MAT292 - Calculus III - Fall 2016

Term Test 2 - November 17, 2016

Time allotted: 70 minutes

Aids permitted: None

Total marks: 50

Full Name: ____
Last First

Student Number: ____

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 11–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

GOOD LUCK!

PART I No explanation is necessary. **(10 marks)**

1. (1 mark) If the origin of the system $\vec{x}'(t) = \begin{pmatrix} a & 2 \\ -2 & 1 \end{pmatrix} \vec{x}(t)$ is a centre, then $a =$ _____________.

2. (2 marks) If y_1 and y_2 form a fundamental set of solutions to $y'' + t^2 y' + 7y = 0$, and $W[y_1, y_2](0) = 2$, then

$$W[y_1, y_2](t) = \underline{\hspace{10cm}}.$$

3. (2 marks) If $y(t) = t^r$ solves $t^2 y''(t) - 4ty'(t) + 4y(t) = 0$ for $t > 0$, then $r =$ _____________.

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4. (3 marks) Consider the ODE

$$y^{(6)} - 5y^{(5)} + 11y^{(4)} - 37y^{(3)} + 32y'' + 10y' = te^t + e^t\cos(2t) + t\sin(3t) + t^2.$$

Hint. $r^6 - 5r^5 + 11r^4 - 37r^3 + 32r^2 + 10r = r(r-1)^3\big((r-1)^2 + 9\big)$.

Then its particular solution will have the following terms: **(Circle all correct options)**

(a) A	**(e)** Et^2e^t	**(i)** $Ie^t\cos(2t)$	**(m)** $M\cos(3t)$	**(q)** $Qe^t\cos(3t)$
(b) Bt	**(f)** Ft^3e^t	**(j)** $Je^t\sin(2t)$	**(n)** $N\sin(3t)$	**(r)** $Re^t\sin(3t)$
(c) Ct^2	**(g)** Gt^4e^t	**(k)** $Kte^t\cos(2t)$	**(o)** $Ot\cos(3t)$	**(s)** $Ste^t\cos(3t)$
(d) Dt^3	**(h)** Ht^5e^t	**(l)** $Lte^t\sin(2t)$	**(p)** $Pt\sin(3t)$	**(t)** $Tte^t\sin(3t)$

where $A, B, \ldots, T \neq 0$.

5. (1 mark) Consider the system of two first-order linear differential equations

$$\vec{x}\,' = A\vec{x},$$

where the matrix A has the eigenvalues $\lambda = \alpha \pm i\beta$ with $\alpha \leqslant 0$ and $\beta > 0$.

(Circle all possible options)

(a) $\lim\limits_{t\to+\infty} \vec{x}(t) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$.

(b) $\lim\limits_{t\to+\infty} \|\vec{x}(t)\| = +\infty$.

(c) $\vec{x}(t)$ keeps orbiting $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ forever, so the limit as $t \to +\infty$ doesn't exist.

6. (1 mark) Consider the second-order differential equation $y'' + y = g(t)$. Give an example of a **bounded function $g(t)$** such that $\boldsymbol{|y(t)|}$ **is not bounded**.

$g(t) =$ ____.

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PART II **Justify** your answers.

7. Consider the following system of differential equations. **(13 marks)**

$$\vec{x}\,'(t) = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \vec{x}(t)$$

(a) **(6 marks)** Find the solution to the system that satisfies the initial condition

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Continued...

(b) (3 marks) Find the solution to the system that satisfies the initial condition

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) (4 marks) Find the special fundamental matrix.

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8. Consider the following initial-value problem. **(13 marks)**

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

where

$$p(t) = \begin{cases} 2 & \text{if } t < 1 \\ 0 & \text{if } t \geqslant 1 \end{cases} \quad , \quad q(t) = \begin{cases} 1 & \text{if } t < 1 \\ 0 & \text{if } t \geqslant 1 \end{cases} \quad , \quad g(t) = \begin{cases} 9e^{2t} & \text{if } t < 1 \\ 0 & \text{if } t \geqslant 1 \end{cases}$$

Find the solution $y(t)$ for all $t \geqslant 0$.

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To be used for the answer to question **8.**

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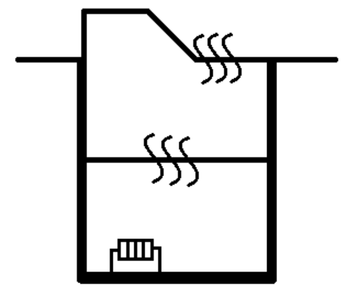
9. Consider an underground facility consisting of two floors one on top of the other with temperatures $x_1(t)$ and $x_2(t)$. **(14 marks)**

- Heat is transferred between the floors at a rate equal to the difference of the temperatures.
- Heat is transferred from the outside to the upper room at a rate proportional to the difference between the outside temperature T and the temperature of the upper floor, with proportionality constant $P > 0$.
- Finally, there is an adjustable temperature regulator heating (or cooling) the lower floor at a constant rate H.



Hint. You can answer all the following questions without finding the solution of the system.

(a) **(3 marks)** What system of first order linear differential equations does the vector $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ satisfy?

(b) **(3 marks)** Find the equilibrium solution $\vec{x}_e = \begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix}$ of the system from **(a)** .

(c) **(3 marks)** Let $u_1(t) = x_1(t) - x_1^e$ and $u_2(t) = x_2(t) - x_2^e$.

What first order linear system of differential equations does $\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ solve? Justify.

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(d) **(2 marks)** For what values of $P > 0$ is $\binom{0}{0}$ a stable critical point of the system from **(c)** ? Justify.

Hint: $(a+b)^{1/2} < a^{1/2} + b^{1/2}$ for $a, b > 0$.

(e) **(3 marks)** If $H = 0$ (i.e. the temperature regulator is off) and for the values of $P > 0$ found in **(d)** , what is $\lim_{t \to \infty} \vec{x}(t)$? Does this make physical sense? Explain why or why not.

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