University of Toronto, Faculty of Applied Science and Engineering

MAT292H1F - Ordinary Differential Equations

Final Exam - December 16, 2019

Examiners: A. Stinchcombe and F. Parsch

Time allotted: 150 minutes

Aids permitted: None

Total marks: 93

Full Name: ______________________________
Last First

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- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers and brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 19 pages (including this title page). Make sure you have all of them.
- You can use pages 14–17 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 14–17.
- You may detach the formula sheet. Work on the formula sheet will NOT be graded.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

Q1-Q10	Q11	Q12	Q13	Q14	Q15	Q16	Q17	Q18	**Total**
20	5	10	10	10	10	10	10	8	**93**

HAVE FUN!

SECTION I Short answer section. Only justify your answer when asked.

1. **(2 marks)** State and classify all equilibria of $y' = \sin(y)$.

   Solution: $y' = 0$ whenever $y = k\pi$ for some integer k. For even k, we get an unstable equilibrium. For odd k, we get a stable equilibrium.

2. **(2 marks)** Consider a linear ODE $y' = f(t, y)$ where f and all its derivatives are continuous. Let y_1 be a solution with $y_1(0) = 3$ and y_2 be a solution with $y_2(0) = 4$. If you know that $\lim_{t\to\infty} y_1(t) = \infty$, what can you conclude about y_2? Justify.

   Solution: The existence and uniqueness theorem applies. Since the ODE is linear, solutions exist for all time. Therefore, two different solutions can't cross. Since $y_2(0) > y_1(0)$, this implies that $y_2(t) > y_1(t)$ for all t. It follows that $\lim_{t\to\infty} y_2(t) = \infty$ as well.

3. **(2 marks)** Consider a 2-dimensional system $\vec{x}\,' = A\vec{x}$. Find a matrix A that gives a **counterexample** to the following statement: *For all solutions, we have either* $\lim_{t\to\infty} |\vec{x}(t)| = \infty$ *or* $\lim_{t\to\infty} |\vec{x}(t)| = 0$

   Solution: Pick any matrix with two imaginary eigenvalues, e.g. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

4. **(2 marks)** Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\}$.

   Using the time-shift property and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ we get

   $$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+1}\right\} = u_2(t)\sin(t-2)$$

5. **(2 marks)** Consider the following PDE known as the wave equation: $u_{tt}(x,t) = a^2 u_{xx}(x,t)$. Assuming that we can separate the variables $u(x,t) = X(x)T(t)$, write down the ODEs that the wave equation produces.

   $T'' + \lambda T = 0, \quad a^2 X'' + \lambda X = 0$

For each of the following statements, do two things: Make a choice if it is TRUE or FALSE. Then give a brief justification of your choice. Remember: TRUE means that the statement is **always** true, and not just true in a special case.

6. **(2 marks)** Every boundary value problem $y'' + a\,y = 0$, $y(0) = 0$, $y(\pi) = 0$ has at least one solution, no matter the value of the constant a. ◯ TRUE ◯ FALSE

Justification: This is true, $y \equiv 0$ is always a solution.

7. **(2 marks)** When solving $ay'' + by' + cy = g(t)$, the method of undetermined coefficients can **NOT** be used if $g(t)$ is a polynomial of degree three or more. ◯ TRUE ◯ FALSE

Justification: This is false. If $g(t)$ is a polynomial of any degree, then we can use $\sum A_i t^i$ as a guessing function and it will work out.

8. **(2 marks)** If the ODE of an IVP $y' = f(t, y)$, $y(0) = c$ is both linear and separable, using either integrating factor or separation of variables will give the same result. ◯ TRUE ◯ FALSE

Justification: This is false. For example, separation can't solve the IVP $y' = y$, $y(0) = 0$ since separation assumes $y \neq 0$. The integrating factor method on the other hand always finds the general solution and can therefore solve any first order linear IVP.

9. **(2 marks)** If $f(t)$ and $g(t)$ are both bounded, then $(f * g)(t)$ is bounded. ◯ TRUE ◯ FALSE

Justification: This is false. Almost anything works as a counterexample. For example, if $f = g = 1$, then $(f * g)(t) = t$.

10. **(2 marks)** The improved Euler method perfectly approximates any first-order linear IVP. ◯ TRUE ◯ FALSE

Justification: This is basically nonsense. Even for $y' = y$ with solution ce^t, improved Euler produces errors.

SECTION II   Long answer section. **Justify** all your answers.

11. (5 marks) Consider a function $f(t)$ of exponential order such that $\lim_{t\to\infty} f(t)$ exists.

In this question, you are asked to prove the *final value theorem*:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s)$$

(a) **(2 marks)** Show that $\mathcal{L}\left\{f'\right\} = sF(s) - f(0)$.

Solution: This proof was done in class.

$$\mathcal{L}\left\{f'\right\}(s) = \int_0^\infty e^{-st} f'(t)\, dt = \lim_{b\to\infty} e^{-st} f(t)\Big|_0^b + \int_0^b se^{-st} f(t)\, dt = -f(0) + s\mathcal{L}\left\{f\right\}(s)$$

(b) **(2 marks)** Show that $\lim_{s\to 0^+} \mathcal{L}\left\{f'\right\} = \left[\lim_{t\to\infty} f(t)\right] - f(0)$.
Hint: for this part, do not use the result from (a).

Solution:

$$\lim_{s\to 0^+} \mathcal{L}\left\{f'\right\} = \lim_{s\to 0^+} \int_0^\infty e^{-st} f'(t)\, dt = \int_0^\infty f'(t)\, dt = \lim_{b\to\infty} \int_0^b f'(t)\, dt = \left[\lim_{b\to\infty} f(t)\right] - f(0)$$

(c) **(1 mark)** Finally, show that $\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s)$.

Solution: Plug in (a) into (b) and add $f(0)$ on both sides.

12. (10 marks) Consider the following initial value problem describing an oscillator:

$$ay'' + by' + cy = g(t), \quad y(0) = 1, \quad y'(0) = -1$$

You are given that the *transfer function* of this oscillator is $H(s) = \dfrac{1}{s^2 + 2s + 2}$.

(a) **(2 marks)** Use the Laplace Transform to convert the IVP into an algebraic equation and solve for $Y(s)$.

Solution: Using the usual rules for derivatives, we arrive at $Y(s) = \dfrac{as - a + b}{as^2 + bs + c} + \dfrac{1}{as^2 + bs + c} G(s)$.

(b) **(2 marks)** Based on the information that you have, state the values of the three coefficients of the ODE. Justify your choice.

$$a = 1 \qquad b = 2 \qquad c = 2$$

The transfer function is the reciprocal of the characteristic polynomial.

(c) **(2 marks)** Find the impulse response $h(t)$.

Solution: Completing the square, we get $H(s) = \frac{1}{(s+1)^2+1}$.

Using the table of Laplace transforms, we get $h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-t}\sin t$.

(d) **(4 marks)** Express the solution of the above IVP in terms of the function $g(t)$.

Solution: Completing the square and using the table as well as the convolution theorem, we get

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+2} + H(s)G(s)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} + H(s)G(s)\right\} \\
&= e^{-t}\cos t + (h * g)(t) = e^{-t}\cos t + \int_0^t e^{-\tau}\sin(\tau)g(t-\tau)\,d\tau
\end{aligned}$$

13. (10 marks) Find the general solution to $(1-t)y'' + ty' - y = -e^t(t-1)^2$ using the following steps.

(a) **(3 marks)** Check if any of the following functions are solutions to the associated homogeneous (complementary) equation: $y_1(t) = 1,\ y_2(t) = t,\ y_3(t) = t^2,\ y_4(t) = e^t,\ y_5(t) = e^{-t}$.

Solution: Only y_2 and y_4 are solutions.

$$\begin{aligned}
(1-t)y_1'' + ty_1' - y_1 &= -1 \neq 0 \\
(1-t)y_2'' + ty_2' - y_2 &= 0 \\
(1-t)y_3'' + ty_3' - y_3 &= t^2 - 2t + 2 \neq 0 \\
(1-t)y_4'' + ty_4' - y_4 &= 0 \\
(1-t)y_5'' + ty_5' - y_5 &= -2te^{-t} \neq 0
\end{aligned}$$

(b) **(3 marks)** Compute the Wronskian of the solutions that you found.

For which value(s) of t is the Wronskian zero?

Solution: $W[t, e^t] = te^t - 1 \cdot e^t = (t-1)e^t$, which is zero only for $t = 1$.

(c) **(4 marks)** Use the variation of parameters formula to find the general solution.

Solution: Since $g(t) = W[t, e^t]$ (note that we must divide through by $1-t$), we have the general solution

$$y = -t\int e^t dt + e^t \int t dt = c_1 t + c_2 e^t - te^t + \frac{1}{2}t^2 e^t.$$

14. (10 marks) Consider a metal rod whose temperature distribution over time is given by $u(x, t)$. There is a heating and cooling device attached along the rod that is controlled by a thermostat set to 20 degrees Celsius.

(a) **(6 marks)** Of the following partial differential equations, only one can govern the physical situation outlined above. Do the following:

- First, choose the one plausible PDE.
- Then, for your choice, explain briefly how the equation matches the physical description.
- Finally, for **each** of the other five equations, give one physical argument why this equation can not govern the situation.

Choose one equation	**Explain why each does/does not match the situation**
○ $u_t = \alpha^2 u_{xx} + \beta^2(u - 20)$	
○ $u_t = \alpha^2 u_{xx} + \beta^2(20 - u)$	
○ $u_t = \alpha^2 u_{xx} + \beta^2(u_x - 20)$	
○ $u_t = \alpha^2 u_{xx} + \beta^2(20 - u_x)$	
○ $u_t = \alpha^2 u_{xx} + \beta^2(u_{xx} - 20)$	
○ $u_t = \alpha^2 u_{xx} + \beta^2(20 - u_{xx})$	

Solution: The only sensible choice is $u_t = \alpha^2 u_{xx} + \beta^2(20 - u)$. The two equations involving u_x can be excluded since a temperature distribution with slope 20 would not trigger the thermostat. The two equations involving u_{xx} can be excluded since the concavity of the temperature distribution doesn't control the thermostat. Finally, to choose between $(20 - u)$ and $(u - 20)$ note that the latter would mean that hot parts would get heated up more, but a thermostat does the exact opposite.

(b) **(4 marks)** For this part, use the one equation that you chose above.

We want to simplify the problem as follows: Substitute $v(x,t) = f(t)(u(x,t) - 20)$ and choose $f(t)$ such that v solves the usual heat equation $v_t = \alpha^2 v_{xx}$.

Using these requirements, reduce this to an ODE only involving f. Then state a solution for f.

Solution: By differentiation rules: $v_t = f'(u - 20) + f u_t$ and $v_{xx} = f u_{xx}$.

$$v_t = \alpha^2 v_{xx} \Leftrightarrow f'(u - 20) + f u_t = \alpha^2 f u_{xx}$$

Using the fact that $u_t = \alpha^2 u_{xx} + \beta^2(20 - u)$, we get

$$\begin{aligned}
&\Leftrightarrow f'(u - 20) + f[\alpha^2 u_{xx} + \beta^2(20 - u)] = \alpha^2 f u_{xx} \Leftrightarrow f'(u - 20) + f\beta^2(20 - u) = 0 \\
&\Leftrightarrow (u - 20)(f' - \beta^2 f) = 0
\end{aligned}$$

$u - 20$ can't always be zero. Therefore $f' - \beta^2 f = 0$ which has solutions $f = ce^{\beta^2 t}$.

15. (10 marks) Matrix exponentials are an important tool to solve ODEs. In this question, we look at another way to compute them.

Let $A = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$ with characteristic polynomial $p(\lambda) = \det(A - \lambda I) = \lambda^2 - 3\lambda + 2$.

(a) **(2 marks)** What are the eigenvalues of A?

Solution: The eigenvalues are on the diagonal, $\lambda = 1$ and $\lambda = 2$.

(b) **(2 marks)** A remarkable theorem of linear algebra, known as the "Cayley-Hamilton Theorem", says that every matrix is the root of its own characteristic polynomial. Verify this fact for A, i.e. check that $p(A) = A^2 - 3 \cdot A + 2 \cdot I = 0$.

Solution: $A^2 - 3A + 2I = \begin{pmatrix} 1 & 12 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} -3 & -12 \\ 0 & -6 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(c) **(2 marks)** It can be shown that there are constants c_0 and c_1 such that $e^{xt} = p(x)q(x,t) + c_0 + c_1 x$, where $p(x)$ is the characteristic polynomial of A.
Using that, why is $e^{\lambda t} = c_0 + c_1\lambda$ when λ is an eigenvalue of A? Why is $e^{At} = c_0 I + c_1 A$?

Solution: This follows directly from $p(\lambda) = 0$ and $p(A) = 0$.

(d) **(4 marks)** Use $e^{\lambda t} = c_0 + c_1\lambda$ and the two known eigenvalues to determine c_0 and c_1. They will depend on t. Now use the second formula to compute $e^{At} = c_0 I + c_1 A$.

Solution: Two linear conditions $c_0 + c_1 = e^t, c_0 + 2c_1 = e^{2t}$. Solving gives $c_0 = 2e^t - e^{2t} = e^t(2 - e^t), c_1 = e^{2t} - e^t = e^t(e^t - 1)$. The matrix exponential is therefore

$$e^{At} = c_0 I + c_1 A = \begin{pmatrix} c_0 + c_1 & 4c_1 \\ 0 & c_0 + 2c_1 \end{pmatrix} = \begin{pmatrix} e^t & 4e^t(e^t - 1) \\ 0 & e^{2t} \end{pmatrix}.$$

16. **(10 marks)** Consider the initial value problem $y'(t) = -y(t) + u_1(t), \quad y(0) = 1,$ where $u_1(t)$ is a unit step function.

(a) **(3 marks)** Solve the initial value problem.

Solution: Taking the Laplace transform gives

$$sY - 1 = -Y + e^{-s}\frac{1}{s} \implies Y = \frac{1}{s+1} + \frac{e^{-s}}{s(s+1)} = \frac{1}{s+1} + e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right).$$

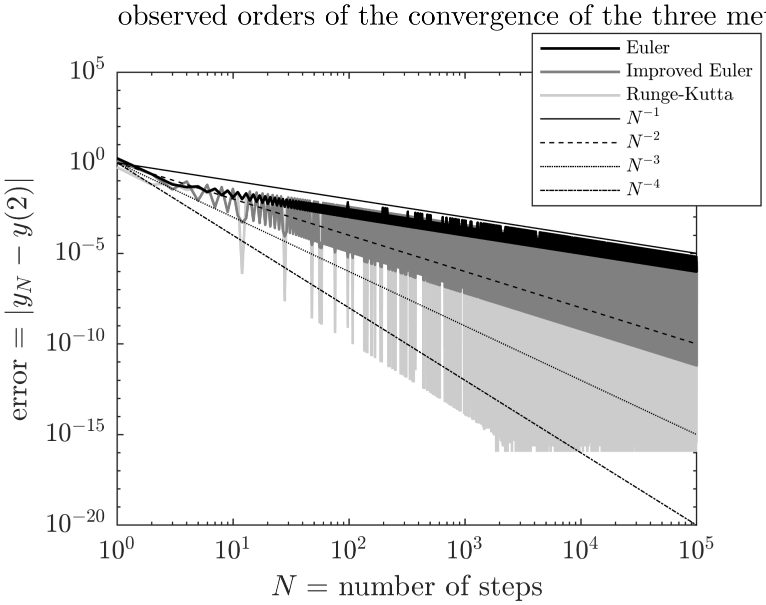
Inverting gives

$$y(t) = e^{-t} + u_1(t)\left(1 - e^{-(t-1)}\right).$$

(b) **(1 mark)** What is the value of $y(2)$?

Solution: $y(2) = 1 - e^{-1} + e^{-2}$.

(c) **(2 marks)** The error in computing $y(2)$ using Euler's method, the improved Euler method, and the Runge-Kutta method are plotted below versus the number of steps N. What are the observed orders of the convergence of the three methods?



Solution: All three methods are first order since the error is bounded by $CN^{-1} = ch$ as can be seen by the plot.

(d) **(2 marks)** Why are the orders of convergence not as expected?

Solution: The orders of convergence are derived using a Taylor series, which is only valid when the right-hand side of the differential equation is sufficiently differentiable. In this case, the right-hand side is not differentiable at $t = 1$ and apparently this reduces all of the methods to first order.

(e) **(2 marks)** Why does the error not go below approximately $2^{-53} \approx 1.11 \cdot 10^{-16}$?

Solution: Evidently IEEE double was used for the calculation (the default in Matlab), which has a machine epsilon of 2^{-53} and the accuracy is limited to this value.

Continued...

17. (10 marks) In this question, we study the relationship between the impulse $\delta(t)$ and the ϵ-impulse:

$$\delta_\epsilon(t) = u_0(t)\frac{1}{\epsilon}e^{-t/\epsilon} \quad \text{for} \quad \epsilon > 0.$$

(a) **(2 marks)** Verify that $\displaystyle\int_{-\infty}^{\infty} \delta_\epsilon(t)\ dt = 1$ and that $\displaystyle\lim_{\epsilon\to 0^+} \delta_\epsilon(t) = 0$ for every $t > 0$.

Solution: $\int_{-\infty}^{\infty} \delta_\epsilon(t)\ dt = \int_0^{\infty} \frac{1}{\epsilon}e^{-t/\epsilon}dt = -e^{-t/\epsilon}\Big|_0^{\infty} = 1$

If $t > 0$ then $e^{-t/\epsilon} \to 0$ as $\epsilon \to 0^+$ and $\frac{1}{\epsilon} \to \infty$. However, either using L'Hopital or the argument that the exponential beats any polynomial, the product has limit $\delta_\epsilon(t) \to 0$.

Now, let's consider an impulsively forced initial value problem.

(b) **(3 marks)** Find a formula for the Laplace transform $Y(s)$ of the solution of the IVP

$$y'' + 5y' + 6y = \delta(t), \quad y(0) = 0, \ \ y'(0) = 0.$$

Solution: $s^2Y + 5sY + 6Y = 1 \Rightarrow Y(s) = \frac{1}{s^2+5s+6}$

(c) **(3 marks)** Find a formula for the Laplace transform $Y_\epsilon(s)$ of the solution of the IVP

$$y_\epsilon'' + 5y_\epsilon' + 6y_\epsilon = \delta_\epsilon(t), \quad y_\epsilon(0) = 0, \ \ y_\epsilon'(0) = 0.$$

Solution: $s^2Y + 5sY + 6Y = \frac{1}{\epsilon}\frac{1}{s+\frac{1}{\epsilon}} \Rightarrow Y(s) = \frac{1}{\epsilon}\frac{1}{s+\frac{1}{\epsilon}}\frac{1}{s^2+5s+6}$

(d) **(2 marks)** Show that $\displaystyle\lim_{\epsilon\to 0^+} Y_\epsilon(s) = Y(s)$.
Solution: Follows directly from plugging in.

18. (a) **(2 marks)** Solve $y' = y^2,\ y(0) = 1$. What is the interval of existence of the solution?

Solution: $y(t) = \frac{1}{1-t}$, which exists for $(-\infty, 1)$.

(b) **(3 marks)** Why does the initial value problem $y' = y^2,\quad y(t_0) = y_0$ have a unique solution for every t_0 and y_0?

Solution: The existence/uniqueness theorem. $f(t, y) = y^2$ is continuous in t, continuous in y, and $\partial f/\partial y = 2y$ is continuous in y for all t and all y.

(c) **(3 marks)** A MAT292 student made the following argument to show that the solution to the initial value problem $y' = y^2,\quad y(0) = 1$ exists for all $t > 0$:

(i) The IVP $y_1' = y_1^2,\quad y_1(0) = 1$ has a unique solution for some interval $t \in [0, h]$.

(ii) The IVP $y_2' = y_2^2,\quad y_2(h) = y_1(h)$ has a unique solution for some interval $t \in [h, 2h]$.

(iii) Repeating, the IVP $y_n' = y_n^2,\quad y_n((n-1)h) = y_{n-1}((n-1)h)$ has a unique solution for some interval $t \in [(n-1)h, nh]$. And so on...

(iv) Since eventually Nh is greater than any t, the solution $y(t)$ exists for any t by 'pasting' together the solutions $y_n, n = 1, \ldots, N$.

What is wrong with this argument?

Solution: The interval of existence of each initial value problem depends on the initial condition, so the value of h should depend on n. The infinite sum $\sum_{n=1}^{\infty} h_n$ can be finite and will be less than one in this case. For example, $\sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} 1/(n+1)^2 = \pi^2/6 - 1 \approx 0.6449$.

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FORMULA SHEET

First-Order Linear Differential Equations. $y' + p(t)y = g(t)$.

- $\mu(t) = e^{\int p(t)\,dt}$
- $y = \dfrac{1}{\mu(t)} \displaystyle\int \mu(t) g(t)\,dt + \dfrac{C}{\mu(t)}$.

Exact First-Order Differential Equations. $M(x,y) + N(x,y)y' = 0$

- Exact if and only if $M_y = N_x$.
- Solution $\Psi(x,y) = C$ where $\Psi_x = M$ and $\Psi_y = N$.

Euler Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $t_n = t_0 + n \cdot h$
- $y_{n+1} = y_n + f(t_n, y_n)h$ $\quad$ or $\quad$ $y'(t_n) = \dfrac{y_{n+1} - y_n}{h}$
- $E_n \leqslant Ch$

Improved Euler Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + k_{n,2}}{2} h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f(t_{n+1}, y_n + k_{n,1}h)$
- $E_n \leqslant Ch^2$

Runge-Kutta Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4}}{6} h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f\left(t_n + \frac{h}{2}, y_n + k_{n,1}\frac{h}{2}\right)$
- $k_{n,3} = f\left(t_n + \frac{h}{2}, y_n + k_{n,2}\frac{h}{2}\right)$
- $k_{n,4} = f(t_{n+1}, y_n + k_{n,3}h)$
- $E_n \leqslant Ch^4$

Euler's Formula. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Limits and Series.

- $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $r < 1$.
- $\displaystyle\exp(At) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$.
- $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}A\right)^n = e^A$.

Variation of Parameters.

$$y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W[y_1, y_2](t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)} dt$$

Laplace Transforms.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$
$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a},$$
$$\mathcal{L}\{\sin(k\,t)\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(k\,t)\} = \frac{s}{s^2+k^2},$$
$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0),$$
$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0),$$
$$\mathcal{L}\{e^{a\,t}\,f(t)\} = F(s-a), \quad \mathcal{L}\{u_a(t)\,f(t-a)\} = e^{-s\,a}\,F(s),$$
$$\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\frac{d^n}{ds^n}F(s),$$
$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} \text{ for } T\text{-periodic } f,$$
$$\mathcal{L}\{f * g\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right\} = F(s)\,G(s),$$
$$\mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}.$$

 The end.