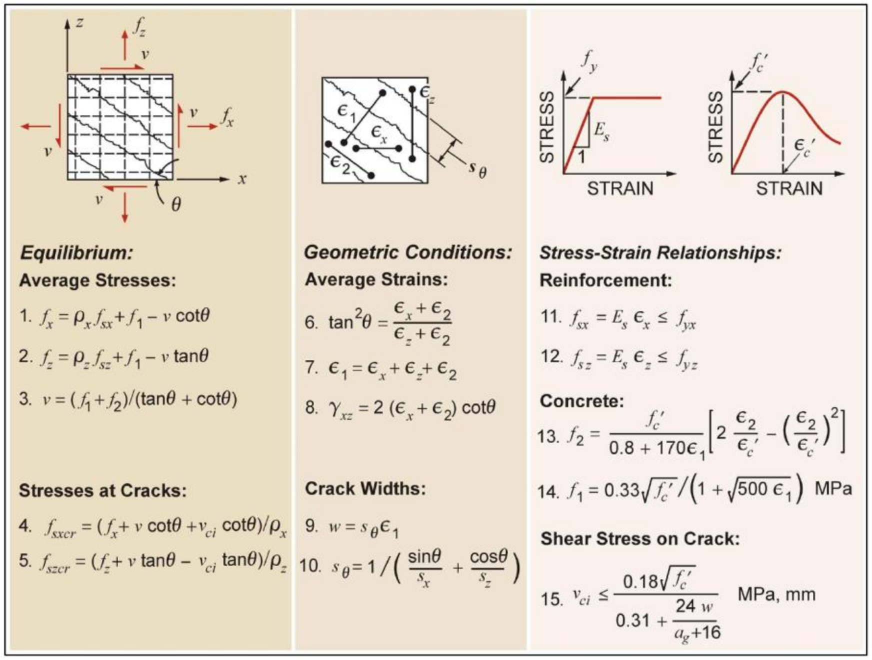
CIV102 – Structures and Materials
An Introduction to Engineering Design

Part 4: Concrete Structures

University of Toronto
Division of Engineering Science
September 2020

Allan Kuan
Michael P. Collins



The fifteen equations which make up the Modified Compression Field Theory for Reinforced Concrete Elements Subjected to Shear. Developed at the University of Toronto in the 1980s, it is the theoretical basis for shear design procedures used by design codes around the world.

CIV102 Course Notes – Part 4: Concrete Structures

Lecture 31 – Building with Stone and Concrete 131
Lecture 32 – Reinforced Concrete: Material Properties 134
Lecture 33 – Reinforced Concrete Members – Design for Flexure 138
Lecture 34 – Reinforced Concrete Members – Design for Shear 143
Lecture 35 – Prestressed Concrete Structures 151

Lecture 31 – Building with Stone and Concrete

Overview:
In this chapter, the unique characteristics of stone and concrete structures are discussed. Stone-like materials are typically strong in compression, have low tensile strength, and have significant self-weight. The theory used to design of arches and towers is presented.

Unique Characteristics of Stone
The existence of stone structures which date back hundreds or thousands of years is evidence of the many strengths of these materials. Unlike organic materials like wood, or metallic materials like iron, stone is very durable and capable of surviving for many years under harsh weather conditions. An example of a stone structure which has survived through over the centuries is the Alcántara Bridge in Spain, shown in Fig. 31.1, which was built in 106 AD, almost two thousand years ago.

Fig. 31.1 - Alcántara Bridge in Spain

Note: Masonry construction refers to the use of stones joined by mortar to build structures. The mortar is used to position the stone blocks together and fill in gaps as needed.

Although stone is strong and durable, it is difficult to transport, and shape. Furthermore, stones suitable for construction are only found in certain geographical regions. A common alternative is to instead use ***concrete***, a Roman invention, which is manmade stone which addresses these shortcomings of natural stone. Concrete, which is made by mixing cement, water, air, fine aggregate (i.e. sand) and coarse aggregate (i.e. larger rocks), can be readily formed into any shape. Transporting its component ingredients is comparatively easy, making it a versatile material commonly used in modern construction. The Romans, however, were also experts of using concrete for their structures. An example of a Roman structure which demonstrates their mastery of the material is the Pantheon in Rome, which was built in 125 AD and still stands despite many wars and natural disasters occurring during its lifetime.

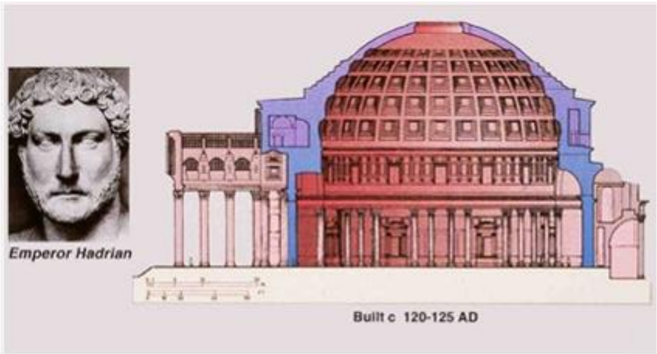


Fig. 31.2 – Pantheon in Rome

Stone-like materials, like concrete, limestone, granite, etc., share many common material properties. They are heavy and tend to be formed into large structures, resulting in a substantial self-weight which must be included in their design. They tend to be strong in compression but weak in tension, and when they fail in tension, they exhibit little to no ductility. Stone structures have essentially the opposite characteristics of slender wires used to carry tensile loads, as they favour compression instead of tension, they are heavy instead of light, and they are brittle instead of tough.

True Theory of Arches

One common application of stone in buildings and bridges is to build arches. When designing arches, the shape must allow the applied loads from above to travel to the supports below without causing any tensile stresses in the structure. Although the Romans were experts at building arches, Robert Hooke later summarized his true theory of arches with the following statement:

'As hangs a flexible cable, so, inverted, stand the touching pieces of an arch'.

Hooke's theory, shown in Fig. 31.3, illustrates how the shape of an arch should be when supporting one, two, three, or four point loads. Under a uniformly distributed load, the ideal shape becomes a parabola, the same way a suspension bridge carries the uniformly distributed weight of a deck using a parabolic cable. The famous Catalan architect, Antoni Gaudí, made use of Hooke's theory of arches when designing his stone cathedrals. Fig. 31.4 shows one of his string models that he used to determine the appropriate shape of his buildings.

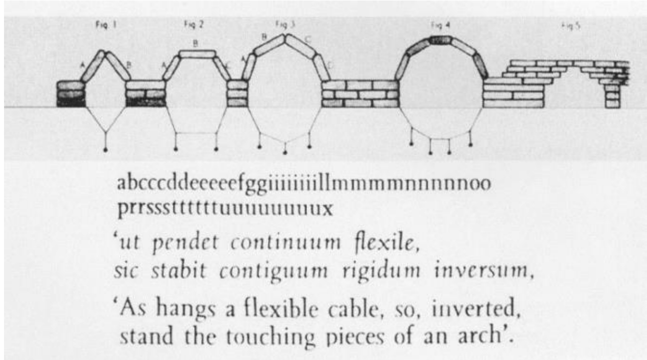


Fig. 31.3 – Hooke's theory of arches

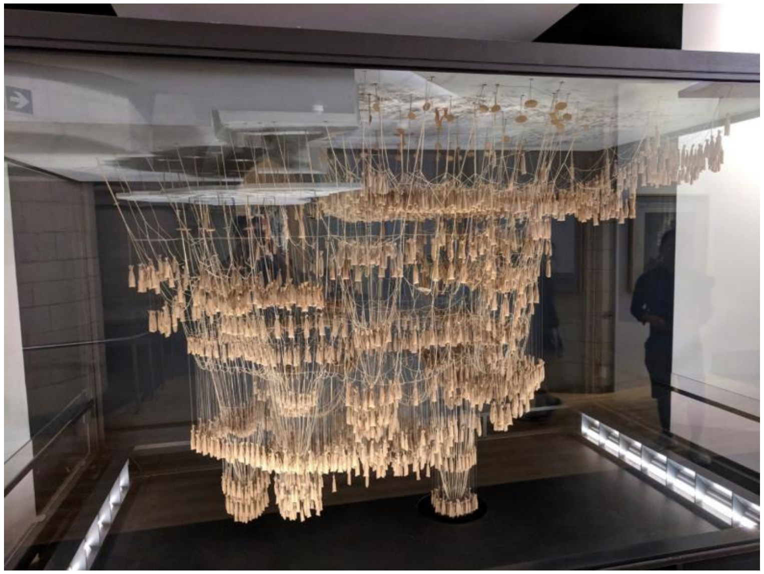


Fig. 31.4 – Model used by Antoni Gaudí to design stone cathedrals. The built shape was obtained by flipping the model upside down

Design for Combined Axial Load and Bending Moment

Although the poor tensile strength of stone reduces the ability of masonry structures to resist large bending moments, their substantial self-weight allows some structures, like towers, to overcome this weakness. Consider the tower shown in Fig. 31.5, which is subjected to large horizontal forces from a severe windstorm. At the base of the structure, there must be a significant bending moment, $\mathbf{M_{base}}$, to prevent the tower from tipping over, and a high horizontal shear force, $\mathbf{V_{base}}$, to prevent it from sliding. There must also be a large vertical reaction force provided by the ground to support the self-weight of the tower, which places the tower in compression.

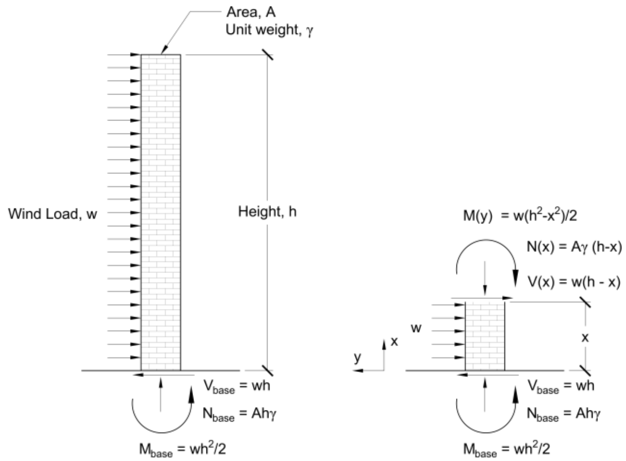


Fig. 31.5 – Stone tower subjected to high wind forces (left) and free body diagram at the base (right)

At the bottom of the tower, the axial force, **N**, is equal to the weight of the tower above. Using the unit density of the material, **γ**, the stress at the base caused by the axial load, $\boldsymbol{\sigma_N}$, can be calculated to be:

$$\sigma_N = -\frac{N}{A} = -\frac{V\gamma}{A} \tag{31.1}$$

In Eq. (31.1), **V** is the volume of the tower, and the minus sign indicates that these stresses are compressive. In addition to the stress from the self-weight, there will also be stresses at the base of the tower due to the presence of the bending moment. These stresses, $\boldsymbol{\sigma_M}$, can be calculated by using Navier's equation:

$$\sigma_M = \frac{My}{I} \tag{31.2}$$

The total stress in the tower at its base can be obtained by simply adding the two effects together if it is linear elastic. Therefore, the stress on the side where the wind blows, which would normally be in tension, is:

$$\sigma = -\frac{N}{A} + \frac{My_{left}}{I} \tag{31.3}$$

And the stress on the side opposite to the wind, which would normally be in compression, is:

$$\sigma = -\frac{N}{A} - \frac{My_{right}}{I} \tag{31.4}$$

Although the tower is subjected to a high moment, the stresses may remain compressive if the tower is heavy enough.

Note: If the cross-sectional area of the tower is constant over its height, **h**, then the stress at the base can be calculated as:

$$\sigma = \frac{V\gamma}{A} = \frac{Ah\gamma}{A} = h\gamma$$

Note: Due to the axial load, the stress in the member at the centroidal axis will no longer be equal to zero. This can be shown in Fig. 31.6.

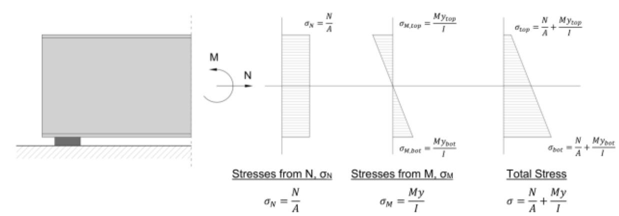


Fig. 31.6 – Stress distribution under combined axial load and bending moment

Lecture 32 – Reinforced Concrete: Material Properties

Overview:
In this chapter, the material properties of concrete, steel, and reinforced concrete are discussed.

Reinforced Concrete
Concrete is the most commonly used building material in the world, having applications in the construction of roads, bridges, buildings, dams, tunnels and more. Besides being both strong and durable, concrete can be formed into any shape, offering a versatility which cannot be matched by other materials like steel and timber.

Although concrete, being a stone-like material, has poor tensile strength, this weakness can be addressed by providing reinforcement in the form of steel bars which are cast into the concrete. Concrete containing internal reinforcement is typically referred to as ***reinforced concrete***, while concrete without reinforcement is typically referred to as ***plain concrete***.

Material Properties of Concrete
The stress-strain response of concrete under axial load is shown in Fig. 32.2. In compression, the stress-strain response is linear until a stress of approximately 40% of the ultimate compressive stress is reached; after this, the behaviour is highly nonlinear. In tension, the behaviour is linear elastic until the concrete fails by cracking, which typically occurs at a stress around 2 to 3 MPa.

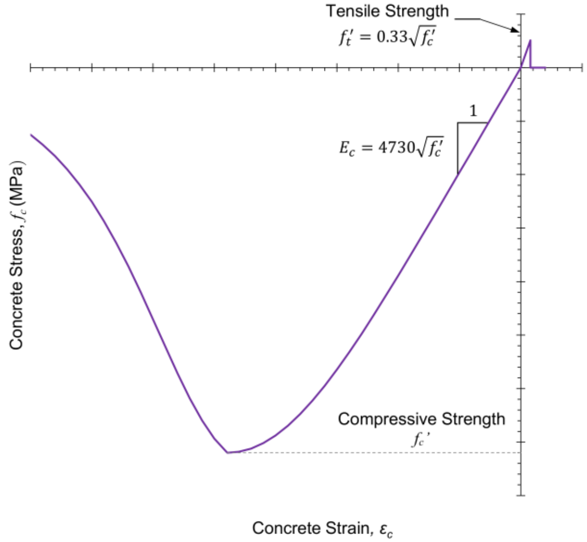


Fig. 32.2 – Stress-strain response of concrete under axial load

When building concrete structures, it is customary to cast small concrete cylinders made from the same concrete, which are tested later to measure the material properties of the concrete present in the actual structure. These cylinders are tested in compression to determine the compressive strength of the concrete, f_c', which is then correlated to other

Fig. 32.1 – Large reinforced concrete beam in the Bahen Centre for Information Technology at the University of Toronto. Details of the internal reinforcement are shown.

Note: Although steel is typically used as reinforcement, other materials can be used as well. Common alternatives include using glass fibre reinforced polymer (GFRP) bars or embedding steel or polymer fibers into the concrete to create Fiber Reinforced Concrete (FRC).

*Note: The notation for stresses for concrete structures differs from the notation used for the rest of the course. Axial stresses are represented as **f** and shear stresses are represented as **v**, instead of σ and τ respectively.*

properties using empirical equations. Concrete which has a compressive strength of less than 40 MPa is ***normal strength concrete***. Special circumstances may necessitate the use of ***high strength concrete***, which has a higher compressive strength that can exceed 100 MPa.

Given the concrete compressive strength, f_c', a common empirical equation to estimate the tensile strength of the concrete, f_t', is:

$$f_t' = 0.33\sqrt{f_c'} \tag{32.1}$$

Note: Performing tensile tests on stone-like materials is very difficult. For this reason, it is more common to correlate the cracking stress with the compressive strength.

Like the tensile strength, the Young's modulus of the concrete, $\mathbf{E_c}$, can also be correlated with the compressive strength of the concrete. Although there are many equations in the literature which express $\mathbf{E_c}$ as a function of f_c'; a common expression used for concrete whose strength is 40 MPa or less is:

$$E_c = 4730\sqrt{f_c'} \tag{32.2}$$

When using Eq. (32.1) and (32.2), f_t' and $\mathbf{E_c}$ will be in units of MPa if f_c' is also in MPa.

Note: Another equation for E_c which appeared in previous versions of the CIV102 notes is the following:

$$E_c = 3320\sqrt{f_c'} + 6900 \tag{32.3}$$

When solving reinforced concrete problems which do not provide E_c in the question, use Eq. (32.2) to estimate E_c if needed.

Material Properties of Reinforcing Steel

To reinforce concrete members, steel reinforcing bars are bent and tied together to form complex, interlocking cages like the one shown in Fig. 32.3. The stress-strain response of these steel reinforcing bars under axial load is shown in Fig. 32.4. Reinforcing steel bars, colloquially referred to as ***rebars***, are made of mild steel and are manufactured to have surface deformations to help anchor them into the surrounding concrete. The Young's modulus of the steel is taken as $\mathbf{E_s} = \mathbf{200,000\ MPa}$, and the yield strength of Canadian reinforcing bars is typically $f_y = \mathbf{400\ MPa}$ in both tension and compression.

Note: Although E_s is 200,000 MPa for all types of steel, different countries use different steel strengths. For example, in USA, it is customary to use steel which has a yield strength of 60 ksi (414 MPa).

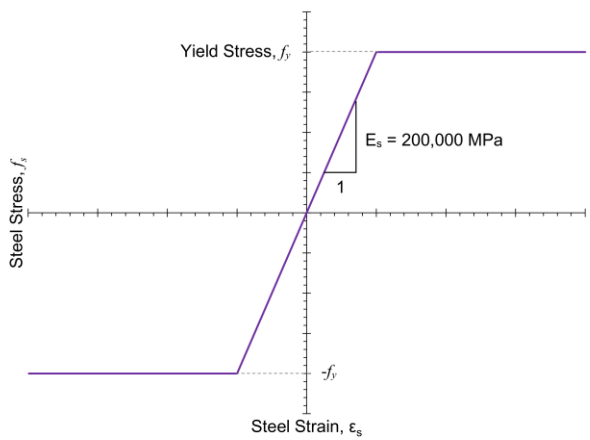


Fig. 32.4 – Stress-strain response of reinforcing steel under axial load

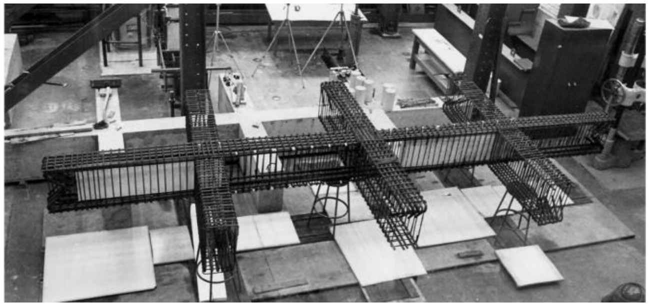


Fig. 32.3 – Reinforcement cage comprised of rebars which are bent and tied together.

Table 32.1 – Common Canadian Reinforcing Bar Information

Designation	Linear Density (kg/m)	Nominal Diameter (mm)	Cross-Sectional Area (mm^2)
10M	0.785	11.3	100
15M	1.570	16.0	200
20M	2.355	19.6	300
25M	3.925	25.2	500
30M	5.495	29.9	700
35M	7.850	35.7	1000
45M	11.775	43.7	1500
55M	19.625	56.4	2500

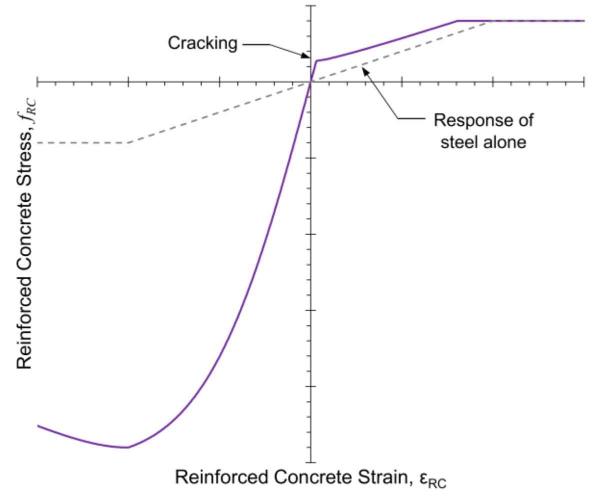


Fig. 32.5 – Stress-strain response of reinforced concrete under axial load

Steel producers fabricate rebar into many standardized sizes. Table 32.1, which is also found in Appendix G, shows the common types of rebar which are available in Canada. The bar number roughly refers to the diameter of the bar, although the actual diameter is usually slightly larger. The smaller bars (10M and 15M) are usually used for reinforcement which needs to be bent into compact shapes, while the larger bars are more difficult to bend and are usually used where straight bars are needed.

Material Properties of Reinforced Concrete

Fig. 32.4 shows the stress-strain response of a concrete member which is reinforced with steel bars. Its response under compression is similar to that of plain concrete because the steel only provides a small increase in compressive strength. In tension however, the response is substantially different, especially after the concrete cracks. When this occurs, the tensile force is carried by the steel instead of by the cracked concrete, which allows substantial tensile forces to be carried by the material. The response of the material becomes also ductile because failure in tension occurs due to the steel yielding instead of the concrete cracking. The presence of steel also affects the pattern of cracking and the sizes of the individual cracks. As the amount of steel in the concrete increases, more cracks form which have smaller widths because the elongation of the member is distributed over many cracks, instead of being localized at one location.

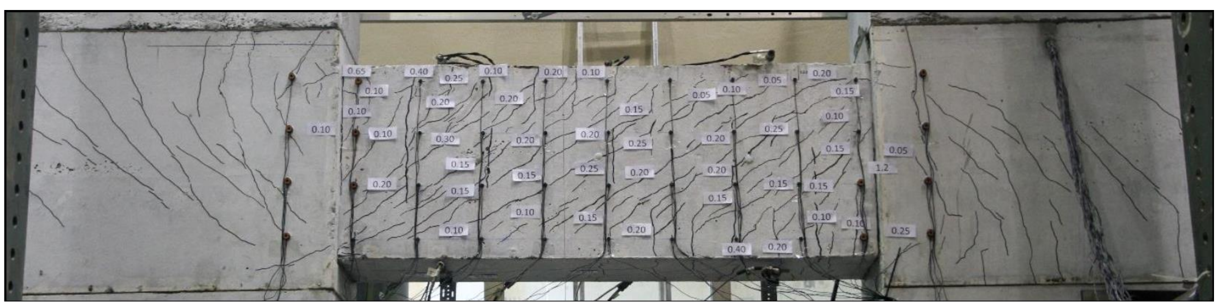
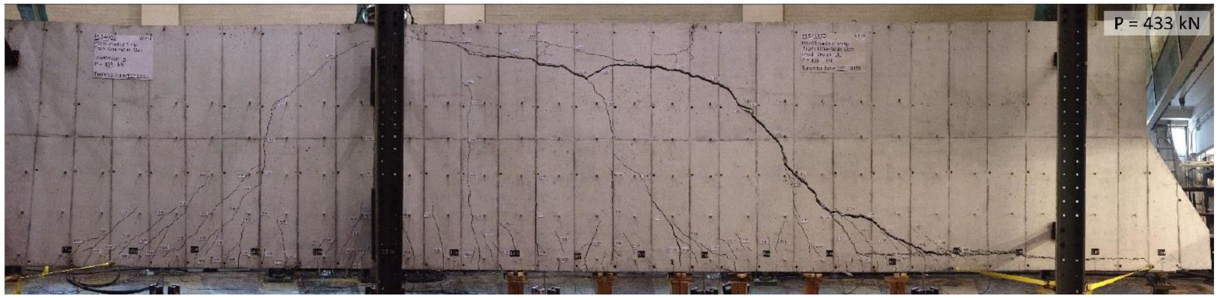


Fig. 32.6 – Heavily reinforced coupling beam tested at the University of Toronto by Fischer et al. Electronic measuring gauges are attached to the surface of the concrete, and labels indicate the measured crack widths in mm.



32.7 – Large slab strip tested at the University of Toronto by Quach et al. The right-hand side contained no vertical reinforcement and failed in a brittle manner.

The heavily reinforced beam shown in Fig. 32.6 illustrates the ability of internal reinforcement to improve the behaviour of concrete members after cracking. This beam, loaded under moment and shear, was capable of carrying significant forces after cracking. Because large amounts of steel were used to reinforce it, the shear and flexural deformations are distributed over many narrow cracks. This member failed in a ductile manner once the steel began to yield.

Fig. 32.7 shows a reinforced concrete beam built without vertical reinforcement in its East (right) span. Because there was no vertical reinforcement, the deformations caused by the shear forces were concentrated at a single wide crack, which ultimately caused its failure. This member failed in a brittle manner once widening and sliding along the crack reduced its load-carrying capacity to be less than the applied load.

The benefits of using steel reinforcement in concrete can be summarized below:

i. It provides tensile capacity in a member after the concrete cracks
ii. It controls the crack widths after cracking occurs, so that tensile deformations are distributed over multiple narrow cracks, instead of fewer wide cracks.

In design, it is customary to use different factors of safety associated with the stresses in the concrete and in the steel. Concrete, whose strength can be affected by factors like how it was cast, variances in the mix and the curing environment, is typically associated with a larger factor of safety compared to steel, which is a typically manufactured in controlled conditions. ***In CIV102, we will account for these by using partial safety factors to reduce the strength of concrete by 0.5 and reduce the strength of steel by 0.6 when designing reinforced concrete members.***

Lecture 33 – Reinforced Concrete Members – Design for Flexure

Overview:

In this chapter, the behaviour of reinforced concrete structures subjected to bending moments is described. A simple procedure which can be used to design the flexural reinforcement or estimate the flexural strength of a reinforced concrete member is presented.

Overview of Flexural Behaviour

The response of a reinforced concrete member subjected to bending moments can be described as having three distinct phases, which are shown in Fig. 33.1. For small loads, the stresses in the concrete will be low and the member will exhibit linear elastic behaviour. During this portion of the member's life, the presence of the reinforcing steel has little influence of the flexural response, and Navier's equation can be used to determine the stresses in the concrete.

For larger loads, the flexural stresses obtained by using Navier's equation will exceed the tensile strength of the concrete, and vertical cracks will form. After this happens, the member will continue to carry bending moments with the concrete carrying the compression forces on one side of the member, and the longitudinal steel reinforcement carrying the tension forces on the other side of the member. During this stage, the compressive stresses in the concrete and the tensile stresses in the steel are relatively low, and they will both behave in a linear elastic manner. This sort of load-carrying mechanism is sometimes referred to as the cracked elastic state of the member.

Under larger loads, the reinforcement will begin to yield, and the concrete will begin to crush, which results in a nonlinear response being observed. If the steel yields before the concrete crushes, then the failure mode will be ductile, and large deformations will occur before the beam finally breaks. If the opposite is true, then the member will fail more suddenly. Analyzing the nonlinear behaviour requires more advanced tools and is beyond the scope of CIV102.

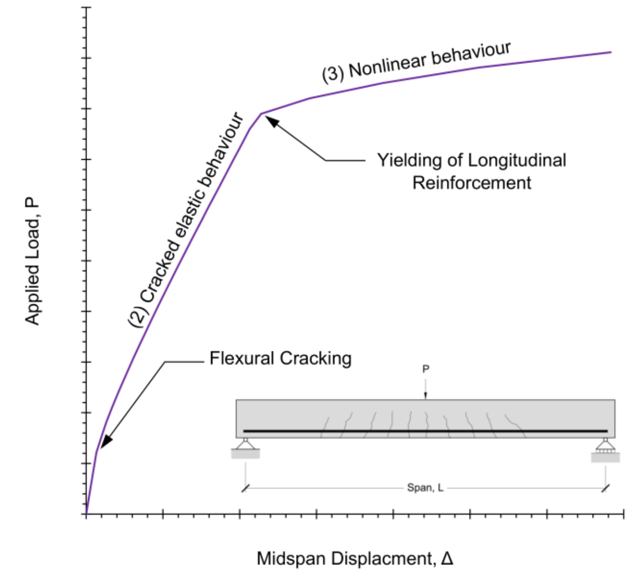


Fig. 33.1 – Typical load-displacement plot of a reinforced concrete beam subjected to bending

The primary task involved when designing for flexure is to determine how much longitudinal steel is needed to carry the bending moments. This steel is provided on the side of the beam experiencing tension, so that the tensile forces in the steel and the compressive forces in the concrete together resist the applied moments.

Cracked Elastic Response

To analyze the flexural behaviour of a reinforced concrete member after it has cracked, the following assumptions are made:

i. Plane sections remain plane, and hence there are longitudinal strains which vary linearly over the height of the member
ii. The concrete cannot carry any tensile stresses
iii. The steel is perfectly bonded to the concrete, so that the concrete and steel experience the same strain at every point

Fig. 33.2 – A reinforced concrete beam tested by Garratt et al. which failed in flexure. Note the large cracks and displacements, as well as the crushed concrete at the top of the beam.

Consider the cracked reinforced concrete beam shown in Fig. 33.2 which bends as it supports the applied loads. The bending moments carried by the member causes vertical cracks to form on the bottom of the beam. To resist the applied loads after cracking occurs, longitudinal steel reinforcement is required. The steel carrying the flexural tension has a total area of $\mathbf{A_s}$ and the distance from the extreme compression fibre of the beam to the centroid of this reinforcement is equal to **d**.

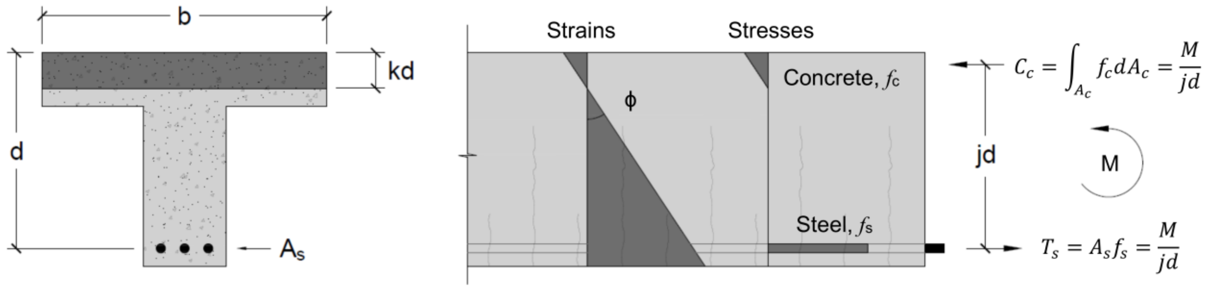


Fig. 33.3 – Schematic of a cracked reinforced concrete beam subjected to bending moments, showing the distribution of strains, stresses and forces

As the cracked beam curves, there will be longitudinal strains which vary linearly over the height of the member. If the curvature, **ϕ**, is known, then the longitudinal strain, **ε**, at a distance **y** from the neutral axis will be equal to:

$$\varepsilon = \phi y \tag{33.1}$$

Although this is the same equation which we used in Lecture 10 for elastic beams, the neutral axis of the cracked member will not align with the centroidal axis of the cross section because the cracked concrete cannot carry any tensile stresses. To determine the location of the new neutral axis of the cracked member, we need to find the corresponding strain conditions that guarantee that the net axial force in the member is zero.

If we define the distance from the extreme compression fibre to the neutral axis as **kd**, then we can express the curvature of the section as:

$$\phi = \frac{\varepsilon_{c,top}}{kd} = \frac{\varepsilon_c + \varepsilon_s}{d} \tag{33.2}$$

In Eq. (33.2), $\boldsymbol{\varepsilon_{c,top}}$ is the concrete strain at the top of the section, and $\boldsymbol{\varepsilon_s}$ is the strain in the steel, which is located a distance **d** from the top of the section. Rearranging Eq. (33.2) results in the following equations for these strains in terms of the curvature and **kd**:

$$\varepsilon_{c,top} = \phi kd \tag{33.3}$$

$$\varepsilon_s = \phi d(1 - k) \tag{33.4}$$

Using Hooke's law, the stresses in the concrete and steel can be obtained once these strains are known. The concrete carries compressive stresses which increase linearly from 0 at the depth of compression to a maximum at the top of the section, and the steel carries a tensile stress at the location of the bars. The net compressive force in the concrete, $\mathbf{C_c}$, can be obtained by integrating the concrete stresses over the cross section, resulting in:

$$C_c = \int_{A_c} f_c dA_c = \int_{A_c} E_c \varepsilon_c dA_c = \frac{1}{2} bkdE_c \varepsilon_{c,top} \quad (33.5)$$

The net tensile force in the steel, $\mathbf{T_s}$, can be obtained by multiplying the steel stresses over the cross section, which results in:

$$T_s = f_s A_s = E_s \varepsilon_s A_s \quad (33.6)$$

For a member subjected to pure bending, there will be no net axial force and therefore the compressive force in the concrete must equal to the tensile force carried by the steel. Therefore, setting Eqs. (33.5) and (33.6) to equal each other and substituting the expressions for $\boldsymbol{\varepsilon_{c,top}}$ and $\boldsymbol{\varepsilon_s}$ into the resulting equation yields:

$$\frac{1}{2}\phi b(kd)^2 E_c = \phi E_s A_s d(1-k) \quad (33.7)$$

Eliminating $\boldsymbol{\phi}$ from Eq. (33.7) and rearranging terms results in the following quadratic equation for **k**:

$$\frac{1}{2}k^2 + k\frac{E_s}{E_c}\frac{A_s}{bd} - \frac{E_s}{E_c}\frac{A_s}{bd} = 0 \quad (33.8)$$

We can abbreviate Eq. (33.8) by defining the modular ratio, **n**, as the following:

$$n = \frac{E_s}{E_c} \quad (33.9)$$

Furthermore, we will define the quantity of longitudinal reinforcement, $\boldsymbol{\rho}$, as:

$$\rho = \frac{A_s}{bd} \quad (33.10)$$

Substituting these new quantities into Eq. (33.8) results it to be rewritten in the following form:

$$\frac{1}{2}k^2 + kn\rho - n\rho = 0 \quad (33.11)$$

Solving for Eq. (33.11) using the quadratic equation results in the following equation for **k**:

$$k = \sqrt{(n\rho)^2 + 2n\rho} - n\rho \quad (33.12)$$

Having found the required value of **k** so that the net axial force carried by the member equals to zero under pure moment, the bending moment carried by the member, **M**, can now be obtained by using fact that the compression

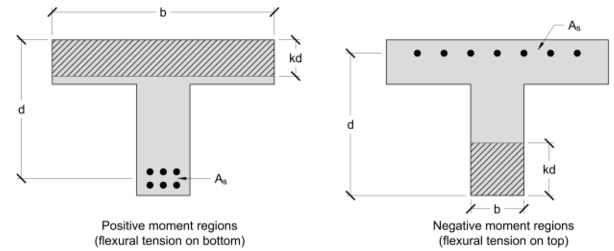


Fig. 33.4 – Definition of b

Note: b is defined as the width of the of the compression side of the member. Therefore, for this T-beam, b is the width of the flange when it is carrying positive moments, and width of the stem when it is carrying negative moments.

force in the concrete, $\mathbf{C_c}$**,** is equal and opposite to the tension force in the steel, $\mathbf{T_s}$**.** Together, they form a couple which has the following properties:

$$M = C_c jd = T_s jd \tag{33.13}$$

In Eq. (33.11), **jd** is the vertical distance between the compressive and tensile forces and is called the ***flexural lever arm***. Because the concrete stresses, are distributed in a triangular pattern over the height **kd**, the resultant compressive force is located at a distance of kd/3 from the top of the member. Therefore, **jd** is equal to:

$$jd = d - \frac{1}{3}kd \rightarrow j = 1 - \frac{1}{3}k \tag{33.14}$$

Knowing the value of the flexural lever arm is important as it provides the necessary link between the bending moment carried by the member and the stress in the reinforcement, f_s. This relationship can be determined by combining Eqs. (33.6) and (33.13):

$$M = A_s f_s jd \ \rightarrow f_s = \frac{M}{A_s jd} \tag{33.15}$$

If the steel is still linear elastic, the strain in the steel can be obtained by using Hooke's law and dividing f_s by the Young's modulus of steel, $\mathbf{E_s}$. Substituting the steel strain into Eq. (33.4), results in the following equation for the curvature of the member:

$$\phi = \frac{M}{A_s E_s j d^2 (1 - k)} \tag{33.16}$$

Substituting Eq. (33.16) in Eq. (33.3) and multiplying the concrete strain by $\mathbf{E_c}$ results in a compact equation for the maximum concrete stress when the member is carrying the bending moment:

$$f_c = \frac{k}{1 - k} \frac{M}{n A_s jd} \tag{33.15}$$

Finally, the maximum moment which can be carried by the member if it fails by yielding of the flexural reinforcement can be determined if we use Eq. (33.13) and let the steel stress f_s equal the yield stress f_y:

$$M_{yield} = A_s f_y jd \tag{33.16}$$

Design Process Summary

The following steps outline a procedure for proportioning the longitudinal reinforcement in a beam to safely resist bending moments which result from applied loads.

i. Obtain the bending moment diagram and determine the moment which must be resisted by the beam, **M**.

ii. Using a provided value of **d**, estimate **k** and **j** to be **k = 3/8** and **j = 7/8**. If it is assumed that the maximum allowable tensile stress in the steel is $\mathbf{0.6}\boldsymbol{f_y}$, then the required area of steel is:

$$A_{s,min} = \frac{M}{0.6 f_y jd}$$

iii. Using the rebar table in Appendix G, select the number of bars needed so that the area of longitudinal steel, $\mathbf{A_s}$, is greater or equal to $\mathbf{A_{s,min}}$.

iv. Calculate the actual value of **k**. Recall that $\mathbf{n = E_s/E_c}$ and $\boldsymbol{\rho} \mathbf{= A_s/bd}$.

$$k = \sqrt{(n\rho)^2 + 2n\rho} - n\rho$$

v. Calculate the actual length of the flexural lever arm, **jd**:

$$jd = d\left(1 - \frac{1}{3}k\right)$$

vi. Check to ensure that the steel stress, $\boldsymbol{f_s}$, does not exceed $0.6\boldsymbol{f_y}$:

$$f_s = \frac{M}{A_s jd} \leq 0.6 f_y$$

vii. Check to ensure that the concrete stress, $\boldsymbol{f_c}$, does not exceed $0.5\boldsymbol{f_c'}$:

$$f_c = \frac{k}{1-k}\frac{M}{nA_s jd} \leq 0.5 f_c'$$

When checking if a design is safe, only steps i and iv to vii need to be performed.

Lecture 34 – Reinforced Concrete Members – Design for Shear

Overview:

In this chapter, the behaviour of reinforced concrete members subjected to shear is discussed. The Simplified Method for shear design in the Canadian concrete design code, CSA A23.3:19, is presented, and its application for both designing members for shear and evaluating the shear strength of existing structures is explained.

Historical Background

Shear stresses in reinforced concrete members can cause failure due to the resulting diagonal tensile and compressive stresses. If a member does not contain adequate amounts of shear reinforcement, vertical bars which run perpendicular to the longitudinal steel, then it may fail suddenly without noticeable signs of distress. The mechanism by which reinforced concrete members carry shear is complex, and there have been significant structural failures in the 20th century due to inadequate design and construction practices. Some notable shear failures include the collapse of the Sleipner A offshore platform in 1991, which imploded while under construction, leading to an estimated cost of $700 million (USD), and the collapse of the De la Concorde overpass in 2006, which killed five people and left six others with serious injuries. Photos of these collapses are shown in Fig. 34.1.

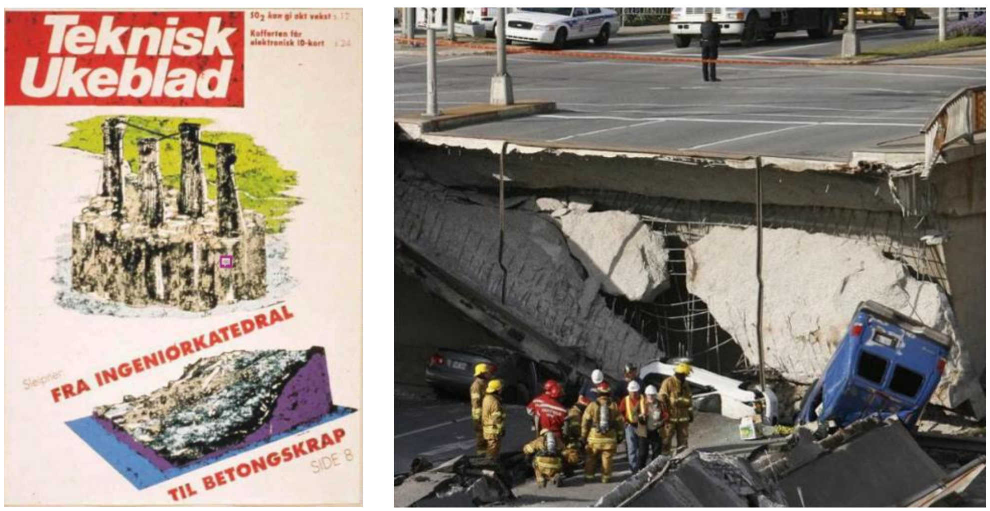


Fig. 34.1 – Collapse of the Sleipner A platform (left) and De la Concorde overpass (right).
The magazine title translates to "From Engineering Cathedral to Concrete Scrap"

Research work performed at the University of Toronto has led to significant advances in the understanding of how reinforced concrete carries shear stresses. Experiments performed on concrete specimens using unique equipment like the Panel Element Tester and Shell Element Tester, shown in Fig. 34.2, have led to the development of the ***Modified Compression Field Theory,*** which serves as the theoretical basis for the Canadian and Australian concrete design codes, the fib Model Code 2010 and the American bridge design code.

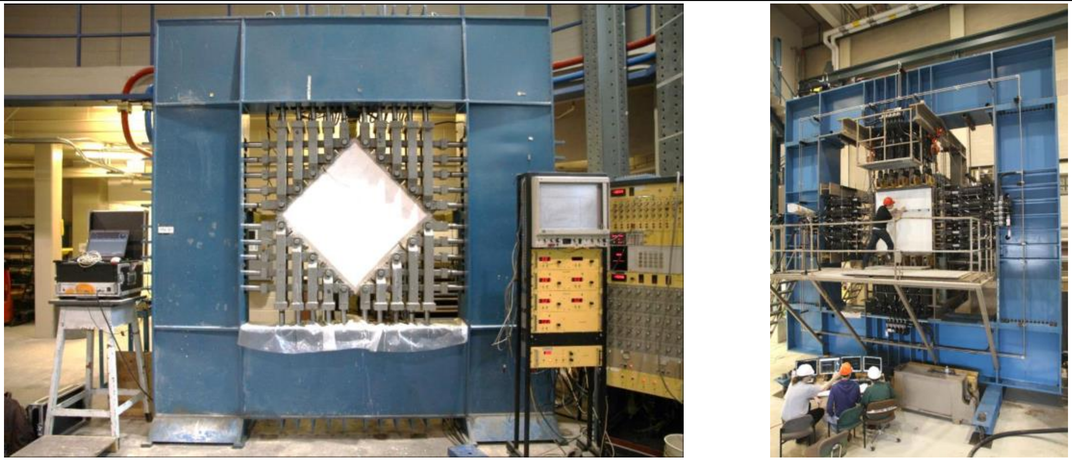


Fig. 34.2 – Panel Element Tester (left) and Shell Element Tester (right) at the University of Toronto

Overview of Shear Behaviour

Recall from Lecture 25 that shear stresses in beams causes diagonal compressive and tensile stresses. Because concrete has a low tensile strength, these diagonal tensile stresses causes diagonal cracks to form. After cracking, the reinforced concrete has two basic mechanisms for carrying shear stresses:

i. **Aggregate Interlock** – shear stresses acting along the crack surfaces, which are rough due to the aggregate embedded in the concrete, along with tensile stresses in the longitudinal steel carry tension across the crack

ii. **Shear Reinforcement** – steel reinforcement which is perpendicular to the longitudinal reinforcement carry tensile stresses which, along with the tensile stresses in the longitudinal steel, carry tension across the crack

The first mechanism is the predominant method of carrying shear for members which do not have shear reinforcement and its strength is strongly influenced by the width of the cracks which form under shear loading. A close-up photo showing interlocking of the aggregate is shown in Fig. 34.3.

Fig. 34.3 – Interlocking of aggregate in a concrete member, preventing sliding along the crack

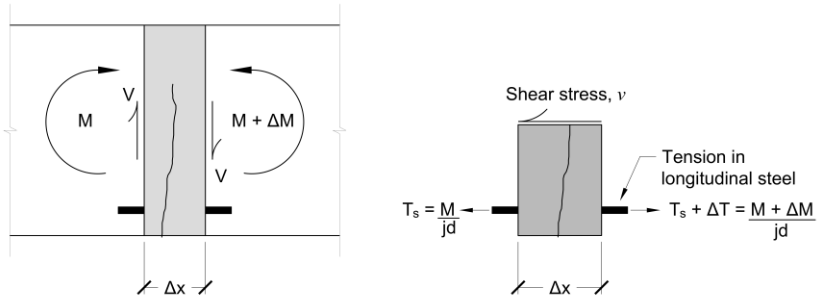


Fig. 34.5 – Derivation of shear stresses in a cracked concrete member

After cracking, the shear stress distribution no longer follows the shape predicted by Jourawski's equation because the tensile capacity of the concrete is severely reduced. Consider a slice of a beam with a width **Δx** shown from elevation view, shown in Fig. 34.5. Recall that the shear force, **V**, is related to the bending moment **M**, by:

$$V = \frac{dM}{dx} \tag{34.1}$$

The change in moment leads to an increase in the tensile forces in the longitudinal steel, which is related to the tension in the steel, $\mathbf{T_s}$, by the flexural lever arm, **jd**. Therefore, Eq. (34.1) can be rewritten in terms of the change in tension force in the longitudinal steel, $\mathbf{\Delta T_s}$:

$$M = T_s jd \rightarrow V = \frac{\Delta T_s jd}{\Delta x} \tag{34.2}$$

The change in tension force in the longitudinal steel is due to the shear stresses in the member acting over the area defined by the web width, $\mathbf{b_w}$, and the length of our slice, **Δx**. Horizontal equilibrium requires that the shear stresses, ***v***, must be defined as:

$$v b_w \Delta x = \Delta T_s \tag{34.3}$$

Substituting Eq. (34.2) in Eq. (34.3) and eliminating **Δx** from the equation results in the following equation for the maximum shear stress in a cracked concrete member, which occurs in its web:

$$v = \frac{V}{b_w jd} \tag{34.4}$$

A shear failure in a member occurs when the shear stress exceeds its shear capacity. As previously noted, the shear capacity is related to the strength offered by the aggregative interlock, $\boldsymbol{v_c}$, and the shear strength offered by the steel shear reinforcement, $\boldsymbol{v_s}$. In heavily reinforced members which contain large amounts of reinforcement, then another

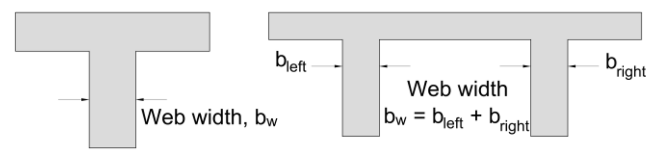


Fig. 34.6 – Effective web width, b_w.

Note: the web width may consider adjacent webs, like in the case of the double-tee beam shown in Fig. 34.6.

possible failure mode is if the diagonal compression from the shear causes crushing to take place. The shear stress which causes this to occur, $\boldsymbol{v_{max}}$, is defined in the Canadian concrete design code as:

$$v_{max} = 0.25f_c' \qquad (34.5)$$

Where $\boldsymbol{f_c'}$ is the compressive strength of the concrete. Putting these concepts together, the shear strength of a member, $\mathbf{V_r}$, is equal to the sum of strength attributed to the concrete, $\mathbf{V_c}$, and the strength attributed to the steel, $\mathbf{V_s}$, and less than $\mathbf{V_{max}}$. To provide an adequate factor of safety in design, the terms associated with the concrete strength are also multiplied by 0.5, and the terms associated with the steel strength are multiplied by 0.6, which results in the following equation:

$$V_r = 0.5V_c + 0.6V_s \leq 0.5V_{max} \qquad (34.6)$$

Finally, when designing for shear, the value of the shear force at the location of a reaction force or concentrated point load is typically not used. This is because the region around these forces is “disturbed” by local compressive stresses, since the forces compress the member where they are applied. Because shear failures are typically associated with diagonal tension, the additional compression in these regions helps to prevent a failure from occurring. Instead of the maximum shear force, the shear force in the beam which is located a distance **d** away from the reaction force or point load is typically used, where **d** is the distance between the longitudinal steel and the compression face of the member used for flexural design. Fig. 34.6 illustrates the shear force which should be used in design.

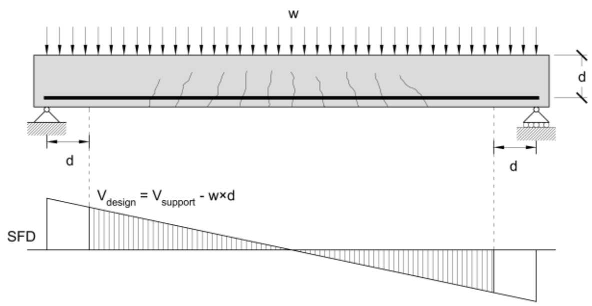


Fig. 34.6 – Design shear force

Shear Capacity for Beams Without Shear Reinforcement

Slabs, which are used commonly used in floors or foundations of a building, are often built without shear reinforcement. Therefore, their shear strength solely depends on the ability of the concrete to carry stresses across the cracks via aggregate interlock. The shear strength is strongly related to how thick the member is, because larger members tend to have wider cracks. Since aggregate interlocking becomes less effective as the cracks get larger, the shear strength, $\boldsymbol{v_c}$, tends to become smaller as the overall depth of the member increases. This is called the ***size effect*** and has been observed in experiments done at the University of Toronto and other institutions. Fig. 34.7 shows the effect of member depth on predicted shear strength using the Canadian code, as well as the shear strengths of many experiments.

Note: $\boldsymbol{v_c}$ and $\boldsymbol{v_s}$ correspond to the shear strengths associated with the concrete and steel respectively in units of MPa. To determine the corresponding shear force, $\boldsymbol{V_c}$ and $\boldsymbol{V_s}$ respectively, these quantities should be multiplied by $b_w jd$ in accordance with Eq. (34.4)

Note: The maximum shear force causing crushing is equal to:

$$V_{max} = 0.25f_c' b_w jd$$

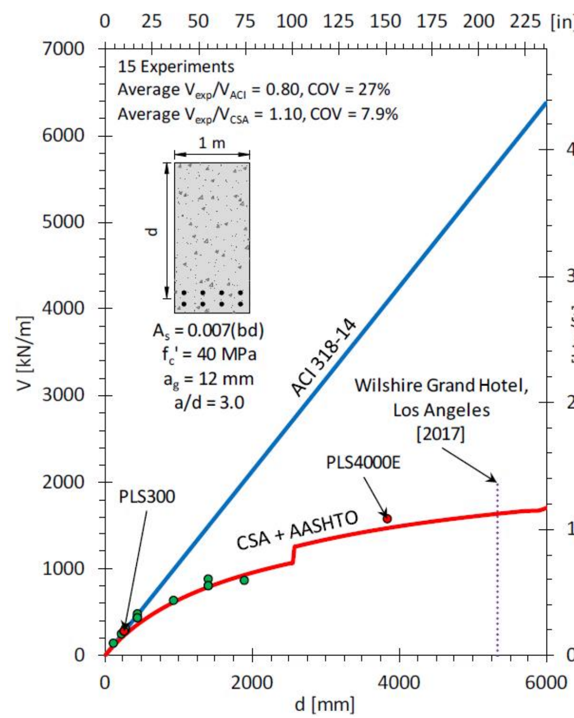


Fig. 34.7 – Size effect for reinforced concrete members subjected to shear. The failure stress of a member decrease as it becomes deeper

The equation for the shear strength of members without shear reinforcement is given in Eq. (34.7). This equation is based on the Modified Compression Field Theory and is used for shear design in Canada.

$$V_c = v_c b_w jd = \frac{230\sqrt{f_c'}}{1000 + 0.9d} b_w jd \tag{34.7}$$

*Note: the shear strength (in MPa) attributed to the concrete, **v_c**, is equal to:*

$$v_c = \frac{230\sqrt{f_c'}}{1000 + 0.9d}$$

In Eq. (34.7), **V_c** is the shear strength in units of force, ***f_c'*** is the compressive strength of the concrete in MPa, **d** is the effective depth in mm, **b_w** is the width of the web and **jd** is the flexural lever arm.

Shear Capacity for Members Containing Shear Reinforcement

Shear reinforcement are bars which are perpendicular to the direction of the longitudinal reinforcement. They are commonly inserted into reinforced concrete members by bending bars to form hoops or U-shapes, which are commonly referred to as ***stirrups*** in North America. Some examples of shear reinforcement are shown in Fig. 34.8. The area of shear reinforcement, **A_v**, refers to the total cross-sectional area of the bars which are oriented vertically, and the primary task of an engineer is to determine their spacing to ensure that the member has adequate shear strength.

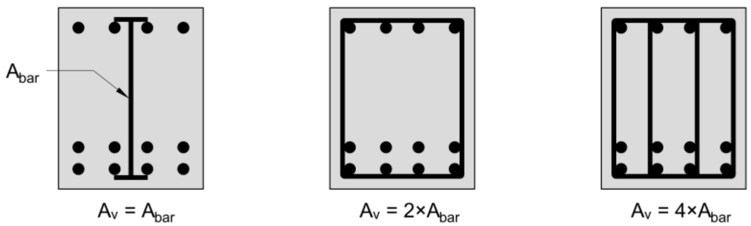


Fig 34.8 – Types of shear reinforcement and corresponding values of A_v

Providing shear reinforcement has two benefits: the bars themselves provide additional shear strength, **V_s**, and they control the size of the diagonal cracks, which improves the aggregate interlocking strength and hence increases **V_c** as well. The shear strength provided by the stirrups can be described by visualizing how the stresses in a cracked reinforced concrete beam are carried, with fields of diagonal compression in the concrete being equilibrated by the tensile stresses in the shear reinforcement, which have an area of **A_v** and spacing **s**. This is shown in the left diagram in Fig. 34.9.

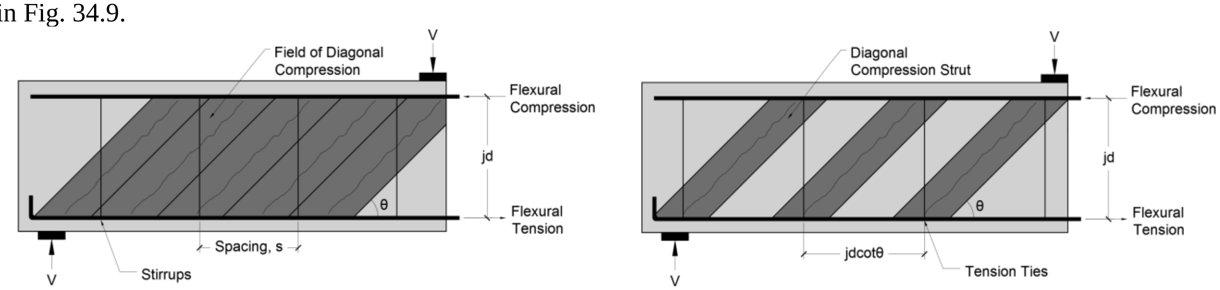


Fig. 34.9 – Diagonal stress fields in a cracked reinforced concrete beam (left) and Simplified truss model for concrete members subjected to shear (right)

The right diagram in Fig. 34.9 shows a simplified depiction of the left diagram, representing the stress fields as a truss with a height of **jd**, and discrete diagonal compression members inclined at an angle of **θ** from the longitudinal axis. Given these geometric constraints, the vertical tension members must be spaced at **jdcotθ** and have a cross sectional area, **A**, equal to the following:

$$\frac{A}{jd\cot\theta} = \frac{A_v}{s} \rightarrow A = \frac{A_v jd\cot\theta}{s} \tag{34.8}$$

If yielding of these vertical tension members governs the failure of the truss, then the maximum shear force which can be carried in the truss, $\mathbf{V_s}$, will occur when the stress in these bars reaches the yield stress, $\boldsymbol{f_y}$:

$$V_s = \frac{A_v f_y jd}{s}\cot\theta \tag{34.9}$$

In the Canadian design code, the angle of the diagonal stresses is assumed to be equal to θ = 35°, resulting in the following equation for $\mathbf{V_s}$:

$$V_s = \frac{A_v f_y jd}{s}\cot 35° \tag{34.10}$$

Note: The shear strength (in MPa) attributed to the steel, $\boldsymbol{v_s}$, is equal to:

$$v_s = \frac{V_s}{b_w jd} = \frac{A_v f_y}{b_w s}\cot 35°$$

For small amounts of shear reinforcement, $\mathbf{V_c}$ is still calculated by using Eq. (34.7). However, if the quantity of shear reinforcement exceeds a threshold value, then the size effect disappears, and an improved equation can be used. This minimum reinforcement requirement is:

$$\frac{A_v f_y}{b_w s} \geq 0.06\sqrt{f_c'} \tag{34.11}$$

If Eq. (34.11) is satisfied, then $\mathbf{V_c}$ is instead calculated as:

$$V_c = v_c b_w jd = 0.18\sqrt{f_c'}b_w jd \tag{34.12}$$

Note: The equation for $\boldsymbol{v_c}$ when at least a minimum amount of shear reinforcement is present is equal to:

$$v_c = 0.18\sqrt{f_c'}$$

Although there are many concepts covered in this chapter, applying them to solve problems is relatively straightforward. There are two primary uses of the equations, which are to (1) evaluate the shear strength of a member or (2) design the shear reinforcement by selecting the appropriate spacing of stirrups. Each process is described in the following summary sections.

Summary - Evaluating the Shear Strength of a Member

This procedure applies when the failure load of a member is needed. Because we are dealing with structural failure, the partial safety factors, 0.5 for concrete and 0.6 for steel, are not used.

i. Solve for the reaction forces and obtain the shear force and bending moment diagrams. Determine the maximum shear force, **V**, which is not located within **d** of a reaction force or applied point load.

ii. Calculate the relevant parameters used for flexural behaviour, such as **n**, **ρ**, **k** and **j**. Recall that:

$$k = \sqrt{(n\rho)^2 + 2n\rho} - n\rho$$

$$j = 1 - \frac{1}{3}k$$

iii. Check if the provided amount of reinforcement meets or exceeds the minimum reinforcement requirement described in Eq. (34.11). Calculate $\mathbf{V_c}$ using the appropriate equation.

$$V_c = \begin{cases} \dfrac{230\sqrt{f_c'}}{1000 + 0.9d} b_w jd & \dfrac{A_v f_y}{b_w s} < 0.06\sqrt{f_c'} \text{ or no stirrups} \\ 0.18\sqrt{f_c'} b_w jd & \dfrac{A_v f_y}{b_w s} \geq 0.06\sqrt{f_c'} \end{cases}$$

iv. If shear reinforcement is present, calculate $\mathbf{V_s}$:

$$V_s = \frac{A_v f_y jd}{s} \cot 35°$$

v. Calculate the shear strength of the member, $\mathbf{V_r}$:

$$V_r = V_c + V_s \leq V_{max} = 0.25 f_c' b_w jd$$

vi. Failure of the member occurs when the applied shear force is equal to the shear resistance.

$$V = V_r$$

Summary – Designing the shear reinforcement of a member

The first two steps of this procedure are the same as the procedure for evaluating the shear strength of a member, which are finding the shear force diagram and determining the flexural properties **k** and **j**. Once these are found, then the required task is determining if shear reinforcement is needed, and if so, the required spacing **s**.

i. Same as step i in the previous procedure.

ii. Same as step ii in the previous procedure.

iii. Check if the shear demand, **V**, exceeds $\mathbf{0.5V_{max}}$. If so, the cross section is too small and needs to be resized. Otherwise, proceed to step iv.

$$0.5 \times V_{max} = 0.125 f_c' b_w jd$$

iv. Check to see if the shear force can be resisted by the $\mathbf{V_c}$ alone. If so, the design is complete.

$$V_r = 0.5 V_c = 0.5 \frac{230\sqrt{f_c'}}{1000 + 0.9d} b_w jd$$

v. If the shear force cannot be carried by $\mathbf{V_c}$ alone, provide the minimum amount of shear reinforcement and check if this provides enough capacity to carry the shear demand. Remember that providing the minimum amount of shear reinforcement permits Eq. (34.12) to be used when calculating $\mathbf{V_c}$ instead of Eq. (34.7)

$$\frac{A_v}{s} = 0.06\sqrt{f_c'}\frac{b_w}{f_y} \rightarrow s = \frac{A_v f_y}{0.06\sqrt{f_c'} b_w}$$

$$V_r = 0.5V_c + 0.6V_s = 0.5 \times 0.18\sqrt{f_c'} b_w jd + 0.6\frac{A_v f_y jd}{s}\cot 35°$$

vi. If the shear capacity is still too low, then a smaller spacing must be obtained to carry the shear force. This spacing, **s**, can be obtained by letting **V** = $\mathbf{V_r}$ and re-arranging the equation, resulting in the following:

$$s = \frac{V - 0.5 \times 230\sqrt{f_c'} b_w jd}{0.6 A_v f_y jd \cot 35°}$$

Lecture 35 – Prestressed Concrete Structures

Overview:

In this chapter, the fundamental behaviour of prestressed concrete members is discussed. The forced-in-the-tendon method to determine the stresses in a prestressed concrete member subjected to bending moments is presented.

Prestressed Concrete:

In many ways, prestressed concrete is similar to reinforced concrete. Steel reinforcement is cast into the concrete and after the concrete cracks under loads, the steel carries tensile forces to increase the structure's load-carrying capacity. Unlike reinforced concrete however, prestressing the reinforcement results in the steel carrying significant tensile forces before the external loads are applied. Because the steel is embedded into the concrete, these tensile forces in the steel cause the concrete to be in a state of compression. Prestressed reinforcement is sometimes referred to as ***active reinforcement*** because the steel is actively in a state of tension, which contrasts with ***passive reinforcement***, which is only engaged once the concrete cracks.

The precompression applied to the concrete by the prestressed reinforcement has a significant impact on its load-deformation response because significant tensile forces are now needed to overcome the precompression and cause cracking. Therefore, under the loads which are expected under typical daily situations, the concrete will remain uncracked and behave in a linear elastic manner. Figure 35.1 illustrates the difference prestressing can make for a beam carrying a uniform load. For loads which are about half of the predicted failure load, which corresponds to a typical factor of safety, the reinforced concrete beam deflects significantly more than the equivalent prestressed beam.

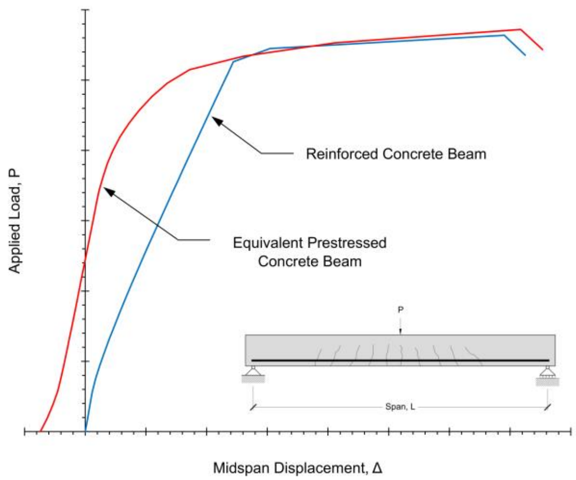


Fig. 35.1 – Comparison of load-deformation response of equivalent reinforced and prestressed concrete members

The stresses in the concrete and steel caused by prestressing are said to be self-equilibrating, meaning that they balance each other out without the influence of external loads. Therefore, the total stresses in the concrete can be calculated as the sum of the stresses cause by prestressing and the stresses caused from axial loads, shear forces, and moments.

Note: Prestressed concrete requires using high-strength steel and high strength concrete. This is so the steel can be stressed to carry large forces to compress the concrete and not yield or rupture. The steel must also be able to sustain these stresses over the lifetime of the structure and minimize losses due to effects like creep. The concrete must have a high compressive strength to avoid crushing under the combined effects of the applied loads and the prestress.

Fig. 35.2 – Skilled worker post-tensioning cables in a concrete member using a hydraulic jack

*Note: There are two primary means of prestressing concrete. The first method, called **pre-tensioning**, involves casting concrete around strands of steel which are being pulled. Once the concrete hardens, the steel is cut from the bed and embedded parts compress the concrete. The second method, called **post-tensioning**, involves casting hollow ducts into the concrete. Once the concrete hardens, steel strands are inserted into the ducts where they are stressed and anchored.*

*Note: The prestressed reinforcement in a prestressed concrete member is sometimes referred to as **tendons**.*

Calculating Stresses in Prestressed Concrete Members with Concentric Tendons

Consider the prestressed member shown in Fig. 35.3, which has a tendon running along its centroidal axis which is stressed to carry a tensile force, **P**, that compresses the concrete. The member is simply supported over a span **L** and carries a uniformly distributed load along its span, **w**, causing a bending moment of $wL^2/8$ at the midspan.

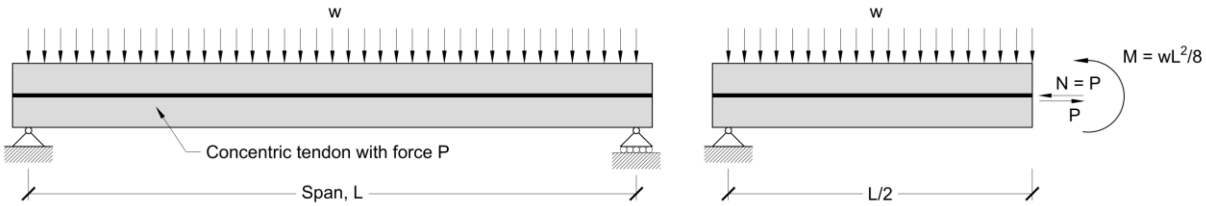


Fig. 35.3 – Prestressed member with a concentric tendon. Elevation view (left) and free body diagram at midspan (right). Note the tension force acting to the right should be aligned with the tendon.

Consider the stresses in the member at the midspan, also shown in Fig. 35.2. In addition to the bending moment, there is also an axial load applied to the section due to the prestressing force. Therefore, if the concrete is behaving in a linear elastic manner, the stresses in the concrete, $\boldsymbol{\sigma_c}$, can be calculated by using the basic definition of stress and Navier's equation:

$$\sigma_{c,top} = -\frac{P}{A} - \frac{My_{top}}{I} \tag{35.1}$$

$$\sigma_{c,bot} = -\frac{P}{A} + \frac{My_{bot}}{I} \tag{35.2}$$

In Eqs (35.1) and (35.2), $\boldsymbol{\sigma_{c,top}}$ and $\boldsymbol{\sigma_{c,top}}$ refer to the concrete stresses at the top and bottom of the beam respectively, **A** is the cross sectional area, **I** is the second moment of area, and $\mathbf{y_{top}}$ and $\mathbf{y_{bot}}$ are the distances to the top and bottom of the beam from the centroidal axis. This method is appropriate for finding the stresses if, using these equations, they are found to be less than the cracking strength of the concrete.

Note: If the stresses exceed the tensile strength of the concrete, it will crack, and more advanced analysis procedures are required to analyze the behaviour.

Calculating Stresses in Prestressed Concrete Members with Eccentric Tendons

Arranging the tendons in a prestressed concrete member so that they do not align with the centroidal axis is an effective strategy when designing for bending moments. Consider the beam shown in Fig. 35.4 which has a straight tendon which is eccentric by a distance **e** from the centroidal axis. In the absence of applied forces, the prestressing force will compress the member at the location of the steel, causing it to curve upwards. This counteracts the downwards displacements caused by gravity loads, as shown in Fig. 35.5.

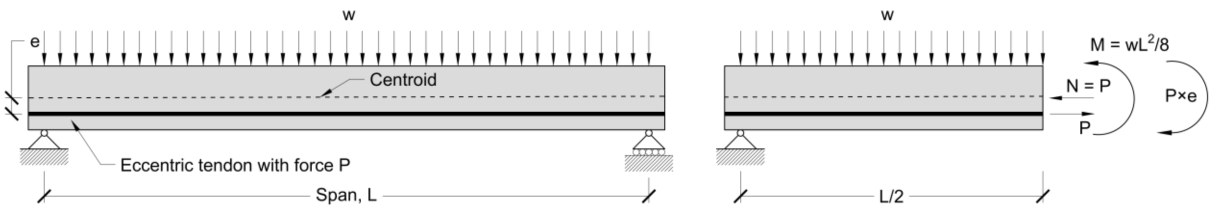


Fig. 35.4 – Prestressed concrete member with eccentric tendons. Elevation view (left) and free body diagram at midspan (right).

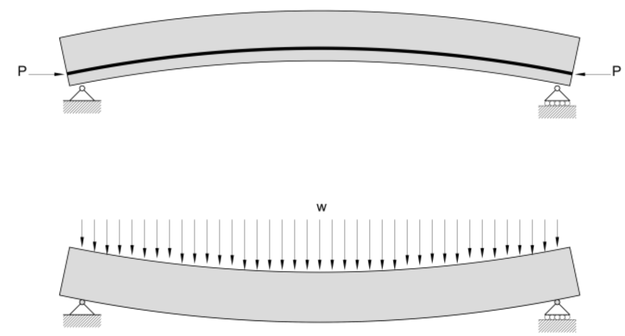


Fig. 35.5 – Curving of prestressed member due to eccentric tendons (top), which opposes the deflections caused by applied loads (bottom)

When the tendon is eccentric, the member curves because the axial force in the concrete, which resists the tensile force in the steel, acts through the centroid of the cross section. Since the two forces are equal and opposite, but are separate by a distance **e**, they produce a couple which counteracts the bending moment caused by the loads. The stresses in concrete can then be taken as the sum of those caused by the prestressing force, those caused by the eccentricity of the tendon, and those caused by the applied loads:

$$\sigma_{c,top} = -\frac{P}{A} + \frac{Pey_{top}}{I} - \frac{My_{top}}{I} \tag{35.3}$$

$$\sigma_{c,bot} = -\frac{P}{A} - \frac{Pey_{bot}}{I} + \frac{My_{bot}}{I} \tag{35.4}$$

Eqs. (35.3) and (35.4) can be used to calculate the stresses in the concrete at any location along the length of the member. Note that in regions where the bending moment is low, like near the supports, the stresses caused by the eccentricity of the tendons may cause the top of the member to crack. Therefore, it is common to have the tendons follow a curved profile so that the eccentricity varies to match the demand from the applied loads.