University of Toronto, Faculty of Applied Science and Engineering

MAT292H1F - Ordinary Differential Equations

Final Exam - December 13, 2016

Examiners: B. Galvão-Sousa and C. Sinnamon

Time allotted: 150 minutes

Aids permitted: None

Total marks: 80

Full Name: ______________________________

Last First

Student ID: ______________________________

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages (including this title page) and a detached formula sheet.

  Make sure you have all of them.
- You can use paged 12–14 to complete a question (**mark clearly**).

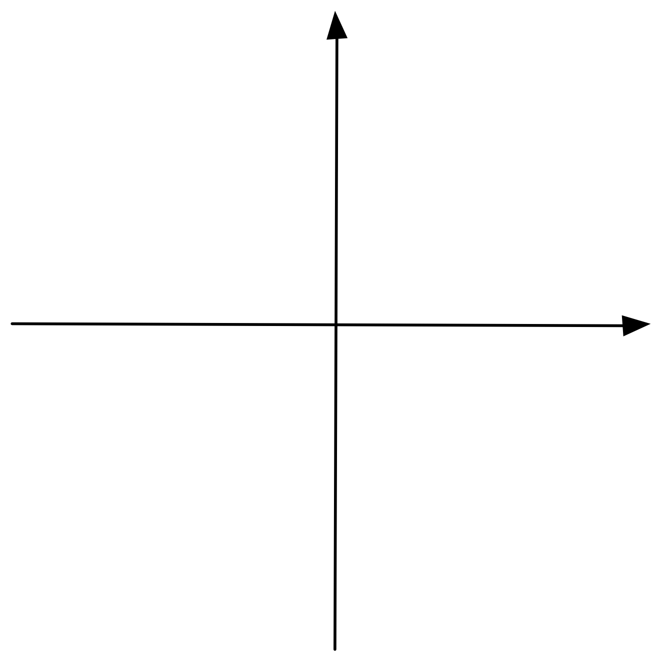
GOOD LUCK!

PART I No explanation is necessary. **(16 marks)**

1. **(2 marks)** If $y'(t) - \frac{1}{t}y(t) = 1$ with $y(1) = 1$, then $y(t) =$ ____________________.

2. **(2 marks)** If $y'(t) = e^y(1-y)(y-2)(y-4)$ with $y(-48) = \pi$ then $\lim_{t\to\infty} y(t) =$ ____________________.

3. **(2 marks)** Sketch a phase portrait for the system $\vec{x}\,'(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \vec{x}(t)$.



4. **(2 marks)** An example of a solution to $\vec{x}\,'(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \vec{x}(t)$ such that $\lim_{t\to\infty} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

$\vec{x}(t) =$ ____.

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5. (4 marks) Consider the following differential equation

$$y^{(9)} - 12y^{(8)} + 55y^{(7)} - 124y^{(6)} + 139y^{(5)} - 16y^{(4)} - 147y^{(3)} + 152y'' - 48y' = 1 + 7e^t - te^{-t} + t^3\cos(2t)$$

where $r^9 - 12r^8 + 55r^7 - 124r^6 + 139r^5 - 16r^4 - 147r^3 + 152r^2 - 48r = r(r-1)^3(r+1)(r-4)^2\left((r-1)^2+2\right)$. When using the Method of Undetermined Coefficients, we assume that the terms in the *particular solution* that are *not in the complementary solution* have the form **(select all that apply)**:

(a) $A\cos 2t$	**(e)** $E\sin 2t$	**(i)** I	**(m)** Me^t	**(q)** Qe^{-t}
(b) $Bt\cos 2t$	**(f)** $Ft\sin 2t$	**(j)** Jt	**(n)** Nte^t	**(r)** Rte^{-t}
(c) $Ct^2\cos 2t$	**(g)** $Gt^2\sin 2t$	**(k)** Kt^2	**(o)** Ot^2e^t	**(s)** St^2e^{-t}
(d) $Dt^3\cos 2t$	**(h)** $Ht^3\sin 2t$	**(l)** Lt^2	**(p)** Pt^3e^t	**(t)** Tt^3e^{-t}

6. (2 marks) If $1 = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$ for $0 < x < 1$, then $A_3 =$ ____________________.

(your answer must be a number)

7. (2 marks) $\mathfrak{L}^{-1}\left\{\frac{e^{-2s}}{s-3}\right\}(t) =$ ____.

PART II **Justify** all your answers.

8. Consider the non-exact differential equation **(16 marks)**

$$2y - xe^x + x\frac{dy}{dx} = 0 \qquad (\star)$$

(a) **(3 marks)** If $\mu(x)$ is an integrating factor that makes $(\star)$ exact, find a differential equation for $\mu(x)$.

(b) **(3 marks)** Solve the differential equation from part **(a)** to find $\mu(x)$ such that $\mu(1) = 1$.

(c) **(2 marks)** Find an exact differential equation with the same solutions as $(\star)$.

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(d) **(6 marks)** Find the general solution of the equation $(\star)$.

Hint. Note that $\displaystyle\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x + C$.

(e) **(2 marks)** Find the solution of $(\star)$ such that $y(1) = 1$.

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9. Consider the differential equation **(16 marks)**

$$\begin{cases} y''(t) + y(t) = u_1(t)\sin(t-1) \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

Let $Y(s) = \mathfrak{L}\{y(t)\}(s)$ be the Laplace tranform of $y(t)$.

(a) **(6 marks)**

$\mathfrak{L}\{y'(t)\}(s) =$ ____.

$\mathfrak{L}\{y''(t)\}(s) =$ ____.

$\mathfrak{L}\{g(t)\}(s) =$ ____.

(b) **(3 marks)** Find $Y(s)$.

(c) **(7 marks)** Use the inverse Laplace transform to find $y(t)$.

Hint. You should use convolution in your answer.

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10. Consider a perfectly insulated rod modelled by the boundary value problem **(16 marks)**

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < \pi,\ t > 0 \\ u(0,t) = 2 & \text{for } t > 0 \\ u(\pi,t) = 1 & \text{for } t > 0 \end{cases}$$

(a) **(4 marks)** Find the steady state solution $v(x)$ (i.e. the solution that doesn't change with time).

Let $w(x,t) = u(x,t) - v(x)$.

(b) **(2 marks)** Show that $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}$.

(c) **(2 marks)** Show that $w(0,t) = w(\pi,t) = 0$

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(d) **(6 marks)** If $w(x,0) = 6\sin(4x)$, find $w(x,t)$.

(e) **(2 marks)** Find $u(x,t)$.

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11. You are consulting for the police on Bernardo's murder. **(16 marks)**

These are the facts about the murder:

(a) The body was found at 9am

(b) The body was found with the temperature of 25^oC (average temperature is 37^oC)

(c) The victim measured 185cm tall (average is 176cm) and weighed 75kg (average is 80kg)

(d) The body was found in his living room, which measured $25m^2$, and the thermostat was set to 22^oC

There are three suspects that were with the victim the previous night

- Francis (height 176cm, weight 65kg) met with the victim at 8pm-10pm
- Arman (height 172cm, weight 64kg) met with the victim at 10pm-midnight
- Craig (height 183cm, weight 69kg) met with the victim at midnight-2am

Recall Newton's Law of Cooling: "The temperature change is proportional to the temperature difference". The average proportionality constant for a human being is $k = \dfrac{\ln 5}{8}$.

Who killed Bernardo? (Your answer should stand in a court of law!)

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