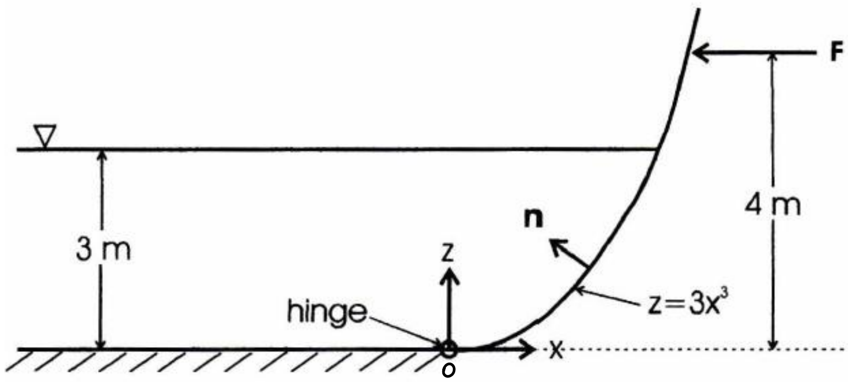
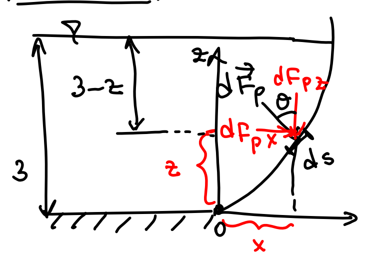
A rectangular channel of width 2 m contains water to a depth of 3 m. The gate at the end of the channel is a parabola, $z = 3x^3$, and a horizontal force, $\vec{F}$, is applied 4 m above the hinge to keep the gate closed. Ignoring the mass of the gate, find the magnitude of the force required <u>using the integration method</u>. Density of water is $\rho_{water} = 1{,}000$ kg/m^3 and gravitational acceleration is g = 10 m/s^2.



Closing moment: $M_{closing} = F \cdot 4$. height.

Opening moment: $M_{opening} = ?$

Method 1:

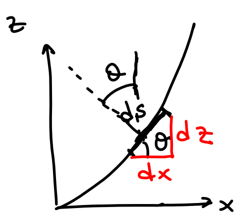


$$dM_o = x\,|dF_{p,z}| + z\,|dF_{p,x}|$$

$$\|d\vec{F}_p\| = dF_p = p\,dA = \underbrace{\rho g\,(3-z)}_{p}\,\underbrace{w\,ds}_{dA}$$

$$dF_{p,x} = dF_p \sin\theta = \rho g (3-z)\, w \sin\theta\, ds$$

$$dF_{p,z} = dF_p \cos\theta = \rho g (3-z)\, w \cos\theta\, ds$$



$$ds\cos\theta = dx$$
$$ds\sin\theta = dz \quad \Rightarrow \quad ds = \frac{dx}{\cos\theta} = \frac{dz}{\sin\theta}$$

$$|dF_{p,x}| = \rho g (3-z)\, w \sin\theta\, ds = \rho g (3-z)\, w\, \cancel{\sin\theta}\,\frac{dz}{\cancel{\sin\theta}} = \rho g (3-z)\, w\, dz$$

$$|dF_{p,z}| = \rho g (3-z)\, w \cos\theta\, d\theta = \rho g (3-z)\, w\, \cancel{\cos\theta}\,\frac{dx}{\cancel{\cos\theta}} = \rho g (3-z)\, w\, dx$$

$$dM_o = x\,|dF_{p,z}| + z\,|dF_{p,x}|$$

$$= x\rho g(3-z)\,w\,dx + z\rho g(3-z)\,w\,dz$$

$z = 3x^3$

$$= x\rho g(3-3x^3)\,w\,dx + z\rho g(3-z)\,w\,dz$$

$$M_{opening} = \int_{x=0}^{x=1} x\rho g(3-3x^3)\,w\,dx + \int_{z=0}^{3} z\rho g(3-z)\,w\,dz$$

$$= \rho g w\int_{x=0}^{1}(3x-3x^4)\,dx + \rho g w\int_{z=0}^{3}(3z-z^2)\,dz$$

$$= \rho g w\left[\frac{3x^2}{2} - \frac{3x^5}{5}\right]_{x=0}^{1} + \rho g w\left[\frac{3z^2}{2} - \frac{z^3}{3}\right]_{z=0}^{3}$$

$$= \rho g w\left[\frac{3}{2} - \frac{3}{5}\right] + \rho g w\left[\frac{27}{2} - \overset{9}{\cancel{\frac{27}{3}}}\right]$$

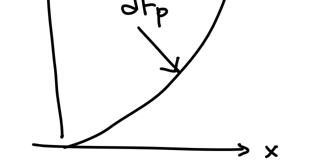
$$= \rho g w\left[15 - \frac{3}{5} - 9\right] = \frac{27}{5}\rho g w$$

$$M_{opening} = M_{closing}$$

$$\frac{27}{5}\rho g w = 4F$$

$$\frac{27}{\cancel{5}}\cdot 1000\cdot \overset{5}{\cancel{10}}\cdot \cancel{2} = \overset{2}{\cancel{4}}F \Rightarrow F = 27{,}000\ N$$

<u>Method 2:</u>



$$dM = \vec{r}\times dF_p$$

$$d\vec{F}_p = -p\,dS\,\vec{n}$$

$$dM_{opening} = dM_y = (\vec{r}\times d\vec{F}_p)\cdot\vec{j}$$

$$= \left[(\vec{r}\times\vec{n})\cdot\vec{j}\right](-p\,dS)$$

Let's parametrize the surface:

$$\left.\begin{array}{l} x = x \\ y = y \\ z = 3x^3 \end{array}\right\} \vec{r}(x,y) = x\vec{i} + y\vec{j} + 3x^3\vec{k} \quad \text{where} \quad \begin{array}{l} 0 \le y \le 2 \text{ (width)} \\ 0 \le x \le 1 \end{array}$$

$$\vec{r}_x = \vec{i} + 9x^2\vec{k}$$

$$\vec{r}_y = \vec{j}$$

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 9x^2 \\ 0 & 1 & 0 \end{vmatrix} = -9x^2\vec{i} + \vec{k}$$

(this $\vec{N}$ is in the $-d\vec{F}_p$ direction. So, we will use this).

$$dM_{opening} = (\vec{r} \times \vec{n}) \cdot \vec{j}\,(-p\,dS)$$

$$= \left[\vec{r} \times \frac{(\vec{r}_x \times \vec{r}_y)}{\cancel{\|\vec{r}_x \times \vec{r}_y\|}}\right] \cdot \vec{j}\,(-p)\,\underbrace{\cancel{\|\vec{r}_x \times \vec{r}_y\|}\,dx\,dy}_{dS}$$

$$= \left[\vec{r} \times (\vec{r}_x \times \vec{r}_y)\right] \cdot \vec{j}\,(-p\,dx\,dy)$$

$$\vec{r} \times (\vec{r}_x \times \vec{r}_y) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & 3x^3 \\ -9x^2 & 0 & 1 \end{vmatrix} = y\vec{i} + (-x - 27x^5)\vec{j} + 9x^2y\,\vec{k}$$

$$dM_{opening} = \underbrace{\left[\vec{r} \times (\vec{r}_x \times \vec{r}_y)\right] \cdot \vec{j}}\,(-p\,dx\,dy)$$

$$= (-x - 27x^5)(-p\,dx\,dy)$$

$$p = \rho g(3 - z) = \rho g(3 - 3x^3)$$

$$z = 3x^3$$

$$dM_{opening} = (x + 27x^5)\,\rho g(3 - 3x^3)\,dx\,dy$$

$$M_{opening} = \iint_S dM_{opening}$$

$$M_{opening} = \int_{x=0}^{1} \int_{y=0}^{2} 3\rho g \left(x - x^4 + 27x^5 - 27x^8 \right) dy\, dx$$

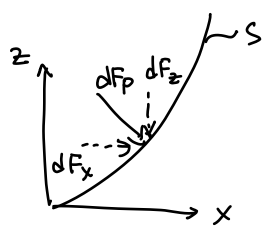
$$= 3\rho g \cdot 2 \cdot \left[\frac{x^2}{2} - \frac{x^5}{5} + \frac{27x^6}{6} - \frac{27x^9}{9} \right]_{x=0}^{1}$$

$$= 6\rho g \Big[\underbrace{\frac{1}{2} - \frac{1}{5} + \frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}} - \frac{\cancel{27}^{\,3}}{\cancel{9}}}_{\frac{18}{10}} \Big] = 6\rho g \cdot \frac{9}{5}$$

$$M_{opening} = M_{closing}$$

$$6\rho g \cdot \frac{9}{5} = F \cdot 4 \;\Rightarrow\; F = 6 \cdot 1000 \cdot 10 \cdot \frac{9}{5} \cdot \frac{1}{4} = 27{,}000\,\text{N}$$

Method 3:



Parametric eqn for the surface S is:

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + 3x^3\vec{k}$$

$$d\vec{F}_p = -p \underset{?}{\vec{n}}\, \underbrace{dS}_{?}$$

$$\left.\begin{aligned} \vec{r}_x &= \vec{i} + 9x^2\vec{k} \\ \vec{r}_y &= \vec{j} \end{aligned}\right\}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 9x^2 \\ 0 & 1 & 0 \end{vmatrix} = -9x^2\vec{i} + \vec{k}$$

$$d\vec{F}_p = -p\,\vec{n}\, dS$$

$$= -p \overbrace{\frac{(\vec{r}_x \times \vec{r}_y)}{\cancel{\|\vec{r}_x \times \vec{r}_y\|}}}^{\vec{n}} \overbrace{\cancel{\|\vec{r}_x \times \vec{r}_y\|}\, dx\, dy}^{dS}$$

$$= -p\, \underbrace{(\vec{r}_x \times \vec{r}_y)}_{-9x^2\vec{i} + \vec{k}}\, dx\, dy$$

$$d\vec{F}_p = -p\left(-9x^2\,\vec{i} + \vec{k}\right)dx\,dy$$

$$p = \rho g(3 - z) = \rho g(3 - 3x^3)$$

$$z = 3x^3$$

$$d\vec{F}_p = \rho g\left(3 - 3x^3\right)\left(9x^2\,\vec{i} - \vec{k}\right)dx\,dy$$

$$dF_{px} = 9x^2 \rho g\left(3 - 3x^3\right)dx\,dy$$

$$dF_{pz} = -\rho g\left(3 - 3x^3\right)dx\,dy$$

$$dM = x\,|dF_z| + z\,|dF_x|$$

$$z = 3x^3$$

$$= x\rho g\left(3 - 3x^3\right)dx\,dy + z \cdot 9x^2 \rho g\left(3 - 3x^3\right)dx\,dy$$

$$dM = 3\rho g\left(x - x^4\right)dx\,dy + 81\rho g\left(x^5 - x^8\right)dx\,dy$$

$$M_{opening} = \int_{x=0}^{1}\int_{y=0}^{2} 3\rho g\left(x - x^4\right)dy\,dx + 81\rho g\int_{x=0}^{1}\int_{y=0}^{2}\left(x^5 - x^8\right)dy\,dx$$

$$= 3\rho g \cdot 2 \cdot \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_{x=0}^{1} + 81\rho g \cdot 2 \cdot \left[\frac{x^6}{6} - \frac{x^9}{9}\right]_{x=0}^{1}$$

$$= 3\rho g \cdot 2\left(\frac{1}{2} - \frac{1}{5}\right) + \overset{27}{\cancel{81}}\rho g \cdot 2 \cdot \left(\frac{1}{\underset{2}{\cancel{6}}} - \frac{1}{\underset{3}{\cancel{9}}}\right)$$

$$= 6\rho g\left(\frac{1}{2} - \frac{1}{5} + \frac{9}{2} - 3\right)$$

$$M_{opening} = 6\rho g \cdot \left(\frac{9}{5}\right)$$

$$M_{opening} = M_{closing} \Rightarrow 6\rho g \cdot \left(\frac{9}{5}\right) = F \cdot 4 \Rightarrow \quad F = 6 \cdot 1000 \cdot 10 \cdot \frac{9}{5} \cdot \frac{1}{4} = 27{,}000\ \underline{N}$$