Second-Order Lab: Second-Order Linear DEs in MATLAB

In this lab, you will learn how to use `iode` to plot solutions of second-order ODEs. You will also learn to classify the behaviour of different types of solutions.

Moreover, you will write your own Second-Order ODE system solver, and compare its results to those of `iode`.

Opening the m-file lab5.m in the MATLAB editor, step through each part using cell mode to see the results.

There are seven (7) exercises in this lab that are to be handed in on the due date of the lab.

Student Information

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Iode for Second-Order Linear DEs with constant coefficients

In the `iode` menu, select the `Second order linear ODEs` module. It opens with a default DE and a default forcing function `f(t) = cos(2t)`. The forcing function can be plotted along with the solution by choosing `Show forcing function` from the `Options` menu.

Use this module to easily plot solutions to these kind of equations.

There are three methods to input the initial conditions:

Method 1. Enter the values for `t0`, `x(t0)`, and `x'(t0)` into the `Initial conditions` boxes, and then click `Plot solution`.

Method 2. Enter the desired slope `x'(t0)` into the appropriate into the `Initial conditions` box, and then click on the graph at the point `(t0,x(t0))` where you want the solution to start.

Method 3. Press down the left mouse button at the desired point `(t0,x(t0))` and drag the mouse a short distance at the desired slope `x'(t0)`. When you release the mouse button, `iode` will plot the solution.

Growth and Decay Concepts

We want to classify different kinds of behaviour of the solutions. We say that a solution:

`grows` if its magnitude tends to `infinity` for large values of `t`, that is, if either the solution tends to $+\infty$ or $-\infty$,

`decays` if its magnitude converges to `0` for large values of `t`,

`decays while oscillating` if it keeps changing sign for large values of `t` and the amplitude of the oscillation tends to zero,

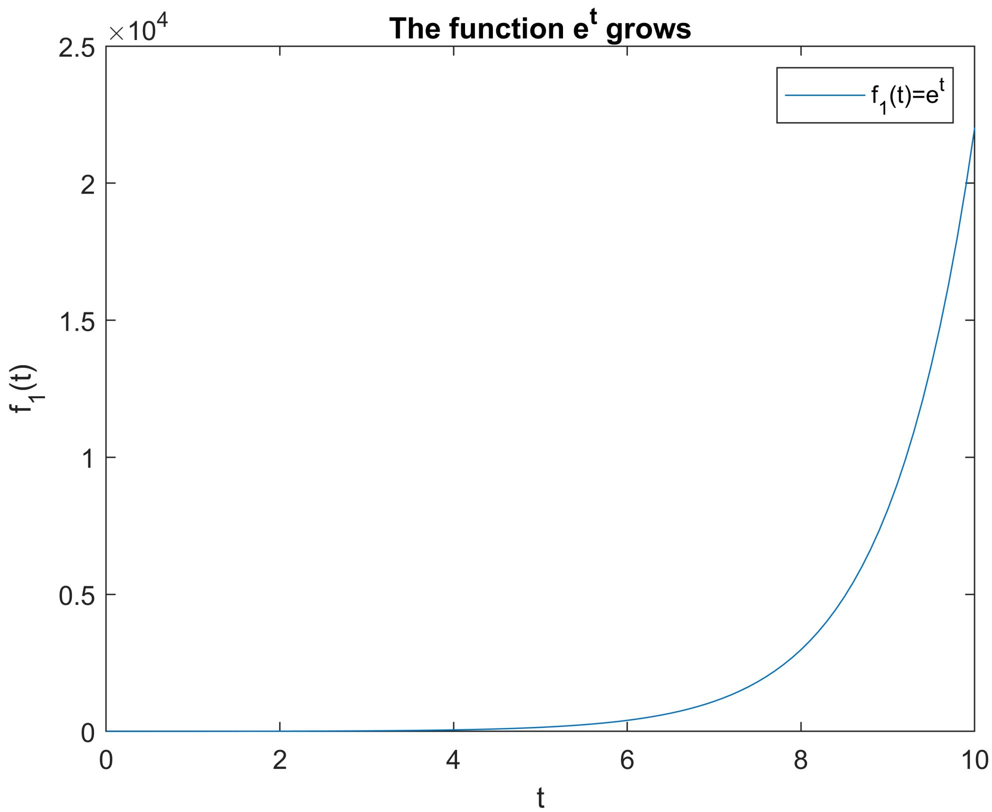
`grows while oscillating` if it keeps changing sign for large values of `t` and the amplitude of the oscillation tends to infinity.

Example

```
t = 0:0.1:10;
```

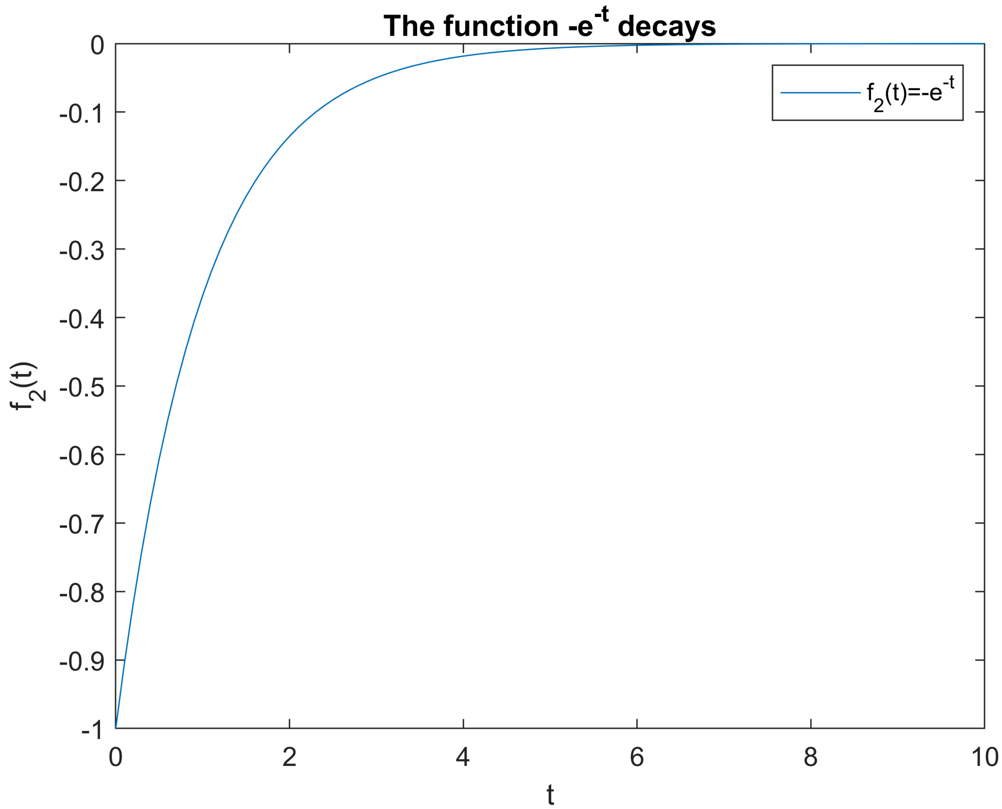
```
% Example 1
figure();
y1 = exp(t);
plot(t,y1)

% Annotate the figure
xlabel('t');
ylabel('f_1(t)');
title('The function e^t grows');
legend('f_1(t)=e^t');
```



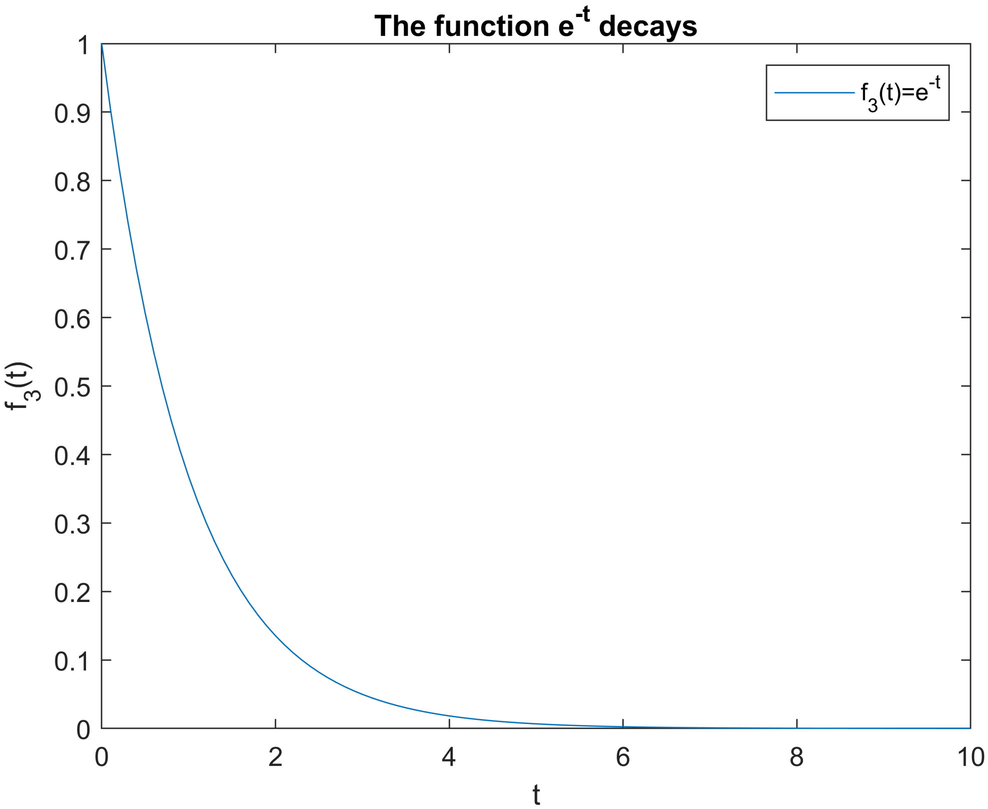

```
% Example 2
figure();
y2 = -exp(-t);
plot(t,y2)

% Annotate the figure
xlabel('t');
ylabel('f_2(t)');
title('The function -e^{-t} decays');
legend('f_2(t)=-e^{-t}');
```



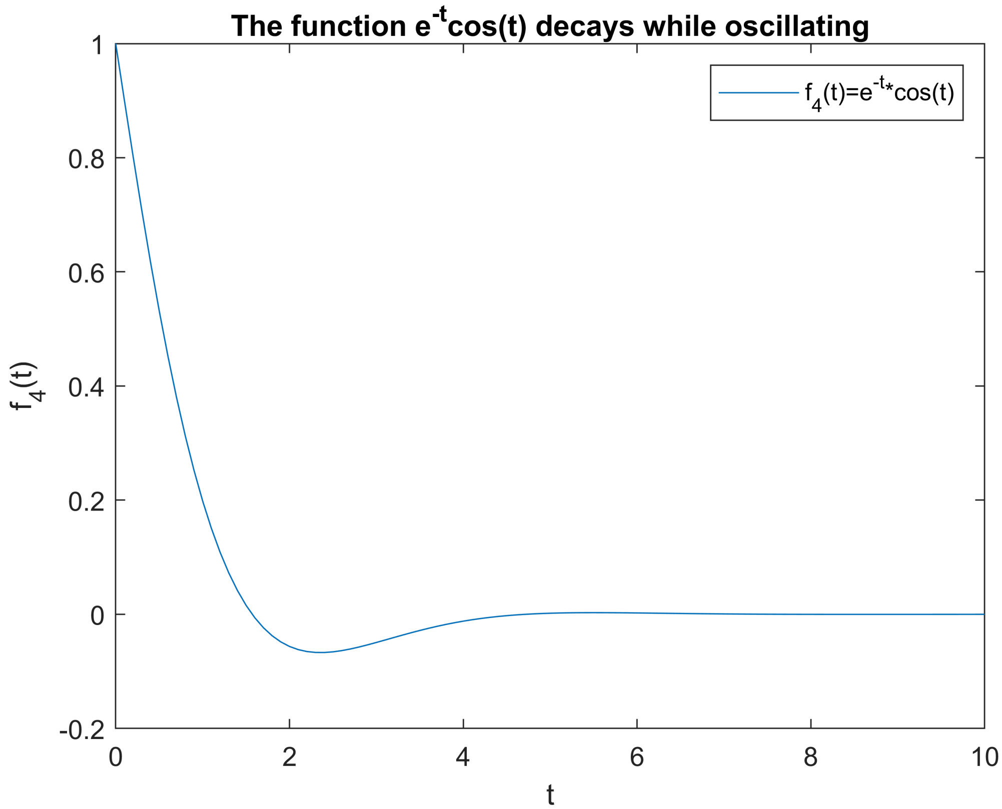

```
% Example 3
figure();
y3 = exp(-t);
plot(t,y3)

% Annotate the figure
xlabel('t');
ylabel('f_3(t)');
title('The function e^{-t} decays');
legend('f_3(t)=e^{-t}');
```



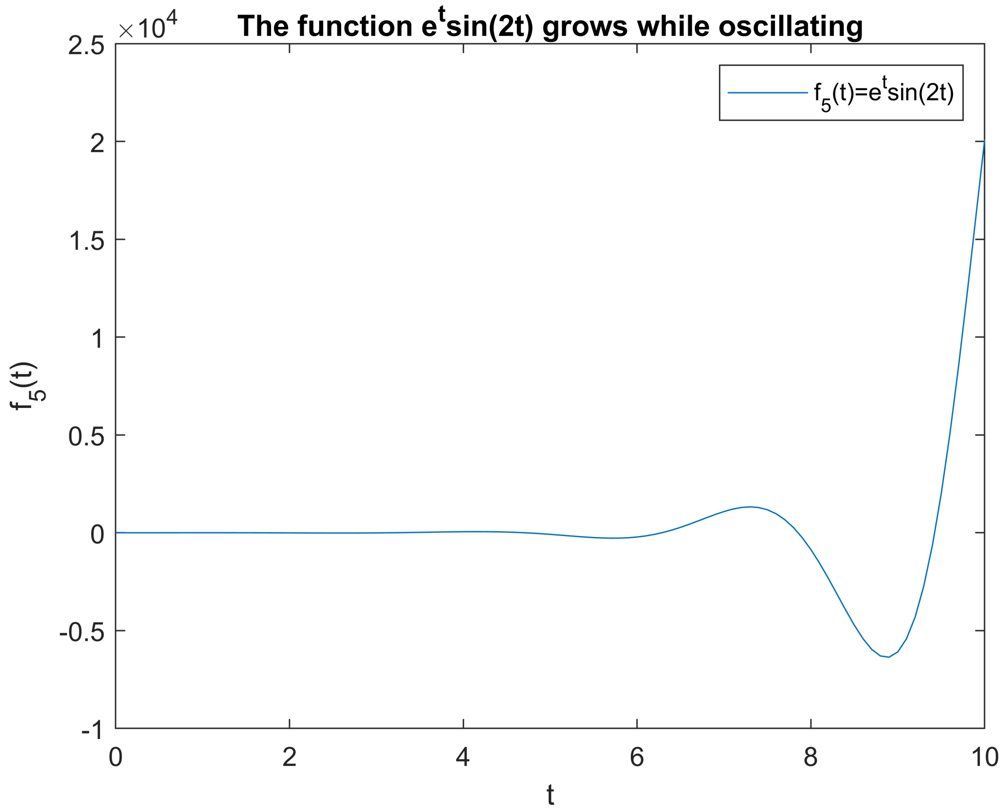

```
% Example 4
figure();
y4 = exp(-t).*cos(t);
plot(t,y4)

% Annotate the figure
xlabel('t');
ylabel('f_4(t)');
title('The function e^{-t}cos(t) decays while oscillating');
legend('f_4(t)=e^{-t}*cos(t)');
```



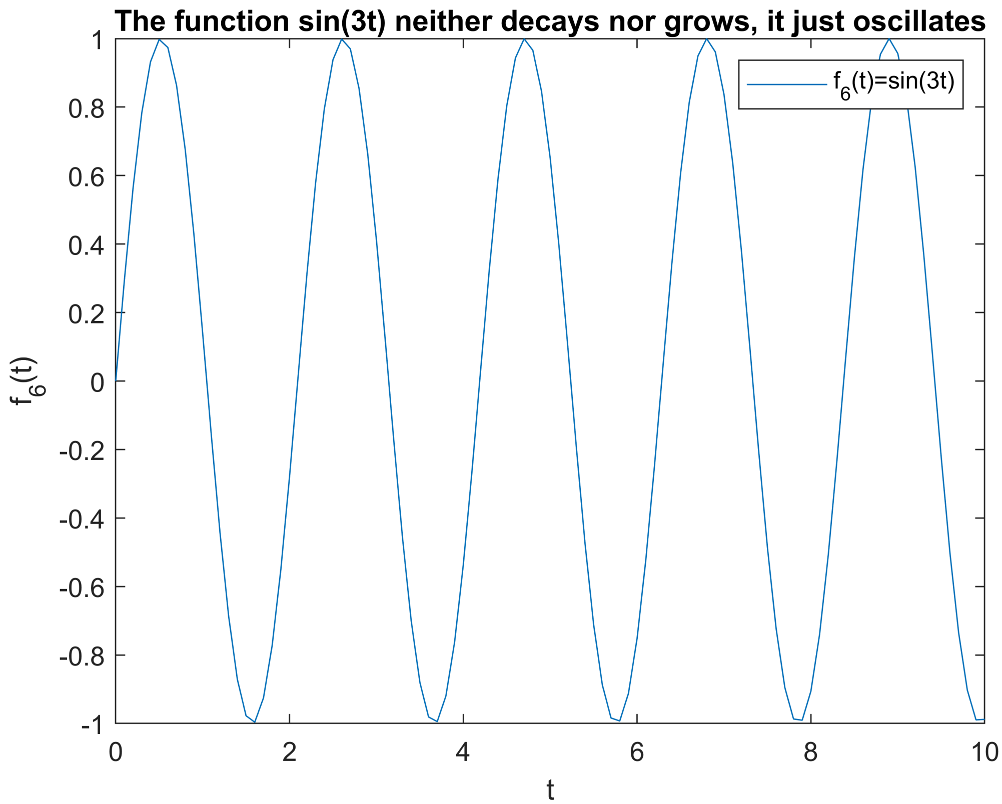

```
% Example 5
figure();
y5 = exp(t).*sin(2*t);
plot(t,y5)

% Annotate the figure
xlabel('t');
ylabel('f_5(t)');
title('The function e^{t}sin(2t) grows while oscillating');
legend('f_5(t)=e^{t}sin(2t)');
```




```
% Example 6
figure();
y6 = sin(3*t);
plot(t,y6)

% Annotate the figure
xlabel('t');
ylabel('f_6(t)');
title('The function sin(3t) neither decays nor grows, it just oscillates');
legend('f_6(t)=sin(3t)');
```




```
% |Remark.| A function which |grows while oscillating| doesn't |grow|,
% because it keeps changing sign, so it neither tends to $+\infty$ nor to
% $-\infty$.
```

Exercise 1

Objective: Use `iode` to solve second-order linear DEs. And classify them.

Details: Consider the ODE:

```
4y'' + 4 y' + 17 y = 0
```

(a) Use `iode` to plot six (6) numerical solutions of this equation with "random" initial data (use Method 3 above) and press-and-drag at various initial points, with some of the slopes being positive and some negative)

Use only initial points in the part of the window where `0<t<1` and `-1<x<1` and take all initial slopes between `-3` and `+3`.

Change the window to `[0,10]x[-3,3]`. Attach a cropped screenshot to your answers file.

(b) Based on the results of (a), state what percentage of solutions decay, grow, grow while oscillating, or decay while oscillating.

decay - 0%

grow - 0%

grow while oscillating - 0 %

decay whule oscillating - 100%

(c) Solve the DE and write the exact solution. Explain why this justifies your answer in (b).

```
y = (e^(-t/2))*(c_1*cos(2*t)+c_2*sin(2*t)). This justifies the observation in part 2
because the (e^(-t/2)) term causes the function to decay toward 0 with increasing t
while the (c_1*cos(2*t)+c_2*sin(2*t)) part of the function causes it to oscillate between
positive and negative values ad infinitum.
```

Exercise 2

Consider the ODE:

```
y'' + sqrt(3) y' - y/4 = 0
```

Repeat (a), (b), (c) from Exercise 1 with this DE.

b)

decay - 0%

grow - 100%

grow while oscillating - 0 %

decay whule oscillating - 0%

c)

```
y = c_1*e^(((2-sqrt(3))/2)*t)+c_2*e^(((-2-sqrt(3))/2)*t). Both terms in this expression
are exponential. The term (2-sqrt(3))/2 is positive while the term (-2-sqrt(3))/2 is
negative, meaning that the former approaches infinity as t approaches infinity while the
latter aproaches 0 as t aproaches infinity. There sum therefore approaches positive or
negative infinity, depending on the sign of c_1. The absence of a sine and cosine terms
confirms that we would not expect this function to oscillate.
```

Exercise 3

Consider the ODE:

```
y'' + sqrt(3) y' + y/4 = 0
```

Repeat (a), (b), (c) from Exercise 1 with this DE.

b)

decay - 100%

grow - 0%

grow while oscillating - 0 %

decay whule oscillating - 0%

c)

```
y = c_1*e^(((sqrt(2)-sqrt(3))/2)*t)+c_2*e^(((-sqrt(2)-sqrt(3))/2)*t)
```

```
In this case, both exponential terms have negative exponenents for positive t since both
(sqrt(2)-sqrt(3))/2 and (-sqrt(2)-sqrt(3))/2 are negative numbers. This means that both
terms approach 0 as t approaches infinity, so we expect decaying behaviour. Again, there
is no sign or cosing term for any oscillating behaviour.
```

Example

Consider the ODE:

```
y'' + 2y' + 10y = 0
```

The solution is

```
y(t) = e^(-t) ( c1 cos(3t) + c2 sin(3t) )
```

From this, it is easy to see that all solutions decay while oscillating.

Similarly, for the equation

```
y'' - 2y' + 10y = 0
```

The solution is

```
y(t) = e^t ( c3 cos(3t) + c4 sin(3t) )
```

which grows while oscillating.

Exercise 4

Consider the fourth-order ODE:

```
y'''' + 2 y''' + 6 y'' + 2 y' + 5 y = 0
```

(a) Find the general solution for this problem. You can use MATLAB to find the roots of the characteristic equation numerically with `roots`.

```
y = [1,2,6,2,5];
roots(y)
```

```
ans = 4×1 complex
  -1.0000 + 2.0000i
  -1.0000 - 2.0000i
   0.0000 + 1.0000i
   0.0000 - 1.0000i
```

General solution:

```
y = e^(-t)*(c_1*sin(2*t)+c_2*cos(2*t))+c_3*sin(t)+c_4*cos(t)
```

(b) Predict what percentage of solutions with random initial data will grow, decay, grow while oscillating, and decay while oscillating. Explain.

100% will only oscillate as t approach. Since the exponential term has a negative coefficient, it will approach 0 while the remaining terms oscillate between two finite numbers (bounded by the positive and negative sum of the absolute values of the c_1 and c_2). Since the dominant term is approaching 0 and the other terms oscillate between positive and negative for all t, we expect the first term in the sume to approach 0. The remaining 2 terms, `c_3*sin(t)+c_4*cos(t), will continue to oscillate freely as t approaches infinity. This means that the expression oscillates as t approaches infinity, however it doesn't grow or decay since those require the amplitude to either approach 0 or infinity while here it is bounded by the sum of c_3 and c_4. If the terms c_3 and c_4 are 0, the amplitude will be bounded by 0, meaning we observe oscillating decay dictated by the first term.`

Exercise 5

Objective: Classify equations given the roots of the characteristic equation.

Details: Your answer can consist of just a short sentence, as `grows` or `decays while oscillating`.

Consider a second-order linear constant coefficient homogeneous DE with `r1` and `r2` as roots of the characteristic equation.

Summarize your conclusions about the behaviour of solutions for randomly chosen initial data when.

(a) `0 < r1 < r2`

`grows`

(b) `r1 < 0 < r2`

`grows`

(c) `r1 < r2 < 0`

`decays`

(d) `r1 = alpha + beta i` and `r2 = alpha - beta i` and `alpha < 0`

`decays while oscillating`

(e) `r1 = alpha + beta i` and `r2 = alpha - beta i` and `alpha = 0`

`oscillates without growth or decay`

(f) `r1 = alpha + beta i` and `r2 = alpha - beta i` and `alpha > 0`

```
grows while oscillating
```

Numerical Methods for Second-Order ODEs

One way to create a numerical method for second-order ODEs is to approximate derivatives with finite differences in the same way of the Euler method.

This means that we approximate the first derivative by:

```
y'(t[n]) ~ ( y[n] - y[n-1] ) / h
```

and

```
y''(t[n])  ~ ( y'(t[n+1]) - y'(t[n]) ) / h ~ ( y[n+1] - 2 y[n] + y[n-1]
) / ( h^2 )
```

By writing these approximations into the ODE, we obtain a method to get `y[n+1]` from the previous two steps `y[n]` and `y[n-1]`.

The method for approximating solutions is:

1. Start with `y[0]=y0`

2. Then we need to get `y[1]`, but we can't use the method, because we don't have two iterations `y[0]` and `y[-1]`(!!). So we use Euler to get

```
y[1] = y0 + y1 h
```

y1 is the slope given by the initial condition

3. Use the method described above to get `y[n]` for `n=2,3,...`.

Exercise 6

Objective: Write your own second-order ODE solver.

Details: Consider the second-order ODE

```
y'' + p(t) y' + q(t) y = g(t)
```

Write a second-order ODE solver using the method described above.

This m-file should be a function which accepts as variables (t0,tN,y0,y1,h), where t0 and tN are the start and end points of the interval on which to solve the ODE, y0, y1 are the initial conditions of the ODE, and h is the stepsize. You may also want to pass the functions into the ODE the way `ode45` does (check MATLAB lab 2). Name the function DE2_<UTORid>.m.

Note: you will need to use a loop to do this exercise.

Exercise 7

Objective: Compare your method with `iode`

Details: Use `iode` to plot the solution of the ODE `y'' + exp(-t/5) y' + (1-exp(-t/5)) y = sin(2*t)` with the initial conditions `y(0) = 1, y'(0) = 0`

Use the window to `[0,20]x[-2,2]` Without removing the figure window, plot your solution (in a different colour), which will be plotted in the same graph.

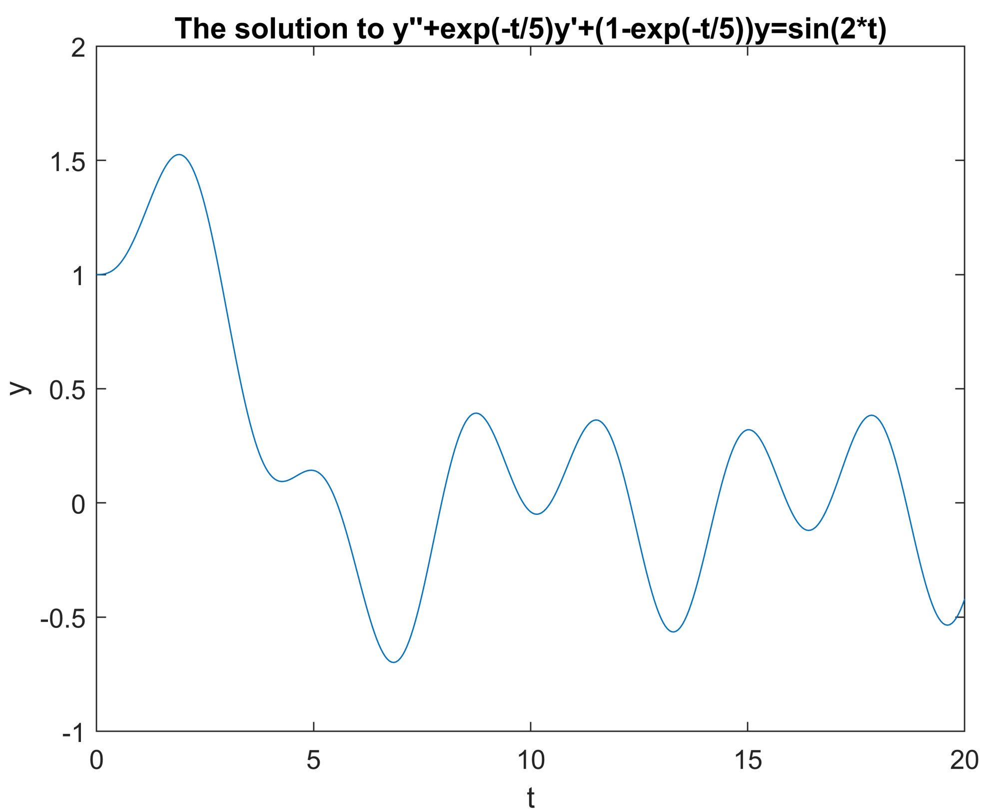
Comment on any major differences, or the lack thereof.

```
f = @(t,y_p,y) -exp(-t/5)*y_p-(1-exp(-t/5))*y+sin(2*t);

y0 = 1;
y1 = 0;
t0 = 0;
tN = 20;
h = 0.05;
t = t0:h:tN;

[t, y] = DE2_purcarur(f,t0,tN,y0,y1,h);

plot(t,y);
ylabel('y');
xlabel('t');
title("The solution to y''+exp(-t/5)y'+(1-exp(-t/5))y=sin(2*t)");
```




```
legend("iode solution", "DE2_ solution")
```