MAT292 - Calculus III - Fall 2016

Term Test 1 - October 20, 2016

Time allotted: 60 minutes

Aids permitted: None

Total marks: 50

Solution

Full Name: ______________________
Last First

Student Number: ______________________

Email: ______________________ @mail.utoronto.ca

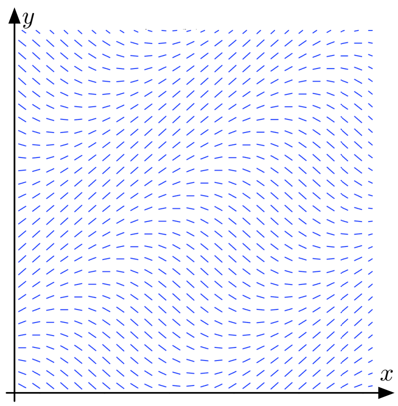
Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 10 pages (including this title page). Make sure you have all of them.
- You can use pages 9–10 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 9–10.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

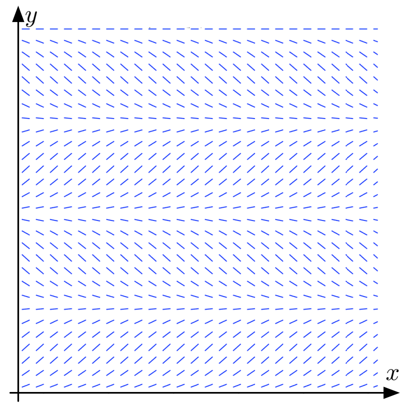
GOOD LUCK!

PART I No explanation is necessary. **(10 marks)**

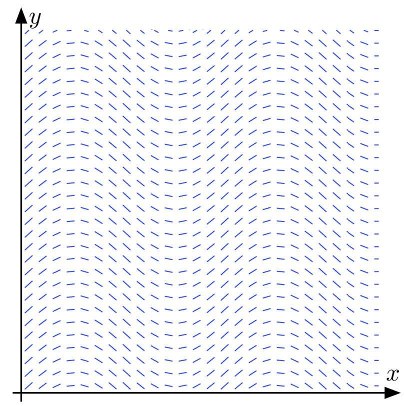
1. **(1 mark)** Consider the following direction fields:



A



B



C

Which of these is the direction field of an **Autonomous Differential Equation** $y' = f(y)$?

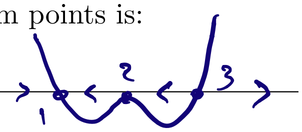
B (because the arrows don't depend on x)

2. **(2 marks)** Consider the following equilibrium solutions:

- $y(t) = 1$ is stable;
- $y(t) = 2$ is semistable;
- $y(t) = 3$ is unstable.

Then an **Autonomous Differential Equation** with these equilibrium points is:

$y' =$ $(y-1)(y-2)^2(y-3)$



3. **(2 marks)** Consider the DE:

$$\begin{cases} (t^2 + y^2 - 9)y' = 2t \\ y(t_0) = y_0 \end{cases}$$

From the Existence and Uniqueness Theorem, there is a unique solution for

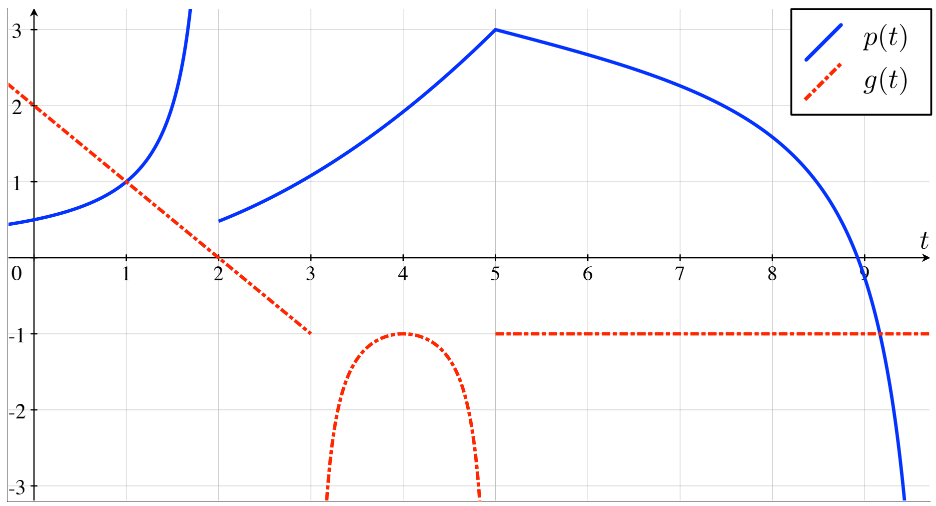
$(t_0, y_0) \in \{$ $(t,y) \in \mathbb{R} \times \mathbb{R} : \quad t^2 + y^2 \neq 9$ $\}$,

and for these values, the solution is defined for

$t \in ($ $t_0 - h$, $t_0 + h$ $)$,

Continued...

For questions **4.** , **5.** and **6.** , consider the Differential Equation $y' + p(t)y = g(t)$ and the following graphs of $p(t)$ and $g(t)$.



4. **(1 mark)** There exists a unique solution satisfying $y(6) = -2$ defined for

$$t \in \left(\underline{\quad 5 \quad} \;,\; \underline{\quad \infty \text{ or } 10 \quad} \right).$$

if 10 on (4), then 10 should be included in (6).

5. **(1 mark)** There exists a unique solution satisfying $y\left(\frac{5}{2}\right) = 2$ defined for

$$t \in \left(\underline{\quad 2 \quad} \;,\; \underline{\quad 3 \quad} \right).$$

6. **(1 mark)** There exists a unique solution satisfying $y(t_0) = -1$ for

$$t_0 \in \left\{ \underline{\; t \in \mathbb{R} : \quad t \neq 2, 3, 5 \;, 10 \;} \right\},$$

7. **(2 marks)** The Differential Equation

$$e^y \cos(x) + 2x + \left(e^y \sin(x) - y\right)\frac{dy}{dx} = 0$$

is exact, so the solution is of the form $\Psi(x, y) = C$ for

$$\Psi(x, y) = \underline{\; e^y \sin(x) + x^2 - \frac{y^2}{2} \;}$$

Continued...

PART II **Justify** your answers.

8. Consider the initial value problem **(13 marks)**

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{2x}{\cos(y) + 2y} \\ y(\pi) = \pi \end{cases}$$

(a) **(7 marks)** Find its solution.

This is a Separable DE, so

$$\int \left(\cos(y) + 2y\right) dy = -\int 2x \, dx \iff \sin(y) + y^2 = -x^2 + C$$

we find C using the initial condition: $\pi^2 = -\pi^2 + C \iff C = 2\pi^2$

The solution is given in implicit form:

$$\boxed{\sin(y) + y^2 = -x^2 + 2\pi^2}$$

(b) **(6 marks)** *Assume that there is a unique number* $-0.46 < D < -0.45$ *with* $2D + \cos(D) = 0$. In terms of D, what is the largest interval on which the solution from part **(a)** is defined?

Hint. To find the interval of definition, look for points where the integral curve has a vertical tangent.

The integral curves are vertical when "$\frac{dy}{dx} = \infty$" or equivalently when $\cos(y) + 2y = 0 \iff y = D$.

This happens for x satisfying $\sin(D) + D^2 = -x^2 + 2\pi^2$

$$\iff x^2 = \underbrace{2\pi^2 - \underbrace{\left(\sin(D) + D^2\right)}_{\leq 1 + (0.46)^2 \leq 2}}_{2\pi^2 - 2 > \pi^2 + \pi^2 - 2 > \pi}$$

$$\iff x = \pm \underbrace{\sqrt{2\pi^2 - \left(\sin(D) + D^2\right)}}_{A}$$

The domain of the solution is

$$\boxed{(-A, A) = \left(-\sqrt{2\pi^2 - (\sin(D) + D^2)}, \sqrt{2\pi^2 - (\sin(D) + D^2)}\right)}$$

Continued...

9. Let $y_n = y(t_n)$ where $t_n = h \cdot n$ be a discrete representation of $y(t)$. **(13 marks)**

(a) **(6 marks)** Apply Euler's method twice to discretize the differential equation

$$\frac{d}{dt}\left(k(t)y'(t)\right) = Q(t)$$

Euler's Method uses the approximation $y'(t_n) \approx \frac{y_{n+1} - y_n}{h}$.

So we get $\frac{k(t_{n+1})\, y'(t_{n+1}) - k(t_n)\, y'(t_n)}{h} = Q(t_n)$

$$\frac{1}{h}\left[k(t_{n+1})\,\frac{y_{n+2} - y_{n+1}}{h} - k(t_n)\,\frac{y_{n+1} - y_n}{h}\right] = Q(t_n)$$

$$\boxed{\frac{1}{h^2}\left[k(t_{n+1})\left(y_{n+2} - y_{n+1}\right) - k(t_n)\left(y_{n+1} - y_n\right)\right] = Q(t_n)}$$

(b) **(4 marks)** If $k(t) = k$, and $Q(t) = 0$, show that $y_{n+1} = \frac{y_{n+2} + y_n}{2}$ for all n.

Then the method becomes:

$$\frac{k}{h^2}\left[\left(y_{n+2} - y_{n+1}\right) - \left(y_{n+1} - y_n\right)\right] = 0$$

$$y_{n+2} - 2y_{n+1} + y_n = 0$$

$$\boxed{y_{n+1} = \frac{y_{n+2} + y_n}{2}}$$

(c) **(3 marks)** Under the same conditions as **(b)**, show that if there exists m such that $y_m \geqslant y_{m+1}$, and $y_m \geqslant y_{m-1}$ then there is a constant c such that $y_n = c$ for all n.

idea. y_m is the average of y_{m-1} and y_{m+1}, so if it is greater or equal than both, then all three numbers must be the same.

$$\left.\begin{array}{l} \boxed{y_{m+1} \leq y_m} = \boxed{\dfrac{y_{m+1} + y_{m-1}}{2}} \Rightarrow \dfrac{y_{m+1}}{2} \leq \dfrac{y_{m-1}}{2} \\ \boxed{y_{m-1} \leq y_m} = \boxed{\dfrac{y_{m+1} + y_{m-1}}{2}} \Rightarrow \dfrac{y_{m-1}}{2} \leq \dfrac{y_{m+1}}{2} \end{array}\right\} \Rightarrow y_{m-1} = y_{m+1}$$

option #1

So $y_{m-1} = y_m = y_{m+1}$. Then

$$y_{m-2} = 2y_{m-1} - y_m = y_m \Rightarrow \dots \Rightarrow y_n = y_m \text{ for } n = 1, \dots, m+1$$

and

$$y_{m+2} = 2y_{m+1} - y_m = y_m \Rightarrow \dots \Rightarrow y_n = y_m \text{ for } n = m+2, \dots$$

We just showed y_n is constant for all n. ∎

option #2

By strong induction with $c = y_m$.

Base Case. $n = m$ ✓

IH_1. Assume $y_n = c$ for $n = m, m+1, \dots, N$.

Then $y_{N+1} = 2y_N - y_{N-1} = 2c - c = c$ ✓

$\Rightarrow y_n = c$ for $n = m, m+1, \dots$

Base Case. $n = m$ ✓

IH_2. Assume $y_n = c$ for $n = N, \dots, m-1, m$

Then $y_{N-1} = 2y_N - y_{N+1} = 2c - c = c$ ✓

$\Rightarrow y_n = c$ for $n = 1, \dots, m$

Continued...

10. To model the spread of an infectious disease, we divide the population in 3 groups: **(14 marks)**

- Susceptible Individuals $S(t)$ = number of people who haven't contracted the disease;
- Infected individuals $I(t)$ = number of people infected;
- "Recovered" individuals $R(t)$ = number people that either died or recovered from the disease and are now immune to it.

Assume that the total population is N (in thousands).

These functions satisfy a nonlinear system of Differential Equations:

$$\frac{dS}{dt} = -\beta \frac{I(t)}{N} S(t)$$
$$\frac{dI}{dt} = \beta \frac{I(t)}{N} S(t) - \gamma I(t)$$

(a) **(3 marks)** Explain why these equations are nonlinear.

Because the unknown functions are $S(t)$ and $I(t)$ and we have terms $I(t)\,S(t)$ which are nonlinear.

We haven't learned how to solve systems of DEs (much less nonlinear systems), so we will simplify it by using some data about infected people. Assume that in this country, the hospitals reported that the number of infected people were

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Infected People	0	7	12	15	16	15	12	7	0

where the number of infected people is in thousands.

(b) **(3 marks)** Using this data (not the DEs), find a function $I(t)$ that approximates this data well.

$I(t) = 16-(t-5)^2$ for $t\in[1,9]$

$I(t) = 16-(t-4)^2$, $t\in[0,8]$

t in months and $t=1$ = January or $t=0$ = January

$I(t) = 16\sin\left(\frac{\pi}{8}t\right)$, $t\in[0,8]$

or

$I(t) = 16\sin\left(\frac{\pi}{8}(t-1)\right)$, $t\in[1,9]$

Continued...

(c) **(3 marks)** What is the initial condition for $S(t)$?

The initial condition is $S(1) = N$. (or $S(0) = N$)

Other initial conditions are possible with proper justification:

- $S(2) = N - 7$ (no one recovered yet)
- $0 = \beta \frac{16}{N} S(5) - \gamma 16$ (from 2nd DE at $t=5$, where $I(t)$ reaches max so $I'(5)=0$)

(d) **(5 marks)** Since you now have a function $I(t)$, find $S(t)$ from the Differential Equations.

$$S'(t) = -\frac{\beta}{N} S(t) I(t) = -\frac{\beta}{N} S(t) \left[16 - (t-5)^2\right]$$

Separable DE

$$\iff \int \frac{1}{S} dS = -\frac{\beta}{N} \int \left[16 - (t-5)^2\right] dt$$

$$\iff \ln S = -\frac{\beta}{N}\left[16t - \frac{(t-5)^3}{3}\right] + A$$

$$\iff S(t) = C e^{-\frac{\beta}{N}\left[16t - \frac{(t-5)^3}{3}\right]}$$

Using the initial condition we obtain

$$N = S(1) = C e^{-\frac{\beta}{N}\left(16 + \frac{4^3}{3}\right)} \iff C = N e^{\frac{\beta}{N}\left(16 + \frac{4^3}{3}\right)}$$

The solution is

$$\boxed{S(t) = N e^{-\frac{\beta}{N}\left[16(t-1) - \frac{(t-5)^3 - 4^3}{3}\right]}}$$