University of Toronto, Faculty of Applied Science and Engineering

**MAT292H1F - Ordinary Differential Equations**

# Final Exam - December 13, 2016

Examiners: B. Galvão-Sousa and C. Sinnamon

Time allotted: 150 minutes

Aids permitted: None

Total marks: 80

Solution

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Last First

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## Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages (including this title page) and a detached formula sheet.

  Make sure you have all of them.
- You can use paged 12–14 to complete a question (**mark clearly**).

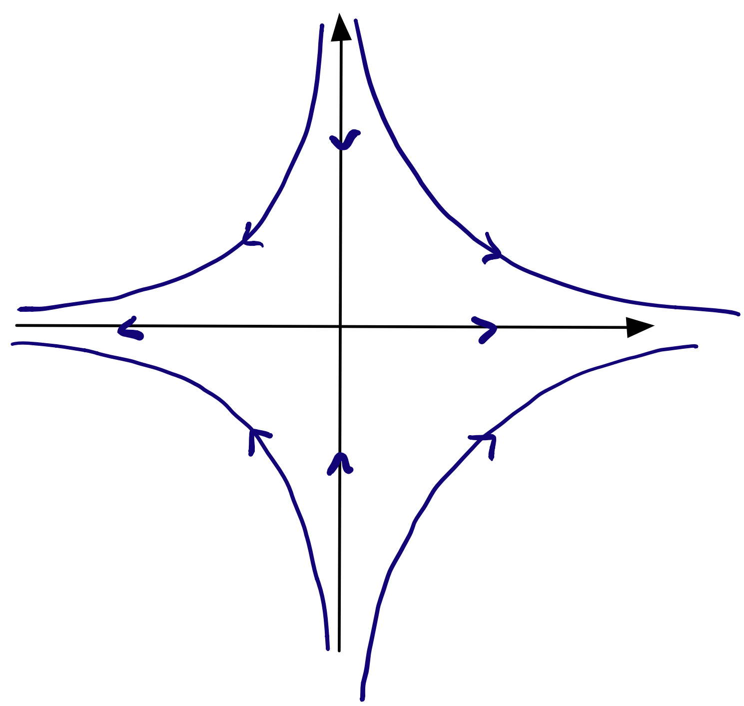
GOOD LUCK!

**PART I** No explanation is necessary. **(16 marks)**

**1.** **(2 marks)** If $y'(t) - \frac{1}{t}y(t) = 1$ with $y(1) = 1$, then $y(t) =$ $t(\ln|t| + 1)$.

**2.** **(2 marks)** If $y'(t) = e^y(1-y)(y-2)(y-4)$ with $y(-48) = \pi$ then $\lim_{t\to\infty} y(t) =$ $4$.

**3.** **(2 marks)** Sketch a phase portrait for the system $\vec{x}\,'(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \vec{x}(t)$.



**4.** **(2 marks)** An example of a solution to $\vec{x}\,'(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \vec{x}(t)$ such that $\lim_{t\to\infty} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

$\vec{x}(t) =$ $c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$ for any $c_2 \in \mathbb{R}$.

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**5.** **(4 marks)** Consider the following differential equation

$$y^{(9)} - 12y^{(8)} + 55y^{(7)} - 124y^{(6)} + 139y^{(5)} - 16y^{(4)} - 147y^{(3)} + 152y'' - 48y' = 1 + 7e^t - te^{-t} + t^3\cos(2t)$$

where $r^9 - 12r^8 + 55r^7 - 124r^6 + 139r^5 - 16r^4 - 147r^3 + 152r^2 - 48r = r(r-1)^3(r+1)(r-4)^2\left((r-1)^2+2\right)$. When using the Method of Undetermined Coefficients, we assume that the terms in the *particular solution* that are *not in the complementary solution* have the form **(select all that apply)**:

| | | | | |
|---|---|---|---|---|
| **(a)** $A\cos 2t$ | **(e)** $E\sin 2t$ | **(i)** $I$ | **(m)** $Me^t$ | **(q)** $Qe^{-t}$ |
| **(b)** $Bt\cos 2t$ | **(f)** $Ft\sin 2t$ | **(j)** $Jt$ | **(n)** $Nte^t$ | **(r)** $Rte^{-t}$ |
| **(c)** $Ct^2\cos 2t$ | **(g)** $Gt^2\sin 2t$ | **(k)** $Kt^2$ | **(o)** $Ot^2e^t$ | **(s)** $St^2e^{-t}$ |
| **(d)** $Dt^3\cos 2t$ | **(h)** $Ht^3\sin 2t$ | **(l)** $Lt^2$ | **(p)** $Pt^3e^t$ | **(t)** $Tt^3e^{-t}$ |

**6.** **(2 marks)** If $1 = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$ for $0 < x < 1$, then $A_3 =$ $\frac{4}{3\pi}$.

(your answer must be a number)

**7.** **(2 marks)** $\mathfrak{L}^{-1}\left\{\frac{e^{-2s}}{s-3}\right\}(t) =$ $u_2(t)\, e^{3(t-2)}$.

Continued...

**PART II** **Justify** all your answers.

**8.** Consider the non-exact differential equation **(16 marks)**

$$2y - xe^x + x\frac{dy}{dx} = 0 \qquad (\star)$$

**(a)** **(3 marks)** If $\mu(x)$ is an integrating factor that makes $(\star)$ exact, find a differential equation for $\mu(x)$.

$$\mu(2y - xe^x) + \mu x y' = 0$$

So $$\frac{\partial}{\partial y}\left[\mu(2y - xe^x)\right] = \frac{\partial}{\partial x}\left[\mu x\right]$$

$$\Leftrightarrow \quad 2\mu = x\mu' + \mu$$

$$\Leftrightarrow \quad \boxed{\mu' = \frac{\mu}{x}}$$

**(b)** **(3 marks)** Solve the differential equation from part **(a)** to find $\mu(x)$ such that $\mu(1) = 1$.

$\frac{\mu'}{\mu} = \frac{1}{x}$ is separable

$$\Rightarrow \int \frac{1}{\mu}\, d\mu = \int \frac{1}{x}\, dx \Rightarrow \ln\mu = \ln x + C$$

$$\Rightarrow \mu = e^C x$$

To make sure $\mu(1) = 1$, we get $\boxed{\mu = x}$

**(c)** **(2 marks)** Find an exact differential equation with the same solutions as $(\star)$.

The original DE is equivalent to

$$\boxed{2xy - x^2 e^x + x^2\frac{dy}{dx} = 0}$$ which is exact

**(d)** **(6 marks)** Find the general solution of the equation ($\star$).

**Hint.** Note that $\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x + C$.

We have $\psi_x = 2xy - x^2 e^x$, so

$$\psi = \int (2xy - x^2 e^x) \, dx = x^2 y - (x^2 - 2x + 2)e^x + h(y)$$

Then

$$\psi_y = x^2 + h'(y) = x^2 \iff h'(y) = 0 \iff h(y) = C$$

The solution is (in implicit form):

$$\boxed{x^2 y - (x^2 - 2x + 2)e^x = C}$$

**(e)** **(2 marks)** Find the solution of ($\star$) such that $y(1) = 1$.

Using this condition, we have

$$1 - (1 - 2 + 2)e^1 = C \iff C = 1 - e$$

The solution is given by

$$x^2 y - (x^2 - 2x + 2)e^x = 1 - e$$

$$\iff \boxed{y = \frac{1 - e + (x^2 - 2x + 2)e^x}{x^2}}$$

**9.** Consider the differential equation **(16 marks)**

$$\begin{cases} y''(t) + y(t) = u_1(t)\sin(t-1) = g(t) \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

Let $Y(s) = \mathfrak{L}\{y(t)\}(s)$ be the Laplace tranform of $y(t)$.

**(a)** **(6 marks)**

$$\mathfrak{L}\{y'(t)\}(s) = s\,Y(s) - y(0) = s\,Y(s).$$

$$\mathfrak{L}\{y''(t)\}(s) = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - 1.$$

$$\mathfrak{L}\{g(t)\}(s) = e^{-s} \frac{1}{s^2+1}.$$

**(b)** **(3 marks)** Find $Y(s)$.

$$s^2 Y(s) - 1 + Y(s) = \frac{e^{-s}}{s^2+1}$$

$$\Leftrightarrow \quad Y(s)(1+s^2) = 1 + \frac{e^{-s}}{s^2+1}$$

$$\Leftrightarrow \quad Y(s) = \frac{e^{-s}}{(s^2+1)^2} + \frac{1}{s^2+1}$$

**(c)** **(7 marks)** Use the inverse Laplace transform to find $y(t)$.

**Hint.** You should use convolution in your answer.

$$Y(s) = \frac{e^{-s}}{(s^2+1)^2} + \frac{1}{s^2+1}$$

We know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) = \sin(t)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s^2+1)^2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+1}\cdot\frac{1}{s^2+1}\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+1}\right\}(t) * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t)$$

$$= \left[u_1(t)\sin(t-1)\right] * \left[\sin(t)\right]$$

So

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \left[u_1(t)\sin(t-1)\right] * \left[\sin(t)\right] + \sin(t)}$$

**10.** Consider a perfectly insulated rod modelled by the boundary value problem **(16 marks)**

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < \pi,\ t > 0 \\ u(0,t) = 2 & \text{for } t > 0 \\ u(\pi,t) = 1 & \text{for } t > 0 \end{cases}$$

**(a)** **(4 marks)** Find the steady state solution $v(x)$ (i.e. the solution that doesn't change with time).

$v''(x) = 0 \iff v(x) = mx + b$

$v(0) = 2 \iff b = 2$

$v(\pi) = 1 \iff m\pi + 2 = 1 \iff m = -1/\pi$

$$\boxed{v(x) = 2 - \frac{x}{\pi}}$$

Let $w(x,t) = u(x,t) - v(x)$.

**(b)** **(2 marks)** Show that $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}$.

$\dfrac{\partial w}{\partial t} = \dfrac{\partial u}{\partial t}$ and $\dfrac{\partial w}{\partial x} = \dfrac{\partial u}{\partial x} + \dfrac{1}{\pi} \Rightarrow \dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 u}{\partial x^2}$

$\Rightarrow \dfrac{\partial w}{\partial t} = \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 w}{\partial x^2}$

**(c)** **(2 marks)** Show that $w(0,t) = w(\pi,t) = 0$

$w(0,t) = u(0,t) - v(0) = 2 - 2 = 0$

$w(\pi,t) = u(\pi,t) - v(\pi) = 1 - 1 = 0$

**(d)** **(6 marks)** If $w(x,0) = 6\sin(4x)$, find $w(x,t)$.

Using Separation of Variables: $w(x,t) = \phi(x) g(t)$

$\Rightarrow \phi(x) g'(t) = \phi''(x) g(t) \iff \dfrac{g'(t)}{g(t)} = \dfrac{\phi''(x)}{\phi(x)} = -\lambda$ constant

Then $g'(t) = -\lambda g(t) \Rightarrow \boxed{g(t) = e^{-\lambda t}}$

And $\begin{cases} \phi''(x) = -\lambda \phi(x) \\ \phi(0) = 0 \\ \phi(\pi) = 0 \end{cases}$ $\overset{\text{formula sheet}}{\Rightarrow}$ $\boxed{\begin{cases} \lambda_n = n^2 & , n = 1, 2, \ldots \\ \phi_n(x) = \sin(nx) \end{cases}}$

Then, by Superposition Principle: $\boxed{w(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}}$

Using the IC, we obtain $w(x,0) = 6\sin(4x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, so

$b_n = \begin{cases} 6 & \text{if } n = 4 \\ 0 & \text{if } n \neq 4 \end{cases}$

The solution is $\boxed{w(x,t) = 6\sin(4x) e^{-16t}}$

**(e)** **(2 marks)** Find $u(x,t)$.

$u(x,t) = w(x,t) + v(x)$

$\iff \boxed{u(x,t) = 2 - \frac{x}{\pi} + 6\sin(4x) e^{-16t}}$

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**11.** You are consulting for the police on Bernardo's murder. **(16 marks)**

These are the facts about the murder:

(a) The body was found at 9am

(b) The body was found with the temperature of $25^o$C (average temperature is $37^o$C)

(c) The victim measured 185cm tall (average is 176cm) and weighed 75kg (average is 80kg)

(d) The body was found in his living room, which measured $25m^2$, and the thermostat was set to $22^o$C

There are three suspects that were with the victim the previous night

- Francis (height 176cm, weight 65kg) met with the victim at 8pm-10pm
- Arman (height 172cm, weight 64kg) met with the victim at 10pm-midnight
- Craig (height 183cm, weight 69kg) met with the victim at midnight-2am

Recall Newton's Law of Cooling: "The temperature change is proportional to the temperature difference". The average proportionality constant for a human being is $k = \frac{\ln 5}{8}$.

Who killed Bernardo? (Your answer should stand in a court of law!)

We only need to use the temperatures.

While Bernardo was alive, his temperature was 37°C. It started decaying after he was murdered.

His core temperature follows Newton's law of Cooling:

$$\boxed{T'(t) = k\left(T_0 - T(t)\right)}$$

where $k = \frac{\ln 5}{8}$ (in $h^{-1}$) and $T_0 = 22°C$ = temperature of the living room

$t$ in hours

Continued...

**(Continuation of solution to 11. )**

We also know that at 9am $(t=9)$, the temperature of the body was $25°C$, so

$$\boxed{T(9)=25}$$

We need to solve

$$\begin{cases} T' = k(22-T) \\ T(9)=25 \end{cases}$$

and then find $t^*$ such that $T(t^*)=37$.

Option #1

$T' = k(22-T) \Rightarrow \frac{T'}{22-T} = k$

$\Leftrightarrow -\ln|22-T| = kt + A$

$\Leftrightarrow 22-T = Be^{-kt}$

$\Leftrightarrow T = 22 - Be^{-kt}$

Option #2

$T' + kT = 22k \Leftrightarrow (e^{kt}T)' = 22ke^{kt}$

$\Leftrightarrow e^{kt}T = 22e^{kt} + C$

$\Leftrightarrow T = 22 + Ce^{-kt}$

$T(9)=25 \Leftrightarrow 3 = -Be^{-9k} \Leftrightarrow B = -3e^{9k}$

So $\boxed{T(t) = 22 + 3e^{k(9-t)}}$ (or $T(t) = 22 + 3e^{9k}e^{-kt}$)

Then $T(t^*) = 37 = 22 + 3e^{k(9-t^*)}$

$\Leftrightarrow 5 = e^{k(9-t^*)} \Leftrightarrow \underbrace{\cancel{\ln 5}}_{8}(9-t^*) = \cancel{\ln 5}$

$\Leftrightarrow \boxed{t^* = 9-8 = 1 \text{ am}}$

Since Bernardo was with Craig at 1am, it was Craig that killed Bernardo.

Continued...