MAT292 - Fall 2017

Term Test 1 - October 23, 2017

Time allotted: 100 minutes | Aids permitted: None

Total marks: 60

Full Name: ______________________________
Last | First

Student Number: ______________________________

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<u>Instructions</u>

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 11–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

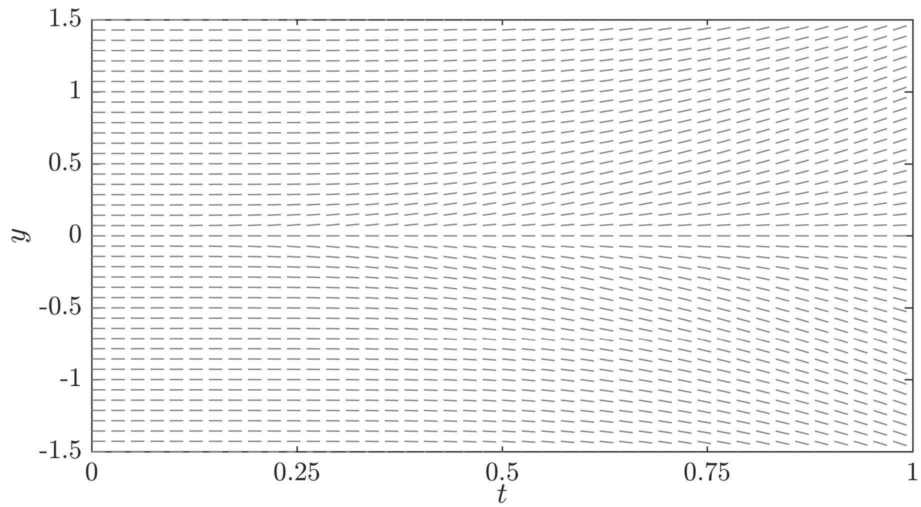
HAVE FUN!

SECTION I No explanation is necessary. **(10 marks)**

1. **(2 marks)** What is a solution to the initial value problem $\frac{dy}{dt} = 5y,\ y(1) = 1$?

$y(t) =$ ______________________________

2. **(2 marks)** The direction field for a differential equation is given below. Sketch a solution y such that $y(0) = 0.5$.



3. **(1 mark)** Is the differential equation corresponding to the direction field above, an automonous differential equation? Answer 'yes' or 'no'. ______________________

4. **(2 marks)** Find A and ϕ so that $y(t) = A\sin(t+\phi)$ is a solution to $\frac{dy}{dt} + y = \sin(t)$.
Hint: $\sin(\theta) + \cos(\theta) = \sqrt{2}\sin(\theta + \pi/4)$. $A =$ ____________, $\phi =$ ____________

5. **(3 marks)** For the automonous differential equation $y' = (1-y^2)y^2$, label the following three equilibriums solutions as stable, unstable, or semi-stable:

$y(t) = -1$ ____________________

$y(t) = 0$ ____________________

$y(t) = 1$ ____________________

SECTION II **Justify** your answers. **(50 marks)**

6. Find all equilibrium solutions to **(5 marks)**

$$\frac{dy}{dt} = \sin\left(\frac{\pi}{y}\right)$$

and classify them as stable, semi-stable, or unstable.

7. We seek the solution $y(t)$ of some differential equation. **(5 marks)**

We run a numerical method (such as Euler, improved Euler, or Runge-Kutta) several times, each time with a different step size Δt, obtaining the following approximations for $y(1)$:

$\Delta t =$	0.08	0.04	0.02	0.01	0.005
$y(1) \approx$	1.6395	1.1602	1.0399	1.0100	1.0025

Guess the (integer) order of the numerical method that we used for this problem. Justify your answer.

8. Solve the following initial value problem for $t > 0$: **(10 marks)**

$$\frac{dy}{dt} = \frac{2\ln t}{t} y + \exp\left((\ln t)^2\right), \quad y(1) = 1.$$

9. Consider the differential equation $\frac{dy}{dt} = \frac{\pi}{2}\sqrt{1-y^2}$ for $y \in [-1, 1]$. **(10 marks)**

(a) **(1 mark)** Find any equilibrium solutions.

(b) **(3 marks)** Find a solution with the initial condition $y(0) = 0$. You may use the fact that $\frac{d}{dy}\sin^{-1}(y) = \frac{1}{\sqrt{1-y^2}}$. *Hint: Be careful! The derivative of any solution is always non-negative.*

(c) **(2 marks)** Does there exist a unique solution to the differential equation for initial conditions $y(t_{\mathrm{s}}) = y_{\mathrm{s}}$ where $t_{\mathrm{s}} \in (-\infty, \infty)$ and $y_{\mathrm{s}} \in (-1, 1)$? What if $y_{\mathrm{s}} = 1$ or $y_{\mathrm{s}} = -1$? Justify your answer.

(d) **(2 marks)** If the solution found in part **(b)** is labelled $y_0(t)$, show that $y_c(t) = y_0(t - c)$ is a solution for any value of c. *Hint: This is a consequence of the differential equation being autonomous. You don't need to have solved part* **(b)** *to answer this quesion.*

(e) **(2 marks)** Write down all of the solutions that satisfy $y(0) = -1$.

10. An out-of-control nuclear reactor has temperature $H(t)$ described by **(10 marks)**

$$\frac{dH}{dt} = \frac{1}{3}kH^4,$$

for some constant $k > 0$. You record the temperature at $t = 1$ and $t = 2$. Let T be the time such that H approaches infinity as t approaches T from below, i.e. $\lim_{t \to T^-} H(t) = \infty$.

(a) **(3 marks)** Express k and T as functions of $H(1)$ and $H(2)$.

(b) **(1 mark)** For $H(1) = 1$ and $H(2) = 2$, give numerical values for k and T.

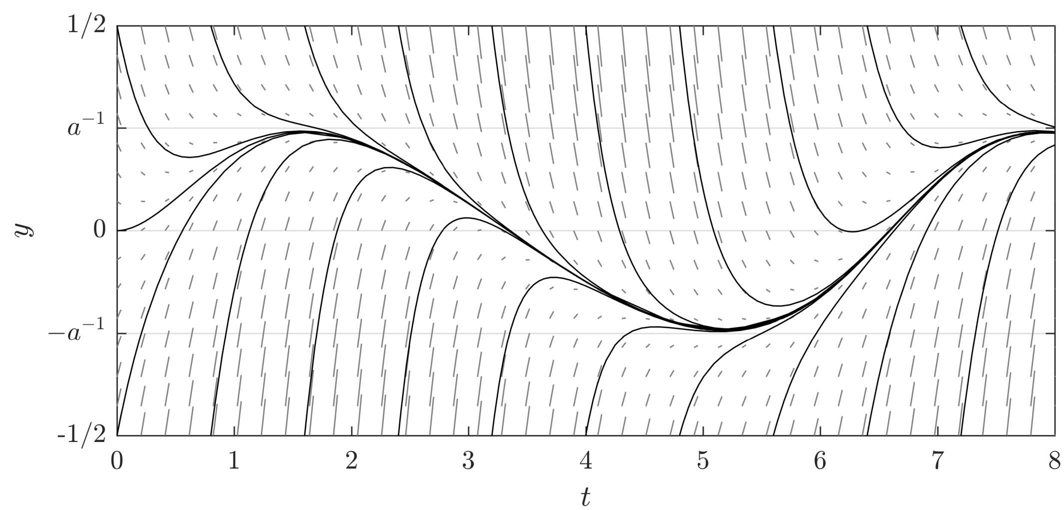
(c) **(3 marks)** The emergency control system can intervene at $t = 2$ by dumping water on the reactor which will instantaneously reduce the reactor temperature. What would the value of T become if $H(2) = 1$? Use the value of $k = \frac{7}{8}$. How low must the temperature be reduced to avoid H becoming infinite?

(d) **(3 marks)** A second intervention used instead of the method from **(c)** is to slowly insert the control rods causing k to decrease as a function of time, $k(t) = \frac{7}{8}\exp(-\alpha(t-2))$ for some $\alpha > 0$. Find a formula for T as a function of α. How large does α need to be so that H will never become infinite? Use the initial condition $H(2) = 2$.

11. Consider the differential equation $\frac{dy}{dt} = -ay + \sin(y+t)$ with $a > 0$. **(10 marks)**

(a) **(2 marks)** Write the formula for Euler's method that updates the approximate solution value y_n at time t_n to y_{n+1} at time $t_n + \Delta t$.

(b) **(2 marks)** Solutions to the differential equation can only get so large: $-1/a \leqslant y(t) \leqslant 1/a$ for large t. However, if Δt is too large, then the approximations from Euler's method will grow without bound. This undesirable situation is known as numerical instability. Without computing, sketch on the direction field below a numerical solution using Euler's method starting from $y(0) = 1/4 = 1/a$. Pick Δt large enough to demonstrate numerical instability. The parameter is $a = 4$ and several solutions have been plotted over the direction field. The length of each line seqment is proportional to $|dy/dt|$.



(c) **(3 marks)** Let's create a numerical method that doesn't have this instability problem. As an approximation, assume that the non-linear term is held constant at its value from the beginning of the timestep, $\sin(y+t) \approx \sin(y_n + t_n)$. Use the method of integrating factors to solve the initial value problem

$$\frac{dv}{dt} = -av + \sin(y_n + t_n)$$
$$v(t_n) = y_n.$$

Hint: Note that $\sin(y_n + t_n)$ *is a constant - just call it* S.

(d) **(1 mark)** Write an update formula using $y_{n+1} = v(t_n + \Delta t)$ using your solution from part **(c)** .

(e) **(2 marks)** The new method does not have the numerical instability of Euler's method. Using the update formula from part **(d)** , show that if $|y_n| \leqslant 1/a$ then $|y_{n+1}| \leqslant 1/a$ for the new method.

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