UNIVERSITY OF TORONTO, FACULTY OF APPLIED SCIENCE AND ENGINEERING

MAT292H1F - Ordinary Differential Equations

Final Exam - December 16, 2019

EXAMINERS: A. STINCHCOMBE AND F. PARSCH

Time allotted: 150 minutes

Aids permitted: None

Total marks: 93

Full Name: ______________________________
Last First

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- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers and brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 19 pages (including this title page). Make sure you have all of them.
- You can use pages 14–17 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 14–17.
- You may detach the formula sheet. Work on the formula sheet will NOT be graded.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

Q1-Q10	Q11	Q12	Q13	Q14	Q15	Q16	Q17	Q18	**Total**
20	5	10	10	10	10	10	10	8	**93**

HAVE FUN!

SECTION I Short answer section. Only justify your answer when asked.

1. **(2 marks)** State and classify all equilibria of $y' = \sin(y)$.

2. **(2 marks)** Consider a linear ODE $y' = f(t, y)$ where f and all its derivatives are continuous. Let y_1 be a solution with $y_1(0) = 3$ and y_2 be a solution with $y_2(0) = 4$. If you know that $\lim_{t\to\infty} y_1(t) = \infty$, what can you conclude about y_2? Justify.

3. **(2 marks)** Consider a 2-dimensional system $\vec{x}\,' = A\vec{x}$. Find a matrix A that gives a **counterexample** to the following statement: *For all solutions, we have either* $\lim_{t\to\infty} |\vec{x}(t)| = \infty$ *or* $\lim_{t\to\infty} |\vec{x}(t)| = 0$

4. **(2 marks)** Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\}$.

5. **(2 marks)** Consider the following PDE known as the wave equation: $u_{tt}(x,t) = a^2 u_{xx}(x,t)$. Assuming that we can separate the variables $u(x,t) = X(x)T(t)$, write down the ODEs that the wave equation produces.

For each of the following statements, do two things: Make a choice if it is TRUE or FALSE. Then give a brief justification of your choice. Remember: TRUE means that the statement is **always** true, and not just true in a special case.

6. **(2 marks)** Every boundary value problem $y'' + a\,y = 0$, $y(0) = 0$, $y(\pi) = 0$ has at least one solution, no matter the value of the constant a. ◯ TRUE ◯ FALSE

Justification:

7. **(2 marks)** When solving $ay'' + by' + cy = g(t)$, the method of undetermined coefficients can **NOT** be used if $g(t)$ is a polynomial of degree three or more. ◯ TRUE ◯ FALSE

Justification:

8. **(2 marks)** If the ODE of an IVP $y' = f(t, y)$, $y(0) = c$ is both linear and separable, using either integrating factor or separation of variables will give the same result. ◯ TRUE ◯ FALSE

Justification:

9. **(2 marks)** If $f(t)$ and $g(t)$ are both bounded, then $(f * g)(t)$ is bounded. ◯ TRUE ◯ FALSE

Justification:

10. **(2 marks)** The improved Euler method perfectly approximates any first-order linear IVP. ◯ TRUE ◯ FALSE

Justification:

SECTION II Long answer section. **Justify** all your answers.

11. (5 marks) Consider a function $f(t)$ of exponential order such that $\lim_{t\to\infty} f(t)$ exists.

In this question, you are asked to prove the *final value theorem*:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s)$$

(a) **(2 marks)** Show that $\mathcal{L}\left\{f'\right\} = sF(s) - f(0)$.

(b) **(2 marks)** Show that $\lim_{s\to 0^+} \mathcal{L}\left\{f'\right\} = \left[\lim_{t\to\infty} f(t)\right] - f(0)$.
Hint: for this part, do not use the result from (a).

(c) **(1 mark)** Finally, show that $\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s)$.

12. (10 marks) Consider the following initial value problem describing an oscillator:

$$ay'' + by' + cy = g(t), \quad y(0) = 1, \quad y'(0) = -1$$

You are given that the *transfer function* of this oscillator is $H(s) = \dfrac{1}{s^2 + 2s + 2}$.

(a) **(2 marks)** Use the Laplace Transform to convert the IVP into an algebraic equation and solve for $Y(s)$.

(b) **(2 marks)** Based on the information that you have, state the values of the three coefficients of the ODE. Justify your choice.

$a =$ __________ $b =$ __________ $c =$ __________

(c) **(2 marks)** Find the impulse response $h(t)$.

(d) **(4 marks)** Express the solution of the above IVP in terms of the function $g(t)$.

Continued...

13. (10 marks) Find the general solution to $(1-t)y'' + ty' - y = -e^t(t-1)^2$ using the following steps.

(a) **(3 marks)** Check if any of the following functions are solutions to the associated homogeneous (complementary) equation: $y_1(t) = 1,\ y_2(t) = t,\ y_3(t) = t^2,\ y_4(t) = e^t,\ y_5(t) = e^{-t}$.

(b) **(3 marks)** Compute the Wronskian of the solutions that you found.
For which value(s) of t is the Wronskian zero?

(c) **(4 marks)** Use the variation of parameters formula to find the general solution.

14. (10 marks) Consider a metal rod whose temperature distribution over time is given by $u(x,t)$. There is a heating and cooling device attached along the rod that is controlled by a thermostat set to 20 degrees Celsius.

(a) **(6 marks)** Of the following partial differential equations, only one can govern the physical situation outlined above. Do the following:

- First, choose the one plausible PDE.
- Then, for your choice, explain briefly how the equation matches the physical description.
- Finally, for **each** of the other five equations, give one physical argument why this equation can not govern the situation.

Choose one equation	**Explain why each does/does not match the situation**
◯ $u_t = \alpha^2 u_{xx} + \beta^2(u - 20)$	
◯ $u_t = \alpha^2 u_{xx} + \beta^2(20 - u)$	
◯ $u_t = \alpha^2 u_{xx} + \beta^2(u_x - 20)$	
◯ $u_t = \alpha^2 u_{xx} + \beta^2(20 - u_x)$	
◯ $u_t = \alpha^2 u_{xx} + \beta^2(u_{xx} - 20)$	
◯ $u_t = \alpha^2 u_{xx} + \beta^2(20 - u_{xx})$	

(b) **(4 marks)** For this part, use the one equation that you chose above.

We want to simplify the problem as follows: Substitute $v(x,t) = f(t)(u(x,t) - 20)$ and choose $f(t)$ such that v solves the usual heat equation $v_t = \alpha^2 v_{xx}$.

Using these requirements, reduce this to an ODE only involving f. Then state a solution for f.

15. (10 marks) Matrix exponentials are an important tool to solve ODEs. In this question, we look at another way to compute them.

Let $A = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$ with characteristic polynomial $p(\lambda) = \det(A - \lambda I) = \lambda^2 - 3\lambda + 2$.

(a) **(2 marks)** What are the eigenvalues of A?

(b) **(2 marks)** A remarkable theorem of linear algebra, known as the "Cayley-Hamilton Theorem", says that every matrix is the root of its own characteristic polynomial. Verify this fact for A, i.e. check that $p(A) = A^2 - 3 \cdot A + 2 \cdot I = 0$.

(c) **(2 marks)** It can be shown that there are constants c_0 and c_1 such that
$e^{xt} = p(x)q(x,t) + c_0 + c_1 x$, where $p(x)$ is the characteristic polynomial of A.
Using that, why is $e^{\lambda t} = c_0 + c_1\lambda$ when λ is an eigenvalue of A? Why is $e^{At} = c_0 I + c_1 A$?

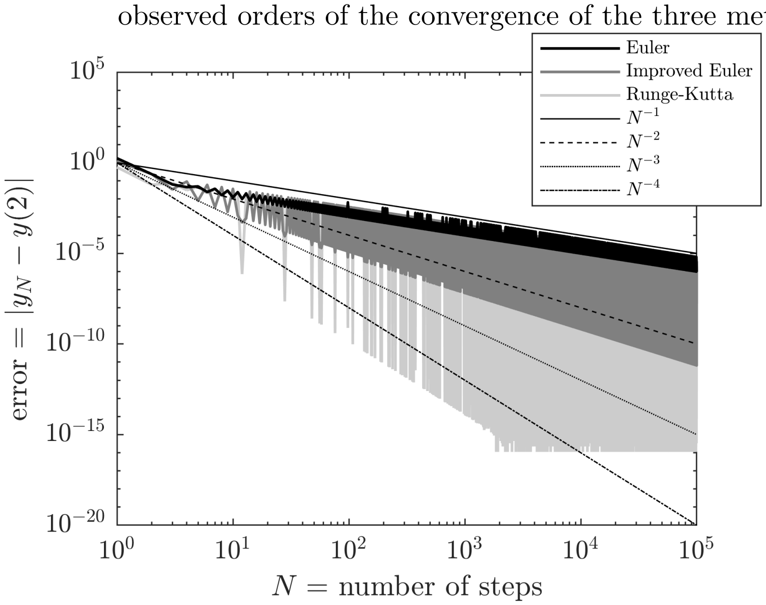
(d) **(4 marks)** Use $e^{\lambda t} = c_0 + c_1\lambda$ and the two known eigenvalues to determine c_0 and c_1. They will depend on t. Now use the second formula to compute $e^{At} = c_0 I + c_1 A$.

16. (10 marks) Consider the initial value problem $y'(t) = -y(t) + u_1(t), \quad y(0) = 1,$ where $u_1(t)$ is a unit step function.

(a) **(3 marks)** Solve the initial value problem.

(b) **(1 mark)** What is the value of $y(2)$?

(c) **(2 marks)** The error in computing $y(2)$ using Euler's method, the improved Euler method, and the Runge-Kutta method are plotted below versus the number of steps N. What are the observed orders of the convergence of the three methods?



(d) **(2 marks)** Why are the orders of convergence not as expected?

(e) **(2 marks)** Why does the error not go below approximately $2^{-53} \approx 1.11 \cdot 10^{-16}$?

17. (10 marks) In this question, we study the relationship between the impulse $\delta(t)$ and the ϵ-impulse:

$$\delta_\epsilon(t) = u_0(t)\frac{1}{\epsilon}e^{-t/\epsilon} \quad \text{for} \quad \epsilon > 0.$$

(a) **(2 marks)** Verify that $\displaystyle\int_{-\infty}^{\infty} \delta_\epsilon(t)\, dt = 1$ and that $\displaystyle\lim_{\epsilon\to 0^+} \delta_\epsilon(t) = 0$ for every $t > 0$.

Now, let's consider an impulsively forced initial value problem.

(b) **(3 marks)** Find a formula for the Laplace transform $Y(s)$ of the solution of the IVP

$$y'' + 5y' + 6y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

(c) **(3 marks)** Find a formula for the Laplace transform $Y_\epsilon(s)$ of the solution of the IVP

$$y_\epsilon'' + 5y_\epsilon' + 6y_\epsilon = \delta_\epsilon(t), \quad y_\epsilon(0) = 0, \quad y_\epsilon'(0) = 0.$$

(d) **(2 marks)** Show that $\displaystyle\lim_{\epsilon\to 0^+} Y_\epsilon(s) = Y(s)$.

18. (a) **(2 marks)** Solve $y' = y^2,\ y(0) = 1$. What is the interval of existence of the solution?

(b) **(3 marks)** Why does the initial value problem $y' = y^2, \quad y(t_0) = y_0$ have a unique solution for every t_0 and y_0?

(c) **(3 marks)** A MAT292 student made the following argument to show that the solution to the initial value problem $y' = y^2, \quad y(0) = 1$ exists for all $t > 0$:

(i) The IVP $y_1' = y_1^2, \quad y_1(0) = 1$ has a unique solution for some interval $t \in [0, h]$.

(ii) The IVP $y_2' = y_2^2, \quad y_2(h) = y_1(h)$ has a unique solution for some interval $t \in [h, 2h]$.

(iii) Repeating, the IVP $y_n' = y_n^2, \quad y_n((n-1)h) = y_{n-1}((n-1)h)$ has a unique solution for some interval $t \in [(n-1)h, nh]$. And so on...

(iv) Since eventually Nh is greater than any t, the solution $y(t)$ exists for any t by 'pasting' together the solutions $y_n, n = 1, \ldots, N$.

What is wrong with this argument?

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FORMULA SHEET

First-Order Linear Differential Equations. $y' + p(t)y = g(t)$.

- $\mu(t) = e^{\int p(t)\,dt}$
- $y = \dfrac{1}{\mu(t)} \displaystyle\int \mu(t)g(t)\,dt + \dfrac{C}{\mu(t)}$.

Exact First-Order Differential Equations. $M(x, y) + N(x, y)y' = 0$

- Exact if and only if $M_y = N_x$.
- Solution $\Psi(x, y) = C$ where $\Psi_x = M$ and $\Psi_y = N$.

Euler Method. $y' = f(t, y)$ $\quad y(t_0) = y_0$.

- $t_n = t_0 + n \cdot h$
- $y_{n+1} = y_n + f(t_n, y_n)h$ $\quad$ or $\quad$ $y'(t_n) = \dfrac{y_{n+1} - y_n}{h}$
- $E_n \leqslant Ch$

Improved Euler Method. $y' = f(t, y)$ $\quad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + k_{n,2}}{2}h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f(t_{n+1}, y_n + k_{n,1}h)$
- $E_n \leqslant Ch^2$

Runge-Kutta Method. $y' = f(t, y)$ $\quad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4}}{6}h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f\left(t_n + \frac{h}{2}, y_n + k_{n,1}\frac{h}{2}\right)$
- $k_{n,3} = f\left(t_n + \frac{h}{2}, y_n + k_{n,2}\frac{h}{2}\right)$
- $k_{n,4} = f(t_{n+1}, y_n + k_{n,3}h)$
- $E_n \leqslant Ch^4$

Euler's Formula. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Limits and Series.

- $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $r < 1$.
- $\displaystyle\exp(At) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$.
- $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}A\right)^n = e^A$.

Variation of Parameters.

$$y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W[y_1, y_2](t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)} dt$$

Laplace Transforms.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(k\,t)\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(k\,t)\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0),$$

$$\mathcal{L}\{e^{a\,t}\,f(t)\} = F(s-a), \quad \mathcal{L}\{u_a(t)\,f(t-a)\} = e^{-s\,a}\,F(s),$$

$$\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\frac{d^n}{ds^n}F(s),$$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} \text{ for } T\text{-periodic } f,$$

$$\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right\} = F(s)\,G(s),$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}.$$

The end.