University of Toronto, Faculty of Applied Science and Engineering

MAT292H1F - Ordinary Differential Equations

Final Exam - December 15, 2018

Examiners: A. Stinchcombe and A. Khovanskii

Time allotted: 150 minutes | Aids permitted: None

Total marks: 100

Full Name: ______________________________
Last | First

Student Number: ______________________________

Email: ______________________________ @mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test is **double-sided**. Make sure you don't skip any problems.
- This test contains 18 pages, including this title page and a formula sheet.
  Make sure you have all of them.
- You can use pages 14–16 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 14–16.
- No calculators, cellphones, or any other electronic gadgets are allowed.
- You may detach the formula sheet. Work on the formula sheet will NOT be graded.

SECTION I No explanation is necessary. **(26 marks)**

For questions 1–6, please fill in the blanks.

1. (2 marks) Find the stable ($y = a$) and unstable ($y = b$) equilibrium points of $y' = e^{2y} - 4e^y + 3$.

$a = \underline{0}$ $b = \underline{\ln 3}$.

2. (2 marks) The solution to the initial value problem $\boldsymbol{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x}(t)$, $\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is

$\boldsymbol{x}(t) = \underline{\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}}$.

3. (2 marks) The solution to the initial value problem $\frac{d^4y}{dt^4} + 7\frac{d^3y}{dt^3} = 0$, $y(0) = 1$, $y'(0) = y''(0) = y'''(0) = 0$ is

$y(t) = \underline{1}$.

4. (2 marks) State a first order autonomous differential equation $y' = f(y)$ for which Euler's method gives exactly correct values (for any stepsize):

$f(y) = \underline{c, c \in \mathbb{R}}$.

5. (2 marks) Assume that the function $z(t) = \sin(t-1)$ satisfies the equation $y''(t)+p(t)y'(t)+q(t)y(t) = 0$ for $1 \leqslant t \leqslant a$. For which value(s) of a does the function $z(t)$ satisfy the boundary condition $y(1) = y(a)$.

$a = \underline{k\pi + 1, k \in \mathbb{Z}, k \geqslant 1}$.

6. (2 marks) For $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, the matrix exponential $e^{At} = \underline{\begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}}$.

For questions 7–13, circle **True** or **False**.

7. (2 marks) The initial value problem $\sin(y' - y) = 0$, $y(0) = 0$ has a unique solution. **True** $\boxed{\textbf{False}}$

8. (2 marks) The solution to $y'(t) = \exp(y)\cos(y)$, $y(0) = 0$, exists for all t. $\boxed{\textbf{True}}$ **False**

9. (2 marks) For all differential equations and all stepsizes h, the improved Euler method is more accurate than the Euler method. **True** $\boxed{\textbf{False}}$

10. (2 marks) The equilibrium point of $\boldsymbol{x}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \boldsymbol{x}$ is stable. $\boxed{\textbf{True}}$ **False**

11. (2 marks) $\mathcal{L}\{\exp(t)\cos(t)\sin(t)\}(s)$ is a rational function of s. $\boxed{\textbf{True}}$ **False**

12. (2 marks) All solutions of $y'' + y = \cos t$ are bounded. **True** $\boxed{\textbf{False}}$

13. (2 marks) The Wronskian $W[y_1, y_2](t)$ for solutions y_1, y_2 of $y'' + p(t)y' + q(t)y = 0$ can not take values -1, 0, and 1 at the points $t = 1$, $t = 2$, and $t = 3$ correspondingly. $\boxed{\textbf{True}}$ **False**

SECTION II **Justify** your answers. **(74 marks)**

14. Find a function $F(x, y)$ and a constant C such that $F(x, y(x)) = C$ is an implicit solution to the initial value problem $(2y + x)y' + (2x + y) = 0$, $y(1) = 1$. **(5 marks)**

Solution: The equation is exact since $\frac{\partial(2y+x)}{\partial x} = \frac{\partial(2x+y)}{\partial y} = 1$. We have to find $F(x, y)$ such that $\frac{\partial F}{\partial x} = 2x+y$ and $\frac{\partial F}{\partial y} = 2y+x$. From the first equation we have $F = x^2+xy+K(y)$. From the second equation we have $K'(y) = 2y$. Thus $F(x, y) = x^2 + xy + y^2$. On each solution $y(x)$ the function $F(x, y)$ is a constant C. Plugging $x = 1$ and $y = 1$ we obtain that $C = 3$. Thus the graph of the solution $y(x)$ belongs to the curve $x^2 + xy + y^2 = 3$.

15. Let $y(t)$ for $-\infty < t < \infty$ be the solution of $ay'' + by' + cy = 0$, where a, b, and c are constants and the initial condition is $y(0) = 0, y'(0) = a^{-1}$. Let $z(t)$ be the impulse response, so that $az'' + bz' + cz = \delta(t)$ and $z(0) = 0 = z'(0)$. **(5 marks)**

a) **(2 marks)** Show that $\mathcal{L}\{y\} = \mathcal{L}\{z\}$.

Solution: Applying the Laplace transform we get: $(as^2 + bs + c)\mathcal{L}\{y\} - aa^{-1} = 0$ and $(as^2 + bs + c)\mathcal{L}\{z\} = 1$. Thus $\mathcal{L}\{y\} = \frac{1}{as^2+bs+c} = \mathcal{L}\{z\}$.

b) **(3 marks)** Is it true that $y(t) = z(t)$ for all real t?

Solution: The identity $\mathcal{L}\{z\} = \mathcal{L}\{y\}$ means that for $t > 0$ the continuous functions y and z have the same values. But for $t < 0$ the function z is equal to zero and the function y is not zero (it is given by the same formula as for $t > 0$). Thus $z(t) \neq y(t)$ (in fact $z(t) = y(t)u_0(t)$).

16. Consider the initial value problem $y' + ay = g(t)$, $y(0) = 0$, where a is a constant. **(9 marks)**
Find the solution $y(t)$ for $t > 0$ and express it in the exact same form using the following three methods:

a) **(3 marks)** the integrating factor method
Solution: Let $\mu(t) = \exp(at)$. Then $(\mu y)' = \mu y' + \mu ay = \mu g(t)$. Thus $\mu y = \int_0^t \exp(a\tau)g(\tau)d\tau$ and $y = \exp(-at)\int_0^t \exp(a\tau)g(\tau)d\tau$.

b) **(3 marks)** using the Laplace transform and the convolution theorem
Solution: Applying the Laplace transform we obtain $\mathcal{L}\{y\}(s+a) = \mathcal{L}\{g\}$, or $\mathcal{L}\{y\} = \mathcal{L}\{h\}\mathcal{L}\{g\}$ where $h = \mathcal{L}^{-1}\{\frac{1}{s+a}\} = \exp(-at)$. According to the convolution theorem
$y = \int_0^t \exp\left(-a(t-\tau)\right) g(\tau)d\tau = \exp(-at)\int_0^t \exp(a\tau)g(\tau)d\tau$.

c) **(3 marks)** the method of variation of parameters: suppose that $y(t) = c(t)y_1(t)$ for y_1 a solution to the homogeneous equation and then solve for $c(t)$.
Solution: Separating variables gives $y_1(t) = \exp(-at)$. Substituting $y = c(t)\exp(-at)$, we find that $c'(t)\exp(-at) = g(t)$.
Therefore $c(t) = \int_0^t \exp(\tau a)g(\tau)d\tau$ and $y(t) = \exp(-at)\int_0^t \exp(a\tau)g(\tau)d\tau$.

17. Consider the initial value problem $y' + ay = \exp(bt)$, $y(0) = 0$, where $b \neq a$ are constants. Find the solution $y(t)$ for $t > 0$ using the following methods: **(9 marks)**

a) **(3 marks)** the method of undetermined coefficients

Solution: Try a particular solution in the form $y_p = C\exp(bt)$ where C is an undetermined constant. We obtain $bC\exp bt + aC\exp(bt) = \exp(bt)$. Thus $C = \frac{1}{a+b}$ and $y_p = \frac{1}{a+b}\exp(bt)$. General solution of the homogeneous equation $y' + ay = 0$ is $C\exp(-at)$, where $C = y(0)$. For y_p we have $y_p(0) = \frac{1}{a+b}$. Thus $y = y_p - \frac{1}{a+b}\exp at = \frac{1}{a+b}(\exp(bt) - \exp(at))$.

b) **(3 marks)** using the Laplace transform

Solution: $\mathcal{L}\{y\}(s+a) = \mathcal{L}\{\exp(bt)\} = \frac{1}{s-b}$. Thus $\mathcal{L}\{y\} = \frac{1}{(s+a)(s-b)} = \frac{1}{a+b}\left(\frac{1}{s-b} - \frac{1}{s+a}\right)$. Thus $y(t) = \frac{1}{a+b}(\exp(bt) - \exp(at))$.

c) **(3 marks)** evaluating the integral in your answer to problem **16** with $g(t) = \exp(bt)$.

Solution: $y = \exp(-at)\int_0^t \exp(a\tau)\exp(b\tau)d\tau = \exp(-at)\frac{1}{a+b}(\exp[(a+b)t] - 1) = \frac{1}{a+b}(\exp(bt) - \exp(at))$.

18. Consider the autonomous system, $x_1'(t) = -x_2(t), \quad x_2'(t) = 4x_1(t).$ **(10 marks)**

a) **(2 marks)** What are the eigenvalues of the coefficient matrix?

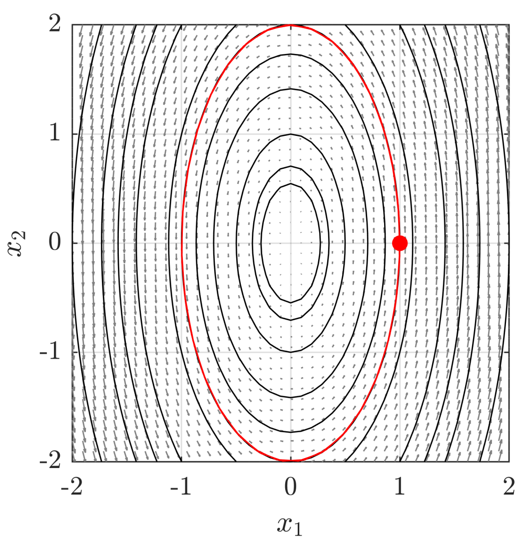
Solution: The characteristic equation is $(-\lambda)(-\lambda) + 4 = 0$ with roots $\lambda = \pm 2i$.

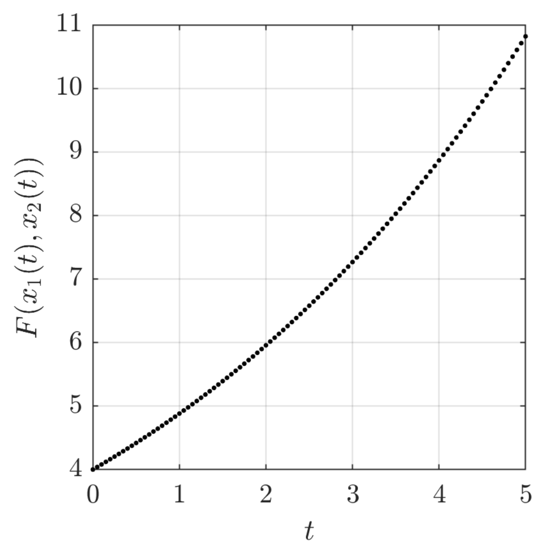
b) **(2 marks)** Does the trajectory starting at $x_1(0) = 1, x_2(0) = 0$ return to its initial value? To decide, calculate the time derivative of $F(x_1, x_2) = 4x_1^2 + x_2^2$.

Solution: $\frac{d}{dt}F(x_1, x_2) = 8x_1x_1' + 2x_2x_2' = -8x_1x_2 + 8x_2x_1 = 0$. Since F, with elliptical contours, is constant on trajectories, the trajectory start at $(1, 0)$ will return to its initial value.

c) **(4 marks)** Sketch the phase plane on axes below, to the left. Include the contours (curves of constant value) of F and the trajectory passing through $(1, 0)$.

Solution:





d) **(2 marks)** The plot above, on the right, shows the value of F evaluated from an Euler's method numerical solution with stepsize $h = 0.1$. Why does F increase instead of remaining constant?

Solution: The numerical method introduces error, which means that F no longer needs to be exactly constant. The error systemically increases F since in Euler's method, you move along the tangent of the elliptical contour of F, which results in the value of the solution at the end of each step having a larger value of F (ellipses are convex).

19. Consider the system of equations $\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ \alpha & 0 \end{pmatrix} \mathbf{x}(t)$ with real parameter α. **(8 marks)**

a) **(3 marks)** For which values of α is $\mathbf{0}$ the unique unstable critical point of the system?

Solution: The characteristic equation is $(2-\lambda)(-\lambda) - \alpha = 0$ with roots $\lambda = 1 \pm \sqrt{1+\alpha}$. For $\alpha = 0$ ($\lambda = 0, 2$), there will be more than one critical point. Otherwise, there is only one critical point. For $\alpha > 0$, $\mathbf{0}$ is a saddle, which is unstable. For $-1 \leqslant \alpha < 0$, $\mathbf{0}$ is an unstable node. For $\alpha < -1$, $\mathbf{0}$ is an unstable spiral. Therefore, for $\alpha \neq 0$, $\mathbf{0}$ is the unique unstable critical point of the system.

b) **(5 marks)** Find the general (real) solution for $\alpha = -2$.

Solution: The eigenvalues are $1 \pm i$ and the eigenvectors are $\begin{pmatrix} -1 \mp i \\ 2 \end{pmatrix}$. The general solution is thus $\mathbf{x} = c_1 e^t \begin{pmatrix} -\cos(t) + \sin(t) \\ 2\cos(t) \end{pmatrix} + c_2 e^t \begin{pmatrix} -\sin(t) - \cos(t) \\ 2\sin(t) \end{pmatrix}$.

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20. An igloo is heated by an oil lamp called a qulliq. Let $y(t)$ represent the temperature of the igloo in degrees Celsius at time t in hours, which is modelled by the initial value problem **(10 marks)**

$$y'(t) = -0.1(y(t) + 50) + 5u_a(t), \quad y(0) = -50.$$

a) **(3 marks)** Describe the assumptions that resulted in this initial value problem.

Solution: When the qulliq is lit, the temperature in the igloo increases at a constant rate of 5 degrees Celsius per hour. When it is unlit, it is not heating the igloo at all. According to Newton's law of cooling, the temperature in the igloo decreases at a rate proportional to the difference between the current temperature of the igloo $y(t)$ and the external temperature of -50 with proportionality constant of 0.1. At time $t = 0$, the lamp is off and the temperature in the igloo is -50 degrees Celsius. At time $t = a > 0$, the qulliq is lit and remains lit for a long time.

b) **(5 marks)** Find $Y(s) = \mathcal{L}\{y(t)\}(s)$ and invert the Laplace transform to find $y(t)$.

Solution: $sY(s) + 50 = -0.1\left(Y(s) + 50\frac{1}{s}\right) + 5\frac{e^{-sa}}{s},$

$$\begin{aligned} Y(s) &= \frac{1}{s+0.1}\left(-5\frac{1}{s} + 5\frac{e^{-sa}}{s} - 50\right) = \frac{-5 - 50s}{s(s+0.1)} + 5\left(\frac{1}{s(s+0.1)}\right)e^{-sa} \\ &= 50\left(\frac{-1}{s} + \left(\frac{0.1}{s(s+0.1)}\right)e^{-as}\right) = 50\left(\frac{-1}{s} + \left(\frac{1}{s} - \frac{1}{s+0.1}\right)e^{-as}\right) \\ y(t) &= 50\left(-1 + u_a(t) - u_a(t)e^{-0.1(t-a)}\right) = 50\left(-1 + u_a(t)(1 - e^{-0.1(t-a)})\right) \end{aligned}$$

c) **(2 marks)** When should the lamp be lit so that the temperature in the igloo will be -25 degrees Celsius at time $t = 24$ hours?

Solution:

$$\begin{aligned} -25 &= 50\left(-1 + u_a(24)(1 - e^{-0.1(24-a)})\right) \\ \frac{1}{2} &= e^{-0.1(24-a)} \\ 10\ln 2 &= 24 - a \\ a &= 24 - 10\ln 2 \end{aligned}$$

21. A qualitative model of the human circadian clock **(8 marks)**

(the body's light-driven, 24-hour time-keeping mechanism) is given by the differential equation

$$y'' + \frac{\pi}{30}y' + \left(\frac{2\pi}{24}\right)^2 y = L(t),$$

in which t is the time in hours since sunrise, $L(t)$ is the light input that drives the clock, and y is the circadian output variable which typically oscillates with a period of 24-hours. The variable y corresponds directly to body temperature, which rhythmically varies by 1 degree Celsius each day.

(a) **(2 marks)** Is the system undamped, underdamped, critically damped, or overdamped?
Solution: The characteristic equation is $\lambda^2 + \frac{\pi}{30}\lambda + (\frac{2\pi}{24})^2 = 0$ with discriminant $\frac{\pi^2}{900} - 4(\frac{2\pi}{24})^2 = \frac{-24\pi^2}{900} < 0$. Therefore, the system is underdamped. A model that exhibits decaying oscillations makes sense for a model of the circadian clock.

(b) **(2 marks)** For constant light input, what is the long-run behaviour of the body temperature?
Solution: Using the analogy of the mass-spring-damper system, a constant force changes the equilibrium. Therefore, in constant light, the body temperature will oscillate and decay in amplitude towards a constant value - the circadian rhythm disappears.

(c) **(2 marks)** Does the light input $L(t) = \frac{1}{2}\left[1 + \sin\left(\frac{2\pi}{24}t\right)\right]$ result in unbounded solutions? Explain.
Solution: No. Since the homogeneous system exhibits a *decaying* oscillation, the solutions are of the form $e^{at}\cos(\omega t)$ and $e^{at}\sin(\omega t)$, which will not resonate with sinusoidal forcing, even with frequency ω, or any 24-hour periodic signal.

(d) **(2 marks)** How would the body temperature behave on Mars with 25-hour long days?
Solution: The body temperature will oscillate with a period of 25-hours. The method of undetermined coefficients requires a form of the particular solution of $c_1\sin(\frac{2\pi}{25}t) + c_2\cos(\frac{2\pi}{25}t) + c_3$. Since the homogeneous solutions decay, only the particular solution will persist leaving a 25-hour periodic solution.

22. A ball has mass m and position $x(t)$, a function of time. **(10 marks)**

In a *potential well*, the ball's position is governed by the differential equation $mx'' = -V'(x)$ for potential $V(x) = x^{2p}$ for positive integer p.

a) **(2 marks)** Find any equilibrium solutions and classify them as stable or unstable.

Solution: The only equilibrium is $x' = 0, x = 0$. It is stable since $-V'(x) = -2px^{2p-1}$ is negative for $x > 0$ and positive for $x < 0$.

b) **(2 marks)** Show that the energy of the ball $E = \frac{1}{2}m(x')^2 + V(x)$ is constant, i.e. $\frac{dE}{dt} = 0$ for $x(t)$ a solution of the differential equation.

Solution: $\frac{dE}{dt} = \frac{1}{2}m(2x'x'') + V'(x)x' = x'(-V(x)) + V'(x)x' = 0$, so E is constant.

c) **(2 marks)** In the limit $p \to \infty$, $V(x) = 0$ for $x \in [-1, 1]$. Explain why the ball is confined within $[-1, 1]$ and why, in the long-run, it spends an equal amount of time near each position $x \in [-1, 1]$. Use the initial condition $x(0) = 0, x'(0) = 1$.

Solution: Any initial condition implies a finite energy, which limits the values of x that are possible, i.e $V(x) < E$. As $p \to \infty$, the potential goes to ∞ outside $[-1, 1]$ and therefore the ball position is limited to be within $[-1, 1]$. Since $V = 0$ in $[-1, 1]$, the ball moves with constant speed and experiences an impulse at $x = \pm 1$ that reserves its direction. Moving with constant speed means that it spends an equal amount of time near each value of x.

d) (3 marks) If the ball is very small, it will not spend an equal amount of time near each point in $[-1, 1]$ in the limit $p \to \infty$. According to quantum physics, the position of a particle is determined from its wave-function $\psi(x)$ as $\int_a^b |\psi(x)|^2 dx =$ the probability of finding the ball in $[a, b]$. In the case of an infinite square well potential ($p \to \infty$), the steady-state wave-function $\psi(x)$ satisfies the differential equation

$$\frac{d^2\psi}{dx^2} = -k^2\psi,$$

with two boundary conditions $\psi(-1) = 0 = \psi(1)$ for a parameter $k > 0$. Solve for $\psi(x)$ and show that only particular values of k (particle energies) are permitted.

Solution: The general solution is $\psi(x) = c_1 \sin(kx) + c_2 \cos(kx)$. The boundary conditions give $c_1 \sin(k) + c_2 \cos(k) = 0$ and $c_1 \sin(-k) + c_2 \cos(-k) = -c_1 \sin(k) + c_2 \cos(k) = 0$. Adding these equations gives $2c_2 \cos(k) = 0$, so $c_2 = 0$ or $k = \frac{\pi}{2} + \pi n$. Subtracting gives $2c_1 \sin(k) = 0$, so $c_1 = 0$ or $k = \pi n$. The solutions are $c_1 \sin(\pi n x)$ for positive integer n (so that $k > 0$) and $c_2 \cos(\pi n x + \frac{\pi}{2}x)$ for non-negative integer n (so that $k > 0$). Any non-zero values are permitted for c_1 and c_2. If k does not have one of these discrete values then there is no solution to the differential equation with those boundary conditions.

Although the particle spends different amounts of time near different x, it becomes uniform as $k \to \infty$.

e) (1 mark) Explain why $f(x) = \frac{\sqrt{15}}{4}(1 - x)(1 + x)$ can be written as a linear combination of solutions $\psi(x)$ from part **d**.

Solution: The function $f(x) = \frac{\sqrt{15}}{4}(1-x)(1+x)$ satisfies $f(-1) = 0 = f(1)$ and can be written in a Fourier series consisting of the solutions to the boundary value problem determined above. Since $f(x)$ is even, only the $\cos(\pi n x + \frac{\pi}{2}x)$ solutions are needed. In particular,

$$f(x) = \sum_{n=0}^{\infty} \frac{8\sqrt{15}(-1)^n}{\pi^3(2n+1)^3} \cos\left(\pi n x + \frac{\pi}{2}x\right),$$

but it is not necessary to compute the Fourier coefficients to answer this question.

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FORMULA SHEET

First-Order Linear Differential Equations. $y' + p(t)y = g(t)$.

- $\mu(t) = e^{\int p(t)\,dt}$
- $y = \dfrac{1}{\mu(t)} \displaystyle\int \mu(t)g(t)\,dt + \dfrac{C}{\mu(t)}$.

Exact First-Order Differential Equations. $M(x,y) + N(x,y)y' = 0$

- Exact if and only if $M_y = N_x$.
- Solution $\Psi(x,y) = C$ where $\Psi_x = M$ and $\Psi_y = N$.

Euler Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $t_n = t_0 + n \cdot h$
- $y_{n+1} = y_n + f(t_n, y_n)h$ or $y'(t_n) = \dfrac{y_{n+1} - y_n}{h}$
- $E_n \leqslant Ch$

Improved Euler Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + k_{n,2}}{2}h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f(t_{n+1}, y_n + k_{n,1}h)$
- $E_n \leqslant Ch^2$

Runge-Kutta Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4}}{6}h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f\left(t_n + \frac{h}{2}, y_n + k_{n,1}\frac{h}{2}\right)$
- $k_{n,3} = f\left(t_n + \frac{h}{2}, y_n + k_{n,2}\frac{h}{2}\right)$
- $k_{n,4} = f(t_{n+1}, y_n + k_{n,3}h)$
- $E_n \leqslant Ch^4$

Euler's Formula. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Limits and Series.

- $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $r < 1$.
- $\displaystyle\exp(At) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$.
- $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}A\right)^n = e^A$.

Variation of Parameters.

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W[y_1, y_2](t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)} dt$$

Laplace Transforms.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$
$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a},$$
$$\mathcal{L}\{\sin(k\,t)\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(k\,t)\} = \frac{s}{s^2+k^2},$$
$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0),$$
$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0),$$
$$\mathcal{L}\{e^{a\,t}\,f(t)\} = F(s-a), \quad \mathcal{L}\{u_a(t)\,f(t-a)\} = e^{-s\,a}\,F(s),$$
$$\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\frac{d^n}{ds^n}F(s),$$
$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} \text{ for } T\text{-periodic } f,$$
$$\mathcal{L}\{f * g\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right\} = F(s)\,G(s),$$
$$\mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}.$$

 The end.