$.

$$y'' + 2\delta y' + y = \cos(\omega t), \quad y(0) = y'(0) = 0, \quad \delta > 0$$

Solution: The resonant frequency ω_r, which produces the maximum amplitude response, is strictly below the natural frequency (here one) according to the formula $\omega_r = \sqrt{\omega_0^2 - 2\delta^2}$. If the system is overdamped, $\delta > 1$, then the maximum amplitude occurs at $\omega = 0$.

(e) A system of ODEs $\vec{x}\,' = A\vec{x}$ has *many* fundamental matrices; ⊗ TRUE ○ FALSE
and only *one* special fundamental matrix for $t_0 = 0$.
Solution: The system of ODEs $\vec{x}\,' = A\vec{x}$ has many fundamental matrices - see Q5b. There is only one fundamental matrix satisfying $X(0) = I$, called the special fundamental matrix.

SECTION II **Justify** your answers. **(28 marks)**

6. **(a)** **(6 marks)** Consider $A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $c \in \mathbb{R}$ is a constant. Find e^{At} using the definition.

Simplify your answer as much as you can.

Hint: Compute A^2 and A^3. By then, you should see a pattern.

Computing powers of A gives

$$A^2 = \begin{bmatrix} 1 & 0 & 2c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 0 & 3c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \dots, \quad A^n = \begin{bmatrix} 1 & 0 & nc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From the definition of the matrix exponential,

$$e^{At} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} & 0 & c\sum_{n=0}^{\infty} \frac{t^n}{n!} n \\ 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!} & 0 \\ 0 & 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{bmatrix}.$$

Recognize that $\sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t$. Re-index the sum to find $\sum_{n=0}^{\infty} \frac{t^n}{n!} n = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} = te^t$.

$$e^{At} = \begin{bmatrix} e^t & 0 & cte^t \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

(b) **(2 marks)** Find the solution to this IVP: $\vec{x}'(t) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}(t) \qquad \vec{x}(0) = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$

$$\vec{x}(t) = e^{At}\vec{x}(0) = \begin{bmatrix} e^t & 0 & cte^t \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3e^t + 3cte^t \\ -3e^t \\ 3e^t \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} 3e^t + 3cte^t \\ -3e^t \\ 3e^t \end{bmatrix}$$

7. The goal of this question is to find the general solution to $ty'' - y' + 4t^3 y = 4t^3 e^{-t^2}$.

You can take for granted that $y_1(t) = \sin(t^2)$ and $y_2(t) = \cos(t^2)$ are solutions to the associated homogeneous equation.

(a) **(2 marks)** For which values of t is $W[y_1, y_2](t) \neq 0$?
Solution:

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(t^2) & \cos(t^2) \\ 2t\cos(t^2) & -2t\sin(t^2) \end{vmatrix} = -2t\sin^2(t^2) - 2t\cos^2(t^2) = -2t$$

The solutions y_1 and y_2 are linearly independent away from $t = 0$.

(b) **(2 marks)** The initial value problem $ty'' - y' + 4t^3 y = 0,\ y(0) = 0,\ y'(0) = 0$ has at least two solutions, namely $y_1 = \sin(t^2)$ and $y_3 \equiv 0$. Why does this not violate the existence-uniqueness theorem?
Solution: The coefficients are $p(t) = -1/t$ and $q(t) = 4t^2$. Since $p(t)$ is not continuous at $t = 0$, the existence-uniqueness theorem does not apply.

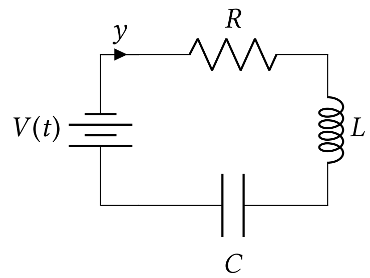
(c) **(3 marks)** Find a particular solution using the variation of parameters formula.
Hint: At some point, the substitution $x = t^2$ might be useful.
Solution: Evaluating the variation of parameters formula with $g(t) = 4t^2 e^{-t^2}$ and the change of variables $x = t^2$, $dx = 2t\,dt$ gives

$$\begin{aligned} Y &= -\sin(t^2)\int \frac{4t^2 e^{-t^2}}{-2t}\cos(t^2)\,dt + \cos(t^2)\int \frac{4t^2 e^{-t^2}}{-2t}\sin(t^2)\,dt \\ &= +\sin(t^2)\int e^{-x}\cos(x)\,dx - \cos(t^2)\int e^{-x}\sin(x)\,dx \\ &= \sin(t^2)\frac{1}{2}e^{-t^2}\left(\sin(t^2) - \cos(t^2)\right) + \cos(t^2)\frac{1}{2}e^{-t^2}\left(\cos(t^2) + \sin(t^2)\right) \\ &= \frac{1}{2}e^{-t^2} \end{aligned}$$

(d) **(1 mark)** State the general solution to the ODE.

$$\boxed{y(t) = c_1\sin(t^2) + c_2\cos(t^2) + \tfrac{1}{2}e^{-t^2}}$$

8. An industrial electric heater is modelled as a series RLC circuit with a voltage source $V(t)$. The differential equation governing the current $y(t)$ is also shown. Suppose that $R = 1$, $L = 2$, and $C = 4$.



$$Ly'' + Ry' + \frac{1}{C}y = V'(t)$$

(a) **(3 marks)** What is the general solution to the associated homogeneous equation?
Denote this complementary solution as y_c.

Solution:

$$2y_c'' + y_c' + \frac{1}{4}y_c = 0 \longrightarrow 2\lambda^2 + \lambda + \frac{1}{4} = 0 \longrightarrow \lambda = -\frac{1}{4} \pm \frac{1}{4}i$$

The real and imaginary parts of $e^{\left(-\frac{1}{4}+\frac{i}{4}\right)t} = e^{-t/4}\cos\left(\frac{t}{4}\right) + ie^{-t/4}\sin\left(\frac{t}{4}\right)$ give real linearly independent solutions. The general solution to the associated homogeneous equation is therefore

$$y_c = c_1 e^{-t/4}\cos\left(\frac{t}{4}\right) + c_2 e^{-t/4}\sin\left(\frac{t}{4}\right).$$

$y_c(t) = c_1 e^{-t/4}\cos\left(\frac{t}{4}\right) + c_2 e^{-t/4}\sin\left(\frac{t}{4}\right)$

(b) **(2 marks)** Is the system undamped, underdamped, critically damped, or overdamped?

Choose here: ○ undamped ⊗underdamped ○ critically damped ○ overdamped

Explain here: Since $R > 0$, the system includes damping and is therefore not undamped. Since the characteristic values (eigenvalues λ) are complex, i.e. have non-zero imaginary parts, the system is underdamped.

(c) **(3 marks)** The voltage supply is going to be slowly turned off so that $V(t) = 12e^{-at}$ for some $a > 0$. Find a particular solution $Y(t)$. Ensure that your particular solution does not contain terms that are solutions to the associated homogeneous equation.

Note that the non-homogeneous term is $V'(t)$, not $V(t)$.

Solution:

$$2y'' + y' + \frac{1}{4}y = -12ae^{-at}$$

Try a solution of the form

$$Y = Ae^{-at}$$

giving

$$2a^2A - aA + \frac{1}{4}A = -12a \longrightarrow A = \frac{-12a}{2a^2 - a + \frac{1}{4}} = \frac{-48a}{8a^2 - 4a + 1}.$$

Therefore, a particular solution is

$$Y = \frac{-48a}{8a^2 - 4a + 1}e^{-at}.$$

$Y(t) = \frac{-48a}{8a^2-4a+1}e^{-at}$

(d) **(2 marks)** Which values for y_0 and y_1 guarantee that the particular solution $Y(t)$ you found in part (c) is the solution to

$$Ly'' + Ry' + \frac{1}{C}y = V'(t), \quad y(0) = y_0, \quad y'(0) = y_1 \quad ?$$

Solution: Using uniqueness of the IVP, we can obtain the initial data from the particular solution above: $y_0 = Y(0) = A = \frac{-48a}{8a^2-4a+1}$ and $y_1 = Y'(0) = -aA = \frac{48a^2}{8a^2-4a+1}$.

$y_0 = \frac{-48a}{8a^2-4a+1}$

$y_1 = \frac{48a^2}{8a^2-4a+1}$

(e) **(2 marks)** The heat produced by the heater is the Ohmic loss in the resistor with power Ry^2. If $y(t) = Y(t)$ then the total heat produced during the shut-off is given by $\int_0^\infty RY^2\,dt$. Determine an expression for the total heat produced in terms of a. To get an effective heater, you would want to maximize this expression by selecting a. **You do not need to actually perform the maximization.**

Solution:

$$\int_0^\infty RY^2\,dt = \frac{(-48)^2a^2}{(8a^2-4a+1)^2}\int_0^\infty e^{-2at}\,dt = \frac{(-48)^2a^2}{(8a^2-4a+1)^2}\frac{1}{2a} = \frac{1152a}{(8a^2-4a+1)^2}$$

See page 9 for details on the maximization.

Expression you would want to maximize: $1152a/(8a^2 - 4a + 1)^2$

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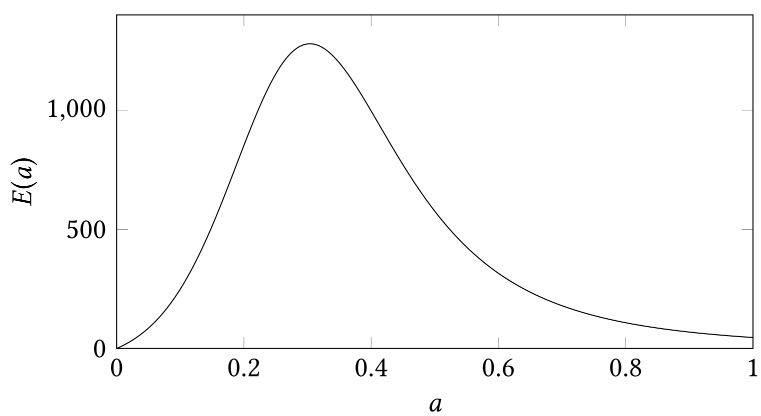
In question 8, we need to maximize

$$E(a) := \frac{1152a}{(8a^2 - 4a + 1)^2},$$

a rational function in a. Since the poles occur at $a = \frac{1}{4} \pm \frac{1}{4}i$, likely at the real value $a = \frac{1}{4}$ (the decay rate of the eigenvalues) the value of the function will be large. Computing the derivative,

$$E'(a) = \frac{1152(-24a^2 + 4a + 1)}{(8a^2 - 4a + 1)^3},$$

reveals that $E'(a) = 0$ for $a = \frac{1}{12} \pm \frac{\sqrt{7}}{12}$. The former of which gives the maximum $E(0.3038) \approx 1279$.



The maximum power delivery to the circuit occurs when the time-scale of the input $1/a \approx 3.29$ is near the intrinsic time-scale of the system 4.

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