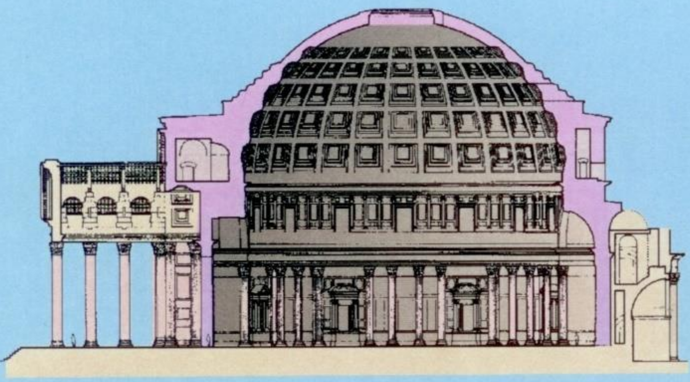
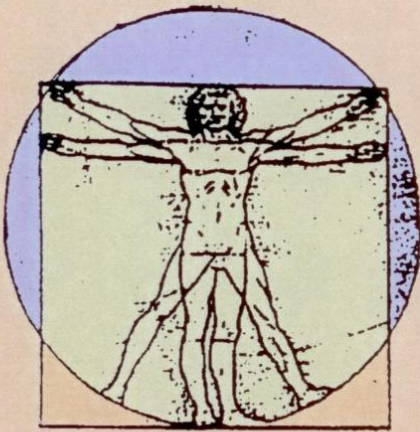
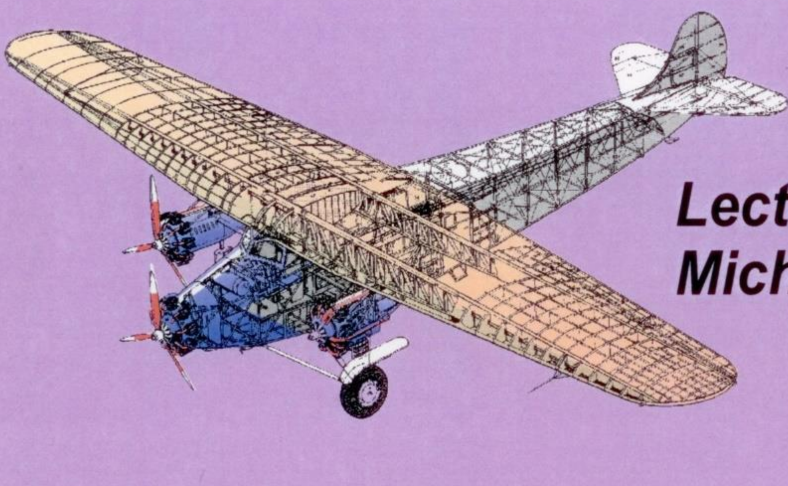


CIV 102F

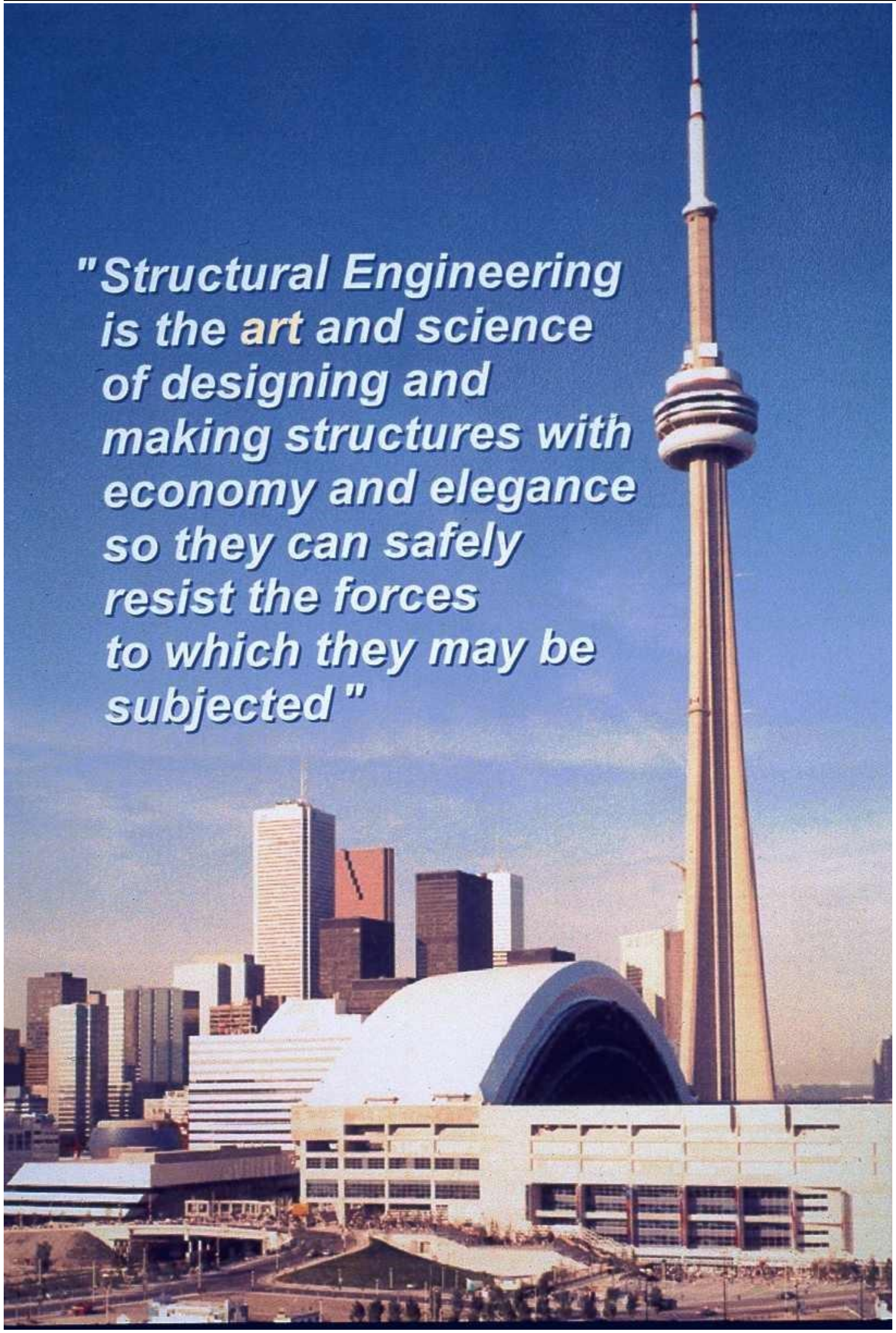
STRUCTURES AND MATERIALS



An Introduction To Engineering Design



Lectures by
Michael P. Collins



CIV102 – Structures and Materials

An Introduction to Engineering Design

University of Toronto
Division of Engineering Science
September 2020

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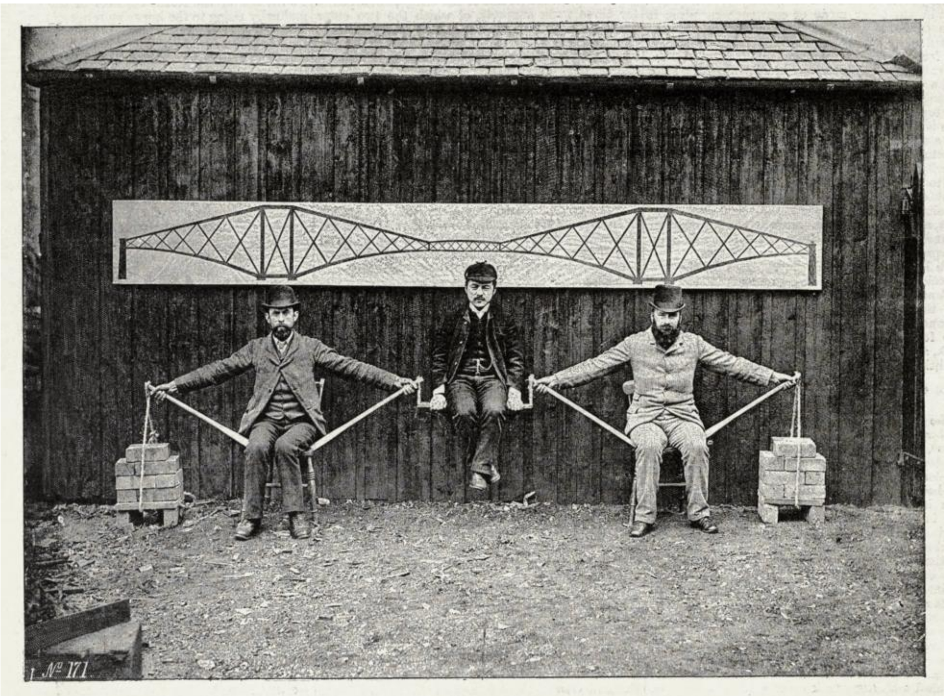


Living demonstration explaining the principles allowing the Firth of Forth railroad bridge to carry load. Designed by Benjamin Baker and John Fowler.

CIV102 Course Notes – Part 1: Basic Concepts

Lecture 1 – The Three Principles of Engineering

The Three Principles of Engineering:

"The four to twelve page Toike Oike we remember – with its notices of School events, reports of meetings, accounts of the exploits of School teams, messages from the Dean and occasional jokes – would scarcely seem to have the potential to convulse the University of Toronto. True the bound volume in the Engineering Society Office did have one issue with a short joke encircled in blue pencil. We understood that this had got the editor suspended – possibly even expelled. It is also true that in our fourth year Engineering Physics wrested control from the former editors, apparently meeting with little organized resistance, and produced Volume XXX (or possibly XXXI – there seems to have been some confusion). Luckily no one attempted to use this experiment as a launching pad for a career in journalism or politics. But one of the editors who remains active has just fearlessly restated the three fundamental Principles of Engineering presented in that 1939 manifesto:

1. *F = M × A*
2. *You can't push on a rope.*
3. *A necessary condition for solving any given Engineering problem is to know the answer before starting*

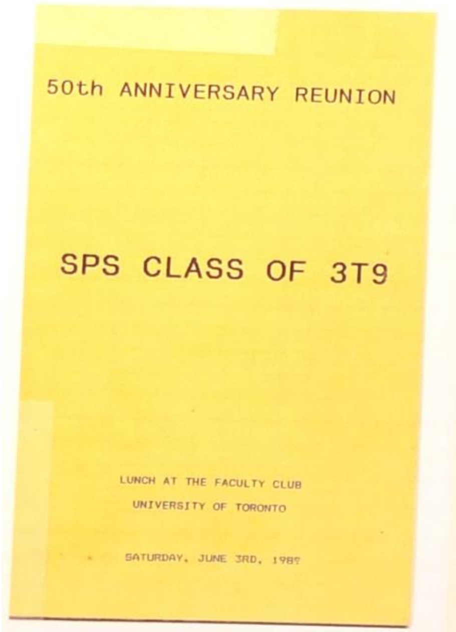


Figure 1.1 – Engineering Manifesto

Structural engineering is a branch of civil engineering which is interested in the analysis and design of structures which must safely carry forces. Some examples of civil structures designed by structural engineers, which are typically built out of steel, concrete or timber, include buildings, bridges, tunnels, dams, and concrete offshore platforms. The principles used in structural engineering are also applicable to disciplines outside of civil engineering, such as aerospace engineering and biomedical engineering.

Before getting into the nuts and bolts of structural engineering, it is worth spending some time exploring the meaning of the ***Three Principles of Engineering*** in greater detail. As their name suggests, these principles apply, generally speaking, to all disciplines of engineering. They are particularly relevant to those practicing structural engineering however, as the collapse of significant structures – including in Canada – have greatly influenced how engineering is practiced today. In fact, the Iron Ring ceremony, a ritual undergone by all engineers trained in Canada to affirm the duties and responsibilities of the profession, has its roots in the collapse of the Quebec Bridge in 1907.

The First Principle of Engineering, **F = M × A**, is Newton's second law of motion. It has practical applications in many branches of physics and engineering: for example, mechanical and aerospace engineers use it to design objects intended to move, such as transportation vehicles, spacecraft, or even robotic drones. Structural engineers on the other hand, typically make use of the special case of Newton's second law, which is when A, the acceleration of a body, is equal to 0. When this condition is satisfied, a system is said to be in a ***state of equilibrium***. The concept of equilibrium is fundamental to structural engineering, which is primarily interested in systems which do not accelerate.

Symbolically, the First Principle of Engineering represents the idea that engineers use mathematical models to understand and shape the world around them. Many branches of engineering do not use Newton's laws of motion specifically, but instead use their own discipline-specific set of tools which are also grounded in physics and math. Practicing engineers must master the application of these tools in their work, while engineers who work in research seek to develop new models and expand the body of knowledge of their respective field of engineering.

The Second Principle of Engineering, **“You can’t push on a rope**”, is very different from the First Principle. From a structural engineering perspective, a rope collapses when pushed because it is too flexible to carry a compression force. The technical reason why this occurs is because the rope ***buckles***. Buckling, which is discussed in great detail later in the course, does not just happen when a rope is being pushed, but occurs when any slender member is being subjected to a large compression force.

Buckling of slender compression members was the cause of failure of the Quebec Bridge, which collapsed while under construction in 1907, killing 75 workers. The bridge, which was designed by the American engineer Theodore Cooper, had a similar shape as the Firth of Forth Railway Bridge which was at the time the longest cantilever truss bridge in the world. Baker and Fowler’s bridge, shown in Fig. 1.2, used very large members to safely carry the high compression forces in the structure. Cooper, who ridiculed the Firth of Forth Railway Bridge for using excessive amounts of steel, instead used comparatively slender members in his design, which is shown in Fig. 1.3. These members, which buckled during construction, caused the failure shown in Fig. 1.4.

Fig. 1.2 – Baker and Fowler’s Firth of Forth Railway Bridge. Two main spans each 1700 feet long. Opened 1890.

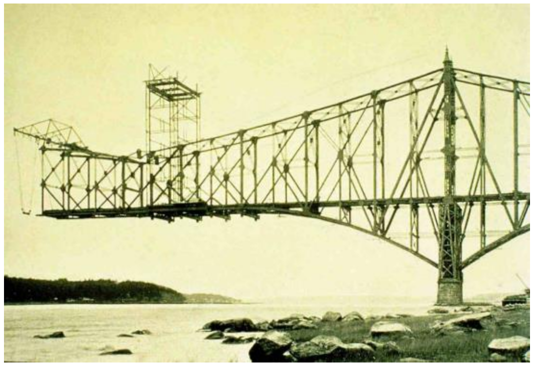


Fig. 1.3 – Cooper’s Quebec Bridge during construction

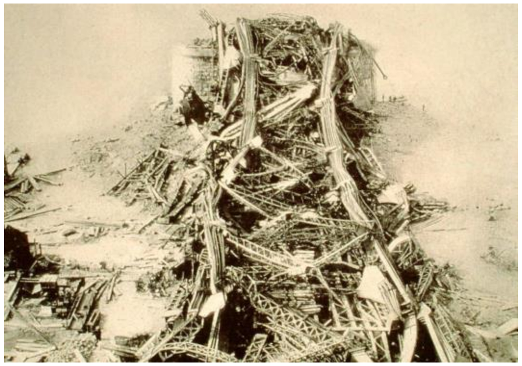


Fig. 1.4 – Cooper’s Quebec Bridge after collapse

Although the First Principle of Engineering celebrates the use of mathematical models in engineering, the Second Principle is a reminder to draw upon common sense and experience when working with the real world. The results of calculations – whether they be simple calculations done by hand or complex simulations performed on a supercomputer – must be checked to ensure that they make sense. Gravity pulls objects down, materials tend to expand when heated, light travels faster than sound, and slender members buckle when pushed with great force. Calculations which suggest otherwise should generally not be trusted.

The Third Principle of Engineering is perhaps best summarized as **"To find the answer, you must know the answer"**. Although seemingly paradoxical, engineering is full of situations where this statement is true. The following example is a common occurrence in structural engineering: a bridge must be designed to carry loads which includes its own self-weight. However, its weight is not known until after the bridge has been designed, which in itself requires knowing its weight at the beginning of the design process. Without experience, resolving this paradox can be challenging or may even result in designs which are dangerously unsafe. Solving this design problem requires having a reasonable idea of what the final design will be before starting – the answer must be known before it is obtained.

The Third Principle of Engineering illustrates the value of experience when practicing engineering. It is also a reminder of the dangers of approaching new problems where one does not have any prior experience to act as a guide. The collapse of the Quebec Bridge, which was by far the longest bridge ever designed by Cooper, can partially be attributed to him straying from the Third Principle and attempting to find the answer without knowing what it should have been.

Lecture 2 – Basic Concepts: Newton, Pulling on Ropes, Units

Overview:

In this chapter, a variety of basic concepts are introduced which permit simple structural systems to be understood from a mechanics-based perspective. Beginning from Newton's laws of motion, the idea of forces – such as the force due to gravity, and tension and compression forces transmitted through structural members – is discussed. By introducing the concept of a moment, a turning action which causes bodies to rotate, the three equations of equilibrium for two-dimensional systems are presented. The chapter concludes with a brief discussion about units.

Newton's Laws of Motion:

Newton's three laws of motion are:

1. "Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it". This can be mathematically expressed as:

$$\sum F = 0 \rightarrow a = 0 \tag{2.1}$$

2. "The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed"

$$F = ma \tag{2.2}$$

3. "To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts"

In Eq. (2.1) and (2.2), **F** is a force applied to the body, **m** is its mass (which is assumed to be constant), and **a** is the translational acceleration of the body. The first law of motion is especially relevant to the field of structural engineering, where the bodies considered are typically not accelerating despite being subjected to numerous forces.

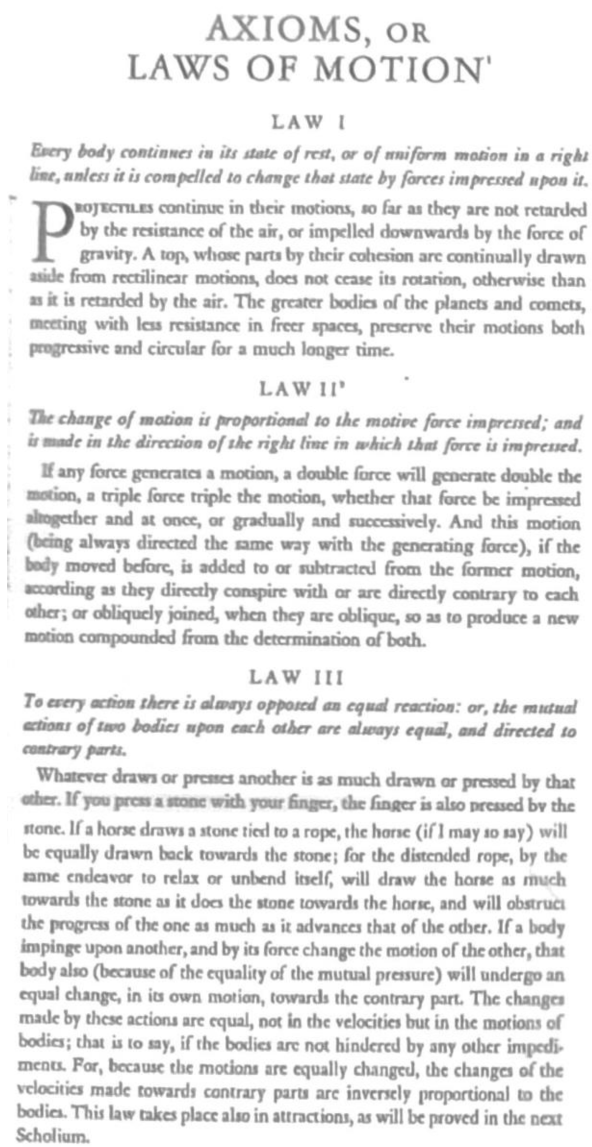
Basic Definitions – Forces:

The acceleration due to gravity caused by the pull of the Earth on a body is defined as **g**. Using Newton's second law, the gravitational force $\mathbf{F_g}$, which pulls an object with mass **m** towards the centre of the earth, is defined as:

$$F_g = mg \tag{2.3}$$

g varies around the world depending on the elevation of the ground, assuming a larger value closer to sea level and a smaller one at high elevations. A typical value of g which is accurate to three significant figures is $\mathbf{g = 9.81\ m/s^2}$. This value is reasonably accurate over a wide range of elevations and will be used for all calculations in this course.

The main purpose of a structural member is to transmit forces from one location to another, like the rod shown in Fig. 2.2 which is being pulled with a force of 100 N on either side. Their behaviour when subjected to forces can be understood by using Newton's first and third laws. Because the two forces are equal, the body does not accelerate and is in a state of translational equilibrium. Furthermore, if the applied forces were caused by two people pulling on the



AXIOMS, OR
LAWS OF MOTION

LAW I

Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Projectiles continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

LAW II

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

LAW III

To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone; for the distended rope, by the same endeavor to relax or unbend itself, will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are inversely proportional to the bodies. This law takes place also in attractions, as will be proved in the next Scholium.

Fig. 2.1 – Newton's three laws of motion

rod, they would feel the rod resist their applied force with an equal and opposite reaction force to maintain this state of equilibrium. A free body diagram drawn through any point along the rod would show that the internal force at every location is a pulling, or tensile, force of 100 N.

Fig. 2.2 – A body carrying 100 N of tension

A member like the one shown in Fig. 2.2 which is carrying a pulling force acting through its axis is said to be in ***tension***. The opposite of tension is ***compression***, which is defined as a pushing force acting through the axis of a member like the one shown in Fig. 2.3. A free body diagram drawn through any point along the rod's length will reveal that the internal force at every location is a pushing, or compression, force of 100 N.

Fig. 2.3 – A body carrying 100 N of compression

Components of Forces in Two Dimensions:

When dealing with two-dimensional systems in the x-y plane, forces will generally produce an effect in both the horizontal (typically taken as x) and vertical (typically taken as y) directions. The actions of a force along these directions are called its x- and y- components respectively. A force **F** which is acting at an angle **θ** relative to the x axis, like the one drawn in Fig. 2.4, has components in the x- and y- direction, $\mathbf{F_x}$ and $\mathbf{F_y}$ respectively, defined as:

$$F_x = F\cos\theta \tag{2.4}$$

$$F_y = F\sin\theta \tag{2.5}$$

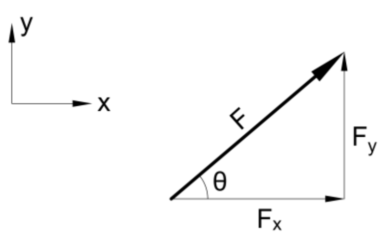


Fig. 2.4 – Components of a force

The magnitude of the force is related to its components by Pythagoras' theorem:

$$F = \sqrt{(F_x)^2 + (F_y)^2} \tag{2.6}$$

Note: It often more convenient to define the x and y components of a force F using the side lengths of a similar triangle whose hypotenuse is parallel to F, instead of defining an angle θ. For example, for the force vector shown below in Fig. 2.5, defining sinθ = (b/c) and cosθ = (a/c) allows us to define the components as:

$$F_x = F\cos\theta = \frac{a}{c}F$$

$$F_y = F\sin\theta = \frac{b}{c}F$$

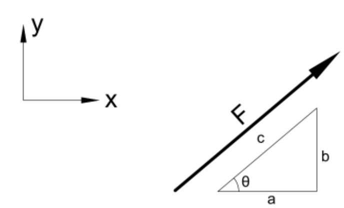


Fig. 2.5 – Determining the components of a force using a similar triangle

Rotational actions – Moments:

In addition to causing translational motion, forces can also cause bodies to rotate. A ***moment*** is the turning effect produced by a force about a reference point when the line of action of the force does not pass through the defined point of reference. The moment $\mathbf{M_i}$ caused by a force about reference point **i** is defined as the product of the magnitude of the force **F** and the perpendicular distance between its line of action and the reference point, $\mathbf{d_i}$:

$$M_i = F \times d_i \tag{2.7}$$

Because a moment is defined based on a reference point, the moment produced by a force will be different when calculated about different reference points. For example, the 5 N force in Fig. 2.6 produces a counterclockwise moment of 5 N × 4 m = 20 Nm about point A and a clockwise moment of 5 N × 6 m = 30 Nm about point B.

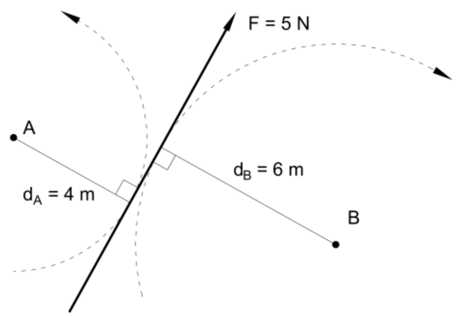


Fig. 2.6 – A 5 N force producing moments about points A and B

A ***couple*** is special class of moments which occurs when two forces with the same magnitude **F** act in the opposite direction of each other while being separated by a perpendicular distance **d**. This produces a pure turning effect about every reference point in the x-y plane:

$$M = F \times d \tag{2.8}$$

A schematic of a couple is shown in Fig. 2.8

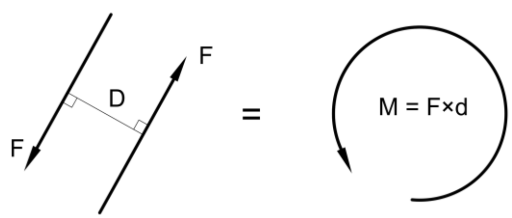


Fig. 2.8 – Definition of a couple

*Note: Like moments, a **torque** is also a form of rotational force. A torque is a special case of a moment which acts through the axis of a prismatic object.*

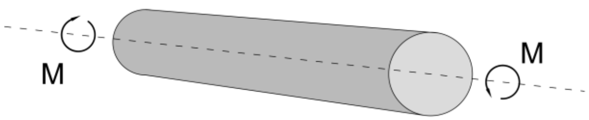


Fig. 2.7 – A moment acting about the axis of a prism, often referred to as a torque.

Equations of Equilibrium:

To consider translational and rotational effects produced by forces, Newton's first law must be extended to include both translational and rotational equilibrium in order to be applicable to two-dimensional systems. Translational equilibrium requires the sum of all forces to equal zero in both the x- and y- directions so that there is no net translational acceleration:

$$\sum F_x = 0 \tag{2.9}$$

$$\sum F_y = 0 \tag{2.10}$$

Rotational equilibrium in the x-y plane also requires that the sum of all moments be equal to zero so that there is no rotational acceleration:

$$\sum M = 0 \tag{2.11}$$

Equations (2.9) to (2.11) are collectively referred to as the ***equations of equilibrium***. For a system which is in equilibrium, these equations are always satisfied regardless of the choice of coordinate system and reference point used to calculate rotational equilibrium.

Equilibrium of Forces which Meet at a Point:

A special case of equilibrium is a system of forces which meet at a point, like the five forces shown in Fig. 2.9. Because each force passes through a common point, rotational equilibrium is guaranteed because the moment produced by each force about the point of intersection is equal to zero. Hence, only the two translational equations of equilibrium, Eq. (2.9) and (2.10), need to be satisfied for the system to be in equilibrium.

Note: Equilibrium of a series of forces which meet at a point can be visualized by graphically rearranging the force vectors so that the tail of one vector connects with the tip of another. If a closed path can be formed by re-arranging the forces in this manner, the system is in equilibrium.

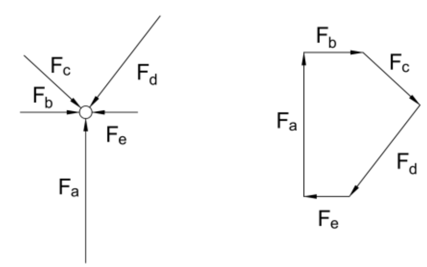


Fig. 2.9 – Five forces which meet at a point. They are in equilibrium because the force vectors can be rearranged to form a closed path. Note their lengths are proportionate to their magnitude.

Frictionless Pulleys:

Pulleys used together with cables are one of the simplest systems used in structural engineering. Consider the circular pulley with radius ***r*** shown in Fig. 2.10 which supports a rope being pulled with a force $\mathbf{T_1}$ on the left and $\mathbf{T_2}$ on the right. When the system is in equilibrium, the pulley will not rotate and hence the sum of moments must equal to zero. If it is assumed that there is no friction in the system, the moments produced by $\mathbf{T_1}$ and $\mathbf{T_2}$ about the centre of the pulley, $\mathbf{M_1}$ and $\mathbf{M_2}$ respectively, can be calculated as:

$$M_1 = T_1 \times r, \text{ acting counterclockwise}$$

$$M_2 = T_2 \times r, \text{ acting clockwise}$$

Taking the sum of all moments and setting them to equal to zero yields the following result:

$$\sum M_o = M_1 + M_2 = T_1 \times r - T_2 \times r = 0 \rightarrow T_1 = T_2 \tag{2.12}$$

Thus, the tension carried by a wire remains constant as it goes around a frictionless pulley. The pulley only serves to redirect the tension force carried by the wire.

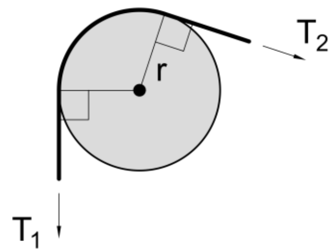


Fig. 2.10 – Free body diagram of an ideal pulley

Dimensions and Units:

In engineering, calculations are done using physical quantities which have units. In order to carry out these calculations correctly, it is necessary to become comfortable working with units and converting between them as needed.

It is important to distinguish between dimensions and units. ***Dimensions*** refer to the measurable physical quantities which describe a property – for example, the dimension of velocity is distance/time. ***Units*** are the means to describe dimensions according to some sort of standard reference. Using our example from before, common units used to measure velocity are metres per second (m/s) or kilometres per hour (km/h).

When performing calculations with physical properties, note the following rules:

1. Two quantities which are added together or subtracted from one other must have the same units
2. Quantities which are multiplied together or divided from each other will have their units multiplied or divided accordingly. For example, a velocity in m/s multiplied by a time in s will result in a distance in units of m.

In engineering, having a sense of what each unit means is necessary to interpret the correctness of a calculation and avoid unrealistic answers. Measurements of length and area are the most intuitive because of our experience working with objects and space in the real world. Units for weight and pressure are generally more difficult to visualize, but can still be interpreted using the following simple examples:

- 1 N is approximately the weight of a small apple
- 1 kN is approximately the weight of a football player
- 1 MPa is approximately the pressure applied to a notebook carrying the weight of an African bush elephant

Table 2.1 contains a list of common conversions which will occur in the course, and is reproduced in **Appendix E**.

Note: In CIV102, calculations will primarily be done using units of N or kN for forces, mm or mm^2 for geometric properties, MPa for stresses and kNm for bending moments. Calculations can be consistently done by working exclusively with units of mm, N and MPa.

Table 2.1 – Sample Unit Conversions

Working with SI units		
Lengths, Strains and Curvatures	Pressures and Stresses	Forces and Moments
1 m = 1,000 mm 1 m^2 = 10^6 mm^2 1 m^3 = 10^9 mm^3 1 mm/m = 10^3 mm/mm 1 rad/m = 10^6 mrad/mm	1 Pa = 1 N/m^2 1 kPa = 1 kN/m^2 1 MPa = 1 MN/m^2 1 MPa = 1 N/mm^2	1 kN = 1,000 N 1 MN = 10^6 N 1 Nm = 1,000 Nmm 1 kNm = 10^6 Nmm
Working with other unit systems and other miscellaneous quantities		
1 foot = 12 inches 1 cubit = 18 inches 1 yard = 3 feet 1 chain = 22 yards 1 furlong = 10 chains 1 mile = 8 furlongs 1 mile = 1,760 yards 1 acre = 10 square chains 1 square mile = 640 acres 1 ha = 10,000 square m	1 inch = 25.4 mm 1 foot = 304.8 mm 1 mile = 1609 m 1 ha = 2.47 acres 1 kg = 2.20 lbs 1 stone = 14.0 lbs 1 lbs/ ft^3 = 16.02 kg/ m^3 100 lbs/ft^3 = 15.72 kN/m^3 1 N = 0.225 lbs (force) 1 kip = 4.45 kN	9.81 m/s^2 = 32.2 feet/s^2 1 kNm = 0.738 kip ft 1 kNm = 8.85 kip in 1 hp = 746 Watt 1 km/h = 0.278 m/s 1 km/h = 0.621 miles/h 1 knot = 1.852 km/h 1 MPa = 145.0 psi 1 kN/m^2 = 20.9 lbs/ft^2

Lecture 3 – Building Bridges

Overview:

In this chapter, the history of bridges is briefly discussed before the topic of suspension bridges is examined in more detail. The mechanics of how cable structures carry load is explained by using the concepts covered in Lecture 2.

Building Bridges:

Bridges are structures which cross over an obstacle such as a river, road, or cliff and hence connect two locations which would otherwise be separated. The earliest bridges were used to cross over rivers and were built by simply felling a large tree and positioning the trunk over the water to span the distance between the two banks. More elaborate bridges were used by the Romans, who crossed over larger rivers by driving wood pieces into the riverbed and using them to support the longer deck, like the bridge shown in Fig. 3.1. Some examples of modern types of bridges are truss bridges, which are fabricated from many steel or timber pieces arranged in a lattice-like configuration, suspension bridges which use steel cables to support a deck over long distances, and arch bridges built from stone or reinforced concrete. Many bridge systems will be introduced and discussed in further detail throughout the course, beginning with suspension bridges in this lecture.

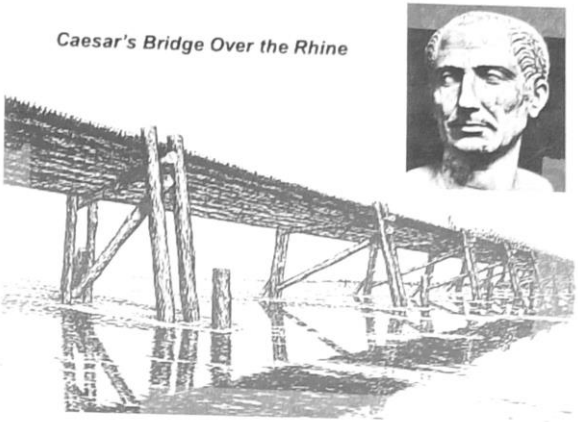


Fig. 3.1 – Caesar's Bridge over the Rhine

Suspension Bridges:

Suspension bridges, like the bridge designed by Thomas Telford shown in Fig. 3.2, use long cables carrying significant tension forces to carry loads across their span. Improvements in construction methods and the increasing quality of steel cables has meant that many of the longest bridges in the world today are suspension bridges. Despite these advancements, the underlying mechanics of how these structures work remains grounded in the basic principles covered in the previous chapter.

Fig. 3.2 – Thomas Telford's Wrought-Iron Suspension Bridge Across Menai Strains, in Wales
177 m span.

Cable Forces in Suspension Bridges

For cable structures which carry hanging loads, the shape of the cables depends on how the loads are distributed. A cable which is not carrying any loads except for its own self-weight is shown in Fig. 3.3. The shape that the cable takes in this situation is called a ***catenary*** and is an important parameter when designing cable structures whose loading is dominated by the self-weight of the cables. A common example of such structures are power lines used to transmit electricity over long distances.

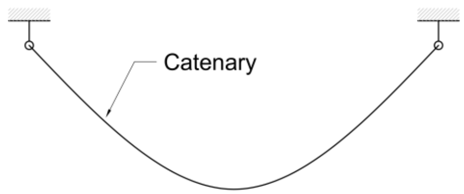


Fig. 3.3 – A hanging cable forming the shape of a catenary under its own weight

When a load which is significant compared to the self-weight of the cable is hung at the midspan, the cable will change shape to form two straight lines. Having three weights will result in four straight lines, five weights will result in six straight lines, and so on. This progression is shown in Fig. 3.4, which shows how the shape of a hanging cable changes as the number of weights hung from it increase from one, to three, to five. Although the shape of the cable remains piecewise linear, the straight segments begin to approximate a smooth curve as the number of weights is increased. If the weights remain constant in value and the spacing between them approaches zero, the load is said to be ***uniformly distributed*** along the length of the structure. When this happens, the slope of the cable will vary linearly along the span and hence assume the shape of a ***parabola***.

Note: Although the shape of a cable under its own weight (a catenary) and a cable supporting a uniformly distributed load (a parabola) appear similar, the shapes are subtly different because of the differences in the loading. For a freely hanging cable, the load per unit length of the cable is constant, whereas for a cable supporting a uniform load, the load per unit length of the span is constant. Because the cable follows a curved profile, the self-weight of the cable is not constant along the length of the span, which results in the shape of the catenary.

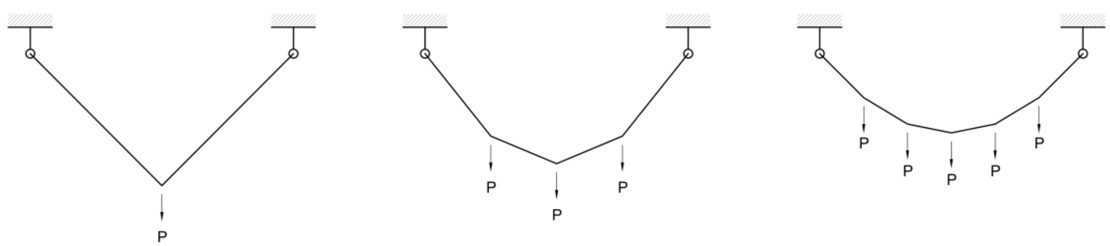


Fig. 3.4 – Change in cable shape as weights are added

The tension in the cable at any location along the span can be determined by drawing a free body diagram. This is done by drawing the free body diagram so that it cuts through the structure at the location of interest. Because the original structure was in equilibrium, the resulting substructures are also in equilibrium, and hence the three equations of equilibrium must be satisfied for each free body diagram. This is illustrated in Fig. 3.5, where the tension in the cable between the first and second loads from the left is investigated by separating the structure at this location. Since the left and right substructures must both be in equilibrium, either may be used to solve for the unknown tension in the wire, which has a vertical component of 3/2P.

Note: When drawing a free body diagram by cutting through a member, the internal forces which were carried by the member at the cut must be drawn into the resulting free body diagrams. This is because the internal forces are necessary for satisfying equilibrium of the structure at that point.

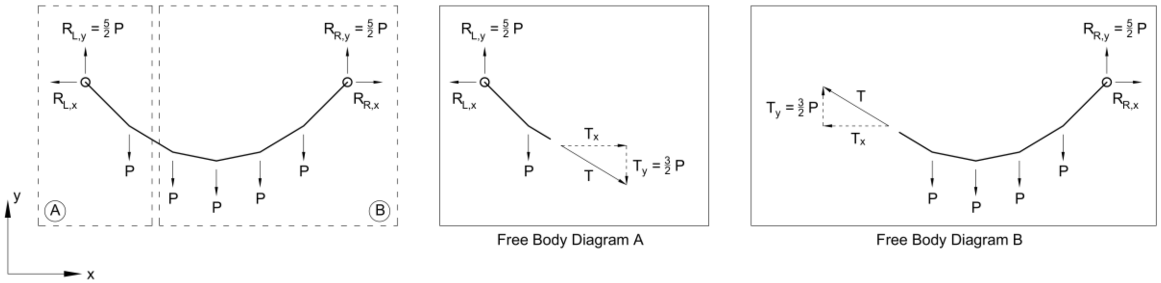


Fig. 3.5 – Analysis of tensile forces in a cable structure

If a series of free body diagrams are drawn, like in Fig. 3.6, the variation in the tension force in the cable can be determined. Due to symmetry, only three free body diagrams need to be drawn to solve for forces in the six straight segments of the structure, which are the forces at locations A, B and C.

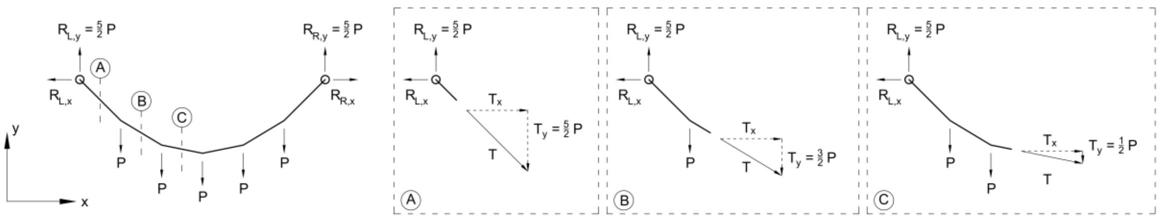


Fig. 3.6 – Variation of tension force in the cable

Determining the vertical component of the tension is a straightforward task if the vertical force carried by the two supports on the ends is known. They can be found by applying the three equations of equilibrium to a free body diagram of the whole structure, which results in reaction force of 5/2P on each side in this example. Once these support forces are known, equilibrium in the vertical direction requires that the vertical component of tension in the cable vary from a maximum of 5/2 P at the support, to a minimum of 1/2P at the midspan. If the spacing of the weights was reduced to approach zero, i.e., the load be uniformly distributed, the vertical component of force would reduce linearly from a maximum at the support to 0 at the midspan.

The horizontal component of force in the cable can be obtained by using the other two equations of equilibrium. It should be noted however that in each of the free body diagrams shown in Fig. 3.6, the horizontal component of tension at the location of interest equals horizontal force supplied by the support. This means that the horizontal component of tension remains constant along the structure.

For a cable carrying tension, its inclination is a function of the relative size of its horizontal and vertical components. Under uniform loading, the vertical component of force in the cable varies linearly along its length while its horizontal component remains constant. Therefore, the slope of the cable will also vary linearly along its span, which results in the cable taking the shape of a parabola.

Note: Recall Fig. 2.4 which is reproduced below as Fig. 3.7. The slope of the force, and hence the cable, is equal to F_y divided by F_x.

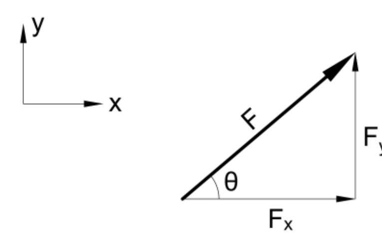


Fig. 3.7 – Components of a force

Lecture 4 – Design of a Suspension Bridge

Overview:

In this chapter, the equations of equilibrium are used to derive the cable forces in a uniformly loaded suspension bridge, which are then used for design. The design of the Golden Gate Bridge is presented as a real-life example of such calculations.

Span: the horizontal distance between the two supports of a bridge.

Drape: The vertical distance between the highest point and lowest points of the bridge.

Reaction Force: The force supplied by a support which holds the structure in equilibrium.

Analysis of Suspension Bridges:

Consider the suspension bridge shown in elevation view in Fig. 4.1 which has a span ***L*** and a drape ***h***. The main cables support the loads carried by the deck, which is attached to the main cables using secondary hanger cables. The bridge supports a uniformly distributed load ***w*** which has units of force per unit length (i.e. kN/m). Using a free body diagram of the whole structure, the vertical reaction forces provided by the towers at the two ends of the bridge can be determined by considering vertical and rotational equilibrium about the centre of the bridge:

$$\sum F_y = 0 \rightarrow R_{L,y} + R_{R_y} - wL = 0 \tag{4.1}$$

$$\sum M_{\text{midspan}} = 0 \rightarrow -R_{L,y} \times \left(\frac{L}{2}\right) + R_{R,y} \times \left(\frac{L}{2}\right) = 0 \tag{4.2}$$

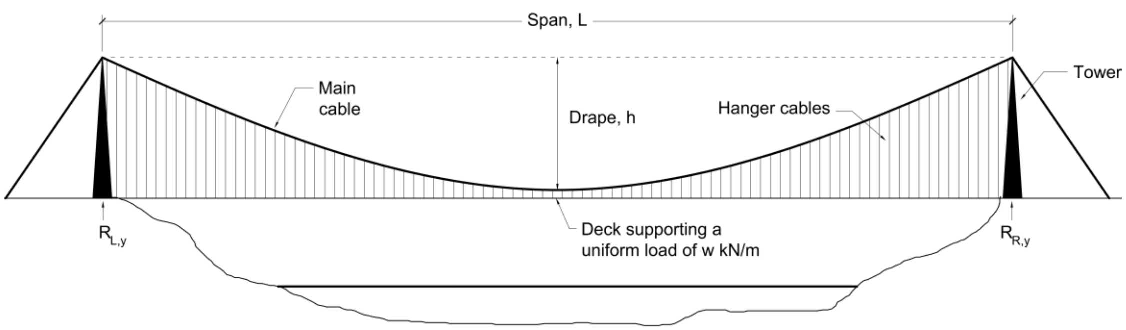


Fig. 4.1 – Elevation (side) view of a suspension bridge

Equations (4.1) and (4.2) form a system of two equations and the two unknown reaction forces $\mathbf{R_{L,y}}$ and $\mathbf{R_{R,y}}$. Solving for the two unknowns yields the following result, that each support carries half of the total load on the bridge:

Note: The result shown in Eq. (4.3) was rigorously proven using the equations of equilibrium. The same result could have been obtained by considering symmetry of the system.

$$R_{L,y} = R_{R,y} = \frac{wL}{2} \tag{4.3}$$

Once the forces in the piers are known, we can also use the equations of equilibrium to learn more about the forces in the cables. Consider a free body diagram of half of the bridge, taken from the left support to the midspan, which is shown in Fig. 4.2.

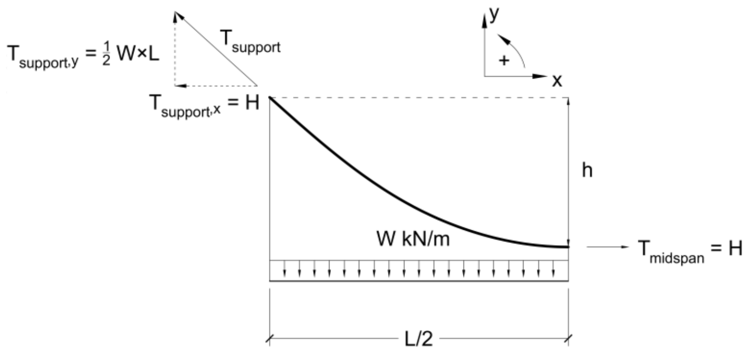
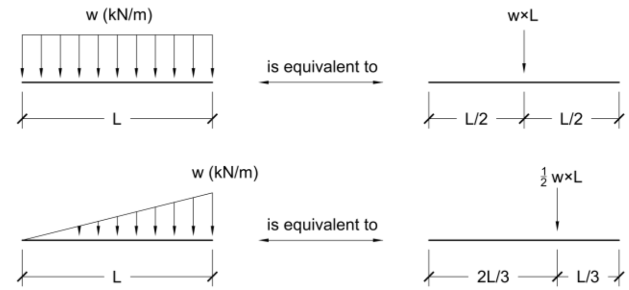


Fig. 4.2 – Analysis of forces in a cable. Note that $T_{midspan}$ has no y-component because the cables are horizontal at midspan.

*Note: A distributed force **w** acting over a length **L** can be replaced by an equivalent point load which has the same magnitude and acts through the centroid of the distributed load. Some common examples are shown below:*



Applying the three equations of equilibrium to the free body diagram yields the following results:

$$\sum F_x = 0 \rightarrow -T_{\text{support, x}} + T_{\text{midspan}} = 0 \tag{4.4}$$

$$\sum F_y = 0 \rightarrow T_{\text{support, y}} - w \times \frac{L}{2} = 0 \tag{4.5}$$

$$\sum M_{\text{support}} = 0 \rightarrow T_{\text{midspan}} \times h - \left(w \times \frac{L}{2}\right) \times \frac{L}{4} = 0 \tag{4.6}$$

Note: In Eq. 4.6, the moments are calculated about the top-left corner of the free body diagram (i.e. at the support). The results of the derivation would not be affected if the moments were instead calculated about any other location.

From Eq. (4.4), we can conclude that the horizontal component of tension, **H**, is constant in the cables. Furthermore, in Eq. (4.5), the vertical component of tension, **V**, is highest at the support and reduces to zero at the midspan. These observations are consistent with the results discussed in the previous chapter. Therefore:

$$T_{\text{y, max}} = T_{\text{support, y}} = \frac{wL}{2} \tag{4.7}$$

Finally, re-arranging for H in Eq. (4.6) yields the following important result:

$$H = \frac{wL^2}{8h} \tag{4.8}$$

The maximum tension in the cables can be determined by calculating the net force from the vertical and horizontal components:

$$T_{max} = \sqrt{\left(T_{\text{x, max}}\right)^2 + \left(T_{\text{y, max}}\right)^2} = \sqrt{\left(\frac{wL^2}{8h}\right)^2 + \left(\frac{wL}{2}\right)^2} \tag{4.9}$$

*Note: The tension in the cables is lowest at the midspan of the bridge because $T_y = 0$ there. At this location, the tension in the cable is simply equal to **H**. Furthermore, the forces discussed in this chapter are the sum of the tensile forces carried by each main cable. Because there are usually two main cables in a suspension bridge, these forces should be divided by two when designing each individual cable.*

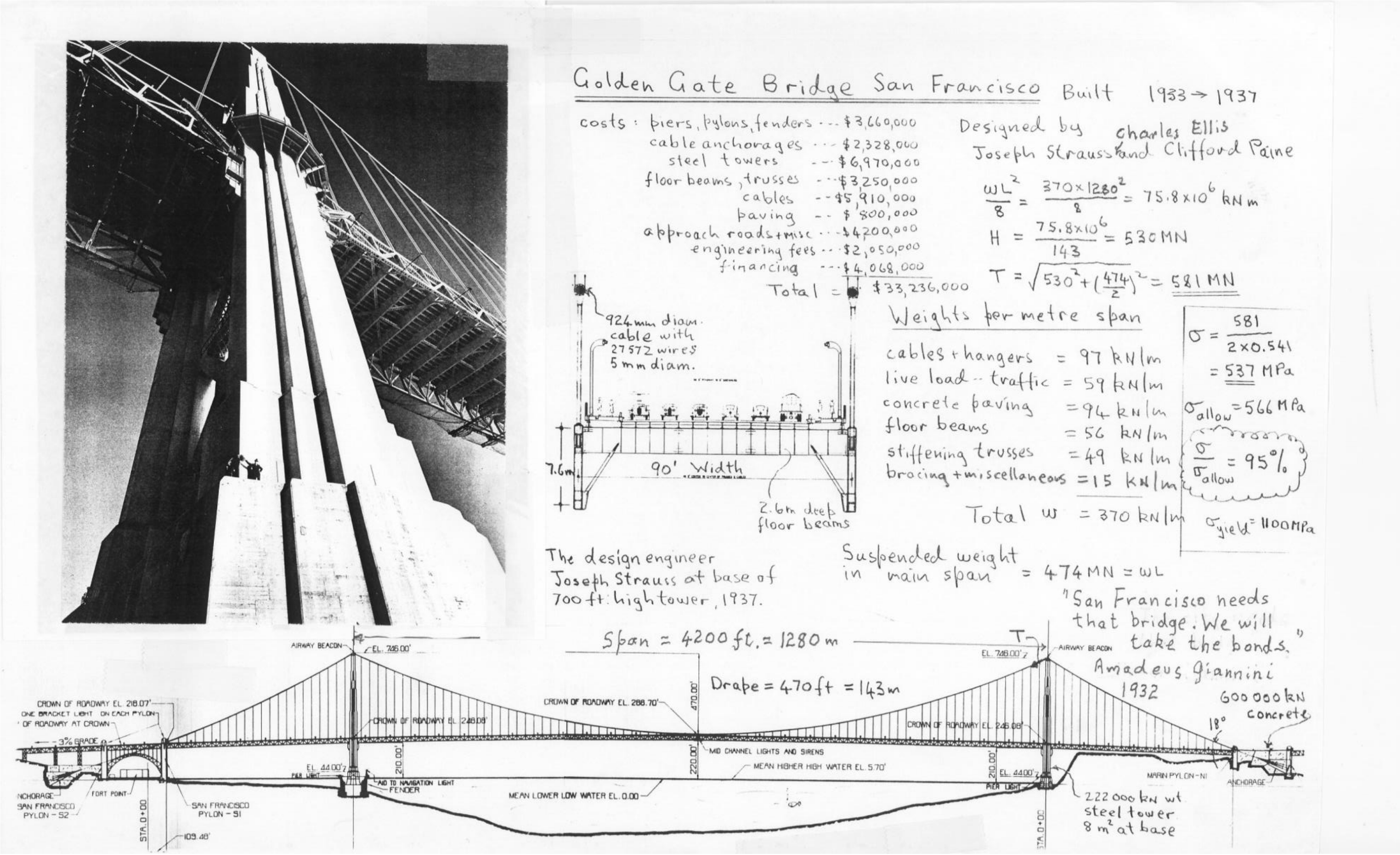


Fig. 4.3 – Design Calculations for the Golden Gate Bridge

Lecture 5 – Stress, Strain, Hooke's Law and Young's Modulus

Overview:
In this chapter, the basics of material behaviour are introduced. Hooke's law for linear elastic springs is discussed for simple structures subjected to tension or compression forces. After introducing the concepts of stress and strain, the Young's Modulus is introduced for relating the two for linear elastic materials. The spring constant for a member is demonstrated to be a product of its material stiffness and geometry.

Hooke's Law:
In Robert Hooke's 1678 paper "Explaining the Power of Springing Bodies" he states, "The particles therefore that compose all bodies I do suppose to owe the greatest part of their sensible or potential extension to a vibrative motion." He suggested that the particles might vibrate back and forth one million times a second and protect their natural space.

After studying the behaviour of materials and springs, Hooke presented his findings as an anagram, "ceiiinosssttuv", which when decoded spells out "ut tensio, sic vis". This is a Latin phrase which translates to "as the extension, so the force". Mathematically, Hooke's law explains that the restoring force in a spring, **F**, is proportionate to its change in length, **Δl**, by a constant **k**:

Note: Although Hooke's law refers to the behaviour of springs, it is applicable to any structure which is subjected to direct tension or compression, such as a cable or a column.

$$F = k\Delta l \tag{5.1}$$

In Eq. (5.1), the spring stiffness **k** has units of force per unit length, such as N/mm. A structure which obeys Hooke's law is said to be ***linear elastic***.

*Note: Because **F** and **Δl** are defined as the force and change in length of in the direction of the axis of a prismatic member, **k** is sometimes referred to as the **axial stiffness** of a member.*

Hooke's Law for Linear Elastic Materials – Stress, Strain and Young's Modulus:
The spring constant of a member depends on how it is shaped, and what material was used to make it. For example, a thin wire is easier to stretch than a thick wire made from the same material, and a rope made from a stiff material like steel is more difficult to stretch than a similarly shaped rope made from a softer material like plant fibre. To understand how the geometry of a member and its material composition individually contributes to its overall stiffness, we will introduce the concepts of stress and strain.

Stress is a physical quantity which describes the internal forces acting on a material. For a force **F** which is carried by a prismatic member with an undeformed cross-sectional area **A**, like the situation shown in Fig. 5.1, the engineering stress **σ** is defined as:

Note: Although the definitions of stress and pressure appear to be similar, pressures refer to forces which are externally applied to a body (i.e. pressure applied to a surface), whereas stresses refer to internal forces which are carried by a structure (i.e. stress in a cable) or a material (i.e. stress in steel).

$$\sigma = \frac{F}{A} \tag{5.2}$$

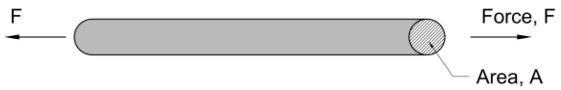


Fig. 5.1 – Terminology used to define stress

Stress has dimensions of force per unit area and is typically described in units of MPa (MN/m^2 or N/mm^2). Because of this, stress can be thought of as the normalized force per unit area experienced by a material.

Strain is a physical quantity which describes how much a material is being deformed. For a prismatic member with original length $\mathbf{L_o}$ being elongated by a length $\mathbf{\Delta l}$, like the situation shown in Fig. 5.2, the engineering strain $\boldsymbol{\varepsilon}$ is defined as:

$$\varepsilon = \frac{\Delta l}{L_o} \tag{5.3}$$

*Note: The **engineering stress** and **engineering strain** are defined using the undeformed geometry of the member. In reality, a member's cross-sectional area and length will change when it is carrying load. The **true stress** and **true strain** are the corresponding definitions of stress and strain when the deformed geometry is used instead. Although the true stress and true strain are more realistic indicators of a material's physical state, they cannot be easily measured, so the engineering definitions are used instead.*

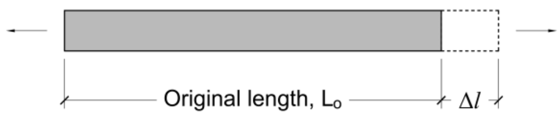


Fig. 5.2 – Terminology used to define strain

Although strain is dimensionless, it is typically described in units of mm/mm, mm/m or even %. Because of this, strain can be thought of as the normalized change in length experienced by a material.

The benefits of using stress and strain to describe the forces and deformations in a material, instead of simply using the force and displacement of a structure, is because it allows the behaviour of structures to be compared even if they are different sizes. For example, a thin wire will intuitively break at a lower *load* than a thicker wire, but failure will occur at the same *stress* if they are both made from the same material.

Just like how Hooke's law states that the force and stretch of a spring are related by a constant, the stress felt by a linear elastic material is proportionately related to its strain by a constant **E**:

$$\sigma = E\varepsilon \tag{5.4}$$

*Note: Eq. (5.4), which relates the stress and strain in the material, is sometimes referred to an example of a constitutive relationship. Because E is a characteristic property of material which relates stress and strain, it is sometimes referred to as the **material stiffness**.*

E is called the ***Young's Modulus***, named after the English scientist Thomas Young. It has the same dimensions as stress and is commonly written in units of MPa. Fig. 5.3 shows the stress-strain relationship for several materials which have different values of **E**.

Note that the strains used in Eq. (5.4) are strains which are associated with the material deforming as it tries to carry stress. Materials may deform for other reasons, and in some circumstances, it may be necessary to distinguish between the strains associated with stress, and the strains caused by other effects. Some examples of strains which do not cause stress and should not be used in Eq. (5.4) are thermal strains caused by temperature effects, or shrinkage strains caused by water loss. Some of these effects are described in later chapters.

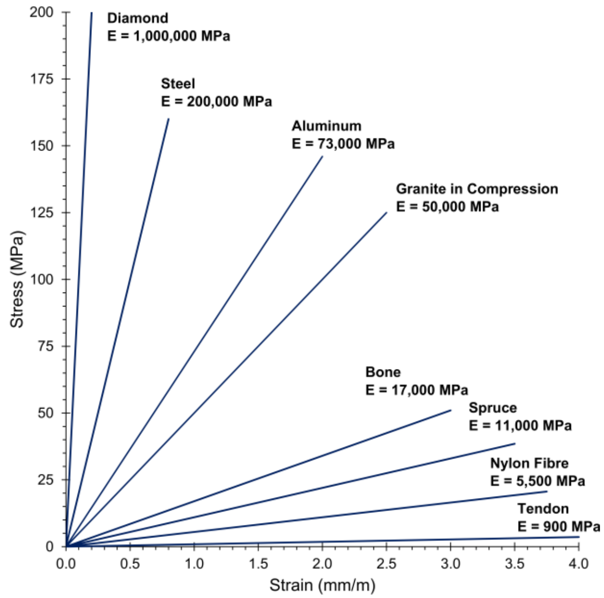


Fig 5.3 – Typical values of E for various materials

Expressing k in Terms of Geometry and Material Stiffness:
Hooke's law for linear elastic springs, Eq. (5.1), and its equivalent for linear elastic materials, Eq. (5.4), resemble each other because they both relate a force-based quantity (**F** or **σ**) and a displacement-based quantity (**Δl** or **ε**) by a stiffness-based quantity (**k** or **E**). Fig. 5.4 shows how all of these quantities are related to each other for linear elastic structures subjected to axial load:

	Force-like quantities	Stiffness-like quantities	Deformation-like quantities
Structures	**Forces, *F*** [N]	**Axial Stiffness, *k*** [N/mm] $\boxed{F = k\Delta l}$	**Displacement, *Δl*** [mm]
	$\boxed{\sigma = \frac{F}{A}}$		$\boxed{\varepsilon = \frac{\Delta l}{L_o}}$
Materials	**Stress, *σ*** [MPa]	**Young's Modulus, *E*** [MPa] $\boxed{\sigma = E\varepsilon}$	**Strain, *ε*** [mm/mm]

Fig. 5.4 – Relationships between structures and materials

If the geometric properties **A** and **L_o**, and material stiffness **E**, are known, then **k** for a member can be calculated by combining equations (5.2) to (5.4) and isolating for **F** and **Δl**. This results in the following relationship:

$$F = \frac{AE}{L_o}\Delta l \rightarrow k = \frac{AE}{L_o} \tag{5.5}$$

Therefore, the axial stiffness of a member **k** is proportionate to its cross-sectional area **A** and material stiffness **E**, and inversely proportionate to its length **L_o**.

*Note: **k** will have units of N/mm if **A** is in mm^2, **E** is in MPa and **L_o** is in mm.*

Lecture 6 – Stress-Strain Response, Resilience, Toughness & Ductility

Overview:

In this chapter, key material properties which are used in the design of structures are discussed. The complete stress-strain relationship for mild steel, a common material used in many steel and reinforced concrete structures, is presented.

Generalized Stress-Strain Behaviour:

As materials are loaded to failure, they generally do not exhibit linear elastic behaviour for their entire life. Gradual accumulation of damage to the microstructure of the material and other material-specific internal effects mean that the stress-strain curve is generally nonlinear. Even materials which look and feel similar may have very different stress-strain properties. For example, Fig. 6.1 shows the stress-strain behaviour of three different types of steel whose stress-strain behaviour differs greatly due to the amount of carbon present.

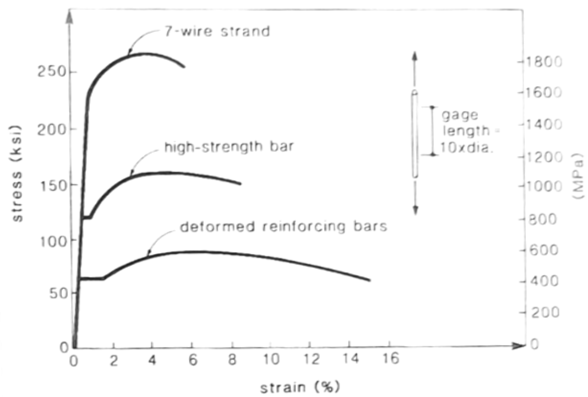


Fig. 6.1 – Stress-strain behaviour of different types of steel

To describe key features of a material's stress-strain curve, engineers have defined ***material properties*** which serve as a useful tool for evaluating and comparing different materials. Common aspects of a material which are described by material properties include its weight, strength, stiffness, ductility and energy absorption capabilities. Fig 6.2, which shows the stress-strain curve of mild steel, illustrates many of the various material properties which are described below:

The ***strength*** of a material describes how much stress it can carry before failure occurs. Multiple definitions of strength exist to recognize the various stages of failure which a material experiences as it is loaded. The ***yield strength*** is defined as the stress which causes yielding to occur. The ***ultimate strength*** is defined as the largest stress which the material experiences before failure. The strength of many materials in tension is different than their strength in compression.

Yielding: the state when a material, usually metals, begins to accumulate permanent deformations. When yielding begins, the strain will continue to increase even if the stress is held constant. The portion of the stress-strain curve which exhibits this behaviour is sometimes referred to as the ***yield plateau.***

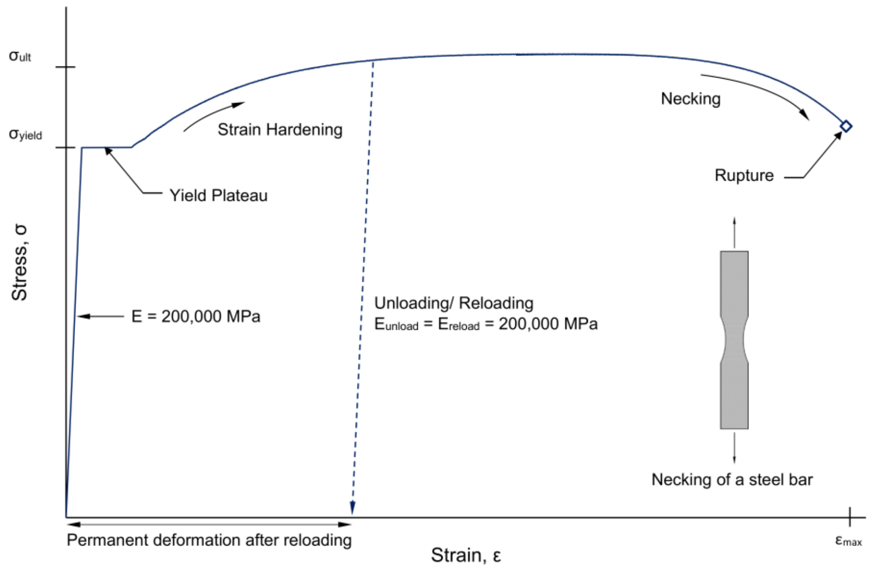


Fig. 6.2 – Stress-strain curve of mild steel in tension

Strain Hardening: the phenomenon where a material gains strength and stiffness when strained beyond its yield point.

Necking: the phenomenon where localization of tensile strains in a material cause the cross-sectional area to become noticeably smaller at one location and resembles the shape of a neck. Usually precedes failure.

*Note: Deformations accumulated when a material is no longer behaving in a linear elastic manner tend to be permanent for many materials. These non-recoverable deformations are sometimes referred to as **plastic deformations** to contrast with the recoverable **elastic deformations**.*

The concept of ***ductility*** has several definitions, but generally refers to much a material can be deformed before it breaks. On a stress-strain curve, this refers to the largest strain a material can carry before fracturing. Materials which are able to sustain significant amounts of permanent deformation before failing are generally referred to as being ***ductile***, while those which cannot are referred to as being ***brittle***.

The slope of the linear elastic region of the stress-strain curve of a material is the ***Young's Modulus***, **E**. Materials which have a large Young's Modulus are generally referred to being ***stiff***, while those with a small Young's Modulus are called ***flexible***. Many materials such as steel tend to follow the slope of the linear elastic region when they are unloaded or reloaded, even after permanent deformations have occurred.

The complete stress-strain behaviour of mild steel (sometimes referred to as low-allow steel) can be roughly described as having three phases. For small strains, steel behaves in a linear elastic manner, and hence the stress and strain are related by the Young's Modulus. Once the stress reaches the yield stress, the material exhibits plastic behaviour along its yield plateau. For even larger strains, the stress-strain relationship is nonlinear, with some strengthening due to strain hardening, followed by softening as necking begins.

Strain Energy:

Strain energy is the energy stored in a structure or material as it is deformed. The strain energy **W** is defined as the area underneath the force-displacement curve of a structure, which is mathematically represented as:

Note: Eq. (6.1) is the integral of the force carried by the structure over the change in length it has experienced.

$$W = \int F d\Delta l \tag{6.1}$$

When a material is loaded while it is behaving in a linear elastic manner, the area underneath the force-displacement curve is a triangle. Thus, the strain energy for a material with axial stiffness **k** which has been elongated by **Δl** from its original length is calculated as:

$$W = \frac{1}{2}F\Delta l = \frac{1}{2}k(\Delta l)^2 \tag{6.2}$$

The area underneath the stress-strain curve of a material also represents an energy-based quantity, which is the strain energy density **U:**

$$U = \int \sigma d\varepsilon \tag{6.3}$$

U can be thought of as the energy stored in the material per unit volume and is typically expressed using units of MJ/m^3. The strain energy density, **U**, is related to the strain energy, **W**, by the following equation:

$$W = U \cdot V_o \tag{6.4}$$

Where $\mathbf{V_o}$ is the original volume of the member before it has been deformed. When a material is behaving in a linear elastic manner, the area underneath the stress-strain curve is a triangle, which results in an alternative equation for the strain energy **W** if Eq. (6.3) and (6.4) are combined:

$$W = \int \sigma d\varepsilon \cdot V_o = \frac{1}{2}\sigma\varepsilon V_o \tag{6.5}$$

Having defined what strain energy is, we can now define the resilience and toughness of a material: the maximum amount of energy which a structure or material can absorb before it exhibits permanent deformations is defined as its ***resilience***. The resilience of a material is calculated as the area under the stress-strain curve in the linear-elastic region. The ***toughness*** of a structure or material is a measure of how much energy it can absorb before breaking. The toughness of a material is hence defined as the area underneath the complete stress-strain curve.

Note: The strain energy density has units of MJ/m^3 if the stress and strain are in units of MPa and mm/mm respectively. This is because 1 MPa = $1MN/m^2$ = $1MNm/m^3$ = $1\ MJ/m^3$.

Note: For a prismatic member, the undeformed volume V_o can be expressed as the product of the undeformed length L_o and the cross-sectional area A. Eq. (6.5) can then be rewritten as:

$$W = \frac{1}{2}\sigma\varepsilon A L_o$$

W will have units of J if σ is in MPa, ε is in mm/mm, A is in mm^2 and L_o is in m.

Thermal Expansion:

Materials tend to expand when heated and contract when cooled; the rate at which this occurs is a unique property of every material. The thermal strains experienced by a material, $\boldsymbol{\varepsilon_{th}}$ are related to the change in temperature **ΔT** by the ***coefficient of thermal expansion*** **α** according to the following equation:

$$\varepsilon_{th} = \alpha\Delta T \tag{6.6}$$

For example, if a 1200 mm long rod made of low alloy steel, which has $\alpha = 12\times10^{-6}$ /°C, was heated by 30°C, then it would experience a thermal strain of $(12\times10^{-6}) \times (30) = +\ 0.0036$ mm/mm, which corresponds to an elongation of 0.432 mm. Thermal strains can be significant for large structures – for example, large suspension bridges can change length by a few metres under large variations in temperature.

Note: If a member is free to expand or contract, then thermal strains do not lead to stresses developing in the material. However, there is some sort of restraint which prevents it from changing sizes, then stresses will begin to develop, and the material may fail. One example where this happens is if a glass container with water inside is placed into a freezer. Under the cooler temperature, the contain shrinks around the water, which is instead expanding as it freezes. Because the ice is preventing it from contracting, stresses begin to accumulate in the glass and it may shatter.

A table of useful material properties for many materials is found below, and is also reproduced in **Appendix A**.

Table 6.1 – Common Material Properties

Average Properties of Some Typical Materials Note that except for density, stiffness and coefficient of thermal expansion, all values have a considerable range

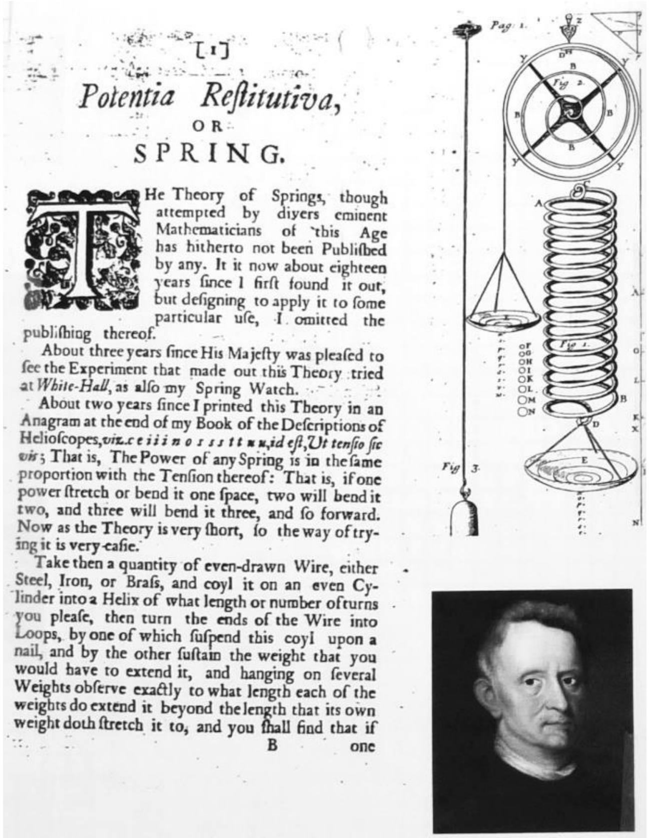
Material	Weight (kN/m^3)	Stiffness E (MPa)	Tensile Strength (MPa)		Compressive Strength (MPa)	Resilience (MJ/m^3)	Toughness (MJ/m^3) tens./comp.	Ductility Max. Elong. (%) Plastic/Elastic	α $10^{-6}/^{\circ}C$	Cost $/kg	Comment
			Yield	Ultimate							
Low Alloy Steel	77	200,000	420	560	420	0.44	135	25/0.21	12	0.60	Used in buildings, bridges, cars, etc.
High Tensile Steel	77	200,000	1650	1860	1650	6.8	55	4/0.83	12	1.50	Wire ropes, cables
High Alloy Steel	77	200,000	700	800	700	1.22	200	25/0.35	12	2.00	Pressure Vessels and tanks
Piano Wire	77	200,000	-	3000	-	22	22	0.2/1.50	12	1.50	Brittle material, not used in structures
Cast Iron	70	150,000	-	110	770	0.04	0.06/6	1/0.7	11	0.50	Traditional cast iron, moulded
Wrought Iron	75	185,000	200	350	200	0.11	90	30/0.11	12	1.00	99% pure iron, hammered, fibrous
Aluminum	27	69,000	40	80	60	0.012	19	40/0.06	24	1.80	Light, ductile, non-corrosive, soft metal
Aluminum Alloy	27	73,000	470	580	500	1.51	50	11/0.64	24	2.50	Used for canoes, aircraft, etc.
Copper	88	124,000	70	230	200	0.02	85	55/0.06	20	7.47	Very ductile metal – rounded curve
Bronze	79	105,000	200	390	350	0.2	40	12/0.19	17	2.80	Tin + copper alloy – stronger
Gold	189	82,000	40	220	180	0.01	80	50/0.05	14	40k	Heavy, expensive metal
Granite	26	52,000	-	11	140	0.001	0.01/0.26	0/0.02	8	0.15	Strongest and most durable building stone
Limestone	25	58,000	-	8	62	0.0006	0.01/0.09	0/0.01	6	0.03	Soft, useful building store
Slate	28	95,000	-	60	100	0.019	0.02/0.10	0/0.06		0.08	Stratified rock with high tensile strength
Brick	19	20,000	-	3	20	0.0002	0.01/0.03	0/0.01	9	0.10	Fired clay
Concrete	24	30,000	-	3	35	0.002	0.01/0.10	0/0.01	9	0.12	Mixture of cement, sand, stone, water
Glass	27	69,000	-	100	200	0.072	0.07/0.8	0/0.15	20	1.50	Solidified liquid sand
Oak	7.5	14,000	75	90	60	0.23	0.3/2.5	0.5/0.47	3	3.2	Strong, tough, heavy hardwood
Spruce	4.4	11,000	55	70	50	0.19	0.2/2.2	0.5/0.50	7	2.0	Light, strong, durable softwood
Tendon	10	900	70	80	-	2.7	4	1/7.8		-	Used as tension ties in mammals
Bone	20	17,000	150	180	180	0.66	1	0.5/0.9		-	Used as struts and beams in mammals
Rubber	9.2	7	-	20	20	15	20	4/300	500	2.0	Strange, useful material – low stiffness
Spider's Silk	10	4,000	-	1400	-	160	170	10/35		-	Most resilient material
Carbon Fibre	15	160,000	-	1800	-	10	10	0.1/1.1		50.0	Carbon fibre composites used in aircraft
Nylon Fibre	11	5,500	-	900	-	74	75	2/16	80	8.00	Excellent if stiffness not required
Kevlar Fibre	14	130,000	-	3600	-	50	60	1/2.7		50.00	Super material in many ways

Lecture 7 – "Explaining the Power of Springing Bodies"

Overview:
Structures may vibrate when subjected to loads which are not stationary, or when disturbed from their equilibrium position. In this chapter, the basic concepts of vibrating spring-mass systems under free vibration are introduced.

Free Vibration of Spring-Mass Systems:
So far, we have primarily considered systems which are in a state of equilibrium and hence do not accelerate. However, structures will generally accelerate when subjected to time-varying loads, or when disturbed from their equilibrium position. Consider the simplest case which was investigated by Hooke (shown in Fig. 7.1), which is a spring carrying a mass. Fig. 7.2, which illustrates his experiment in more detail, shows a linear elastic spring with a stiffness **k** attached to a mass **m**. The mass is considered to be significantly larger than the mass of the spring, which can be considered as weightless. For simplicity, the system will be analyzed without considering gravity, which will be re-introduced later in the chapter.

Suppose the mass **m** is pulled downwards from its resting position and then released. The mass will then vibrate up and down before eventually returning to its resting position. This is called ***free vibration*** and would theoretically continue forever were it not for factors such as air resistance and internal friction. The vertical movement of the mass from its equilibrium position can be mathematically described using the time-varying function **x(t)**, and its vertical acceleration as it vibrates can be defined using another time-varying function **a(t)**. Note that the displacement **x(t)** is measured from the undeformed length of the spring. Both **x(t)** and **a(t)** are positive in the downwards direction.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Potentia Reſtitutiva,
OR
SPRING.

THe Theory of Springs, though attempted by divers eminent Mathematicians of this Age has hitherto not been Publiſhed by any. It is now about eighteen years ſince I firſt found it out, but deſigning to apply it to ſome particular uſe, I omitted the publiſhing thereof.

About three years ſince His Majeſty was pleaſed to ſee the Experiment that made out this Theory tried at *White-Hall*, as alſo my Spring Watch.

About two years ſince I printed this Theory in an Anagram at the end of my Book of the Deſcriptions of Helioſcopes, *viz.* c e i i i n o s s s t t u u, *id eſt, Ut tenſio ſic vis*; That is, The Power of any Spring is in the ſame proportion with the Tenſion thereof: That is, if one power ſtretch or bend it one ſpace, two will bend it two, and three will bend it three, and ſo forward. Now as the Theory is very ſhort, ſo the way of trying it is very eaſie.

Take then a quantity of even-drawn Wire, either Steel, Iron, or Braſs, and coyl it on an even Cylinder into a Helix of what length or number of turns you pleaſe, then turn the ends of the Wire into Loops, by one of which ſuſpend this coyl upon a nail, and by the other ſuſtain the weight that you would have to extend it, and hanging on ſeveral Weights obſerve exactly to what length each of the weights do extend it beyond the length that its own weight doth ſtretch it to, and you ſhall find that if

B one

Fig. 7.1 – Excerpt from Robert Hooke's 1678 paper "Explaining the Power of Springing Bodies"

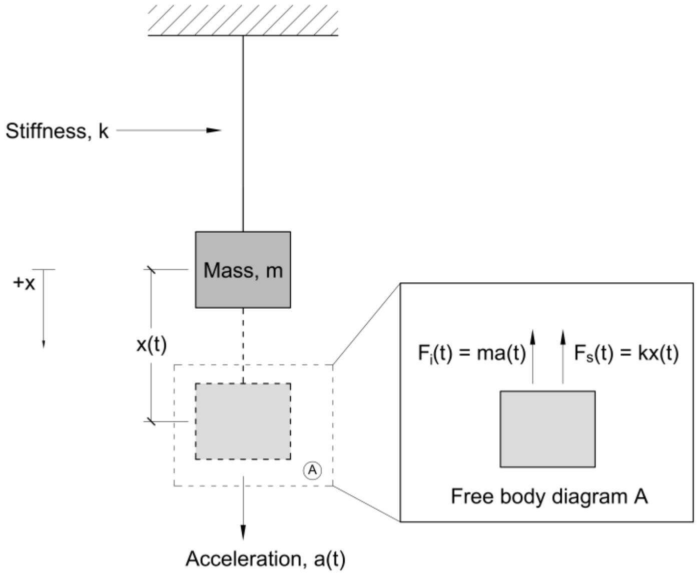


Fig. 7.2 – Analysis of a vibrating spring-mass system

Free body diagram A in Fig. 7.2 shows the forces applied to the mass if it is travelling with both an acceleration and displacement acting downwards. The forces which resist this motion are the inertial force, **F_i(t)** and the spring force **F_s(t)**. As these are the only forces acting on the body, the two forces sum to zero, which produces the following equation:

$$F_i(t) + F_s(t) = 0 \rightarrow ma(t) + kx(t) = 0 \tag{7.1}$$

Note: When the mass is vibrating, the inertial force is its resistance to being accelerated. Using Newton's second law, $F_i = ma(t)$.

The acceleration **a(t)** is defined as the second derivative of the displacement **x(t)** with respect to time. This allows Eq. (7.1) to be written as just a function of **x(t)** and its derivatives:

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0 \tag{7.2}$$

Note: Eq. (7.2) is called a ***differential equation*** *because it relates a function, x(t), to one or more of its derivatives. Solving a differential equation means obtaining the unknown function x(t). In this case, the acceleration a(t) is related to x(t) by:*

$$a(t) = \frac{d^2x(t)}{dt^2}$$

Differential equations such as Eq. (7.2) have a useful property called ***uniqueness****. This means that if a function x(t) satisfies the equation, and other conditions, then it is the only correct solution. The existence and uniqueness of solutions to differential equations is discussed further in calculus courses.*

Eq. (7.2) can be solved by assuming an answer for **x(t)**, and then verifying that our assumed function satisfies the differential equation. Consider the following function which is a sinusoid with amplitude of vibration **A**, angular frequency **ω_n**, and phase shift **ϕ**:

$$x(t) = A\sin(\omega_n t + \phi) \tag{7.3}$$

The second derivative of **x(t)** with respect to time is then:

$$\frac{d^2x(t)}{dt^2} = -A\omega^2\sin(\omega_n t + \phi) \tag{7.4}$$

Substituting Eq. (7.3) and (7.4) into Eq. (7.2) and simplifying yields the following requirement for **ω_n**:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{7.5}$$

The frequency of vibration of the spring-mass system when freely vibrating is related to the stiffness of system **k** and the magnitude of its mass **m** but is independent of other factors such as the amplitude of vibration and the initial disturbance. **ω_n**, is commonly referred to as the ***natural frequency*** of the system because it represents how quickly the system oscillates under free vibration, and is purely defined by the system's inherent mechanical properties **k** and **m**. Stiffer systems, which have a large value of **k**, will hence have a high natural frequency and vibrate quickly. On the other hand, systems with a higher mass have more inertia and will vibrate more slowly.

Note: ω_n has units of rad/s. Expressing the natural frequency in units of cycles per second requires converting ω_n to f_n using Eq. (7.6).

The natural frequency expressed in terms of cycles per second (hz), ***f_n***, and the natural period, **T_n**, can be defined as:

$$f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{7.6}$$

Note: the period T is the time elapsed as one full cycle takes place.

$$T_n = \frac{1}{f_n} = 2\pi\sqrt{\frac{m}{k}} \tag{7.7}$$

Note that when solving Eq. (7.2), we have not determined the values of the amplitude of vibration **A** or the phase shift **ϕ**. These parameters can be solved if the displacement and acceleration corresponding to t = 0 are known (i.e. x(t=0) = x_o and a(t=0) = a_o respectively). x_o and a_o are referred to as initial conditions.

Note: Solving for A and ϕ using the initial conditions, x_o and a_o will not be required in CIV102.

Consideration of Gravity:

Although we neglected the presence of gravity when defining and solving Eq. (7.2), consider the effect of adding the gravitational force to the free body diagram in Fig. 7.2, which acts downwards. The sum of forces can then be written as:

$$F_i(t) + F_s(t) = F_g = mg \tag{7.8}$$

Introducing the displacement **x(t)** and the acceleration **a(t)** results in a slightly modified version of Eq. (7.2):

$$m\frac{d^2x(t)}{dt^2} + kx(t) = mg \tag{7.9}$$

This can be solved using a sinusoid like before which oscillates around a value of x = Δ_o instead of x = 0 like before:

$$x(t) = A\sin(\omega_n t + \phi) + \Delta_o \tag{7.10}$$

Substituting Eq. (7.10) into Eq. (7.9) results in the same equation for $\boldsymbol{\omega_n}$ as before and produces the following condition for $\boldsymbol{\Delta_o}$:

$$\mathrm{k}\Delta_o = mg \tag{7.11}$$

Therefore, the system does not oscillate about the undeformed length of the wire, but instead oscillates about the resting position under the weight of the mass, which is $\boldsymbol{\Delta_o}$, the elongation of the spring due to the gravitational force. The inclusion of gravity does not influence the frequency of vibration, and hence our equations for $\boldsymbol{\omega_n}$, $\boldsymbol{f_n}$ and $\mathbf{T_n}$ are still valid.

A plot of the position of the mass over time, **x(t)**, is shown in Fig. 7.3. When reading the plot, note that downwards displacements are taken as positive (as defined in Fig. 7.2).

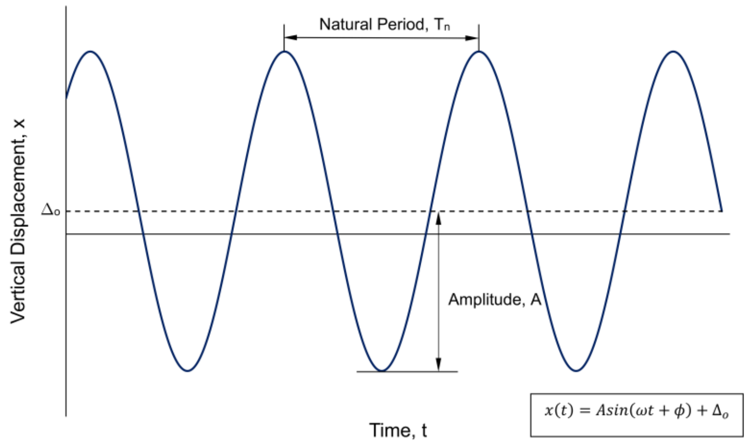


Fig. 7.3 – Displacement of a spring-mass system under free vibration

Note: The response of structures under time-varying or dynamic loads is discussed in Lecture 19.

Other methods for calculating ω_n

The natural frequency $\boldsymbol{f_n}$ is an important parameter which can be used to determine if a structure is susceptible to time-varying loads. Although Eq. (7.6) can be used to calculate it, determining the stiffness **k** can be difficult if the structure has a complex geometry. A more convenient approach is to instead define the natural frequency in terms of the vertical displacement of the structure under gravity loads. This can be done by re-arranging Eq. (7.11) to express **k** in terms of Δ_o:

$$k = \frac{mg}{\Delta_o} \tag{7.12}$$

Substituting Eq. (7.12) into Eq. (7.6) results in the following expression for $\boldsymbol{f_n}$:

$$f_n = \frac{1}{2\pi}\sqrt{\frac{mg}{\Delta_o}\cdot\frac{1}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_o}} \tag{7.13}$$

If the acceleration due to gravity is taken as 9,810 mm/s^2 and Δ_o is in units of mm, then $\boldsymbol{f_n}$ can be calculated as:

Note: Eq. (7.14) requires Δ_o to be in units of mm, otherwise the resulting answer will be incorrect.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9{,}810}{\Delta_o}} \cong \frac{15.76}{\sqrt{\Delta_o}} \tag{7.14}$$

Thus, the natural frequency can be conveniently calculated for a structure if its static displacement, Δ_o is known.

Lecture 8 – Factors of Safety: Dead vs. Live Load, Brittle vs. Tough

Overview:

Although previous lectures have introduced the necessary tools to design structures in idealized conditions, uncertainties in the expected loads on the structure and the strength of the materials used must be considered to avoid failure. This lecture describes the concept of working stress design, which employs factors of safety to carry out design safely.

Dead and Live Loads:

When designing structures to safely carry loads, it is common to distinguish between the different kinds of loads which can be expected. In CIV102, we will primarily focus on dead loads and live loads.

Dead loads are loads which remain constant over the lifetime of the structure. Examples of dead load include the self-weight of the structure, and the self-weight of nonstructural components which are attached to the structure (sometimes referred to as superimposed dead loads).

Live loads are loads which vary over time and are primarily attributed to factors relating to the usage of the structure by people. Examples of live loads include the weight of a crowd of people, the weight of vehicular traffic on a bridge, or the weight of objects which are not permanently attached to the structure such as furniture. The weight of a tightly packed crowd of people is a significant live load, which has traditionally been approximated as 100 lbs per square foot, which converts to a load of about 5 kPa.

Other types of loads which are commonly considered when designing structures include wind loads, snow loads, and earthquake loads.

Structural Failure:

Failure occurs when the stresses in the structure caused by the applied loads, σ_{demand}, equals or exceed the strength of the materials, σ_{capacity}:

$$\sigma_{\text{demand}} \geq \sigma_{\text{capacity}} \tag{8.1}$$

Although this concept is relatively straightforward, consideration must be made to account for uncertainty in the loads, as well as uncertainty in the strengths of the materials used to build the structure. A dangerous situation may occur if the loads are higher than expected and/or the strength of the materials is lower than specified. This variation in the capacity and demand is illustrated in Fig. 8.1, where the two curves represent the probability distributions of the demand (red) and capacity (blue). The height of the curves represents the likelihood of the capacity or demand being a certain value.

Fig. 8.1 describes a situation where the expected capacity exceeds the expected demand on average. However, there is substantial overlap between the two curves due to the variability in the both the applied loads and the strength of the materials. In this overlapping region, Eq. (8.1) is satisfied and failure occurs. Therefore, simply designing so that the expected strength is on average higher than the expected demand is not a sufficient method to ensure that the resulting structure is safe.

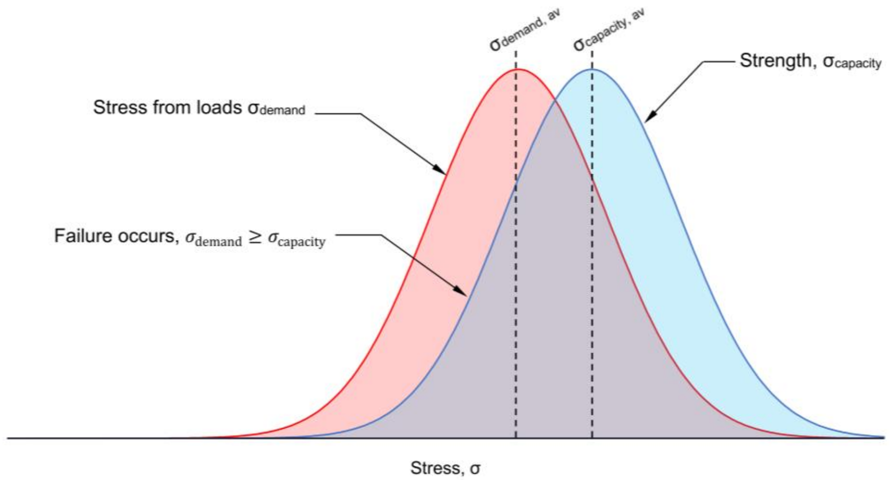


Fig. 8.1 – A comparison of applied stresses vs. the strength of the material. Although the average demand is less than the average capacity, there is substantial overlap between the two curves where failure occurs.

Note: Failure is not a straightforward concept to define, as it depends on the criteria used to determine what constitutes failure. Common metrics used to define failure include when the material begins to experience permanent deformations ($\sigma = \sigma_{yield}$), when the material breaks ($\sigma = \sigma_{ult}$), or when the deformations of a structure exceed an acceptable value ($\Delta \geq \Delta_{max}$).

Factors of Safety

To account for the uncertainty in the loads and strength of the materials, the concept of a ***factor of safety*** (FOS) is used in engineering design. The factor of safety is a measure of the capacity in the system relative to its demand. In structural engineering, the capacity refers to the strength of the materials and the demand refers to the stresses caused by applied loads. If the factor of safety is less than 1.0, then the demand exceeds the capacity and failure occurs.

$$FOS = \frac{\text{Capacity}}{\text{Demand}} \tag{8.2}$$

In practice, factors are safety are employed to reduce the permissible demand in order to reduce the likelihood of failure occurring to an acceptable level. This is known as ***working stress design***, in which the maximum allowable stress in the structure, $\boldsymbol{\sigma_{allow}}$, is calculated as:

$$\sigma_{allow} = \frac{\sigma_{\text{fail}}}{FOS} \tag{8.3}$$

In Eq. (8.3), $\boldsymbol{\sigma_{fail}}$ is the stress which causes the structure to fail. The benefit of using factors of safety is illustrated when comparing Fig. 8.2, shown on the following page, to Fig. 8.1. By employing a factor of safety to limit the stress permitted in the structure to be $\boldsymbol{\sigma_{allow}}$, the area where the two curves overlap has been significantly reduced, which reduces the likelihood of a failure taking place.

Note: Values of the factor of safety are chosen in order to limit the probability of failure to be less than an acceptable value. Using working stress design, the factor of safety depends both on the quality of the material and the danger of the failure mode in question. Larger factors of safety are employed against failure mechanisms which are sudden and cause more catastrophic consequences.

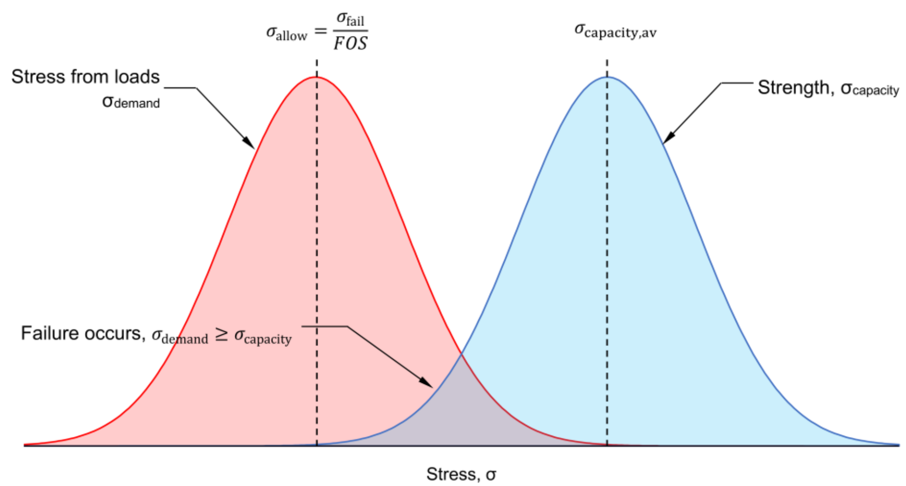


Fig. 8.2 – Reduced likelihood of failure by employing a factor of safety

Professor William Macquorn Rankine of the University of Glasgow gave the following advice about Factors of Safety in his classic "Manual of Civil Engineering" 1862 edition

143. A **Factor of Safety** (*A. M.*, 247) when not otherwise specified, means the ratio in which the breaking load exceeds the working load.

In fixing factors of safety, a distinction is to be drawn between a *dead load*—that is, a load which is put on by imperceptible degrees, and which remains steady, such as the weight of the structure itself, and a *live load*—that is, a load which is put on suddenly, or accompanied with vibration, such as a swift train travelling over a railway bridge, or a force exerted in a moving machine.

Comparing together the experiments of Mr. Fairbairn and of the commission on the strength of iron, and the rules followed in the practice of engineers, the following table gives a fair summary of our knowledge respecting factors of safety :—

	Dead Load.	Live Load.
For perfect materials and workmanship,	2	4
For good ordinary materials and workmanship :—		
in Metals,	3	6
in Timber,	4 to 5	8 to 10
in Masonry,	4	8

Name	Date Completed	Span (m)	σ_u (MPa)	F.O.S	Designer	L/h
Brooklyn	1883	486	1100	5.0	Roebling	12.45
Golden Gate	1937	1280	1517	2.68	Strauss	8.94
Akashi Kaikyo	1998	1991	1770	2.25	Bridge [illegible]	10.04

Fig. 8.3 – Suggested values of safety factors and historic values of safety factors used in suspension bridges

Examples of Safety Factors in Engineering Design:
Fig. 8.3 shows some suggested values for factors of safety as recommended by William Rankine, as well as the actual factors of safety used in the Brooklyn Bridge, the Golden Gate Bridge, and more recently the Akashi Kaikyo Bridge. The factors of safety suggested by Rankine are very large (up to 10!) compared to those employed in modern design, which are typically around 2.0. This is due to advances in design/construction practices and improvements in predicting the loads which structures may be subjected to over their lifetime.

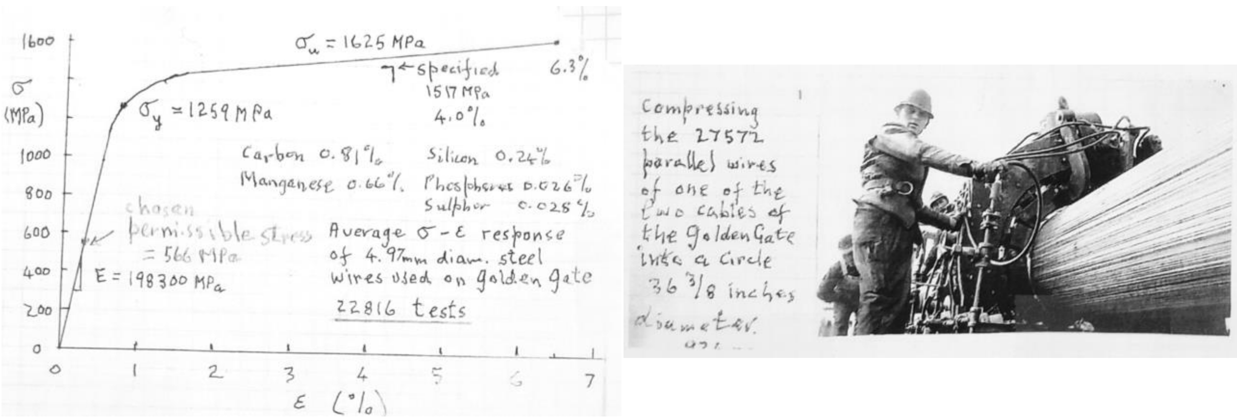


Fig. 8.4 – Stress-strain characteristics of steel used in the main cables of the Golden Gate Bridge

Lecture 9 – Weight a Moment! What is I?

Overview:

In the previous chapters, we have discussed the basic equations for stress and strain which are sufficient for studying members subjected to direct tension. However, they are inadequate for describing the behaviour of members which bend when carrying bending moments or when buckling under compression forces. In this chapter, the fundamentals of rotational motion are discussed. Although structures rarely experience significant rotations, the concepts used to describe rotational motion are analogous to those needed to explain the behaviour of structures when they bend.

Relating a Moment to the Angular Acceleration of a Point Mass

Consider the system shown in Fig. 9.1, which shows a point mass **m** attached to a pivot point by means of a weightless, rid rod with length **y**. If a moment **M** is applied to the system, the mass will spin around the axis of rotation with an angular acceleration of **α** radians per second squared.

*Note: the distance used for rotational motion, taken here as **y**, is often taken as **r** in other courses.*

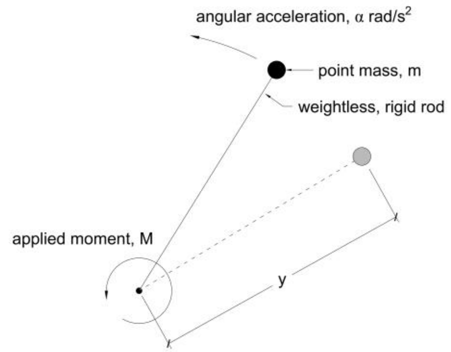


Fig. 9.1 – Point mass rotating as a result of an applied moment

The relationship between the moment and the angular acceleration can be determined by considering the effective translational force applied to the mass and the corresponding translational acceleration. Recall from Lecture 2 that a moment is the product of a force, **F**, and a perpendicular distance to a reference point, which is referred to here as **y**:

$$M = Fy \tag{9.1}$$

The translational acceleration of the mass, **a**, is related to the angular acceleration, **α**, in the same way that the length of a circular arc is equal to the product of the radius and the angle traversed:

$$a = \alpha y \tag{9.2}$$

Now that the force and acceleration are known, they can be related to each other using Newton's second law of motion, which is F = m×a. This allows Eq. (9.1) and (9.2) to be combined:

$$M = Fy = (ma)y = m\alpha y^2 \tag{9.3}$$

Grouping the mass and length terms produces the following result:

$$M = (my^2)\alpha \tag{9.4}$$

The term my^2 describes the resistance of the mass to rotation and is analogous to the concept of inertia used in Newton's laws of motion. Rewriting Eq. (9.4) to be applicable to more complex situations than the simple example shown in Fig. (9.1) results in the fundamental equation of rotational motion:

$$M = I_m\alpha \tag{9.5}$$

Note: $\sum$ is used to denote the summation of terms (typically over n terms). Eq. (9.3) can be expanded to be:

$$\sum \Delta m_i y_i^2 = \Delta m_1 y_1^2 + \Delta m_2 y_2^2 + \cdots + \Delta m_n y_n^2$$

In Eq. (9.5), I_m is the ***moment of inertia*** and has dimensions of mass × length squared.

Defining the moment of inertia, I_m

The resistance of a point mass to rotation about an axis is defined as its moment of inertia, $\mathbf{I_m}$. For a small object which has mass $\mathbf{m_i}$ and is located a distance $\mathbf{y_i}$ away from the axis of rotation, its individual moment of inertia $\mathbf{I_{m,i}}$ is defined as:

$$I_{m,i} = m_i y_i^2 \tag{9.6}$$

Although Eq. (9.6) allows us to calculate the moment of inertia for a point mass relative to an axis of rotation, how can we extend it to consider objects whose mass is not distributed at a single point? Consider the body shown in Fig. 9.2 which has been subdivided into many discrete point masses $\Delta\mathbf{m_i}$ which are each located a distance of $\mathbf{y_i}$ relative to the axis of rotation. The moment of inertia of the body is the sum of the moments of inertia of each point mass, $\mathbf{I_{m,i}}$, over the whole body:

$$I_m \cong \sum I_{m,i} = \sum \Delta m_i y_i^2 \tag{9.7}$$

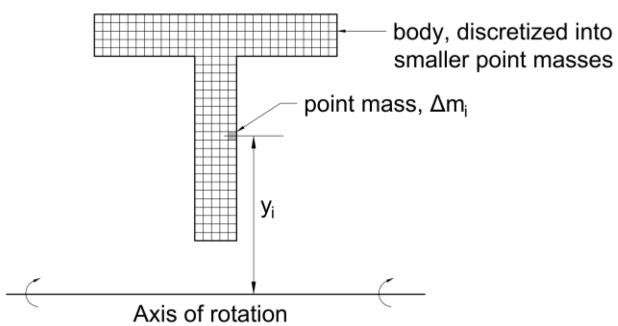


Fig. 9.2 – Calculation of I_m for a two-dimensional body which has been discretized into smaller pieces

Eq. (9.7) is an approximation for $\mathbf{I_m}$, as the value of $\mathbf{I_m}$ obtained using the equation depends on how finely we have subdivided the original body. The approximation becomes more accurate as we subdivide the body into smaller pieces, which in turn reduces the size of the individual point masses. If we take the limit as $\Delta \mathbf{m_i}$ approaches zero, then the summation can be instead replaced by an integral and $\mathbf{I_m}$ can be calculated exactly.

$$I_{\mathrm{m}} = \lim_{\Delta m_i \to 0} \sum \Delta m_i y_i^2 = \int_M y^2 dm \tag{9.8}$$

Eq. (9.8) is the definition of the moment of inertia for a finite body, which is the sum of the moments of inertia contributed by the infinitesimally small point masses **dm** over the entire mass **M**.

In the special case where the body under consideration is a two-dimensional object having a uniform density with units of perhaps kg/m^2, then **dm** can be written as the product of the density **ρ** and a differential area **dA**:

$$dm = \rho dA \tag{9.9}$$

Equations (9.8) and (9.9) can hence be combined to produce the following result:

Note: Eq (9.6) is valid only if ρ does not vary over the area of the body. If it does, then it must be included inside of the integral.

$$I_m = \int_M y^2 dm = \rho \int_A y^2 dA \tag{9.10}$$

The moment of inertia is thus the product of the density of the material multiplied by an integral term which consists of purely geometric properties. The integral term is known as the ***second moment of area, I***, which has dimensions of length^4.

$$I = \int_A y^2 dA \tag{9.11}$$

Physical Interpretation of the Moment of Inertia:

By examining Eq. (9.6) to Eq. (9.8), the following properties can be understood about the moment of inertia, $\mathbf{I_m}$. These properties are also true for the second moment of area, **I**, if the references to mass instead refer to area.

1. $\mathbf{I_m}$ depends on the location and orientation of the axis of rotation, as this affects the term $\mathbf{y_i}$.
2. Masses which are located further away from axis of rotation tend to have a larger contribution to $\mathbf{I_m}$ compared to masses which are located closer to the axis of rotation.

To illustrate these properties, consider the steel I-beam shown in Fig. (9.3). Its designation, W530×92, refer to its nominal height of 530 mm and weight of 92 kilograms per metre of length. The central image in the figure shows the member being rotated about its y-axis, and the image on the left shows the member being rotated about its x-axis. It can be seen that the member has a substantially larger second moment of area taken about its x-axis compared to its y-axis because the majority of its area is distributed far away from the axis of rotation when it is aligned about its x-axis.

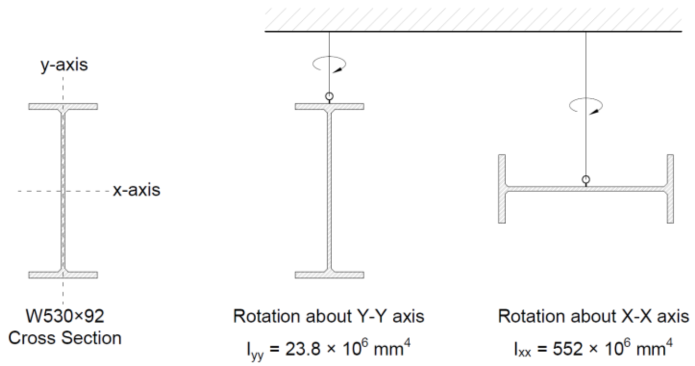


Fig. 9.3 – Sample values of I for a W530×92 I-beam

Example Calculation of I: Rectangle

A simple example of applying Eq. (9.11) is to find the second moment of area of a rectangle about its own centroid, which is located at its mid-height. We begin the process by expressing a small area of the rectangle, **dA**, as the product of the rectangle's width **b** and a small thickness **dy**, as shown in Fig. 9.4.

$$dA = bdy \tag{9.12}$$

We can now calculate the second moment of area by substituting Eq. (9.12) into the definition of **I** and then integrating over the height of the rectangle, which is from y = -h/2 to y = h/2:

$$I = \int_{-\frac{h}{2}}^{\frac{h}{2}} by^2 dy = \frac{1}{3} by^3 \Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{1}{3} b \left(\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3 \right) \tag{9.13}$$

Evaluating Eq. (9.13) results in a simple expression for I of a rectangle which is rotating about an axis at its mid-height:

$$I = \frac{bh^3}{12} \tag{9.14}$$

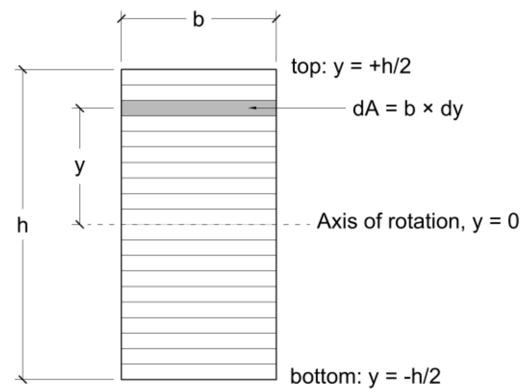


Fig. 9.4 – Derivation of the second moment of area of a rectangle about its centroid. Each "slice" of area has a thickness of dy.

Lecture 10 – Introduction to the Bending of Beams

Overview:
In this lecture, the principles discussed in previous chapters are used to derive equations describing the behaviour of members which bend.

Fundamental Assumption for Bending: Plane Sections Remain Plane
Consider a series of vertical lines on a beam which are drawn a distance $\mathbf{L_o}$ apart. As the beam is bent, it will curve to form the arc of a circle. These lines will remain straight but will rotate so that on one side of the beam, say the top, they are slightly further apart and on the other side, the bottom, they are slightly closer apart. At the centroid of the beam, these lines retain a separation of $\mathbf{L_o}$. This phenomenon was described by Robert Hooke in 1678 using the phrase "plane sections remain plane" and is illustrated in Fig. 10.1. One way to quantify how much the beam has bent is to measure the relative angle between two vertical lines, $\boldsymbol{\theta}$, and divide this angle by the original distance between them, which is $\mathbf{L_o}$. The resulting quantity is called the average curvature. In general, the ***curvature*** $\boldsymbol{\phi}$ is more rigorously defined as the change of this angle $\boldsymbol{\theta}$ along the length of the member, **x**.

$$\phi = \frac{d\theta}{dx} \tag{10.1}$$

Another property of the curvature is that the quantity $1/\phi$ is equal to radius of the circle formed by the beam after it has been curved. This quantity is known as the ***radius of curvature***.

Bending of the member produces strains in the member because the distance between the vertical lines is no longer equal to the original spacing $\mathbf{L_o}$ except for at the centroid. Consider the spacing of points A and B which are drawn on the beam in Fig. 10.1 and located a distance **y** above the centroidal axis. After the beam has been bent with a curvature $\boldsymbol{\phi}$, the change in angle between A and B is equal to $\theta_{AB} = \phi \times L_o$, and the distance between the points to the centre of the circle is equal to $y + 1/\phi$. Using this information, the distance between points A and B after the beam has been curved, $\mathbf{L_{AB}'}$, can be calculated as:

$$L'_{AB} = (\phi L_o) \cdot \left(y + \frac{1}{\phi}\right) = \phi y L_o + L_o \tag{10.2}$$

Using the deformed length, we can now calculate the strain of the member between points A and B which are located a distance **y** above the centroidal axis:

$$\varepsilon(y) = \frac{\Delta l}{L_o} = \frac{L'_{AB} - L_o}{L_o} \tag{10.3}$$

Substituting Eq. (10.2) into Eq. (10.3) yields the fundamental relationship between the strains in the member, $\boldsymbol{\varepsilon}$, and its curvature, $\boldsymbol{\phi}$:

$$\varepsilon(y) = \phi y \tag{10.4}$$

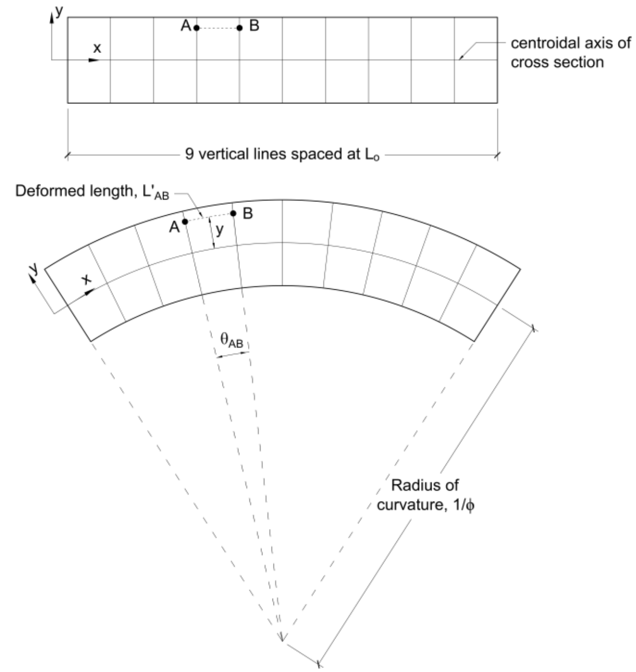


Fig. 10.1 – Figure illustrating Robert Hooke's 1678 hypothesis that when members are subjected to pure bending, "Plane Sections Remain Plane". The vertical lines drawn on the side of the member remain straight when it is curved.

The strains in a member subjected to pure bending are not constant over the cross section as was the case for pure tension. Rather, they vary linearly along the height, from a maximum tensile strain on one side of the member to a maximum compressive strain on the other. The strain at the height of the centroid, which is the axis about which each section rotates, is equal to zero. These observations are shown on Fig. 10.2.

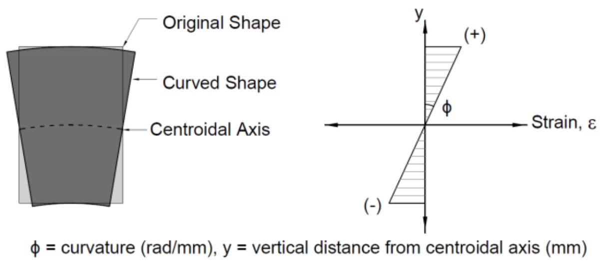


Fig. 10.2 – Strain profiles caused by pure bending. Note that for a beam made of a linear elastic material, ε = 0 at the centroidal axis (y = 0).

Flexural Stiffness – Determining the Relationship between Bending Moment and Curvature:

When we were studying the behaviour of members subjected to axial force, we were interested in calculating the axial stiffness of the member, **k**, which related the tension in the member to its elongation. For members which bend, we are instead interested in the relationship between the moment carried by a member, **M**, and its curvature, **ϕ**. Just like how we used the definitions of stress and strain to derive the axial stiffness of a member, we will do the same to derive the flexural stiffness based on our assumption that plane sections remain plane.

To begin, we can first determine the stresses in a member which has a curvature **ϕ** if we know the Young's modulus of the material, **E**:

$$\sigma = E\varepsilon \rightarrow \sigma(y) = E\phi y \tag{10.2}$$

Eq. (10.2) states that like the strains **ε**, the stresses **σ** also vary linearly across the height of the cross section. To understand how the distribution of stresses is related to the bending moment, consider a thin slice of the cross section with area **ΔA.** The stresses in this slice of the cross-sectional area act uniformly over **ΔA** if **ΔA** is relatively small, producing a force **ΔF** which can be calculated using our definition of stress:

$$\sigma = \frac{F}{A} \rightarrow \Delta F = \sigma(y)\Delta A \tag{10.3}$$

Because the force **ΔF** does not necessarily pass through the centroidal axis of the member, it will produce a turning effect. The moment **ΔM** caused by **ΔF** acting a distance **y** away from the centroidal axis can be calculated using our definition of a moment:

$$M = F \cdot d \rightarrow \Delta M = \Delta F \cdot y \tag{10.4}$$

Substituting the Eq. (10.2) and (10.3) into (10.4) yields the following expression for **ΔM**:

Note: The process of calculating the resulting bending moment from the linearly varying strains is shown in Fig. 10.3. The flexural stresses, shown in the second figure, act over a differential area of the cross section, dA, producing a force dF. These forces, shown in the third figure which shows a slice of the beam from elevation view, produce a moment on the left side which equilibrates the applied bending moment on the right side.

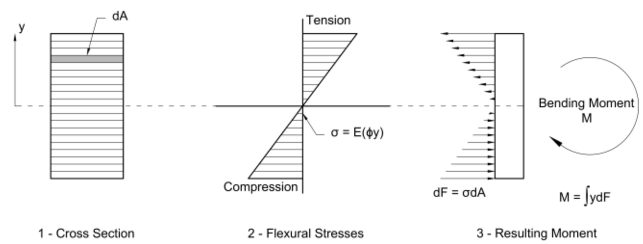


Fig. 10.3 – Summary of how the bending moment carried by a member is determined if the distribution of strains is known from the curvature.

$$\Delta M = \phi E y^2 \Delta A \tag{10.5}$$

We can now find the total moment carried by the member by summing the turning effects caused by each piece of area in the cross section. If we used slices with a nonzero thickness, this will be an approximation of the true moment because the resulting sum will depend on the size of the slides considered. When the thickness of each slice becomes infinitesimally small, **ΔA** approaches zero and the summation sign is instead replaced with an exact integral:

$$M = \lim_{dA \to 0} \sum \phi E y^2 \Delta A = \int_A \phi E y^2 dA \tag{10.6}$$

In Eq. (10.6), we can move **ϕ** out of the integral because the curvature of the member is constant over the cross section and not a function of **A**. If the member is made of the same material over the whole cross section, we can also remove **E** from the integral, which results in the following expression for **M**:

$$M = \phi E \int_A y^2 dA \tag{10.7}$$

By examining Eq. (10.7), we recognize that the integral of y^2 over the area of the cross section is the definition of the second moment of area, **I**, which was introduced in Lecture 9. Substituting this property into the equation yields the final result:

$$M = EI \cdot \phi \tag{10.8}$$

In Eq. (10.8), **EI** is the ***flexural stiffness*** of the member. Like the axial stiffness **k**, which was derived in Lecture 5, the flexural stiffness **EI** also relates a force-based quantity, the bending moment, to a displacement-based quantity, the curvature. Similarly, it is a function of both the stiffness of the material, **E**, and the geometric stiffness provided by the shape of the cross section, **I**. This comparison is shown in Fig. 10.4.

	Force-like quantities	Stiffness-like quantities	Deformation-like quantities
Structures which stretch or contract	**Axial Force, *F*** [N]	**Axial Stiffness** [N/mm] ⟷ $\boxed{F = k\Delta l}$	**Displacement, *Δl*** [mm]
Structures which bend	**Bending Moment, M** [Nmm]	**Flexural Stiffness** [Nmm²] ⟷ $\boxed{M = EI\phi}$	**Curvature, *ϕ*** [rad/mm]

Fig. 10.4 – A comparison of the axial and flexural stiffnesses of a member

Lecture 11 – Statically Determinate Structures

Overview:

Structural engineering is primarily concerned about determining how structures transfer loads from one location to another. For most civil structures, this involves transmitting vertical loads (i.e. gravity loads) or horizontal loads (i.e. wind loads or earthquake loads) to the ground. In this chapter, the basics of structural analysis are introduced, beginning with the determination of reaction forces for statically determinate structures.

Supports:

Supports are the elements which hold up the structure and transmit the forces carried by the structure to the ground below. Examples of supports include bearing pads, foundations, and hinges, which all transmit some degree of force and/or moment to support the structure and prevent it from accelerating. The forces/moments which are supplied by supports to hold up the structure are called ***reaction forces***.

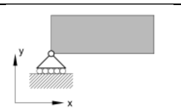
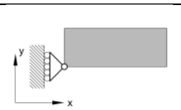
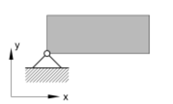
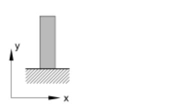
Reaction forces are closely related to the level of restraint which a support can provide. For example, a hinge support which is well-anchored to the ground will be able to prevent attached structure from moving translationally but will freely swivel. Hence, a hinge can provide reaction forces which resist translational movement but cannot provide any moment to prevent rotation. The key principle is that increasing the amount of restraint provided by the support increases, increases its ability provide a reaction force along that that degree of freedom, and vice-versa.

Note: A simple example of a hinge support are the hinges which fasten a door to a door frame. These hinges prevent the door from translating, but do not provide any resistance to the door being swung open.

In structural engineering, we typically define three common types of supports which are called ***rollers***, ***pins*** and ***fixed ends***. Solving any structural engineering problem typically first involves calculating the reaction forces which these supports provide to the structure. Table 11.1 describes each type of support, its permitted degrees of freedom, the support reactions which can be supplied to the attached structure.

Note: Real supports are unable to perfectly restrain a structure like the idealized pins, rollers and fixed ends described in Table 11.1. Choosing which ideal support best reflects realistic conditions requires engineering judgement and experience.

Table 11.1 – Types of supports and their reaction forces

Name	**Symbol**	**Permitted Degrees of Freedom**	**Restrained Degrees of Freedom**	**Support Reactions**
Roller		Δx, θ_{xy}	$\Delta y = 0$	F_y
		Δy, θ_{xy}	$\Delta x = 0$	F_x
Pin		θ_{xy}	$\Delta x = \Delta y = 0$	F_x, F_y
Fixed end		None	$\Delta x = \Delta y = \theta_{xy} = 0$	F_x, F_y, M_{xy}

*Note: The **degrees of freedom**, when used to refer to geometric situations such as 2-D space or 3-D space, are the variables required to describe the position and orientation of a body. Three degrees of freedom are needed to define a non-deformable body in 2-D space. These are:*

1. *Position in the x-direction*
2. *Position in the y-direction*
3. *Rotational orientation in the x-y plane*

A body which deforms may require more degrees of freedom to describe its position, orientation, and deformed shape.

Solving for Reaction Forces – Free Body Diagrams

Solving for the reaction forces requires an understanding of how the loads carried by the structure are distributed to the supports on which it sits. Consider the structure shown in Fig. 11.1 which is a beam supported by a pin and roller and carrying three masses, m_1, m_2 and m_3:

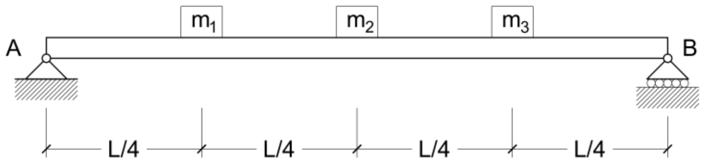


Fig. 11.1 – Simply supported beam carrying three weights

A complete understanding how the structure transmits the load from the three masses to the ground below can be obtained by drawing a series of free body diagrams. Five free body diagrams can be drawn which each describes the interaction between an applied load or support and the structure. The sixth free body diagram is of the structure itself being subjected to the various applied loads and reaction forces caused by the supports and weights. These free body diagrams are shown below in Fig. 11.2:

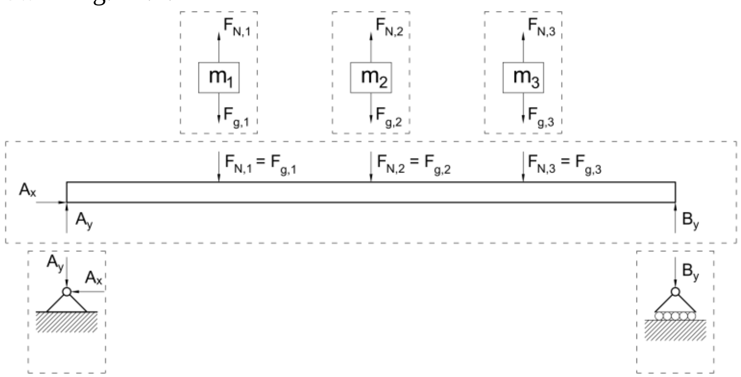


Fig. 11.2 – Free body diagrams demonstrating how the weight of the three loads is transferred to the ground

In Fig. 11.2, $\mathbf{F_N}$ is the normal force supplied by the beam to hold up the masses, $\mathbf{F_g}$ is the force of gravity acting on each mass and $\mathbf{A_x}$, $\mathbf{A_y}$ and $\mathbf{B_y}$ are the reaction forces. The self-weight of the beam is ignored. When drawing these free body diagrams, the following two rules have been used:

1. According to Newton`s third law of motion, the force applied to the structure by an applied load or support is equal and opposite to the force applied by the structure to the applied load or support.
2. When drawing in a force which is unknown, like an undetermined reaction force, any assumed direction will suffice. The assumed direction will not affect the solution as long as the equilibrium equations are consistent with the drawn free body diagram.

Note: If the assumed direction is incorrect, then the resulting value obtained by solving the equations of equilibrium will be a negative number.

Because the system as a whole is in equilibrium, each subsystem will also be in equilibrium, and hence the equilibrium equations must be satisfied for each free body diagram. As noted in Lecture 3, these equations are:

$$\sum F_x = 0 \tag{11.1}$$

$$\sum F_y = 0 \tag{11.2}$$

$$\sum M = 0 \tag{11.3}$$

These equations can then be used to determine the unknown reaction forces, $\mathbf{A_x}$, $\mathbf{A_y}$ and $\mathbf{B_y}$, once the appropriate free body diagrams have been drawn. Typically, the most useful free body diagram to consider is the free body of the structure itself being subjected to the applied loads and reaction forces.

Statically Determinate Structures

Structures whose reaction forces can be directly solved using the three equations of equilibrium are called ***statically determinate***. Statically determinate structures have the property where the reaction forces are purely a function of the size, quantity, location, and direction of the applied loads, and are unrelated to the stiffness of the structure. This occurs if the number of unknown reaction forces is equal to the number of equilibrium equations. Most simple structures are statically determinate if their supports provide a total of three reaction forces.

Structures which have fewer reaction forces than the number of equilibrium equations are called ***mechanisms***. This is because they are unstable and will accelerate when subjected to an applied load.

Structures which have more reaction forces than the number of equilibrium equations are ***statically indeterminate***. The reaction forces cannot be directly solved using the equilibrium equations alone, and hence must consider other factors such as the stiffness of the structure and positioning of the applied loads. The degree of indeterminacy is a measure of how statically indeterminate a structure is and is equal to the number of reaction forces minus the number of equilibrium equations.

Examples of the three situations can be found in Fig. 11.3.

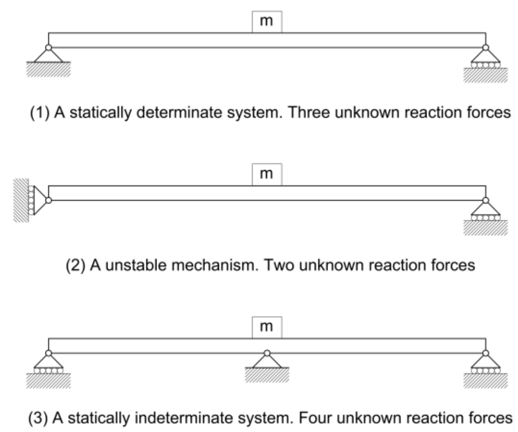


Fig. 11.3 – Examples of a statically determinate structure, a mechanism, and a statically indeterminate structure.

Note: Many building structures are statically indeterminate. Solving for their reaction forces and internal stresses requires more advanced analysis methods than those covered in CIV102.

Example: Structures with an Internal Hinge:

Some structures are built with an internal hinge which connects two substructures together. Because a hinge freely rotates and is unable to resist moment, it has the effect of reducing the indeterminacy of the structure by one for each internal hinge. The following example illustrates how to account for hinges in a structure when solving for the reaction forces.

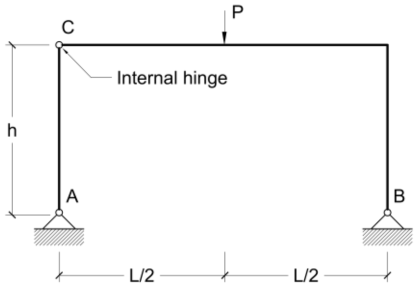


Fig. 11.4 – Example of a frame structure containing an internal hinge

Consider the frame shown above in Fig. 11.4 which is carrying a point load, P, acting downwards on its top beam. The frame is supported on two pins, resulting in 2 × 2 = 4 unknown reaction forces. Although this might suggest that the structure is statically indeterminate, we can take advantage of the internal hinge to solve for these unknown forces. To do this, two free body diagrams which cut through the hinge are drawn, which reveals the two internal hinge forces. Because each free body diagram is in equilibrium, we have a total of six equilibrium equations (three from each free body diagram) which we can use to solve for the four reaction forces and two internal hinge forces. These free body diagrams are shown below in Fig. 11.5.

Note: When the frame is cut and separated at the internal hinge, the hinge forces must be drawn in opposite directions on the two free body diagrams. This is to ensure that the forces cancel out when the frame is "put back together".

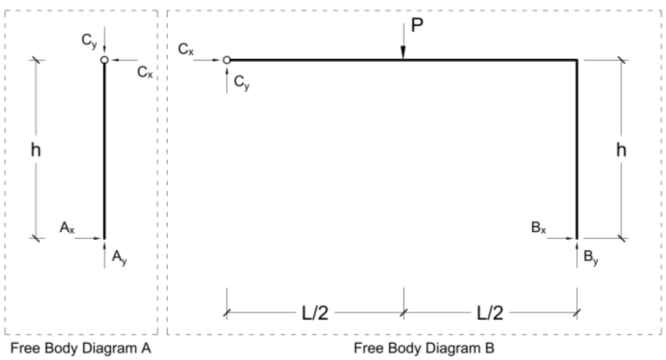


Fig. 11.5 – Free body diagrams of the frame after it has been separated at the hinge

The three equilibrium equations which correspond to Free Body Diagram A are shown below. Note that the moment equation is taken about point A, the left support.

$$\sum F_x = 0 \rightarrow A_x - C_x = 0 \tag{11.4}$$

$$\sum F_y = 0 \rightarrow A_y - C_y = 0 \tag{11.5}$$

$$\sum M_A = 0 \rightarrow C_x \times h = 0 \tag{11.6}$$

The three equilibrium equations corresponding to Free Body Diagram B are shown below, with the moment equation being taken about point B, the right support.

$$\sum F_x = 0 \rightarrow B_x + C_x = 0 \tag{11.7}$$

$$\sum F_y = 0 \rightarrow B_y + C_y - P = 0 \tag{11.8}$$

$$\sum M_B = 0 \rightarrow C_x \times h + C_y \times L - P \times \frac{L}{2} = 0 \tag{11.9}$$

Because we have six equations (Eq. (11.4) to (11.9)) and six unknowns (A_x, A_y, B_x, B_y, C_x and C_y), we can solve for each force and hence the system is statically determinate.