MAT292 - Fall 2018

Term Test 2 - November 12, 2018

Time allotted: 100 minutes | Aids permitted: None

Total marks: 65

Full Name: ____
Last | First

Student Number: ____

Email: ______________________________________ @mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test contains 10 pages (including this title page). Make sure you have all of them.
- You can use pages 9–10 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 9–10.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

HAVE FUN!

SECTION I No explanation is necessary. **(10 marks)**

1. **(4 marks)** Each curve in the phase plane is a solution to which differential equation $\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}$?

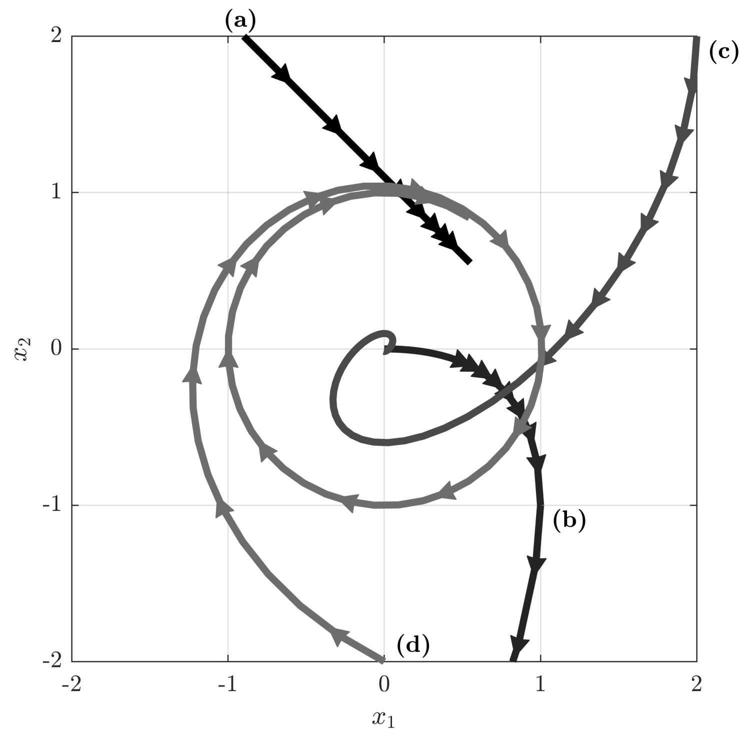
i) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

ii) $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

iii) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

iv) $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

v) none of the above



(a) ii

(b) iii

(c) iv

(d) v

2. **(2 marks)** Let y_1 and y_2 be solutions to $y''(t) + 7y = 0$ with initial values $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$. Compute the Wronskian of $y_1(t)$ and $y_2(t)$: $W[y_1, y_2](t) =$ -1

3. **(1 mark)** Find γ so that $y = cte^{2t}$ (for some constant c) is a solution to $y'' - 3y' + \gamma y = e^{2t}$.
$\gamma =$ 2

4. **(2 marks)** Find a and b so that $\boldsymbol{x}(t) = e^{at}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution of the system $\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 1 & 2b \\ 2 & 3b \end{pmatrix}\boldsymbol{x}$.

$a =$ -1 $b =$ -1

5. **(1 mark)** For $\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\boldsymbol{x}$, $\boldsymbol{x}_0 = \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find $\boldsymbol{x}_1$, the result of applying Euler's method with step size $h = 1$.

$\boldsymbol{x}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

SECTION II **Justify** your answers. **(55 marks)**

6. Solve the following initial value problem with two different methods, **(10 marks)**

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \frac{1}{3}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}, \quad x_1(0) = 1,\ x_2(0) = 2.$$

(a) **(5 marks)** The eigenvalue method.

Solution: The characteristic polynomial is $(\frac{1}{3} - \lambda)(-\frac{1}{3} - \lambda) - \frac{8}{9} = 0$ with roots $\lambda = \pm 1$. Eigenvectors are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $\lambda = 1$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $\lambda = -1$. Since $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, the solution to the initial value problem is $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^t\begin{pmatrix} 2 \\ 1 \end{pmatrix} - e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2e^t - e^{-t} \\ e^t + e^{-t} \end{pmatrix}$.

(b) **(5 marks)** Let $z_1 = x_1 + x_2$ and $z_2 = x_1 - 2x_2$. Solve separate differential equations for z_1 and z_2 and then determine $x_1 = (2z_1 + z_2)/3$ and $x_2 = (z_1 - z_2)/3$.

Solution:

$z_1' = (x_1 + x_2)' = x_1' + x_2' = \frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{2}{3}x_1 - \frac{1}{3}x_2 = x_1 + x_2 = z_1$.

Since $z_1(0) = x_1(0) + x_2(0) = 3$, $z_1(t) = 3e^t$.

$z_2' = (x_1 - 2x_2)' = x_1' - 2x_2' = \frac{1}{3}x_1 + \frac{4}{3}x_2 - \frac{4}{3}x_1 + \frac{2}{3}x_2 = -x_1 + 2x_2 = -z_2$.

Since $z_2(0) = x_1(0) - 2x_2(0) = -3$, $z_2(t) = -3e^{-t}$.

Therefore, $x_1 = (2z_1 + z_2)/3 = 2e^t - e^{-t}$ and $x_1 = (z_1 - z_2)/3 = e^t + e^{-t}$.

7. Show that for two solutions $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ to the system of differential equations $\frac{d\boldsymbol{x}}{dt} = P(t)\boldsymbol{x}$, that $\boldsymbol{x}_1(t) + \boldsymbol{x}_2(t)$ is a solution. Is it necessary that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be linearly independent? **(5 marks)**

Solution: Using linearity of the derivative and matrix multiplication,

$$\frac{d}{dt}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \frac{d\boldsymbol{x}_1}{dt} + \frac{d\boldsymbol{x}_2}{dt} = P(t)\boldsymbol{x}_1 + P(t)\boldsymbol{x}_2 = P(t)(\boldsymbol{x}_1 + \boldsymbol{x}_2),$$

so that $\boldsymbol{x}_1 + \boldsymbol{x}_2$ is a solution to the differential equation. It is **not** necessary that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be linearly independent.

8. Consider the differential equation $y'' - 2\alpha y' + (\alpha^2 - \alpha + 1)y = 0$ with parameter $\alpha \in \mathbb{R}$. **(10 marks)**

(a) **(2 marks)** For which values of α are solutions (except $y \equiv 0$) i) growing amplitude oscillations, and ii) decaying amplitude oscillations.

Solution: The characteristic equation is $\lambda^2 - 2\alpha\lambda + (\alpha^2 - \alpha + 1) = 0$ with roots $\lambda = \alpha \pm \sqrt{\alpha - 1}$. For $0 < \alpha < 1$, there are only growing amplitude oscillations (i). For $\alpha < 0$, there are only decaying amplitude oscillations (ii).

(b) **(3 marks)** For all α, find the general real solution. Consider the cases of distinct real, repeated real, and complex conjugate pairs of eigenvalues separately.

Solution:

$\alpha < 1: \quad y = c_1 e^{\alpha t} \cos(\sqrt{1-\alpha}t) + c_2 e^{\alpha t} \sin(\sqrt{1-\alpha}t)$

$\alpha > 1: \quad y = c_1 \exp\left((\alpha + \sqrt{\alpha - 1})t\right) + c_2 \exp\left((\alpha - \sqrt{\alpha - 1})t\right)$

$\alpha = 1: \quad y = c_1 e^t + c_2 t e^t$

(c) **(5 marks)** Using the method of undetermined coefficients **for all** α find a particular solution of $y'' - 2\alpha y' + (\alpha^2 - \alpha + 1)y = e^t$.

Solution: Try $y_{\mathrm{p}} = Ae^t$ so that $y_{\mathrm{p}}' = y_{\mathrm{p}}'' = Ae^t$ and
$y_{\mathrm{p}}'' - 2\alpha y_{\mathrm{p}}' + (\alpha^2 - \alpha + 1)y_{\mathrm{p}} = A(1 - 2\alpha + \alpha^2 - \alpha + 1)e^t = A(\alpha^2 - 3\alpha + 2)e^t$.
So we have a particular solution for $A = 1/(\alpha^2 - 3\alpha + 2)$.
This form fails for $\alpha = 1$ and $\alpha = 2$.

For $\alpha = 2$, try $y_{\mathrm{p}} = Ate^t$ so that $y_{\mathrm{p}}' = Ae^t(1+t), y_{\mathrm{p}}'' = Ae^t(2+t)$ and $y_{\mathrm{p}}'' - 2\alpha y_{\mathrm{p}}' + (\alpha^2 - \alpha + 1)y_{\mathrm{p}} = Ae^t(2 + t - 4(1+t) + 3t) = Ae^t(-2)$.
So we have a particular solution for $A = -\frac{1}{2}$.

For $\alpha = 1$, try $y_{\mathrm{p}} = At^2e^t$ so that $y_{\mathrm{p}}' = Ae^t(2t + t^2), y_{\mathrm{p}}'' = Ae^t(2 + 4t + t^2)$ and $y_{\mathrm{p}}'' - 2\alpha y_{\mathrm{p}}' + (\alpha^2 - \alpha + 1)y_{\mathrm{p}} = Ae^t(2 + 4t + t^2 - 2(2t + t^2) + t^2) = Ae^t(2)$.
So we have a particular solution for $A = \frac{1}{2}$.

Continued...

9. If a solution y_1 is known for the differential equation $y'' + p(t)y' + q(t)y = 0$, **(10 marks)** then one can find the general solution using the Wronskian $W[y_1, y_2] = y_1 y_2' - y_2 y_1'$ as follows.

(a) **(2 marks)** Show that $\left(\frac{y_2}{y_1}\right)' = \frac{W[y_1,y_2]}{y_1^2}$.

Solution: Using the quotient rule $\left(\frac{y_2}{y_1}\right)' = \frac{y_2' y_1 - y_2 y_1'}{y_1^2} = \frac{W[y_1,y_2]}{y_1^2}$.

(b) **(2 marks)** Show that $W[y_1, y_2]$ satisfies $W' + p(t)W = 0$ and therefore $W[y_1, y_2] = c_1 \exp\left(-\int_0^t p(\tau)d\tau\right)$.

Solution: Using the sum and product rules and the differential equation,

$$\begin{aligned} W' &= (y_1 y_2' - y_2 y_1')' \\ &= y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' \\ &= y_1(-p y_2' - q y_2) - y_2(-p y_1' - q y_1) \\ &= -p(y_1 y_2' - y_2 y_1') \\ &= -pW, \end{aligned}$$

as required.

(c) **(6 marks)** Check that $y_1 = t^2$ is a solution of $t^2 y'' - 2y = 0$. Use the Wronskian to find a second linearly independent solution. What is the general solution?

Solution:

Since $y_1 = t^2$, $y_1' = 2t, y_1'' = 2$ and $t^2 y'' - 2y = t^2 2 - 2t^2 = 0$.

Since $p(t) \equiv 0$, $W[y_1, y_2] = C$, a constant.

Therefore, $(y_2/t^2)' = C/t^4$, which implies $y_2/t^2 = -\frac{C}{3}/t^3 + K$ or $y_2 = -\frac{C}{3}\frac{1}{t} + Kt^2$. A second linearly independent solution is $y_2 = 1/t$ and the general solution is $y = c_1 t^2 + c_2/t$.

10. Consider the system of equations $\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. **(10 marks)**

(a) **(4 marks)** Find the eigenvalues of A. What are the equilibrium solution(s)? Are they stable?

Solution: The characteristic equation is $\lambda^2 + 1 = 0$ and so the eigenvalues of A are $\lambda = \pm i$. Since A is invertible, the only equilibrium solution is $\boldsymbol{x} = \mathbf{0}$. It is stable, but not asymptotically stable.

(b) **(4 marks)** Find solutions $\boldsymbol{x}_1(t) = \begin{pmatrix} a_{1,1}(t) \\ a_{2,1}(t) \end{pmatrix}$ and $\boldsymbol{x}_2(t) = \begin{pmatrix} a_{1,2}(t) \\ a_{2,2}(t) \end{pmatrix}$ with initial conditions $\boldsymbol{x}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{x}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Solution: An eigenvector for $\lambda = i$ is $\begin{pmatrix} i \\ 1 \end{pmatrix}$ and therefore one complex solution is $e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i\cos(t) - \sin(t) \\ \cos(t) + i\sin(t) \end{pmatrix}$. The real general solution is thus $\boldsymbol{x}(t) = c_1 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$.

So $\boldsymbol{x}_1(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ and $\boldsymbol{x}_2(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$ have the required initial conditions.

(c) **(2 marks)** Explain why the solution $\boldsymbol{x}(t)$ with the initial data $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is equal $B(t)\boldsymbol{x}_0$, where $B(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$. (Note that the matrix $B(t)$ is called $\exp(At)$).

Solution: The general solution can be written as

$$\boldsymbol{x}(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Evaluating at $t = 0$ gives

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 = B(0)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = I \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

So the general solution is $\boldsymbol{x}(t) = B(t)\boldsymbol{x}_0$

11. Aerial refueling is a dangerous procedure in which the receiver aircraft approaches a tanker aircraft from below. The receiver aircraft has altitude $h(t)$ beginning at $h(0) = h_0$ and no vertical speed $h'(0) = 0$. The time until 'docking' is T. The altitude $h(t)$ is modelled with the differential equation **(10 marks)**

$$mh'' = U\left(1 - \frac{t}{T}\right) - \gamma h' - mg.$$

(a) **(4 marks)** Explain the meaning of each term in the differential equation.

Solution: The differential equation is Newton's second law.

mh'' is the inertia term for aircraft mass m.

$U\left(1 - \frac{t}{T}\right)$ is the lift force, which is assumed to decrease linearly from U at $t = 0$ (at the start of the refueling procedure) to zero at $t = T$ (at the time of docking).

$-\gamma h'$ is the drag from presumably air resistance, which is assumed to be proportional to velocity.

$-mg$ is the downward force of gravity.

(b) **(6 marks)** For a 'soft-landing', $h'(T) = 0$. In some units, $m = 1$, $\gamma = 2$, $g = 1$, and $T = 5$. Find U so that there is a soft-landing. How far apart were the aircraft initially?

Solution: The homogeneous (complementary) problem $h_\text{c}'' + 2h_\text{c}' = 0$ has the solution $h_\text{c} = c_1 + c_2 e^{-2t}$. Try the particular solution $y_\text{p} = At^2 + Bt$ noting that the constant solution solves the homogeneous problem.

Substituting gives $h_\text{p}'' + 2h_\text{p}' = 2A + 2(2At + B) = U - 1 - \frac{U}{T}t$,

Equating coefficients gives $A = -\frac{U}{4T}$ and $B = \frac{1}{2}(U-1) - A = \frac{1}{2}(U-1) + \frac{U}{4T}$.

Applying the initial conditions $h_0 = c_1 + c_2$ and $0 = -2c_2 + B$, so $c_2 = B/2, c_1 = h_0 - B/2$.

Applying the condition $h'(T) = 0$ determines U:

$$\begin{aligned}
0 = h'(5) &= -2c_2 e^{-2T} + 2AT + B \\
&= (-2)B/2e^{-10} + 10A + B \\
&= (1 - e^{-10})B + 10A \\
&= (1 - e^{-10})\left[\frac{1}{2}(U-1) + \frac{U}{20}\right] - \frac{U}{2}
\end{aligned}$$

Solving gives $U = (10 - 10e^{-10})/(1 - 11e^{-10})$.

The initial separation is

$$\begin{aligned}
h(T) - h_0 = c_1 + c_2 e^{-2\cdot 5} + A \cdot 5^2 + B \cdot 5 - h_0 &= -B/2 + B/2e^{-10} + 25A + 5B \\
&= \left(9 + e^{-10}\right)\frac{B}{2} + 25A = \frac{\left(9 + e^{-10}\right)}{2}\left(\frac{1}{2}(U-1) + \frac{U}{20}\right) - \frac{5}{4}U \\
&= \left[\frac{\left(9 + e^{-10}\right)/4}{1 - e^{-10}} - \frac{5}{4}\right]U = \left[\frac{\left(9 + e^{-10}\right)/4}{1 - e^{-10}} - \frac{5}{4}\right]\frac{10 - 10e^{-10}}{1 - 11e^{-10}} = \frac{10 + 15e^{-10}}{1 - 11e^{-10}}.
\end{aligned}$$

Page for scratch work or for clearly-labelled overflow from previous pages

Continued...

Page for scratch work or for clearly-labelled overflow from previous pages

 The end.