MAT292 - Fall 2018

Term Test 1 - October 25, 2018

Time allotted: 100 minutes | Aids permitted: None

Total marks: 65

Full Name: ______________________________
Last First

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test contains 10 pages (including this title page). Make sure you have all of them.
- You can use pages 9–10 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 9–10.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

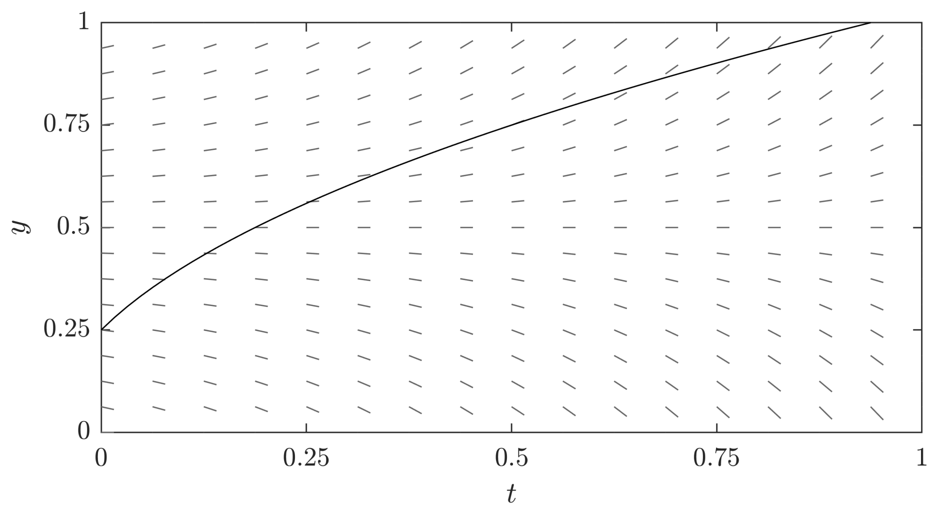
HAVE FUN!

SECTION I No explanation is necessary. **(10 marks)**

1. **(2 marks)** Give the general solution to differential equation $\frac{dy}{dt} = y$: $y(t) =$ $\underline{\quad ce^t \quad}$

2. **(1 mark)** Is the function $y(t) = e^t - 1$, a solution to the differential equation $\frac{dy}{dt} = 1 + \max(-2y, y)$? Answer 'yes' or 'no'. <u>yes</u>

3. **(2 marks)** The direction field for a differential equation is given below. What are the equilibrium solution(s) to this differential equation? $\underline{\quad y = 1/2 \quad}$



4. **(1 mark)** Is the black curve a solution to the differential equation? Answer 'yes' or 'no'. <u>no</u>

5. **(1 mark)** What is the order of the differential equation $y^4 + (1 - y^2)y''' + \sin(y'') + 1 = 0$? <u>third</u>

6. **(1 mark)** Is $e^y y' + e^{-t+y} y = 0$, a linear or a non-linear differential equation? <u>linear</u>

7. **(1 mark)** Does the initial value problem $y' = \sin(\pi e^{t+y})$, $y(0) = 0$ have a unique solution on $t \in (-\infty, \infty)$? Answer 'yes' or 'no'. <u>yes</u>

8. **(1 mark)** Give an example of a first order, non-linear, autonomous ordinary differential equation. $\underline{\quad y' = y^2 \quad}$

SECTION II **Justify** your answers. **(55 marks)**

9. The solution to the initial value problem $dy/dt = 4/(1+t^2)$, $y(0) = 0$ has the value $y(1) = \pi$. **(5 marks)**

(a) **(2 marks)** Use Euler's method to estimate the value of π with $\Delta t = 1$ and $\Delta t = 0.5$.

Solution: For $\Delta t = 1$, $\pi \approx 0 + 1 \cdot \frac{4}{1+0^2} = 4.000$.
For $\Delta t = 0.5$, $\pi \approx 0 + 0.5 \cdot \frac{4}{1+0^2} + 0.5 \cdot \frac{4}{1+0.5^2} = 3.600$.

(b) **(2 marks)** How small would you have to make Δt to expect to get two correct digits to $\pi \approx 3.14159265358979$?

Solution: Euler's method is first order, so each reduction in Δt by a factor of $\frac{1}{2}$ roughly halves the error. Using s reductions: $|4.000 - 3.600|2^{-s} < 10^{-2}$ gives $s \approx 6$. So $\Delta t \approx 0.5 \cdot 2^{-6}$ would be required to get two correct digits. In fact, $\Delta t = 2^{-7}$ gives $\pi \approx 3.149395$ with two accurate digits.

(c) **(1 mark)** Let π_N be the approximation using $N = 1/\Delta t$ steps of Euler's method. We can extrapolate a more accurate value by combining π_N and π_{2N} as $\hat{\pi}_N := 2\pi_{2N} - \pi_N$. What is $\hat{\pi}_1$?

Solution: $\hat{\pi}_1 = 2\pi_2 - \pi_1 = 2 \cdot 3.600 - 4.000 = 3.2$. Side note: $\hat{\pi}_8$ is two digits accurate.

10. Solve the following initial value problem using the integrating factor method: **(10 marks)**

$$\sin(t)\frac{dy}{dt} + \cos(t)y = \sin(t)\cos(t), \quad y\left(\frac{\pi}{2}\right) = 0.$$

Solution: An integrating factor for this first-order, linear differential equation is

$$\mu(t) = \exp\left(-\int \frac{\cos(t)}{\sin(t)}\, dt\right) = \sin(t).$$

Therefore,

$$\frac{d}{dt}\left(\sin(t)y(t)\right) = \sin(t)\cos(t)$$

with general solution

$$y(t) = -\frac{C + \cos^2(t)/2}{\sin(t)}$$

The solution to the initial value problem is

$$y(t) = -\frac{\cos^2(t)}{2\sin(t)}$$

after selecting the constant of integration $C = 0$ so that $y(\pi/2) = 0$.

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11. Consider the differential equation $2ty + (t^2 - y^2)\frac{dy}{dt} = 0$. **(10 marks)**

(a) **(4 marks)** Find value(s) of c so that $y = ct$ is a solution to the differential equation. Does this contradict *the* existence/uniqueness theorem for the initial condition $y(0) = 0$?

Solution: Substituting $y = ct$ gives $2ct^2 + (t^2 - c^2t^2)c = 0$, which has three solutions $c = 0, c = \pm\sqrt{3}$. This does not contradict the existence/uniqueness theorem since $\frac{dy}{dt} = \frac{2ty}{y^2-t^2}$ is not continuous at $(0, 0)$. In particular, its value along $y = ct$ depends on c: $\lim\limits_{t\to 0} \frac{2ct^2}{c^2t^2-t^2} = \frac{2c}{c^2-1}$.

(b) **(6 marks)** Find an implicit solution to the exact differential equation.

Solution: This is exact since $\partial_y(2ty) = 2t = \partial_t(t^2 - y^2)$. From $\partial_t\phi = 2ty$, we get $\phi(t, y) = t^2y + F(y)$ and therefore $\partial_y\phi = t^2 + F'(y) = t^2 - y^2$. Therefore, $F(y) = -y^3/3$ and implicit solutions to the differential equation are given by $\phi(t, y) = t^2y - y^3/3 = C$.

12. Consider the differential equation $y' = y^2(y^2-1)$. **(10 marks)**

(a) **(1 mark)** Find all equilibrium solutions.

Solution: Equilibrium solutions are defined by $g(y) = y^2(y^2-1) = 0$ giving -1, 0 and $+1$.

(b) **(3 marks)** Determine which of the equilibrium solutions are stable, unstable, or semi-stable.

Solution: The solution $y = -1$ is stable (because about -1 function, $g(y)$ changes sign from plus to minus when y is increasing). Similarly 0 is semi-stable (($g(y)$ is negative about the origin) and $+1$ is unstable.

(c) **(2 marks)** Let $y_1(t), y_2(t)$ be the solutions with initial conditions $y_1(0) = -1/2,\ y_2(0) = 1/2$. Without writing formulas for these solutions, find the following limits:

i) $\lim\limits_{t\to-\infty} y_1(t)$ **Solution:** 0 ii) $\lim\limits_{t\to\infty} y_1(t)$ **Solution:** -1

iii) $\lim\limits_{t\to-\infty} y_2(t)$ **Solution:** 1 iv) $\lim\limits_{t\to\infty} y_2(t)$ **Solution:** 0

(d) **(2 marks)** Let $y_3(t)$ be the solution with initial condition $y_3(0) = 2$. Explain why the solution $y_3(t)$ is only defined on $-\infty < t < T$ for $T = \int_2^\infty y^{-2}(y^2-1)^{-1}\, dy$.

Solution: For every t satisfying $-\infty < t < T$ the solution $y_3(t)$ is increasing and is defined by formula

$$t = \int_2^{y_3(t)} \frac{dy}{y^2(y^2-1)}$$

As $y_3 \to \infty$, t has a finite value.

(e) **(1 mark)** Find the following limits:

i) $\lim\limits_{t\to-\infty} y_3(t)$ **Solution:** 1 ii) $\lim\limits_{t\to T} y_3(t)$ **Solution:** ∞

(f) **(1 mark)** For $y > 2$, the function $f(y) = 1/(y^4 - y^2) = y^{-4} \cdot 1/(1-y^{-2})$ satisfies $y^{-4} < f(y) < \frac{4}{3}y^{-4}$. Show that the number T satisfies $1/24 < T < 1/18$.

Solution: By integrating we get $-1/3y^{-3}|_2^\infty < T < 4/3(-1/3y^{-3}|_2^\infty)$. Or $1/24 < T < 1/18$.

13. Consider the differential equation $\frac{dy}{dt} = 3y^{2/3}$. **(10 marks)**

(a) **(3 marks)** Find the general solution for $y > 0$ and for $y < 0$. Show that $y \equiv 0$ is a solution.

Solution: In the domain $y > 0$ and in the domain $y < 0$ the function y does not attain the value 0. Thus in the both domains we can divide by $y^{2/3}$. We will get the equations $1/3y^{-2/3}y' = 1$, $y > 0$ and $1/3y^{-2/3}y' = 1$, $y < 0$. Thus in the first domain we get $y^{1/3} = 1$, $y > 0$. So $y = (t-a)^3, t > a$. In the second domain we get $y^{1/3} = 1$, $y < 0$. Thus $y = (t-b)^3, t < b$. If instead of $y(t)$ one plugs $y(t) \equiv 0$ the right and the left sides of the ODE vanish. Thus $y(t) \equiv 0$ satisfies the ODE.

(b) **(1 mark)** Find the solution $y_0^+(t)$ in the domain $y > 0$ that satisfies $\lim_{t \to 0^+} y_0^+(t) = 0$.

Solution: According to **(a)** we get $y_0^+(t) = t^3$, where $t > 0$.

(c) **(2 marks)** Let $y_a^+(t) = y_0^+(t-a)$ for $t > a$, where y_0^+ is the function defined in **(b)**. Show that $y_a^+(t)$ is the solution for the domain $y > 0$ and that $\lim_{t \to a^+} y_a^+(t) = 0$.

Solution: $y_a^+ = (t-a)^3$ where $t > a$. $dy_a^+/dt(t) = dy_0^+(t-a)/dt \cdot 1 = 3y^{2/3}(t-a) = 3(y_a^+)^{2/3}$, $\lim_{t \to a^+} y_a^+(t) = \lim_{t \to 0^+} y_0^+(t) = 0$ by continuity of y_0^+.

(d) **(4 marks)** Let $y_b^-(t)$ be the solution in the domain $y < 0$ for $t < b$ satisfying $\lim_{t \to b^-} y_b^-(t) = 0$. With reference to $y_a^+(t)$ and $y_b^-(t)$, show that there are an infinite number of solutions to the differential equation satisfying $y(1) = 1$. Why does this not contradict *the* existence/uniqueness theorem?

Solution: We have $y_b^- = (t-b)^3$ where $t < b$. In the domain $y > 0$ we have the unique solution $y(t) = t^3 = y_0^+(t)$ of the ODE such that $y(1) = -1$. This solution is defined for $t > 0$ (for negative t the function $y(t) = t^3$ also satisfies the ODE but it does not belong to the domain $y > 0$). We see that for any $b \leqslant 0$ the function $y(t) = y_b^-(t)$ for $t < 0$ and $y_0^+(t)$ for $t > 0$ is a solution of the ODE satisfying the condition $y(1) = -1$.

This does not contradict the existence and uniqueness theorem: in the domain $y > 0$ we do have the unique solution. We got many solutions in the plane (t, y) because on the line $y = 0$ the function $y^{2/3}$ is not smooth.

14. A *well-mixed* tank of constant volume contains bacteria in water. **(10 marks)**

Water is pumped into and out of the tank at equal rates. The concentration $C(t)$ of bacteria in the tank is modelled by the differential equation $C' = \alpha C(C_* - C) + rC_0 - rC$, with non-negative parameters α, C_0, C_*, r.

(a) **(3 marks)** Explain the meaning of each term and parameter in the differential equation.

Solution: $\alpha C(C_* - C)$ accounts for logisitic, resource constrained, growth of the bacteria with growth rate α and carrying capacity C_*. rC_0 accounts for in-flow at rate r with bacteria concentration C_0 and rC accounts for out-flow at rate r with bacteria concentration C.

(b) **(3 marks)** Find the equilibrium solutions and state their stability.

Solution: There are two equilibrium solutions:

$$C = \frac{C_*}{2} - \frac{r}{2\alpha} \pm \frac{1}{2\alpha}\sqrt{(\alpha C_* - r)^2 + 4\alpha r C_0}.$$

The smaller one is unstable and the large one is stable since $-\alpha C^2 + (\alpha C_* - r)C + rC_0$ is a downward opening parabola.

(c) **(4 marks)** Suppose that $C_0 = 0$ and that the system is used to produce the bacteria. Consider α and C_* to be fixed, but r can be varied. What value of r maximizes the equilibrium rate that bacteria is removed from the tank?

Solution: For $C_0 = 0$, the equilibria are $C = 0$ and $C = C_* - r/\alpha$. The stable, steady production rate is $r(C_* - r/\alpha)$ provided $r < C_*\alpha$ and zero otherwise. Maximizing the parabola in r, shows that the maximum production occurs with $r = \alpha C_*/2$.

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