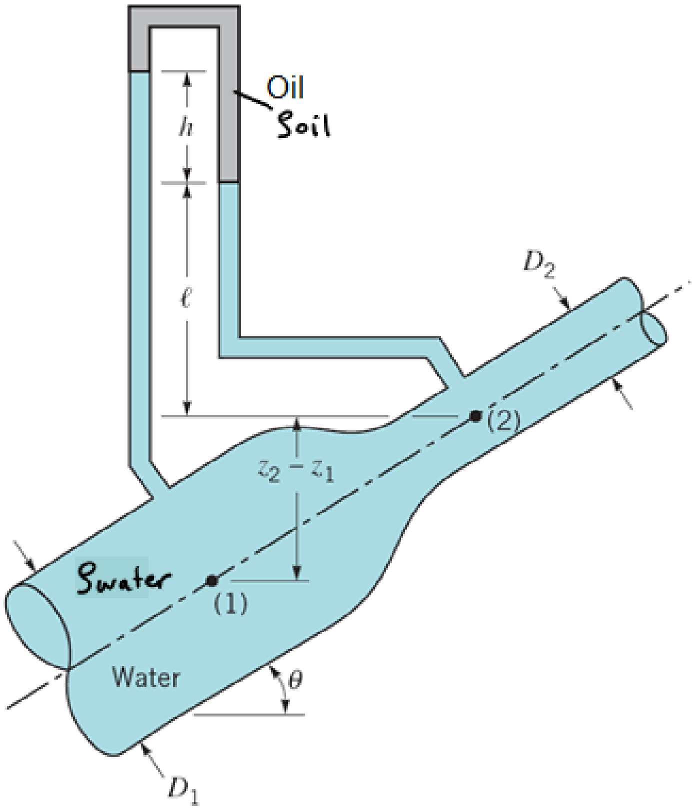
Question:

Water flows through a pipe reducer as shown in the figure. The static pressures at (1) and (2) are measured by the U-tube manometer containing oil with density ρ_{oil}. Determine the expression for the manometer reading (h) in terms of the density of water (ρ_{water}), the density of oil (ρ_{oil}), gravitational acceleration (g), volume flow rate ($\dot{V}$) and the area ratio (A_2/A_1). (Assume steady, inviscid, incompressible flow and uniform velocity profiles over the cross-sections of the pipe.)



Solution:

With the assumptions of steady, inviscid, incompressible flow, the Bernoulli equation can be written as

$$p_1 + \frac{1}{2}\rho_w V_1^2 + \rho_w g z_1 = p_2 + \frac{1}{2}\rho_w V_2^2 + \rho_w g z_2 \qquad ①$$

The continuity equation provides a second relationship between V_1 and V_2 if we assume the velocity profiles are uniform at those two locations, for the incompressible fluid we can write:

$$\underbrace{\dot{\forall}}_{\substack{\uparrow \\ \text{volumetric} \\ \text{flow rate}}} = A_1 V_1 = A_2 V_2$$ ②

From continuity eqn (eqn. 2), $V_1 = \frac{A_2}{A_1} V_2$

By combining Bernoulli eqn. with the continuity eqn., we can write;

$$p_1 - p_2 = \rho_w g (z_2 - z_1) + \frac{1}{2} \rho_w V_2^2 \left(1 - \left(\frac{A_2}{A_1} \right)^2 \right)$$ ③

The pressure difference is measured by the manometer and can be determined using the hydrostatic pressure variation with elevation.

$$p_1 - \rho_w g (z_2 - z_1) - \cancel{\rho_w g l} - \rho_w g h + \rho_{oil} g h + \cancel{\rho_w g l} = p_2$$

$$\boxed{p_1 - p_2 = \rho_w g (z_2 - z_1) + g h (\rho_w - \rho_{oil})}$$

Then we can place the pressure difference found in the above equation (3)

$$\cancel{\rho_w g(z_2 - z_1)} + gh(\rho_w - \rho_{oil}) = \cancel{\rho_w g(z_2 - z_1)} + \frac{1}{2}\rho_w V_2^2\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)$$

$$h = \frac{1}{2g}\,\frac{\rho_w}{(\rho_w - \rho_{oil})}\,V_2^2\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)$$

$$\text{since } V_2 = \frac{\overset{\text{Volumetric flow rate}}{\dot{V}}}{\underset{\text{Area at cross-section 2}}{A_2}}$$

$$\boxed{h = \frac{1}{2g}\,\frac{\rho_w}{(\rho_w - \rho_{oil})}\left(\frac{\dot{V}}{A_2}\right)^2\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}$$

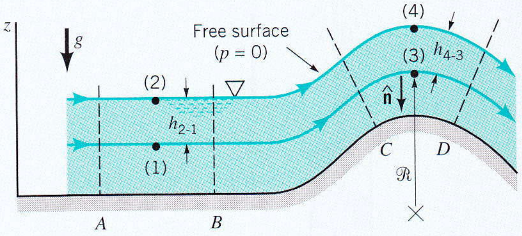
tion across straight streamlines is hydrostatic.

sure, or a combination of both. In many instances the streamlines are nearly straight ($\mathcal{R} = \infty$) so that centrifugal effects are negligible and the pressure variation across the streamlines is merely hydrostatic (because of gravity alone), even though the fluid is in motion.

EXAMPLE 3.5 Pressure Variation in a Flowing Stream

GIVEN Water flows in a curved, undulating waterslide as shown in Fig. E3.5*a*. As an approximation to this flow, consider the inviscid, incompressible, steady flow shown in Fig. E3.5*b*. From section *A* to *B* the streamlines are straight, while from *C* to *D* they follow circular paths.

■ FIGURE E3.5*a* **(Photo courtesy of Schlitterbahn® Waterparks.)**



■ FIGURE E3.5*b*

FIND Describe the pressure variation between points (1) and (2) and points (3) and (4).

SOLUTION

With the above assumptions and the fact that $\mathcal{R} = \infty$ for the portion from *A* to *B*, Eq. 3.14 becomes

$$p + \gamma z = \text{constant}$$

The constant can be determined by evaluating the known variables at the two locations using $p_2 = 0$ (gage), $z_1 = 0$, and $z_2 = h_{2-1}$ to give

$$p_1 = p_2 + \gamma(z_2 - z_1) = p_2 + \gamma h_{2-1} \qquad \text{(Ans)}$$

Note that since the radius of curvature of the streamline is infinite, the pressure variation in the vertical direction is the same as if the fluid were stationary.

However, if we apply Eq. 3.14 between points (3) and (4) we obtain (using $dn = -dz$)

$$p_4 + \rho \int_{z_3}^{z_4} \frac{V^2}{\mathcal{R}}(-dz) + \gamma z_4 = p_3 + \gamma z_3$$

With $p_4 = 0$ and $z_4 - z_3 = h_{4-3}$ this becomes

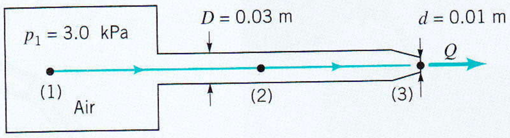
$$p_3 = \gamma h_{4-3} - \rho \int_{z_3}^{z_4} \frac{V^2}{\mathcal{R}} dz \qquad \text{(Ans)}$$

To evaluate the integral, we must know the variation of V and $\mathcal{R}$ with z. Even without this detailed information we note that the integral has a positive value. Thus, the pressure at (3) is less than the hydrostatic value, γh_{4-3}, by an amount equal to $\rho \int_{z_3}^{z_4} (V^2/\mathcal{R})\, dz$. This lower pressure, caused by the curved streamline, is necessary to accelerate the fluid around the curved path.

COMMENT Note that we did not apply the Bernoulli equation (Eq. 3.13) across the streamlines from (1) to (2) or (3) to (4). Rather we used Eq. 3.14. As is discussed in Section 3.8, application of the Bernoulli equation across streamlines (rather than along them) may lead to serious errors.

EXAMPLE 3.8 Flow from a Tank—Pressure

GIVEN Air flows steadily from a tank, through a hose of diameter $D = 0.03$ m, and exits to the atmosphere from a nozzle of diameter $d = 0.01$ m as shown in Fig. E3.8. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure.



■ **FIGURE E3.8**

FIND Determine the flowrate and the pressure in the hose.

SOLUTION

If the flow is assumed steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) as

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3$$

With the assumption that $z_1 = z_2 = z_3$ (horizontal hose), $V_1 = 0$ (large tank), and $p_3 = 0$ (free jet), this becomes

$$V_3 = \sqrt{\frac{2p_1}{\rho}}$$

and

$$p_2 = p_1 - \tfrac{1}{2}\rho V_2^2 \qquad (1)$$

The density of the air in the tank is obtained from the perfect gas law, using standard absolute pressure and temperature, as

$$\rho = \frac{p_1}{RT_1}$$
$$= [(3.0 + 101)\text{ kN/m}^2]$$
$$\times \frac{10^3 \text{ N/kN}}{(286.9 \text{ N}\cdot\text{m/kg}\cdot\text{K})(15 + 273)\text{K}}$$
$$= 1.26 \text{ kg/m}^3$$

Thus, we find that

$$V_3 = \sqrt{\frac{2(3.0 \times 10^3 \text{ N/m}^2)}{1.26 \text{ kg/m}^3}} = 69.0 \text{ m/s}$$

or

$$Q = A_3V_3 = \frac{\pi}{4}d^2V_3 = \frac{\pi}{4}(0.01\text{ m})^2(69.0\text{ m/s})$$
$$= 0.00542 \text{ m}^3\text{/s} \qquad \textbf{(Ans)}$$

The pressure within the hose can be obtained from Eq. 1 and the continuity equation (Eq. 3.19)

$$A_2V_2 = A_3V_3$$

Hence,

$$V_2 = A_3V_3/A_2 = \left(\frac{d}{D}\right)^2 V_3$$
$$= \left(\frac{0.01\text{ m}}{0.03\text{ m}}\right)^2(69.0\text{ m/s}) = 7.67\text{ m/s}$$

and from Eq. 1

$$p_2 = 3.0 \times 10^3 \text{ N/m}^2 - \tfrac{1}{2}(1.26\text{ kg/m}^3)(7.67\text{ m/s})^2$$
$$= (3000 - 37.1)\text{N/m}^2 = 2963 \text{ N/m}^2 \qquad \textbf{(Ans)}$$

COMMENTS Note that the value of V_3 is determined strictly by the value of p_1 (and the assumptions involved in the Bernoulli equation), independent of the "shape" of the nozzle. The pressure head within the tank, $p_1/\gamma = (3.0\text{ kPa})/(9.81\text{ m/s}^2)(1.26\text{ kg/m}^3) = 243$ m, is converted to the velocity head at the exit, $V_2^2/2g = (69.0\text{ m/s})^2/(2 \times 9.81\text{ m/s}^2) = 243$ m. Although we used gage pressure in the Bernoulli equation ($p_3 = 0$), we had to use absolute pressure in the perfect gas law when calculating the density.

In the absence of viscous effects the pressure throughout the hose is constant and equal to p_2. Physically, the decreases in pressure from p_1 to p_2 to p_3 accelerate the air and increase its kinetic energy from zero in the tank to an intermediate value in the hose and finally to its maximum value at the nozzle exit. Since the air velocity in the nozzle exit is nine times that in the hose, most of the pressure drop occurs across the nozzle ($p_1 = 3000\text{ N/m}^2$, $p_2 = 2963\text{ N/m}^2$, and $p_3 = 0$).

Since the pressure change from (1) to (3) is not too great [i.e., in terms of absolute pressure $(p_1 - p_3)/p_1 = 3.0/101 = 0.03$], it follows from the perfect gas law that the density change is also not significant. Hence, the incompressibility assumption is reasonable for this problem. If the tank pressure were considerably larger or if viscous effects were important, the above results would be incorrect.

Fluids in the News

Hi-tech inhaler The term inhaler often brings to mind a treat-

liquid to divide into a mist of micron-scale droplets. The device

3.85

3.85 Water flows from the pipe shown in Fig. P3.85 as a free jet and strikes a circular flat plate. The flow geometry shown is axisymmetrical. Determine the flowrate and the manometer reading, *H*.

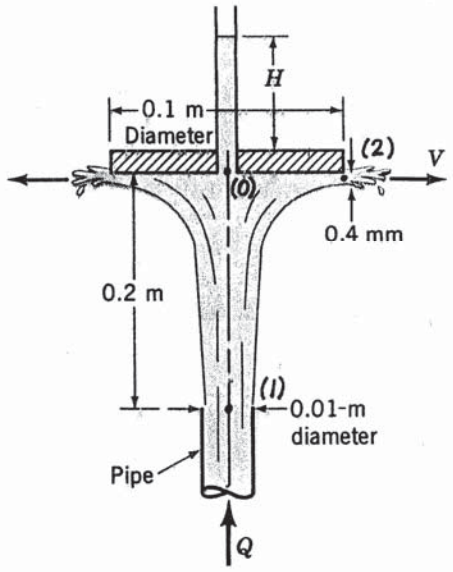


FIGURE P3.85

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2, \text{ where } p_1=0,\ p_2=0,\ z_1=0, \text{ and } z_2=0.2m$$

Thus,

$$\frac{V_1^2}{2g}=\frac{V_2^2}{2g}+z_2 \quad \text{where} \quad A_1V_1=A_2V_2=Q \qquad (1)$$

or $V_1=\frac{A_2}{A_1}V_2=\frac{\pi D_2 h}{\frac{\pi}{4}D_1^2}V_2=\frac{4D_2h}{D_1^2}V=\frac{4(0.1m)(4\times10^{-4}m)}{(0.01m)^2}V_2=1.6V_2$

Hence, Eq. (1) gives

$(1.60V_2)^2=V_2^2+2(9.81\frac{m}{s^2})(0.2m)$ or $V_2=1.59\frac{m}{s}$

so that

$$Q=A_2V_2=\pi(0.1m)(4\times10^{-4}m)(1.59\frac{m}{s})=\underline{\underline{2.00\times10^{-4}\frac{m^3}{s}}}$$

Also,

$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_0}{\gamma}+\frac{V_0^2}{2g}+z_0$, where $V_0=0$, $z_0=0.2m$, $V_1=1.60V_2$

or $V_1=1.60(1.59\frac{m}{s})=2.54\frac{m}{s}$, and $p_1=0$

Thus,

$$H=\frac{p_0}{\gamma}=\frac{V_1^2}{2g}-z_0=\frac{(2.54\frac{m}{s})^2}{2(9.81\frac{m}{s^2})}-0.2m=\underline{\underline{0.129\ m}}$$