Guided Practice by Chapter

1 Foundations

Organization of this book

The material for this course is aimed at guiding you in developing a solid understanding of the principles and practice of physics. The following Guided Practice sets provide a variety of questions and problems that allow you to apply and sharpen your understanding of physics.

Each chapter set contains specific aids and exercises targeted to the physics discussed in the corresponding chapter. These categories are ordered so that the earlier materials support the later ones:

1. **Review questions.** A set of simple questions probes your understanding of the basic material. You should be able to answer these questions without trouble after having read the corresponding chapter.
2. **Developing a feel.** These estimation problems are designed to exercise your ability to grasp the scope of the world around you, following the approach outlined in the section of the same title in the chapter.
3. **Worked and guided problems.** This section presents a series of paired example problems: a worked example followed by a similar problem presented with only a few guiding hints and questions. The idea is that if you understand the methods of the worked example, then you should be able to adapt those methods (and perhaps add a twist) to solve the guided problem. At the beginning of this section you will find a copy of any **Procedure boxes** from the chapter.
4. **Answers and solutions** to Review Questions and Guided Problems are at the end of the chapter.

Review Questions

Answers to these questions can be found at the end of this chapter.

1.1 The scientific method

1. What is a common definition of *physics*? That is, what is physics about?
2. Briefly describe the scientific method and what it involves.
3. Name some skills that are useful in doing science.
4. Describe the difference between the two types of reasoning involved in doing science.

1.2 Symmetry

5. What does symmetry mean in physics?
6. What are two types of symmetry that are demonstrated in the reproducibility of experimental results?

1.3 Matter and the universe

7. In physics, what is the definition of *universe*?
8. What does expressing a value to an order of magnitude mean? Why would we express values in this way?
9. To what order of magnitude should you round the numbers 2900 and 3100? Explain why your answers are different for the two numbers.
10. Physicists study phenomena that extend over what range of sizes? Over what range of time intervals?

1.4 Time and change

11. What does the phrase *arrow of time* mean?
12. What principle that relates events depends on the arrow of time? State this principle, and briefly explain what it means.

1.5 Representations

13. When solving physics problems, what are the advantages of making simplified visual representations of the situations?
14. What is the purpose of the *Concepts* part of each chapter of this book? What is the purpose of the *Quantitative Tools* part of each chapter?

1.6 Physical quantities and units

15. What two pieces of information are necessary to express any physical quantity?
16. What are the seven SI base units and the physical quantities they represent?
17. What concept does density represent?
18. What is the simplest way to convert a quantity given in one unit to the same quantity given in a different unit?

1.7 Significant digits

19. Explain the difference between number of digits, number of decimal places, and number of significant digits in a numerical value. Illustrate your explanation using the number 0.037 20.
20. What is the difference between leading zeros and trailing zeros? Which ones are considered significant digits?
21. How many significant digits are appropriate in expressing the result of a multiplication or division?
22. How many significant digits are appropriate in expressing the result of an addition or subtraction?

1.8 Solving problems

23. Summarize the four-stage problem-solving procedure used in this book.
24. When checking calculations, what do the letters in the acronym VENUS stand for?

1.9 Developing a feel

25. What are the benefits of making order-of-magnitude estimates?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

(You determine the order of magnitude of any quantity by writing it in scientific notation and rounding the coefficient in front of the power of ten to 1 if it is equal to or less than 3 or to 10 if it is greater than 3. Then write the answer as a power of ten without a coefficient in front. Also remember to express your answer in SI units.)

1. The width of your index finger (C, H)
2. The length of a commercial airliner (E, A)
3. The height of a stack containing 1,000,000 one-dollar bills (G, M)
4. The area of your bedroom floor (J, X)
5. The number of jelly beans needed to fill a 1-gallon jar (F, L)
6. The mass of the air in a typical one-family house (J, D, N)
7. The mass of water needed to fill the passenger compartment of a midsize car (P, I)
8. The mass of a mountain (K, V, S, P)
9. The mass of water in a city swimming pool (T, P)
10. The number of cups of coffee consumed yearly in the United States (R, W)
11. The number of people in the world who are eating at the instant you are reading this question (Q, Z)
12. The number of automobile mechanics in California (B, U, O, Y)

Hints

A. What is the distance between rows?
B. What is the population of California?
C. What is the width of a finger in inches?
D. How many bedroom-size rooms are there in a typical one-family house?
E. How many rows of seats are there in an airliner?
F. What are the dimensions of a jelly bean?
G. How thick is a ream of paper?
H. How many inches make 1 meter?
I. What are the length, width, and height of a car's passenger compartment in meters?
J. What are the length, width, and height of your bedroom in feet?
K. What size and shape could model a mountain?
L. What is the volume of a gallon jar in SI units?
M. How many sheets of paper are there in a ream?
N. What is the mass density of air?
O. How many hours of maintenance does a car need each year?
P. What is the mass density of water?
Q. What is the population of the world?

R. What is the adult population of the United States?
S. How does the mass density of rock compare with that of water?
T. What are the dimensions of a city pool?
U. How many cars are there in California?
V. What volume results from your model?
W. How many cups of coffee does the average American drink daily?
X. How many square feet are there in 1 square meter?
Y. How many hours does an automobile mechanic work each year?
Z. What fraction of your day do you spend eating?

Key (all values approximate)

A. 1 m; **B.** 4×10^7 people; **C.** 0.5 in.; **D.** 8 rooms; **E.** 4×10^1 rows; **F.** $(1 \times 10^{-2}\ \text{m}) \times (1 \times 10^{-2}\ \text{m}) \times (2 \times 10^{-2}\ \text{m})$; **G.** 5×10^{-2} m; **H.** 4×10^1 in./m; **I.** 2 m × 2 m × 1 m; **J.** $(1 \times 10^1\ \text{ft}) \times (2 \times 10^1\ \text{ft}) \times (1 \times 10^1\ \text{ft})$; **K.** a cone 1 mile high with a 1-mile base radius; **L.** 4 quarts ≈ 4 L = $4 \times 10^{-3}\ \text{m}^3$; **M.** 5×10^2 sheets; **N.** 1 kg/m^3; **O.** 6 h; **P.** $1 \times 10^3\ \text{kg/m}^3$; **Q.** 7×10^9 people; **R.** 2×10^8 people; **S.** 5 × larger; **T.** $(7\ \text{m}) \times (2 \times 10^1\ \text{m}) \times (2\ \text{m})$; **U.** 3×10^7 cars; **V.** $4 \times 10^9\ \text{m}^3$; **W.** 1 cup; **X.** $1 \times 10^1\ \text{ft}^2/\text{m}^2$; **Y.** 2×10^3 h; **Z.** 0.1

Worked and Guided Problems

Procedure: Solving Problems

Although there is no set approach when solving problems, it helps to break things down into several steps whenever you are working a physics problem. Throughout this book, we use the four-step procedure summarized here to solve problems. For a more detailed description of each step, see Section 1.8.

1. **Getting started.** Begin by carefully analyzing the information given and determining in your own words what question or task is being asked of you. Organize the information by making a sketch of the situation or putting data in tabular form. Determine which physics concepts apply, and note any assumptions you are making.
2. **Devise plan.** Decide what you must do to solve the problem. First determine which physical relationships or equations you need, and then determine the order in which you will use them. Make sure you have a sufficient number of equations to solve for all unknowns.
3. **Execute plan.** Execute your plan, and then check your work for the following five important points:

   Vectors/scalars used correctly?
   Every question asked in problem statement answered?
   No unknown quantities in answers?
   Units correct?
   Significant digits justified?

   As a reminder to yourself, put a checkmark beside each answer to indicate that you checked these five points.
4. **Evaluate result.** There are several ways to check whether an answer is reasonable. One way is to make sure your answer conforms to what you expect based on your sketch and the information given. If your answer is an algebraic expression, check to be sure the expression gives the correct trend or answer for special (limiting) cases for which you already know the answer. Sometimes there may be an alternative approach to solving the problem; if so, use it to see whether or not you get the same answer. If any of these tests yields an unexpected result, go back and check your math and any assumptions you made. If none of these checks can be applied to your problem, check the algebraic signs and order of magnitude.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 1.1 Solar hydrogen

The mass of the Sun is 1.99×10^{30} kg, its radius is 6.96×10^{8} m, and its composition by mass is 71.0% hydrogen (H). The mass of a hydrogen atom is 1.67×10^{-27} kg. Calculate (*a*) the average mass density and (*b*) the average number density of the hydrogen atoms in the Sun.

❶ **GETTING STARTED** To calculate mass and number densities we need to determine the volume of the Sun. We assume the Sun is perfectly spherical so that we have a formula for its volume. We are given the mass of the Sun and the percentage of hydrogen, so we can determine the mass of hydrogen in the Sun and the number of atoms required to provide this mass.

❷ **DEVISE PLAN** Mass density is mass per unit volume, $\rho = m/V$, and number density is number per unit volume, $n = N/V$. The (assumed spherical) solar volume is $V = \frac{4}{3}\pi R^3$. The solar radius is given, so we need either the number of hydrogen atoms or the mass of hydrogen in the Sun in order to proceed. The mass of hydrogen is 71.0% of the Sun's mass, so we use that value to compute the mass density first. Then the number N of hydrogen atoms is the mass of all the hydrogen atoms divided by the mass of a single atom. We can use this number to calculate the number density n, or we can simply note that n is equal to the mass density of hydrogen divided by the mass of a single hydrogen atom.

❸ **EXECUTE PLAN**
(*a*) For the mass density, we have

$$\rho_{\mathrm{H}} = \frac{m_{\mathrm{H}}}{\frac{4}{3}\pi R_{\mathrm{Sun}}^3} = \frac{0.710 m_{\mathrm{Sun}}}{\frac{4}{3}\pi R_{\mathrm{Sun}}^3} = \frac{(0.710)(1.99 \times 10^{30}\ \mathrm{kg})}{\frac{4}{3}\pi(6.96 \times 10^{8}\ \mathrm{m})^3}$$

$$= 1.00 \times 10^{3}\ \mathrm{kg/m^3}.$$

(*b*) The number density of the hydrogen atoms is

$$n_{\mathrm{H}} = \frac{\rho_{\mathrm{H}}}{m_{\mathrm{H\,atom}}} = \frac{1.00 \times 10^{3}\ \mathrm{kg/m^3}}{1.67 \times 10^{-27}\ \mathrm{kg}} = 5.99 \times 10^{29}\ \mathrm{atoms/m^3}.$$

Using VENUS to check, we have
Vectors/scalars: all quantities are scalars ✔
Every question answered: mass density, ✔ number density ✔
No unknown quantities in answers: none ✔
Units correct: $\mathrm{kg/m^3}$ for mass density, ✔ $\mathrm{atoms/m^3}$ for number density ✔
Significant digits: three in each answer because all given quantities have three ✔

❹ **EVALUATE RESULT** We calculated a hydrogen mass density equal to the mass density of water. Because hydrogen is a gas, you may think that this mass density is unreasonably large and that the answer should be about equal to the value found for helium gas in Exercise 1.6, about 0.2 $\mathrm{kg/m^3}$. However, because the gas in the Sun is highly compressed, a mass density several orders of magnitude larger is not unreasonable. But this value is the mass density of water! Does that make any sense? Well, water vapor certainly has a much smaller mass and number density than liquid water. If the hydrogen atoms in the Sun were squeezed together as closely as the molecules in liquid water, we might expect their mass density to be of the same order of magnitude. We might also compare our answer to the average mass density of the Sun obtained by using the data provided in the problem statement. Assuming a spherical Sun, we obtain

$$\rho_{\mathrm{Sun}} = \frac{m_{\mathrm{Sun}}}{\frac{4}{3}\pi R_{\mathrm{Sun}}^3} = 1.4 \times 10^{3}\ \mathrm{kg/m^3}.$$

So it seems reasonable that our hydrogen mass density is of the same order of magnitude as the average mass density.

Our result for the number density is also several orders of magnitude larger than the number density of helium found in Exercise 1.6, which is what we might expect for a very dense object like the Sun.

Note: When we are dealing with quantities completely outside our everyday experience, a quick check in a reference book or a couple of independent online sources might be needed to remove any lingering doubts. Doing so, we obtain an average solar mass density consistent with the calculation above.

Guided Problem 1.2 Solar oxygen

Oxygen atoms make up 0.970% of the Sun's mass, and each one has a mass of 2.66×10^{-26} kg. Calculate the average mass density of oxygen in the Sun and the average number density of the oxygen atoms. Use information given in Worked Problem 1.1 as needed.

1 GETTING STARTED

1. How much of the plan of Worked Problem 1.1 can be used?

2 DEVISE PLAN

2. What is the definition of mass density? Number density?
3. Do you have enough information to compute these values?

3 EXECUTE PLAN

4. What is the mass of oxygen in the Sun?
5. How is the number density of oxygen related to its mass density and to the mass of an oxygen atom?

4 EVALUATE RESULT

6. Are your answers consistent with the ones in Worked Problem 1.1?

Worked Problem 1.3 Rod volume

A cylindrical rod is 2.58 m long and has a diameter of 3.24 in. Calculate its volume in cubic meters.

1 GETTING STARTED We know that the volume of a cylinder is length times cross-sectional area. We are given the length, and we can use the diameter value given to calculate area. Because the diameter is in inches, however, we must convert to the SI equivalent.

2 DEVISE PLAN First we convert inches to meters via the conversion factors 25.4 mm = 1 in. (Equation 1.5) and 1 m = 1000 mm. Then we use the SI values for length (ℓ) and area (A) in the formula for volume: $V = A\ell = \pi R^2\ell$.

3 EXECUTE PLAN In meters, the radius, (3.24 in.)/2, is

$$1.62 \text{ in.} \times \frac{25.4 \text{ mm}}{1 \text{ in.}} \times \frac{1 \text{ m}}{1000 \text{ mm}} = 4.115 \times 10^{-2} \text{ m}.$$

Note: We use four significant digits here. Because the values given all have three significant digits, the final answer must have three also. In intermediate steps, though, we carry an extra digit to avoid accumulating rounding errors.

The volume of the rod is thus

$$V = \pi R^2\ell = \pi(4.115 \times 10^{-2} \text{ m})^2(2.58 \text{ m}) = 1.37 \times 10^{-2} \text{ m}^3.$$

Using VENUS to check, we have
Vectors/scalars: all quantities are scalars ✔
Every question answered: volume ✔
No unknown quantities in answer: none ✔
Units correct: cubic meters ✔
Significant digits: three in answer because all given quantities have three ✔

4 EVALUATE RESULT Although this rod is about 2.5 m long, its radius is only 1.62 in., which is about 40 mm, or 0.040 m. We can make an order-of-magnitude estimate for comparison by treating the rod as a rectangular block of square cross section 8 cm on a side, with length 3 m. The result is $(8 \times 10^{-2} \text{ m})(8 \times 10^{-2} \text{ m})(3 \text{ m}) \approx 10^{-2} \text{ m}^3$. Therefore a volume of about 10^{-2} m^3 is reasonable.

Guided Problem 1.4 Box volume

A box measures 1420 mm by 2.75 ft by 87.8 cm. Express its volume in cubic meters.

1 GETTING STARTED

1. Is the approach of Worked Problem 1.3 useful?

2 DEVISE PLAN

2. What is the relationship between the given dimensions and the volume of the box?
3. Which of the given quantities do you need to convert to SI units?
4. Can you use any of the values given in SI units in the form given?

3 EXECUTE PLAN

5. What are your conversion factors?

4 EVALUATE RESULT

Worked Problem 1.5 Working with digits

Express the result of each calculation both to the proper number of significant digits and as an order of magnitude:

(*a*) $(42.003)(1.3 \times 10^4)(0.007\,000)$

(*b*) $(42.003)(13{,}000)(0.007\,000)$

(*c*) $\dfrac{170.08\pi}{32.6}$

(*d*) $113.7540 - 0.08$

❶ **GETTING STARTED** We have two products, a quotient, and a difference involving quantities that have different numbers of significant digits. The number of significant digits in each answer depends on the number of significant digits in the given values.

❷ **DEVISE PLAN** To express each answer to the proper number of significant digits, we use the rules given in the text. The number of significant digits in a product or quotient is the same as the number of significant digits in the input quantity that has the *fewest* significant digits. The number of decimal places in a sum or difference is the same as the number of decimal places in the input quantity that has the *fewest* decimal places.

To express our answers as orders of magnitude, we write each in scientific notation, round the coefficient either down to 1 (coefficients ≤ 3) or up to 10 (coefficients > 3), and then write the answer as a power of ten without the coefficient.

❸ **EXECUTE PLAN**
(*a*) In $(42.003)(1.3 \times 10^4)(0.007000)$, the first factor has five significant digits, the second has two significant digits, and the third has four significant digits. Therefore the product can have only two significant digits: 3.8×10^3. The coefficient 3.8 is greater than 3, and so we round it to 10, making the order-of-magnitude answer $10 \times 10^3 = 10^4$.

(*b*) This time the middle factor has five significant digits, so we adjust our calculation to allow four (due to the third factor) significant digits: 3.822×10^3. Of course, the order of magnitude is unchanged.

(*c*) In $170.08\pi/32.6$, the denominator has three significant digits, the first factor in the numerator has five significant digits, and π (=3.14159. . .) has as many significant digits as our calculator shows. So the result can have only three significant digits: 16.4. To express this value as an order of magnitude, we must write it in scientific notation, 1.64×10^1. Because 1.64 is less than 3, we round it to 1, making the order-of-magnitude answer $1 \times 10^1 = 10^1$.

(*d*) The value that has the fewest decimal places is 0.08, meaning the difference must be reported to two decimal places: 113.67. To express this result as an order of magnitude, we write 1.1367×10^2 and round the 1.1367 down to 1, making the order-of-magnitude answer $1 \times 10^2 = 10^2$.

Checking by VENUS gives
Vectors/scalars: all quantities are scalars ✔
Every question answered: all results reported both to the correct number of significant digits and as an order of magnitude ✔
No unknowns in answers: none ✔
Units correct: no units given ✔
Significant digits: all significant digits are correct ✔

❹ **EVALUATE RESULT** We can check that each answer has the correct order of magnitude.

(*a*) $(42.003)(1.3 \times 10^4)(0.007\,000)$ is about $40 \times 13{,}000 \times 0.01 = 5200$, order of magnitude $5.2 \times 10^3 \approx 10 \times 10^3 = 10^4$, consistent with our answer.

(*b*) The same result holds.

(*c*) $(200\pi)/30 = 600/30 = 20 = 2.0 \times 10^1 \approx 1 \times 10^1 = 10^1$

(*d*) $100 - 0 = 100 = 10^2$

Guided Problem 1.6 Digits on your own

Express the result of each calculation both to the proper number of significant digits and as an order of magnitude:

(*a*) (205)(0.0041)(489.623)

(*b*) $\dfrac{(190.8)(0.407\,500)}{\pi}$

(*c*) 6980.035 + 0.2

1 GETTING STARTED

1. Are the techniques of Worked Problem 1.5 useful? Sufficient?

2 DEVISE PLAN

2. For parts *a* and *b*, how many significant digits does each number contain?
3. For part *c*, which value limits the number of decimal places allowed in the answer?

3 EXECUTE PLAN

4. How do you convert each answer to an order-of-magnitude number?

4 EVALUATE RESULT

Worked Problem 1.7 Oceans

Make an order-of-magnitude estimate of the percent of Earth's mass that is contained in the oceans.

1 GETTING STARTED About 70% of Earth's surface is covered by oceans, but the entire volume of Earth contributes to its mass. To obtain a percentage we need to know the mass of the oceans and the mass of Earth. The latter we might look up or recall, but the former requires a computation based on mass density and volume. We must devise a simple model for the volume of the oceans and perhaps also for Earth.

2 DEVISE PLAN A spherical Earth coated with a thin shell of water covering 70% of the surface seems a reasonable first try. The volume of the oceans is then 70% of the surface area of this sphere multiplied by the ocean depth. The mass of the oceans then involves the mass density of ocean water, the radius of Earth squared, and the average depth of the oceans. We need to estimate the value of each of these and then divide the mass of the oceans by the mass of Earth in order to obtain the desired percentage. That means we must estimate the mass of Earth, too, and this is also related to the radius of Earth. Some factors may cancel out if we express both masses in terms of mass densities.

3 EXECUTE PLAN The mass of Earth, including oceans, is $m_E = \rho_E V_E = \rho_E(\frac{4}{3}\pi R_E^3)$. The area of the oceans is $0.70A_E$. Assuming the oceans have an average depth d, we can approximate their volume as surface area A_o times depth d: $V_o = A_o d = 0.70A_E d$. Their mass is therefore $m_o = \rho_o V_o = \rho_o(0.70A_E d) = \rho_o(0.70)(4\pi R_E^2)d$. The fraction f of Earth's mass contained in the oceans is

$$f = \frac{m_o}{m_E} = \frac{\rho_o(0.70)(4\pi R_E^2)d}{\rho_E(\frac{4}{3}\pi R_E^3)} = 2.1\frac{d}{R_E}\frac{\rho_o}{\rho_E}.$$

There are still four quantities to estimate, but at least all the squaring and cubing of values are eliminated! The ocean's average depth d is about a mile, or 1.6 km. The radius of Earth is about 4000 mi, or 6400 km. The mass density of the ocean is about the same as that of fresh water. The mass density of Earth's solid surface materials (things like rocks) must be a few times greater than the mass density of water because rocks sink readily. Earth's interior must have a considerably larger mass density than this because gravity compresses matter near the center. So we can estimate that the average mass density of Earth is about five times that of water, and therefore the ratio ρ_o/ρ_E is about $1/5$. This makes the fraction f

$$f = (2.1)\left(\frac{1.6\text{ km}}{6400\text{ km}}\right)\tfrac{1}{5} = 1.1 \times 10^{-4} \approx 10^{-4}.$$

Therefore about $1/100$ of 1% of Earth's mass is contained in the oceans.

Check by VENUS:

Vectors/scalars: all quantities are scalars ✔
Every question answered: the percent is calculated ✔
No unknowns in answers: all quantities known or estimated ✔
Units correct: the answer has no units, it is a percentage ✔
Significant digits: order-of-magnitude estimate ✔

4 EVALUATE RESULT The depth of the oceans is much less than the radius of Earth, and water is considerably less dense than the solid material of Earth. We therefore should expect the oceans to contain a very small percent of Earth's mass. If you look up the actual value, you find that our estimate is within a factor of 2 or so.

Guided Problem 1.8 Roof area

One suggestion for reducing the use of fossil fuels is to cover the roofs of all buildings in the United States with solar collectors. Make an order-of-magnitude estimate of the combined surface area, in square kilometers, of all these solar collectors.

❶ GETTING STARTED

1. What simple shape can you use to approximate the United States?
2. Is it reasonable to assume that buildings are primarily in cities?

❷ DEVISE PLAN

3. What is the approximate area of the United States?
4. What percent of that area is occupied by cities?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

Answers to Review Questions

1. Physics is the study of matter and motion in the universe. It is about understanding the unifying patterns that underlie all phenomena in nature.
2. The scientific method is an iterative process that develops validated theories to explain our observations of nature. It involves observing some phenomenon, formulating a hypothesis from the observations, making predictions based on the hypothesis, and validating the predictions by running experiments to test them.
3. Some useful skills are interpreting observations, recognizing patterns, making and recognizing assumptions, reasoning logically, developing models, and using the models to make predictions.
4. Inductive reasoning is arguing from the specific to the general; deductive reasoning is arguing from the general to the specific.
5. Symmetry means that the appearance of an object, process, or law is not changed by a specific operation, such as rotation or reflection.
6. Translational symmetry in space, in which different observers at different locations get the same value for a given measurement, and translational symmetry in time, in which an observer gets the same value for a given measurement taken at different instants, are two types.
7. The *universe* is the totality of matter and energy plus the space and time in which all events occur.
8. An order of magnitude is a value rounded to the nearest power of ten. Using orders of magnitude gives you a feel for a quantity and is a key skill in any quantitative field.
9. The order of magnitude of 2900 is 10^3, and the order of magnitude appropriate for 3100 is 10^4. This is because the first digit 3 is used as the demarcation between rounding up and rounding down. On a logarithmic scale, the base-10 logarithm of 3 is about halfway between the logarithm of 1 and the logarithm of 10.
10. The size scale ranges from the subatomic (10^{-16} m or smaller) to the size of the universe (10^{26} m or larger). The time scale ranges from a hundredth of an attosecond (10^{-20} s) or smaller to the age of the universe (10^{17} s).
11. Time flows in a single, irreversible direction, from past to present to future.
12. The principle of causality states that if an event A causes an event B, all observers see A happening before B. This means that if event A is observed to occur after event B, then A cannot be the cause of B.
13. Making simplified visual representations such as sketches, graphs, or tables helps you to establish a clear mental image of the situation, relate it to past experience, interpret its meaning and consequences, focus on essential features, and organize more relevant information than you can keep track of in your head.
14. The *Concepts* part develops the conceptual framework of the topics covered in the chapter. The *Quantitative Tools* part develops the mathematical framework for these topics.
15. A numerical value and an appropriate unit of measurement are needed.
16. The SI base units are meter for length, second for time, kilogram for mass, ampere for electric current, kelvin for temperature, mole for quantity of substance, and candela for luminous intensity.
17. Density is the concept of how much there is of some substance in a given volume.
18. Multiply the quantity by a conversion factor, a fraction in which the numerator is a number and the desired unit and the denominator is the equivalent value expressed in the given unit.
19. The number of digits is all the digits written to express a numerical value. The number of decimal places is the number of digits to the right of the decimal point. The number of significant digits is the number of digits that are reliably known. The number 0.037 20 has 6 digits, the last 4 of them significant, and 5 decimal places.
20. Leading zeros are any that come before the first nonzero digit in a number. Trailing zeros are any that come after the last nonzero digit. No leading zero is significant. All trailing zeros to the right of the decimal point are significant. Trailing zeros to the left of the decimal point may or may not be significant.
21. The number of significant digits in the result should be the same as the number of significant digits in the input quantity that has the fewest significant digits.
22. The number of decimal places in the result should be the same as the number of decimal places in the input quantity that has the fewest decimal places.
23. Getting started: Identify problem, visualize situation, organize relevant information, clarify goal.
    Devise plan: Figure out what to do by developing a strategy and identifying physical relationships or equations you can use.
    Execute plan: Proceed stepwise through the plan, performing all the mathematical, algebraic, and computational operations necessary to reach the goal; check calculations.

Evaluate result: Consider whether the answer is reasonable, makes sense, reproduces known results in limiting or special cases, or can be confirmed by an alternative approach.

24. **V**ectors and scalars used correctly? **E**very question answered? **N**o unknown quantities in answers? **U**nits correct? **S**ignificant digits justified?
25. They allow you to develop a feel for a problem without getting too involved in the details. They help you explore relationships between physical quantities, consider alternative approaches, make simplifying assumptions, and evaluate answers obtained by more rigorous methods.

Answers to Guided Problems

Guided Problem 1.2

$$\rho_{\mathrm{O}} = \frac{0.0097 m_{\mathrm{Sun}}}{\frac{4}{3}\pi R_{\mathrm{Sun}}^3} = 13.7\ \mathrm{kg/m^3};$$

$$n_{\mathrm{O}} = \frac{\rho_{\mathrm{O}}}{m_{\mathrm{O\ atom}}} = 5.14 \times 10^{26}\ \mathrm{atoms/m^3}$$

Guided Problem 1.4 $V = Wh\ell = 1.05\ \mathrm{m^3}$

Guided Problem 1.6 (*a*) 4.1×10^2, 10^3; (*b*) 24.75, 10^1; (*c*) 6980.2, 10^4

Guided Problem 1.8 About $10^5\ \mathrm{km^2}$, assuming that buildings cover about 1%–2% of the land and that the United States is a rectangle with dimensions 1000 mi × 3000 mi (1600 km × 4800 km)

Guided Practice by Chapter

2 Motion in One Dimension

Review Questions

Answers to these questions can be found at the end of this chapter.

2.1 From reality to model

1. A friend constructs a graph of position as a function of frame number for a film clip of a moving object. She then challenges you to construct a graph identical to hers. She tells you that she measured distances in millimeters, but what two additional pieces of information must she give you?
2. How do you determine whether an object is moving or at rest?

2.2 Position and displacement

3. Suppose you have a film clip showing the motion of several objects. If you want to calibrate distances for a graph, what information do you need to know about at least one of the objects?
4. Explain how the information you named in Review Question 3 can be used to calibrate a graph.
5. What is the purpose of using the phrase *x component of* when describing some physical quantities?
6. How do you represent displacement in a drawing or diagram?

2.3 Representing motion

7. What does *interpolation* mean with regard to plotting data points?
8. Describe how to determine the x component of the position of an object at a specific instant, given (a) a graph of position x as a function of time t and (b) an equation for $x(t)$.

2.4 Average speed and average velocity

9. In an $x(t)$ curve, what is the significance of a steep slope as opposed to a gentle slope? What is the significance of a curve that slopes downward as you move from left to right along the time axis as opposed to a curve that slopes upward as you move from left to right along that axis?
10. What does it signify physically if the x component of an object's average velocity is negative over some time interval?

2.5 Scalars and vectors

11. What are the defining characteristics of a scalar? Of a vector?
12. What is the relationship between the magnitude of a vector and the vector's x component?
13. What is the purpose of the unit vector $\hat{\imath}$?

2.6 Position and displacement vectors

14. What is the mathematical meaning of the symbol Δ, the Greek capital letter delta?
15. Is displacement a scalar or a vector? Is distance a scalar or a vector?
16. Can distance be negative? Can distance traveled be negative?
17. Under what conditions is the x component of the displacement negative?
18. How is distance traveled calculated when motion occurs in three segments: first in one direction along the x axis, then in the opposite direction, and finally in the initial direction?
19. Describe the graphical procedure for adding two vectors and for subtracting one vector from another.
20. If you multiply a vector $\vec{A}$ by a scalar c, is the result a scalar or a vector? If the result is a scalar, what is its magnitude? If the result is a vector, what are its magnitude and direction? What if the scalar $c = 0$?

2.7 Velocity as a vector

21. Is average speed a scalar or a vector? Is average velocity a scalar or a vector?
22. What properties does velocity have that make it a vector?
23. How is an object's average velocity related to its displacement during a given time interval?

2.8 Motion at constant velocity

24. What is the shape of the $x(t)$ curve for an object moving at constant velocity? What is the shape of the $v_x(t)$ curve for this object?
25. An object travels at a constant velocity of 10 m/s north in the time interval from $t = 0$ to $t = 8$ s. What additional information must you know in order to determine the position of the object at $t = 5$ s?
26. In a velocity-versus-time graph, what is the significance of the area under the curve for any time interval $t_f - t_i$?

2.9 Instantaneous velocity

27. What feature of the $x(t)$ curve for a moving object gives the object's x component of velocity at a given instant?
28. Under what conditions is the average velocity over a time interval equal to the instantaneous velocity at every instant in the interval?
29. What mathematical relationship allows you to compute the x component of an object's velocity at some instant, given the object's x component of position as a function of time?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The height of a 20-story apartment building (D)
2. The distance light travels during a human life span (B, N)
3. The displacement (from your mouth) of an (indigestible) popcorn kernel as it passes through your body, and the distance traveled by the same kernel (F, O)
4. The time interval within which a batter must react to a fast pitch before it reaches home plate in professional baseball (C, H)
5. The time interval needed to drive nonstop from San Francisco to New York City by the most direct route (G, K)
6. The distance traveled when you nod off for 2 s while driving on the freeway (K)
7. The average speed of an airliner on a flight from San Francisco to New York City (G, Q)
8. The average speed of a typical car in the United States in one year (not just while it's running) (E)
9. The time interval for a nonstop flight halfway around the world from Paris, France, to Auckland, New Zealand (J and item 7 above)
10. The number of revolutions made by a typical car's tires in one year (L, E)
11. The maximum speed of your right foot while walking (A, M, P)
12. The thickness of rubber lost during one revolution of a typical car tire (I, R, L, S)

Hints

A. What is your average walking speed?
B. What is the speed of light?
C. What is the speed of a fastball thrown by a professional pitcher?
D. What is the height of each story in an apartment building?
E. What distance does a typical car travel during one year?
F. When you are sitting upright, how far above the chair seat is your mouth?
G. What is the distance between San Francisco and New York City?
H. What is the distance from the pitcher's mound to home plate?
I. What thickness of rubber is lost during the lifetime of a car tire?
J. What is the circumference of Earth?
K. What is a typical freeway speed?
L. What is the circumference of a car tire?
M. For what time interval is your right foot at rest if you walk for 2 min?
N. What is a typical human life span?
O. What is the length of the digestive tract in an adult person?
P. If you walk 10 m in a straight line, what is the displacement of your right foot?
Q. How much elapsed time does a flight from San Francisco to New York City require?
R. How many miles of service does a car tire provide?
S. How many revolutions does a car tire make in traveling 1 m?

Key (all values approximate)

A. 2 m/s; **B.** 3×10^8 m/s; **C.** 4×10^1 m/s; **D.** 4 m; **E.** 2×10^7 m; **F.** 1 m; **G.** 5×10^6 m; **H.** 2×10^1 m; **I.** 1×10^{-2} m; **J.** 4×10^7 m; **K.** 3×10^1 m/s; **L.** 2 m; **M.** 1 min; **N.** 2×10^9 s; **O.** 7 m; **P.** 1×10^1 m; **Q.** 2×10^4 s; **R.** 8×10^7 m; **S.** 0.5 rev/m

Worked and Guided Problems

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 2.1 Shopping hunt

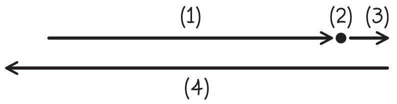
A frantic shopper performs the following sequence of movements along a supermarket aisle in search of a special item:

(1) He moves eastward at a constant speed of 3.0 m/s for 10 s,
(2) stops for 5.0 s,
(3) walks slowly in the same direction at a constant speed of 0.50 m/s for 20 s, and
(4) immediately turns around and rushes westward at a constant speed of 4.0 m/s for 9.0 s.

(*a*) Plot the shopper's position as a function of time during the 44-s interval. (*b*) What is the x component of his average velocity from $t = 0$ to $t = 35$ s? From $t = 0$ to $t = 44$ s? (*c*) What distance does he travel during the 44-s interval? (*d*) What is his average speed during this interval?

❶ GETTING STARTED We know that the shopper first moves in a straight line in one direction, then stops for a while, then moves in the same direction again at a slower speed, and then switches direction. To visualize the motion, we make a sketch that represents the velocity in each of the four segments (Figure WG2.1).

Figure WG2.1



We know that the speed and direction of travel affect the shopper's position at any given instant. Let us say that all the motion is along an x axis. Our first task is to plot the $x(t)$ curve from the starting position, based on the information about his speed and direction of motion from that instant. In addition, we need to determine his average velocity during two time intervals, the distance traveled in 44 s, and the average speed in the 44-s interval.

We summarize the given information to see which quantities we know. We need unambiguous symbols for the quantities in the various segments. We shall use the segment number as a subscript to identify the final instant and final position of each segment; for example, t_3 is the final instant of the third segment, and at this instant the shopper is at position x_3. Choosing the positive x direction to be eastward, we can list this information from the problem statement:

$$v_{x,1} = +3.0 \text{ m/s during interval from } t_0 = 0 \text{ to } t_1 = 10 \text{ s}$$
$$v_{x,2} = 0 \text{ during interval from } t_1 = 10 \text{ s to } t_2 = 15 \text{ s}$$
$$v_{x,3} = +0.50 \text{ m/s during interval from } t_2 = 15 \text{ s to } t_3 = 35 \text{ s}$$
$$v_{x,4} = -4.0 \text{ m/s during interval from } t_3 = 35 \text{ s to } t_4 = 44 \text{ s.}$$

Although it would be tempting to label the initial instant t_i rather than t_0, we recognize that there are several segments to this motion and that the initial instant of motion in one segment is the final instant of motion in the previous segment. Thus t_i in our general kinematic formulas has different values in the different segments and is not always equal to zero. To avoid confusion, therefore, we designate our starting instant by something other than t_i; we choose t_0. Likewise we choose t_4 rather than t_f as the ending instant of the fourth segment to avoid confusion with the symbol t_f used in our general equations as the final instant of each segment.

Because one task is to plot position as a function of time from the starting position, we choose the starting coordinate to be $x_0 = 0$.

❷ DEVISE PLAN In order to make our position-versus-time graph for part *a*, we need to determine the shopper's final position for each segment of the motion. The initial position for each successive segment is equal to the final position for the previous segment. We can connect the start and end positions in any segment with a straight line (constant slope) because the velocity is constant in each segment. For motion at constant velocity, we can use Eq. 2.19 to calculate the position x_f at the end of any time interval:

$$x_f = x_i + v_x(t_f - t_i). \tag{1}$$

For part *b*, we can divide the displacement between the two specified instants by the corresponding time interval to get the x component of the average velocity (Eq. 2.14):

$$v_{x,\text{av}} = \frac{x_f - x_i}{t_f - t_i}. \tag{2}$$

For part *c*, we get the distance traveled in 44 s by adding the distances traveled in the four segments, noting that distance traveled is a magnitude. For part *d*, the average speed is the distance traveled in 44 s divided by that time interval. Of our four tasks, the only "hard" part is getting the displacement in each time interval for our position-versus-time graph.

❸ EXECUTE PLAN (*a*) To plot position versus time, we need to calculate the position at the end of each segment of the motion. To use Eq. 1, we note that in the first segment the initial position is $x_i = x_0 = 0$, the velocity in the x direction is $v_x = v_{x,1} = +3.0$ m/s, the initial instant is $t_i = t_0 = 0$, and the final instant is $t_f = t_1 = 10$ s. Substituting these values into Eq. 1, we compute the shopper's final position $x_f = x_1$ after 10 s (segment 1)

$$x_1 = x_0 + v_{x,1}(t_1 - t_0) = 0 + (+3.0 \text{ m/s})(10 \text{ s} - 0) = +30 \text{ m.}$$

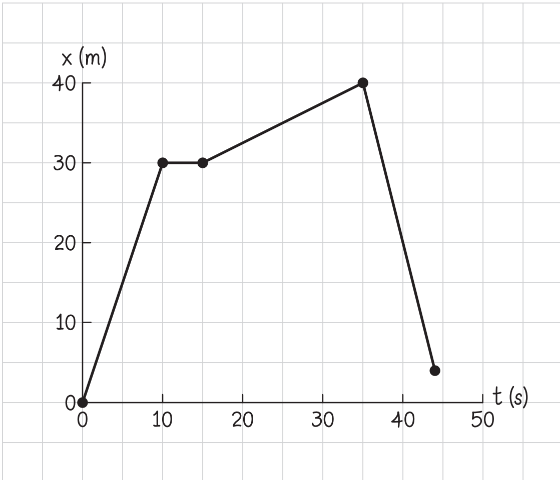
We repeat this procedure for the other three segments, using the final position of the preceding segment as the initial position of the current segment:

$$x_2 = x_1 + v_{x,2}(t_2 - t_1) = +30 \text{ m} + (0)(15 \text{ s} - 10 \text{ s}) = +30 \text{ m}$$
$$x_3 = +30 \text{ m} + (+0.50 \text{ m/s})(35\text{s} - 15 \text{ s}) = +40 \text{ m}$$
$$x_4 = +40 \text{ m} + (-4.0 \text{ m/s})(44 \text{ s} - 35 \text{ s}) = +4.0 \text{ m.}$$

PRACTICE

Now we plot these positions in Figure WG2.2 and connect them with straight lines. ✔

Figure WG2.2



(*b*) From Eq. 2, the x component of the average velocity for the first 35 s, which runs to the end of segment 3, is the x component of the displacement during that time interval divided by the time interval:

$$v_{x,03} = \frac{x_3 - x_0}{t_3 - t_0} = \frac{+40\text{ m} - 0}{35\text{ s} - 0} = +1.1\text{ m/s}. \quad ✔$$

This result is positive because the overall travel is eastward. The x component of the shopper's average velocity over the whole trip (to the end of segment 4) is

$$v_{x,04} = \frac{x_4 - x_0}{t_4 - t_0} = \frac{+4.0\text{ m} - 0}{44\text{ s} - 0} = +0.091\text{ m/s}. ✔ \tag{3}$$

(*c*) The distance traveled in 44 s is the sum of the distances traveled in the various segments. In this case, because each segment involves motion in a single direction, the distance traveled is the same as the distance—that is, the magnitude of the displacement. We can use the symbol d with appropriate subscripts to represent the distances traveled in the segments:

$$\begin{aligned} d_{04} &= d_{01} + d_{12} + d_{23} + d_{34} \\ &= |x_1 - x_0| + |x_2 - x_1| + |x_3 - x_2| + |x_4 - x_3| \\ &= |+30\text{ m} - 0| + |+30\text{ m} - (+30\text{ m})| + \\ &\quad\ |+40\text{ m} - (+30\text{ m})| + |+4.0\text{ m} - (+40\text{ m})| \\ &= 30\text{ m} + 0 + 10\text{ m} + 36\text{ m} = 76\text{ m}. ✔ \end{aligned}$$

(*d*) The average speed is the distance traveled in 44 s divided by that time interval:

$$\begin{aligned} v_{\text{av}} = v_{04} &= \frac{d_{04}}{\Delta t_{04}} = \frac{d_{04}}{t_4 - t_0} = \frac{76\text{ m}}{44\text{ s}} \\ &= 1.7\text{ m/s (between } t_0 \text{ and } t_4). ✔ \end{aligned} \tag{4}$$

Remember that the checkmarks (✔) in our calculations show that VENUS, the mnemonic for checking our work, has been applied.

❹ **EVALUATE RESULT** It is not unreasonable for a frantic shopper to move 76 m in 44 s. This is covering a distance that is a little shorter than the length of a football field in a time interval somewhat less than a minute, which is within the realm of possibility for a person pressed for time.

The signs of our position values agree with the positions we drew in Figure WG2.1: All four x values are on the positive side of the x axis (because the shopper always stays east of the origin). Notice that the average speed of 1.7 m/s (Eq. 4) is very different from the magnitude of the average velocity, 0.091 m/s (Eq. 3). This is to be expected because average speed is based on distance traveled, which is the sum of the magnitudes of the individual displacements, while average velocity is based on displacement, whose magnitude is much less than the distance traveled because of the shopper's backtracking.

Guided Problem 2.2 City driving

You need to drive to a grocery store that is 1.0 mi west of your house on the same street on which you live. There are five traffic lights between your house and the store, and on your trip you reach all five of them just as they change to red. While you are moving, your average speed is 20 mi/h, but you have to wait 1 min at each light. (*a*) How long does it take you to reach the store? (*b*) What is your average velocity for the trip? (*c*) What is your average speed?

❶ GETTING STARTED

1. Draw a diagram that helps you visualize all the driving and stopped segments and your speed in each segment. Does it matter where the traffic lights are located?
2. You start and stop, speed up and slow down. What does the average speed signify in this case, and how is it related to your displacement in each segment?

❷ DEVISE PLAN

3. During how long a time interval are you moving? During how long a time interval are you stopped at the lights?
4. What is your displacement for the trip?
5. How can you apply the answers to questions 3 and 4 to obtain your average velocity?
6. What is the distance traveled, and how is it related to your average speed?
7. How are average velocity and average speed related in this case?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

8. Are your answers plausible, and is your result within the range of your expectations?

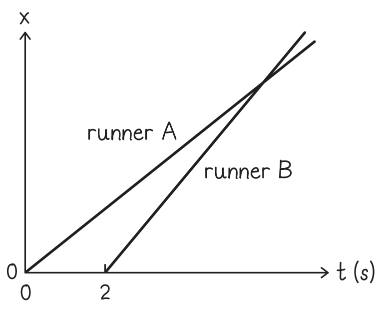
Worked Problem 2.3 Head start

Two runners in a 100-m race start from the same place. Runner A starts as soon as the starting gun is fired and runs at a constant speed of 8.00 m/s. Runner B starts 2.00 s later and runs at a constant speed of 9.30 m/s.

(*a*) Who wins the race? (*b*) At the instant she crosses the finish line, how far is the winner ahead of the other runner? (In your calculations, take the race length, 100 m, to be an exact value.)

❶ GETTING STARTED Although runner A has a 2.00-s head start, runner B runs faster than runner A. We want to know who first reaches the finish line, which is at $x_f = +100$ m. We begin by making a position-versus-time graph for both runners. Because both runners have constant speeds, their $x(t)$ curves are straight lines, with both starting at the same position ($x_i = 0$) but at the different initial instants $t_{A,i}$ and $t_{B,i}$ (Figure WG2.3). Note that the faster runner's curve has the steeper slope.

Figure WG2.3



We see from our drawing that at some instant the two $x(t)$ curves intersect, meaning that the runners are at the same position at that instant, but we do not know whether this instant is before or after the slower runner A crosses the 100-m mark on the track.

❷ DEVISE PLAN Because we want to know who crosses the finish line first, we can determine the instant t_f at which each runner reaches the position $x_f = +100$ m using Eq. 2.18: $x_f - x_i = v_x(t_f - t_i)$.

PRACTICE

The runner who has the lower value for t_f is the winner. Once we know who wins and at what instant she reaches the finish line, we can find out where the other runner is at that instant and then determine the distance between them. Although both runners start at $x_i = 0$, their start times are not the same, and we must account for this. When runner A leaves the starting position ($x_{A,i} = 0$), the timer's stopwatch reads $t_{A,i} = 0$. When runner B leaves the same starting position ($x_{B,i} = 0$), the stopwatch reads $t_{B,i} = 2.00$ s.

❸ **EXECUTE PLAN** (*a*) We need to solve for the instant t_f at which each runner reaches the finish line at $x_f = 100$ m. Each runner moves in the direction of the positive x axis, so the x component of the velocity for each of them is positive: $v_x = +v$, where v is the particular runner's speed. Using $\Delta t = t_f - t_i$, we can rearrange Eq. 2.18 to express t_f, the desired quantity, in terms of the other quantities:

$$x_f - x_i = v_x \Delta t = (+v)(t_f - t_i)$$

$$t_f = t_i + \frac{x_f - x_i}{v}.$$

The reading on the timer's stopwatch for runner A as she reaches the 100-m mark is

$$t_{A,f} = t_{A,i} + \frac{x_{A,f} - x_{A,i}}{v_{Ax}} = 0 + \frac{(+100\text{ m}) - 0}{+8.00\text{ m/s}} = 12.5\text{ s}.$$

As runner B crosses the finish line, the stopwatch reads

$$t_{B,f} = t_{B,i} + \frac{x_{B,f} - x_{B,i}}{v_{Bx}} = 2.00\text{ s} + \frac{(+100\text{ m}) - 0}{+9.30\text{ m/s}} = 12.8\text{ s}.$$

Runner A wins because she crosses the finish line first. ✔

(*b*) To determine their distance apart at the instant runner A crosses the finish line, when the stopwatch reads $t_{A,f} = 12.5$ s, we have to compute $x_{B,f}$, the position of runner B at that instant. From Eq. 2.19, we have

$$\begin{aligned} x_{B,f} &= x_{B,i} + v_{Bx}(t_{A,f} - t_{B,i}) \\ &= 0 + (+9.30\text{ m/s})(12.5\text{ s} - 2.00\text{ s}) = +97.6\text{ m}. \end{aligned}$$

Their distance apart when runner A crosses the finish line is

$$d = |x_{B,f} - x_{A,f}| = |+97.6\text{ m} - (+100\text{ m})| = 2.4\text{ m}. ✔$$

❹ **EVALUATE RESULT** The values we obtained for $t_{A,f}$ and $t_{B,f}$ are both positive, as they must be because negative values would represent instants before the race started. The time intervals of approximately 12 s to run 100 m are reasonable.

We can also solve the second part of this problem in a different way. Runner B is catching up with runner A at the rate of $9.30\text{ m/s} - 8.00\text{ m/s} = 1.30\text{ m/s}$. Runner A has a 2.00-s head start, and so she is $(8.00\text{ m/s})(2.00\text{ s}) = 16.0$ m ahead when runner B starts. Runner A wins the race with a time of 12.5 s; runner B has then been running for $12.5\text{ s} - 2.0\text{ s} = 10.5$ s and has therefore closed the 16.0-m gap by $(1.30\text{ m/s})(10.5\text{ s}) = 13.6$ m; that leaves her $16.0\text{ m} - 13.6\text{ m} = 2.4$ m behind runner A. That result is the same as the one we obtained above.

Guided Problem 2.4 Race rematch

If the two runners in Worked Problem 2.3 want to cross the finish line together in a rematch, and runner A again starts at the instant the starting gun is fired, how many seconds should runner B delay in order to catch runner A right at the finish line?

❶ **GETTING STARTED**

1. Make a graph that shows the new situation, in which both runners reach 100 m at the same instant t_f.

❷ **DEVISE PLAN**

2. What type of motion occurs in this race? What equations can you use to describe this motion?
3. What physical quantity must you determine? What algebraic symbol is associated with this variable?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

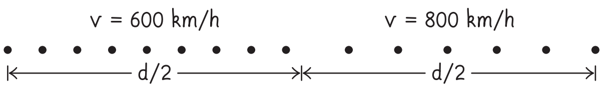
4. After solving for the desired starting time for runner B, how can you use the result of Worked Problem 2.3 to evaluate your answer?

Worked Problem 2.5 Airplane velocity

An airplane travels in a straight line from one airport to another. It flies at a constant speed of 600 km/h for half the distance and then at a constant speed of 800 km/h for the remainder of the distance in order to reach the destination at the scheduled time. What is its average velocity for the entire trip?

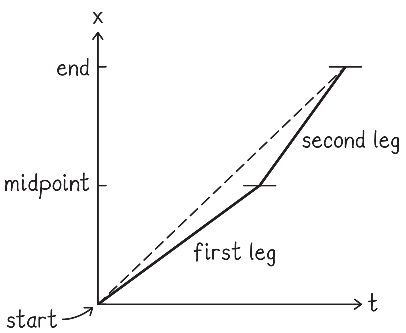
❶ **GETTING STARTED** To picture the situation, we draw a motion diagram. We are not given the distance between the airports, and so we have to use the variable d to represent this distance, as shown in Figure WG2.4.

Figure WG2.4



To get the average velocity, we must divide the displacement by the time interval required for the trip, but neither of these values is given! So we have to devise a way to relate these two unknown quantities to the average velocity. The distance traveled in each segment of the trip is the same, but the segment with the slower speed requires a longer time interval. With this insight, we can draw a position-versus-time graph for the plane's motion (Figure WG2.5) that shows the two segments, where we choose the positive x axis to be in the direction of the plane's motion. We include a dashed line to represent a plane that leaves the first airport at the same instant our plane leaves, travels at the average speed of our plane for its whole trip, and arrives at the second airport at the same instant our plane arrives. The slope of this dashed line represents the average velocity that we seek.

Figure WG2.5



❷ **DEVISE PLAN** If we know the x component of displacement, we can get the x component of the average velocity from Eq. 2.14:

$$v_{x,\text{av}} \equiv \frac{\Delta x}{\Delta t} = \frac{x_\text{f} - x_\text{i}}{t_\text{f} - t_\text{i}}.$$

The x component of the displacement is the sum of the x components of the two (equal) displacements during the two segments of the flight. We can use the symbol x_m to represent the x coordinate of the midpoint position:

$$\Delta x = x_\text{f} - x_\text{i} = (x_\text{f} - x_\text{m}) + (x_\text{m} - x_\text{i}) = \Delta x_2 + \Delta x_1 = \Delta x_1 + \Delta x_2.$$

Furthermore, we know that $\Delta x_1 = \Delta x_2 = +\frac{1}{2}d$. The overall time interval is the sum of the two (unequal) time intervals: $\Delta t = \Delta t_1 + \Delta t_2$. The x component of the velocity during each segment is equal to the change in position divided by the time interval for that segment. Knowing this allows us to express the unknown time intervals in terms of the equal (but unknown) distances.

❸ **EXECUTE PLAN** For the entire trip:

$$v_{x,\text{av}} = \frac{\Delta x}{\Delta t} = \frac{\Delta x_1 + \Delta x_2}{\Delta t_1 + \Delta t_2}.$$

Although the time intervals are unknown, we can express them in terms of the two equal x components of the displacements:

$$\Delta t_1 = \frac{\Delta x_1}{v_{x,1}} \quad \text{and} \quad \Delta t_2 = \frac{\Delta x_2}{v_{x,2}}.$$

Substituting these expressions for Δt into the preceding equation gives

$$v_{x,\text{av}} = \frac{\Delta x_1 + \Delta x_2}{\left(\dfrac{\Delta x_1}{v_{x,1}}\right) + \left(\dfrac{\Delta x_2}{v_{x,2}}\right)} = \frac{(+\frac{1}{2}d) + (+\frac{1}{2}d)}{\left(\dfrac{+\frac{1}{2}d}{v_{x,1}}\right) + \left(\dfrac{+\frac{1}{2}d}{v_{x,2}}\right)}$$

$$= \frac{1}{\left(\dfrac{\frac{1}{2}}{v_{x,1}}\right) + \left(\dfrac{\frac{1}{2}}{v_{x,2}}\right)} = \frac{2}{\left(\dfrac{1}{v_{x,1}}\right) + \left(\dfrac{1}{v_{x,2}}\right)}.$$

The x components of the velocities in both segments are positive—that is, equal to the speeds: $v_{x,1} = +v_1$ and $v_{x,2} = +v_2$. Substituting in the numerical values gives us

$$v_{x,\text{av}} = \frac{2}{\left(\dfrac{1}{+600\ \text{km/h}}\right) + \left(\dfrac{1}{+800\ \text{km/h}}\right)} = +686\ \text{km/h}.$$

Because the problem asks for the average velocity, which is a vector, we multiply the x component of the velocity by the unit vector to get the desired result:

$$\vec{v}_{\text{av}} = v_{x,\text{av}}\hat{\imath} = (+686\ \text{km/h})\hat{\imath}. \checkmark$$

❹ EVALUATE RESULT We expect the x component of velocity to be positive because the velocities in both segments are in the positive x direction. The value of the answer is plausible because it lies between the two velocities at which the airplane actually flies.

Using the two constant speeds 600 km/h and 800 km/h means we ignored the short time interval needed to take off and reach cruising speed and the short time interval needed to slow down for landing. These two time intervals are much shorter than the time interval needed for the flight, and so our simplification is a reasonable one.

Guided Problem 2.6 Wrong way

A helicopter pilot at an airport is told to fly a distance d at speed v directly east to pick up a stranded hiker. However, he sees no hiker when he arrives. Radioing the control tower, he learns that the hiker is actually a distance d west of the airport. The pilot turns around and heads straight for the hiker at a speed that is 50% faster than his speed during the initial segment of this wrong-way trip. (*a*) What is the x component of the helicopter's average velocity for the trip from takeoff to rendezvous with the hiker? Assume the x axis points east and has its origin at the airport. (*b*) What is the helicopter's average velocity for the trip?

❶ GETTING STARTED

1. Draw a diagram representing the motion during the trip, indicating the position of the airport, the turning point, and the rendezvous location.
2. Label the eastward and westward segments with symbols you can use to set up appropriate equations. Then indicate how your chosen symbols relate to the given information, d and v.
3. Add labeled vector arrows to your diagram to indicate the velocity in each segment.
4. Would a graph of $x(t)$ be useful?

❷ DEVISE PLAN

5. How does this problem differ from Worked Problem 2.5? How are the two problems similar?
6. Because no numerical values are given, what physical quantities do you expect to be represented by algebraic symbols in your answer?
7. Be careful about the signs of the x component of the displacement and the x component of the velocity in each segment of the trip. Is Δx in the eastward segment positive or negative? Is v_x in this segment positive or negative? What about Δx and v_x in the westward segment? How can these quantities be expressed in terms of v and d?
8. What time interval is needed to fly the eastward segment? What time interval for the westward segment? If you don't get positive values for both intervals, check your signs for Δx and v_x.
9. How can you combine this information to determine the x component of the average velocity and then the average velocity in vector form?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

10. Is the sign on $v_{x,\text{av}}$ what you expect based on the displacement of the helicopter?

Worked Problem 2.7 Uphill putt

The position of a golf ball moving uphill along a gently sloped green is given as a function of time t by $x(t) = p + qt + rt^2$, with the positive direction up the green and with $p = +2.0$ m, $q = +8.0$ m/s, and $r = -3.0$ m/s^2.

(a) Calculate the x component of the velocity and the speed of the ball at $t = 1.0$ s and at $t = 2.0$ s.

(b) What is the ball's average velocity during the time interval from 1.0 s to 2.0 s?

❶ **GETTING STARTED** We are asked to calculate velocity, a vector. Equation 2.25 defines velocity as the time derivative of position: $\vec{v} = d\vec{r}/dt$. It is easiest to work with components of vectors and then express the final result as a vector. In this case, the components are x and v_x, which we can use to construct the vectors $\vec{x} = x\hat{\imath}$ and $\vec{v} = v_x\hat{\imath}$. According to Eq. 2.22, the x component of the velocity equals the time derivative of the ball's x coordinate:

$$v_x(t) = \frac{dx}{dt}. \tag{1}$$

The problem statement gives this x coordinate as a function of time, and so we have the needed information.

❷ **DEVISE PLAN** For part a, we need to get v_x through our knowledge of the mathematical expression for the golf ball's position $x(t)$. We can then evaluate v_x at the specified instants to get numerical values. For part b, the x component of the average velocity can be determined using Eq. 2.14:

$$v_{x,\text{av}} = \frac{\Delta x}{\Delta t} = \frac{x_\text{f} - x_\text{i}}{t_\text{f} - t_\text{i}}. \checkmark \tag{2}$$

All we need to compute in this equation is Δx because we know that the time interval Δt is 1.0 s.

❸ **EXECUTE PLAN** (a) To determine the x component of the velocity as a function of time, we use Eq. 1:

$$\begin{aligned} v_x(t) &= \frac{d}{dt}[p + qt + rt^2] = q + 2rt \\ &= +8.0\ \text{m/s} + 2(-3.0\ \text{m/s}^2)t \\ &= +8.0\ \text{m/s} - (6.0\ \text{m/s}^2)t. \checkmark \end{aligned}$$

Evaluating this at the specified instants yields

$$\begin{aligned} v_x(1.0\ \text{s}) &= +8.0\ \text{m/s} - (6.0\ \text{m/s}^2)(1.0\ \text{s}) = +2.0\ \text{m/s} \checkmark \\ v_x(2.0\ \text{s}) &= +8.0\ \text{m/s} - (6.0\ \text{m/s}^2)(2.0\ \text{s}) = -4.0\ \text{m/s}. \checkmark \end{aligned}$$

The desired speeds are the magnitudes of these velocity components, 2.0 m/s and 4.0 m/s. ✔

(b) For the average velocity, we need the initial and final coordinates of the motion in the specified time interval for use in Eq. 2:

$$\begin{aligned} x_\text{i} = x(1.0\ \text{s}) &= 2.0\ \text{m} + (8.0\ \text{m/s})(1.0\ \text{s}) - (3.0\ \text{m/s}^2)(1.0\ \text{s})^2 \\ &= +7.0\ \text{m} \\ x_\text{f} = x(2.0\ \text{s}) &= 2.0\ \text{m} + (8.0\ \text{m/s})(2.0\ \text{s}) - (3.0\ \text{m/s}^2)(2.0\ \text{s})^2 \\ &= +6.0\ \text{m}. \end{aligned}$$

The average velocity is then

$$\begin{aligned} \vec{v}_\text{av} &= v_{x,\text{av}}\hat{\imath} = \left(\frac{x_\text{f} - x_\text{i}}{t_\text{f} - t_\text{i}}\right)\hat{\imath} = \frac{(+6.0\ \text{m}) - (+7.0\ \text{m})}{(2.0\ \text{s}) - (1.0\ \text{s})}\hat{\imath} \\ &= (-1.0\ \text{m/s})\,\hat{\imath} \quad \text{(between 1.0 s and 2.0 s).} \checkmark \end{aligned}$$

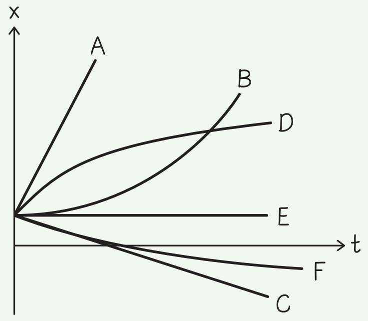
❹ **EVALUATE RESULT** The negative signs for the x component of the velocity at 2.0 s and the x component of the average velocity mean that the ball, whose initial velocity was positive, changed direction during the motion because of the upward slope of the green. This is consistent with the fact that the position x at 2.0 s is a smaller positive number than the position x at 1.0 s. It is also reassuring that the x component of the average velocity computed from the displacement for the interval from 1.0 s to 2.0 s is between the values we calculated for the instantaneous velocity at 1.0 s and 2.0 s.

PRACTICE

Guided Problem 2.8 You're it!

Six children—call them A, B, C, D, E, and F—are running up and down the street playing tag. Their positions as a function of time are plotted in Figure WG2.6. The street runs east-west, and the positive x direction is eastward. Use Figure WG2.6 to answer these questions: (*a*) In which direction is each child moving? (*b*) Which, if any, of the children are running at constant velocity? Of those who are, is the x component of their velocity positive or negative? (*c*) Which, if any, of the children are not moving at constant velocity? Are they speeding up or slowing down? (*d*) Which child has the highest average speed? The lowest average speed? (*e*) Does any child pass another child during the time interval shown in the graph?

Figure WG2.6



❶ **GETTING STARTED**

1. Use the curves in Figure WG2.6 to describe the motion of each child in words.

❷ **DEVISE PLAN**

2. How can you determine, from an $x(t)$ curve, the direction of a child's motion at any instant, and how is that information related to velocity and to speed?
3. What does motion at constant velocity look like on a position-versus-time graph? What does a changing-velocity situation look like on the graph?
4. How can you tell which child has the highest speed and which the lowest speed?
5. What feature in the graph would indicate that one child has passed another? In other words, what is the relationship of their positions at the instant one passes another?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

Answers to Review Questions

1. If your graph is to be identical, she must tell you what she chose as her reference axis and her origin.
2. An object is moving if its position changes with time. It is at rest if its position remains the same as time passes.
3. You need to know the actual size of the object.
4. Use one dimension of the object's actual size to determine what real-world length 1 mm on a frame of the film clip corresponds to. If the object is, for instance, a car 3 m long and the car length on the frame is 2 mm, you know that every 1-mm distance moved by the objects in the film clip represents a real-world distance of 1.5 m. Then use the conversion factor (1.5 m)/(1 mm) to convert any distance each object moves on the film clip to a real-world distance in meters, and plot the distances (or quantities derived from them) on your graph.
5. The phrase reminds you that those quantities are vectors and are measured with respect to some specific x axis.
6. Displacement is a vector, so it is represented by an arrow drawn from the initial position to the final position.
7. *Interpolation* means drawing a smooth curve through a set of data points plotted on a graph. The region of the curve between any two adjacent points tells us, for the infinite number of points between the two plotted points, the numerical value of the quantity being plotted.
8. (*a*) Locate the instant on the time axis and draw a vertical line that passes through that value and intersects the curve. Then draw a horizontal line that passes through the intersection point and extends to the position axis. (*b*) Solve the equation for x, substitute the specific time value wherever the symbol t occurs in the equation, and calculate the numerical value of x.
9. A steep slope indicates a larger speed than that associated with a gentle slope. A downward slope indicates velocity in the negative x direction; an upward slope indicates velocity in the positive x direction.
10. The object moved in the negative x direction during that time interval.
11. A scalar is a mathematical quantity that is completely specified by a number and a unit of measure. A vector is a quantity that must be specified by a direction in addition to a number and a unit of measure.
12. The magnitude of a vector is a number and a unit of measure equal to the absolute value of the x component of the vector.
13. The purpose of $\hat{\imath}$ is to define the direction of the positive x axis.

14. The symbol Δ means "the change in" the variable that immediately follows the symbol Δ. It is the difference between the final value of that variable and its initial value.
15. Displacement is a vector; distance is a scalar.
16. Distance can never be negative because it is defined as the absolute value of the difference between two positions. Distance traveled can never be negative because it is a sum of distances (all positive) between successive positions in the segments of a trip.
17. The x component of the displacement is negative when the direction of the displacement is opposite the direction of the positive x axis.
18. Distance traveled is found by adding the three distances (which are all positive by definition).
19. You add two vectors by placing the tail of the second vector at the tip of the first vector. The sum is the vector that runs from the tail of the first vector to the tip of the second. You subtract one vector from another by reversing the direction of the vector being subtracted and then adding this reversed vector to the other vector.
20. The result is a vector of magnitude $|c\vec{A}| = |c||\vec{A}| = |c|A$. Its direction is the same as the direction of $\vec{A}$ if c is positive and opposite the direction of $\vec{A}$ if c is negative. If c is zero, the result is the *zero vector*, which has no direction.
21. Average speed is a scalar; average velocity is a vector.
22. Velocity has a magnitude (the moving object's speed) and a direction (which way the object is going).
23. The average velocity during some time interval is the displacement divided by the time interval.
24. The $x(t)$ curve is a straight line with a constant slope that is not zero, and the $v_x(t)$ curve is a horizontal line.
25. You need to know its position at some instant during this interval, usually the initial position.
26. The area under the curve gives the x component of the displacement during the time interval.
27. The x component of velocity at a given instant is the slope of the tangent to the $x(t)$ curve at that instant.
28. They are equal when the velocity is constant.
29. The derivative of the position with respect to time $v_x = dx/dt$.

Answers to Guided Problems

Guided Problem 2.2 (a) 8 min; (b) 7.5 mi/h, west; (c) 7.5 mi/h
Guided Problem 2.4 1.8 s
Guided Problem 2.6 (a) $-0.43v$; (b) $-0.43v\hat{\imath}$, with $\hat{\imath}$ pointing west
Guided Problem 2.8 (a) A, B, D east; C, F west; E standing still; (b) A, C, E; x component of velocity is positive for A, negative for C, zero for E; (c) B speeding up; D, F slowing down; (d) A highest average speed, E lowest average speed; (e) B passes D

Guided Practice by Chapter

3 Acceleration

Review Questions

Answers to these questions can be found at the end of this chapter.

3.1 Changes in velocity

1. What is the difference between velocity and acceleration?
2. Does *nonzero acceleration* mean the same thing as *speeding up*?
3. Does the acceleration vector always point in the direction in which an object is moving? If so, explain why. If not, describe a situation in which the direction of the acceleration is not the same as the direction of motion.
4. How is the curvature of an $x(t)$ curve related to the sign of the x component of acceleration?

3.2 Acceleration due to gravity

5. A cantaloupe and a plum fall from kitchen-counter height at the same instant. Which hits the floor first?
6. Is it correct to say that a stone dropped from a bridge into the water speeds up as it falls because the acceleration due to gravity increases as the stone gets closer to Earth?

3.3 Projectile motion

7. You toss a rock straight up. Compare the acceleration of the rock at the instant just after it leaves your hand with its acceleration at the instant just before it lands back in your hand, which has remained at the point of release.
8. You throw a ball straight up. What is the ball's acceleration at the top of its trajectory?

3.4 Motion diagrams

9. List the information that should be included in a motion diagram.
10. What is the purpose of a motion diagram?

3.5 Motion with constant acceleration

11. What can you say about a train's acceleration if its $v(t)$ curve is (*a*) a straight line that is not parallel to the t axis and (*b*) a horizontal line that is parallel to the t axis?
12. For an object experiencing constant acceleration, the expression for position as a function of time is $x(t) = x_i + v_{x,i}t + \frac{1}{2}a_x t^2$. Explain, in terms of the area under the $v(t)$ curve for the object, why the acceleration term includes the factor $\frac{1}{2}$.
13. For constant acceleration, describe the relationship among displacement, initial and final velocities, and acceleration when the time variable is algebraically eliminated.

3.6 Free-fall equations

14. You throw a ball straight up and then hold your hand at the release position. Compare the time interval between the release of the ball and its arrival at its highest position to the time interval between leaving its highest position and returning to your hand.
15. (*a*) For an object released from rest, describe how the distance the object falls varies with time. (*b*) Describe how the object's speed varies with time.

3.7 Inclined planes

16. How is distance traveled related to the amount of time needed to travel that distance for a ball rolling down an inclined plane?
17. In what way does the motion of an object rolling down an inclined plane resemble that of an object in free fall?
18. On which of the following, if any, does the magnitude of the acceleration of a ball rolling down an inclined plane depend: angle of incline, speed of ball, direction of motion?

3.8 Instantaneous acceleration

19. For what type of motion is it important to distinguish between instantaneous and average acceleration?
20. What does each of the following represent: (*a*) slope of an $x(t)$ curve at a given point on the curve, (*b*) curvature of an $x(t)$ curve at a given point on the curve, (*c*) slope of a $v(t)$ curve at a given point on the curve, (*d*) area under an $a(t)$ curve, (*e*) area under a $v(t)$ curve?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The time interval needed for a hailstone to fall to the ground from the altitude of a cruising airliner (H, F, O, E, G)
2. The speed of a ball released from the top of a 100-story skyscraper just before the ball hits the ground (H, F, R, O, V)
3. The speed at which a hailstone hits the ground after falling from the altitude of a cruising airliner (H, F, R, O, G)
4. The magnitude of the average acceleration of a fast sports car as it accelerates from rest to freeway cruising speed (D, P, O, C)
5. The magnitude of the average acceleration of a passenger car cruising on the freeway when it has to make an emergency stop (D, A, P, K)
6. The magnitude of a car's average acceleration moving at a typical city speed when it crashes into a stalled dump truck (R, M, K, S)
7. The magnitude of your average acceleration while getting up to speed in a sprint run (D, O, Q, T)

8. The magnitude of your average acceleration while getting up to speed on a bicycle (D, I, O, U)
9. The magnitude of your acceleration while landing feet first on the ground after jumping off a wall and falling for 1 s (R, N, J)
10. The magnitude of the average acceleration of an airliner speeding up for takeoff (D, L, B)

Hints

A. How many seconds does it take a car to stop from cruising speed?
B. How many seconds does it take an airliner to lift off after starting from rest?
C. What minimum time interval is needed to reach freeway cruising speed?
D. What expression relates change in velocity, average acceleration, and the time interval over which the velocity change occurs?
E. How are distance traveled, acceleration, and time interval related for this type of motion?
F. What type of motion is this?
G. What is the cruising altitude of an airliner?
H. What must you assume about the effect of air resistance?
I. What is your maximum speed on a bicycle?
J. What is your speed just before your feet hit the ground?
K. What is the final speed?
L. What is an airliner's takeoff speed?
M. What is the typical speed of a car traveling on an uncongested city street?
N. When your feet hit the ground, what maximum vertical displacement could the center of your body travel before stopping?
O. What is the initial speed?
P. What is the cruising speed of a car on a freeway?
Q. What is your sprinting speed?
R. How are initial speed, final speed, acceleration, and displacement related for this type of motion?
S. What distance do you travel while a car accelerates during a crash?
T. How many seconds does it take you to reach sprinting speed?
U. How many seconds does it take you to reach top speed on a bicycle?
V. What is the height of a 100-story skyscraper?

Key (all values approximate)

A. 4 s; **B.** 3×10^1 s; **C.** 4 s; **D.** Eq. 3.1: $a_{x,\text{av}} = \dfrac{\Delta v_x}{\Delta t}$; **E.** for an object starting from rest, (distance traveled) $=$ |displacement| $= |a_x(\Delta t)^2/2|$; **F.** free-fall motion; **G.** 1×10^4 m; **H.** assume that ignoring air resistance does not affect your calculation; **I.** 7 m/s; **J.** 1×10^1 m/s; **K.** 0; **L.** 6×10^1 m/s; **M.** 1×10^1 m/s; **N.** 1 m; **O.** assume $v_\text{i} = 0$; **P.** 3×10^1 m/s; **Q.** 5 m/s; **R.** Eq. 3.13: $v_{x,\text{f}}^2 = v_{x,\text{i}}^2 + 2a_x\Delta x$; **S.** 1 m; **T.** 2 s; **U.** 5 s; **V.** 4×10^2 m

Worked and Guided Problems

Procedure: Analyzing motion using motion diagrams

When you solve motion problems, it is important to begin by making a diagram that summarizes what you know about the motion.

1. Use dots to represent the moving object at equally spaced time intervals. If the object moves at constant speed, the dots are evenly spaced; if the object speeds up, the spacing between the dots increases; if the object slows down, the spacing decreases.
2. Choose an x (position) axis that is convenient for the problem. Most often this is an axis that (*a*) has its origin at the initial or final position of the object and (*b*) is oriented in the direction of motion or acceleration.
3. Specify the position and velocity at all relevant instants. In particular, specify the *initial conditions*—position and velocity at the beginning of the time interval of interest—and the *final conditions*—position and velocity at the end of that time interval. Also specify all positions where the velocity reverses direction or the acceleration changes. Label any unknown parameters with a question mark.
4. Indicate the acceleration of the object between all the instants specified in step 3.
5. To consider the motion of more than one object, draw separate diagrams side by side, one for each object, using one common x axis.
6. If the object reverses direction, separate the motion diagram into two parts, one for each direction of travel.

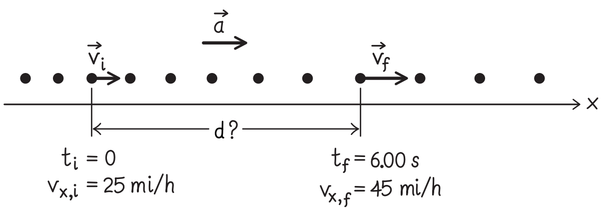
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 3.1 Speeding up

A woman driving at the speed limit in a 25-mi/h zone enters a zone where the speed limit is 45 mi/h. She accelerates at a constant rate and reaches the new speed limit in 6.00 s. What distance does the car travel during that acceleration?

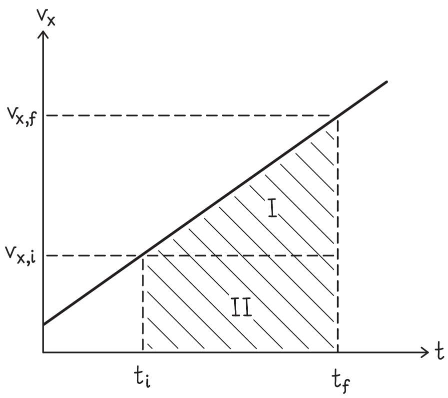
❶ GETTING STARTED This is a constant-acceleration problem. Let the car's direction of motion be the positive x direction. We sketch a motion diagram (Figure WG3.1) and on it indicate the initial and final velocities v_i and v_f, the time interval $\Delta t = t_f - t_i$, and the unknown distance d the car travels in the time interval Δt.

Figure WG3.1



❷ DEVISE PLAN Because the car moves in one direction only, the distance d it travels is equal to the absolute value of its displacement: $d = |\Delta x|$. We must therefore determine Δx. Knowing that displacement is equal to the area under a constant-acceleration $v(t)$ curve, we sketch one that matches Figure 3.17*b* (Figure WG3.2):

Figure WG3.2



Here the shaded area representing Δx as the car's speed changes from $v_{x,i}$ to $v_{x,f}$ in the time interval Δt.

❸ EXECUTE PLAN The shaded area is most easily analyzed as a triangle (I) plus a rectangle (II). The areas are

$$\mathrm{I} = \tfrac{1}{2}(v_{x,f} - v_{x,i})(t_f - t_i); \qquad \mathrm{II} = v_{x,i}(t_f - t_i).$$

Combining areas I and II, we obtain the car's displacement:

$$\Delta x = \tfrac{1}{2}(v_{x,i} + v_{x,f})\Delta t.$$

After we convert units, this expression allows us to calculate

$$d = |\Delta x|$$

to three significant digits:

$$\frac{25.0 \text{ mi}}{\text{h}} \times \frac{1609 \text{ m}}{1 \text{ mi}} \times \frac{1 \text{ h}}{3600 \text{ s}} = 11.2 \text{ m/s};$$

$$\frac{45.0 \text{ mi}}{\text{h}} \times \frac{1609 \text{ m}}{1 \text{ mi}} \times \frac{1 \text{ h}}{3600 \text{ s}} = 20.1 \text{ m/s}$$

$$d = |\Delta x| = \tfrac{1}{2}(11.2 \text{ m/s} + 20.1 \text{ m/s})(6.00 \text{ s}) = 93.9 \text{ m.} ✔$$

❹ **EVALUATE RESULT** Accelerating from 25 mi/h to 45 mi/h in just less than 94 m (a bit more than 300 ft) is about what you might expect based on your experience driving cars.

Guided Problem 3.2 Slowing down

You are driving at 45 mi/h in a 30-mi/h zone. Spotting a police officer ahead, you brake at a constant rate to 27 mi/h, traveling 275 ft in so doing. What time interval do you need to reach your final speed?

❶ **GETTING STARTED**

1. What quantities do you know? What quantity must you compute?
2. Draw a motion diagram. What kind of motion is involved?

❷ **DEVISE PLAN**

3. How is this problem similar to Worked Problem 3.1? How is it different?
4. What are the numerical values of the known quantities in SI units?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

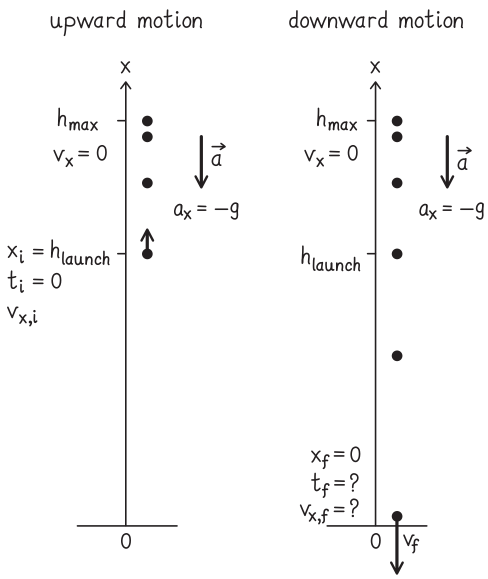
Worked Problem 3.3 Average and instantaneous

You love projectile motion, and you throw baseballs, rocks, and other small objects almost every day. It occurs to you that, for constant-acceleration motion, there might be a special relationship between average velocity and instantaneous velocity. Standing on a platform with your shoulders at height h_{launch} above the ground, you launch a ball upward at speed v_i and observe that the ball rises to a height h_{max} above the ground and then falls to the ground.

❶ **GETTING STARTED** This is a context-rich problem because there is no explicit question. And there are no numbers either! We must therefore generate as much information as we can about average and instantaneous velocities, and hope that a relationship between average velocity and instantaneous velocity reveals itself. We begin with a motion diagram and then write the motion equations.

Figure WG3.3 is the motion diagram for the ball. We separate the motion into upward and downward portions for clarity in the diagram; this separation is not required in the algebraic analysis, however, because the acceleration is the same throughout the motion.

Figure WG3.3



❷ **DEVISE PLAN** The instantaneous velocity values other than $v_{x,i}$ can be obtained from an equation such as Eq. 3.12 with $-g$ in place of a_x:

$$v_x(t) = v_{x,i} - gt. \tag{1}$$

However, we might also use Eq. 3.13, which involves instantaneous velocity and displacement but not a time interval, again with $-g$ in place of a_x:

$$v_{x,f}^2 = v_{x,i}^2 - 2g\Delta x. \tag{2}$$

There is also Eq. 3.11 for position:

$$x(t) = x_i + v_{x,i}t + \tfrac{1}{2}a_x t^2,$$

but this expression does not contain any velocity information other than the initial velocity. We tentatively reject it.

The time interval over which the motion takes place, with t_i set at 0 and the subscript on t_f dropped, is

$$\Delta t = t_f - t_i = t, \tag{3}$$

and t can be obtained from Eq. 1.

Our task is to obtain a relationship between instantaneous velocity v and average velocity v_{av}. Average velocity is defined as the displacement Δx divided by the time interval Δt over which the displacement occurs (see Chapter 2). We should be able to combine the three equations above with our knowledge of the ball's initial and final positions to get an expression for average velocity in terms of instantaneous velocities.

❸ **EXECUTE PLAN** The displacement is the difference in the ball's final and initial positions, both of which are known. We choose the origin at ground level and up as the positive x direction: $x_i = h_{launch}$, $x_f = 0$:

$$\Delta x = x_f - x_i = 0 - h_{launch} = -h_{launch}.$$

We use Eq. 2 to get the ball's final speed at the instant just before it hits the ground:

$$v_{x,f}^2 = v_{x,i}^2 - 2g\Delta x = v_{x,i}^2 + 2gh_{launch}$$

$$|v_{x,f}| = \sqrt{v_{x,i}^2 + 2gh_{launch}}. \tag{4}$$

We do not yet know the initial speed needed for Eq. 4, but it must be related to the ball's maximum height h_{max} because a faster throw should make the ball go higher. Let's use Eq. 2 again to obtain this relationship, evaluating between the initial height h_{launch} and the maximum height h_{max} (because the ball's speed is zero at h_{max}):

$$0 = v_{x,i}^2 - 2g(h_{max} - h_{launch}),$$

so that

$$|v_{x,i}| = v_i = \sqrt{2g(h_{max} - h_{launch})}.$$

Substituting this expression for $v_{x,i}$ into Eq. 4 gives for the final speed of the ball at the instant before it hits the ground:

$$\begin{aligned}|v_{x,f}| &= \sqrt{v_{x,i}^2 + 2gh_{launch}} = \sqrt{2g(h_{max} - h_{launch}) + 2gh_{launch}}\\ &= \sqrt{2gh_{max} - 2gh_{launch} + 2gh_{launch}}\\ |v_{x,f}| &= \sqrt{2gh_{max}}.\end{aligned}$$

Note that this is a speed, which means we should take the positive square root. We know, though, from our choice of positive x axis upward that the final velocity is directed downward and hence is negative.

We can get the time interval Δt by combining Eqs. 1 and Eq. 3:

$$\Delta t = t = \frac{v_{x,f} - v_{x,i}}{-g}.$$

The average velocity in terms of our expressions for Δx and Δt is

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{-h_{launch}}{\left(\dfrac{v_{x,f} - v_{x,i}}{-g}\right)} = \frac{gh_{launch}}{v_{x,f} - v_{x,i}}. \tag{5}$$

This is nice, but is there a way to get the right side entirely in terms of instantaneous velocities? Yes, there is. If we substitute $-h_{launch}$ for Δx in Eq. 2 and rearrange, we have

$$gh_{launch} = \tfrac{1}{2}(v_{x,f}^2 - v_{x,i}^2).$$

PRACTICE

Substituting the term on the right into Eq. 5 gives

$$v_{x,\mathrm{av}} = \frac{\frac{1}{2}(v_{x,\mathrm{f}}^2 - v_{x,\mathrm{i}}^2)}{(v_{x,\mathrm{f}} - v_{x,\mathrm{i}})} = \tfrac{1}{2}(v_{x,\mathrm{f}} + v_{x,\mathrm{i}}).\ ✔ \tag{6}$$

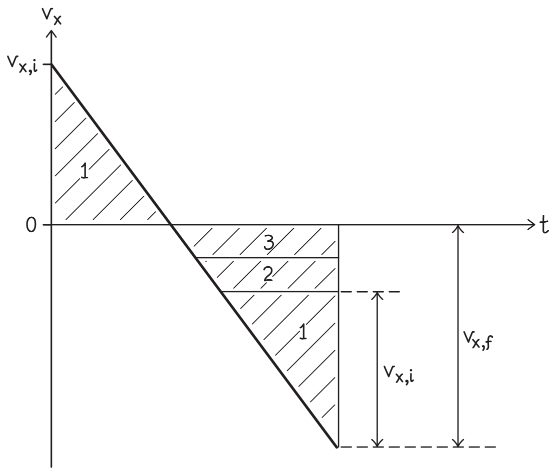
This expression tells us that the average velocity during a chosen time interval is equal to the numerical average of the initial and final instantaneous velocities for that interval. In this example, because $v_{x,\mathrm{f}}$ is negative and of larger magnitude than v_i, which we know from Eq. 4, the average velocity is negative.

This relationship between average and instantaneous velocities seems to be quite general because it does not rely on any values specific to this problem. Of course, it applies only to cases of constant acceleration because Eqs. 1 and 2 are only valid for constant acceleration.

❹ **EVALUATE RESULT** There are no numbers to check, so we seek another way to solve the problem. Perhaps we can visualize this general result graphically. What sort of graph should we draw? A graph of instantaneous velocity versus time comes to mind for several reasons. First, our answer involves instantaneous velocity. Second, from this graph we can obtain displacement (area under the curve) and hence determine average velocity. Third, because velocity is a linear function of time in constant-acceleration problems (Eq. 3.12), this graph is much easier to sketch than the quadratic graph of position versus time (Eq. 3.11). (The acceleration-versus-time graph is even easier to sketch because it is a straight line parallel to the time axis, but it provides less insight.)

Figure WG3.4 shows the velocity-versus-time graph for the ball. Because we chose the positive direction to be upward in our motion diagram, the $v(t)$ curve is a straight line with negative slope (equal to $-g$). The curve extends along the time axis until the ball strikes the ground.

Figure WG3.4



Consider the labeled areas between the $v(t)$ curve and the t axis in Figure WG3.4. Suppose we ask, What (constant) average velocity would produce the same displacement in the same time interval as the varying velocity we had in this problem? As we saw in Chapter 2, at constant velocity the $v(t)$ curve is a straight line parallel to the t axis. In order to produce the same displacement in a given time interval, the area under the constant-velocity curve between any two values t_f and t_i must match the area under the actual $v(t)$ curve for our tossed ball between those same values of t_f and t_i.

Note that the two areas labeled 1 are equal in size and opposite in sign. This means that they cancel when added, and so the net area under the curve is area 2 + area 3. Both these areas are negative in our example, leading to a combined negative area, as we expect for an object with negative average velocity.

Suppose we choose the dividing line between areas 2 and 3such that these areas have the same height. By construction, area 2 + area 3 has height $-(|v_{x,\mathrm{f}}| - |v_{x,\mathrm{i}}|)$. Note that $|v_{x,\mathrm{i}}| = +v_{x,\mathrm{i}}$, while $|v_{x,\mathrm{f}}| = -v_{x,\mathrm{f}}$, so we can write the height as $-(-v_{x,\mathrm{f}} - v_{x,\mathrm{i}}) = (v_{x,\mathrm{f}} + v_{x,\mathrm{i}})$. Note that this is a negative height, as expected. Splitting the height equally between areas 2 and 3 gives each height $\frac{1}{2}(v_{x,\mathrm{f}} + v_{x,\mathrm{i}})$. Now imagine moving area 2 to the location shown in Figure WG3.5. The result is a rectangle 2 + 3 of height $\frac{1}{2}(v_{x,\mathrm{f}} + v_{x,\mathrm{i}})$ and length Δt. Thus the area under the $v(t)$ curve, which is the ball's displacement, is equal to the area of rectangle 2 + 3:

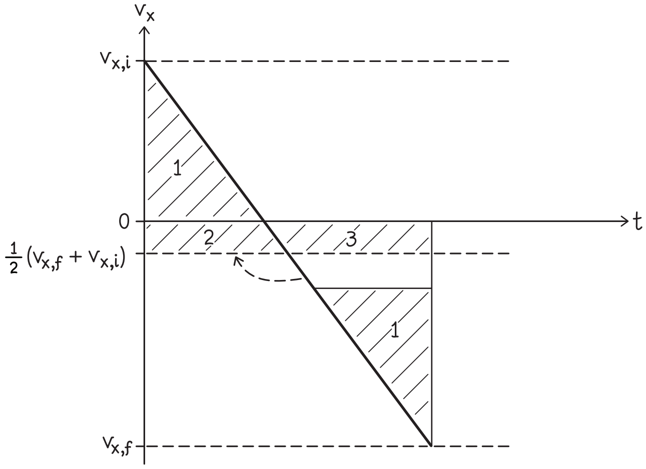
$$\Delta x = \tfrac{1}{2}(v_{x,\mathrm{f}} + v_{x,\mathrm{i}})\Delta t,$$

which means

$$v_{x,\mathrm{av}} = \frac{\Delta x}{\Delta t} = \tfrac{1}{2}(v_{x,\mathrm{f}} + v_{x,\mathrm{i}}),\ ✔ \tag{6}$$

PRACTICE

Figure WG3.5



which is the same as Eq. 6. This same geometrical construction works for initial and final velocities that are either negative or positive (try it!), as long as the curve is a line of constant slope (constant acceleration).

Because our graphical and algebraic methods lead to the same results, we have confidence in the answer.

Guided Problem 3.4 Double launch

You throw a ball upward from height h_{launch} as in Worked Problem 3.3 and it rises to height $h_{\text{max}} = 2h_{\text{launch}}$ before falling to the ground. At the instant you release the ball, a friend standing on a platform at height h_{max} simultaneously throws an identical ball with half the upward velocity you used. Determine (*a*) which ball hits the ground first, (*b*) the final speed of each ball, and (*c*) the time interval between the instant one ball lands and the instant the other ball lands.

❶ GETTING STARTED

1. Draw a motion diagram and a velocity-versus-time graph for each ball.
2. Which technique(s) of Worked Problem 3.3 can you use?

❷ DEVISE PLAN

3. Where is your friend's ball at the instant your ball reaches its maximum height h_{max}?

❸ EXECUTE PLAN

4. Express your answers in terms of the given information.

❹ EVALUATE RESULT

Worked Problem 3.5 Inclined track

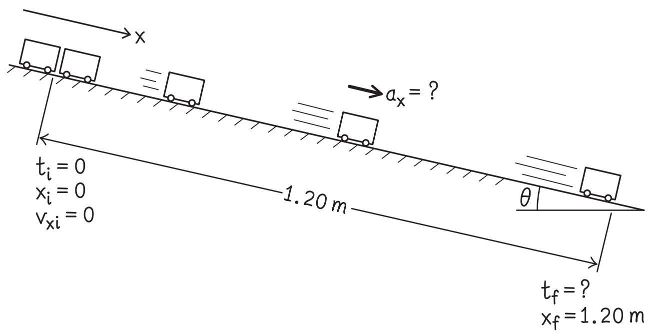
Your physics instructor prepares a laboratory exercise in which you will use a modern version of Galileo's inclined plane to determine acceleration due to gravity. In the experiment, an electronic timer records the time interval required for a cart initially at rest to descend 1.20 m along a low-friction track inclined at some angle θ with respect to the horizontal.

(*a*) In preparation for the experiment, you must obtain an equation from which you can calculate g on the basis of these measurements. What is that equation?

(*b*) To make it possible to check the students' measurements quickly, the instructor breaks the class into five groups and assigns one value of θ to each group. If no mistakes are made, these five θ values yield time intervals of 0.700, 0.800, 0.900, 1.00, and 1.20 s. What are the five θ values?

❶ GETTING STARTED The cart undergoes constant acceleration, from rest, on an inclined plane. We know how to analyze this type of motion, and we know how the acceleration at any given incline angle is related to the acceleration g due to gravity. We sketch a motion diagram (Figure WG3.6), representing a cart moving down an inclined plane, and choose the positive x direction as pointing down the track.

Figure WG3.6



❷ **DEVISE PLAN** The equation asked for in part *a* might be based on Eq. 3.20, $a_x = +g \sin\theta$, but you will not be measuring a_x values directly in this experiment. You will measure displacements Δx and time intervals Δt, which means we need an expression that gives acceleration in terms of these two variables. Equation 3.11 comes to mind, which we can manipulate so that a_x is expressed in terms of Δx and Δt. In part *b*, we can use this result to obtain the five θ values.

❸ **EXECUTE PLAN** (*a*) Equation 3.11 yields $a_x = 2(x_f - x_i)/t^2 = 2\Delta x/t^2$. Because in the derivation for Eq. 3.11 t_i was taken to be zero, the t^2 is actually $(\Delta t)^2$, so that we have

$$a_x = \frac{2\Delta x}{(\Delta t)^2}.$$

Substituting this expression for a_x in Eq. 3.20 yields

$$\frac{2\Delta x}{(\Delta t)^2} = g \sin\theta,$$

from which we obtain the expression for g to be used in the experiment:

$$g = \frac{2\Delta x}{(\Delta t)^2 \sin\theta}. \checkmark \qquad (1)$$

(*b*) Manipulation of Eq. 1 gives the instructor

$$\sin\theta = \frac{2\Delta x}{g(\Delta t)^2}$$

$$\theta = \sin^{-1}\left(\frac{2\Delta x}{g(\Delta t)^2}\right). \qquad (2)$$

Before evaluating this expression five times, the instructor calculates the constant quantity $2\Delta x/g = 0.2449$ and stores it in her calculator. Substitution of $\Delta t = 0.700$, 0.800, 0.900, 1.00, and 1.20 s into Eq. 2 yields the angles of incline she assigned to the five groups: 30.0°, 22.5°, 17.6°, 14.2°, and 9.79°. ✔

❹ **EVALUATE RESULT** The numerical values for the angles are reasonable: Larger angles are associated with smaller time intervals. Even the shortest interval is considerably longer than the time interval needed for an object to fall freely from a height of 1.2 m, as expected.

Guided Problem 3.6 Another inclined track

At another university, in a laboratory exercise similar to that described in Worked Problem 3.5, students measure the angle of incline θ of a low-friction track, the (nonzero) initial and final speeds of a cart as it descends between two positions on the track, and the distance between those two positions. For an angle of incline of 10.0°, one group obtains the values $v_i = 0.820$ m/s and $v_f = 1.65$ m/s for a distance of 0.608 m. On the basis of these data, what value do these students obtain for g?

❶ **GETTING STARTED**

1. Draw a motion diagram. What kind of motion are we dealing with?
2. Choose an appropriate x direction and origin.

❷ **DEVISE PLAN**

3. How is this problem similar to Worked Problem 3.5? How is it different?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

4. Consider whether small or large differences in measured data would be needed to produce a result of 9.80 m/s^2.

Answers to Review Questions

1. Velocity measures displacement (change in position) per unit of time; acceleration measures change in velocity per unit of time.
2. No. *Accelerating* means moving with changing velocity. This includes speeding up as well as any other nonconstant velocity, such as slowing down.
3. No. An object that is slowing down along a straight path has an acceleration vector that points in the direction opposite the direction of motion.
4. The x component of acceleration is positive if the curvature is upward and negative if the curvature is downward. No curvature means no acceleration.
5. They hit at the same instant. If we ignore air resistance, all freely falling objects fall with the same constant acceleration.
6. No. While the stone is falling through the air, its acceleration is a constant 9.8 m/s^2 straight down. Its velocity (and, in this case, its speed) increases because of this constant acceleration. (Once in the water, the stone experiences a different acceleration because the water resistance is much greater than the air resistance, which is negligibly small during the stone's fall.)
7. Because air resistance is negligible for a rock tossed in the air, the acceleration is 9.8 m/s^2 downward at both instants. The acceleration due to gravity is constant near Earth's surface, and the rock experiences that acceleration at all instants during its round trip.
8. The acceleration at the top is 9.8 m/s^2 downward, just as it is during the rest of the flight. The fact that $v = 0$ at the top of the trajectory doesn't mean $a = 0$. The ball is headed upward the instant before reaching the top and headed downward an instant later. This change in velocity direction means that the ball must be accelerating at the top, and the only acceleration operating here is that due to gravity.
9. A motion diagram should use dots to represent the position of the moving object at equally spaced time intervals. The spacing of the dots is crucial: Small spacing means low speed, large spacing means high speed. The diagram should include numerical labels (including units) for known quantities as well as information about the direction of travel (represented by an arrow), if known. It should also include a coordinate axis showing a defined positive direction and, for constant-acceleration situations, an arrow showing the acceleration direction.
10. A motion diagram summarizes the information known about a motion. It provides a visual representation that helps with the mathematical solution to a kinematics problem. The diagram organizes your information and helps you break down the problem into parts.
11. (*a*) Its acceleration is constant. (*b*) Its acceleration is zero.
12. The $\frac{1}{2}$ is included because the area of a triangle is one-half base times height, and with a linear $v(t)$ curve the area under any segment involves a triangle and (in most cases) a rectangle. These areas could be combined into a trapezoid, but doing so would obscure the fact that the triangular portion represents the slope of the $v(t)$ curve (and the slope gives us the object's acceleration).
13. For constant acceleration, the difference in the squares of the final and initial x components of velocity is equal to twice the product of the displacement and the acceleration (Eq. 3.13).
14. Both time intervals are the same. The $x(t)$ curve representing the trajectory is symmetrical about the instant at which the ball reaches its highest position.
15. (*a*) Quadratically (Eq. 3.11); (b) linearly (Eq. 3.12).
16. The ratio of the distance traveled to the square of the elapsed time is constant for a ball released from rest rolling down an inclined plane.
17. In both cases, the acceleration is constant and the motion follows a straight path. The motion diagrams are similar, although the numerical values of the acceleration differ.
18. The acceleration magnitude depends on only the angle of incline (Eq. 3.20).
19. Motion with nonconstant acceleration.
20. (*a*) Velocity at that value of t; (*b*) the second time derivative of position, which is the acceleration; (*c*) acceleration at that value of t; (*d*) the integral of a with respect to t, which is the change in velocity during the time interval (Eq. 3.27); (*e*) the integral of v with respect to t, which is the displacement during the time interval (Eq. 3.28).

Answers to Guided Problems

Guided Problem 3.2 5.2 s

Guided Problem 3.4 (*a*) Friend's ball first; (*b*) yours: $|v_f| = \sqrt{4gh_{\text{launch}}}$, friend's: $|v_f| = \sqrt{4.5gh_{\text{launch}}}$; (*c*) $\Delta t = (2 - \sqrt{2})\sqrt{\frac{h_{\text{launch}}}{g}}$

Guided Problem 3.6 9.71 m/s^2

Guided Practice by Chapter

4 Momentum

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

4.1 Friction

1. Describe several ways of minimizing friction between a surface and an object moving on that surface. Is it possible for a surface to be completely frictionless?

4.2 Inertia

2. Two standard carts collide on a horizontal, low-friction track. How does the change in the velocity of one cart compare with that of the other?
3. Carts A and B collide on a horizontal, low-friction track. Cart A has twice the inertia of cart B, and cart B is initially motionless. How does the change in the velocity of A compare with that of B?

4.3 What determines inertia?

4. An iron cube and an iron sphere contain the same volume of material. How do their inertias compare?
5. Two objects of identical volume and shape are made of different materials: iron and wood. Are their inertias identical?

4.4 Systems

6. What is a system?
7. What is the key difference between an extensive quantity and an intensive quantity?
8. What four processes can change the value of an extensive quantity in a system?
9. What does it mean if an extensive quantity is a conserved quantity?

4.5 Inertial standard

10. Using a pair of standard carts and a baseball, how would you determine the inertia of the baseball?
11. Can the inertia of any object be negative?
12. Which has greater inertia: 1 kg of feathers or 1 kg of lead?

4.6 Momentum

13. Which has greater momentum: a flying bumblebee or a stationary train? Which has greater inertia?
14. A 3-g bullet can knock a wooden block off a fence as easily as a 140-g baseball can. How is this possible?
15. Can the momentum of any object be negative?

4.7 Isolated systems

16. What is the meaning of the word *interaction* in physics?
17. For a given system, what is the distinction between external and internal interactions, and why is the distinction important?
18. What is an isolated system, and of what use is an isolated system?

4.8 Conservation of momentum

19. Are these two statements equivalent? (*a*) The momentum of an isolated system remains constant in time. (*b*) Momentum is conserved.
20. Suppose the 1-kg international inertial standard cylinder were lost or destroyed. How would this affect conservation of momentum?
21. What is impulse, and how is it related to momentum?
22. Compare and contrast the *conservation of momentum* and the *momentum law*.

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The inertia of a suitcase filled with physics textbooks (H, L, Q)
2. The ratio of the inertias of an object and a bowling ball if the object, initially at rest on a bowling alley, doesn't move noticeably after being hit by the bowling ball (C, M, B)
3. The magnitude of the momentum of a tennis ball as it crosses the net (F, R)
4. The magnitude of the momentum of a bowling ball moving toward the pins (A, M)
5. The magnitude of the momentum of a baseball pitched by a major-league pitcher (I, S)
6. The magnitude of the momentum of a typical marathon runner (D, O, V)
7. The magnitude of the change in momentum of a baseball bat that reverses the velocity of the baseball of question 5 (G, N)
8. The magnitude of the change in momentum of a stationary bowling ball hit by the tennis ball of question 3 (B, G, N)
9. The magnitude of the change in momentum of a car that leaves the freeway to stop for gas (E, J, T)
10. The magnitude of the change in velocity of a car that hits a stationary deer (E, P, K, U, N)

Hints

A. What is the inertia of a bowling ball?
B. How does the object of less inertia behave after a collision with an object of greater inertia?
C. What is the largest nonzero speed that is too small to be noticed during a few seconds of observation?
D. What is the inertia of a marathon runner?
E. What is the inertia of a typical car?
F. What is the inertia of a tennis ball?
G. What is the magnitude of the change in momentum of the object with less inertia?
H. What is the inertia of a single physics textbook?
I. What is the inertia of a baseball?
J. What is the speed of the car while on the freeway?
K. What is the inertia of a typical deer?
L. What is the volume of a typical suitcase?
M. What is the typical speed of a bowling ball moving toward the pins?
N. What do you know about the momentum of this system of two objects?
O. What time interval is required to run a marathon?
P. What is the likely speed of a car on a road with deer crossings?
Q. What is the volume of a single physics book?
R. What is the speed of a tennis serve?
S. What is the speed of a pitched baseball?
T. What is the speed of a car when fueling?
U. With what speed would the deer rebound? (First consider a moving deer bouncing off a stationary car by analogy to question 8.)
V. What is the length of a marathon?

Key (all values approximate)

A. 7 kg; **B.** it bounces back, approximately reversing its velocity; **C.** 1×10^{-3} m/s; **D.** 6×10^{1} kg; **E.** 2×10^{3} kg; **F.** 6×10^{-2} kg; **G.** because the motion reverses, about twice the magnitude of the object's initial momentum; **H.** 3 kg; **I.** 0.2 kg; **J.** 3×10^{1} m/s; **K.** 5×10^{1} kg; **L.** 0.1 m^3; **M.** 7 m/s; **N.** it remains approximately constant; **O.** 3 h or more; **P.** less than half freeway speed, perhaps 1×10^{1} m/s; **Q.** 3×10^{-3} m^3; **R.** 5×10^{1} m/s; **S.** 4×10^{1} m/s; **T.** 0; **U.** at a speed somewhat slower than twice the initial speed of the car; **V.** 4×10^{1} km

Worked and Guided Problems

Procedure: Choosing an isolated system

When you analyze momentum changes in a problem, it is convenient to choose a system for which no momentum is transferred into or out of the system (an isolated system). To do so, follow these steps:

1. Separate all objects named in the problem from one another.
2. Identify all possible interactions among these objects and between these objects and their environment (the air, Earth, etc.).
3. Consider each interaction individually and determine whether it causes the interacting objects to accelerate. Eliminate any interaction that does not affect (or has only a negligible effect on) the objects' accelerations during the time interval of interest.
4. Choose a system that includes the object or objects that are the subject of the problem (for example, a cart whose momentum you are interested in) in such a way that none of the remaining interactions cross the system boundary. Draw a dashed line around the objects in your choice of system to represent the system boundary. None of the remaining interactions should cross this line.
5. Make a system diagram showing the initial and final states of the system and its environment.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

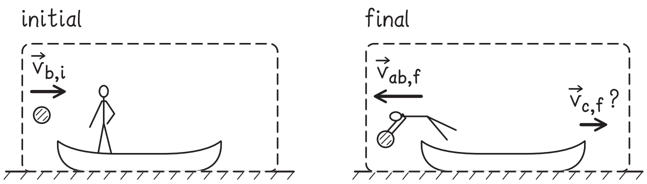
Worked Problem 4.1 Jump ship

In a game of canoe-ball on a calm lake, an athlete stands at the front end of a canoe at rest, facing the shore. A player on shore throws a 1.8-kg ball that arrives at the canoe with a speed of 2.5 m/s. The rules of the game demand that the athlete hold on to the ball and immediately dive into the water, which he does, horizontally, off the front of the canoe. His inertia is 60 kg, and the inertia of the canoe is 80 kg. Friends on shore determine that his speed in the dive is 1.2 m/s. How fast is the canoe moving after he jumps off?

❶ GETTING STARTED Because we are given the canoe's inertia, we can determine its speed once we know the magnitude of its momentum ($v = p/m$). The crucial point is that the momentum exchange is both between the canoe and the athlete and between the ball and the athlete. However, some interaction between the water and the canoe is also possible. Because the event we are analyzing happens during a short time interval and because we know that water is "slippery," we choose to ignore any effect that resistance from the water might have on the canoe's motion. With this simplification, the system comprising the ball, canoe, and athlete is isolated.

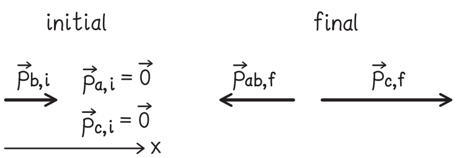
We draw a system diagram (Figure WG4.1) including the relevant velocity information and using subscripts to keep track of our objects: b for ball, a for athlete, c for canoe. We also use the subscripts i for initial quantities and f for final quantities to keep track of the temporal order of events.

Figure WG4.1



❷ DEVISE PLAN Because momentum is a conserved quantity and because the system is isolated, the system's initial momentum must be equal to the system's final momentum. We notice that the momentum is not zero before the catch because the ball is in motion. We also note that, after the dive begins, the athlete and ball can be treated as a single object because they move with a common velocity. We label this composite object with the subscript ab. We have to account for all the contributions to the system's momentum in the initial state (just before the dive) and in the final state (just after the dive), and set the two equal. It is best to draw initial and final momentum representations to keep track of all the objects in the system and to show the direction we choose for the positive x axis (Figure WG4.2). We draw the final momentum vector for the canoe a bit larger than the vector for the athlete plus ball because the system momentum must be in the same direction and of the same magnitude in the initial and final pictures. Note that we choose the direction of the incoming ball as the positive x direction.

Figure WG4.2



With this figure as a guide, we write an equation setting the initial and final momenta equal to each other:

$$\vec{p}_{\text{a,i}} + \vec{p}_{\text{b,i}} + \vec{p}_{\text{c,i}} = \vec{p}_{\text{ab,f}} + \vec{p}_{\text{c,f}}$$
$$\vec{0} + m_{\text{b}}\vec{v}_{\text{b,i}} + \vec{0} = (m_{\text{a}} + m_{\text{b}})\vec{v}_{\text{ab,f}} + m_{\text{c}}\vec{v}_{\text{c,f}}.$$

Because we know all the inertias, the three initial velocities, and the final velocity of the athlete-ball combination, we have all we need to get the final speed of the canoe.

❸ EXECUTE PLAN Recalling that vector components are signed quantities whose signs depend on our choice for the positive x direction, which is in the direction of the incoming ball, we write the equation in component form:

$$m_{\text{b}}v_{\text{b}\,x,\text{i}} = (m_{\text{a}} + m_{\text{b}})v_{\text{ab}\,x,\text{f}} + m_{\text{c}}v_{\text{c}\,x,\text{f}}.$$

In terms of speeds, this becomes

$$m_{\text{b}}v_{\text{b,i}} = (m_{\text{a}} + m_{\text{b}})(-v_{\text{ab,f}}) + m_{\text{c}}v_{\text{c,f}}.$$

PRACTICE

We isolate our unknown, $v_{c,f}$, and then substitute values for the quantities we know:

$$m_c v_{c,f} = m_b v_{b,i} - (m_a + m_b)(-v_{ab,f})$$

$$v_{c,f} = \frac{m_b v_{b,i} + (m_a + m_b)v_{ab,f}}{m_c} \tag{1}$$

$$v_{c,f} = \frac{(1.8\text{ kg})(2.5\text{ m/s}) + (60\text{ kg} + 1.8\text{ kg})(1.2\text{ m/s})}{80\text{ kg}}$$

$$= \frac{78.7\text{ kg}\cdot\text{m/s}}{80\text{ kg}} = 0.983\text{ m/s}.$$

The final speed of the canoe, to two significant digits, is

$$v_{c,f} = 0.98\text{ m/s. ✔}$$

❹ **EVALUATE RESULT** The numerical value of our result is not unreasonable; that is, it is of the order of magnitude we expect for the speed of a human-powered boat. If there were significant water resistance, it would reduce the canoe's speed, but this reduction would take significantly longer than the jump, as you may know from pushing a boat off. For this reason the assumption that water resistance can be neglected during the time interval of interest is justified.

We should also check to see whether our algebraic answer behaves in the way we expect. Equation 1 shows that the canoe's final speed is a sum of positive terms. If Eq. 1 allowed a negative answer for $v_{c,f}$, we would have to rethink our choice of signs for velocity components. The canoe's final x component of momentum $p_{c\,x,f}$ must be positive for this reason: Because the initial momentum of the system is positive, the final momentum must also be positive (but the final momentum of the athlete-ball combination is negative). This is consistent with our intuition that the boat moves to the right in Figure WG4.1 as the athlete dives to the left.

Furthermore, we see that Eq. 1 implies that the canoe's final speed would increase if the ball had greater inertia, if it came in at a higher speed, if the athlete had greater inertia, or if he dove at a higher speed. This matches our qualitative understanding that each of these changes would increase either the initial (positive) momentum of the system or the final (negative) momentum of the athlete-ball combination.

Guided Problem 4.2 A pie in the face

At a carnival, a super-sized, 1.0-kg cream pie is thrown with a speed of 5.0 m/s into the face of a 60-kg clown at rest on roller skates. After the pie strikes his face and sticks there, how fast does the clown roll backward?

❶ **GETTING STARTED**

1. Select an isolated system and sketch a system diagram for the pie-clown collision.
2. What object(s) is (are) in motion before the collision? After the collision?

❷ **DEVISE PLAN**

3. What do you know about the momentum of your system?
4. Express your answer to question 3 quantitatively and count the unknowns.

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

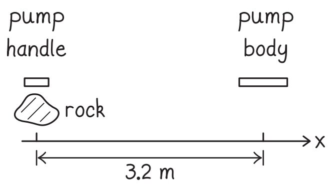
5. Is your answer unreasonably large or small?

Worked Problem 4.3 Exploding bicycle pump

You are riding your bike at a steady 4.0 m/s when suddenly you swerve to avoid a pothole. Your bicycle pump falls from its bracket, strikes a rock at the road's edge, and explodes into three pieces. You watch the pump body sail ahead of you, then hit the ground and stop 0.50 s after the explosion. The pump was horizontal when it exploded, and its length was aligned parallel to the road. Retracing your path, you measure the distance from the pump body to the rock as 3.2 m. Just beside the rock is the pump handle. These two pieces were easy to find, but the third piece—a thin metal spring—is going to be hard to spot. If the inertias are 0.40 kg for the pump body, 0.25 kg for the handle, and 0.20 kg for the spring, where should you look for the spring?

❶ GETTING STARTED We begin with a sketch of the situation (Figure WG4.3). We are given position information for two of the three pieces, but that alone will not locate the third piece. We also have the inertia of each piece, and we know that the velocity of the pump before the explosion was the same as that of the bike.

Figure WG4.3



❷ DEVISE PLAN We consider the three pieces into which the pump disintegrates as our system. For the time interval between the instant after the pump hits the rock and the instant before the pieces land, the system is isolated and so the momentum of the three-piece system does not change. The best we can do, given the information at hand, is to assume that all three pieces hit the ground 0.50 s after the explosion. This allows us to use Eq. 4.19:

$$\vec{p}_{\rm f} = \vec{p}_{\rm i} \quad \text{(isolated system).}$$

There are three objects that move as one (pump) in the initial state but separately (body, handle, spring) in the final state. During the time interval of interest, it is reasonable to assume that all objects move parallel to the road, and so we choose the positive x direction to be the direction of initial travel, from the rock toward the pump body in Figure WG4.3. We arbitrarily locate the origin of the x axis at the rock. The resulting component equation is

$$p_{{\rm p}x,{\rm i}} = p_{{\rm b}x,{\rm f}} + p_{{\rm h}x,{\rm f}} + p_{{\rm s}x,{\rm f}}$$

$$(m_{\rm b} + m_{\rm h} + m_{\rm s})v_{{\rm p}x,{\rm i}} = m_{\rm b}v_{{\rm b}x,{\rm f}} + m_{\rm h}v_{{\rm h}x,{\rm f}} + m_{\rm s}v_{{\rm s}x,{\rm f}}. \quad (1)$$

We know the x component of the initial velocity, but we do not know the x components of the final velocities, and so we need more information. Because the pump handle lies beside the rock, we conclude that the x component of the velocity of the pump handle was zero immediately after the explosion. The pump body lies 3.2 m from the rock, and we know that it traveled this far in 0.50 s. We can determine the x component of its final velocity immediately after the explosion from this information. That leaves only the x component of the final velocity of the spring unknown, which we should be able to compute from the momentum equation. With that information we can compute how far the spring traveled in 0.50 s, and hence where to look for it.

❸ EXECUTE PLAN The x component of the velocity of the pump body after the explosion is

$$v_{{\rm b}x,{\rm f}} = \frac{+3.2\ {\rm m}}{0.50\ {\rm s}} = +6.4\ {\rm m/s}.$$

Next we solve Eq. 1 for $v_{{\rm s}x,{\rm f}}$ and substitute our known values:

$$(m_{\rm b} + m_{\rm h} + m_{\rm s})v_{{\rm p}x,{\rm i}} = m_{\rm b}v_{{\rm b}x,{\rm f}} + m_{\rm h}v_{{\rm h}x,{\rm f}} + m_{\rm s}v_{{\rm s}x,{\rm f}}$$

$$\begin{aligned} v_{{\rm s}x,{\rm f}} &= \frac{(m_{\rm b} + m_{\rm h} + m_{\rm s})v_{{\rm p}x,{\rm i}} - m_{\rm b}v_{{\rm b}x,{\rm f}} - m_{\rm h}v_{{\rm h}x,{\rm f}}}{m_{\rm s}} \\ &= \frac{(0.40\ {\rm kg} + 0.25\ {\rm kg} + 0.20\ {\rm kg})(+4.0\ {\rm m/s})}{0.20\ {\rm kg}} \\ &\quad - \frac{(0.40\ {\rm kg})(+6.4\ {\rm m/s}) + (0.25\ {\rm kg})(0)}{0.20\ {\rm kg}} \\ &= +4.2\ {\rm m/s}. \end{aligned}$$

The spring traveled for 0.50 s at 4.2 m/s, and so we should find it by looking at about

$$x_{{\rm s},{\rm f}} = 0 + (+4.2\ {\rm m/s})(0.50\ {\rm s}) = +2.1\ {\rm m}$$

from the rock in the initial direction of travel. ✔

PRACTICE

❹ EVALUATE RESULT The pump was moving in the x direction prior to the explosion, and so the momentum of the pieces should carry at least some of them down the road in the direction of the initial motion. The spring is located at a position between the body and the handle, which is not unreasonable. The value is of the correct order of magnitude.

Guided Problem 4.4 Space maneuvers

The commander of the starship *Enterprise* is stranded in his orbital shuttle craft, which rests a few kilometers from its docking station. Fortunately, he has two cargo pods that can be ejected (in a direction away from the docking station) with an explosive charge, and he hopes that carrying out this maneuver will move the shuttle craft back to the docking station. Pod 1 has inertia m_1, pod 2 has inertia $m_2 < m_1$, and both pods eject with speed v. Which will get him to the docking station more quickly: (a) ejecting first pod 1 and then pod 2, (b) ejecting first pod 2 and then pod 1, or (c) ejecting both at the same instant?

❶ GETTING STARTED

1. Choose an isolated system for option a, and sketch a system diagram showing the initial and final conditions of the system, building your drawing up as you read each sentence of the problem statement. Do you need a separate diagram for each option?
2. Identify the unknown quantity you are after. What physical principle can be applied to determine this quantity?

❷ DEVISE PLAN

3. Do you expect that the order in which the pods are jettisoned makes a difference?
4. How many moving objects are there in choice a, choice b, and choice c?
5. Add a representation of the momentum vectors before and after the explosive charges are fired in each of your system diagrams.
6. Write the appropriate equation(s) for each case and count the unknowns.

❸ EXECUTE PLAN

❹ EVALUATE RESULT

7. Rank your three answers in order of increasing speed and determine whether or not the order is reasonable based on your conceptual understanding of momentum.

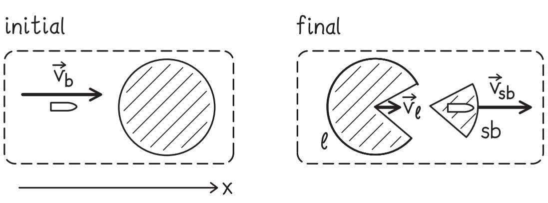
Worked Problem 4.5 Forensic physics

Your friend from law enforcement claims that, in certain cases, a piece of an object hit by gunfire can move toward the shooter rather than away from the shooter. You decide to investigate whether this is possible by firing a target rifle at some melons. Suppose that in one of your tests an 8.0-g bullet is fired at 400 m/s toward a 1.20-kg melon several meters away, splitting the melon into two pieces of unequal size. The bullet lodges in the smaller piece and propels it forward (that is, in the direction the bullet originally traveled) at 9.2 m/s. If the combined inertia of this piece and the lodged bullet is 0.45 kg, determine the final velocity of the larger piece.

❶ GETTING STARTED As usual, we begin by identifying an isolated system. What about one made up of the bullet and melon? There may be some interaction with the ground or the support upon which the melon sits, but if we choose our time interval to start immediately before the bullet strikes the melon and end immediately after the smaller piece breaks loose, we can ignore this interaction. During this very short time interval, we can treat the system as isolated.

Next we sketch a system diagram showing the initial and final conditions of the system (Figure WG4.4). We arbitrarily choose the direction in which the smaller piece moves (to the right in the figure) as our positive x direction. We don't know which way the larger piece moves, and so we draw it as going forward, too. Our calculation will let us know whether this guess is right or wrong. The momentum of the system remains constant during this collision because the system is isolated.

Figure WG4.4



❷ **DEVISE PLAN** We can translate the information given into an equation that requires no change in the momentum of the system. There are two objects in the system in the final condition, but they are different from the ones in the initial condition. We designate the inertia of the bullet as m_b and the x component of its velocity as v_{bx}, and we note that the intact melon had an initial speed $v_{m,i} = 0$ and inertia m_m. We have

$$\Delta\vec{p} = 0 \Rightarrow \vec{p}_i = \vec{p}_f$$

$$m_b v_{bx,i} + m_m v_{mx,i} = (m_s + m_b)v_{sbx,f} + m_\ell v_{\ell x,f}$$

where the subscript s denotes the smaller piece, the subscript ℓ denotes the larger piece, and the subscript sb denotes the combination of the bullet and the smaller piece. We are looking for $v_{\ell x,f}$, and because all our other variables have known values, the planning is finished.

❸ **EXECUTE PLAN** We isolate the desired unknown and then insert numerical values. Recall that all of the velocity components in our diagram are in the positive direction.

$$m_\ell v_{\ell x,f} = m_b v_{bx,i} + m_m v_{mx,i} - (m_s + m_b)v_{sbx,f}$$

$$v_{\ell x,f} = \frac{m_b(+|\vec{v}_{b,i}|) - (m_s + m_b)(+|\vec{v}_{sb,f}|)}{m_\ell}$$

$$= \frac{(0.0080\text{ kg})(400\text{ m/s}) - (0.45\text{ kg})(9.2\text{ m/s})}{1.20\text{ kg} - (0.45\text{ kg} - 0.0080\text{ kg})}$$

$$= -1.240\text{ m/s} = -1.2\text{ m/s.} ✔$$

❹ **EVALUATE RESULT** The negative sign means that the larger piece moves in the negative x direction, which means that our initial guess for the direction of this motion was wrong. Note, however, that the algebra nicely informed us of the incorrect assumption without pain or confusion. The larger piece really does move back toward the rifle. The magnitude of this velocity is fairly small, as we would expect, but it certainly would be noticeable.

Guided Problem 4.6 Bullet impact

A rifle is fired twice in succession at a target that is free to slide on a frozen pond, and both bullets embed in the target. In terms of the relevant inertias and the rifle's muzzle velocity, compute the velocity of the target after the first impact and after the second impact.

❶ **GETTING STARTED**

1. Should you choose one system for the entire process or a separate system for each impact?
2. Draw the system diagram(s) you need.
3. What physical principle governs the target's motion?

❷ **DEVISE PLAN**

4. Carefully consider the time sequence of events when deciding how to apply the labels "initial" and "final."
5. Remember to account for the inertia of the embedded bullets.
6. What equation(s) can you write to describe the physics in each segment of the motion?

❸ **EXECUTE PLAN**

7. Must you solve each equation separately, or can you combine them into one equation for the target's final speed?

❹ **EVALUATE RESULT**

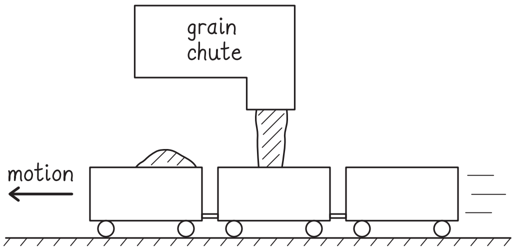
8. Are the sign and magnitude of your answer what you expect?

Worked Problem 4.7 Grain dump slows freight

The problem on your new job at the grain elevator is that, as a series of coupled, empty freight cars coast past the grain-loading chute, the grain added to the cars slows the train down. The result is that there is more grain in the rear cars (which pass under the chute when the train is moving slowly) than in the front cars (which pass under the chute when the train is moving quickly). Your boss does not want to incur the expense of a locomotive to move the train at a constant speed. She wants to know how much more grain ends up in the last car than in the first car, and which variables affect the amount of grain in different cars. She hands you a chart of typical coasting speeds, loading rates, empty-car inertias, numbers of cars to be loaded, and so on. You start by assuming that the cars, when empty, are all identical and that all motion occurs along a horizontal straight line (the train tracks).

❶ GETTING STARTED We begin as usual with a sketch (Figure WG4.5). Note that the sketch shows the situation at an instant when a car in the middle of the train is being loaded rather than some instant when the first car or last car is taking on grain. Choosing this intermediate instant should make the sketch relevant to our analysis throughout the loading process.

Figure WG4.5



There is a lot going on: Grain is falling into the cars, increasing their inertia; the motion is slowing, allowing more grain into each car as the train moves forward, and the additional grain in the cars changes the train's motion more noticeably with time. We have no numerical values as clues, and so we will have to derive some general expressions. The amount of grain in each car will have some relationship to the speed of that car as it passes under the chute, and so we must obtain a relationship involving speed or velocity. We will also need to know how the time interval a given car spends being loaded depends on how much time has passed since the first car began loading. The problem statement implies that the rate of grain release from the chute is constant. This implication is consistent with the fact that the rear cars end up with more grain, and so we assume a constant loading rate. The final goal is to obtain an expression for how long a time interval each car spends under the chute. Given that and a constant loading rate, we can determine the amount of grain in each car and so compare the amounts in the first and last cars.

❷ DEVISE PLAN The system of cars plus grain is isolated if the chute and the track produce no horizontal impulse on the system. We therefore assume that the grain falls vertically into each car and that friction between the track rails and the wheels of the cars is negligible. Under these conditions, the momentum of the system must remain constant, which means that the reduction in velocity exactly balances the increase in the train's inertia. Our analysis begins with N initially empty cars (combined inertia m_i) at the instant $t = 0$ coasting with velocity $\vec{v}_i$.

Equation 4.19 is a good starting point:

$$\vec{p}_i = \vec{p}_f$$
$$m_i\vec{v}_i = m_f\vec{v}_f$$

We seek a general relationship that is valid throughout the loading process, so we represent the final instant by an arbitrary instant t:

$$m_i\vec{v}_i = m(t)\vec{v}(t). \tag{1}$$

Both $m(t)$ and $\vec{v}(t)$ are unknown, which means we need another equation involving either inertia or velocity.

The grain is added in small uniform increments, and the variables that contribute to the momentum of each piece of the system (inertias, velocities) continually change with time. Perhaps we can divide the loading process into very short time intervals, then analyze the inertia change during each interval and integrate over time to obtain an expression for the inertia as a function of time. Substituting the expression we obtain into Eq. 1, we can isolate $v(t)$. Then we must extract information about the time interval each car spends under the chute. This is equivalent to asking, during what time interval does a car move a distance equal to its length ℓ? That suggests we integrate $v(t)$ with respect to time to obtain displacement. We can then define an x axis running along the track and isolate t for the cases $\Delta x = \ell, 2\ell, \ldots, N\ell$. Given these loading time intervals, we can determine the amount of grain in each car, and we will see which variables have an effect on the loading process.

❸ EXECUTE PLAN We denote the constant loading rate by $dm/dt = \lambda$ (kg/s). Because we know the initial inertia of the train m_i, we can integrate to obtain an expression for its final inertia at an arbitrary instant t:

$$\int_{m_i}^{m(t)} dm = \int_0^t \lambda dt \Rightarrow m(t) - m_i = \lambda t$$

$$m(t) = m_i + \lambda t.$$

PRACTICE

Solving Eq. 1 for $\vec{v}(t)$ and then substituting $m_i + \lambda t$ for $m(t)$, we obtain a general expression for $\vec{v}(t)$:

$$\vec{v}(t) = \frac{m_i \vec{v}_i}{m_i + \lambda t}.$$

Defining the positive x direction to be along the track in the direction of the initial motion allows us to express this as a component equation:

$$v_x(t) = \frac{m_i v_{x,i}}{m_i + \lambda t}.$$

What remains is to extract from this expression the time interval each car spends under the chute. Each car has length ℓ, and so we need an expression for the displacement that can be set equal to an integer number of car lengths. Noting that displacement is the integral of velocity with respect to time, we obtain

$$\Delta x = \int v_x(t) dt = m_i v_{x,i} \int \frac{dt}{(m_i + \lambda t)}.$$

We can either look this integral up or calculate it with a substitution of variables: $u = (m_i + \lambda t)$; $du = \lambda dt$. The result is a natural logarithm:

$$\Delta x = \frac{m_i v_{x,i}}{\lambda} \ln(m_i + \lambda t) + C.$$

To eliminate the constant of integration, we pick specific limits of integration. We start the integral at $t = 0$, as the first car just reaches the loading area, and end it just as each car leaves the loading area. That means we need several upper limits, one for each car.
For now we use t to represent an arbitrary final instant:

$$\Delta x = \frac{m_i v_{x,i}}{\lambda} \ln(m_i + \lambda t) \Big|_0^t = \frac{m_i v_{x,i}}{\lambda} \ln\left(\frac{m_i + \lambda t}{m_i}\right).$$

We can extract the time variable by isolating the logarithm and then taking an exponential on both sides:

$$\frac{\lambda \Delta x}{m_i v_{x,i}} = \ln\left(\frac{m_i + \lambda t}{m_i}\right)$$

$$e^{\lambda \Delta x / m_i v_{x,i}} = \left(\frac{m_i + \lambda t}{m_i}\right) = 1 + \frac{\lambda t}{m_i}$$

$$\Delta t = t - 0 = t = \frac{m_i}{\lambda}\left(e^{\lambda \Delta x / m_i v_{x,i}} - 1\right). \quad (2)$$

Now we can obtain the time interval Δt required to fill one car, two cars, or all the cars in the train by substituting the appropriate length ($\ell, 2\ell, \ldots, N\ell$) for the displacement Δx. Suppose each of the N cars in the train has an inertia m_{car} when empty. Then $Nm_{car} = m_i$. The first car is under the chute for a time interval Δt_1, found with $\Delta x = \ell$:

$$\Delta t_1 = \frac{Nm_{car}}{\lambda}\left(e^{\lambda \ell / Nm_{car} v_{x,i}} - 1\right).$$

The amount of grain loaded during an interval Δt is given by

$$\Delta m = \int dm = \int_0^t \lambda\, dt = \lambda \Delta t.$$

The amount loaded into the first car is thus

$$\Delta m_1 = Nm_{car}\left(e^{\lambda \ell / Nm_{car} v_{x,i}} - 1\right).$$

For each subsequent car, we obtain the time interval from $t = 0$ until the car leaves the loading area, then subtract the time interval used to fill the previous cars. Thus the second car is under the chute for a time interval $\Delta t_2 - \Delta t_1$, where Δt_2 is the time interval required for the train to travel a distance $\Delta x = 2\ell$:

$$\Delta t_2 = \frac{Nm_{\text{car}}}{\lambda}(e^{\lambda 2\ell/Nm_{\text{car}}v_{x,\text{i}}} - 1)$$

$$\Delta t_2 - \Delta t_1 = \frac{Nm_{\text{car}}}{\lambda}(e^{\lambda 2\ell/Nm_{\text{car}}v_{x,\text{i}}} - e^{\lambda\ell/Nm_{\text{car}}v_{x,\text{i}}}).$$

The amount of grain added to the second car is therefore

$$\Delta m_2 = \lambda(\Delta t_2 - \Delta t_1) = Nm_{\text{car}}(e^{\lambda 2\ell/Nm_{\text{car}}v_{x,\text{i}}} - e^{\lambda\ell/Nm_{\text{car}}v_{x,\text{i}}}).$$

For the Nth car, we have

$$\Delta t_N - \Delta t_{N-1} = \frac{Nm_{\text{car}}}{\lambda}(e^{\lambda N\ell/Nm_{\text{car}}v_{x,\text{i}}} - e^{\lambda(N-1)\ell/Nm_{\text{car}}v_{x,\text{i}}})$$

$$\Delta m_N = \lambda(\Delta t_N - \Delta t_{N-1})$$

$$= Nm_{\text{car}}(e^{\lambda N\ell/Nm_{\text{car}}v_{x,\text{i}}} - e^{\lambda(N-1)\ell/Nm_{\text{car}}v_{x,\text{i}}}).$$

The ratio of the amount of grain in the last car to the amount in the first car is therefore

$$\frac{\Delta m_N}{\Delta m_1} = \frac{(e^{\lambda N\ell/Nm_{\text{car}}v_{x,\text{i}}} - e^{\lambda(N-1)\ell/Nm_{\text{car}}v_{x,\text{i}}})}{e^{\lambda\ell/Nm_{\text{car}}v_{x,\text{i}}} - 1} = e^{\lambda(N-1)\ell/Nm_{\text{car}}v_{x,\text{i}}} \checkmark \quad (3)$$

This expression answers both of your boss's questions. It gives the ratio of the amounts of grain in the last and first cars and shows that the variables affecting the amount in each car are the number N of cars in the train, length ℓ of each car, loading rate λ, inertia m_{car} of an empty car, and the initial x component of the train's velocity $v_{x,\text{i}}$. ✔

❹ **EVALUATE RESULT** We assumed that the tracks are straight and horizontal, the cars are identical when empty, and the effects of friction are negligibly small. None of these assumptions is unreasonable in this context. We assumed that the grain falls vertically and that it enters the cars at a constant rate. Given that the chute is of a fixed size and shape, it is not unreasonable to assume a constant loading rate, at least over time intervals of a few minutes—more long than enough to get several cars past the chute. This assumption is reasonable but not mandatory. If the loading rate were a controlled variable, however, it could be adjusted to match the speed of the train and your boss would have no problem.

We see from Eq. 2 that the loading time interval increases as expected for cars farther and farther from car 1, as Δx increases. Note also that Δt goes to zero in Eq. 2 as $\Delta x \to 0$ (because it takes no time to move no distance).

The ratio of Eq. 3 grows as the number N of cars increases. This is to be expected because eventually the train slows to a crawl. Let's look at some data. Suppose $N = 4$ cars, $\ell = 15$ m, $m_{\text{car}} = 2.7 \times 10^4$ kg, $v_{x,\text{i}} = 1.0$ m/s, and $\lambda = 1.5 \times 10^3$ kg/s. Substituting these numbers into Eq. 3 yields a ratio of 1.9; that is, the last car has almost twice the grain of the first car! Even with 3 cars, the third car will contain 50% more grain than the first. It looks like the company should control the loading rate to match the train velocity. You may have a promotion coming when you suggest this and provide the time-distance relationship needed to adjust the loading rate.

Guided Problem 4.8 Rocket speed

Suppose a rocket that has a combined rocket + fuel inertia m_i starts from rest and then expels fuel at a rate dm/dt. The speed of the fuel as it exits the rocket at any instant t is the difference between the forward speed of the rocket at this instant and the constant nozzle speed of the ejected fuel v_{fuel} (that is, v_{fuel} is the speed with which the fuel is ejected when the rocket is at rest). Use conservation of momentum to show that, once enough fuel has been expelled to reduce the combined rocket + fuel inertia to m_f, the change in the rocket's speed $\Delta v_{\text{rocket}} = v_{\text{rocket,f}} - v_{\text{rocket,i}}$ is

$$v_{\text{rocket,f}} - v_{\text{rocket,i}} = v_{\text{fuel}} \ln\frac{m_\text{i}}{m_\text{f}}.$$

Ignore any gravity effects. (This classic rocketry formula was first worked out by Russian engineer K. Tsiolkovskii in 1897.)

1 GETTING STARTED

1. As in Worked Problem 4.7, this is a process with continuous changes in inertia and velocity of parts of a system. How can the approach of Worked Problem 4.7 be modified to suit these circumstances? Consider analyzing a short time interval dt and then integrating.
2. Draw one sketch at arbitrary instant t showing the rocket moving at velocity v_{rocket}. The rocket has inertia $m + dm$, where dm represents a tiny amount of fuel still on board but about to be expelled. This is the initial instant. Then draw a second sketch showing the tiny amount of fuel and the rocket as separate objects at instant $t + dt$, a short time later.
3. The rocket moves at a slightly higher velocity ($v_{\text{rocket}} + dv_{\text{rocket}}$) in the second sketch. What is the velocity of the fuel element dm in the second sketch?

2 DEVISE PLAN

4. Notice that there are no numbers here but that all of the symbols except v_{rocket} (and dv_{rocket}) can be considered known quantities.
5. Write the equation that compares the momentum at instant t with the momentum at instant $t + dt$.
6. Examine your equation for each situation—with the fuel of inertia dm on board and with it expelled. Can some terms be canceled?
7. Separate variables to get all v terms on the left side of the equation and all m terms on the right side, then integrate. When you integrate, what sign is associated with dm?

3 EXECUTE PLAN

4 EVALUATE RESULT

8. Does the expression you obtain behave as you might expect? For example, does the velocity increase with time?
9. Make sure any assumptions you made are not unreasonable. Were any assumptions needed about the mechanism of fuel ejection or about the type of fuel?

Answers to Review Questions

1. No one has yet developed a completely frictionless surface, but there are many ways to minimize friction: smooth or polish the surfaces of contact; apply a slippery substance, such as grease or oil; place the object on rollers or mount it on wheels with very good bearings; and float the object on a cushion of air, as is done with low-friction tracks.
2. The changes in velocity are equal in magnitude but of opposite sign.
3. The magnitude of the velocity change of A is half that of B because the ratio of the magnitudes of velocity changes is the inverse of the ratio of the inertias. The two velocity changes are in opposite directions.
4. The inertias are identical because shape has no effect on an object's inertia. The inertia of an object is determined *entirely* by the type of material of which the object is made and by the amount of that material contained in the object.
5. Inertia depends on the material, so they are not identical. Experience suggests that the iron object has greater inertia.
6. A system is an object or group of objects that we can separate in our minds from the surrounding environment.
7. An extensive quantity depends on the extent (size) of the system that you choose; an intensive quantity does not.
8. Input, output, creation, and destruction can change the value.
9. It means that the extensive quantity cannot be created or destroyed. Therefore only two processes can change the value of the quantity in a system: input and output.
10. Fasten the baseball to one standard cart and arrange a collision between this cart and the other standard cart. Measure the change in velocity of each object; then use the fact that the ratio of these changes in velocity is the inverse of the ratio of the inertias of the two objects. Subtract 1 kg from the inertia of the cart with the baseball to obtain the baseball's inertia.
11. No. Inertia is always positive. If one of the objects involved in a collision had negative inertia, *both* objects would have an increase in velocity. Such collisions have never been observed.
12. Their inertias are equal.
13. The bumblebee has greater momentum. The train has zero velocity, and so it has zero momentum. The train has much greater inertia than the bumblebee, however.
14. Even though the inertia of the baseball is greater than the inertia of the bullet, their momenta (and the results of the collisions) can be comparable if the speed of the bullet is proportionately larger than the speed of the baseball.
15. No. Only scalars can be negative, and momentum is a vector. However, the x component of momentum can be negative if the momentum points in the negative x direction.
16. For purposes of this chapter, an interaction is two objects acting on each other such that at least one of them is accelerated.
17. Internal interactions are between two objects in a system, whereas external interactions are between an object inside the system and an object outside the system. The distinction is important because external interactions can change the momentum of a system, while internal interactions cannot.
18. An isolated system is one for which there are no external interactions. Because the momentum of an isolated system does not change with time, we can use information about the momentum at one instant to determine the momentum at a later instant.
19. The statements are not equivalent, but *a* depends on *b*. Statement *b* means that momentum cannot be created or destroyed. Statement *a* applies to only an isolated system, and it applies *because b* is true. By definition, there are no transfers of momentum across the boundary of an isolated system, and so none of the four mechanisms that might change the system's momentum—input, output, creation, and destruction—are available.
20. Conservation of momentum wouldn't be affected in the least. The 1-kg standard is a convention only.
21. *Conservation of momentum* is a verbal statement of the fact that momentum can be neither created nor destroyed. The *momentum law* is a

mathematical statement given by Eq. 4.18, $\Delta\vec{p} = \vec{J}$, saying that any change in momentum is due to transfers of momentum. Because it contains no terms for the creation or destruction of momentum, the momentum law embodies the conservation of momentum.

22. Impulse, $\vec{J}$, represents the transfer of momentum between the environment and a system. It is a vector with the same units as momentum, $\text{kg} \cdot \text{m/s}$, and, in the nomenclature of Section 4.4, it accounts for the "input" and "output" of the system. Impulse is related to the change in momentum of a system by Eq. 4.18: $\vec{J} = \Delta\vec{p}$.

Answers to Guided Problems

Guided Problem 4.2 8.2×10^{-2} m/s

Guided Problem 4.4 *c*, ejecting both at once to attain the highest speed for the shuttle:

$$v_\text{f} = \frac{(m_1 + m_2)}{m_\text{shuttle}} v$$

Guided Problem 4.6 After the first impact: $v_\text{target} = \dfrac{m_\text{bullet}}{(m_\text{target} + m_\text{bullet})} v_\text{muzzle}$;

after the second impact: $v_\text{target} = \dfrac{2m_\text{bullet}}{(m_\text{target} + 2m_\text{bullet})} v_\text{muzzle}$

Guided Practice by Chapter

5 Energy

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

5.1 Classification of collisions

1. What is relative velocity, and how does it differ from relative speed?
2. Explain the order and meaning of the subscripts in the relative velocity symbol $\vec{v}_{12}$.
3. What is the main differentiating characteristic of (*a*) elastic collisions, (*b*) inelastic collisions, and (*c*) totally inelastic collisions?

5.2 Kinetic energy

4. What is kinetic energy?
5. What is the sign of the kinetic energy of an object that is moving in the negative x direction?
6. Is it possible for an object's kinetic energy to be negative or zero?

5.3 Internal energy

7. What is meant by the *state of an object*?
8. What is the difference between a reversible process and an irreversible process?
9. In an inelastic collision, what is the relationship between the system's kinetic energy and internal energy?
10. How do you calculate the change in internal energy in an inelastic collision?

5.4 Closed systems

11. What is a closed system? Is it the same as an isolated system?
12. What is energy conversion, and how is it different from energy transfer?

5.5 Elastic collisions

13. How does the proof that kinetic energy remains constant in an elastic collision (Eqs. 5.4 through 5.13) depend on the elastic nature of the collision?
14. Explain why the relative velocity of two objects changes sign in an elastic collision.

5.6 Inelastic collisions

15. What is the definition of coefficient of restitution?
16. What is the numerical value of the coefficient of restitution for an elastic collision and for a totally inelastic collision?
17. Explain why Eq. 5.19 has a minus sign in front of the fraction.
18. In an elastic collision between a pair of objects where you know the inertias and initial velocities, you can use the fact that the momentum and kinetic energy of the system remain constant to determine the motion of both objects afterward—two equations, two unknowns. In an inelastic collision, however, the kinetic energy does not remain constant. Can you predict the final velocities of both objects in this case, given the inertias and initial velocities?

5.7 Conservation of energy

19. Are you creating energy when you throw a baseball?
20. Can there be a change in the physical state of a system if there is no change in its kinetic energy?
21. Without direct knowledge of how to compute internal energy values, how can we determine the change in the internal energy of a closed system?

5.8 Explosive separations

22. Where does the kinetic energy increase come from in an explosive separation?
23. Is it possible for an explosive separation to be elastic?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The relative velocity of an airliner cruising east seen from a train speeding west (K, R)
2. The relative speed of the Moon in orbit as seen from Earth (E, W)
3. The kinetic energy of a tennis ball as it crosses the net after service (P, I)
4. The kinetic energy of a small car driving along the freeway (G, L)
5. The kinetic energy of an airliner moving at cruising speed (S, K)
6. The speed of the center of a tennis racquet just before the ball is served (P, H, C, X)
7. The change in the kinetic energy of a golf ball hit for a long drive from the tee (A, F, O, T)
8. The chemical energy released during the explosive separation of a professional fireworks shell (N, D, Z)
9. The kinetic energy converted to internal energy when a switching engine couples with a rail car initially at rest (V, J, Y)
10. The amount of useful energy your car can obtain from 1 gallon of gasoline at freeway speed (B, G, M, U, Q)

Hints

A. What distance does the golf ball travel in the air?
B. While cruising on a freeway, how much speed does a car lose in 5 s if it is shifted into neutral?
C. What is the coefficient of restitution for the ball-racquet collision?
D. What is the maximum radius of the exploding sphere?
E. What time interval is needed for one Moon orbit?
F. For how long a time interval is the golf ball in the air?
G. What is the inertia of a typical small car?
H. Considering the inertia of the ball relative to the inertia of the arm plus racquet, how much does the speed of the racquet change during the collision?
I. What is the inertia of a tennis ball?
J. What is the rail car's inertia?
K. What is the speed of a cruising airliner?
L. What is the speed of the car on the freeway?
M. How much kinetic energy is lost by a coasting car in 5 s?
N. What is the inertia of the fireworks payload?
O. What is the speed of a golf ball leaving the tee?
P. What is the speed of a tennis ball after service?
Q. What is the fuel consumption of a typical car on the freeway?
R. What is the speed of a speeding train?
S. What is the inertia of an airliner?
T. What is the inertia of a golf ball?
U. How many kilometers does a car travel in 5 s at freeway speed?
V. What is the switching engine's inertia?
W. What is the radius of the Moon's orbit?
X. How does the velocity of the ball relative to that of the racquet compare before and after the collision?
Y. What is the engine's speed before coupling?
Z. How long a time interval is required for the sphere to expand?

Key (all values approximate)

A. 1×10^2 m; **B.** 4 m/s; **C.** $e \approx 1$; **D.** 3×10^1 m; **E.** 1 month, or 3×10^1 days; **F.** 3 s; **G.** 1×10^3 kg; **H.** very little, because the inertia ratio is large; **I.** 6×10^{-2} kg; **J.** 2×10^4 kg; **K.** 2×10^2 m/s; **L.** 3×10^1 m/s; **M.** 1×10^5 J; **N.** 2 kg; **O.** 4×10^1 m/s; **P.** 4×10^1 m/s; **Q.** 0.1 L/km; **R.** 3×10^1 m/s in the United States; much faster in some other countries; **S.** 1×10^5 kg; **T.** 5×10^{-2} kg; **U.** 0.2 km; **V.** 8×10^4 kg; **W.** 4×10^8 m; **X.** relative speed remains roughly the same, but the direction of the velocity reverses; **Y.** 0.4 m/s; **Z.** 1 s

Worked and Guided Problems

Procedure: Choosing a closed system

When we analyze energy changes, it is convenient to choose a system for which no energy is transferred to or from the system (a closed system). To do so, follow this procedure:

1. Make a sketch showing the initial and final conditions of the objects under consideration.
2. Identify all the changes in state or motion that occur during the time interval of interest.
3. Choose a system that includes all the objects undergoing these changes in state or motion. Draw a dashed line around the objects in your chosen system to represent the system boundary. Write "closed" near the system boundary to remind yourself that no energy is transferred to or from the system.
4. Verify that nothing in the surroundings of the system undergoes a change in motion or state that is related to what happens inside the system.

Once you have selected a closed system, you know that its energy remains constant.

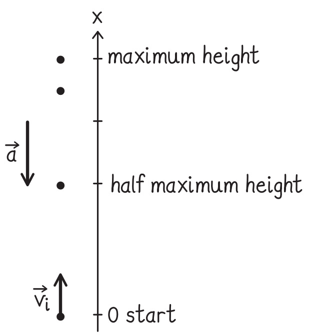
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 5.1 Lofty arrow

You shoot a 0.12-kg arrow vertically upward at 40 m/s. Calculate the arrow's kinetic energy (*a*) at the start of its flight, immediately after it leaves the bow, and (*b*) when it reaches half of its maximum height. (*c*) Use energy arguments to estimate the position of the arrow when it has half its original speed.

❶ **GETTING STARTED** This is a kinematics problem with an energy twist. We can use the kinematics we learned in Chapter 3 first to calculate the velocity (and thus the speed) of the arrow at any position for parts *a* and *b* and then to calculate the position at any speed for part *c*. In addition, because we now know the connection between speed and kinetic energy, we can connect position and kinetic energy. The arrow in motion is not a closed system because of the gravitational interaction, but the arrow + Earth system is closed. A system diagram would include Earth and the arrow but would not provide much additional information. Instead, taking the kinematics approach, we draw a motion diagram (Figure WG5.1).

Figure WG5.1



❷ **DEVISE PLAN** The definition of kinetic energy should be sufficient to tackle part *a* because both the inertia and the initial velocity are known. Parts *b* and *c* are more involved. In free fall, acceleration is downward and constant. With our choice in Figure WG5.1 to have the positive *x* axis pointing upward, we have $a_x = -g$. The relationship between the *x* components of velocity and position that does not explicitly involve time is Eq. 3.13:

$$v_{x,\mathrm{f}}^2 = v_{x,\mathrm{i}}^2 + 2a_x(x_\mathrm{f} - x_\mathrm{i}).$$

This equation may be applied between any two points in the motion, and so we can use it to determine either the arrow's height at a specified speed or its speed at a specified height. This plus the definition of kinetic energy should take care of parts *b* and *c*.

❸ **EXECUTE PLAN**

(*a*) $$K_\text{start} = \tfrac{1}{2}mv_\text{start}^2$$

$$= \tfrac{1}{2}(0.12\text{ kg})(40\text{ m/s})^2 = 96\text{ J}. \checkmark$$

(*b*) Before we can determine the value of *K* at the instant the arrow reaches half its maximum height, we must obtain the distance to the top of the flight. We know the initial position (0) and the *x* components of the initial velocity (+40 m/s), the final velocity (0; why?), and the acceleration (-9.8 m/s^2; why minus?), and so we can write

$$v_{x,\text{top}}^2 = v_{x,\text{start}}^2 + 2a_x(x_\text{top} - x_\text{start})$$

$$x_\text{top} = x_\text{start} + \frac{v_{x,\text{top}}^2 - v_{x,\text{start}}^2}{2a_x} = 0 + \frac{(0)^2 - (40\text{ m/s})^2}{2(-9.8\text{ m/s}^2)}$$

$$= 82\text{ m}.$$

We now obtain the squared velocity at half this maximum height, $x_\text{half} = x_\text{top}/2$:

$$v_{x,\text{half}}^2 = v_{x,\text{start}}^2 + 2a_x(x_\text{half} - x_\text{start})$$

$$= (+40\text{ m/s})^2 + 2(-9.8\text{m/s}^2)(41\text{ m} - 0) = 796\text{ m}^2/\text{s}^2.$$

The arrow's speed at half its maximum height is therefore

$$v_{\text{half}} = \left|\sqrt{v_{x,\text{half}}^2}\right| = \left|\sqrt{796\ \text{m}^2/\text{s}^2}\right| = 28\ \text{m/s}.$$

(Notice that this speed is *not* half the initial speed.) The kinetic energy at this point is thus

$$K_{\text{half}} = \tfrac{1}{2}mv_{\text{half}}^2 = \tfrac{1}{2}(0.12\ \text{kg})(796\ \text{m}^2/\text{s}^2) = 48\ \text{J}. \checkmark$$

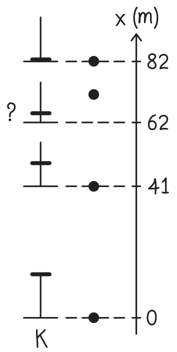
This is half the initial kinetic energy. Having gained half its maximum height, half the kinetic energy has been converted to internal energy of the arrow + Earth system.

(*c*) We know from part *b* that at half its maximum height the arrow's speed is 28 m/s. This means that the position where the speed is 20 m/s, half the initial value, must be above the halfway point. Let's make a guess. Because the velocity in $K = \frac{1}{2}mv^2$ is squared, reducing v to half its initial value reduces K to a quarter of its initial value. We know that $v = 0$ at the top of the arrow's trajectory (see Section 3.3). From part *b* we know that when the arrow is at half its maximum height, half of its kinetic energy has been converted to internal energy. Perhaps at three-quarters of the distance to the top of the flight, three-quarters of the kinetic energy has been converted to internal energy, as shown in the energy bars of Figure WG5.2. Thus, we estimate that the height at which the velocity is +20 m/s, half the original velocity, is about

$$(0.75)(82\ \text{m}) = 61\ \text{m} \checkmark$$

above the starting position.

Figure WG5.2



4 **EVALUATE RESULT** Comparing the answers to parts *a* and *b* shows that the arrow's speed decreases as the arrow rises, as expected. A maximum distance of 82 m is a reasonable height for an arrow, consistent with the 40-m/s initial speed. We assumed that the arrow was in free fall so that we could use the free-fall value for acceleration, and this is a reasonable assumption for an arrow on its upward flight. The rest of the solution follows from kinematics and the definition of kinetic energy.

In part *c*, we were told to use energy arguments to determine a position. If we had not been restricted that way, we could have used our kinematics equation to obtain the position at which the arrow was moving at +20 m/s, and so let's do that now as a check:

$$v_{x,(c)}^2 = v_{x,\text{start}}^2 + 2a(x_{(c)} - x_{\text{start}})$$

$$x_{(c)} = x_{\text{start}} + \frac{v_{x,(c)}^2 - v_{x,\text{start}}^2}{2a} = 0 + \frac{(20\ \text{m/s})^2 - (40\ \text{m/s})^2}{2(-9.8\ \text{m/s}^2)}$$

$$= 61\ \text{m},$$

in nice agreement with our energy-based guess.

PRACTICE

Guided Problem 5.2 Throwing a punch

A boxer delivers a hard blow to the chin of his opponent. The inertia of the boxer's hand (with glove) and forearm is 3.0 kg, and the inertia of the opponent's head is 6.5 kg. You learned in neurobiology class about 10 J of extra internal energy will render an opponent wobbly-kneed, and you guess that about half of the converted energy will end up in the opponent's head. Assuming a coefficient of restitution $e = 0.20$, with what speed does the boxer's fist have to contact the opponent's head in order to deliver the punch?

1 GETTING STARTED

1. Do the boxer's hand, glove, and forearm form an isolated or closed system? What if the opponent's head is included?
2. What kind of diagram might be useful?

2 DEVISE PLAN

3. Using your diagram as a guide, write the relevant equation(s).
4. Make sure you have enough information to solve for the requested answer.
5. Does the opponent's head remain stationary as the punch is delivered?

3 EXECUTE PLAN

6. What value should you use for the converted energy?

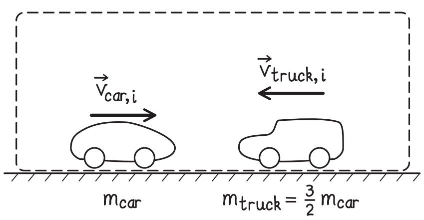
4 EVALUATE RESULT

Worked Problem 5.3 Poor parking

A car traveling at 3.0 m/s whizzes into an empty parking space. At the same instant, a truck that has an inertia 50% greater than that of the car is using the empty parking space to take a shortcut through the parking lot. Both vehicles are coasting (in other words, both have zero acceleration), and just before they hit head-on, the truck is moving at 4.0 m/s. (*a*) If during the collision 75% of the initial kinetic energy of the car-truck system is converted to internal energy, what are the final velocities of the two vehicles? (*b*) What is the coefficient of restitution for the collision?

1 GETTING STARTED Our first step is to draw a system diagram and then add the information about initial velocities (Figure WG5.3). For the short duration of the collision, the car and truck form an isolated system. Because we are told that a given percentage of kinetic energy is converted, we can also assume that the system is closed. We are not given inertia values but are told only that the truck's inertia is 50% greater than that of the car. We hope we do not need the actual values, but we put the information we have in the diagram to help us think clearly about the situation.

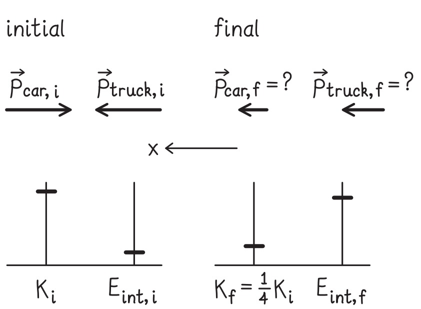
Figure WG5.3



We are asked to determine the two "final velocities," which we assume means "final velocities immediately after the collision" so that any interaction with the environment can be neglected. This information is contained in the momenta of the vehicles. Because kinetic energy is lost, we know that the collision is inelastic.

2 DEVISE PLAN The momentum of the two-vehicle system after the collision is the same as the momentum before the collision. Thus we should draw a diagram representing the initial and final momenta to help plan our approach (Figure WG5.4). To get the directions of the final motion, we shall have to deal with the signs of the momenta with reference to an axis. Let us choose to point our positive *x* axis toward the left and include it in our diagram. We have drawn both final momenta as being in the positive *x* direction because the truck has the greater initial momentum.

Figure WG5.4



PRACTICE

Along the x axis we chose, we have $v_{\text{car }x,\text{i}} = -v_{\text{car,i}}$ and $v_{\text{truck }x,\text{i}} = +v_{\text{truck,i}}$, where $v_{\text{car,i}} = 3.0\text{ m/s}$ and $v_{\text{truck,i}} = 4.0\text{ m/s}$. From this, we know that the x components of the initial momenta are $p_{\text{car }x,\text{i}} = m_{\text{car}}\, v_{\text{car }x,\text{i}}$ and $p_{\text{truck }x,\text{i}} = m_{\text{truck}} v_{\text{truck }x,\text{i}}$, We also know that, from immediately before to immediately after the collision, the momentum of our system is constant. From this information, we can write

$$p_{\text{car }x,\text{f}} + p_{\text{truck }x,\text{f}} = p_{\text{car }x,\text{i}} + p_{\text{truck }x,\text{i}}.$$

We know only the two quantities on the right, however, and so we need more information. The energy of the system does not change under our assumptions, but the final kinetic energy is reduced substantially as kinetic energy is converted to internal energy. We illustrate this with energy bars sketched on the system diagram, and we write the information in equation form:

$$\Delta E = 0, \quad \text{but} \quad K_\text{f} = \tfrac{1}{4}K_\text{i}.$$

Using this kinetic energy equation to obtain the final velocities, we can calculate the coefficient of restitution from the expression

$$e = -\frac{v_{\text{car }x,\text{f}} - v_{\text{truck }x,\text{f}}}{v_{\text{car }x,\text{i}} - v_{\text{truck }x,\text{i}}}.$$

❸ **EXECUTE PLAN** (*a*) Conservation of momentum gives us

$$m_{\text{car}}v_{\text{car }x,\text{f}} + m_{\text{truck}}v_{\text{truck }x,\text{f}} = m_{\text{car}}v_{\text{car }x,\text{i}} + m_{\text{truck}}v_{\text{truck }x,\text{i}}.$$

That the truck's inertia is given in terms of the car's inertia allows us to get rid of the m_{truck} factors and then simplify further by dividing through by m_{car}:

$$m_{\text{car}}v_{\text{car }x,\text{f}} + \tfrac{3}{2}m_{\text{car}}v_{\text{truck }x,\text{f}} = m_{\text{car}}v_{\text{car }x,\text{i}} + \tfrac{3}{2}m_{\text{car}}v_{\text{truck }x,\text{i}}$$

$$v_{\text{car }x,\text{f}} + \tfrac{3}{2}v_{\text{truck }x,\text{f}} = v_{\text{car }x,\text{i}} + \tfrac{3}{2}v_{\text{truck }x,\text{i}}. \tag{1}$$

The final velocities are unknown, and so we have one equation with two unknowns, which means we do not yet have enough information to solve the problem. We need one more equation, so we turn to our energy equation:

$$\begin{aligned}
&K_\text{f} = \tfrac{1}{4}K_\text{i}\\
&\tfrac{1}{2}m_{\text{car}}v^2_{\text{car }x,\text{f}} + \tfrac{1}{2}m_{\text{truck}}v^2_{\text{truck }x,\text{f}}\\
&\qquad = \tfrac{1}{4}\left(\tfrac{1}{2}m_{\text{car}}v^2_{\text{car }x,\text{i}} + \tfrac{1}{2}m_{\text{truck}}v^2_{\text{truck }x,\text{i}}\right)\\
&\tfrac{1}{2}m_{\text{car}}v^2_{\text{car }x,\text{f}} + \tfrac{1}{2}\left(\tfrac{3}{2}m_{\text{car}}\right)v^2_{\text{truck }x,\text{f}}\\
&\qquad = \tfrac{1}{4}\left[\tfrac{1}{2}m_{\text{car}}v^2_{\text{car }x,\text{i}} + \tfrac{1}{2}\left(\tfrac{3}{2}m_{\text{car}}\right)v^2_{\text{truck }x,\text{i}}\right].
\end{aligned}$$

Multiplying through by 4 and canceling m_{car}, we get

$$2v^2_{\text{car }x,\text{f}} + 3v^2_{\text{truck }x,\text{f}} = \tfrac{1}{2}v^2_{\text{car }x,\text{i}} + \tfrac{3}{4}v^2_{\text{truck }x,\text{i}}. \tag{2}$$

This result combined with Eq. 1 gives us two equations containing two unknowns. To solve them, we express $v_{\text{car }x,\text{f}}$ in Eq. 1 in terms of $v_{\text{truck }x,\text{f}}$:

$$\begin{aligned}
v_{\text{car }x,\text{f}} &= v_{\text{car},x,\text{i}} + \tfrac{3}{2}v_{\text{truck }x,\text{i}} - \tfrac{3}{2}v_{\text{truck }x,\text{f}}\\
&= (-v_{\text{car,i}}) + \tfrac{3}{2}(+v_{\text{truck,i}}) - \tfrac{3}{2}v_{\text{truck }x,\text{f}}\\
&= (-3.0\text{ m/s}) + \tfrac{3}{2}(+4.0\text{ m/s}) - \tfrac{3}{2}v_{\text{truck }x,\text{f}}\\
&= +3.0\text{ m/s} - \tfrac{3}{2}\,v_{\text{truck }x,\text{f}}.
\end{aligned} \tag{3}$$

PRACTICE

Now we use this result in Eq. 2 and solve for $v_{\text{truck }x,\text{f}}$:

$$2v^2_{\text{car }x,\text{f}} + 3v^2_{\text{truck }x,\text{f}} = \tfrac{1}{2}v^2_{\text{car }x,\text{i}} + \tfrac{3}{4}v^2_{\text{truck }x,\text{i}}$$

$$2(+3.0\ \text{m/s} - \tfrac{3}{2}v_{\text{truck }x,\text{f}})^2 + 3v^2_{\text{truck }x,\text{f}}$$
$$= \tfrac{1}{2}(-3.0\ \text{m/s})^2 + \tfrac{3}{4}(+4.0\ \text{m/s})^2$$

$$(18\ \text{m}^2/\text{s}^2) - (18\ \text{m/s})\,v_{\text{truck }x,\text{f}} + \tfrac{18}{4}v^2_{\text{truck }x,\text{f}} + 3v^2_{\text{truck }x,\text{f}}$$
$$= \frac{9.0\ \text{m}^2/\text{s}^2}{2} + (12\ \text{m}^2/\text{s}^2)$$

$$(18\ \text{m}^2/\text{s}^2) - (18\ \text{m/s})\,v_{\text{truck }x,\text{f}} + \tfrac{30}{4}v^2_{\text{truck }x,\text{f}}$$
$$= \frac{33\ \text{m}^2/\text{s}^2}{2}$$

$$5.0v^2_{\text{truck }x,\text{f}} - (12\ \text{m/s})\,v_{\text{truck }x,\text{f}} + (1.0\ \text{m}^2/\text{s}^2)$$
$$= 0.$$

Solving this quadratic equation yields

$$v_{\text{truck }x,\text{f}} = +2.3\ \text{m/s} \quad \text{or} \quad +0.086\ \text{m/s}.$$

Substituting these values into Eq. 3 and solving for $v_{\text{car }x,\text{f}}$ gives

$$v_{\text{truck }x,\text{f}} = +2.3\ \text{m/s},\ v_{\text{car }x,\text{f}} = -0.45\ \text{m/s}$$

or

$$v_{\text{truck }x,\text{f}} = +0.086\ \text{m/s},\ v_{\text{car }x,\text{f}} = +2.9\ \text{m/s}.$$

We must choose the result that corresponds to the physical situation. The first result says that the truck keeps moving in the positive x direction and the car keeps moving in the negative x direction, implying that they pass through each other, a physically impossible result. The second solution has the truck continuing but the car switching direction, which is plausible. Consequently, with our having assigned leftward as the positive x direction, the physically correct velocities are

$$v_{\text{truck }x,\text{f}} = +0.086\ \text{m/s} \quad (0.086\ \text{m/s to the left}) ✔$$
$$v_{\text{car }x,\text{f}} = +2.9\ \text{m/s} \quad (2.9\ \text{m/s to the left}). ✔$$

(*b*) The coefficient of restitution is

$$e = -\frac{v_{\text{car }x,\text{f}} - v_{\text{truck }x,\text{f}}}{v_{\text{car }x,\text{i}} - v_{\text{truck }x,\text{i}}}$$
$$= -\frac{+2.9\ \text{m/s} - (+0.086\ \text{m/s})}{-3.0\ \text{m/s} - (+4.0\ \text{m/s})} = +\frac{2.8}{7.0} = 0.40. ✔$$

4 **EVALUATE RESULT** The final speeds are reasonable given the initial speeds (which are a bit high for a parking space!). We made sure the velocity directions are realistic when we chose the appropriate root of the quadratic equation. The kinetic energy of the system does decrease substantially, as expected, with the truck moving very slowly after the collision. Because the magnitude of the change in momentum for the two vehicles is equal but the inertias are not, the truck should have the smaller change in velocity. Its velocity change is $(+0.086\ \text{m/s}) - (+4.0\ \text{m/s}) = -3.9\ \text{m/s}$, while the car's is $(+2.9\ \text{m/s}) - (-3.0\ \text{m/s}) = 5.9\ \text{m/s}$, just as we expected. The coefficient of restitution is reassuringly less than 1 for this inelastic collision.

Guided Problem 5.4 Burrowing bullet

A gun that has a muzzle velocity of 600 m/s is used to fire a 12.0-g bullet horizontally into a 4.00-kg block of wood initially at rest. The bullet passes completely through the block, with negligible loss of inertia to either object. After the collision, the block slides at 1.20 m/s in the direction of the bullet's motion. What is the change in internal energy in the bullet-block system? Ignore any effects due to friction between the block and the surface over which it slides.

❶ GETTING STARTED

1. Which objects should you include in your system? Draw a system diagram for these objects.
2. Which conservation laws apply here: momentum, energy, or both? How could you represent what happens to these quantities graphically?

❷ DEVISE PLAN

3. Write equations relating the initial and final values. Does any equation have a single unknown (and hence can be solved immediately)?
4. What is the relationship between the change in internal energy and the change in kinetic energy?

❸ EXECUTE PLAN

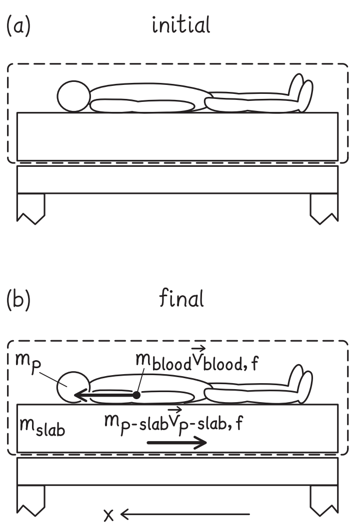
❹ EVALUATE RESULT

Worked Problem 5.5 Explosive beat

You decide to build a *ballistocardiograph* that works on the following principle: A patient lies flat on a slab that floats on a cushion of air so that it can move freely in the horizontal direction. When the heart pumps blood horizontally in one direction, the slab and patient move in the opposite direction. For a resting patient, the heart pumps blood preferentially toward the head. The resulting recoil speed can be measured and can be correlated with the medical information you are after, which is the heart's ability to pump blood. You anticipate having patients as large as 1.0×10^2 kg. You know that a normal heart, each time it pumps, converts 2.0 mJ of chemical energy to kinetic energy and moves about 50 g of blood. You also know you can buy velocity sensors that can detect the slab's speed to a sensitivity of 1.0×10^{-5} m/s, and you want the measured speed to be comfortably higher than this—say, by a factor of 10. What is the highest practical value for the slab's inertia if its maximum speed with a patient lying on it is to be 1.0×10^{-4} m/s? Assume that any friction experienced by the slab sliding over the air cushion is negligible and that the slab and patient move as a single unit.

❶ GETTING STARTED The heart is converting internal energy to the kinetic energy of the blood moving in one direction and the kinetic energy of the slab and patient moving in the opposite direction. This is an explosive event and a signal that we should probably use conservation of momentum in our analysis, which means our first step is to choose an isolated system and draw a system diagram for it (Figure WG5.5*a*). The "initial" diagram is at an instant between heartbeats, when the patient, the slab, and the small amount of blood about to be pumped can be thought of as motionless. The initial system momentum is therefore zero. We approximate that all the pumped blood moves toward the patient's head. After we add the unknown final velocity information, Figure WG5.5(*b*) shows the notation we shall use. We suspect we may need a coordinate system, and so we arbitrarily choose our positive *x* axis pointing to the left. The "final" diagram shows the small volume of blood moving toward the head and the patient and slab recoiling. We suspect that the amount of internal energy converted to kinetic energy will give us information about the speeds.

Figure WG5.5



❷ **DEVISE PLAN** Conservation of momentum gives us, for the isolated system of blood, patient, and slab,

$$\vec{p}_f = \vec{p}_i$$

$$m_{\text{blood}}\vec{v}_{\text{blood,f}} + (m_{\text{patient}} + m_{\text{slab}})\vec{v}_{\text{patient+slab,f}} = 0. \tag{1}$$

We want to know the value of m_{slab} that will generate the minimum detectable value for $|\vec{v}_{\text{patient+slab,f}}| = 1.0 \times 10^{-4}\,\text{m/s}$. Because we have two unknowns, m_{slab} and $\vec{v}_{\text{blood,f}}$, we need another equation. Knowing that the change in the internal energy is $\Delta E_{\text{int}} = -2.0\ \text{mJ} = -2.0 \times 10^{-3}\ \text{J}$, we should be able to use conservation of energy for this closed system to get another equation:

$$\Delta K + \Delta E_{\text{int}} = 0$$

$$K_f - K_i = -\Delta E_{\text{int}}$$

$$\left[\tfrac{1}{2}m_{\text{blood}}v^2_{\text{blood}\,x,\text{f}} + \tfrac{1}{2}(m_{\text{patient}} + m_{\text{slab}})v^2_{\text{patient+slab}\,x,\text{f}}\right] - 0$$
$$= -\Delta E_{\text{int}} \tag{2}$$

This gives us two equations with everything known except two values: m_{slab} and $v_{\text{blood}\,x,\text{f}}$. We should first use Eq. 1 to express $v_{\text{blood}\,x,\text{f}}$ in terms of m_{slab} and then substitute into Eq. 2.

❸ **EXECUTE PLAN** Notice that the patient and slab move together as one object. It is convenient to use the subscript ps to replace the unwieldy patient+slab. This will require that we solve first for the combined inertia of patient and slab and then subtract the known maximum inertia of the patient to obtain the desired inertia of the slab. With this substitution, Eq. 1 gives

$$m_{\text{blood}}v_{\text{blood}\,x,\text{f}} = -m_{\text{ps}}v_{\text{ps}\,x,\text{f}}$$

$$v_{\text{blood}\,x,\text{f}} = -\frac{m_{\text{ps}}}{m_{\text{blood}}}v_{\text{ps}\,x,\text{f}}.$$

Then Eq. 2 gives

$$\tfrac{1}{2}m_{\text{blood}}v^2_{\text{blood}\,x,\text{f}} + \tfrac{1}{2}m_{\text{ps}}v^2_{\text{ps}\,x,\text{f}} = -\Delta E_{\text{int}}$$

$$\tfrac{1}{2}m_{\text{blood}}\left(-\frac{m_{\text{ps}}}{m_{\text{blood}}}v_{\text{ps}\,x,\text{f}}\right)^2 + \tfrac{1}{2}m_{\text{ps}}v^2_{\text{ps}\,x,\text{f}} = -\Delta E_{\text{int}}$$

$$\frac{1}{2}\frac{m^2_{\text{ps}}}{m_{\text{blood}}}v^2_{\text{ps}\,x,\text{f}} + \tfrac{1}{2}m_{\text{ps}}v^2_{\text{ps}\,x,\text{f}} = -\Delta E_{\text{int}}$$

$$\left(\frac{1}{2}\frac{v^2_{\text{ps}\,x,\text{f}}}{m_{\text{blood}}}\right)m^2_{\text{ps}} + \left(\tfrac{1}{2}v^2_{\text{ps},x,\text{f}}\right)m_{\text{ps}} + \Delta E_{\text{int}} = 0. \tag{3}$$

Because two coefficients involve the same quantity, we divide by that quantity:

$$\frac{1}{m_{\text{blood}}}m^2_{\text{ps}} + m_{\text{ps}} + \frac{2\Delta E_{\text{int}}}{v^2_{\text{ps}\,x,\text{f}}} = 0.$$

We solve this quadratic equation for m_{ps} using the quadratic formula:

$$m_{\text{ps}} = \frac{-1 \pm \sqrt{(1)^2 - 4\left(\dfrac{1}{m_{\text{blood}}}\right)\left(\dfrac{2\Delta E_{\text{int}}}{v^2_{\text{ps}\,x,\text{f}}}\right)}}{2\left(\dfrac{1}{m_{\text{blood}}}\right)}.$$

Substituting the known numerical values gives

$$m_{\text{ps}} = \frac{-1 \pm \sqrt{1 - 4\left[\dfrac{1}{5.0 \times 10^{-2}\ \text{kg}}\right]\dfrac{2(-2.0 \times 10^{-3}\ \text{J})}{(1.0 \times 10^{-4}\ \text{m/s})^2}}}{2\left[\dfrac{1}{5.0 \times 10^{-2}\ \text{kg}}\right]},$$

which simplifies to

$$m_{\text{ps}} = \frac{-1 \pm \sqrt{1 + 3.2 \times 10^7}}{4.0 \times 10^1\ \text{kg}^{-1}}.$$

PRACTICE

The positive solution to this equation is $m_{ps} = 1.4 \times 10^2$ kg. We now can obtain the maximum design value for the slab's inertia:

$$m_{slab} = m_{ps} - m_{patient} = 140 \text{ kg} - 100 \text{ kg} = 40 \text{ kg.} \checkmark$$

❹ **EVALUATE RESULT** This is not an especially low inertia, and so the answer suggests that such a device might be feasible. Looking at the math confirms that this is a maximum inertia for the slab because a larger slab would require either less patient inertia or a smaller value of v_{ps}. The assumption of an isolated system is not unreasonable, given the air-cushion design. The assumption that all 50 g of blood moves toward the head is an oversimplification, but it is consistent with the problem statement, and in the absence of more accurate blood flow information we have no better alternative.

(Bonus task: You should convince yourself that the slab moves back and forth as the heart beats rather than moving farther and farther in one direction.)

Guided Problem 5.6 Useful approximations

In Worked Problem 5.5, we had to solve a quadratic equation (Eq. 3). Solving such equations can be a bother when you wish to make a quick calculation. The procedure used in Worked Problem 5.5 can be bypassed in explosive separations if the inertia of one object is much greater than that of the other(s). Consider the explosive separation that breaks an object initially at rest into two fragments, one of which has much greater inertia than the other: $m_1 \ll m_2$. In the explosive separation, some of the internal energy of the system is converted to kinetic energy in the fragments. Show that the following are good approximations for the division of energy between the fragments:

$$\tfrac{1}{2} m_2 v_2^2 \approx \frac{m_1}{m_2} \Delta K \tag{A}$$

$$\tfrac{1}{2} m_1 v_1^2 \approx \Delta K, \tag{B}$$

where v_1 is the speed of the fragment of inertia m_1 and v_2 is the speed of the fragment of inertia m_2. Equation B tells us that almost all of the internal energy converted to kinetic energy goes to the fragment that has less inertia. Only a small fraction m_1/m_2 of the internal energy goes to the fragment that has greater inertia.

❶ **GETTING STARTED**

1. Is the approach of Worked Problem 5.5 relevant?
2. Draw a system diagram for the initial and final situations. It may be necessary to use velocities rather than speeds when dealing with momentum.

❷ **DEVISE PLAN**

3. Is it possible to follow the procedure of Worked Problem 5.5 using the simpler notation m_1, v_1 and m_2, v_2?
4. In your new version of Eq. 3 of Worked Problem 5.5, isolate the kinetic energy of the fragment with greater inertia, $\frac{1}{2} m_2 v_2^2$.
5. Which is true: $m_1/m_2 \gg 1$ or $m_1/m_2 \ll 1$?

❸ **EXECUTE PLAN**

6. Remember to approximate, especially if you are adding or subtracting a small number to or from a very large number.
7. You should now be able to do the step that gives Eq. A.
8. To derive Eq. B, repeat the derivation of Worked Problem 5.5 but this time in terms of the speed v_1 of the object with less inertia m_1.

❹ **EVALUATE RESULT**

9. Use the numerical values given in Worked Problem 5.5 in Eq. A to estimate the maximum practical inertia for the slab. Is the value you calculate consistent with the numerical result obtained in Worked Problem 5.5?

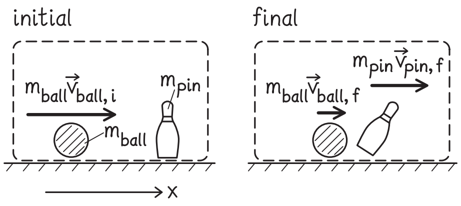
Worked Problem 5.7 Designer bowling

A 7.5-kg bowling ball rolls down the lane toward the 1.5-kg head pin at 5.0 m/s. The bowling alley owner wants to know how fast that pin can go flying, so that the end of the lane can be reinforced properly. You mumble confidently about coefficient of restitution, inertia, and maximum speed, and as the boss leaves she mentions that she wants this information yesterday. Ignore friction between the alley floor and the ball.

❶ **GETTING STARTED** What have we got to work with? Well, for the short time interval of the collision, the ball and pin form an isolated system whose momentum must remain constant. The system may or may not be closed, but it seems probable that the less energy transferred out of the ball-pin system, the more likely the pin will attain its maximum speed. There is no obvious source of energy to be transferred into the system, so assuming a closed system should be a good approximation. There is also no obvious mechanism for an explosive separation, so the coefficient of restitution must be between 1 and 0. We could simply compute the final velocity in terms of an unknown e and then try different values for the coefficient of restitution to see which yields the highest speed.

We draw a system diagram (Figure WG5.6), as usual, with initial and final pictures to help clarify what is going on and allow us to assign symbols for the variables and establish the direction of a positive x axis.

Figure WG5.6



❷ **DEVISE PLAN** Conservation of momentum tells us that

$$\vec{p}_{\text{f}} = \vec{p}_{\text{i}}$$

$$m_{\text{ball}}v_{\text{ball}\,x,\text{f}} + m_{\text{pin}}v_{\text{pin}\,x,\text{f}} = m_{\text{ball}}v_{\text{ball}\,x,\text{i}} + m_{\text{pin}}v_{\text{pin}\,x,\text{i}}, \tag{1}$$

where we assume that the original motion of the ball is in the positive x direction. We know both inertias and both initial velocities, taking $v_{\text{pin}\,x,\text{i}}$ to be 0. Thus we have two unknowns, $v_{\text{ball x,f}}$ and $v_{\text{pin}\,x,\text{f}}$, but we also have a second equation to use:

$$e = \frac{v_{\text{rel,f}}}{v_{\text{rel,i}}} = -\frac{v_{\text{ball}\,x,\text{f}} - v_{\text{pin}\,x,\text{f}}}{v_{\text{ball}\,x,\text{i}} - v_{\text{pin}\,x,\text{i}}} = \text{trial value} \tag{2}$$

Thus, assuming a known trial value for e, we have two equations and two unknowns, the x components of the final velocities.

❸ **EXECUTE PLAN** A logical step toward solution is to manipulate Eq. 2 to look more like Eq. 1 (with all the final values on the left and all the initial values on the right) and then look for a way to eliminate an unknown:

$$m_{\text{ball}}v_{\text{ball}\,x,\text{f}} + m_{\text{pin}}v_{\text{pin}\,x,\text{f}} = m_{\text{ball}}v_{\text{ball}\,x,\text{i}} + m_{\text{pin}}v_{\text{pin}\,x,\text{i}} \tag{1}$$

$$v_{\text{ball}\,x,\text{f}} - v_{\text{pin}\,x,\text{f}} = -e(v_{\text{ball}\,x,\text{i}} - v_{\text{pin}\,x,\text{i}}) \tag{2}$$

Noting that there are different signs on the two final x components of pin velocity, but the same sign on the two final x components of ball velocity, we plan to subtract Eq. 2 from Eq. 1 to eliminate $v_{\text{ball}\,x,\text{f}}$. To make this work, we multiply Eq. 2 by m_{ball}:

$$m_{\text{ball}}v_{\text{ball}\,x,\text{f}} - m_{\text{ball}}v_{\text{pin}\,x,\text{f}} = -em_{\text{ball}}(v_{\text{ball}\,x,\text{i}} - v_{\text{pin}\,x,\text{i}})$$

and subtract this equation from Eq. 1:

$$0 + (m_{\text{ball}} + m_{\text{pin}})v_{\text{pin}\,x,\text{f}}$$

$$= m_{\text{ball}}(1 + e)v_{\text{ball}\,x,\text{i}} + (m_{\text{pin}} - em_{\text{ball}})v_{\text{pin}\,x,\text{i}}$$

$$v_{\text{pin}\,x,\text{f}} = \frac{m_{\text{ball}}(1 + e)v_{\text{ball}\,x,\text{i}} + (m_{\text{pin}} - em_{\text{ball}})v_{\text{pin}\,x,\text{i}}}{(m_{\text{ball}} + m_{\text{pin}})}.$$

Noting that the initial x component of the pin velocity is zero, we obtain

$$v_{\text{pin}\,x,\text{f}} = \frac{m_{\text{ball}}(1 + e)\,v_{\text{ball}\,x,\text{i}} + 0}{(m_{\text{ball}} + m_{\text{pin}})} = \frac{m_{\text{ball}}v_{\text{ball}\,x,\text{i}}}{(m_{\text{ball}} + m_{\text{pin}})}(1 + e)$$

$$= \frac{7.5\text{ kg }(5.0\text{ m/s})}{(7.5\text{ kg} + 1.5\text{ kg})}(1 + e).$$

The speed we seek is the magnitude of this x component of velocity.

PRACTICE

It is now clear that the larger the value of e, the higher the final speed of the pin. Because the largest reasonable value of e is 1, we compute the maximum speed as

$$v_{\text{pin,f}} = |v_{\text{pin }x\text{,f}}| = |(4.17\text{ m/s})(1 + 1)| = 8.34\text{ m/s} = 8.3\text{ m/s.} \checkmark$$

The owner should design for this speed.

❹ EVALUATE RESULT The speed we obtain for the pin is almost 70% faster than the speed of the incoming ball, but that is reasonable given that the ball has greater inertia and keeps rolling in the same direction. We therefore expect the pin to recoil faster than the incoming ball. Because the ball-pin system is closed, we can also check our answer using energy conservation. For an elastic collision, the kinetic energy is constant and so

$$\tfrac{1}{2}m_{\text{ball}}v_{\text{ball,f}}^2 + \tfrac{1}{2}m_{\text{pin}}v_{\text{pin,f}}^2 = \tfrac{1}{2}m_{\text{ball}}v_{\text{ball,i}}^2 + \tfrac{1}{2}m_{\text{pin}}v_{\text{pin,i}}^2$$

$$\tfrac{1}{2}m_{\text{pin}}v_{\text{pin,f}}^2 = \tfrac{1}{2}m_{\text{ball}}v_{\text{ball,i}}^2 - \tfrac{1}{2}m_{\text{ball}}v_{\text{ball,f}}^2 + \tfrac{1}{2}m_{\text{pin}}v_{\text{pin,i}}^2.$$

There are still two unknown final speeds, so we compute one of them using Eq. (2).

$$v_{\text{ball }x\text{,f}} = v_{\text{pin }x\text{,f}} - e(v_{\text{ball }x\text{,i}} - v_{\text{pin }x\text{,i}})$$

$$v_{\text{ball }x\text{,f}} = +8.34\text{ m/s} - 1(+5.0\text{ m/s} - 0)$$

$$v_{\text{ball }x\text{,f}} = +3.34\text{ m/s},\ v_{\text{ball,f}} = 3.3\text{ m/s}$$

Using the values computed above for $e = 1$, we have

$$\tfrac{1}{2}(1.5\text{ kg})v_{\text{pin,f}}^2$$

$$= \tfrac{1}{2}(7.5\text{ kg})(5.0\text{ m/s})^2 - \tfrac{1}{2}(7.5\text{ kg})(3.34\text{ m/s})^2 + 0$$

$$v_{\text{pin,f}}^2 = 69.2\text{ m}^2/\text{s}^2$$

$$v_{\text{pin,f}} = 8.3\text{ m/s},$$

which is the result we obtained.

Guided Problem 5.8 Fast service

Champion table-tennis players can swing the paddle at speeds of 20 m/s. How fast is the ball going after one of these serves?

❶ GETTING STARTED

1. Choose a system and draw a system diagram.
2. Is your system isolated? Closed? Which conservation laws are appropriate to describe it?
3. Draw before and after pictures, and label all known and unknown quantities. It may seem at first like you don't have enough information, but plunge ahead anyway.

❷ DEVISE PLAN

4. What is the initial velocity of the ball?
5. Based on what you expect the properties of a Ping-Pong ball to be, which type of collision is this likely to be: elastic, inelastic, or totally inelastic?
6. No inertia values are given. Assuming the hand and forearm are part of the paddle, how does the inertia of the ball compare with the inertia of the paddle? How will the paddle's final velocity after the collision compare with its initial velocity before the collision?
7. Write the relative velocity equation. Do you need another equation?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

8. How does the final speed of the ball compare with the initial speed of the paddle?
9. Which type of collision did you assume? Is that assumption justified (or at least not unreasonable)? How would your result change if you modified the coefficient of restitution a bit?

Answers to Review Questions

1. Relative velocity is the difference between the velocity of a reference object and the velocity of an object being studied. We are free to choose which object is the object of study and which is the reference object, but we must distinguish which is which because relative velocity is a vector. Relative speed is the magnitude of the relative velocity and is the same regardless of which object is the object of study and which is the reference object.
2. The last subscript represents the object whose velocity we are studying. The first subscript represents the object we are measuring relative to (the reference object).
3. (*a*) The relative speed does not change. (*b*) The relative speed changes. (*c*) The final relative speed is zero (the objects stick together).
4. The kinetic energy of an object is the energy associated with the motion of the object. It can be computed as one-half the product of the object's inertia and the square of its speed.
5. The kinetic energy of a moving object is always positive, regardless of the direction in which the object is moving.
6. Kinetic energy can never be negative. It is zero if an object is stationary.
7. The *state* of an object is its condition as specified by the complete set of the physical variables that describe the object, such as shape and temperature.
8. In a reversible process, there is no permanent change in the objects' physical states; in an irreversible process, there is a permanent change in the physical states.
9. The internal energy increases by an amount equal to the decrease in the kinetic energy so that the energy in the system is unchanged.
10. We have no direct way of computing changes in internal energy at this point, but in the absence of energy transfer across the system boundary, the change in internal energy is the negative of the change in kinetic energy that takes place during the collision. Thus determining the change in kinetic energy and taking its negative is one way to determine the internal energy change.
11. A closed system is one in which no energy is transferred across its boundary. This is not the same as an isolated system (one in which no momentum is transferred across its boundary).
12. Energy conversion is energy changing from one form to another, such as from chemical energy to thermal energy; it may, but need not, involve more than one object. Energy transfer is the exchange of energy from one object to another; it may, but need not, involve the energy converting from one form to another.
13. The relative speeds are the same before and after the collision, a fact expressed in Eq. 5.4. This is true only in elastic collisions.
14. Imagine that you are one object in the collision. Before the collision, the distance between you and the other object is decreasing, but after the collision, the distance between you and the other object is increasing. So, the relative velocity between you and the other object changes sign.
15. The coefficient of restitution is the ratio of the relative speed of two objects after their collision to their relative speed before the collision.
16. Elastic: $e = 1$; totally inelastic: $e = 0$.
17. The minus sign is needed to make e positive. Because relative velocity changes sign after a collision, the fraction $v_{12\,x,\mathrm{f}}/v_{12\,x,\mathrm{i}}$ is always negative, and so the minus sign in front of the fraction makes e always positive.
18. No. The final velocities cannot be predicted unless additional information is available—the value of the coefficient of restitution, for instance, or the change in internal energy.
19. No. You are merely converting some of the internal energy of your muscles to the kinetic energy of the ball and your hand.
20. Yes. Internal energy may be transferred from one object in the system to another or may be converted from one form to another (say, chemical to thermal), which means a change in the physical state.
21. Conservation of energy (Eq. 5.23) states that the energy of a closed system remains constant. Thus if we can compute ΔK for the system, ΔE_{int} must be its equal and opposite value, as in Eq. 5.24.
22. Some of the internal energy of the exploding objects is converted to kinetic energy.
23. No. Because the relative speed of the objects increases (from zero before the separation to a nonzero value after), $e = 1$ can never be true. In another approach, you can reason that the kinetic energy of the system changes in an explosive separation. Because the kinetic energy does not change in elastic collisions, no explosive separation can be elastic.

Answers to Guided Problems

Guided Problem 5.2 5.0 m/s
Guided Problem 5.4 1.92×10^3 J
Guided Problem 5.8 40 m/s

Guided Practice by Chapter

6 Principle of Relativity

Review Questions

Answers to these questions can be found at the end of this chapter.

6.1 Relativity of motion

1. A truck moves at constant velocity on a freeway. Describe how the truck's motion is seen (*a*) by an observer in a car moving with the same constant velocity as the truck and (*b*) by an observer in a car moving at any other constant velocity.
2. On a breezy day, riding a bicycle into the wind can feel like riding into a windstorm. Riding in the direction of the wind on the same breezy day, however, you can sometimes feel no wind at all.

6.2 Inertial reference frames

3. During a given time interval, an observer in inertial reference frame 1 measures an object's change in momentum to be $\Delta\vec{p} = 0$. What is the value of $\Delta\vec{p}$ measured during the same time interval by an observer in inertial reference frame 2, which is moving east at 10 m/s relative to reference frame 1?
4. What is the law of inertia, and of what use is it?
5. An observer sees an isolated object undergo a change in momentum. Is the observer in an inertial reference frame?
6. (*a*) If an object's acceleration is zero in one inertial reference frame, is it zero in all other inertial reference frames? (*b*) If an object is at rest in one inertial reference frame, is it at rest in all other inertial reference frames?

6.3 Principle of relativity

7. State and explain the principle of relativity.
8. Is there any experiment you can do in your own reference frame to determine whether your reference frame is moving at a constant velocity? If yes, can you determine the value of that velocity?
9. If an object is moving forward in one inertial reference frame, it is always possible to find another inertial reference frame in which it is moving backward. If the object is accelerating forward in the first reference frame, which way is it accelerating in the second reference frame?
10. Someone says to you, "Momentum isn't a conserved quantity! All I have to do is change to a different inertial reference frame and the momentum of the system I'm looking at is different from what it was in my first reference frame." How should you respond?
11. Two observers witness an elastic collision from different inertial reference frames. What do they disagree on and what do they agree on regarding kinetic energy? Regarding momentum?

6.4 Zero-momentum reference frame

12. What is the zero-momentum reference frame for a system? How can you use it to solve a problem if the only velocity data you have for the system were measured in a different inertial reference frame?
13. (*a*) Compare the initial and final values of the magnitude of the momentum of a particle involved in an elastic collision when viewed from the zero-momentum reference frame. (*b*) What happens to the direction of the particle's momentum?

6.5 Galilean relativity

14. What are the Galilean transformation equations (Eqs. 6.4 and 6.5) used for?
15. Suppose observers C and D both measure a train's velocity but from different inertial reference frames. C reports velocity $\vec{v}_{Ct}$, and D reports $\vec{v}_{Dt}$. What other information must C have in order to determine whether or not D's measurement is accurate?

6.6 Center of mass

16. How does an object's inertia differ when measured in two different inertial reference frames?
17. How does the position of a system's center of mass measured using one reference point differ from the position measured using some other reference point?
18. In an isolated system that contains moving parts, how does the system's center of mass move?
19. What is the momentum of a system of objects as measured by someone moving along with the center of mass of the system?

6.7 Convertible kinetic energy

20. Is the convertible kinetic energy of a system of objects the same in all inertial reference frames?
21. Which part of a system's kinetic energy changes when an observer switches from one inertial reference frame to another: the convertible part or the nonconvertible part?
22. Can all the initial kinetic energy of two colliding objects be converted to internal energy?
23. (*a*) What is the maximum amount of kinetic energy that can be converted to internal energy in a collision? (*b*) In which type of collision is this maximum conversion possible?

6.8 Conservation laws and relativity

24. (*a*) If the change in momentum of a system is zero in one inertial reference frame, is it zero in all other inertial reference frames? (*b*) If the change in momentum of a system has a specific magnitude and direction in one inertial reference frame, does it have the same magnitude and direction in all other inertial reference frames?
25. (*a*) If the change in energy of a system is zero in one inertial reference frame, is it zero in all inertial reference frames? (*b*) If the change in energy of a system has a specific magnitude in one inertial reference frame, does it have the same magnitude in all other inertial reference frames?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The velocity of the zero-momentum reference frame relative to the Earth reference frame for a head-on collision between a bug and an 18-wheel truck traveling at highway speed (Z, X, R)
2. The velocity of the zero-momentum reference frame relative to the Earth reference frame for a collision between a parked, fully loaded furniture delivery truck and a car traveling on a city street (Z, M, C)
3. The velocity of the zero-momentum reference frame relative to the Earth reference frame for a collision in which a large bird, while flying, is hit from behind by a bullet (E, I, V, P)
4. The reduced inertia of the Earth-Moon system (AA, J, Y)
5. The location of the center of mass of a five-car, one-locomotive train (BB, G, L)
6. The location of the center of mass of your body when you are standing (S, K, T, W)
7. The velocity of the center of mass of a system of 16 billiard balls (including the cue ball) just after the first break shot (A, F, O, U)
8. The kinetic energy converted to internal energy during a head-on collision between a bug and an 18-wheel truck traveling at highway speed (Z, X, D, R)
9. The maximum convertible kinetic energy for a collision between a parked, fully loaded furniture delivery truck and a car traveling on a city street (Z, M, AA, N, C)
10. The maximum convertible kinetic energy for a high-school wrestler slammed to the mat (B, H, Q)

Hints

A. Is the system of balls an isolated system during the break-shot collision?
B. What is an appropriate inertia for the wrestling mat?
C. What is a typical speed of a car driving on a city street?
D. What is the inertia of a bug?
E. What is the inertia of a large bird?
F. What is the speed of the cue ball approaching the 15 other balls?
G. What is the ratio of the inertia of a train car to the inertia of a locomotive?
H. With what speed does the wrestler hit the mat?
I. What is the inertia of a bullet?
J. What is the inertia of Earth?
K. Is your density roughly uniform?
L. How do the lengths of a locomotive and a train car compare?
M. How do the inertias of the two vehicles compare?
N. What is the reduced inertia of the car-truck system?
O. How does the center-of-mass velocity of the billiard balls change during the collision?
P. What is the speed of a bullet in flight?
Q. What is the reduced inertia of the system?
R. What is a typical highway speed of an 18-wheel truck?
S. How tall are you?
T. What is your basic shape when standing?
U. What is the center-of-mass speed of the billiard balls before the collision?
V. What is the speed of a large bird in flight?
W. Where is the center of mass of a uniform cylinder?
X. How do the inertias of the bug and 18-wheel truck compare?
Y. What is the inertia of the Moon?
Z. How is the velocity of the zero-momentum reference frame related to individual velocities and inertias?
AA. How is the reduced inertia related to individual inertias?
BB. How is the center of mass related to individual inertias?

Key (all values approximate)

A. yes, with the cue ball in the system; **B.** the mat is supported by Earth, so the appropriate inertia is huge; **C.** 1×10^1 m/s; **D.** a large bug: 5×10^{-4} kg; **E.** 2 kg; **F.** 1×10^1 m/s; **G.** 1:10; **H.** 4 m/s; **I.** 2×10^{-2} kg; **J.** 6×10^{24} kg; **K.** yes, **L.** they are about equal; **M.** the truck has about five times the inertia of the car; **N.** 5/6 of car inertia, so 1×10^3 kg; **O.** it does not change; **P.** 4×10^2 m/s; **Q.** about the same as the wrestler's inertia—say, 8×10^1 kg; **R.** 3×10^1 m/s; **S.** 2 m; **T.** cylindrical; **U.** from Eq. 6.26, $(10\ \text{m/s})/16 = 0.6$ m/s; **V.** 6 m/s; **W.** halfway along its length; **X.** the bug has almost zero inertia relative to the inertia of the 18-wheel truck; **Y.** 7×10^{22} kg; **Z.** sum of products of individual inertias and velocities divided by sum of inertias, Eq. 6.26; **AA.** product divided by sum, Eq. 6.39; **BB.** sum of products of individual inertias and positions divided by sum of inertias, Eq. 6.24

PRACTICE

Worked and Guided Problems

Procedure: Applying Galilean relativity

In problems dealing with more than one reference frame, you need to keep track not only of objects, but also of reference frames. For this reason, each quantity is labeled with two subscripts. The first subscript denotes the observer; the second denotes the object of interest. For example, if we have an observer on a train and also a car somewhere on the ground but in sight of the train, then $\vec{a}_{Tc}$ is the train observer's measurement of the acceleration of the car. Once you understand this notation and a few basic operations, working with relative quantities is easy and straightforward.

observer A's measurement of velocity of car:

$\vec{v}_{Ac}$

observer ······ object of interest

1. Begin by listing the quantities given in the problem, using this double-subscript notation.
2. Write the quantities you need to determine in the same notation.
3. Use subscript cancellation (Eq. 6.13) to write an equation for each quantity you need to determine, keeping the first and the last subscripts on each side the same. For example, in a problem where you need to determine $\vec{v}_{Tc}$ involving a moving observer B, write

$$\vec{v}_{Tc} = \vec{v}_{TB} + \vec{v}_{Bc}.$$

4. If needed, use subscript reversal (Eq. 6.15) to eliminate any unknowns.
5. Use the kinematics relationships from Chapters 2 and 3 to solve for any remaining unknowns, making sure you stay in one reference frame.

You can use this procedure and the subscript operations for any of the three basic kinematic quantities (position, velocity, and acceleration).

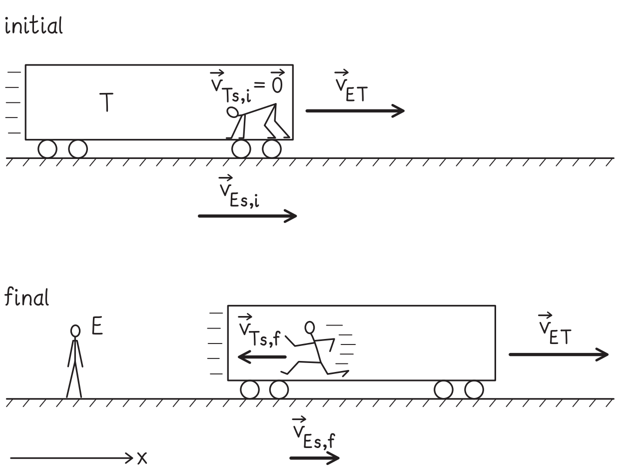
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 6.1 Running a train

On a train carrying the university team to a track meet, a sprinter in one of the cars practices his starts by running down the aisle. In the reference frame of the train, the sprinter starts from rest and runs toward the rear of the car. After 2.0 s, he has accelerated to a speed of 10 m/s. If the train is moving at a constant 30 m/s relative to the Earth reference frame, what does an observer standing alongside the tracks measure for the sprinter's initial velocity, final velocity, and acceleration?

❶ GETTING STARTED We must translate information from the train reference frame T to the Earth reference frame E, which suggests Galilean transformations for velocity and acceleration. As usual, we begin by making a sketch, paying attention to the directions of the vectors (Figure WG6.1).

Figure WG6.1



Because the train does not accelerate, T is an inertial reference frame, as is the Earth reference frame E. We are given enough information to calculate the sprinter's initial velocity $\vec{v}_{\text{Ts,i}}$, final velocity $\vec{v}_{\text{Ts,f}}$, and acceleration $\vec{a}_{\text{Ts}}$ measured by an observer at rest in the train reference frame, and we need to determine what the trackside observer measures for these three quantities, which we denote as $\vec{v}_{\text{Es,i}}$, $\vec{v}_{\text{Es,f}}$, and $\vec{a}_{\text{Es}}$ because they are being measured in the Earth reference frame.

❷ **DEVISE PLAN** As Figure WG6.1 shows, we define the positive x direction as the direction in which the train is moving. The train is therefore moving with a velocity that has x component $v_{\text{ET}x} = +30\,\text{m/s}$ as measured by the observer in the Earth reference frame. (And to a passenger on the train, the trackside observer moves along the x axis with a velocity that has x component $v_{\text{TE}x} = -30\,\text{m/s}$.)

We need to transform velocity information from the T reference frame to the E reference frame, and so we use Eq. 6.14 with the subscripts appropriate to this problem:

$$\vec{v}_{\text{Es}} = \vec{v}_{\text{ET}} + \vec{v}_{\text{Ts}}. \tag{1}$$

For the sprinter's acceleration, we note from Eq. 6.11 that the two inertial reference frames give the same result:

$$\vec{a}_{\text{Es}} = \vec{a}_{\text{Ts}}.$$

We can use Eq. 1 for both the sprinter's initial velocity and his final velocity measured by the trackside observer. To calculate the acceleration this observer measures, we do not have enough information to take the time derivative of the velocity because we know the sprinter's velocity at only two instants. Thus all we can do is either compute his average acceleration or assume his acceleration is constant. In either case, we have for the acceleration measured by the observer in the Earth reference frame

$$\vec{a}_{\text{Es}} = \frac{\Delta\vec{v}_{\text{Es}}}{\Delta t}.$$

❸ **EXECUTE PLAN** We first rewrite Eq. 1 in terms of the x components:

$$v_{\text{Es}x} = v_{\text{ET}x} + v_{\text{Ts}x}.$$

For the x component of the initial velocity we have

$$v_{\text{Es}\,x,\text{i}} = (+30\text{ m/s}) + (0) = +30\text{ m/s}, \checkmark$$

and for the x component of the final velocity we have

$$v_{\text{Es}\,x,\text{f}} = (+30\text{ m/s}) + (-10\text{ m/s}) = +20\text{ m/s}. \checkmark$$

The x component of the acceleration is then

$$a_{\text{Es}x} = \frac{\Delta v_{\text{Es}x}}{\Delta t} = \frac{(+20\,\text{m/s}) - (+30\,\text{m/s})}{2.0\,\text{s}} = -5.0\,\text{m/s}^2. \checkmark$$

❹ **EVALUATE RESULT** The positive signs for the x components of the velocities mean that the observer in the Earth reference frame sees the sprinter moving in the positive x direction even though the sprinter is running toward the rear of the car. One way to think about this is to imagine that both the sprinter and the rear wall of the car are headed in the positive x direction but that the speed at which the rear wall moves in this direction is higher than the speed at which the sprinter moves in this direction. The minus sign for acceleration in our coordinate system means that the acceleration is directed toward the rear of the car.

We can check for consistency by calculating the x component of the acceleration with the information measured in the train reference frame:

$$a_{\text{Ts}x} = \frac{\Delta v_{\text{Ts}x}}{\Delta t} = \frac{(-10\,\text{m/s}) - (0)}{2.0\,\text{s}} = -5.0\,\text{m/s}^2.$$

This value agrees with the value obtained using data from the Earth reference frame. The assumption of constant acceleration is not unreasonable, given the limited data at hand. Whether or not the sprinter maintains constant acceleration, the value we obtained is his average acceleration for the 2.0-s interval.

Guided Problem 6.2 Safe passage

You are driving at 35 m/s on the highway when you come up behind a truck traveling at 25 m/s in the same direction. You'd like to have 20 m of space between the front of your car and the back of the truck when you move into the passing lane and 20 m of space between the back of your car and the front of the truck when you switch back into your original lane. If the truck is 10 m long, how many seconds do you need in the passing lane if you keep your speed constant at 35 m/s?

❶ GETTING STARTED

1. Draw a sketch showing the velocities of the truck and car, with arrows to indicate magnitudes and directions. In what reference frame is each velocity in your sketch measured?
2. How does the passing look to the truck's driver? Which is the simpler way to work this problem: from the point of view of the truck's reference frame or from the point of view of the Earth reference frame?
3. Redo your sketch in the appropriate reference frame, if necessary, depending on your answer to question 2.
4. What should you assume for the length of the car?

❷ DEVISE PLAN

5. How is time interval related to displacement for motion at constant speed? In the reference frame you have chosen, which vehicle is undergoing a displacement?
6. Translate your sketch into algebraic equations that contain the desired time interval.

❸ EXECUTE PLAN

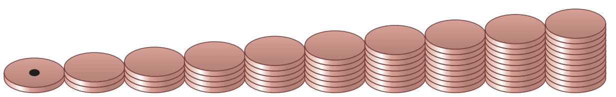
❹ EVALUATE RESULT

7. Is the size of your answer reasonable? Consider that most drivers do not allow such margins in actual practice.
8. Does the length of the car matter?

Worked Problem 6.3 Stacked pennies

You have ten stacks of pennies lined up as in Figure WG6.2—one penny in the first stack, two in the second stack, three in the third, and so on. Where along this row is the center of mass of the system consisting of all the pennies?

Figure WG6.2



❶ GETTING STARTED We suspect that the center of mass of the system is located somewhere in one of the middle stacks, and we know there is a formula to calculate the location involving inertias and positions (Eq. 6.24). We are not told any inertias or dimensions, but we can assume that all the pennies have identical inertias and dimensions. We will use symbols for the inertia and diameter of each penny and answer the question based on that.

❷ DEVISE PLAN Let us call the inertia of each penny m and the diameter of each penny d. (Using a variable to identify an unknown is a better practice than assigning a real number, both because using variables is a more general approach and because numbers can confuse the issue when we derive a final expression.) For the origin of our coordinate system, we arbitrarily choose the center of the leftmost penny, indicated by the black dot in Figure WG6.2. The lack of numerical values means our answer will be in terms of m and d (unless these values cancel in the algebra). We number the stacks from left to right: $n = 1$ for the leftmost stack, $n = 2$ for the stack to its right, and so on. If we call the inertia of any one stack m_n, the position of the center of mass is

$$x_{\text{cm}} = \frac{\sum m_n x_n}{\sum m_n}.$$

We have to figure out the inertia m_n of each stack and its distance x_n from our origin. Let us choose to the right as the positive direction of the x axis. We know that there is one more penny on each stack as we go from left to right and that the position increases by a distance $x = d$ as we move rightward from one stack to the next. We need appropriate equations for m_n and x_n as functions of m, d, and n.

❸ **EXECUTE PLAN** We note that the number of pennies in a stack equals the stack number n. This means $m_n = nm$. We see that the $n = 1$ stack (containing one penny) is located at $x_1 = 0$ and that the center of each of the other stacks is located a distance $x_n = (n - 1)d$ from the origin. With ten stacks, Eq. 6.24 becomes

$$x_{cm} = \frac{\sum m_n x_n}{\sum m_n} = \frac{\sum_{n=1}^{10} (nm)[(n-1)d]}{\sum_{n=1}^{10} nm} = \frac{md \sum_{n=1}^{10} n(n-1)}{m \sum_{n=1}^{10} n}$$

$$= d\frac{\frac{1}{3}(10-1)(10)(10+1)}{\frac{1}{2}(10)(10+1)} = \frac{2}{3}(10-1)d = 6d. \checkmark$$

This result places the center of mass of the system at the center of the seventh stack, six stacks to the right of the first one. If it is not obvious that the seventh stack is at $x = 6d$, start with your finger on the origin in Figure WG6.2 and count d's to the seventh stack. Note that the center-of-mass location is independent of the inertia m of the pennies.

❹ **EVALUATE RESULT** The center is somewhere near the middle of this distribution, which is what we expect. Notice that there are $10 + 9 + 8 = 27$ pennies to the right of the seventh stack, but only $1 + 2 + 3 + 4 + 5 + 6 = 21$ pennies to the left of it. However, the pennies to the right of stack 7 are closer to the calculated center of mass, which means this is not an unreasonable result.

Guided Problem 6.4 Balancing a wheel

You have just had a new set of tires put on your car, and now the mechanic has to balance the wheels. For one of the wheels, the center of mass once the tire is installed is located 1.0 mm from the center of the axle. The mechanic corrects for this by crimping a small lead bar on the wheel rim 200 mm away from the axle. With this added inertia, the center of mass is at the center of the axle. If the wheel with the tire installed has an inertia of 10 kg, what is the inertia of the lead bar?

❶ **GETTING STARTED**

1. Draw a diagram that shows the information given, and add an x axis so that you can specify locations. Where is a convenient place to put the origin of your coordinate system?

❷ **DEVISE PLAN**

2. The goal is to restore the center of mass to the center of the axle so that $x_{cm} = x_{axle}$. What is the value of the x coordinate of the axle with your chosen origin?
3. Write the equation for the location of the center of mass of the balanced wheel, using the labels you put in your drawing. Convince yourself that you now have everything you need to solve this equation.

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

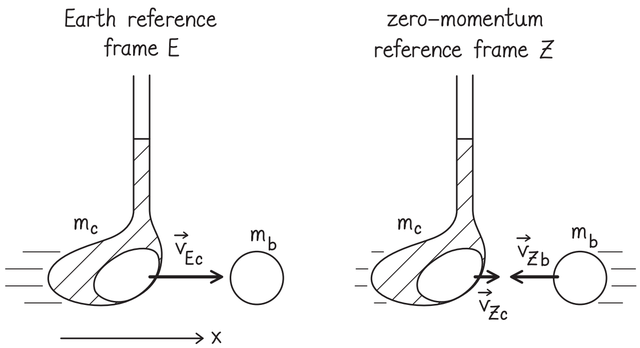
4. Should the numerical value you obtained for the lead bar inertia be negative or positive?
5. Does your calculated inertia value seem reasonable?

Worked Problem 6.5 Fore!

The 0.100-kg head of a golf club is moving at 45 m/s when it strikes a stationary 0.050-kg golf ball. (*a*) What is the kinetic energy of this two-object system in the Earth reference frame? (*b*) In the zero-momentum reference frame, what are the velocities of the club and ball just before they collide? (*c*) What is their relative velocity in the zero-momentum reference frame? (*d*) How much of the kinetic energy of the system can be converted to internal energy? (*e*) What is the translational kinetic energy of the system in the zero-momentum reference frame and in the Earth reference frame?

❶ **GETTING STARTED** Let us make two drawings, one in each reference frame, partly to establish labels for various quantities and partly to define an x axis (Figure WG6.3). We label the Earth reference frame E, the zero-momentum reference frame Z, and use c for club variables and b for ball variables. Notice that Figure WG6.3 represents the initial situation; it is not necessary to draw the final situation because none of the unknowns requires analysis of the situation after the collision.

Figure WG6.3



In drawing our diagram for the zero-momentum reference frame, we must be sure to show that the relative velocity measured in this reference frame is the same as the value measured in the Earth reference frame. The momenta of the club and ball should add up to zero in our zero-momentum reference frame, so the club's velocity arrow should be shorter than the ball's velocity arrow because the club's inertia is greater than that of the ball.

❷ DEVISE PLAN

(*a*) The kinetic energy of the system is the sum of the kinetic energies of the objects.

(*b*) We first use the fact that $v_{\text{EZ}x}$, the value an observer in the Earth reference frame measures for the *x* component of the velocity of the zero-momentum reference frame, is the same as $v_{\text{Ecm}x}$, the value an observer in the Earth reference frame measures for the *x* component of the velocity of the center of mass of the system, which is given by Eq. 6.26:

$$v_{\text{Ecm}x} = \frac{m_c v_{\text{Ec}x} + m_b v_{\text{Eb}x}}{m_c + m_b}.$$

We know all the quantities on the right side of this equation, and so we can calculate $v_{\text{Ecm}x}$. Next we use this Earth-measured center-of-mass velocity in Eq. 6.14 to calculate the *x* components of the velocities of the club and ball measured in the zero-momentum reference frame, using the equality $v_{\text{ZE}x} = -v_{\text{EZ}x} = -v_{\text{Ecm}x}$:

$$v_{\text{Zb}x} = v_{\text{ZE}x} + v_{\text{Eb}x} = -v_{\text{Ecm}x} + v_{\text{Eb}x}$$

$$v_{\text{Zc}x} = v_{\text{ZE}x} + v_{\text{Ec}x} = -v_{\text{Ecm}x} + v_{\text{Ec}x}.$$

(*c*) The relative velocity is the same in both reference frames and so can be evaluated in either.

(*d*) This result gives us enough information to compute the kinetic energy in the zero-momentum reference frame as well as the convertible kinetic energy, which is the kinetic energy of the ball and club in the zero-momentum reference frame.

(*e*) We need to measure the translational kinetic energy K_{cm} of the system in both reference frames, using the expression

$$K_{\text{cm}} = \tfrac{1}{2}(m_c + m_b)(v_{\text{cm}})^2$$

in each one.

❸ EXECUTE PLAN

(*a*) The kinetic energy of the club-ball system measured in the Earth reference frame is

$$\begin{aligned} K_E &= \tfrac{1}{2}m_c v_{\text{Ec}x}^2 + \tfrac{1}{2}m_b v_{\text{Eb}x}^2 \\ &= \tfrac{1}{2}(0.100\ \text{kg})(45\ \text{m/s})^2 + \tfrac{1}{2}(0.050\ \text{kg})(0)^2 \\ &= 1.0 \times 10^2\ \text{J}. ✔ \end{aligned}$$

(*b*) The value an observer in the Earth reference frame measures for the *x* component of the velocity of the zero-momentum reference frame is the same as the value that Earth observer measures for the *x* component of the velocity of the center of mass of the system:

$$v_{\text{Ecm}x} = \frac{m_c v_{\text{Ec}x} + m_b v_{\text{Eb}x}}{m_c + m_b} = \frac{(0.100\ \text{kg})(+45\ \text{m/s})}{0.100\ \text{kg} + 0.050\ \text{kg}} = +30\ \text{m/s}.$$

The x components of the velocities of club and ball measured by an observer in the zero-momentum reference frame are therefore

$$v_{\mathrm{Zb}x} = -v_{\mathrm{Ecm}x} + v_{\mathrm{Eb}x} = -30\,\mathrm{m/s} + 0 = -30\,\mathrm{m/s}\ \checkmark$$

$$v_{\mathrm{Zc}x} = -v_{\mathrm{Ecm}x} + v_{\mathrm{Ec}x} = -30\,\mathrm{m/s} + 45\,\mathrm{m/s} = +15\,\mathrm{m/s}.\ \checkmark$$

(*c*) We get the x component of relative velocity in the zero-momentum reference frame from our transformed velocities in part *b*:

$$v_{\mathrm{Zbc}x} = v_{\mathrm{Zc}x} - v_{\mathrm{Zb}x} = (+15\,\mathrm{m/s}) - (-30\,\mathrm{m/s}) = +45\,\mathrm{m/s}.\ \checkmark$$

(*d*) The convertible kinetic energy is the kinetic energy of the system measured in the zero-momentum reference frame:

$$\begin{aligned} K_{\mathrm{conv}} = K_{\mathrm{Z}} &= \tfrac{1}{2}m_{\mathrm{c}}(v_{\mathrm{Zc}x})^2 + \tfrac{1}{2}m_{\mathrm{b}}(v_{\mathrm{Zb}x})^2 \\ &= \tfrac{1}{2}(0.100\,\mathrm{kg})(+15\,\mathrm{m/s})^2 + \tfrac{1}{2}(0.050\,\mathrm{kg})(-30\,\mathrm{m/s})^2 \\ &= 34\,\mathrm{J}.\ \checkmark \end{aligned}$$

(*e*) The translational kinetic energy of this system in the zero-momentum reference frame is easy to calculate because the system's center of mass is at rest in this reference frame:

$$K_{\mathrm{Zcm}} = \tfrac{1}{2}(m_{\mathrm{c}} + m_{\mathrm{b}})(v_{\mathrm{Zcm}x})^2 = \tfrac{1}{2}(0.150\,\mathrm{kg})(0)^2 = 0.\ \checkmark$$

The system has no translational kinetic energy in the zero-momentum reference frame because the system's center of mass is not translating in that reference frame! In the Earth reference frame, the translational kinetic energy is

$$\begin{aligned} K_{\mathrm{Ecm}} &= \tfrac{1}{2}(m_{\mathrm{c}} + m_{\mathrm{b}})(v_{\mathrm{Ecm}x})^2 \\ &= \tfrac{1}{2}(0.150\,\mathrm{kg})(30\,\mathrm{m/s})^2 = 68\,\mathrm{J}.\ \checkmark \end{aligned}$$

❹ **EVALUATE RESULT** The signs and magnitudes of the x components of the velocities in the zero-momentum reference frame match our expectations shown in Figure WG6.3. The x component of the relative velocity should be the same in all reference frames. We computed it to be $v_{\mathrm{Zbc}x} = 45\,\mathrm{m/s}$ in the zero-momentum reference frame. It should be the same in the Earth reference frame:

$$v_{\mathrm{Ebc}x} = v_{\mathrm{Ec}x} - v_{\mathrm{Eb}x} = +45\,\mathrm{m/s} - 0 = +45\,\mathrm{m/s}.$$

We can check our calculation of convertible energy by using the definition of convertible kinetic energy with our Earth values:

$$K_{\mathrm{conv}} = K_{\mathrm{E}} - K_{\mathrm{Ecm}} = 101.3\,\mathrm{J} - 67.5\,\mathrm{J} = 34\,\mathrm{J}.$$

This energy is the amount that could be converted to internal energy in the collision, and it should be (and is) the same in both reference frames. This does not mean that all of it is converted, however. Because of the design and construction of the golf ball, we expect the collision to be nearly elastic, which means that little of the system's kinetic energy is converted to internal energy.

Guided Problem 6.6 Crunch!

Compare the energy lost in a totally inelastic collision between a moving car and a stationary car of the same inertia with the energy lost in a totally inelastic collision between a moving car and a bridge abutment.

1 GETTING STARTED

1. Even if you think they might not be needed, it is good practice to make sketches of the two crashes. What scalar and/or vector variables should you include? It is generally good to include any variables that you use in your equations. (You can add these symbols to your sketch as you go along.)
2. What amount of kinetic energy is convertible (for instance, in deforming a car and making noise)?

2 DEVISE PLAN

3. Because the collision is totally inelastic in both cases, what fraction of the available convertible energy is converted to internal energy?
4. Write the equation for convertible kinetic energy (Eq. 6.40). What quantity do you need to determine in this equation?
5. Which variable in this equation is different in the two cases? What is its value in each case?
6. Suppose in each case the moving car has inertia m_{moving} and what it runs into has inertia m_{rest}. Write an expression for the reduced inertia in this general case.

3 EXECUTE PLAN

7. Are there too many unknowns? Perhaps if you make a ratio of the converted energies in the two cases some variables will cancel.

4 EVALUATE RESULT

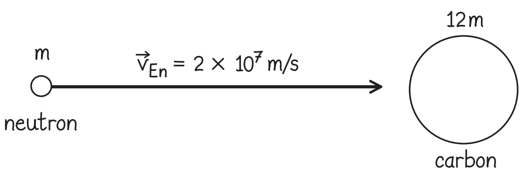
8. Based on the result, which would you rather collide with: a parked car or a concrete wall? Do your results agree with your common sense?
9. Were any assumptions needed? If so, check that they are not unreasonable.

Worked Problem 6.7 Hot reactors

Fast-moving neutrons in the core of a nuclear fission reactor are slowed down by collisions with carbon nuclei in the graphite moderator in the core. The ratio of the inertia of a carbon nucleus to the inertia of a neutron is about 12:1. If a neutron moving at 2.0×10^7 m/s strikes a stationary carbon nucleus head-on in an elastic collision, what are the velocities of the two particles after the collision?

1 GETTING STARTED Figure WG6.4 is a drawing that displays the information given. Because we may need either velocity or momentum equations, it is convenient to show the neutron's initial velocity with a labeled vector arrow and to indicate the inertias m and $12m$.

Figure WG6.4



We could answer the question by using conservation of momentum and kinetic energy as we did in Chapter 5, but the algebra might be intimidating. This is because we are looking at the collision from the reference frame in which the carbon nucleus is at rest (the Earth reference frame E), our analysis will be easier if we move to the zero-momentum reference frame Z.

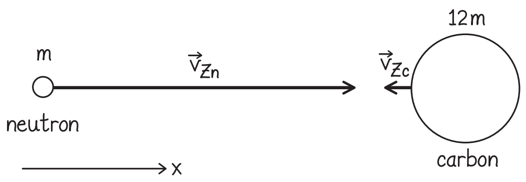
2 DEVISE PLAN We anticipate a Galilean transformation to the zero-momentum reference frame. We can use Eq. 6.26 to get the velocity of the zero-momentum reference frame measured by an observer in the Earth reference frame:

$$\vec{v}_{\text{EZ}} = \vec{v}_{\text{cm}} = \frac{m_{\text{n}}\vec{v}_{\text{n}} + m_{\text{c}}\vec{v}_{\text{c}}}{m_{\text{n}} + m_{\text{c}}}.$$

Because the momenta sum to zero in the zero-momentum reference frame, the collision looks like what is shown in Figure WG6.5, where we have included an x axis so that we can assign signs to our vectors. Because this is the zero-momentum reference frame, the momentum in the positive x direction must cancel the momentum in the negative x direction. Therefore we know that, because of the large differences in inertias, the speed of the carbon nucleus must be much lower than the speed of the neutron in order to have $|m_{\text{n}}\vec{v}_{\text{n}}| = |m_{\text{c}}\vec{v}_{\text{c}}|$.

PRACTICE

Figure WG6.5



We can compute the initial velocity of each particle in the zero-momentum reference frame using Eq. 6.14 with the appropriate subscripts:

$$\vec{v}_{\text{Zn}} = \vec{v}_{\text{ZE}} + \vec{v}_{\text{En}} = -\vec{v}_{\text{EZ}} + \vec{v}_{\text{En}} = -\vec{v}_{\text{cm}} + \vec{v}_{\text{En}}$$

$$\vec{v}_{\text{Zc}} = \vec{v}_{\text{ZE}} + \vec{v}_{\text{Ec}} = -\vec{v}_{\text{EZ}} + \vec{v}_{\text{Ec}} = -\vec{v}_{\text{cm}} + \vec{v}_{\text{Ec}}.$$

Because this is an elastic collision, the final relative velocity in the x direction is the negative of the initial value. This means that in the zero-momentum reference frame the momentum of each particle changes direction by 180°. We should have enough information to determine the final velocity of each particle after the collision. We then do another transformation using Eq. 6.14 to determine the final velocities measured in the Earth reference frame.

❸ **EXECUTE PLAN** We first compute the transformation to the zero-momentum reference frame by determining the x component of the velocity of the center of mass measured in the Earth reference frame:

$$v_{\text{cm}\,x} = \frac{m(+v_{\text{n}}) + 12m(0)}{m + 12m} = +\tfrac{1}{13}v_{\text{n}}.$$

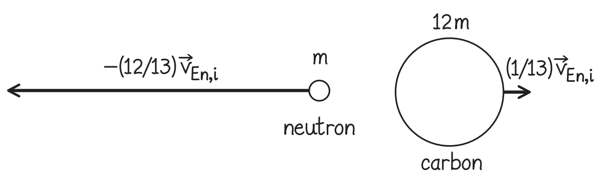
The x components of the initial velocities of the two particles in the zero-momentum reference frame are then

$$v_{\text{Zn}\,x,\text{i}} = -v_{\text{cm}\,x} + v_{\text{En}\,x,\text{i}} = -\tfrac{1}{13}v_{\text{n}} + v_{\text{n}} = +\tfrac{12}{13}v_{\text{n}}$$

$$v_{\text{Zc}\,x,\text{i}} = -v_{\text{cm}\,x} + v_{\text{Ec}\,x,\text{i}} = -\tfrac{1}{13}v_{\text{n}} + 0 = -\tfrac{1}{13}v_{\text{n}}.$$

The negative sign for $v_{\text{Zc}\,x,\text{i}}$ means that the carbon nucleus is moving in the negative x direction. In the zero-momentum reference frame, the initial momenta are equal in magnitude and opposite in direction, and the same is true after the collision, when the momenta have both changed direction by 180° (Figure WG6.6).

Figure WG6.6



To answer the posed question, we must transform back to the Earth reference frame, again using Eq. 6.14:

$$v_{\text{En}\,x,\text{f}} = v_{\text{EZ}\,x} + v_{\text{Zn}\,x,\text{f}} = +\tfrac{1}{13}v_{\text{n}} + \left(-\tfrac{12}{13}v_{\text{n}}\right) = -\tfrac{11}{13}v_{\text{n}}$$

$$v_{\text{Ec}\,x,\text{f}} = v_{\text{EZ}\,x} + v_{\text{Zc}\,x,\text{f}} = +\tfrac{1}{13}v_{\text{n}} + \left(+\tfrac{1}{13}v_{\text{n}}\right) = +\tfrac{2}{13}v_{\text{n}}.$$

The neutron reverses direction and speeds away from the carbon nucleus at a velocity of

$$-\tfrac{11}{13}(2.0 \times 10^7\ \text{m/s}) = -1.7 \times 10^7\ \text{m/s}, ✔$$

and the carbon nucleus moves in the positive x direction at a velocity of

$$+\tfrac{2}{13}(2.0 \times 10^7\ \text{m/s}) = +3.1 \times 10^6\ \text{m/s}. ✔$$

PRACTICE

4 **EVALUATE RESULT** The neutron has lost only 15% of its initial speed, and the post-collision speed of the carbon nucleus is much lower than that of the neutron. This is what we expect for a collision between, say, a golf ball and a basketball, whose inertias differ by a factor of ten or so.

Guided Problem 6.8 Slowing neutrons

In a nuclear reactor, a few uranium nuclei spontaneously split into the nuclei of lighter elements plus several fast neutrons. These neutrons are slowed by the *moderator,* which makes them more likely to be absorbed by other uranium nuclei. That causes them to split and increases the rate of the nuclear reaction. As noted in Worked Problem 6.7, this is usually achieved through elastic collisions between the neutrons and atomic nuclei. The moderator used to slow the neutrons is made of small atoms (like hydrogen and carbon) rather than larger ones (like iron and lead). Why?

1 **GETTING STARTED**

1. This collision problem is similar to Worked Problem 6.7, but the important question is: How is it different?
2. What is the desired outcome of the collision between the neutrons and the atoms in the moderator? What physical quantity is associated with this goal?

2 **DEVISE PLAN**

3. How can you modify the Worked Problem 6.7 analysis to allow a variable inertia m_T for the target nucleus?
4. Do you need any additional information to do the analysis?
5. Determine the neutron's velocity after the collision in terms of m_n and m_T.

3 **EXECUTE PLAN**

6. What is the smallest value possible for the ratio m_T/m_n? (Hint: A hydrogen nucleus "target" is generally a single proton.) What happens to the velocity of the recoiling neutron when this ratio has its smallest value?
7. What is the largest value of m_T one might reasonably have? What happens to the velocity of the recoiling neutron when m_T has this largest value?

4 **EVALUATE RESULT**

8. Compare your results with your expectations for a collision between two everyday objects whose inertias are in the same ratio as the inertias of the neutron and target.

Answers to Review Questions

1. (*a*) The observer in the car moving with the same velocity as the truck sees the truck as not moving. (*b*) The observer in the car moving at any constant velocity different from the truck velocity sees the truck as moving at some constant velocity that is different from the observer's velocity.
2. In a reference frame attached to the wind, the air is at rest. When you ride into the wind, your velocity measured in this reference frame is higher than your velocity measured in the Earth reference frame. The higher your speed measured in the wind reference frame, the more you feel like you are riding through a storm. When you are moving in the direction of the wind and your speed happens to be the same as the wind speed, you sense the air as being at rest; in other words, you feel no wind at all.
3. The observer in reference frame 2 also measures $\Delta\vec{p} = 0$ because the measured value is zero in *any* inertial reference frame.
4. The law of inertia: In an inertial reference frame, any isolated object at rest remains at rest and any isolated object in motion keeps moving with constant velocity. This law is useful in determining whether a reference frame is inertial: You observe as many isolated objects as possible, and if any of them violates the law, you are observing from a noninertial reference frame. If no objects violate the law, you are observing from an inertial reference frame.
5. The observer is in a noninertial reference frame because the law of inertia does not hold.
6. (*a*) Yes. The acceleration is the same in all inertial reference frames. (*b*) No. An observer in some other inertial reference frame can measure a nonzero velocity for the object. The only thing you can say about this measured nonzero velocity is that it is constant.
7. The principle of relativity: The laws of the universe are the same in all inertial reference frames. This means that it is not necessary to invent different laws to explain results obtained by observers moving at any constant velocity with respect to each other; one set of laws is universally valid.
8. No, because one inertial reference frame is physically indistinguishable from any other inertial reference frame. You can determine whether or not your reference frame is inertial, but you can only measure its velocity relative to the velocity of other inertial reference frames.
9. Because the second reference frame is inertial, the object's acceleration in it is the same as the acceleration in the first reference frame.
10. Conservation of momentum doesn't mean that the momentum of the system must have the same numerical value in all inertial reference frames. What it does mean is that momentum can be neither created nor destroyed, so that any changes in system momentum must be accounted for in terms of impulse (momentum input or output). Because changes in system momentum are identical in all inertial reference frames, momentum conservation is equally valid in all of them.
11. They disagree on the kinetic energy of each object and on the amount of kinetic energy in the system, but they agree that the kinetic energy of the system does not change as a result of the collision. They also disagree about the momentum of each object and the system momentum, but they agree that the momentum of the system does not change.

12. The zero-momentum reference frame is the inertial reference frame in which the momentum of a system is zero. It can be identified by computing the system momentum in some inertial reference frame. If the system momentum is zero in your chosen reference frame, you have the zero-momentum reference frame for the system. If your computed momentum is not zero, compute the velocity of the zero-momentum reference frame by dividing the system momentum you calculated in your chosen reference frame by the system inertia. Subtract this velocity from each object's velocity to obtain the velocity of each object in the zero-momentum reference frame.
13. (*a*) The magnitude does not change. (*b*) The direction reverses.
14. They give us a means to relate measurements made in one inertial reference frame to those made in another.
15. C must know D's velocity $\vec{v}_{CD}$ measured in C's reference frame. If D's measurement is accurate, then the three quantities are related to one another as in Eq. 6.14: $\vec{v}_{Ct} = \vec{v}_{CD} + \vec{v}_{Dt}$.
16. The inertia is the same in *all* inertial reference frames because inertias are determined from Eq. 4.2, $m_o = -(\Delta v_{sx}/\Delta v_{ox})m_s$, and changes in velocity are identical in different inertial reference frames.
17. There is no difference. The numerical value of the *x* component of position may change with the specific reference point (origin) chosen, but the actual position of the center of mass of a system is a property of the system that is independent of the reference point.
18. The system momentum is constant because the system is isolated. The center of mass moves with the momentum of the system, which means it moves with constant momentum. (Put another way, the center of mass is at rest in the zero-momentum reference frame for the system.) If the inertia of the system is also unchanging, the center of mass moves at constant velocity.
19. The momentum is zero because the center-of-mass velocity is that of the zero-momentum reference frame. (Compare Eqs. 6.23 and 6.26.)
20. Yes. This was established for a particular collision discussed in the text, but it is also logically required that energy that appears in the form of thermal energy or crumpled fenders must not depend on the reference frame of the inertial observer.
21. The nonconvertible kinetic energy changes, which is the part associated with the motion of the center of mass.
22. Not in general. The translational kinetic energy of the system's center of mass is nonconvertible energy, and so it cannot be converted to internal energy. If you happen to be using the zero-momentum reference frame, though, this translational kinetic energy is zero, so in this case all of the kinetic energy of the system is convertible and hence can be converted.
23. (*a*) The maximum amount is all of the convertible kinetic energy: $\frac{1}{2}\mu v_{rel,i}^2$. (*b*) This maximum conversion happens in totally inelastic collisions because $e = 0$.
24. (*a*) Yes, because "zero change in momentum" means the system is isolated, and a system that is isolated in one inertial reference frame is isolated in all inertial reference frames. (*b*) Yes. The change in the momentum of a system is always the same in any inertial reference frame, even though the momentum magnitudes and directions of objects in the system are not always the same, and even though the magnitude and direction of the system momentum may vary from one reference frame to another.
25. (*a*) Yes, because $\Delta E = 0$ means the system is closed, and a system that is closed in one inertial reference frame is closed in all inertial reference frames. (*b*) Yes. The change in energy of a system is the same in all inertial reference frames even though the energy changes of the objects in the system are not always the same, and even though the energy of the system may vary from one reference frame to another.

Answers to Guided Problems

Guided Problem 6.2 5.0 s plus 0.1 s for each meter length of the car
Guided Problem 6.4 5×10^{-2} kg
Guided Problem 6.6 K_{conv} hitting abutment ≈ $2K_{conv}$ hitting parked car
Guided Problem 6.8 Small nuclei make the best targets because the recoil velocity of the neutron gets lower as the target inertia decreases.

Guided Practice by Chapter

7 Interactions

Review Questions

Answers to these questions can be found at the end of this chapter.

7.1 The effects of interactions

1. A pool ball collides head-on and elastically with a second pool ball initially at rest. Do any of these properties of the system made up of the two balls change during the interaction: (*a*) momentum, (*b*) kinetic energy, (*c*) the sum of all forms of energy in the system? (Assume no significant interaction between the pool table and the balls.)
2. In an elastic collision between two objects, does their relative speed stay the same during the interaction?
3. Consider a system made up of two objects that collide with each other. Describe the difference between what happens to the system's kinetic energy when the collision is (*a*) totally inelastic, (*b*) inelastic, and (*c*) elastic.
4. Summarize the characteristics of an interaction.

7.2 Potential energy

5. Describe the difference between kinetic and potential energy.

7.3 Energy dissipation

6. Give an example of an interaction that converts kinetic energy to thermal energy.
7. How does energy of motion differ from energy of configuration? How does coherent energy differ from incoherent energy?

7.4 Source energy

8. What is the ultimate source of virtually all the energy we use?
9. What distinguishes dissipative interactions from nondissipative ones?
10. Think of several interactions that occur in everyday life. For each interaction, describe which category of energy is converted and to which category it is converted.

7.5 Interaction range

11. Describe the two models physicists use to represent interactions between objects without requiring direct touching.

7.6 Fundamental interactions

12. Which of the four fundamental interactions do we commonly observe in everyday life? Why not all of them?
13. Which fundamental interaction exerts the most control (*a*) in chemical processes and (*b*) in biological processes?
14. The strength of the gravitational interaction is minuscule compared with the strength of the electromagnetic interaction. Yet we can study the interactions of most ordinary objects without considering electromagnetic interactions, while it is essential that we include gravitational interactions. Give a reason why this is so.

7.7 Interactions and accelerations

15. How are the acceleration and inertia of an object 1 related to the acceleration and inertia of an object 2 when the objects collide (*a*) elastically and (*b*) inelastically?
16. List any assumptions made in deriving the relationship between ratios of acceleration and inertias for two interacting objects.

7.8 Nondissipative interactions

17. Explain why mechanical energy remains constant when nondissipative interactions take place in a closed system.
18. Explain how we know that, in a closed system in which all interactions are nondissipative, potential energy is a function of position only.
19. An object subject to only nondissipative interactions moves from point A to point B and then back to A. How does the object's initial kinetic energy compare with its final kinetic energy?

7.9 Potential energy near Earth's surface

20. In Eq. 7.19, is Δx a horizontal or vertical displacement? Does it matter?
21. For an object placed at a given position relative to the ground, is the value of the object's gravitational potential energy either always positive or always negative?

7.10 Dissipative interactions

22. A car collides totally inelastically with a parked car. In the (isolated) system made up of the two cars, is all the initial kinetic energy dissipated during the collision?
23. In a collision between two objects, how much of the convertible kinetic energy is converted to thermal energy, and how much is temporarily stored as potential energy during the collision (*a*) when $e = 1$ and (*b*) when $e = 0$?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The energy dissipated when you clap your hands once (S, K, A)
2. The energy dissipated when you throw a tennis ball against a wall (E, X, D)
3. The thermal energy released by burning three logs in a fireplace (I, P, AA)
4. The upward acceleration of Earth you cause by jumping off a chair (G, L, U)
5. The acceleration of a rifle when a bullet is fired from it (T, Z, Q)
6. The energy supplied by your muscles when you climb a flight of stairs (H, C, Y, L)
7. The energy supplied by an escalator when it carries 30 people up one flight (L, Y, C, F)
8. The maximum potential energy stored when two protons collide head-on if, before the collision, each is moving at 3×10^7 m/s (J, W)
9. The maximum elastic potential energy stored when you perform a bungee jump (L, R, O, Y, W)
10. The gravitational potential energy stored as a 30-story building is constructed (M, V, N, B, Y)

Hints

A. What is the coefficient of restitution for the collision when you clap your hands?
B. What is the inertia of a 30-story building?
C. Does the speed change on the way up?
D. What is the coefficient of restitution?
E. What is the inertia of a tennis ball?
F. What is the height between floors of a building?
G. What is the inertia of Earth?
H. What is the vertical height of a flight of stairs?
I. How much energy is released by breaking one chemical bond?
J. What is the inertia of a proton?
K. What is the inertia of one hand and forearm?
L. What is the inertia of a typical person?
M. What is the height of a 30-story building?
N. What fraction of the volume of a 30-story building is structural concrete and steel?
O. What is the maximum height difference from bottom to top of one bungee jump?
P. What is the size of an atom?
Q. Over what length does the bullet accelerate along the rifle barrel?
R. Is the inertia of the bungee cord negligible in this situation?
S. What is the speed of each hand just before the two touch each other?
T. What is the ratio $m_{\text{rifle}}/m_{\text{bullet}}$?
U. What is your acceleration in free fall?
V. Where is the vertical location of this building's center of mass?
W. What is the minimum kinetic energy in the zero-momentum reference frame?
X. What is the speed of the ball just before it hits the wall?
Y. How does the gravitational potential energy of the system change?
Z. With what speed does the bullet leave the rifle?
AA. How many atoms are there in a fireplace log?

Key (all values approximate)

A. 0; **B.** 1×10^7 kg; **C.** generally no; **D.** 0.7; **E.** 0.1 kg; **F.** 6 m; **G.** 6×10^{24} kg; **H.** 3 m; **I.** 1×10^{-19} J; **J.** 2×10^{-27} kg; **K.** 2 kg; **L.** 7×10^1 kg; **M.** 1×10^2 m; **N.** 1/8; **O.** 1×10^2 m; **P.** 1×10^{-10} m; **Q.** 1 m; **R.** yes; **S.** 1 m/s; **T.** 4×10^2, or 400:1; **U.** 1×10^1 m/s^2; **V.** less than halfway up—say, 4×10^1 m above ground; **W.** 0, because in this reference frame there must be at least one instant at which $v_{\text{rel}} = 0$; **X.** 2×10^1 m/s; **Y.** increases as object rises, decreases as object falls; **Z.** 4×10^2 m/s; **AA.** 1×10^{27} atoms

Worked and Guided Problems

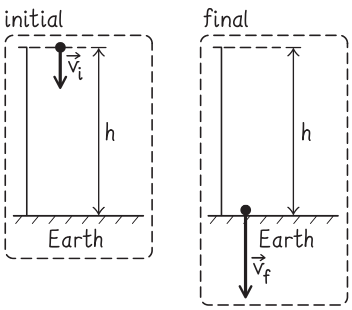
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 7.1 Watch out below!

A student standing at the top of a wall throws a ball straight down, giving it speed v_i as it leaves her hand. Use energy methods to determine the speed v_f at which the ball strikes the pavement a distance h below her hand. Express your answer in terms of v_i and h.

❶ **GETTING STARTED** We define our system as being the ball and Earth so that the gravitational interaction can be expressed in terms of potential energy. The initial system diagram displays the given information, and the final system diagram shows the sought quantity, the final speed v_f (Figure WG7.1).

Figure WG7.1



The change in Earth's kinetic energy while the ball is falling is negligible. We know that the ball speeds up because of the acceleration due to gravity, and we are required to use energy methods to determine by how much. If we let the inertia of the ball be m, we know that the ball starts out with initial kinetic energy $\frac{1}{2}mv_i^2$. As the ball travels downward, the system's gravitational potential energy decreases as this energy gets converted to kinetic energy by the gravitational interaction. Ignoring any dissipative interactions (like air resistance), we can assume that the system's mechanical energy (kinetic plus potential) does not change.

❷ **DEVISE PLAN** Because we have assumed that the mechanical energy of the system is constant, we can set the final mechanical energy equal to the initial value:

$$K_f + U_f = K_i + U_i. \tag{1}$$

We should include the kinetic energy of each object in our system, but we have assumed that Earth's kinetic energy does not change, so the initial and final Earth kinetic energies will simply cancel and need not be explicitly included.

Each gravitational potential energy term depends on the height, and we know neither the initial nor final height. However, if we collect terms, then only the height difference will be needed, and we do know that. We also know the initial kinetic energy term for the ball, which leaves only one unknown: the final kinetic energy of the ball. Because the kinetic energy formula is $\frac{1}{2}mv^2$, we can determine the final speed if we know the final kinetic energy. The ball's inertia m is not given, so we must be sure our answer does not contain this unknown quantity.

❸ **EXECUTE PLAN** Using the expression for gravitational potential energy near Earth's surface, $U^G = mgx$, we write Eq. 1 in the form

$$\tfrac{1}{2}mv_f^2 + mgx_f = \tfrac{1}{2}mv_i^2 + mgx_i.$$

Isolating the final kinetic energy, we obtain

$$\tfrac{1}{2}mv_f^2 = \tfrac{1}{2}mv_i^2 + mg(x_i - x_f). \tag{2}$$

The difference between the initial and final positions should be the height h, but to ensure that we make no error, we add specific values for x_i and x_f to our drawing (Figure WG7.2). The x axis has to point up because that was an assumption made in the derivation of $U^G = mgx$. We know that the zero level of the potential energy is an arbitrary choice, and so we choose the position at which the ball was released. With this choice, $x_f = -h$ because the ball's final position is a distance h in the negative x direction. We can now solve Eq. 2 for v_f:

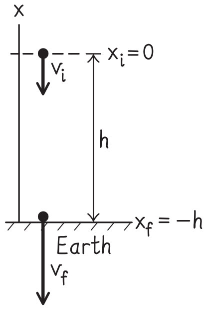
$$\tfrac{1}{2}mv_f^2 = \tfrac{1}{2}mv_i^2 + mg[0 - (-h)]$$

$$mv_f^2 = mv_i^2 + 2mgh$$

$$v_f = \sqrt{v_i^2 + 2gh}. ✔$$

PRACTICE

Figure WG7.2



❹ **EVALUATE RESULT** Because the distance h is a magnitude (as is g) and v_i^2 is always positive, the quantity under the square-root sign is positive. This is reassuring because the final speed must be a real number, not an imaginary one. We expect v_f to increase if either v_i or h increases, and our expression predicts this behavior. We suspect that the actual final speed of a real ball would be lower than this because of air resistance, but our result should be fairly accurate if the ball is more like a baseball than a Ping-Pong ball and if the wall is not too tall.

Guided Problem 7.2 Safe play

A soft rubber ball is held at a height h above the floor and then released. The coefficient of restitution for the ball-floor collision is 0.25. To what height does the ball rise on its first bounce, expressed as a fraction of h?

❶ **GETTING STARTED**

1. Begin by choosing your system and drawing a system diagram that includes the given information and the quantity you seek (labeled with a question mark).

❷ **DEVISE PLAN**

2. What is the numerical value of the ball's speed at the highest point the ball reaches on its first bounce?
3. What kinds of energy conversions take place during the fall, in the ball's collision with the floor, and in the bounce?

❸ **EXECUTE PLAN**

4. With what speed does the ball strike the floor?
5. What is the ball's speed immediately after it bounces off the floor?
6. How much kinetic energy is dissipated in the ball-floor collision?
7. What is the mechanical energy of the system as the ball leaves the floor? What is the mechanical energy at the ball's highest bounce distance above the floor?

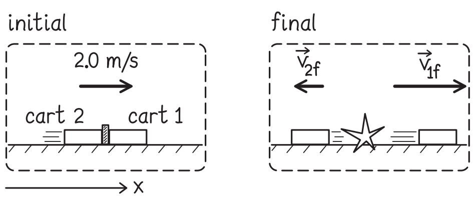
❹ **EVALUATE RESULT**

Worked Problem 7.3 Moving explosion

Two 1.0-kg carts coupled together are initially moving to the right at 2.0 m/s on a horizontal, low-friction track. The coupling contains an explosive charge that is ignited remotely, releasing 18 J of energy. Half of the released energy is dissipated into incoherent energy, such as noise, thermal energy, and damage to the carts. (*a*) What are the velocities of the carts just after the explosive separation? (*b*) Draw a bar diagram showing the energy distribution before and after the explosive separation for the system composed of the two carts and the explosive coupling.

❶ **GETTING STARTED** We are told that the carts are coupled together and initially moving to the right at 2.0 m/s. Given that half of the energy released by the explosive charge is dissipated, we conclude that the other half goes into increasing the kinetic energy of the system. We assume that the rear car, which we can call cart 2, is pushed to the left by the collision (slowing it down) and that the leading cart (call it cart 1) is pushed to the right (speeding it up). Figure WG7.3 shows the system diagram of this moving explosive separation based on these assumptions, with the positive x direction chosen to the right. (Although we expect $\vec{v}_{2f}$ to be in the negative x direction, we do not know for sure. The solution will tell us whether our assumption is right.)

Figure WG7.3



PRACTICE

Because this is an explosive separation, we expect to use conservation of momentum. We are also given energy information, which we must incorporate into our analysis.

❷ **DEVISE PLAN** It is often easier to analyze collisions in the zero-momentum reference frame. We are given information in the Earth reference frame, however, so we need to work with Galilean transformations. In the zero-momentum reference frame Z, the system momentum is always zero, and so we know it is zero both before and after the explosive separation:

$$m_1 v_{Z1x,\mathrm{i}} + m_2 v_{Z2x,\mathrm{i}} = (m_1 + m_2)v_{Zx,\mathrm{i}} = 0 \tag{1}$$

$$m_1 v_{Z1x,\mathrm{f}} + m_2 v_{Z2x,\mathrm{f}} = 0 \tag{2}$$

where $v_{Zx,\mathrm{i}}$ represents the initial x component of the velocity of the system in the zero-momentum reference frame. Because the inertias are not zero, we conclude from Eq. 1 that

$$v_{Z1x,\mathrm{i}} = v_{Z2x,\mathrm{i}} = v_{Zx,\mathrm{i}} = 0.$$

Because we have two unknowns, $v_{Z1x,\mathrm{f}}$ and $v_{Z2x,\mathrm{f}}$, we need another independent equation, and so we turn to the energy information. The energy released is energy initially locked up in the chemical bonds of the explosive charge, which means it is a type of source energy. We are given how much of this source energy is released and how much is dissipated to incoherent energy (an amount that is the same in any reference frame). For simplicity, we combine all of the released incoherent energy (noise, deformation, and so on) and label it as thermal energy. Because the sum of all the forms of energy in the system is constant in this explosive separation, the energetics in the Z reference frame are described by

$$\Delta K_Z + \Delta E_{Zs} + \Delta E_{Z\mathrm{th}} = 0. \tag{3}$$

Equations 1–3 should give us all the information we need to calculate the final velocities in the zero-momentum reference frame. Then we do a Galilean transformation to obtain the velocities in the Earth reference frame.

❸ **EXECUTE PLAN**
(*a*) In the Earth reference frame, the x component of the initial velocity of the system's center of mass is given by Eq. 6.26:

$$v_{\mathrm{cm}\,x} = \frac{m_1 v_{1x,\mathrm{i}} + m_2 v_{2x,\mathrm{i}}}{m_1 + m_2}.$$

Because $m_1 = m_2 = m$, this expression simplifies to

$$v_{\mathrm{cm}\,x} = \frac{m v_{1x,\mathrm{i}} + m v_{2x,\mathrm{i}}}{m + m} = \frac{v_{1x,\mathrm{i}} + v_{2x,\mathrm{i}}}{2}$$

$$= \frac{(+2.0\ \mathrm{m/s}) + (+2.0\ \mathrm{m/s})}{2} = +2.0\ \mathrm{m/s}.$$

In the zero-momentum reference frame the final momentum (like the initial momentum) must be zero, and so from Eq. 2 we have

$$m_1 v_{Z1x,\mathrm{f}} + m_2 v_{Z2x,\mathrm{f}} = m v_{Z1x,\mathrm{f}} + m v_{Z2x,\mathrm{f}} = 0$$

$$v_{Z1x,\mathrm{f}} = -v_{Z2x,\mathrm{f}}. \tag{4}$$

Let us use E to represent the quantity of source energy released, $E = 18$ J. Because the source energy decreases as the chemical bonds in the explosive charge break, the source energy term in Eq. 3 becomes $\Delta E_{Zs} = -E$. Because half of the released source energy is dissipated to thermal energy, the thermal energy term in Eq. 3 is $\Delta E_{Z\mathrm{th}} = +\frac{1}{2}E$. Equation 3 then becomes

$$\Delta K_Z + \Delta E_{Zs} + \Delta E_{Z\mathrm{th}} = 0$$

$$\left[\left(\tfrac{1}{2}m_1 v_{Z1x,\mathrm{f}}^2 + \tfrac{1}{2}m_2 v_{Z2x,\mathrm{f}}^2\right) - \left(\tfrac{1}{2}m_1 v_{Z1x,\mathrm{i}}^2 + \tfrac{1}{2}m_2 v_{Z2x,\mathrm{i}}^2\right)\right] + (-E) + \left(\tfrac{1}{2}E\right) = 0$$

$$\left(\tfrac{1}{2}m v_{Z1x,\mathrm{f}}^2 + \tfrac{1}{2}m v_{Z2x,\mathrm{f}}^2\right) - \left(\tfrac{1}{2}m(0)^2 + \tfrac{1}{2}m_2(0)^2\right) = +\tfrac{1}{2}E.$$

Next we substitute from Eq. 4 to get

$$\tfrac{1}{2}m(-v_{Z2x,\mathrm{f}})^2 + \tfrac{1}{2}mv_{Z2x,\mathrm{f}}^2 = \tfrac{1}{2}E$$

$$v_{Z2x,\mathrm{f}} = +\sqrt{\frac{E}{2m}} \quad \text{or} \quad -\sqrt{\frac{E}{2m}}.$$

The positive root implies that cart 2 passes through cart 1 in the zero-momentum reference frame instead of being pushed away, and so we must discard this root as physically impossible. With the negative root, we have

$$v_{Z2x,\mathrm{f}} = -\sqrt{\frac{E}{2m}}$$

$$v_{Z1x,\mathrm{f}} = -v_{Z2x,\mathrm{f}} = -\left(-\sqrt{\frac{E}{2m}}\right) = +\sqrt{\frac{E}{2m}}.$$

In a procedure analogous to that used in Worked Problem 6.5, the Galilean transformation to the Earth reference frame is

$$\begin{aligned} v_{E1x,\mathrm{f}} &= v_{Z1x,\mathrm{f}} + v_{\mathrm{cm}\,x} = +\sqrt{\frac{E}{2m}} + v_{\mathrm{cm}\,x} \\ &= +\sqrt{\frac{18\ \mathrm{J}}{(2)(1.0\ \mathrm{kg})}} + 2.0\ \mathrm{m/s} = +3.0\ \mathrm{m/s} + 2.0\ \mathrm{m/s} \\ &= +5.0\ \mathrm{m/s}\ ✔ \end{aligned}$$

and

$$\begin{aligned} v_{E2x,\mathrm{f}} &= v_{Z2x,\mathrm{f}} + v_{\mathrm{cm}\,x} = -\sqrt{\frac{E}{2m}} + v_{\mathrm{cm}\,x} \\ &= -3.0\ \mathrm{m/s} + 2.0\ \mathrm{m/s} \\ &= -1.0\ \mathrm{m/s}.\ ✔ \end{aligned}$$

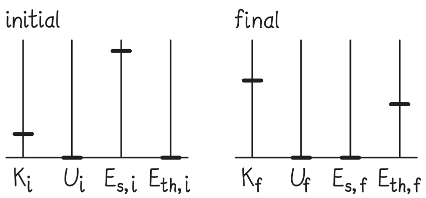
Because to the right is the positive *x* direction in our coordinate system, cart 1 moves to the right after the explosive separation and cart 2 moves to the left, just as we assumed when we began our solution.

(*b*) The energy diagrams should contain a bar for each energy category: kinetic, potential, source, and thermal (Figure WG7.4). We know that $E_{\mathrm{s,i}} = E = 18$ J, $E_{\mathrm{s,f}} = 0$, $E_{\mathrm{th,i}} = 0$, and $E_{\mathrm{th,f}} = \frac{1}{2}E = 9.0$ J. We leave both *U* bars empty because there is no interaction in this system to temporarily store potential energy. The initial and final kinetic energies are

$$\begin{aligned} K_{\mathrm{i}} &= \tfrac{1}{2}m_1(v_{1\mathrm{i}})^2 + \tfrac{1}{2}m_2(v_{2\mathrm{i}})^2 \\ &= \tfrac{1}{2}(1.0\ \mathrm{kg})(2.0\ \mathrm{m/s})^2 + \tfrac{1}{2}(1.0\ \mathrm{kg})(2.0\ \mathrm{m/s})^2 = 4.0\ \mathrm{J} \\ K_{\mathrm{f}} &= \tfrac{1}{2}m_1(v_{1\mathrm{f}})^2 + \tfrac{1}{2}m_2(v_{2\mathrm{f}})^2 \\ &= \tfrac{1}{2}(1.0\ \mathrm{kg})(5.0\ \mathrm{m/s})^2 + \tfrac{1}{2}(1.0\ \mathrm{kg})(1.0\ \mathrm{m/s})^2 = 13\ \mathrm{J}. \end{aligned}$$

These values yield the bar diagram shown in Figure WG7.4. ✔

Figure WG7.4



❹ **EVALUATE RESULT** We can check our results in several ways. It is reassuring that we obtained a center-of-mass velocity of $v_{\text{cm}\,x} = +2.0\ \text{m/s}$ because the two carts are initially locked together and moving at that velocity.

Let us make several other checks, although you normally might not do all of them. We used conservation of momentum in the zero-momentum reference frame. We can verify that this law is satisfied in the Earth reference frame with our calculated numbers:

$$\begin{aligned} p_{x,\text{i}} &= m_1 v_{1x,\text{i}} + m_2 v_{2x,\text{i}} \\ &= (1.0\ \text{kg})(+2.0\ \text{m/s}) + (1.0\ \text{kg})(+2.0\ \text{m/s}) \\ &= +4.0\ \text{kg}\cdot\text{m/s} \\ p_{x,\text{f}} &= m_1 v_{1x,\text{f}} + m_2 v_{2x,\text{f}} \\ &= (1.0\ \text{kg})(+5.0\ \text{m/s}) + (1.0\ \text{kg})(-1.0\ \text{m/s}) \\ &= +4.0\ \text{kg}\cdot\text{m/s}. \end{aligned}$$

We can verify that the system's energy remains constant by calculating its change in the Earth reference frame:

$$\begin{aligned} \Delta K + \Delta E_\text{s} + \Delta E_\text{th} &= (13\ \text{J} - 4.0\ \text{J}) + (0 - 18\ \text{J}) + (9.0\ \text{J} - 0) \\ &= 0. \end{aligned}$$

We can also check consistency in the kinetic energy:

$$\begin{aligned} K &= \tfrac{1}{2}(m_1 + m_2)(v_{\text{cm}\,x})^2 + K_\text{conv} \\ &= \tfrac{1}{2}(m_1 + m_2)(v_{\text{cm}\,x})^2 + \tfrac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(v_{2x} - v_{1x})^2, \end{aligned}$$

where for K_conv we have substituted the expression given in Eq. 6.38. Because the inertias are equal, we can simplify to

$$\begin{aligned} K_\text{f} &= \tfrac{1}{2}(m + m)(v_{\text{cm}\,x})^2 + \tfrac{1}{2}\frac{m^2}{m + m}(v_{2x,\text{f}} - v_{1x,\text{f}})^2 \\ &= m(v_\text{cm})^2 + \tfrac{1}{4}m(v_{2x,\text{f}} - v_{1x,\text{f}})^2 \\ &= (1.0\ \text{kg})(2.0\ \text{m/s})^2 + \tfrac{1}{4}(1.0\ \text{kg})[(-1.0\ \text{m/s}) - (+5.0\ \text{m/s})]^2 \\ &= 4.0\ \text{J} + 9.0\ \text{J} = 13\ \text{J} \end{aligned}$$

in agreement with our calculation for the bar diagram in part *b*.

Guided Problem 7.4 An unequal match

On a low-friction track, two carts coupled together are initially moving to the right at 2.0 m/s. The rear cart has an inertia of 2.0 kg, and the lead cart's inertia is one-third of that. An explosive charge attached to the coupling is ignited remotely, releasing 27 J of energy. One-third of that energy is dissipated to incoherent energy, such as noise, thermal energy, and damage to the carts. Use the Earth reference frame for the following tasks. (*a*) Determine the velocities of the carts just after the explosive separation. (*b*) Draw a bar diagram showing the energy distribution before and after the explosive separation.

❶ **GETTING STARTED**

1. How is this problem similar to Worked Problem 7.3? How does it differ from Worked Problem 7.3?
2. What is a good choice for your system?
3. Draw a system diagram showing all relevant information.
4. Which system quantities remain constant during the explosive separation?

❷ **DEVISE PLAN**

5. Write the equations describing the conservation laws for the quantities named in question 4, as viewed in the Earth reference frame.
6. How many unknowns do you have? Do you have a sufficient number of equations to solve for them?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

7. Check your numerical answer for consistency in at least two ways.

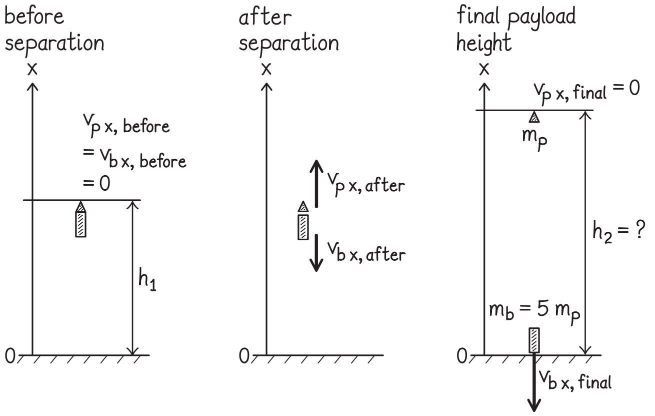
Worked Problem 7.5 Exploding rocket

A rocket is made up of an unpowered payload of inertia m_{payload} and a powered booster connected to the payload by an explosive coupler. The rocket is fired straight up, uses all its fuel, and reaches the peak of its trajectory at height h_1 above the ground. The coupler is detonated at the peak of the trajectory. The quantity of energy released by the explosion is $E > 0$, one-quarter of which is dissipated to incoherent energy. The explosion separates the booster from the payload, and the latter shoots straight up along its original line of motion. Immediately after the separation, the inertia of the booster is five times that of the payload. What maximum height h_2 above the ground does the payload reach after the explosion? Assume that all the incoherent energy is in the form of thermal energy, and report your answer in terms of h_1, E, and m_{payload}.

1 GETTING STARTED This problem involves three stages of motion that are best analyzed separately. First there is the trip from ground to h_1, which needs no analysis because h_1 is given information. The remaining two stages are the explosive separation and the post-separation motions of the booster and payload under the influence of Earth's gravitational field.

First we choose an isolated and closed system and draw a system diagram. A good choice is the system composed of booster, explosive coupler, payload, and Earth. In our diagram (Figure WG7.5), we show the initial location of the rocket (the ground), the rocket at height h_1 just before separation, the booster and payload at h_1 just after separation, and the payload at its maximum height h_2. Because we are considering gravitational potential energy, we choose to point the x axis vertically upward, with the origin at ground level.

Figure WG7.5



The rocket is at the top of its trajectory when the explosive separation occurs, which means that its velocity at the instant of separation is zero. So, we already have complete information about the momentum and energy of the objects in our system at height h_1, and the initial journey from ground to h_1 need not be analyzed! Because the payload travels straight up after the separation, conservation of momentum tells us that the booster must be thrown downward by the explosion. As the payload moves up, its interaction with Earth's gravitational field causes it to slow down as its kinetic energy gets converted to gravitational potential energy until its velocity is zero at the final maximum height we seek. This last portion of the motion, from h_1 to h_2, is simply projectile motion of the payload, which we can analyze with the techniques of Chapter 3 or with energy methods, as we wish.

2 DEVISE PLAN Because the height h_2 the payload reaches is determined by the payload velocity right after separation, we first examine the explosion. In the explosion, an amount E of source energy stored in the coupler is converted to other forms ($\Delta E_s = -E$), with one-quarter dissipated as thermal energy ($\Delta E_{\text{th}} = +E/4$). Because the booster and payload do not have a chance to move much during the explosion, they are at essentially the same location just before and just after the explosion. Thus we can consider the gravitational potential energy to remain unchanged during the explosion, $\Delta U^G = 0$. Therefore, the expression for the changes in energy for the rocket-Earth system during the explosion can be written as

$$\Delta K_{\text{rocket-Earth}} + \Delta E_s + \Delta E_{\text{th}} = 0. \tag{1}$$

Because the rocket has zero speed and consequently zero kinetic energy just before the explosion, we have

$$\Delta K_{\text{rocket-Earth}} = K_{\text{after}} - K_{\text{before}} = K_{\text{after}} - 0.$$

Therefore Eq. 1 gives us an expression for the kinetic energy just after the explosive separation:

$$K_{\text{after}} + \Delta E_s + \Delta E_{\text{th}} = 0$$

$$K_{\text{after}} = -\Delta E_s - \Delta E_{\text{th}}.$$

We must then determine how much of this kinetic energy is associated with the payload because

$$\Delta K_{\text{rocket-Earth}} = K_{\text{after}} = K_{\text{payload, after}} + K_{\text{booster, after}}. \tag{2}$$

To determine how much kinetic energy is associated with the payload, we consider the explosive separation of the booster and payload separately. During the explosion we can consider the payload-booster system to be isolated and so its momentum remains constant, or $\Delta\vec{p}_{\text{payload}} + \Delta\vec{p}_{\text{booster}} = 0$. We can then use this relation to figure out the relationship between the velocities of the payload and the booster just after the explosive separation. Given the payload velocity just after the separation, we can determine the kinetic energy of the payload. After the separation, the mechanical energy in the payload-Earth system should remain constant because now there is no longer any source energy being dissipated to incoherent forms. For the mechanical energy of the payload-Earth system, we therefore can say

$$\Delta K_{\text{payload-Earth}} + \Delta U^G_{\text{payload-Earth}} = 0. \tag{3}$$

Finally, we realize that Earth's inertia is much greater than that of any rocket. This means that only a vanishingly small fraction of the gravitational potential energy goes into changing Earth's kinetic energy (see Developing a Feel 4), which implies

$$\Delta K_{\text{payload-Earth}} = \Delta K_{\text{payload}} + \Delta K_{\text{Earth}} = \Delta K_{\text{payload}} + 0.$$

Because this is so, the only kinetic energy we need to consider is the payload's. From it we can determine the change in gravitational potential energy and the payload's final height h_2.

❷ **EXECUTE PLAN** We first determine the sum of the kinetic energies of the booster and payload right after separation, beginning with Eq. 1:

$$\Delta K_{\text{rocket-Earth}} + \Delta E_{\text{s}} + \Delta E_{\text{th}} = 0$$

$$\Delta K_{\text{rocket-Earth}} + (-E) + (+E/4) = 0$$

$$\Delta K_{\text{rocket-Earth}} = \tfrac{3}{4}E.$$

Substituting from Eq. 2 gives

$$\Delta K_{\text{rocket-Earth}} = K_{\text{payload, after}} + K_{\text{booster, after}} = \tfrac{3}{4}E$$

$$\tfrac{1}{2}m_{\text{payload}}v^2_{\text{payload, after}} + \tfrac{1}{2}m_{\text{booster}}v^2_{\text{booster, after}} = \tfrac{3}{4}E. \tag{4}$$

To determine how this kinetic energy is distributed between payload and booster, we use the fact that the momentum of the payload-booster system is constant during the explosion to determine the relationship of their velocities just after the explosive separation:

$$p_{\text{payload }x\text{, after}} + p_{\text{booster }x\text{, after}} = p_{\text{payload }x\text{, before}} + p_{\text{booster }x\text{, before}}$$

$$m_{\text{payload}}(+v_{\text{payload, after}}) + m_{\text{booster}}(-v_{\text{booster, after}}) = 0 + 0$$

$$v_{\text{booster, after}} = \frac{m_{\text{payload}}}{m_{\text{booster}}}v_{\text{payload, after}},$$

where we have used the velocity directions shown in Figure WG7.5. Substituting this result into Eq. 4 and using the fact that, after separation, $m_{\text{booster}} = 5m_{\text{payload}}$, we obtain

$$\tfrac{1}{2}m_{\text{payload}}v^2_{\text{payload, after}} + \tfrac{1}{2}m_{\text{booster}}\left(\frac{m_{\text{payload}}}{m_{\text{booster}}}v_{\text{payload, after}}\right)^2 = \tfrac{3}{4}E$$

$$\left(\tfrac{1}{2}m_{\text{payload}}v^2_{\text{payload, after}}\right)\left(1 + \frac{m_{\text{payload}}}{m_{\text{booster}}}\right) = \tfrac{3}{4}E$$

$$\left(\tfrac{1}{2}m_{\text{payload}}v^2_{\text{payload, after}}\right)\left(1 + \frac{m_{\text{payload}}}{5m_{\text{payload}}}\right) = \tfrac{3}{4}E$$

$$\left(\tfrac{1}{2}m_{\text{payload}}v^2_{\text{payload, after}}\right)\left(\tfrac{6}{5}\right) = \tfrac{3}{4}E$$

$$\tfrac{1}{2}m_{\text{payload}}v^2_{\text{payload, after}} = \tfrac{15}{24}E = \tfrac{5}{8}E.$$

PRACTICE

Now we are ready to determine the quantity we are asked for—h_2, the maximum height reached by the payload after separation. We use Eq. 3 (the mechanical energy for the payload-Earth system remains constant), remembering that the subscript final means values at the maximum payload height h_2 and after means values just after separation (note that, for our choice of origin for the x axis, x_{final} and x_{after} are equivalent to h_2 and h_1, respectively):

$$\Delta K_{\text{payload}} + \Delta U^G_{\text{payload-Earth}} = 0$$

$$\left[\tfrac{1}{2}m_{\text{payload}}v^2_{\text{payload, final}} - \tfrac{1}{2}m_{\text{payload}}v^2_{\text{payload, after}}\right]$$

$$+ \left[m_{\text{payload}}gx_{\text{final}} - m_{\text{payload}}gx_{\text{after}}\right] = 0$$

$$\left[\tfrac{1}{2}m_{\text{payload}}(0)^2 - \tfrac{5}{8}E\right] + \left[m_{\text{payload}}g(+h_2) - m_{\text{payload}}g(+h_1)\right] = 0$$

$$m_{\text{payload}}gh_2 = m_{\text{payload}}gh_1 + \tfrac{5}{8}E$$

$$h_2 = h_1 + \tfrac{5}{8}\frac{E}{m_{\text{payload}}g}. \checkmark$$

❸ **EVALUATE RESULT** If E is increased, we expect the payload to go higher, which is what our algebraic result predicts. If m_{payload} is increased, we expect the opposite, and this, too, is consistent with our result. Further, the solution suggests that the amount of energy released in the explosive separation, E, appears to be about double the energy required to lift the payload through a height difference $h_2 - h_1$, so the order of magnitude of the result is correct. The assumption $m_{\text{rocket}} \ll m_{\text{Earth}}$ is reasonable.

One caveat is that we have used an expression for gravitational potential energy that is valid only near Earth's surface; our solution is applicable to only low-flying rockets, a situation we will correct in Chapter 13.

Guided Problem 7.6 Booster impact

A toy rocket comprises an unpowered, 1.0-kg payload and a powered booster connected by an explosive coupler. The rocket is fired straight up and reaches a speed of 90 m/s as the last of its fuel is exhausted 100 m above the ground. At this instant, the coupler is detonated, and the two parts separate from each other. The energy released by the explosive separation is 3000 J, with 1000 J of it converted to incoherent energy. Immediately after the explosive separation, the booster has an inertia of 2.0 kg. Use energy and momentum methods to calculate the speed at which the booster strikes the ground.

❶ **GETTING STARTED**

1. How is this problem similar to Worked Problem 7.5? How does it differ in information and in specific task?
2. Choose your system (or systems) and draw a version of Figure WG7.5 that contains the information supplied in this problem, using a question mark to identify the quantity you need to calculate.

❷ **DEVELOP PLAN**

3. Write an expression for the kinetic energy of the payload-booster system right after the separation.
4. Which law can you use to establish a relationship between $\vec{v}_{\text{payload}}$ and $\vec{v}_{\text{booster}}$ just after the separation?
5. What is the booster's x coordinate just after the separation, and what is it just as the booster hits the ground?
6. What is the booster's change in potential energy? How is this change related to its change in kinetic energy?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

Worked Problem 7.7 Somewhat springy potential energy

A 4.0-kg block takes part in only one interaction (with a fixed object) that causes potential energy to be stored in the block-object system as the block moves along an x axis. At $t = 0$, the block is at $x_0 = +2.0$ m and moving in the negative x direction at 2.0 m/s. The amount of potential energy stored depends on the block's position along the axis and is given by $U(x) = ax^2 + bx + c$, where $a = +1.0\ \text{J/m}^2$, $b = -2.0$ J/m, and $c = -1.0$ J. (*a*) Draw a diagram of this potential energy as a function of x, and show the value of the block's mechanical energy on your diagram. (*b*) What is the block's speed after it has traveled 3.0 m from x_0? (*c*) At what value of x is the block's velocity zero? (*d*) Describe what the block does after it passes through that value of x.

❶ **GETTING STARTED** The block and fixed object form a closed, nondissipative system because the only interaction is associated with a potential energy. We can get the potential energy and kinetic energy when the block is at one position and thus determine its mechanical energy at that position. Knowing that the mechanical energy remains constant during the motion allows us to sketch the graph for part *a*. We then need only to determine the potential energy when the block is at any other position in order to obtain the block's kinetic energy and hence its speed there. This covers parts *b* and *c*, and hopefully also provides insight about part *d*.

❷ **DEVISE PLAN** We are given the speed $v_0 = 2.0\ \text{m/s}$ at $x_0 = +2.0\ \text{m}$, from which we can calculate the kinetic energy $K_0 = K(x_0) = \frac{1}{2}mv_0^2$ and the potential energy $U_0 = U(x_0)$. No sources of energy are involved and no energy is dissipated during the motion, and so the mechanical energy remains constant. Consequently we know that for any position x,

$$E_{\text{mech}} = K(x_0) + U(x_0) = K(x) + U(x).$$

Knowing either $K(x)$ or $U(x)$ at any x allows us to calculate the other value:

$$K(x) = E_{\text{mech}} - U(x)$$
$$U(x) = E_{\text{mech}} - K(x).$$

After the block travels 3.0 m in the negative x direction from $x_0 = +2.0\ \text{m}$, it is at $x_b = -1.0\ \text{m}$, where we use the subscript b to remind ourselves that this value is needed to solve part *b*. We can now readily calculate its kinetic energy at that position.

At the instant when the block's velocity is zero at x_c (subscript c because this position is needed to solve part *c*), its kinetic energy is zero: $K(x_c) = 0$. We then get the potential energy from $U(x_c) = E_{\text{mech}} - K(x_c) = E_{\text{mech}} - 0$, and calculate the value of x_c that gives the block that amount of potential energy.

A similar procedure determines the block's motion thereafter, for part *d*.

❸ **EXECUTE PLAN**
(*a*) We can substitute values of x into the $U(x)$ equation to produce the graph. However, to help us in our thinking (and review some tools for more complex problems), it is good to find out whether there are any extrema (maxima and/or minima) in the potential energy curve by determining the value of x at which the derivative of U with respect to x is zero:

$$\frac{d}{dx}U(x) = \frac{d}{dx}(ax^2 + bx + c) = 2ax + b$$

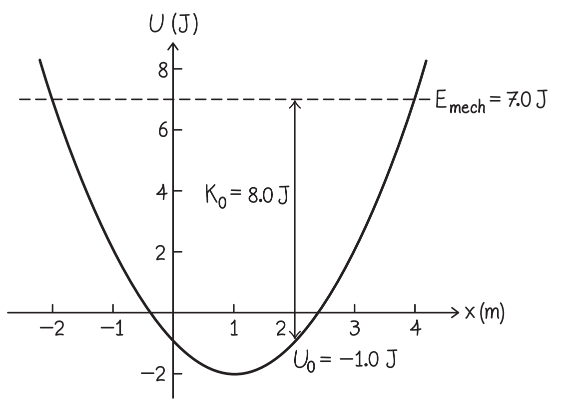
$$\frac{dU}{dx} = 0 = 2ax + b = 2(+1.0\ \text{J/m}^2)x + (-2.0\ \text{J/m}).$$

This equation has a single solution, $x = +1.0\ \text{m}$, and at that value of x the potential energy is

$$U(+1.0\ \text{m}) = (1.0\ \text{J/m}^2)(+1.0\ \text{m})^2 - (2.0\ \text{J/m})(+1.0\ \text{m}) - 1.0\ \text{J}$$
$$= -2.0\ \text{J}.$$

Because the second derivative of the potential energy with respect to x is positive ($+2.0\ \text{J/m}^2$) everywhere, the potential energy curve is concave up. The one extremum is therefore a minimum. To pin down the shape of the curve further, we compute the value of U at a few other values of x: $U(-2.0\ \text{m}) = +7.0\ \text{J}$, $U(-1.0\ \text{m}) = +2.0\ \text{J}$, and $U(0) = -1.0\ \text{J}$. Symmetry gives us the shape on the other side of the minimum, and we end up with the graph shown in Figure WG7.6.

Figure WG7.6



PRACTICE

The graph shows that U decreases as the block travels in the negative x direction. Because $E_{mech} = U + K$ stays constant, decreasing U means increasing K, and increasing K means increasing v; in other words, the block speeds up as it moves from $x = +2.0$ m to $x = +1.0$ m. At $x = +1.0$ m, U has its lowest value, and so K must have its highest value. As the block continues to the left of $x = +1.0$ m, it slows down because U increases and so K and v must decrease.

The kinetic energy at $x_0 = +2.0$ m is

$$K_0 = \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}(4.0\text{ kg})(2.0\text{ m/s})^2 = 8.0\text{ J}.$$

The mechanical energy when the block is at $x_0 = +2.0$ m is then

$$E_{mech} = \tfrac{1}{2}mv_0^2 + U_0 = (8.0\text{ J}) + (-1.0\text{ J}) = 7.0\text{ J}.$$

Because the system's mechanical energy is constant, this is also the value of E_{mech} at any other value of x. We include a dashed horizontal line on the graph to illustrate this constant value of E_{mech}. ✔

(*b*) After the block has traveled 3.0 m from x_0, moving in the negative x direction, it is at $x_b = -1.0$ m. We can determine its speed at this position once we get the kinetic energy:

$$K_b = K(x_b) = E_{mech} - U(x_b) = 7.0\text{ J} - (+2.0\text{ J}) = +5.0\text{ J}$$

$$\tfrac{1}{2}mv_b^2 = K_b$$

$$v_b = \sqrt{\frac{2K_b}{m}} = \sqrt{\frac{2(5.0\text{ J})}{4.0\text{ kg}}} = 1.6\text{ m/s}. \checkmark$$

(*c*) We determine x_c, the position at which the block's velocity is zero, by using the fact that its kinetic energy is also zero at this position:

$$K_c + U_c = E_{mech}$$

$$0 + (ax^2 + bx + c) = E_{mech}$$

$$(1.0\text{ J/m}^2)x_c^2 + (-2.0\text{ J/m})x_c + (-1.0\text{ J}) = 7.0\text{ J}$$

$$x_c^2 - (2.0\text{ m})x_c - (8.0\text{ m}^2) = 0.$$

This expression can be factored, giving

$$[x_c + (2.0\text{ m})][x_c - (4.0\text{ m})] = 0.$$

The solutions to this equation are $x_c = -2.0$ m and $x_c = +4.0$ m, which agree with the values we read from Figure WG7.6 (the two values of x at which the curve intersects the $E_{mech} = 7.0$ J dashed line). Because the block is moving in the negative x direction, the solution we want here is $x_c = -2.0$ m. ✔

(*d*) At $x_c = -2.0$ m, the block has zero velocity because its kinetic energy is zero. Its acceleration is not zero at this location; indeed, its acceleration is always in the direction of decreasing potential energy. For the block at $x_c = -2.0$ m, this means directed along the positive x axis. Because the direction of motion reverses here, this position is called a *turning point* of the motion.

As the block moves in the positive x direction, it picks up speed as the potential energy decreases along the curve in Figure WG7.6, until it reaches $x = +1.0$ m. After it moves through this position of minimum potential energy, the block slows down as kinetic energy is converted to potential energy. The block eventually again has zero velocity at the other value we obtained in part *c*, $x_c = +4$ m, which is the second turning point of the motion. The block then again moves in the direction of decreasing potential energy, which now is to the left. ✔

❹ **EVALUATE RESULT** We can check whether our calculated answers are consistent with our graph. The kinetic energy K_b at $x_b = -1.0$ m in the graph (the amount of energy that must be added to the potential energy at this point to reach the E_{mech} line) is 5.0 J, which is what we calculated in part *b*. We can see from where the E_{mech} line crosses the potential energy curve that the left turning point is at $x = -2.0$ m as we calculated.

PRACTICE

Guided Problem 7.8 Over the hill

A particle takes part in just one interaction with a fixed object. The interaction causes potential energy to be stored in the particle-object system as the particle moves along an x axis. The particle is released from rest at $x_0 = -3.0$ m, and the amount of potential energy stored (in joules) is given by $U(x) = ax + bx^2 + cx^3$, where $a = +12$ J/m, $b = +3.0$ J/m^2, and $c = -2.0$ J/m^3. (*a*) Make a graph of the potential energy that also displays the particle's mechanical energy. (*b*) In which direction does the particle move initially? (*c*) Describe the particle's motion after it leaves x_0. (*d*) At which positions along the curve you drew in part *a* is the particle speeding up, and at which positions is it slowing down? (*e*) What is the particle's kinetic energy at $x = -1.0$ m, $x = +1.0$ m, and $x = +3.0$ m?

1 GETTING STARTED

1. Which portions of Worked Problem 7.7 are relevant?
2. How can you use calculus to determine the extrema of the potential energy function?
3. What happens to the amount of potential energy stored as x gets larger in the positive direction and as x gets larger in the negative direction?

2 DEVISE PLAN

4. Calculate the potential energy at a few values of x. [Hint: What values are useful for part *e*?] What are the values of the potential energy at the extrema?
5. What is the kinetic energy of the particle at the instant it is released? How can you use this information to determine the (constant) mechanical energy of the system?
6. What determines in which direction the particle tends to move once it is released?

3 EXECUTE PLAN

7. Where is the potential energy increasing, and where is it decreasing?

4 EVALUATE RESULT

Answers to Review Questions

1. (*a*) The momentum remains constant. (*b*) The kinetic energy dips briefly as some or all of it is converted to internal energy during the collision. (*c*) The sum of all forms of energy in the system remains constant.
2. No. The relative speed must be nonzero before the collision, but it changes to zero at some instant during the interaction.
3. In all cases a portion of the kinetic energy of the system is converted to internal energy. In part *a*, none of this converted energy reappears as kinetic energy after the collision. In part *b*, some of this energy reappears as kinetic. In part *c*, all of it reappears as kinetic energy after the collision.
4. An interaction involves two objects and results in a change of motion, a change of physical state, or both. If a change of motion is involved, the ratio of the x components of the accelerations of the objects is equal to the negative of the inverse ratio of their inertias. Both the momentum and kinetic energy of each object generally change as a result of the interaction, but if the system is isolated, its momentum does not change. If the system is closed, its energy does not change but its kinetic energy changes during the interaction as some portion is converted to internal energy. When the interaction is an elastic collision, all of the converted kinetic energy ultimately reappears as kinetic energy after the interaction.
5. Kinetic energy is the energy any moving object has because of its motion. Potential energy is energy stored in a system due to reversible changes in the configuration state of the system.
6. Many answers are possible. Two possibilities are bending (coherent kinetic energy) a paper clip back and forth, which heats it up (incoherent thermal energy), and rubbing your hands together to warm them.
7. Energy of motion is energy due to the motion of objects or molecules in a system. Energy of configuration is energy associated with the locations of objects or molecules in a system. Coherent energy is associated with either motions or configurations that are organized, so that individual objects or molecules in the system either all move with a common velocity or are all displaced by a small amount along a common direction. Incoherent energy is associated with either motions or configurations that are disorganized, so that objects or molecules in the system move randomly or are displaced randomly.
8. Solar radiation generated by nuclear reactions in the Sun is the ultimate source.
9. Nondissipative interactions cause reversible changes; dissipative interactions cause irreversible changes.
10. Many answers are possible. In the interaction between a car and the gasoline in its fuel tank, source energy (chemical energy of the gasoline) is converted to thermal energy, to the car's kinetic energy, and, if the car is climbing a hill, to gravitational potential energy. When you rub your hands together to warm them, chemical energy in the molecules that make up the muscles you use to move your hands is converted to kinetic energy, which in turn is converted to thermal energy by friction.
11. In the field model, every object is surrounded by a field, and it is with this field that other objects interact. In the gauge-particle model, two objects interact by exchanging fundamental particles called gauge particles.
12. Only the gravitational and electromagnetic interactions have a long enough range to be noticed on the macroscopic scale. The range of the strong and weak interactions is too short to cause changes that we can observe on the macroscopic scale.
13. (*a*) The electromagnetic interaction is most important. (*b*) Again, the electromagnetic interaction is most important.
14. Although the electromagnetic interaction is much stronger, it can be either attractive or repulsive depending on the signs of the electrically charged particles involved in the interaction. Because there are equal numbers of the two types of particles in any macroscopic object, the particles tend to arrange themselves to produce electrical neutrality and to minimize the effect of the interaction. In contrast, the gravitational interaction is always attractive, and no cancellation diminishes its cumulative effect. Because we live on the surface of Earth, the very large effect of the gravitational interaction between macroscopic objects and Earth is generally not negligible.

15. (*a*) $a_{1x}/a_{2x} = -m_2/m_1$. (*b*) Again $a_{1x}/a_{2x} = -m_2/m_1$ because Eq. 7.6 holds for all interactions.

16. Two assumptions were made—that the system of interacting objects is isolated, and the inertias of the objects are not changed by the interaction.

17. For a closed system, energy must not change, so the changes in the four categories of energy—kinetic, potential, source, and thermal—must cancel. Both source energy and thermal energy always involve dissipation, so if all interactions are nondissipative, only kinetic and potential energy changes are possible, and they must add to zero. Mechanical energy is the sum of kinetic and potential energies and hence remains constant in such situations.

18. Nondissipative interactions cause only reversible changes. A movie recording the object(s) in such a system must appear the same when run either forward or backward. That means the kinetic energy of each object must have the same value at a given position regardless of whether the object is moving forward or backward at that position (or not moving at all). But mechanical energy remains constant in such systems and is equal to the sum of kinetic and potential energies. If the kinetic energy must be expressible as a function of position, then so must the potential energy.

19. Because of the reversibility of changes due to nondissipative interactions, the initial and final kinetic energies must be equal at the common position (point A).

20. Vertical. Yes, it does matter because changes in gravitational potential energy are zero for horizontal displacements.

21. No, the gravitational potential energy can be either positive or negative, depending on which position you choose for your $U^G = 0$ position.

22. No, only the convertible part of the system's energy is converted. The inconvertible part remains as system kinetic energy. Some final nonzero velocity (and hence final nonzero kinetic energy) is needed to keep the system's momentum constant.

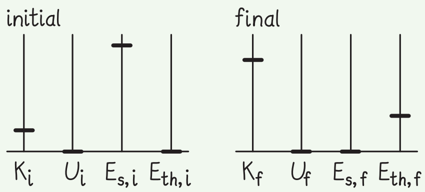
23. (*a*) Equation 7.26, $\Delta E_{th} = -\Delta K = \frac{1}{2}\mu v_{12i}^2(1 - e^2)$, tells us that none of the convertible energy is converted to thermal energy (which means the collision is elastic). All of it is temporarily stored as potential energy. (*b*) Equation 7.26 tells us that all of the convertible kinetic energy is ultimately converted to thermal energy, but all or part of it may still be temporarily stored as potential energy.

Answers to Guided Problems

Guided Problem 7.2 $h/16$

Guided Problem 7.4 (*a*) rear $= -0.12$ m/s, lead $= +8.4$ m/s;

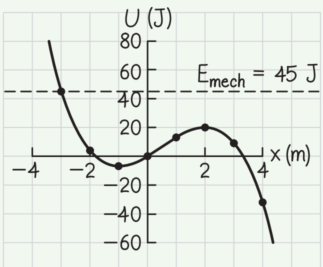
(*b*)



Guided Problem 7.6 $v_f = 78$ m/s

Guided Problem 7.8

(*a*)



x	U(x)
−3	45
−2	4
−1	−7
0	0
1	13
2	20
3	9
4	−32

(*b*) toward positive *x*; (*c*) accelerates toward positive *x*, then accelerates toward negative *x* between $x = -1.0$ m and $x = +2.0$ m, then accelerates toward positive *x* again; (*d*) speeding up from $x = -3.0$ m to $x = -1.0$ m and from $x = +2.0$ m to large *x* values, slowing down between $x = -1.0$ m and $x = +2.0$ m; (*e*) $K(x = -1.0 \text{ m}) = +52$ J, $K(x = +1.0 \text{ m}) = +32$ J, $K(x = +3.0 \text{ m}) = +36$ J

Guided Practice by Chapter

8 Force

Review Questions

Answers to these questions can be found at the end of this chapter.

8.1 Momentum and force

1. Does a force exerted on an object always increase the object's speed? If yes, explain why. If no, provide an example in which an exerted force does not increase the object's speed.
2. What are the magnitude and direction of the vector sum of the forces exerted on a car traveling on a straight downward slope of a highway at a constant 100 km/h?
3. What is the relationship between momentum and force?

8.2 The reciprocity of forces

4. Is there any circumstance in which an object 1 may exert a force on an object 2 without an equal, reciprocal force being exerted by object 2 on object 1?
5. A cup filled with coffee sits on a table. Do the downward force the cup exerts on the table and the downward force the coffee exerts on the bottom of the cup form an interaction pair?

8.3 Identifying forces

6. What is the principal difference between a contact force and a field force?
7. Characterize each of the following as a contact force or a field force: (*a*) the force that causes a book sliding across a polished floor to eventually slow down; (*b*) the force that causes a wine glass to fall downward after it has been knocked off a table; (*c*) the force that causes one magnet to repel another; (*d*) the force exerted by the wind on a sailboat; (*e*) the force that causes an object attached to one end of a stretched rubber band to move toward the object attached to the other end.

8.4 Translational equilibrium

8. Can an object that is in translational equilibrium have any forces exerted on it?
9. How can you determine, by observing an object, whether or not it is in translational equilibrium?

8.5 Free-body diagrams

10. Describe the three main elements in a free-body diagram.
11. Give at least two reasons it is important to draw a free-body diagram to analyze physical situations.
12. Why should an arrow representing the vector sum of the forces exerted on an object *not* appear in the free-body diagram for the object?

8.6 Springs and tension

13. How is the change in the length of a spring related (*a*) to the magnitude F_{os} of the force exerted by an object on the spring and (*b*) to the magnitude F_{so} of the force exerted by the spring on the object?
14. What is an elastic force?
15. Under what condition(s) is a pulling force exerted by a hand on one end of a rope, spring, or thread transmitted undiminished to a block attached to the other end?

8.7 Equation of motion

16. Knowing all the forces exerted on an object gives you direct information about which aspect of the object's motion?
17. What condition allows you to convert the relationship $\sum\vec{F} = d\vec{p}/dt$ to the relationship $\sum\vec{F} = m\vec{a}$?
18. State Newton's three laws of motion in your own words.

8.8 Force of gravity

19. (*a*) Which gravitational force exerted by Earth has the larger magnitude: that exerted on an apple or that exerted on a sack of apples? (*b*) How do you reconcile your answer to part *a* with the observation that, when they are in free fall, the apple and the sack of apples accelerate at the same rate?
20. An object at rest has zero acceleration. Does this mean that Eq. 8.17 ($F^G_{\text{Eo}x} = -mg$) does not accurately represent the force of gravity exerted on the object?

8.9 Hooke's law

21. How does the numerical value of the spring constant k for a spring under compression compare with the value of k for the same spring when it is stretched?
22. Equation 8.20 relates the force exerted by a spring to the displacement of the spring's free end: $(F_{\text{by spring on load}\,x}) = -k(x - x_0)$. What is the meaning of the symbol x_0?
23. What is the physical significance of the slope of the curve in Figure 8.18, a graph that plots the displacement of the free end of a spring as a function of the x component of the force exerted on the spring?

8.10 Impulse

24. State the impulse equation in words.
25. Suppose that one or more of the forces exerted on an object is not constant, but that you do have information about how each force exerted on the object varies as a function of time during the time interval of interest. How can this information be used to compute the impulse?

8.11 Systems of two interacting objects

26. What distinguishes an internal force from an external force?
27. Explain what is special about the motion of the center of mass of a system made up of two interacting objects when some external force is exerted on the system.

8.12 Systems of many interacting objects

28. What is the numerical value of the vector sum of all internal forces exerted in a system composed of many objects?
29. Explain why the motion of rigid objects is usually easier to analyze than the motion of deformable objects.

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The magnitude of the vector sum of forces needed to give a bus the acceleration seen in a sports car that can go from 0 to 60 mi/h in 5 s (K, O)
2. The magnitude of the average force exerted by a bat on a baseball hit straight toward the pitcher (E, W, J, U)
3. The magnitude of the vector sum of forces exerted on an airliner accelerating for takeoff (P, I, G)
4. The magnitude of the gravitational force exerted by Earth on an elephant (S)
5. The magnitude of the upward force exerted by the air on an airliner flying horizontally at constant velocity (A, P, V)
6. Over and above the force required to stand on the floor, the magnitude of the extra force exerted by a professional basketball player on the floor in order to leap for a slam dunk (H, C, X, R, L)
7. The spring constant of the spring in a ballpoint pen (F, T)
8. The spring constant of an automobile spring (D, Z)
9. The magnitude of the impulse required to get a minivan up to freeway speed (N, Y)
10. The magnitude of the impulse required to stop a speeding train in 60 s (B, M, Q)

Hints

A. Is the airliner in translational equilibrium?
B. What is the speed of a speeding train?
C. What has happened to the basketball player's kinetic energy when he is at the top of the jump?
D. By what distance does a car fender drop when an adult of average inertia sits on the car?
E. What is the baseball's incoming velocity?
F. Through what distance is the spring compressed when you click the pen?
G. What is the speed just as the airliner's wheels lift off the ground?
H. How does Newton's third law of motion apply to this situation?
I. How far along the runway does the airliner move before the wheels lift off the ground?
J. What is the baseball's outgoing velocity?
K. What is the magnitude of the acceleration of a sports car?
L. Over what time interval is the force exerted?
M. What is the inertia of a train?
N. What is freeway speed?
O. How does the inertia of a bus compare with the inertia of a sports car?
P. What is the inertia of an airliner?
Q. What portion of the momentum must change in the time interval?
R. What is the inertia of a professional basketball player?
S. What is the inertia of an elephant?
T. What is the magnitude of the force you must exert to click the pen?
U. For what time interval is the ball in contact with the bat?
V. What other force is exerted vertically on the airliner?
W. What is the inertia of a baseball?
X. What is the height of the basketball player's jump?
Y. What is the inertia of a minivan?
Z. What is the inertia of an average adult?

Key (all values approximate)

A. yes; **B.** 3×10^1 m/s; **C.** it has all been converted to gravitational potential energy; **D.** 4×10^1mm; **E.** 4×10^1 m/s toward batter; **F.** 5 mm; **G.** 9×10^1 m/s; **H.** it tells us that $F_{\text{by player on floor}} = F_{\text{by floor on player}}$; **I.** 1 km; **J.** a bit faster than the incoming velocity—say, 5×10^1 m/s away from batter; **K.** 5 m/s^2; **L.** 0.4 s; **M.** 5×10^6 kg; **N.** 3×10^1 m/s; **O.** $m_{\text{bus}} \approx 10m_{\text{car}}$; **P.** 1×10^5 kg; **Q.** the entire momentum, 2×10^8 kg·m/s, must go to zero; **R.** 9×10^1 kg; **S.** 6×10^3 kg; **T.** 3 N; **U.** 2×10^{-2} s; **V.** the gravitational force; **W.** 0.2 kg; **X.** each part of the body rises 0.7 m; **Y.** 2×10^3 kg; **Z.** 8×10^1 kg

Worked and Guided Problems

Procedure: Drawing free-body diagrams

1. Draw a center-of-mass symbol (a circle with a cross) to indicate the object you wish to consider*—this object is your system. Pretend the object is by itself in empty space (hence the name *free body*). If you need to consider more than one object in order to solve a problem, draw a separate free-body diagram for each.
2. List all the items in the object's environment that are in contact with the object. These are the items that exert *contact forces* on the object. *Do not add these items to your drawing*! If you do, you run the risk of confusing the forces exerted *on* the object with those exerted *by* the object.
3. Identify all the forces exerted *on* the object by objects in its environment. (For now, omit from consideration any force not exerted along the object's line of motion.) In general, you should consider (*a*) the *gravitational field force* exerted by Earth on the object and (*b*) the *contact force* exerted by each item listed in step 2.
4. Draw an arrow to represent each force identified in step 3. Point the arrow in the direction in which the force is exerted and place the tail at the center of mass. If possible, draw the lengths of the arrows so that they reflect the relative magnitudes of the forces. Finally, label each arrow in the form

$$\vec{F}^{\text{type}}_{\text{by on}},$$

where "type" is a single letter identifying the origin of the force (c for contact force, *G* for gravitational force), "by" is a single letter identifying the object exerting the force, and "on" is a single letter identifying the object subjected to that force (this object is the one represented by the center of mass you drew in step 1).
5. Verify that all forces you have drawn are exerted **on** and not **by** the object under consideration. Because the first letter of the subscript on $\vec{F}$ represents the object exerting the force and the second letter represents the object on which the force is exerted, every force label in your free-body diagram should have the same second letter in its subscript.
6. Draw a vector representing the object's acceleration *next to* the center of mass that represents the object. Check that the vector sum of your force vectors points in the direction of the acceleration. If you cannot make your forces add up to give you an acceleration in the correct direction, verify that you drew the correct forces in step 4. If the object is not accelerating (that is, if it is in translational equilibrium), write $\vec{a} = \vec{0}$ and make sure your force arrows add up to zero. If you do not know the direction of acceleration, choose a tentative direction for the acceleration.
7. Draw a reference axis. If the object is accelerating, it is often convenient to point the positive *x* axis in the direction of the object's acceleration.

When your diagram is complete it should contain only the center-of-mass symbol, the forces exerted *on* the object (with their tails at the center of mass), an axis, and an indication of the acceleration of the object. Do not add anything else to your diagram.

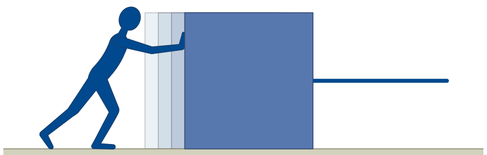
* Representing, say, a car by a single point might seem like an oversimplification, but because we are interested only in the motion of the car as a whole, any details (like its shape or orientation) do not matter. Any unnecessary details you add to your drawing only distract from the issue at hand.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 8.1 Moving a block

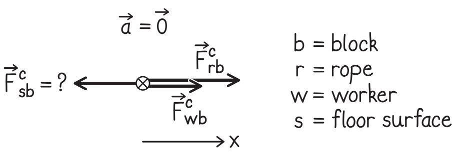
A forklift is attempting to pull a 1400-kg block of granite across a warehouse floor with a rope, aided by a push from a worker (Figure WG8.1). The tension in the rope is 2500 N, and the worker exerts a horizontal push of 200 N. The block slides across the floor at a constant speed of 1.5 m/s. What is the magnitude of the frictional force exerted on the block?

Figure WG8.1



❶ **GETTING STARTED** We are asked to calculate the magnitude of a frictional force. We note that, in addition to the frictional force, there are two other horizontal contact forces exerted on the block: the pulling force exerted by the rope and the pushing force exerted by the worker. A relationship that involves force and motion is needed, so we will likely need Eq. 8.6, $\sum\vec{F} = m\vec{a}$, Newton's second law when inertia is constant. A free-body diagram (Figure WG8.2) helps us organize the information. The directions of the pulling and pushing forces must be the same, so we arbitrarily draw those to the right in the figure. The frictional force opposes slipping between the block and the floor surface. If we were to stop pushing and pulling, friction should cause the block to slow and stop, so friction should be directed to the left, opposite the velocity of the block. We note that this velocity is constant, which means the acceleration is zero.

Figure WG8.2



❷ **DEVISE PLAN** We know the tension in the rope and the magnitude of the worker's pushing force. We know that the acceleration is zero, and we know the direction of the frictional force. We arbitrarily choose the positive x direction to be the direction of motion (to the right in Figure WG8.2). The three forces are horizontal, so we need only their x components to relate their values:

$$\sum F_x = 0. \tag{1}$$

One equation is sufficient to solve for the one unknown frictional force.

❸ **EXECUTE PLAN** Using Figure WG8.2 as a guide, we express the left side of Newton's second law as the sum of the x components of the individual forces. By comparing the directions of the forces with the positive x direction, we see that the x components of the pushing and pulling forces are positive and that of the frictional force is negative. If we substitute the magnitude of the forces, Eq. 1 becomes

$$F^c_{rb\,x} + F^c_{wb\,x} + F^c_{sb\,x} = 0$$

$$+F^c_{rb} + F^c_{wb} + (-F^c_{sb}) = 0 \tag{2}$$

$$F^c_{sb} = F^c_{rb} + F^c_{wb} = 2500\text{ N} + 200\text{ N} = 2700\text{ N.} \checkmark$$

❹ **EVALUATE RESULT** If we substitute this answer into Eq. 2, the signs in the equation are consistent with the force directions shown in Figure WG8.2. No assumptions were required to solve this problem, nor did we require all the numerical values that were given.

PRACTICE

Guided Problem 8.2 Car springs

As an automotive engineer, you have been assigned the task of designing a set of springs for a car. If each of the four springs must compress a distance of 30 mm when the 640-kg car body rests on them, what value of the spring constant k should each spring have?

1 GETTING STARTED

1. What interactions does the car body take part in?
2. Draw a free-body diagram of the car body.

2 DEVISE PLAN

3. After the springs stop bouncing, what is the body's acceleration? Add that information to your free-body diagram.
4. Decide on an appropriate direction for the positive x axis and indicate it on the diagram.
5. Using your free-body diagram as a guide, write the component version of the appropriate force equation.

3 EXECUTE PLAN

6. Solve for the force exerted by each spring on the body.
7. How is this force related to the spring constant?

4 EVALUATE RESULT

8. Examine your expression for k. If the inertia of the car were increased, what would happen to k? Is this reasonable?
9. Did you make any assumptions?

Worked Problem 8.3 Truckin' on

A tractor is pulling two trailers. Trailer 1 has an inertia of 4000 kg, and trailer 2 has an inertia of 6000 kg. The tractor pulls its load with a force of 2.5 kN when it starts from rest. (*a*) What is the magnitude of the tractor's acceleration? (*b*) What is the tension in the coupling between the two trailers?

1 GETTING STARTED Figure WG8.3 identifies the objects involved: tractor t, trailer 1, and trailer 2. Because two objects are being pulled, we may need to draw a free-body diagram for each one. What shall we put in our free-body diagrams? The motion is horizontal, so we concern ourselves only with horizontal forces. The only force exerted on trailer 2 is the forward pull exerted by the coupling with trailer 1. The forces exerted on trailer 1 are the 2.5-kN force exerted by the tractor pulling forward and the force exerted by the coupling with trailer 2. We ignore air resistance because the truck starts from rest and so is moving at low speed. We also assume the wheels are well greased so that bearing friction does not impede the motion. As long as the tractor and two trailers move together, they must have a common acceleration, and it is in the direction in which the tractor pulls (which we make the positive x direction). We can therefore use the same acceleration symbol a for the tractor and both trailers, and our free-body diagrams are as shown in Figure WG8.4.

Figure WG8.3

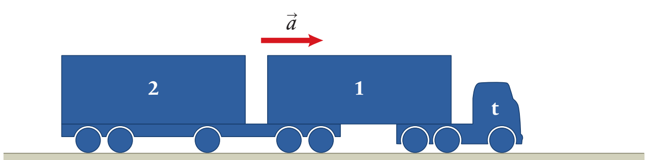
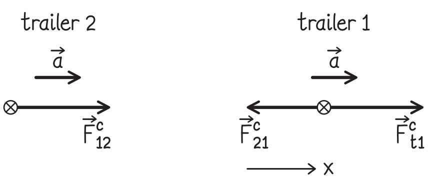


Figure WG8.4



2 DEVISE PLAN Because this problem deals with acceleration and force, we start with Newton's second law. With two free-body diagrams we obtain two equations, one from each diagram:

$$\sum F_{1x} = m_1 a_{1x} = m_1 a_x$$

$$\sum F_{2x} = m_2 a_{2x} = m_2 a_x.$$

The unknown common acceleration $a = |\vec{a}| = |a_x|$ and the unknown coupling tension $\mathcal{T} = |\vec{F}^{\text{tensile}}| = |\vec{F}^{c}_{12}| = |\vec{F}^{c}_{21}|$ will appear in both equations. We know the magnitude of the tractor's pulling force: $F^{c}_{t1} = 2.5$ kN. Two equations with two unknowns are all we need.

PRACTICE

3 EXECUTE PLAN

(*a*) Our second-law equations give us

$$\sum F_{1x} = F^{c}_{21x} + F^{c}_{t1x} = m_1 a_x = -F^{\text{tensile}} + F^{\text{pull}} = m_1(+a)$$

$$\sum F_{2x} = F^{c}_{12x} = m_2 a_x = +F^{\text{tensile}} = m_2(+a). \quad (1)$$

Adding these two equations eliminates the tensile force and allows us to solve for the acceleration of each object:

$$m_2 a + m_1 a = (+F^{\text{tensile}}) + (-F^{\text{tensile}}) + (F^{\text{pull}})$$

$$(m_1 + m_2)a = F^{\text{pull}}$$

$$a = \frac{F^{\text{pull}}}{m_1 + m_2} = \frac{2.5 \times 10^3 \text{ N}}{4000 \text{ kg} + 6000 \text{ kg}} = 0.25 \text{ m/s}^2. \checkmark \quad (2)$$

(*b*) We compute the tension $\mathcal{T} = |\vec{F}^{\text{tensile}}|$ in the coupling by substituting the expression we derived for a in Eq. 2 into Eq. 1:

$$\mathcal{T} = m_2\left(\frac{F^{\text{pull}}}{m_1 + m_2}\right) \quad (3)$$

$$\mathcal{T} = \frac{m_2}{m_1 + m_2} F^{\text{pull}} = \frac{6000 \text{ kg}}{4000 \text{ kg} + 6000 \text{ kg}}(2.5 \times 10^3 \text{ N})$$

$$= 1.5 \times 10^3 \text{ N}. \checkmark$$

4 EVALUATE RESULT Notice from Eq. 2 that the acceleration is the force exerted by the tractor divided by the inertia of the trailers, which is what we would expect if we had lumped the two trailers together as one object to get the acceleration.

The magnitude of the tension we obtained for the coupling, 1.5×10^3 N, is numerically equal to the magnitude of the tensile force required to accelerate just trailer 2 and should be less than the magnitude of the force exerted by the tractor, 2.5×10^3 N, which has to get both trailers to accelerate at the same rate.

Our algebraic expression for tension in the coupling (Eq. 3) states that the tension is some fraction of the 2.5-kN force exerted by the tractor, as expected. It also predicts that as the inertia of trailer 1 decreases to zero, the tension in the coupling is just the force the tractor pulls with, which is to be expected if trailer 1 becomes just a connector. As m_2 goes to zero, the tension also goes to zero because no force is needed to accelerate a trailer of zero inertia.

We did ignore friction and air resistance, which is reasonable at low speed.

Guided Problem 8.4 Rubber-band leash

A child drags a toy dog around at constant speed, using a rubber band as a leash. If the spring constant of the rubber band is 20 N/m and the frictional force between the toy and the floor is 0.734 N, how much does the rubber band stretch?

1 GETTING STARTED

1. What physical quantity is appropriate to represent the unknown you seek?
2. Which object, the rubber band or the toy, should be the subject of a free-body diagram?
3. Don't forget to include an x axis on your free-body diagram to indicate the positive direction.

2 DEVISE PLAN

4. What relates the tensile force exerted on the object in your free-body diagram to the distance the rubber band stretches?
5. How does the number of equations you have compare with the number of unknowns?

3 EXECUTE PLAN

6. Solve your equations to obtain an algebraic expression for the desired unknown.
7. Substitute the values given.

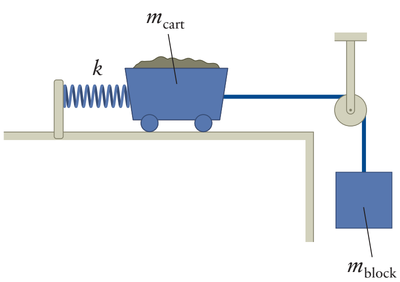
4 EVALUATE RESULT

8. Does your algebraic expression make sense as you imagine a larger or smaller frictional force? What about a larger or smaller spring constant?
9. Did you make any simplifying assumptions that would significantly affect your answer?

Worked problem 8.5 Retarding a mine cart

In Figure WG8.5, a hanging block suspended by a rope hung over a pulley is used to accelerate a loaded mine cart along rail tracks to the edge of a cliff, where the load is dumped. To help reduce its acceleration as it nears the edge, the cart is attached to a horizontal spring of spring constant k, and the other end of the spring is attached to a wall. Ignoring friction and the small amount of stretch in the rope, determine the acceleration of the cart when the spring is stretched a distance d beyond its relaxed length.

Figure WG8.5



1 GETTING STARTED The rope tends to accelerate the cart to the right in Figure WG8.5, but the stretched spring tends to accelerate it to the left. If the spring is stretched only a little, the force it exerts on the cart is small, and the pull of the rope causes the cart to accelerate to the right. If the spring is stretched a lot, the magnitude of the force it exerts on the cart may be greater than the magnitude of the force exerted by the rope, accelerating the cart to the left.

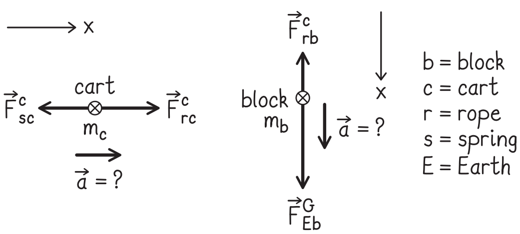
The block is subject to both the upward tug of the rope and the downward tug of gravity, and so we do not know which way it is accelerating. However, unless the rope goes slack, stretches, or breaks, the magnitude of the acceleration of the cart and the block must be the same. This common magnitude of acceleration is horizontal for the cart and vertical for the block, and the directions of these accelerations must allow the rope to remain taut and of constant length.

Even though the cart moves horizontally and the block moves vertically, each moves in only one dimension, which means this is just two one-dimensional problems coupled by the tension in the rope.

2 DEVISE PLAN We begin with a free-body diagram of the cart because it is the cart's acceleration we are asked to calculate (Figure WG8.6). The cart is subject to two horizontal forces, one exerted by the rope and one by the spring. We know the value of neither, and so, counting the acceleration we seek, there are three unknowns!

The expression we must obtain for the acceleration involves the symbols given in the problem statement. We will be able to substitute for the unknown spring force using $F^c_{\text{sc}\,x} = -k(x - x_0) = -kd$, where x is the position of the cart end of the spring at any instant when the spring is either stretched or compressed and x_0 is the position of the cart end of the spring when the spring is relaxed. Because we have no obvious substitution for the force exerted by the rope, we need another equation, which suggests that we draw a free-body diagram of the other object, the block [Figure WG8.6]. We show the acceleration in both free-body diagrams but add a question mark to indicate that we are unsure of its direction. However, we know that if the block accelerates downward, the cart accelerates to the right and if the block accelerates upward, the cart accelerates to the left. Thus both accelerations are either along the positive x directions we have chosen (to the right and downward) or along the negative x directions. This allows us to use a single symbol, a_x, for the x component of the acceleration of either object.

Figure WG8.6



We are now prepared to analyze this problem using forces, generating equations of motion for each free-body diagram. Hooke's law (Eq. 8.20) is also involved because it allows us to substitute kd for the unknown force magnitude F^c_{sc}. Reciprocity of forces allows us to write $|\vec{F}^c_{\text{cr}}| = |\vec{F}^c_{\text{rc}}| = |\vec{F}^c_{\text{rb}}| = |\vec{F}^c_{\text{br}}| = \mathcal{T}$, where $\mathcal{T}$ is the tension in the rope (a scalar). Note that the middle equality, $|\vec{F}^c_{\text{rc}}| = |\vec{F}^c_{\text{rb}}|$, is due to the ability of a rope to transmit force undiminished from one end to the other and so depends on the rope having a negligibly small inertia.

The free-body diagram for the block allows us to relate the tension (magnitude of the tensile force) in the rope to the known variables m_{b} and g and to the unknown acceleration $\vec{a}$ or, equivalently, its x component a_x.

3 EXECUTE PLAN The sum of the x components of all forces exerted on the cart is

$$\sum F_{cx} = F^c_{\text{sc}x} + F^c_{\text{rc}x} = m_c a_x.$$

Because we don't know the direction of the acceleration, we choose not to substitute a sign and magnitude for that quantity. However, we can substitute for the force components, using Figure WG8.6 as a guide:

$$-F^c_{sc} + F^c_{rc} = m_c a_x$$

$$-kd + \mathcal{T} = m_c a_x. \quad (1)$$

Repeating the process using Figure WG8.6 for the block, we obtain

$$\sum F_{bx} = F^G_{Ebx} + F^c_{rbx} = m_b a_x$$

$$+F^G_{Eb} - F^c_{rb} = +m_b a_x$$

$$+m_b g - \mathcal{T} = m_b a_x. \quad (2)$$

We combine Eqs. 1 and 2 to eliminate the unknown tension.

$$+m_b a_x + m_c a_x = +m_b g - \mathcal{T} - kd + \mathcal{T}$$

$$(m_b + m_c)a_x = m_b g - kd$$

$$a_x = \frac{m_b g - kd}{m_b + m_c}. \checkmark$$

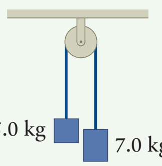
This expression for the x component of acceleration can be either positive or negative, which means that the acceleration of the cart may be either to the right ($a_x > 0$) or to the left ($a_x < 0$).

❸ **EVALUATE RESULT** Our expression for a_x tells us that the acceleration of the cart can be to the left or to the right, depending on the size of kd relative to the magnitude $m_b g$. If either the stretch distance d or the spring constant k is small enough to make $kd < m_b g$, our expression gives a positive value for a_x and the cart accelerates in the positive x direction. If either d or k is large enough to make $kd > m_b g$, we get a negative value for a_x and the cart accelerates in the negative x direction. The acceleration becomes very small if the inertia of the cart is very large, as expected.

Guided Problem 8.6 Atwood machine

A simple device for fully or partially counterbalancing objects, the Atwood machine, consists of two blocks connected by a cable draped across a stationary pulley that rotates freely. The pulley serves only to change the direction of the cable without increasing or decreasing the tension in it. For the simple Atwood machine depicted in Figure WG8.7, what are (*a*) the magnitude of acceleration of each block and (*b*) the tension in the cable after the blocks are released? Assume the blocks are released from rest, and ignore any friction in the pulley.

Figure WG8.7



❶ **GETTING STARTED**

1. Draw a free-body diagram for each block.
2. In which direction should you draw the acceleration vector in each diagram? What is an appropriate direction for the x axis in each diagram?
3. Are the accelerations of the two blocks related to each other?

❷ **DEVISE PLAN**

4. How many equations are needed to solve this problem?
5. For each free-body diagram you can write one equation, paying attention to the signs for each term.
6. Can you use Newton's third law to eliminate any unknowns?

❸ **EXECUTE PLAN**

7. Isolate the unknowns, and obtain expressions for the acceleration and tension.
8. Substitute the values given, and compute numerical values for the unknowns.

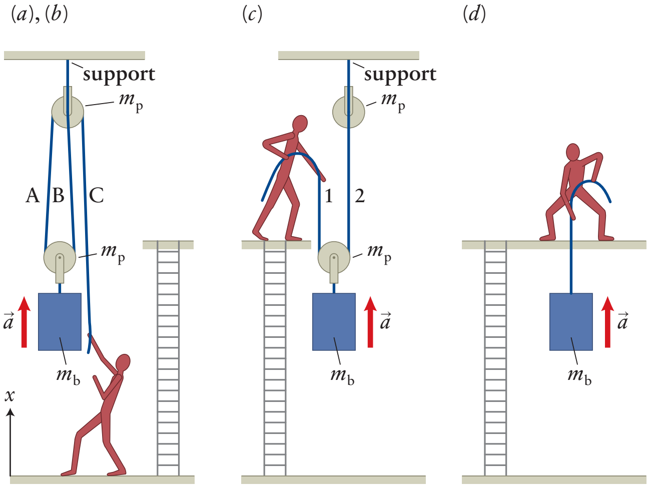
❹ **EVALUATE RESULT**

9. Do all magnitudes come out positive? If not, determine the source of your error.
10. Examine your expression for a. If the inertia of either block is increased or decreased, what happens to the value of a? Is this reasonable?
11. Examine your expression for $\mathcal{T}$. If the inertia of either block is increased or decreased, what happens to the value of $\mathcal{T}$? Is this reasonable?
12. Compare the numerical values you obtained to known values, such as the acceleration in free fall and the gravitational force on each block. Are the results reasonable?

Worked Problem 8.7 Pulley power

The block-and-tackle system in Figures WG8.8*a* and WG8.8*b* is composed of two pulleys connected by a looped rope, with the upper pulley suspended from a fixed support attached to the ceiling. Starting at the point where the person holds the rope, the rope goes over the upper pulley (rope C), goes under the lower pulley (rope A), is attached to the upper pulley (rope B), which is linked to the ceiling by the support. The object to be lifted from the ground to the platform is hung from the lower pulley. If the pulleys have negligible inertia and good bearings, no energy is lost to turning the pulleys or to friction. If the inertia of the rope is also negligible, the tension is the same everywhere along the rope.

Figure WG8.8



Assume that a block of inertia m_b is the object being lifted and that each pulley has negligible inertia, $m_p \ll m_b$. (*a*) With what force magnitude must a person pull on rope segment C to give the block an upward acceleration of magnitude *a*? (*b*) As the block moves, what is the tension $\mathcal{T}_{support}$ in the support that attaches the upper pulley to the ceiling? (*c*) Suppose the free end of the rope falls off the upper pulley and the person cannot reach the ceiling to rewrap the rope. He can climb to the platform and use the lower pulley and the rope (with the fixed end of the rope still attached to the ceiling through the support). He positions himself above the block and lower pulley (Figure WG8.8*c*) and lifts the combination by pulling straight up on the free end of the rope. If he wants to give the block an upward acceleration of magnitude *a*, how does the magnitude of the force he must exert compare with the force magnitude determined in part *a*? (*d*) Now suppose he discards the pulleys, ties the rope around the block (Figure WG8.8*d*), and lifts it with the same acceleration $\vec{a}$ by pulling upward on the rope. Compare the force he must exert in this case with your answers to parts *a* and *c*.

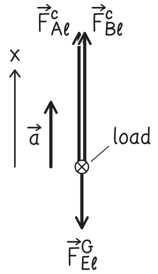
❶ GETTING STARTED (*a*) The lower pulley and the block are raised (as a single unit) by rope segments A and B. The problem statement allows us to assume that the tension $\mathcal{T}$ in the rope is transmitted undiminished throughout the rope, so that this tension equals the magnitude of the pulling force exerted by the worker on segment C, which is what we must calculate in part *a*. Let's call this tension in the three rope segments $\mathcal{T}_{segment}$ to distinguish it from $\mathcal{T}_{support}$, the tension in the ceiling support we need in part *b*.

(*b*) To determine the tension $\mathcal{T}_{support}$ in the support attaching the system to the ceiling, we use the fact that in an object under tension, the opposing forces creating the tension have equal magnitudes. The two tensile forces here are $\vec{F}^c_{cs}$, the upward force exerted by the ceiling on the support, and $\vec{F}^c_{ps}$, the downward force exerted by the upper pulley on the support. If we know either force magnitude, we know $\mathcal{T}_{support}$. Because we know nothing about the force exerted by the ceiling, we'll have to work with the force exerted by the pulley on the support.

(*c*), (*d*) The free-body diagram for the lower pulley-block system used with Newton's second law gives us what we need to answer part *c*. A similar approach should apply to the block alone in part *d*.

❷ DEVISE PLAN (*a*) Let us call the combination of the lower pulley and block our *load* and say that it has an upward acceleration of magnitude *a*. Rope segments A and B exert on the load upward forces of magnitudes $F^c_{A\ell}$ and $F^c_{B\ell}$, and Earth exerts a gravitational force of magnitude $m_\ell g = (m_p + m_b)g$. We need to draw the free-body diagram for the load and then apply Newton's second law to get the magnitude of the downward pulling force the worker exerts on segment C. This magnitude equals the common tension $\mathcal{T}_{segment}$ in the rope segments: $\mathcal{T}_{segment} = F^c_{wC}$. The free-body diagram for the load is shown in Figure WG8.9.

Figure WG8.9



(*b*) We need to determine the force $\vec{F}^c_{ps}$ exerted by the upper pulley on the support because the magnitude of this force equals the tension $\mathcal{T}_{support}$ in the support. From the reciprocity of forces we know that $\vec{F}^c_{ps}$ has the same magnitude as the force $\vec{F}^c_{sp}$ exerted by the support on the pulley. So, if we calculate the magnitude F^c_{sp}, we have $\mathcal{T}_{support}$. The support exerts an upward force of magnitude F^c_{sp} on the upper pulley. The three rope segments exert on the upper pulley downward forces of magnitude $F^c_{Ap} = F^c_{Bp} = F^c_{Cp} = \mathcal{T}_{segment}$. Because this pulley does not rise or fall, its acceleration in the vertical direction is zero, which is information we'll need when we work with the second-law equation.

(*c*) The approach we used in part *a* should work for this part also, but now we need concern ourselves only with the lower pulley plus block. To avoid confusion, we label the two rope segments 1 and 2 for this case (Figure WG8.8*c*). The worker exerts on segment 1 an upward force of magnitude F^c_{w1}, and this magnitude is again equal to the tension in either rope segment.

(*d*) The same approach should apply, this time focusing on only the block. Here we use F^c_{wr} for the magnitude of the force exerted by the worker on the (only) rope.

❸ **EXECUTE PLAN** (*a*) Summing the *x* components of forces exerted on the load gives us

$$\sum F_{\ell x} = m_\ell a_x$$

$$F^c_{A\ell x} + F^c_{B\ell x} + F^G_{E\ell x} = (m_p + m_b)(+a). \quad (1)$$

Because the tensile forces in the three rope segments all have the same magnitude and because this magnitude is equal to the magnitude F^c_{wC} of the pulling force the worker must exert on rope segment C, we have

$$|F^c_{wC\,x}| = |F^c_{A\ell x}| = |F^c_{B\ell x}|.$$

Using Figure WG8.9 as a guide, we can substitute F^c_{wC} for the first two terms in Eq. 1, along with a value for the gravitational force component:

$$F^c_{wC} + F^c_{wC} - m_\ell g = m_\ell a$$

$$2F^c_{wC} = m_\ell a + m_\ell g = (m_\ell)(a + g)$$

$$F^c_{wC} = \tfrac{1}{2}(m_p + m_b)(a + g). \checkmark \quad (2)$$

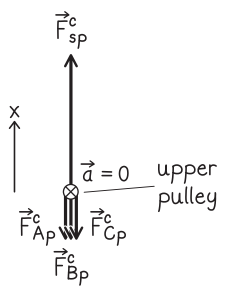
Remember, this force magnitude F^c_{wC} is equal to the tension $\mathcal{T}_{segment}$ in the rope segments. We'll need this information in our solution for part *b*.

(*b*) To get the tension in the support, our plan is to calculate F^c_{sp}, the magnitude of the force exerted by the support on the upper pulley, because this magnitude is equal to $\mathcal{T}_{support}$. The free-body diagram for the upper pulley is shown in Figure WG8.10. Summing force components gives

$$\sum F_{px} = m_p a_{p\,x}$$

$$F^c_{spx} + F^c_{Apx} + F^c_{Bpx} + F^c_{Cpx} + F^G_{Epx} = m_p(0).$$

Figure WG8.10



PRACTICE

As noted when we devised our plan for part b, $F^c_{Ap} = F^c_{Bp} = F^c_{Cp} = \mathcal{T}_{segment}$, where $\mathcal{T}_{segment}$ is the tension in any rope segment. Substituting $\mathcal{T}_{segment}$ for the terms that represent forces exerted by the three rope segments and taking into account the direction in which these forces are exerted give us

$$F^c_{sp} - \mathcal{T}_{segment} - \mathcal{T}_{segment} - \mathcal{T}_{segment} - m_p g = 0$$

$$F^c_{sp} = 3\mathcal{T}_{segment} + m_p g.$$

Because we know from part a that $\mathcal{T}_{segment} = F^c_{wC} = \frac{1}{2}(m_p + m_b)(a + g)$, we can write

$$F^c_{sp} = \mathcal{T}_{support} = 3\left[\tfrac{1}{2}(m_p + m_b)(a + g)\right] + m_p g$$

$$\mathcal{T}_{support} = \tfrac{3}{2}(m_p + m_b)(a + g) + m_p g. \checkmark \tag{3}$$

(c) The physical situation and free-body diagram for the load are illustrated in Figure WG8.11: Rope segments 1 and 2 pull upward on the load with a common tension equal to the force magnitude F^c_{w1} exerted by the worker on rope segment 1: $F^c_{w1} = |F^c_{1\ell x}| = |F^c_{2\ell x}|$. The free-body diagram is just like the diagram for the load in part a, and so the mathematical solution is the same here:

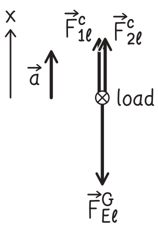
$$\Sigma F_{\ell x} = m_\ell a_x$$

$$F^c_{1\ell x} + F^c_{2\ell x} + F^G_{E\ell x} = m_\ell(+a)$$

$$F^c_{w1} + F^c_{w1} - m_\ell g = m_\ell a$$

$$F^c_{w1} = \tfrac{1}{2}(m_p + m_b)(a + g). \checkmark$$

Figure WG8.11



With only one pulley, the person must pull upward with the same magnitude of force he exerts downward on the rope in part a.

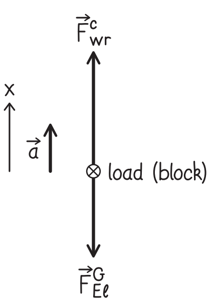
(d) When both pulleys are removed and the person pulls straight up on the load (which now consists of only the block), the free-body diagram has only two forces as shown in Figure WG8.12. The required force magnitude F^c_{wr} is

$$\Sigma F_{\ell x} = F^c_{r\ell x} + F^G_{E\ell x} = m_b a_x$$

$$F^c_{wr} - m_b g = m_b(+a)$$

$$F^c_{wr} = m_b(a + g).$$

Figure WG8.12



PRACTICE

We see that if $m_p \ll m_b$, we have

$$F^c_{wC} = F^c_{w1} = \tfrac{1}{2}F^c_{wr} = \tfrac{1}{2}m_b(a + g). \tag{4}$$

Twice as much force is needed to lift the block without using pulleys. ✔

❶ **EVALUATE RESULT** Equation 4 shows the mechanical advantage of the block and tackle: This simple system of a rope and two pulleys allows you to raise the load with approximately half the effort of a direct lift! Equation 4 also reveals that the advantage of using two pulleys, with one fixed to the ceiling, must be in something other than the required force. The advantage is one of direction: You should find it easier to pull down (rather than up) to lift a heavy object.

The $+a$ term in Eq. 2 implies that you have to pull harder if you wish the load to have a greater acceleration upward, which is exactly what we expect. Equation 3 implies that the more inertia we have to lift against gravity, the larger the support force has to be, which is reasonable.

We did ignore any effort to turn the pulleys, a topic we will return to in Chapter 11. Real pulleys are often of negligible inertia and nearly frictionless, so the conditions of this problem are reasonable. We also approximated the three rope segments to be vertical even though Figure WG8.8*a* shows segments B and C as slightly angled. Because they are nearly vertical, our answer should be fairly accurate.

Guided Problem 8.8 Support force revisited

Redo part *b* of Worked Problem 8.7. This time, though, calculate $\mathcal{T}_{\text{support}}$ by using Eq. 8.44 in the form $\sum F_{\text{ext}\,x} = ma_{\text{cm}\,x}$, where the system comprises the two pulleys and the block.

❶ **GETTING STARTED**

1. Sketch a system diagram and list the inertias and accelerations of the individual parts of the system.
2. What external forces are exerted on the objects in the system? (Hint: Why is it hard to raise the block?)
3. Are the forces exerted by rope segments A, B, and C internal or external forces? In other words, do they represent interactions between objects in the system or interactions between the system and objects external to the system?

❷ **DEVISE PLAN**

4. Draw a free-body diagram with an appropriate x axis for your system.
5. In the second-law equation, $\sum F_{\text{ext}\,x} = ma_{\text{cm}\,x}$, what is m?
6. What is the acceleration of the system's center of mass, given that two of the objects in the system are accelerating and one is not?
7. Write the sum of the external forces exerted on the system, based on your free-body diagram, and determine the sign and magnitude of each force.

❸ **EXECUTE PLAN**

8. Solve for the tension in the support $\mathcal{T}_{\text{support}}$.

❹ **EVALUATE RESULT**

9. Does your expression agree with the one obtained in part *b* of Worked Problem 8.7? If it does not, you might not have correctly identified all the external forces exerted by Earth (an external object) on the system components.

Answers to Review Questions

1. No. An opposing force of equal magnitude might also be exerted on the object. For example, a book at rest on a table experiences two forces, the downward gravitational force exerted by Earth and the upward contact force exerted by the table, and yet no change in motion occurs. Also, a force exerted on an object can decrease the object's speed, as when a crate sliding along the floor comes to rest as a result of the frictional force exerted on it.
2. The fact that the velocity is constant tells you that the vector sum of forces is zero. No direction need be specified for a zero vector.
3. The vector sum of all forces exerted on an object equals the time rate of change in the object's momentum.
4. No. Forces always form interaction pairs. Whenever object 1 exerts a force on object 2, object 2 exerts an equal and opposite force on object 1.
5. No. The forces in an interactive pair are exerted on the two objects of the pair. The forces asked about here are $F_{\text{by cup on table}}$ and $F_{\text{by coffee on cup}}$. Because these forces do not involve the same two objects, they do not form an interaction pair. (Each force is one member of an interaction pair, though, and you should be able to name the other member of each pair.)
6. A contact force is one that arises when two objects physically touch each other (which, as seen in Chapter 7, means the surfaces of the two objects must approach within a few atomic diameters of each other in order to create an interaction). A field force is one due to an interaction between two objects that occurs even when the objects are not touching each other. Contact forces require physical contact, while field forces do not.

7. (*a*) Contact (a frictional force); (*b*) field (the gravitational force exerted by Earth); (*c*) field (the magnetic force); (*d*) contact (air touching the sails); (*e*) contact (the pulling force exerted by the rubber band)
8. Yes. The only requirement for translational equilibrium is that the vector sum of the forces being exerted on the object is zero.
9. Watch for changes in its momentum (and use an inertial reference frame to make your observations). An object in translational equilibrium experiences no change in momentum.
10. First, a dot to represent the object that is the focus of the diagram. Second, a vector arrow, with its tail on the dot, for each force exerted on the object. Each arrow is labeled with an **F** carrying a superscript that represents the type of force and a subscript that represents first the object *by* which the force is exerted and then the object *on* which the force is exerted. And third, a separate vector arrow indicating the direction of the acceleration of the object that is the focus of the diagram.
11. Drawing a free-body diagram allows you to separate and focus on a particular object in a system of interacting objects and to account for and keep track of the forces exerted on that object. The goal is to ensure that in analyzing the problem you do not overlook any forces that *are* exerted on the object or include any forces that are *not* exerted on the object.
12. Because all the forces included in the vector sum should already be shown on the diagram. Thus adding an arrow representing the vector sum might lead to double counting of forces.
13. (*a*) As the magnitude of the force exerted on the spring increases, the spring length changes in proportion to the magnitude of the force. If the force is compressing the spring, the length decreases. If the force is stretching the spring, the length increases. (*b*) The magnitude of the force exerted by the spring on the object is identical to that exerted by the object on the spring, so the change in length is still proportional to the magnitude of the force exerted by the spring on the object.
14. An elastic force is a force exerted by an object that is reversibly deformed (either stretched or compressed).
15. The force is undiminished when the gravitational force on the rope, spring, or thread is much less than the pulling force being exerted and much less than the gravitational force on the block—in other words, when the rope, spring, or thread has negligible inertia.
16. Acceleration. There is in general no direct relationship between the vector sum of forces and any other aspect of motion, such as velocity or position.
17. The inertia of the object must be constant.
18. First law: In an inertial reference frame, an isolated object (one on which no forces are exerted) has a constant velocity. Second law: The time rate of change of an object's momentum equals the vector sum of the forces exerted on the object. Third law: When two objects A and B interact, the force exerted by A on B is equal in magnitude to the force exerted by B on A, but the two forces point in opposite directions.
19. (*a*) The force of gravity exerted on an object is proportional to the object's inertia. Because the sack of apples has larger inertia than a single apple, the force of gravity exerted on the sack is larger. (*b*) The free-fall acceleration (g) is the same for both the sack of apples and a single apple because the acceleration of each object is the vector sum of the forces exerted on the object divided by the object's inertia. Thus the effect of larger inertia cancels.
20. No. Equation 8.17 was derived for an object in free fall, but it holds for any object near Earth's surface, even one at rest. Objects subject to multiple forces may have accelerations that differ from g, but they still experience the same gravitational force mg.
21. For a given spring, the numerical value of k is the same for both stretching and compression.
22. The symbol represents the position of the free end when the spring is relaxed (neither stretched nor compressed).
23. The slope represents $1/k$, the inverse of the spring constant k.
24. The impulse delivered to an object equals the product of the vector sum of the forces exerted on the object and the time interval over which the forces are exerted.
25. It is necessary to take into account the time dependence of the variable force (or forces). This can be done by direct integration, as in Eq. 8.26, or by determining the average value for each variable force and including that value in Eq. 8.25.
26. An internal force is exerted between two objects that are part of the chosen system. An external force is exerted by an object external to the system on an object that is part of the system.
27. The center of mass accelerates as if both objects were located at the system's center of mass and the external force were applied at this location.
28. The vector sum is zero.
29. All parts of a rigid object experience the same acceleration, but for a deformable object different parts of the object have different velocities and different accelerations. For a rigid object, the acceleration of each part is the same as the acceleration of the center of mass, but the same is not true for a deformable object.

Answers to Guided Problems

Guided Problem 8.2 5.2×10^4 N/m
Guided Problem 8.4 37 mm
Guided Problem 8.6 (*a*) $a = g/6 = 1.6\ \text{m/s}^2$; (*b*) $\mathcal{T} = 57$ N
Guided Problem 8.8 $\mathcal{T}_{\text{support}} = \frac{3}{2}(m_p + m_b)(a + g) + m_p g$ reduces to $\mathcal{T}_{\text{support}} = \frac{3}{2}m_b(a + g)$ for $m_p = 0$

Guided Practice by Chapter

9 Work

Review Questions

Answers to these questions can be found at the end of this chapter.

9.1 Force displacement

1. What is the meaning of the term *work*?
2. Suppose you exert an external force on a system. Is work necessarily done on the system?

9.2 Positive and negative work

3. (*a*) A headwind pushes on a bicycle coasting eastward, slowing the bicycle down. What is the sign of the work done by the wind on the bicycle? (*b*) If the same wind pushes in the same direction on a bicycle initially at rest and moves the bicycle westward, what is the sign of the work done by the wind on the bicycle?
4. You throw a ball up into the air and then catch it. How much work is done by gravity on the ball while it is in the air?
5. Can the work done by a spring ever be negative?

9.3 Energy diagrams

6. In a system in which all the energy is mechanical, the initial and final energies are $K_i = 10.0$ J, $K_f = 3.0$ J, $U_i = +4.0$ J, $U_f = +6.0$ J. Sketch an energy diagram for the system.
7. A cart is at rest but free to move on a low-friction track. When you push and release the cart, it travels at constant speed to the right until it strikes a spring fixed to the end of the track. After a short time interval in contact with the spring, the cart moves back toward you at the same constant speed it had as it approached the spring. Sketch energy diagrams for the cart during these time intervals: (*a*) from the initial resting state until traveling at constant speed to the right, (*b*) from traveling at constant speed to the right until the spring is fully compressed, (*c*) from spring being fully compressed until traveling to the left at constant speed, (*d*) from the initial resting state until traveling to the left at constant speed.

9.4 Choice of system

8. Why should you avoid choosing a system that has a frictional force exerted across the system boundary?
9. When computing energies, should the gravitational interaction be associated with work, with potential energy, or with both?
10. What is the utility of analyzing an energy problem using more than one choice of system?

9.5 Work done on a single particle

11. State the relationship between work and energy in words.
12. Does the energy law (Eq. 9.1) apply to closed systems? To systems that are not closed?
13. Equations 9.8 and 9.9 are valid only for a particle. Why?
14. Discuss the similarities and differences in the momentum law (Eq. 4.18, $\Delta\vec{p} = \vec{J}$) and the energy law (Eq. 9.1, $\Delta E = W$).
15. In what sense is a crate a particle? Under what circumstances would it be inappropriate to treat a crate as a particle in working a physics problem?

9.6 Work done on a many-particle system

16. How does the energy law (Eq. 9.1) applied to a single particle differ from the energy law applied to a many-particle system? How does the work equation (Eq. 9.9) applied to a single particle differ from the work equation applied to a many-particle system?
17. Why is Eq. 9.18 restricted to nondissipative forces?
18. How does the change in kinetic energy of a many-particle system differ from the work done on the system by its environment?

9.7 Variable and distributed forces

19. When you plot the force exerted on a particle as a function of the particle's position, what feature of the graph represents the work done on the particle?
20. (*a*) In $W_{bs} = \frac{1}{2}k(x - x_0)^2$, the expression obtained in Example 9.8 for the work done by a brick in compressing or stretching a spring, where does the factor $\frac{1}{2}$ come from? (*b*) If the force exerted by a load on a spring is $F_{\ell s x} = k(x - x_0)$ (Eq. 8.18) and the force displacement is $x - x_0$, why isn't W_{bs} just $F(x - x_0) = k(x - x_0)\,(x - x_0) = k(x - x_0)^2$?
21. You use Eq. 9.27 or Eq. 9.28 to calculate the thermal energy dissipated by friction as a crate slides across the floor and eventually comes to rest. Could this thermal energy be called the work done on the crate by the frictional force?

9.8 Power

22. How is instantaneous power related to average power?
23. When is the instantaneous power delivered to an object equal to the magnitude of the vector sum of the forces $\sum F_{ext}$ exerted on the object multiplied by the object's speed?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The minimum work required to lift a large bag of groceries from floor to tabletop (D, J)
2. The energy dissipated when you stand on the top of an empty soda can and flatten the can (B, L, T)
3. The minimum work the engine must do to bring a fully loaded, 18-wheel truck from rest to freeway speed (A, K, Q)
4. The spring constant of a bungee cord (E, H, N, V)
5. The minimum power required to pitch a fastball like a professional baseball player (C, I, S)
6. The power provided to a car by the engine while cruising at freeway speed (F, M, U, Y)
7. The average electrical power used in your home during a typical day (R, G, W)
8. The electrical energy used in a typical U.S. home during a year (R, G, W, O, X)
9. The amount of chemical (source) energy needed to supply electrical power for residential consumption in the United States during one year (R, G, W, O, X, P, Z)

Hints

A. What is the inertia of a loaded 18-wheeler?
B. What is the gravitational force exerted by Earth on you?
C. What is the typical speed of a major league fastball?
D. What is the gravitational force exerted on a large bag of groceries?
E. What is the height of a typical bungee jump?
F. How many kilometers per liter of gasoline do you get cruising at freeway speed?
G. How much power is used during peak consumption hours?
H. What is the unstretched length of a bungee cord?
I. What is the inertia of a baseball?
J. What is the height of a typical table?
K. What is a typical freeway speed?
L. Through what displacement is the force exerted?
M. How much chemical (source) energy is available in 1 L of gasoline?
N. What is the relevant energy change?
O. How many seconds are in 1 year?
P. What is the number of residences in the United States?
Q. Does all the work done by the engine go into changing the truck's kinetic energy?
R. How much power is consumed by a typical appliance?
S. How long a time interval does a pitcher's forward throwing motion require?
T. How does the force you exert on the can relate to the gravitational force exerted by Earth on you?
U. What percentage of the energy in the gasoline is transferred to the drive wheels of the car?
V. What is a typical inertia of a bungee jumper?
W. During what percentage of a full day do you consume electricity at the peak consumption rate?
X. How is the amount of energy consumed related to the average power?
Y. How long a time interval is required to travel 100 km?
Z. What is the efficiency of source energy conversion, including transmission losses?

Key (all values approximate)

A. 3×10^4 kg; **B.** 7×10^2 N; **C.** 4×10^1 m/s; **D.** 1×10^2 N; **E.** 8×10^1 m; **F.** 1×10^1 km/L; **G.** 6×10^3 W; **H.** 3×10^1 m; **I.** 0.2 kg; **J.** 1 m; **K.** 3×10^1 m/s; **L.** 0.1 m; **M.** 4×10^7 J; **N.** gravitational potential energy becomes spring potential energy; **O.** 3×10^7 s; **P.** 90 million; **Q.** we assume it does in order to determine the minimum work; **R.** 100 W for lamps and most small appliances; 1000 W for refrigerators, microwave ovens, irons, toasters, bathroom heaters; 5000 W for central air conditioners; **S.** 0.5 s; **T.** about equal in magnitude; **U.** less than 20%; **V.** 7×10^1 kg; **W.** less than 30%; **X.** $\Delta E = P_{av}\Delta t$; **Y.** 1 h; **Z.** about 10%

PRACTICE

Worked and Guided Problems

Procedure: Drawing energy diagrams

1. Specify the **system** under consideration by listing the components inside the system.
2. **Sketch** the system in its initial and final states. (The initial and final states may be defined for you by the problem, or you may have to choose the most helpful states to examine.) Include in your sketch any external forces exerted on the system that undergo a nonzero force displacement, and draw a dot at the point of application of each force.
3. Determine any nonzero **changes in energy** for each of the four categories of energy, taking into account the four basic energy-conversion processes illustrated in Figure 7.13:

   a. Did the speed of any components of the system change? If so, determine whether the system's kinetic energy increased or decreased and draw a bar representing ΔK for the system. For positive ΔK, the bar extends above the baseline; for negative ΔK, it extends below the baseline. (For some problems you may wish to draw separate ΔK bars for different objects in the system; if so, be sure to specify clearly the system component that corresponds to each bar and verify that the entire system is represented by the sum of the components.)

   b. Did the configuration of the system change in a reversible way? If so, draw a bar representing the change in potential energy ΔU for the system. If necessary, draw separate bars for the changes in different types of potential energy, such as changes in elastic and gravitational potential energy.

   c. Was any source energy consumed? If so, draw a bar showing ΔE_s. Source energy usually decreases, making ΔE_s negative, and so the bar extends below the baseline. Keep in mind that conversion of source energy is always accompanied by generation of thermal energy (Figure 7.13*c* and *d*).

   d. Does friction occur within the system, or is any source energy consumed? If so, draw a bar showing ΔE_{th}. In nearly all cases we consider, the amount of thermal energy increases, and so ΔE_{th} is positive.

4. Determine whether or not any **work** W is done by external forces on the system. Determine whether this work is negative or positive. Draw a bar representing this work, making the length of the bar equal to the sum of the lengths of the other bars in the diagram.

   If no work is done by external forces on the system, leave the bar for work blank, then go back and adjust the lengths of the other bars so that their sum is zero.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 9.1 Trash that

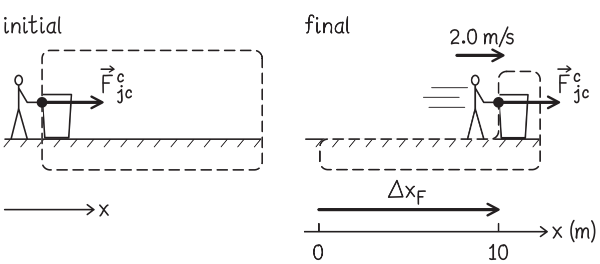
A janitor starts to push a 75-kg trash can that is initially at rest across a level surface by exerting a constant horizontal force of 50 N. After moving 10 m, the trash can has a speed of 2.0 m/s. (*a*) What are the magnitude and direction of the frictional force exerted by the surface on the can? (*b*) What power does the janitor provide at the 10-m mark?

PRACTICE

❶ GETTING STARTED

(*a*) The work done by a force equals the magnitude of the force multiplied by the force displacement. The pushing force and the frictional force are both exerted over a known displacement to change the can's kinetic energy. The can accelerates, so the janitor must provide a force larger than the frictional force. The frictional force opposes the sliding motion, so it is directed in the opposite direction to the janitor's push. We could take the can as our system. However, that would mean the force of friction is exerted at the system boundary (and then we don't know how much of the dissipated energy ends up inside the system). So instead we choose the can and the floor as our system. Figure WG9.1 shows the can and floor system in its initial and final states, the force displacement Δx_F, and the single applied external force (friction is internal!). We need to keep track of components, so we arbitrarily choose the positive x axis along the direction of motion of the can (to the right in Figure WG9.1)

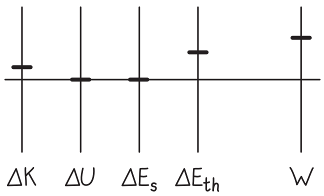
Figure WG9.1



(b) Because we know the constant force exerted by the janitor and the can's speed once it has moved 10 m, we can use the product of force and speed to obtain the power delivered at the 10-m position.

❷ **DEVISE PLAN** The surface does not move and so does not contribute to the system kinetic energy. Because the can is rigid, its contribution to the system kinetic energy is computed using the can's center-of-mass motion. The center of mass of the can experiences exactly the same displacement as any other point on the can, which is the force displacement. There is no potential energy to consider (no vertical motion to involve gravity, no springs) and no source energy in our system. The energy law allows us to account for the changes in kinetic energy, thermal energy, and work. We draw an energy diagram (Figure WG9.2) to summarize this information. We have an expression for each of these terms: The change in kinetic energy can be computed from given information, the frictional force is responsible for the change in thermal energy, and the janitor's force is responsible for the work done.

Figure WG9.2



(a) We know the change in kinetic energy, the displacement, and the magnitude of the force exerted by the janitor. Thus the only unknown is the one we are asked to calculate: The magnitude of the frictional force. (b) Because the only external force that has a force displacement is exerted by the janitor, and that force is constant, the power at any instant is given by Eq. 9.35, $P = F_{\text{ext}\,x}v_x$. We know both the force and the speed at the requested location, so we can obtain the power.

❸ **EXECUTE PLAN**
(a) We begin with the energy law, Eq. 9.1:

$$\Delta E = \Delta K + \Delta U + \Delta E_s + \Delta E_{th} = W.$$

The thermal energy change ΔE_{th} is caused by the frictional force, so we can employ Eq. 9.27 to make the relationship explicit:

$$\Delta E_{th} = -F^{f}_{sc\,x}\Delta x_{cm}.$$

Similarly, we use Eq. 9.9 to express W in terms of the force exerted by the janitor and then substitute both expressions into the energy law:

$$\Delta K + 0 + 0 + (-F^{f}_{sc\,x}\Delta x_{cm}) = F^{c}_{jc\,x}\Delta x. \qquad (1)$$

Because the can is rigid, $\Delta x_{cm} = \Delta x$. Substituting this and our usual expression for kinetic energy into Eq. 1 and rearranging terms gives

$$\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = F^{c}_{jc\,x}\Delta x + F^{f}_{sc\,x}\Delta x$$

$$\tfrac{1}{2}mv_f^2 - 0 = F^{c}_{jc\,x}\Delta x + F^{f}_{sc\,x}\Delta x$$

$$F^{f}_{sc\,x} = \frac{\tfrac{1}{2}mv_f^2}{\Delta x} - F^{c}_{jc\,x} = \frac{\tfrac{1}{2}(75\text{ kg})(2.0\text{ m/s})^2}{10\text{ m}} - (+50\text{ N}) = -35\text{ N}.$$

Because we chose to the right as the positive x direction in Figure WG9.1, the negative value for $F^{f}_{sc\,x}$ tells us that the direction of the frictional force is to the left. The magnitude of the frictional force is 35 N. ✔

(b) The power delivered by the janitor is

$$P = F_{\text{ext}\,x}v_x = (+50\text{ N})(+2.0\text{ m/s}) = 1.0 \times 10^2\text{ W}. ✔$$

❹ **EVALUATE RESULT** The frictional force is directed to the left and has a magnitude smaller than the force exerted by the janitor, as we expected.

We can also check our answer with kinematics because constant forces imply constant acceleration. The acceleration of the trash can is related to the initial and final speeds and the displacement by (see Example 3.4)

$$a_x = \frac{v_{x,f}^2 - v_{x,i}^2}{2\Delta x} = \frac{(2.0\text{ m/s})^2 - 0^2}{2(10\text{ m})} = +0.20\text{ m/s}^2.$$

PRACTICE

If we take just the can as our system, the vector sum of forces exerted on the can must therefore be

$$\sum F_{\text{ext}\,x} = ma_x = (75\text{ kg})(+0.20\text{ m/s}) = +15\text{ N}.$$

Knowing this value allows us to solve for the frictional force:

$$\sum F_{\text{ext}\,x} = F^{\text{c}}_{\text{jc}\,x} + F^{\text{f}}_{\text{sc}\,x} = +50\text{ N} + F^{\text{f}}_{\text{sc}\,x} = +15\text{ N}$$

$$F^{\text{f}}_{\text{sc}\,x} = +15\text{ N} - 50\text{ N} = -35\text{ N}$$

in agreement with our previous result.

The power we obtained is that of a 100-W light bulb. The amount of power required to lift a textbook at a constant 0.33 m/s is about 10 W, so this janitor is working about ten times as hard to accelerate the heavy trash can to 2.0 m/s by the end of the 10-m trip. Doing 1 chin-up per second, however, requires an average power of more than 200 W for a 60-kg individual, so the 100 W we obtained is not unreasonable.

Guided Problem 9.2 Delivering a piano

A moving company decides it's easier to hoist a 150-kg piano up to a second-floor apartment with a pulley rather than negotiate a narrow stairway. The piano is hoisted with a rope to the level of a window 5.3 m above the ground and temporarily brought to rest. How much work is done on the piano (*a*) by the rope and (*b*) by the force of gravity? (*c*) If it takes 1.0 min to hoist the piano, what average power must be supplied by the hoist?

1 GETTING STARTED

1. What is a good choice of system? Should Earth be included in the system? Sketch the system in its initial and final states.
2. Can the piano be treated as a rigid object? How is the motion of its center of mass related to the motion of any other part of it?
3. Draw an energy diagram. Note that the piano is at rest both before and after it is hauled up.
4. Are there external forces exerted on the system? If so, what is the force displacement for each?

2 DEVISE PLAN

5. How is the change of the system's energy related to the work done on the system by external forces? What formula expresses this relationship?
6. Is it reasonable to ignore any friction in the pulley so you can say that no energy is dissipated?
7. Is there enough information to solve for the unknowns?

3 EXECUTE PLAN

8. Calculate the requested results.
9. Does the person running the hoist do work on the piano?

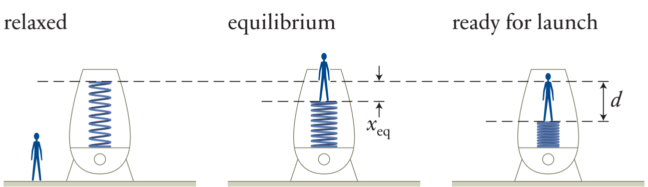
4 EVALUATE RESULT

10. How does your calculated power compare with the power needed to walk up the stairs to the first floor in the same 1-min time interval?
11. Does the power calculated in part *c* seem reasonable for lifting a piano?

Worked Problem 9.3 Human cannonball

A circus performer of inertia m is launched into the air by a "cannon" that contains a spring platform for which the spring constant is k. The performer climbs into the cannon, compressing the spring. Then the spring is additionally compressed (by a winch) so that the initial position of the platform is a distance d below the position of the platform when the spring is relaxed (Figure WG9.3). (*a*) What maximum speed does the performer attain after launch? (*b*) How high above the relaxed position of the spring platform does the performer fly? Ignore any dissipative interactions, and analyze both parts of the problem using two systems: one that allows you to solve the problem using potential energy but not work and one that allows you to solve the problem using work but not potential energy.

Figure WG9.3



1 GETTING STARTED For a performer launched from rest by a compressed spring, we must determine the performer's maximum speed and maximum height. We must treat the performer as a particle or rigid object because there is no information that allows us to do otherwise. The release of the spring causes the performer to shoot straight up, and at the highest point of his trajectory he has zero instantaneous velocity as his direction of travel reverses. Because the force exerted by the spring is not constant, using a force approach to solve this problem would give us nonconstant acceleration, which means we cannot use constant-acceleration formulas to get the maximum height.

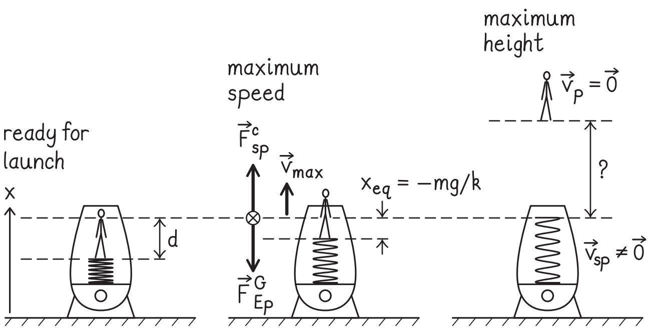
In choosing our systems, we must consider all the forces that might affect the performer's motion. We are told to ignore dissipative forces, and so it appears that the force exerted by the spring and the gravitational force are the crucial ones. This means we have two interaction pairs: spring-performer and Earth-performer. In order to use potential energy for the gravitational and spring forces, we must include both objects from each pair in our system. We can make this system closed and isolated by including the cannon and the ground underneath it. So system 1 is cannon, spring, performer, and Earth.

To solve the problem by analyzing the work done on the performer, we recall that only forces external to a system can do work on the system. Thus we define system 2 as being only the performer, so that the forces exerted on him are all external forces.

We know that the performer's speed is zero both initially and at his greatest height, and we know the formulas for the elastic and gravitational potential energies and for the work done by each force. The problem requires us to think about two different final instants during the motion: In part *b* we have $t_f =$ the instant the performer reaches his greatest height, but in part *a* we have $t_f =$ the instant he has his highest speed. We therefore analyze parts *a* and *b* separately, using the same initial position but a different final position in each case. But where, exactly, does the performer achieve the maximum speed? There are two forces that affect the speed of the performer: the gravitational force and the spring force. The gravitational force is constant and directed downward. The spring force is not constant and is directed upward. In order for the performer to accelerate upward, the spring force must exceed the gravitational force. As long as this is true, the performer gains speed by accelerating upward. The spring force gets smaller as the performer rises and the spring begins to relax. At some point the spring force must become smaller than the gravitational force so that the performer no longer accelerates upward but begins to accelerate downward, slowing his speed. Thus the maximum speed occurs when the spring force equals the gravitational force so that the performer is instantaneously in equilibrium. This is the position labeled "equilibrium" in Figure WG9.3 (though in that figure the performer is not moving).

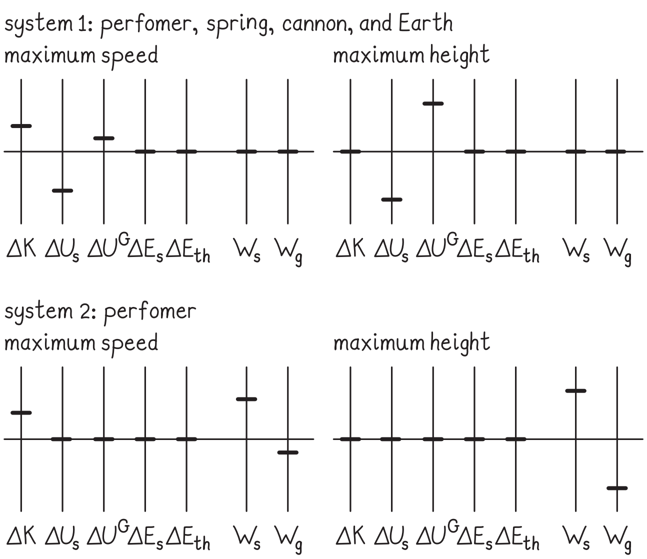
Figure WG9.4 shows a time sequence of events, including the instant of launch, the equilibrium position of maximum speed, and the position of maximum height.

Figure WG9.4



Now we produce an energy diagram for each system, beginning with the performer ready for launch and including the position of maximum speed as well as the position of maximum height (Figure WG9.5). Note that we separate the potential energy into two parts (elastic potential energy U_{sp} in the spring and the performer's gravitational potential energy U^G relative to Earth) and we do the same for work (work W_{sp} done by the spring on the performer and work W_g done by gravity on him).

Figure WG9.5



❷ **DEVISE PLAN** Using Figure WG9.5 as a guide, we write for the change in system 1 energy from launch to maximum speed

$$\Delta E_{\text{sys 1}} = \Delta K + \Delta U + \Delta E_{\text{s}} + \Delta E_{\text{th}} = 0$$
$$\Delta K + \Delta U_{\text{sp}} + \Delta U^{G} + 0 + 0 = 0$$
$$\Delta K + \Delta U_{\text{sp}} + \Delta U^{G} = 0. \quad (1)$$

For the maximum height computation, both the initial and final kinetic energies of the performer are zero. The spring and platform may continue to move up and down after the performer's feet leave the platform. However, any kinetic energy involved in the final motion of the spring and platform can be expressed as remaining spring potential energy because at the turning points of their motion the energy is entirely potential. This gives

$$\Delta E_{\text{sys 1}} = \Delta K + \Delta U + \Delta E_{\text{s}} + \Delta E_{\text{th}} = 0$$
$$\Delta U_{\text{sp}} + \Delta U^{G} = 0. \quad (2)$$

We have enough information to solve these equations.

For system 2, the energy accounting is still guided by Figure WG9.5, but it looks a bit different because now we are not looking at any potential energy changes in the system. For maximum speed we have

$$\Delta E_{\text{sys 2}} = \Delta K + \Delta U + \Delta E_{\text{s}} + \Delta E_{\text{th}} = W$$
$$\Delta K + 0 + 0 + 0 = W$$
$$\Delta K = W_{\text{g}} + W_{\text{sp}}. \quad (3)$$

For maximum height, we do not have to concern ourselves with any energy remaining in the system due to the motion of the spring because the spring and platform are not part of system 2. This makes the accounting a bit easier:

$$\Delta E_{\text{sys 2}} = \Delta K + \Delta U + \Delta E_{\text{s}} + \Delta E_{\text{th}} = W$$
$$0 + 0 + 0 + 0 = W$$
$$W_{\text{sp}} + W^{G} = 0 \quad (4)$$

Because the spring force and the gravitational force are exerted over different force displacements, we track them separately. The gravitational force is exerted during the entire motion, but the spring force is exerted only during the time interval from the last instant at which the spring is in its launch (maximum compression) position to the instant when it reaches equilibrium for Eq. 3, and the position at which performer contact is lost for Eq. 4.

❸ **EXECUTE PLAN**

(*a*) For either system 1 or system 2 maximum speed, the initial position is $x_{\text{i}} = -d$, and the final position is found by imposing the equilibrium condition (that the spring force and gravitational force cancel). The free-body diagram (see Figure WG9.4) can be used to solve for the final position. With the positive x axis directed upward and the origin at the relaxed position, both $\vec{x}_{\text{f}}$ and $\vec{g}$ have negative x components:

$$\sum \vec{F} = \vec{0}$$
$$\vec{F}^{\text{c}}_{\text{sp}} + \vec{F}^{G}_{\text{Ep}} = \vec{0}$$
$$-k(\vec{x}_{\text{f}}) + m(\vec{g}) = \vec{0}$$
$$-kx_{\text{f}} - mg = 0$$
$$x_{\text{f}} = x_{\text{eq}} = -\frac{mg}{k}.$$

The initial speed is zero, so for system 1 we can solve Eq. 1 for the final speed (maximum speed):

$$\Delta K + \Delta U_{sp} + \Delta U^G = 0$$

$$\tfrac{1}{2}mv_{max}^2 - \tfrac{1}{2}mv_i^2 + \tfrac{1}{2}kx_{eq}^2 - \tfrac{1}{2}kx_i^2 + mg(x_{eq}) - mg(x_i) = 0$$

$$\tfrac{1}{2}mv_{max}^2 - 0 + \tfrac{1}{2}k\left[\left(\frac{-mg}{k}\right)^2 - (-d)^2\right] + mg\left[\left(\frac{-mg}{k}\right) - (-d)\right] = 0$$

$$v_{max}^2 = \frac{k}{m}\left[d^2 - \left(\frac{m^2g^2}{k^2}\right)\right] - 2g\left(d - \frac{mg}{k}\right).$$

This expression can be simplified algebraically by combining the second and fourth terms on the right-hand side and then noting that the expression can be factored:

$$v_{max}^2 = \frac{k}{m}\left(d^2 - \frac{2mgd}{k} + \frac{m^2g^2}{k^2}\right)$$

$$v_{max}^2 = \frac{k}{m}\left(d - \frac{mg}{k}\right)^2$$

$$v_{max} = \sqrt{\frac{k}{m}}\left(d - \frac{mg}{k}\right). ✔$$

We now compute the maximum speed for system 2, using Eq. 3:

$$\Delta K = W_g + W_{sp}$$

$$\tfrac{1}{2}mv_{max}^2 - 0 = -mg\left(d - \frac{mg}{k}\right) + \tfrac{1}{2}k\left(d^2 - \frac{m^2g^2}{k^2}\right)$$

$$v_{max}^2 = -2g\left(d - \frac{mg}{k}\right) + \frac{k}{m}\left(d^2 - \frac{m^2g^2}{k^2}\right)$$

$$v_{max} = \sqrt{\frac{k}{m}}\left(d - \frac{mg}{k}\right). ✔$$

(*b*) For the maximum height calculation we must first determine the position at which performer contact is lost. The spring and platform exert a force on the performer given by Hooke's law—that is, a force whose magnitude is proportional to the stretch or compression of the spring, and whose direction is opposite to the displacement of the spring from its relaxed position. However, the performer is not attached to the platform, so the force applied to the performer can be in only the upward direction. That means the performer can experience a force from the spring only when the platform is below the position labeled "relaxed" in Figure WG9.3. We designate the final position of the spring and platform $x_{sp,f}$ and the final position of the performer $x_{p,f}$. The initial position of both objects is still the same, $x_i = -d$. Substituting Eq. 9.23 for the spring potential energy and Eq. 7.19, $\Delta U^G = mg\Delta x$, for the gravitational potential energy in Eq. 2, we have

$$\Delta E_{sys\,1} = \Delta U_{sp} + \Delta U^G = 0$$

$$\tfrac{1}{2}kx_{sp,f}^2 - \tfrac{1}{2}kx_{sp,i}^2 + mgx_{p,f} - mgx_{p,i} = 0$$

$$0 - \tfrac{1}{2}k(-d)^2 + mgh - mg(-d) = 0$$

$$mgh = \tfrac{1}{2}kd^2 - mgd$$

$$h = \frac{kd^2}{2mg} - d. ✔$$

PRACTICE

Now we repeat for system 2, noting that the spring force varies with position but the gravitational force does not. Equation 4 gives us

$$W_{sp} + W_g = 0$$

$$\int_{x_{sp,i}}^{x_{sp,f}} F^c_{sp\,x}dx + F^G_{Ep\,x}(x_{p,f} - x_{p,i}) = 0$$

$$+\tfrac{1}{2}k(-d)^2 - mg[+h - (-d)] = 0$$

$$\tfrac{1}{2}kd^2 - mg(h + d) = 0$$

$$h = \frac{kd^2}{2mg} - d. \checkmark$$

❹ EVALUATE RESULT Having obtained the same answers with two systems gives us confidence in the results. As a further check, consider how the height depends on the spring constant: A larger k gives a greater height, as expected. We also expect the performer to rise higher if his inertia is smaller. The only place the performer's inertia appears in our expression for h is in the denominator, which is consistent with our expectation.

Our expression for the maximum speed tells us that d must exceed mg/k in order to get a positive result. This is reasonable because mg/k is the distance by which the spring would compress if the performer were simply standing on the platform; certainly that would not be enough for any launch! Additionally, our expression for h tells us that, to give a sensible result, d must exceed not mg/k but rather $2mg/k$. For small compressions beyond equilibrium, it is reasonable that the performer would not be launched at all—that is, would not lose contact with the platform, but would simply ride up and down with the spring and platform.

Guided Problem 9.4 Thrill seeking

A 50-kg woman goes bungee-jumping off a bridge that is 15 m above the ground. The bungee cords attached to her feet have a relaxed length of 5.5 m, and she is 1.8 m tall. What must the spring constant of the cords be if she is to just miss touching the ground with her head when she jumps?

❶ GETTING STARTED

1. What system is appropriate for this problem? Sketch the initial and final situations, showing such details as the relaxed length of the cords, the woman's height, the bridge height, and an x axis.
2. As usual, several approaches are possible, some easier than others. Try energy methods first.
3. Make an energy diagram and a situation diagram for your system.
4. What assumptions do you make?

❷ DEVISE PLAN

5. Write a general expression for the changes in energy during an appropriate time interval for the jump.
6. Check that the number of unknowns and the number of equations are the same.

❸ EXECUTE PLAN

7. Do the algebra and isolate the spring constant.
8. Substitute your numerical values, including units.

❹ EVALUATE RESULT

9. Examine your expression for the spring constant. What happens to k as the woman's inertia increases? As the bridge height increases?
10. Does the numerical answer make sense? Be sure it is the woman's head that just misses the ground and not her feet!
11. Are all assumptions you made reasonable?

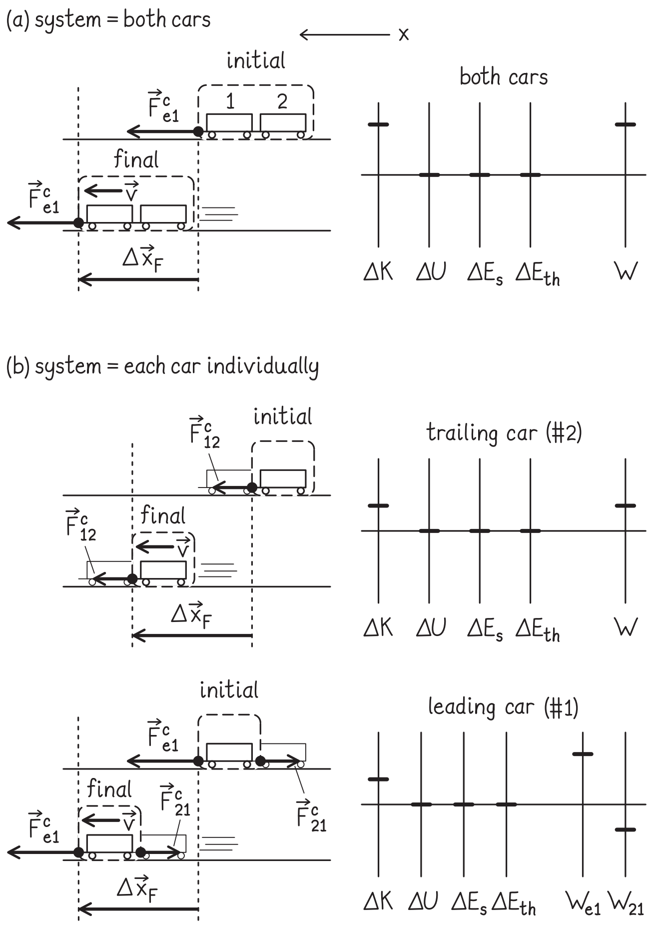
Worked Problem 9.5 Working on the railroad

A train engine pulls a boxcar of inertia 12×10^3 kg, which in turn pulls a boxcar of inertia 17×10^3 kg. The train accelerates from rest to 11.1 m/s in 3.0 min. (*a*) How much work is done by the engine on the two-car combination? (*b*) How much work is done on each car individually? Ignore any friction effects.

❶ GETTING STARTED The work done by a force equals the magnitude of the force multiplied by the force displacement. We need to determine all forces exerted by any external objects on each car, and how far its point of application moves during the common 3.0-min time interval. The only displacement is horizontal, in the direction of motion of the train, which we will call the positive x direction. Thus only horizontal forces need be considered. Because the train cars are connected and move together, each point on each car has the same displacement as the engine. Therefore each applied force has the same force displacement, $\Delta\vec{x}_F$.

For part *a* we need to calculate the work done by the engine on the two-car combination, and so we take the two cars as our system. If we ignore any dissipative forces, there is one external horizontal force on the system, the pulling contact force $\vec{F}^{c}_{e1}$ exerted by the engine on the two-car combination. For part *b* we treat each car separately. Figure WG9.6 contains an initial and final sketch of each system, together with an energy diagram for each system.

Figure WG9.6



For the system consisting of the trailing car (car 2), there is also one external force that does work: the force exerted by the leading car (car 1). We call this force $\vec{F}^{c}_{12}$. For the system consisting of the leading car (car 1), there are two external forces that do work: the force exerted by the engine on car 1 ($\vec{F}^{c}_{e1}$) and the force exerted by car 2 on car 1 ($\vec{F}^{c}_{21}$). All three systems share some properties: No system has an internal source of energy, each system gains kinetic energy during the 3.0-min time interval of interest, and each experiences work done by external forces. Because we ignore any friction effects, we expect no thermal energy changes in any of the systems.

❷ **DEVISE PLAN** For each of the three systems we begin with the energy law, which in each case can be expressed as

$$\Delta E = \Delta K = W.$$

This expression tells us that the work done on each system depends on *only* the change in the velocity of that system. It does not depend on the time interval over which the force is exerted or on the magnitude of the acceleration. We can compute the change in kinetic energy directly in each case.

❸ EXECUTE PLAN Each system has zero initial kinetic energy, and each has a specific final kinetic energy. The difference of these values is the work done by external forces on each system.

(*a*) For the system of two cars, we obtain

$$W = \Delta K = K_f - K_i = \tfrac{1}{2}(m_1 + m_2)v^2 - 0$$

$$W = \tfrac{1}{2}(12 \times 10^3 \text{ kg} + 17 \times 10^3 \text{ kg})(11.1 \text{ m/s})^2$$

$$W = 1.8 \times 10^6 \text{ J.} \checkmark$$

(*b*) Here we must apply the energy law to each system (car) separately.
For the leading car (car 1):

$$W = \Delta K = K_f - K_i = \tfrac{1}{2}(m_1)v^2 - 0$$

$$W = \tfrac{1}{2}(12 \times 10^3 \text{ kg})(11.1 \text{ m/s})^2$$

$$W_1 = 7.4 \times 10^5 \text{ J} \checkmark$$

For the trailing car (car 2):

$$W = \Delta K = K_f - K_i = \tfrac{1}{2}(m_2)v^2 - 0$$

$$W = \tfrac{1}{2}(17 \times 10^3 \text{ kg})(11.1 \text{ m/s})^2$$

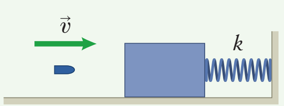
$$W_2 = 1.0 \times 10^6 \text{ J.} \checkmark$$

❹ EVALUATE RESULT These two work values add up (to within rounding error) to the work done on the two-car system by the engine: $W_1 + W_2 = W$. This is reassuring. The only assumption we made is that the force exerted by the engine is the only external force exerted on the two-car system, and that assumption is valid because we were told to ignore friction effects. Based on our knowledge of the energy law, we should have recognized from the beginning that the time interval given was an extraneous piece of information.

Guided Problem 9.6 Fast as a speeding bullet

Because a fired bullet moves so quickly, determining its speed directly is difficult. Here is an indirect way. You fire the bullet (of inertia m_{bullet}) into a block of inertia m_{block} placed against a spring of spring constant k (Figure WG9.7), and the bullet becomes embedded in the block. If the spring compresses to a maximum distance d, what was the speed of the bullet? Ignore friction between the block and the surface on which it rests.

Figure WG9.7



❶ GETTING STARTED

1. How do you expect the compression distance to be related to the speed of the bullet? In other words, what should happen to the compression distance as the speed of the bullet is increased?
2. What can you say about the coefficient of restitution?
3. What kind of energy conversions occur after the collision?
4. Is it reasonable to consider the collision and the compression of the spring separately?

❷ DEVISE PLAN

5. How can you determine the speed of the block with the bullet embedded in it as the spring begins to compress?
6. For what system is the mechanical energy constant from the time immediately after the collision until the spring is fully compressed?

❸ EXECUTE PLAN

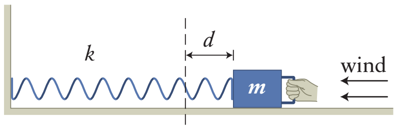
❹ EVALUATE RESULT

7. Does your expression for v_{bullet} show that the speed varies as you expect with changes in d, k, m_{bullet}, and m_{block}?

Worked Problem 9.7 Spring forward

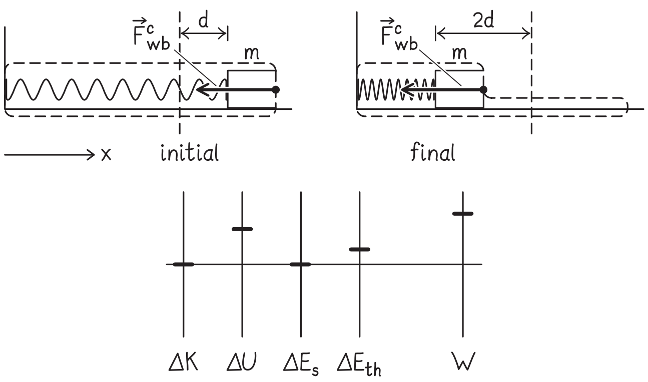
One end of a spring of force constant k is attached to a wall, and the other end is attached to a block of inertia m sitting on a horizontal desk (Figure WG9.8). You pull on the block to stretch the spring a distance d past its relaxed position, and then hold the block at that position. A steady wind exerts on the block a constant force directed to the left, opposite the direction in which you moved the block. When you release the block, it moves to the left, experiencing a frictional force of magnitude F^{f}_{sb} until equilibrium is reached with the spring compressed a distance $2d$ from its relaxed position. What is the magnitude of the force exerted by the wind on the block?

Figure WG9.8



❶ **GETTING STARTED** Because three horizontal forces are exerted on the block (by the spring, friction, and the wind), we might be tempted to solve this as a force problem. We note, however, that the spring exerts a variable force, which means the acceleration is not constant and we cannot use constant-acceleration kinematics. Because we can calculate the potential energy of the spring and the amount of energy dissipated by the friction interaction, this problem might be solved by energy methods. The force of the wind is constant, and we can readily calculate any work done by this force. Choosing as our system the block, spring, and desk makes both the frictional and spring forces internal forces, so the only work done will be by the wind. Also, at maximum compression, the block's kinetic energy is zero, as it is the instant before you release it. Figure WG9.9 illustrates the initial and final situations and shows an energy diagram.

Figure WG9.9



❷ **DEVISE PLAN** We begin with the energy law, $\Delta E = W$ (Eq. 9.1), with W being the work done on the system by the wind force:

$$\Delta E = W$$

$$\Delta K + \Delta U_{spring} + \Delta E_s + \Delta E_{th} = W. \tag{1}$$

Our goal is to determine F^{c}_{wb}, the magnitude of the contact force exerted by the wind on the block, and we know from the definition of work that here $W = F^{c}_{wb}\,\Delta x_F$, where Δx_F is the displacement of the point of application of the force exerted by the wind. Because the wind exerts a force on the right face of the block, Δx_F is equal to the displacement Δx of the block (which, in turn, is equal to the displacement of the end of the spring). Thus, if we know each term on the left in Eq. 1, we can determine F^{c}_{wb}. There is no source energy for this system, and the initial and final speeds of the block are both zero, which makes $\Delta K = 0$. All we need to do is substitute into Eq. 1 the algebraic expressions for the spring's potential energy and the change in the amount of thermal energy dissipated by friction. We choose an x axis along the direction of motion of the block, with the positive direction pointing to the right in Figure WG9.9.

❸ **EXECUTE PLAN** The potential energy of a spring displaced from its relaxed length is given by Eq. 9.23, $\Delta U_{spring}(x) = \frac{1}{2}k(x - x_0)^2$, and the energy dissipated by friction (Eq. 9.28) is the magnitude of the frictional force F^{f}_{db} exerted by the desk on the block multiplied by the distance traveled (which in this case is $|\Delta x|$). With x_0 representing the block's position with the spring relaxed, Eq. 1 becomes

$$0 + \tfrac{1}{2}k(x_f - x_0)^2 - \tfrac{1}{2}k(x_i - x_0)^2 + 0 + |F^{f}_{db\,x}|\,|\Delta x| = F^{c}_{wb\,x}\Delta x$$

$$\tfrac{1}{2}k(x_f - x_0)^2 - \tfrac{1}{2}k(x_i - x_0)^2 + |F^{f}_{db\,x}|\,|x_f - x_i| = F^{c}_{wb\,x}(x_f - x_i).$$

Now we substitute in values from Figure WG9.9:

$$\tfrac{1}{2}k(-2d - 0)^2 - \tfrac{1}{2}k(+d - 0)^2 + |F^{\text{f}}_{\text{db}\,x}|\,|-2d - (+d)|$$

$$= F^{\text{c}}_{\text{wb}\,x}[-2d - (+d)]$$

$$\tfrac{1}{2}k(4d^2) - \tfrac{1}{2}kd^2 + F^{\text{f}}_{\text{db}}(3d) = -F^{\text{c}}_{\text{wb}}(-3d)$$

$$\tfrac{3}{2}kd^2 + 3F^{\text{f}}_{\text{db}}d = 3F^{\text{c}}_{\text{wb}}d$$

$$F^{\text{c}}_{\text{wb}} = \tfrac{1}{2}kd + F^{\text{f}}_{\text{sb}}. \checkmark$$

❹ **EVALUATE RESULT** We expect that as F^{f}_{sb} increases, the wind needs to blow harder in order to achieve the same final spring compression. Indeed, if F^{f}_{db} increases, F^{c}_{wb} increases, as predicted by our equation. As k decreases, less wind is needed to achieve the same compression, which is also reflected in our result.

The variation with d involves two opposing effects. As the distance d the spring is stretched increases, more initial elastic potential energy $(\frac{1}{2})(kd^2)$ is stored in the spring. However, an even greater amount $(\frac{1}{2})k(2d)^2 = 2kd^2$ must be stored in the maximally compressed spring. The larger d is, the greater the difference in spring potential energy that must be supplied by the external work done by the wind force. This is in agreement with our result.

Guided Problem 9.8 Darting about

A dart gun uses a compressed spring ($k = 2000$ N/m) to shoot a 0.035-kg dart horizontally at a target 1.0 m away. The spring is originally compressed 25 mm before the dart is released. (*a*) With what speed does the dart leave the gun? (*b*) If the dart penetrates the target to a distance of 10 mm, what average force is exerted on the dart to stop it?

❶ **GETTING STARTED**

1. How are the two questions related?
2. There are several approaches to solving this problem, some easier than others.
3. Choose an appropriate system and make an energy diagram.
4. Based on your chosen approach, which general equation is appropriate?

❷ **DEVISE PLAN**

5. Are there external forces exerted on your system? Should you worry about gravity in a horizontal flight over a short distance?
6. Check that you have enough equations for your unknowns.
7. How can you use the dart speed to determine the change in the dart's kinetic energy as it comes to a stop in the target?
8. What causes the dart to slow down? What formula relates the change in kinetic energy to the distance of penetration?

❸ **EXECUTE PLAN**

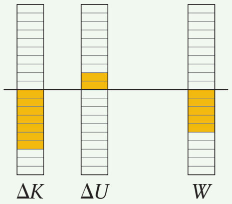
❹ **EVALUATE RESULT**

9. Examine the algebraic expression for the dart's speed. What happens as k gets bigger? Is this what you expect? What about when m (the inertia of the dart) gets bigger?
10. If the frictional force is the same at any depth, does the penetration depth you expect vary properly when making k or m bigger or smaller?
11. Is it reasonable to ignore the effect of gravity on the dart?

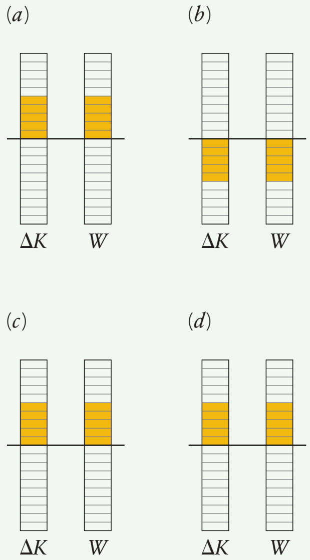
Answers to Review Questions

1. Work is the change in the energy of a system due to external forces.
2. No. In order for work to be done on a system, an external force must be exerted on the system and *the point of application of that force must move*. If your force is exerted on a concrete wall, for example, there may be no observable displacement, which means you do no work on the wall.
3. (*a*) Negative, because the wind force points in one direction (toward the west) and the force displacement (picture it as the point where the wind "touches" the front of the bicycle) is in the opposite direction. (*b*) Positive, because the direction of the force displacement is the same as the direction of the force.
4. None, because the negative work done by gravity on the ball as it rises is canceled by the positive work done by gravity on the ball as it falls back into your hand.
5. Yes, as when you step on a spring scale. As you compress the spring, it exerts an upward force on you. You are moving downward, though, and so the point of application of the force exerted by the spring on you is displaced downward.

6. Mechanical energy is kinetic and potential energy; there must be no changes in thermal or source energy for this system. $\Delta K = K_f - K_i = +3.0\text{ J} - (+10.0\text{ J}) = -7.0\text{ J}$. $\Delta U = U_f - U_i = +6.0\text{ J} - (+4.0\text{ J}) = +2.0\text{ J}$. The sum of these changes must equal the work done, which allows us to draw the work bar on the system's energy diagram:



7. With the cart as the system, both your hand and the spring exert external forces during the motion, and both involve nonzero force displacements. Because the cart returns at the same speed, no thermal energy is dissipated along the track or at the spring. Because your hand is outside the system, there is no source energy to consider, and because the spring is outside the system, there is no potential energy to consider. Thus only kinetic energy and work energy bars must be drawn. No numbers are given, so the resulting diagrams cannot be precisely scaled, but they must look like this:



8. First, the point of application of a frictional force is not well defined but involves contact across many points on two surfaces. This makes it impossible to determine the force displacement. Second, the work done by friction changes the thermal energy of not just the system but also the other object in the interaction pair that is outside the system; you do not generally know how to compute the exact split in this thermal energy, so it is not possible to attribute a specific thermal energy change to the system.
9. Never with both. Which form you use depends on your choice of system. If both parts of the interaction pair (Earth and an object) are included in your system, you must use potential energy rather than work. If only half of the interaction pair is in your system, then you must use work rather than potential energy.
10. Using more than one viewpoint gives us a better understanding of what energy transfers and conversions take place. Also, comparing the different results provides a consistency check.
11. The work done on a system, which is the product of the external force exerted on the system and the displacement of its point of application, is equal to the change in the system's energy.
12. The energy law applies to any system. In the case of a closed system, there are no external forces to do work on the system, so the energy of the system remains constant ($\Delta E = 0$). For a system that is not closed, external forces can do work, so the energy of the system may change by the amount of this work ($\Delta E = W$). In either case the energy law ($\Delta E = W$) applies.
13. In deriving this equation, it was necessary to assume that the only possible change in energy was a change in kinetic energy: $\Delta E = \Delta K$. This is not true in general, but it is true for a particle (which has no internal structure and hence cannot have internal energy changes).
14. Both laws express conservation principles. Neither energy nor momentum can be created or destroyed, but either can be transferred into or out of a system. The momentum law states that any change in the momentum of a system must be due to such a transfer, called impulse. The energy law states that any change in the energy of a system must be due to such a transfer, called work. For an isolated system, no momentum transfer is allowed, while for a closed system, no energy transfer is allowed. The laws are thus similar. The momentum law, though, involves vector values. The equation requires careful attention to vector components of momentum and impulse. The energy law involves scalars; there are no components of energy or work to account for, even though computation of work or energy might require attention to components of vectors during some of the intermediate steps.

PRACTICE

15. As long as the internal energy changes are negligible, we can think of the crate as a particle. If there is substantial internal energy change (in a collision, for example), the crate cannot be treated as a particle.
16. The energy law has the same form, applied in the same way in both cases: $\Delta E = W$. The work equation has different forms in the two cases. For the single particle, there is only one displacement possible, so all external forces exerted on the particle have the same force displacement, and the work equation is the vector sum of forces multiplied by this common displacement: $W = (\sum F_x)\Delta x$. For a many-particle system, different particles may experience different external forces and different force displacements, so the work equation must keep track of each separate force and displacement:

$$W = \sum_n (F_{\text{ext}\,nx} \Delta x_{Fn}).$$

17. Because dissipative forces generally dissipate energy on both sides of the system boundary in such a way that the precise amount of energy transferred into or out of the system is not calculable.
18. Because energy may appear in forms other than kinetic, the work done on a many-particle system is not necessarily equal to the change in the system's kinetic energy. The work done is the sum of the external forces times the displacement of the point of application of each force (Eq. 9.18), and the change in kinetic energy is the sum of the external forces times the displacement of the center of mass of the system (Eq. 9.14).
19. The area under the curve is the magnitude of the work done on the particle.
20. (*a*) The factor $\frac{1}{2}$ comes from the integration formula $\int x^m\, dx = x^{m+1}/(m + 1)$ with $m = 1$. (*b*) Because the force exerted by the spring is a function of how much the spring is stretched or compressed, the same must be true of the force exerted on the spring. Because $F_{\text{on spring}}$ varies, the simple formula $W = F(x - x_0)$ cannot be used. However, the factor $\frac{1}{2}$ in the expression for $W_{\text{on spring}}$ is reasonable in the sense that we expect the average force to be $(\frac{1}{2})k(x - x_0)$, and so $W_{\text{on spring}}$ is $(\frac{1}{2})k(x - x_0)(x - x_0)$.
21. No. The frictional force dissipates energy in both the crate and the floor. Equation 9.28 was derived for a block-surface system, meaning that it contains both the energy change experienced by the block and the energy change experienced by the surface over which the block moves. We do not know how much of this energy to attribute to the block alone, so we cannot in good conscience call this the "work done by the frictional force on the block." We cannot even call it the "work done by the frictional force on the block-surface system" because the frictional force is not external to this system.
22. Instantaneous power is power delivered at any given instant, and average power is power delivered over a time interval Δt. Instantaneous power is average power in the limit that the energy change occurs over a time interval (Δt) that approaches zero. Average power is a ratio of energy change to time interval, while instantaneous power is a derivative of energy with respect to time.
23. When $\sum F_{\text{ext}}$ is constant.

Answers to Guided Problems

Guided Problem 9.2 (*a*) 7.8×10^3 J; (*b*) -7.8×10^3 J; (*c*) 1.3×10^2 W

Guided Problem 9.4 2.5×10^2 N/m

Guided Problem 9.6 $v_{\text{bullet}} = \dfrac{\sqrt{k(m_{\text{block}} + m_{\text{bullet}})}}{m_{\text{bullet}}} d$

Guided Problem 9.8 (*a*) 6.0 m/s; (*b*) 63 N

Guided Practice by Chapter

10 Motion in a Plane

Review Questions

Answers to these questions can be found at the end of this chapter.

10.1 *Straight* is a relative term

1. A plane flying east drops a package (without a parachute) to workers at a logging camp. What does the path of the package look like to a logger standing on the ground? What does the path look like to the pilot? Where is the plane relative to the logger when the package lands at his feet: east of where he stands, west of where he stands, or directly overhead?
2. A passenger in a speeding train drops a peanut. Which is greater: the magnitude of the acceleration of the peanut as measured by the passenger, or the magnitude of the acceleration of the peanut as measured by a person standing next to the track?

10.2 Vectors in a plane

3. Why does describing motion that is not along a straight line require two reference axes?
4. How does adding two vectors that lie in a plane and are neither parallel nor antiparallel to each other differ from adding two vectors that lie along a straight line?
5. Describe how to subtract a vector 1 from a vector 2 when the two do not lie along a straight line.
6. Is vector addition commutative? Is vector subtraction commutative?
7. Describe the procedures for adding and subtracting two vectors.
8. What does it mean when the instantaneous velocity of an object at instant t_1 is not parallel to the average velocity during a short time interval $\Delta t = t_2 - t_1$?
9. Describe the effect of the parallel and perpendicular components of the acceleration of an object on the object's velocity.

10.3 Decomposition of forces

10. What is the meaning of "normal" and "tangential" components of a contact force?
11. Sketch a free-body diagram for a block resting on an inclined surface using (*a*) vector forces and (*b*) normal (y) and tangential (x) force components.
12. When choosing a rectangular coordinate system for an object moving in two dimensions and being accelerated during that motion, why should you choose a system in which one axis lies along the direction of the acceleration?

10.4 Friction

13. Compare and contrast the normal force with the force of static friction and the force of kinetic friction.
14. You push horizontally on a crate at rest on the floor, gently at first and then with increasing force until you cannot push harder. The crate does not move. What happens to the force of static friction between crate and floor during this process?
15. You are trying to push a heavy desk, and it's not budging. A friend remarks, "You're not pushing it hard enough to overcome its inertia." What's wrong with that statement, and what's really going on?

10.5 Work and friction

16. Why is it all right to choose a system for which static friction occurs at the system boundary but a bad idea to do so for kinetic friction?
17. Describe one case in which a force of static friction does no work and one case in which a force of static friction does work. Make your examples different from the ones given in the text.
18. Explain what is wrong with this statement: The force of friction always opposes motion.

10.6 Vector algebra

19. In polar coordinates, can r be negative? If not, why not?
20. Suppose the angle between a vector $\vec{B}$ and the y axis of a rectangular coordinate system is θ. Write an expression for B_x in terms of the magnitudes of B and θ.
21. How are the component vectors $\vec{A}_x$ and $\vec{A}_y$ of vector $\vec{A}$ and the x and y components A_x and A_y of $\vec{A}$ related?

10.7 Projectile motion in two dimensions

22. A baseball player hits a fly ball that has an initial velocity for which the horizontal component is 30 m/s and the vertical component is 40 m/s. What is the speed of the ball at the highest point of its flight?
23. What is the shape of the path of an object launched at an angle to the vertical, assuming that only the force of gravity is exerted on the object?
24. Does the maximum height achieved by a projectile depend on both the x and y components of its launch velocity? Does the horizontal range depend on both components?

10.8 Collisions and momentum in two dimensions

25. How do the principles and equations describing collisions in two dimensions differ from the principles and equations describing collisions in one dimension?
26. Why is it possible to completely determine the outcome of a one-dimensional collision of two objects given only the inertias and initial velocities of each object and the coefficient of restitution, but it is not possible to do this in two dimensions?

10.9 Work as the product of two vectors

27. Under what circumstances could the scalar product of two vectors be zero?
28. Describe how to calculate the work done by a constant nondissipative force as an object moves along a path from position 1 to position 2, and identify which quantities in your statement are scalars and which are vectors.
29. Describe how to compute the work done by a variable nondissipative force as an object moves from position 1 to position 2.

10.10 Coefficients of friction

30. Discuss the difference between contact area and effective contact area, and explain why this distinction is relevant for frictional forces exerted by two surfaces on each other.
31. A fellow student tells you that, according to a measurement he did, the concrete surface of the road has a coefficient of kinetic friction of 0.77. What's wrong with this statement?
32. Often a wedge-shaped doorstop won't hold a door open unless you kick it until it sits tightly under the door. What does forcing the wedge into a tight fit accomplish?
33. In a panic situation, many drivers make the mistake of locking their brakes and skidding to a stop rather than applying the brakes gently. A skidding car often takes longer to stop. Why?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The sum of the displacement vector from New York to Cheyenne, WY and the displacement vector from Cheyenne, WY to Los Angeles (O, D, I)
2. The work done in one day by a moving sidewalk conveyor in a major airport (AA, H, T, W)
3. The maximum tension you can generate in a rope tied between two trees by pushing steadily against the rope midpoint (CC, Z, E, T, P)
4. The stopping distance for a police car in hot pursuit on the freeway under dry conditions (E, J, L, S)
5. The maximum range of a rifle bullet fired horizontally over level ground from a standing position (G, M, Q)
6. The maximum horizontal range of a thrown baseball (C, R)
7. The average force of static friction exerted on a person by a moving sidewalk conveyor in a major airport (AA, H, T, BB, A)
8. The maximum angle of inclination for which your physics book will not slip down a tilted wooden table (U, Y)
9. The maximum distance a hockey puck can slide on smooth horizontal ice (N, B, F, K, V, X)

Hints

A. How is the work done by constant external forces related to the force displacement?
B. What is the inertia of a hockey puck?
C. What is the average speed of a major league fastball?
D. What does the sum of these displacements represent?
E. What is a reasonable estimate of the coefficient of static friction?
F. What is the inertia of a hockey stick plus the inertia of a player's upper body?
G. What is the initial height of the rifle bullet above the ground?
H. What are the speeds of a person walking and a person walking on a moving sidewalk?
I. What is the displacement vector from Cheyenne to Los Angeles?
J. What is a hot-pursuit freeway speed?
K. At what speed is the hockey stick being swung just before it hits the puck?
L. What is the magnitude of the acceleration produced by the vector sum of the forces exerted on the car when the brakes are applied?
M. For how long is the bullet in the air?
N. How could you determine the launch speed of the puck?
O. How do you apply the graphical method to obtain the sum of two vectors?
P. About how far can you displace the center of the rope by pushing?
Q. What is the bullet's speed as it exits the rifle barrel?
R. How should the x and y components of the launch velocity compare with each other?
S. Given the average (constant) acceleration and the car's initial speed, how can you get the car's displacement from the instant the brakes are applied to the instant the car stops?
T. What is the inertia of a person?
U. What is the coefficient of static friction for a book cover on wood?
V. What is the coefficient of kinetic friction between puck and ice?
W. How many people ride a moving sidewalk in a major airport in one day?
X. How does the puck's displacement relate to the other information you know?
Y. How is the magnitude of the force of static friction related to the magnitude of the normal force?
Z. What is the most useful free-body diagram?
AA. Ignoring dissipation, how is work related to kinetic energy change for an object subject to an external force?
BB. What is the length of a moving sidewalk in an airport?
CC. What is a reasonable length for a rope tied between two trees?

Key (all values approximate)

A. Eq. 9.9, $W = \sum F_x \Delta x_F$; **B.** 0.2 kg; **C.** 4×10^1 m/s; **D.** the displacement from New York to Los Angeles; **E.** less than 1.0 for most situations, but use 1.0 as a reasonable estimate; **F.** 2×10^1 kg; **G.** less than 2 m; **H.** walking, 1 m/s, walking on moving sidewalk, 2 m/s; **I.** 2×10^3 km west southwest; **J.** 5×10^1 m/s; **K.** 2×10^1 m/s; **L.** use a free-body diagram to show that $a \approx g$; **M.** 0.6 s, the time interval it takes to fall from the initial height; **N.** treat stick-puck hit as an elastic collision; **O.** place the relevant vectors head to tail on a scale diagram; the vector sum is a vector from the tail of the first vector to the head of the last; **P.** 1 m; **Q.** 5×10^2 m/s; **R.** they should be equal for maximum range; **S.** Eq. 1 in Example 3.4: $\Delta x = (v_{x,f}^2 - v_{x,i}^2)/2a_x$; **T.** 7×10^1 kg; **U.** similar to wood on wood, about 0.5; **V.** 0.1; **W.** 7×10^3; **X.** consider the relationship between the initial kinetic energy of the puck and the energy dissipated by friction; **Y.** Eq. 10.54, $F_{12}^s \leq \mu_s F_{12}^n$; **Z.** free-body diagram of a small piece of rope at its midpoint; **AA.** $W = \Delta K$; **BB.** 4×10^1 m; **CC.** 3×10^1 m because longer ropes cannot be pulled tight enough that it doesn't sag much under its own weight

Worked and Guided Problems

Procedure: Working with frictional forces

1. Draw a free-body diagram for the object of interest. Choose your x axis parallel to the surface and the y axis perpendicular to it, then decompose your forces along these axes. Indicate the acceleration of the object.
2. The equation of motion in the y direction allows you to determine the sum of the y components of the forces in that direction:

$$\sum F_y = ma_y.$$

   Unless the object is accelerating in the normal direction, $a_y = 0$. Substitute the y components of the forces from your free-body diagram. The resulting equation allows you to determine the normal force.
3. The equation of motion in the x direction is

$$\sum F_x = ma_x.$$

   If the object is not accelerating along the surface, $a_x = 0$. Substitute the x components of the forces from your free-body diagram. The resulting equation allows you to determine the frictional force.
4. If the object is not slipping, the normal force and the force of static friction should obey Inequality 10.54.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

PRACTICE

Worked Problem 10.1 Mosquito trail

The position of a mosquito along a curved path is given by

$$\vec{r} = (at^3 - bt)\hat{\imath} + (c - dt^4)\hat{\jmath}, \tag{1}$$

where $a = 20.0$ mm/s^3, $b = 50.0$ mm/s, $c = 60.0$ mm, and $d = 70.0$ mm/s^4. (*a*) Plot the mosquito's trajectory in the time interval from $t = -2.00$ s to $t = +2.00$ s, and interpret your graph. (*b*) Calculate the mosquito's position, velocity, and acceleration at $t = +2.00$ s. (*c*) What is the orientation of the line tangent to the curved path at this instant?

❶ GETTING STARTED We are given position as a function of time in vector form (Eq. 1). The trajectory we are asked for in part *a* is the curve describing the mosquito's path in two-dimensional space. To obtain the trajectory we must extract the x and y position values from Eq. 1 and plot y versus x. Position, velocity, and acceleration are related by derivatives. We can obtain the position needed for part *b* directly from Eq. 1, and we can take time derivatives of Eq. 1 to obtain the velocity and acceleration at $t = +2.00$ s. We know that the velocity of any object is always tangent to its trajectory, so for part *c*, the orientation of the tangent line should match the direction of the velocity we computed in part *b*.

❷ DEVISE PLAN The trajectory is a plot of the y component of the position versus the x component of the position, and we have the information about each component at any instant in Eq. 1. (*a*) To draw this curve, we need a table of the position components for several values of t. We can choose specific values of t and substitute those values into the expressions for the x and y components of position:

$$x(t) = at^3 - bt \tag{2}$$

$$y(t) = c - dt^4. \tag{3}$$

(*b*) We know that velocity and position are related by $\vec{v} = d\vec{r}/dt$, and acceleration and velocity by $\vec{a} = d\vec{v}/dt$. We need to evaluate these time derivatives at $t = +2.00$ s. (*c*) The tangent to the trajectory at $t = +2.00$ s is in the same direction as the velocity at that instant. We can therefore specify the orientation of the tangent line by computing the angle relative to the x axis that satisfies $\tan\theta = v_y/v_x$.

3 EXECUTE PLAN

(*a*) We construct a table for the x and y components of the mosquito's position at instants that are close enough so that we can see how to interpolate. To get a rough idea of the shape, we begin with a few equally spaced instants of time: $t = -2.00$ s, -1.00 s, 0, $+1.00$ s, and $+2.00$ s. The results are

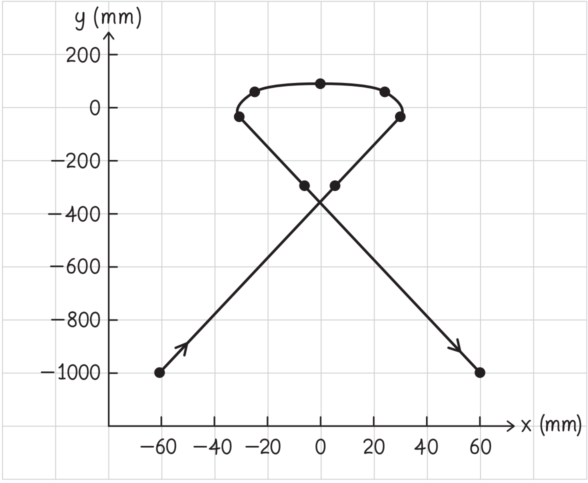
t (s)	x (mm)	y (mm)
−2.00	−60.0	−1060
−1.00	+30.0	−10.0
0	0	+60.0
+1.00	−30.0	−10.0
+2.00	+60.0	−1060

This choice of times provides a uniform spread of x values, but the y values are rather widely spread to be useful for interpolation. Now that we know the range of values, we can set the scales of the axes on our graph (which will be quite different for x and y). We also see the need to fill in a few more time values to specify the variation of y, perhaps one extra time between the central table rows and at least two extra values between the outer rows (where y grows dramatically). Inserting various values for t into Eqs. 2 and 3, we end up with

t (s)	x (mm)	y (mm)
−2.00	−60.0	−1060
−1.50	+7.50	−294
−1.00	+30.0	−10.0
−0.500	+22.5	+55.6
0	0	+60.0
+0.50	−22.5	+55.6
+1.00	−30.0	−10.0
+1.50	−7.50	−294
+2.00	+60.0	−1060

These data points yield the curve shown in Figure WG10.1.

Figure WG10.1



PRACTICE

The physical interpretation of Figure WG10.1 is that there are three basic motions. First, the mosquito flies in from the left at high speed, indicated by the large spacing between $t = -2.00$ s and $t = -1.00$ s. That these dots get closer and closer together around $t = -1.00$ s tells us that the mosquito slows down. The more closely spaced horizontal portion indicates a slow speed as the mosquito backtracks around the origin. Then it flies off to the right at increasing speed. ✔

(*b*) Equation 1 gives us the mosquito's position at $t = +2.00$ s. The velocity at that instant is

$$\vec{v} = \frac{d}{dt}[(at^3 - bt)\hat{\imath} + (c - dt^4)\hat{\jmath}]$$

$$= (3at^2 - b)\hat{\imath} - 4dt^3\hat{\jmath}$$

and the acceleration is

$$\vec{a} = \frac{d}{dt}[(3at^2 - b)\hat{\imath} - 4dt^3\hat{\jmath}]$$

$$= 6at\,\hat{\imath} - 12dt^2\hat{\jmath}.$$

At $t = +2.00$ s, these equations give

$$\vec{r}(+2.00\text{ s}) = [(20.0\text{ mm/s}^3)(2.00\text{ s})^3 - (50.0\text{ m/s})(2.00\text{ s})]\hat{\imath}$$

$$+ [60.0\text{ m} - (70.0\text{ m/s}^4)(2.00\text{ s})^4]\hat{\jmath}$$

$$= +(60.0\text{ mm})\hat{\imath} - (1.06 \times 10^3\text{ mm})\hat{\jmath}\ ✔$$

$$\vec{v}(+2.00\text{ s}) = +(190\text{ mm/s})\hat{\imath} - (2.24 \times 10^3\text{ mm/s})\hat{\jmath}\ ✔$$

$$\vec{a}(+2.00\text{ s}) = (240\text{ mm/s}^2)\hat{\imath} - (3.36 \times 10^3\text{ mm/s}^2)\hat{\jmath}.\ ✔$$

(*c*) The orientation of the line tangent to the trajectory at $t = +2.00$ s is the same as the orientation of the velocity vector at that instant because the velocity always points along the direction of motion. To determine the orientation of a vector in a plane, we use $\theta = \tan^{-1}(v_y/v_x)$. Taking our v_x and v_y values from the velocity expression we derived in part *b*, we have

$$\theta = \tan^{-1}\left(\frac{-2240\text{ mm/s}}{190\text{ mm/s}}\right) = -85.2°.$$

This is the angle between the positive *x* axis and the line tangent to the trajectory. ✔

❹ **EVALUATE RESULT** Judging from Figure WG10.1, the answers are not unreasonable. The mosquito's speed is fast, mostly in the $\pm\, y$ direction, and it is accelerating in this general direction. We could have constructed a larger table with more values of *t*, but the curve we obtained seems to be smooth enough not to hide any surprises.

We can also compare the calculated angle of the tangent line at $t = +2.00$ s with the information in our *xy* table. At $t = +2.00$ s, the mosquito's horizontal position is $x = 60.0$ mm. We see from Figure WG10.1 that the tangent to the trajectory at that *x* position does have a negative slope, consistent with our negative angle of $-85.2°$. However, is the size of the angle consistent with the graph? The angle between the tangent to the curve of Figure WG10.1 at $x = 60.0$ mm and the *x* axis looks to be approximately $-45°$ rather than $-85°$. We have to be careful, though! Notice that the scales for the *x* and *y* axes are different. If we had used the same scale on both axes, the angle would look like $-85°$. On the other hand, the resulting graph would contain mostly blank space, with the entire trajectory squeezed into a narrow vertical band along the central portion of the *x* axis. Having so much blank space in a graph would be a clear sign that some rescaling is needed.

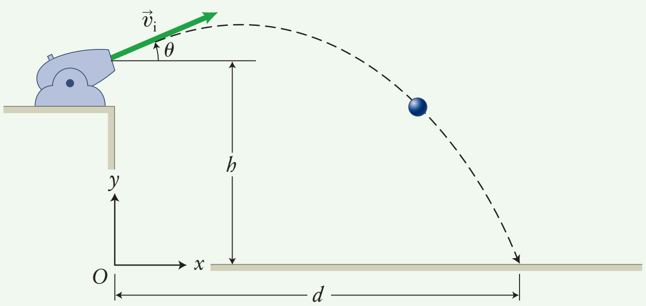
Guided Problem 10.2 Cannonball trajectory

A cannonball fired from the top of a cliff at instant $t = 0$ has an initial velocity of magnitude v_i and is aimed at an angle θ above the horizontal (Figure WG10.2). (*a*) Someone who wants to know the path of the ball after it leaves the cannon asks you for an equation describing the motion. Using the parameters given in Figure WG10.2, derive the equation

$$y(x) = y_i + (\tan\theta)(x - x_i) - \frac{g}{2(v_i \cos\theta)^2}(x - x_i)^2. \qquad (1)$$

(*b*) If the ball lands a distance d from the cliff base, what is the height h from which the cannon ball is fired? Assume the distance between the cliff edge and the cannon's muzzle tip is negligibly small.

Figure WG10.2



❶ **GETTING STARTED**

1. What type of motion is this?
2. What is the acceleration of the ball after it leaves the cannon? What are the sign and magnitude of a_x and of a_y? If the acceleration is constant, which kinematics equations can you use to derive Eq. 1?

❷ **DEVISE PLAN**

3. Write separate expressions for $x(t)$ and $y(t)$, the ball's position coordinates as a function of time.
4. For Eq. 1, you want the y coordinate as a function of x rather than of t. Can you solve for t in terms of $(x - x_i)$? Make the appropriate replacement of variables in the equation for y.
5. Can the x and y components of the initial velocity be expressed as functions of v_i and θ? Substitute this information as needed.

❸ **EXECUTE PLAN**

6. Do the algebra to finish deriving the trajectory equation.
7. For part *b*, choose a suitable origin for your coordinate system. It should be different from the random origin shown in Figure WG10.2.
8. For your choice of origin, replace the generic symbols x_i, y_i, x, and y with symbols specific to part *b*, such as d and h.

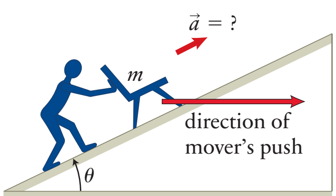
❹ **EVALUATE RESULT**

9. Does Eq. 1 have the expected dependence of y on x?
10. Does the value of h behave as you would expect as v_i, θ, and d change?

Worked Problem 10.3 Moving day

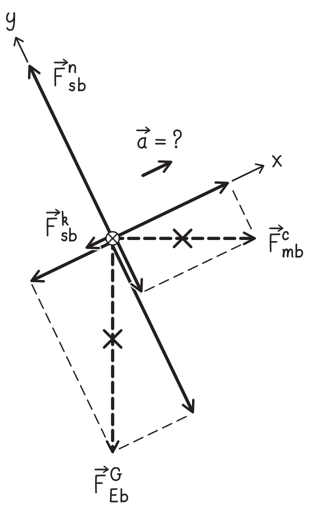
A mover pushes a bench of inertia m up a ramp that extends from the ground to the deck of a moving van and makes an angle θ with the ground (Figure WG10.3). None too smartly, he pushes horizontally with a force that has a magnitude equal to half the magnitude of the gravitational force exerted on the bench. The coefficient of kinetic friction between the ramp and the bench is μ_k. (*a*) Derive an expression for the acceleration of the bench in terms of θ and μ_k. (*b*) What must the coefficient of kinetic friction be (expressed in terms of θ) if the speed of the bench is constant? (*c*) Write an expression for the work done by the mover on the bench as the bench moves a distance d up the ramp.

Figure WG10.3



❶ GETTING STARTED The bench is the object we must focus on, and several forces are exerted on it: the mover's pushing force, the gravitational force, the frictional force, and the normal force exerted by the ramp surface. We must remember that the pushing force is directed horizontally, not up the ramp. The bench moves only parallel to the ramp, so the component of acceleration perpendicular to the ramp is zero. This information is summarized in the free-body diagram of the bench (Figure WG10.4).

Figure WG10.4



Because the direction of the pushing force (horizontal) is not the same as the direction of the motion, we must take care to get the force components right. We choose a coordinate system tilted like the ramp surface and show components for any vector that is not parallel to an axis. We construct our components by drawing arrows for them along the *x* and *y* directions and then crossing out the original vectors. Because the speed is constant up the ramp in part *b*, we know the acceleration is zero in that part, but it is probably not zero in part *a*. However, the acceleration might be directed either up or down the ramp. The question mark at the acceleration in Figure WG10.4 reinforces our uncertainty on this point.

❷ DEVISE PLAN

(*a*) We have no numerical values, but using the free-body diagram to construct the vector sum of forces should give us an expression for the acceleration.

Because they each have a component that opposes the *x* component $F^c_{\text{mb}\,x}$ of the force exerted by the mover, both the frictional force $\vec{F}^k_{\text{sb}}$ and the gravitational force $\vec{F}^G_{\text{Eb}}$ affect the acceleration. Applying some geometry and the information given about some of these forces, we should be able to use $\sum F_{\text{b}y}$ to express the remaining forces in terms of θ and μ_k and then use $\sum F_{\text{b}x}$ to obtain an expression for the acceleration. We can eliminate one unknown because the magnitude of the frictional force depends on the magnitude of the normal force according to the relationship $F^k_{\text{sb}} = \mu_k F^n_{\text{sb}}$.

Because the normal force is in the direction of the *y* axis, we will employ the vector sum of forces in the *y* direction to calculate it. There is no acceleration in the *y* direction, and so $\sum F_{\text{b}y} = ma_y = m(0) = 0$. In addition to the normal force, the gravitational force and the force exerted by the mover have components in the *y* direction. The only remaining variable is the acceleration, which we symbolize as a_x. We can isolate this variable to obtain the expression we need, and the expression should reveal the correct direction for the acceleration.

(*b*) Constant velocity requires zero acceleration. If we set this expression for acceleration equal to zero, it should be possible to obtain an expression for the coefficient of friction.

(*c*) The work done by a constant external force can be computed as the scalar product of the force and the force displacement. These quantities are not parallel to each other, so we must carefully keep track of components. Because we have information about both the magnitude and direction of each of these quantities, the work done can be computed from Eq. 10.35 regardless of the direction of the acceleration.

❸ EXECUTE PLAN (*a*) With our choice of axes orientation, the acceleration is along the *x* axis, and we start by analyzing the vector sum of the forces in the *x* direction. Because four forces are exerted on the bench, there should be four terms initially in the sum. We are given that $F^c_{\text{mb}} = \frac{1}{2}F^G_{\text{Eb}} = \frac{1}{2}mg$, and so

$$\sum F_{\text{b}x} = ma_x$$

$$F^k_{\text{sb}\,x} + F^n_{\text{sb}\,x} + F^c_{\text{mb}\,x} + F^G_{\text{Eb}\,x} = ma_x$$

$$-F^k_{\text{sb}} + 0 + (+F^c_{\text{mb}}\cos\theta) + (-F^G_{\text{Eb}}\sin\theta) = ma_x$$

$$-\mu_k F^n_{\text{sb}} + \tfrac{1}{2}mg\cos\theta - mg\sin\theta = ma_x. \quad (1)$$

The vector sum of the forces in the y direction gives us

$$\sum F_{\text{b}y} = ma_y$$

$$F^k_{\text{sb}y} + F^n_{\text{sb}y} + F^c_{\text{mb}y} + F^G_{\text{Eb}y} = m(0)$$

$$0 + (+F^n_{\text{sb}}) + (-F^c_{\text{mb}} \sin\theta) + (-F^G_{\text{Eb}} \cos\theta) = 0$$

$$F^n_{\text{sb}} = \tfrac{1}{2}mg\sin\theta + mg\cos\theta.$$

Substituting this value for F^n_{sb} in Eq. 1 yields

$$-\mu_k\left(\tfrac{1}{2}mg\sin\theta + mg\cos\theta\right) + \tfrac{1}{2}mg\cos\theta - mg\sin\theta = ma_x$$

$$a_x = \tfrac{1}{2}g\cos\theta - \mu_k g\cos\theta - \tfrac{1}{2}\mu_k g\sin\theta - g\sin\theta$$

$$a_x = g\cos\theta\left(\tfrac{1}{2} - \mu_k\right) - g\sin\theta\left(1 + \tfrac{1}{2}\mu_k\right). \tag{2}$$

Thus the acceleration is in either the positive or negative x direction, depending on the relative size of the terms in Eq. 2. Expressing the result in vector form, we have

$$\vec{a} = \left[g\cos\theta\left(\tfrac{1}{2} - \mu_k\right) - g\sin\theta\left(1 + \tfrac{1}{2}\mu_k\right)\right]\hat{\imath}. ✔$$

(b) If the bench's speed is constant as it goes up the ramp, a_x must be zero. To determine the value of μ_k for which this is true, we set a_x equal to zero in Eq. 2 and solve for μ_k:

$$0 = g\cos\theta\left(\tfrac{1}{2} - \mu_k\right) - g\sin\theta\left(1 + \tfrac{1}{2}\mu_k\right)$$

$$= \tfrac{1}{2}g\cos\theta - \mu_k g\cos\theta - \tfrac{1}{2}\mu_k g\sin\theta - g\sin\theta$$

$$\mu_k\cos\theta + \tfrac{1}{2}\mu_k\sin\theta = \tfrac{1}{2}\cos\theta - \sin\theta$$

$$\mu_k(2\cos\theta + \sin\theta) = \cos\theta - 2\sin\theta$$

$$\mu_k = \frac{\cos\theta - 2\sin\theta}{2\cos\theta + \sin\theta}. ✔$$

(c) Because the bench is rigid and moves along a straight line, the displacement of the location where the mover exerts his force, the force displacement, is the same as the center-of-mass displacement—namely, a distance d up the ramp. The work done by the mover's force is

$$W = \vec{F}\cdot\Delta\vec{r}_F = \vec{F}^c_{\text{mb}}\cdot\vec{d} = F^c_{\text{mb}}d\cos\theta = \tfrac{1}{2}mgd\cos\theta. ✔$$

❹ **EVALUATE RESULT** Because of the plus and minus signs in Eq. 2, a_x can be positive or negative depending on the relative sizes of μ_k and θ. Because μ_k is always associated with a minus sign in our expression, it tends to make the acceleration negative (down the ramp), as we expect. For small angles, $\cos\theta$ is positive and much larger than $\sin\theta$, making the acceleration positive for small values of μ_k. (If $\mu_k > \frac{1}{2}$, the acceleration must be negative.) As θ increases, $\cos\theta$ decreases, which causes a decrease in $g\cos\theta(\frac{1}{2} - \mu_k)$, the only positive term in 2, relative to the increasing $\sin\theta$ term. Thus increasing θ drives a_x to negative values, which is what we expect: As the slope gets steeper, we expect the bench to slow down rather than speed up. The value of μ_k in part b balances these effects to achieve zero acceleration. Note that the work done by the mover is positive because $\cos\theta$ is positive for any inclination between 0 and 90°. This is true regardless of the magnitude or direction of the bench's acceleration, as long as it moves up the ramp.

Guided Problem 10.4 Push that shed

Three winter anglers on an ice-fishing trip push a 120-kg shed out onto the ice. They are somewhat careless and push in different directions. As seen from above, these forces are $\vec{F}^c_{1s} = 32$ N directed 30° east of north, $\vec{F}^c_{2s} = 55$ N directed due north, and $\vec{F}^c_{3s} = 41$ N directed 60° west of north. What are the magnitude and direction of the shed's acceleration?

1 GETTING STARTED

1. Because this is a force problem involving three vectors, consider which is more appropriate: graphical addition or addition by components.
2. Because forces are involved, a free-body diagram of the shed is useful. Is it possible to draw only the forces in the plane of the ice? If so, why is it a good idea to set things up this way?

2 DEVISE PLAN

3. Choose two coordinate axes. Does it matter which directions you choose for your axes, given that the direction of the acceleration is unknown? Is one orientation of your coordinate system more appropriate than other orientations, given the information supplied?
4. Write the two components of Newton's second law for forces parallel to the ice.
5. Do you have enough information to compute the value of each component of the three forces in your coordinate system?

3 EXECUTE PLAN

6. Add components to obtain an x component and y component for the vector sum of the forces.
7. How are the components of the acceleration related to the force components?
8. Can you compute the magnitude and direction of the acceleration (relative to north) given the x and y components of the acceleration?

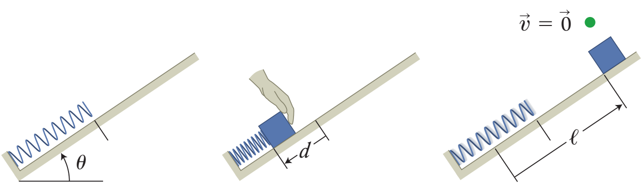
4 EVALUATE RESULT

9. Did you make any assumptions? If so, decide whether they are reasonable or should be reconsidered.
10. Is the acceleration direction consistent with what your intuition tells you about the overall effect of the three forces?
11. Is the magnitude of the acceleration reasonable?

Worked Problem 10.5 Spring up that incline

The ramp in Figure WG10.5 has a rough surface and makes an angle θ with the horizontal. The spring attached to the support at the bottom of the ramp has a force constant k. You place a block of inertia m against the free end of the spring and then push the block against the spring until the block has moved a distance d from its position when the spring is at its relaxed length. When you release the block from rest, it slides up the ramp and eventually leaves the spring behind, stopping finally a distance ℓ from its position just before release. Derive an expression for ℓ in terms of k, d, m, θ, and the coefficient of kinetic friction μ_k between block and ramp.

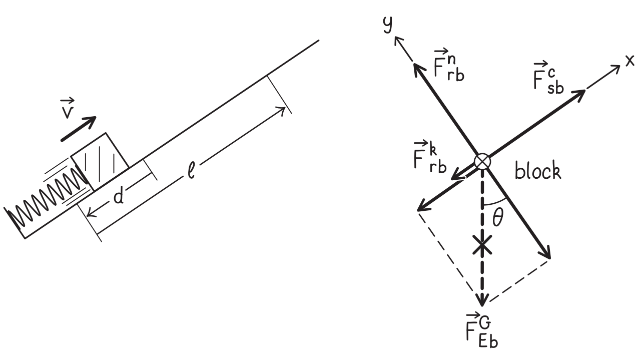
Figure WG10.5



Spring relaxed | Spring compressed | Spring pushes block; block slides to halt

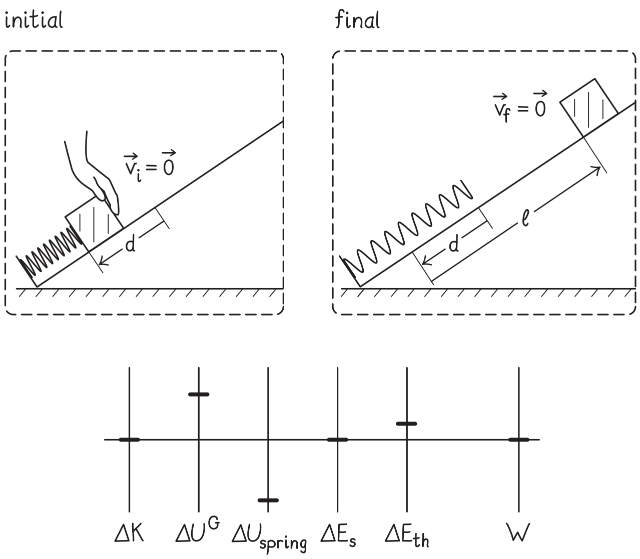
1 GETTING STARTED Four forces are exerted on the block: the constant force of gravity, the variable spring force, the constant normal force, and the dissipative frictional force. (The phrase *rough surface* in the problem statement is what tells us we must consider friction.) A good first step is to draw a free-body diagram that shows all four forces (Figure WG10.6). To get an expression for the magnitude of the force of kinetic friction, we need to know the magnitude of the normal force. We include in our diagram a tilted coordinate system to help us calculate this force and the components of any vector not parallel to one of our coordinate axes.

Figure WG10.6



Because the spring force varies with position, we should investigate whether we can use work-energy methods, accounting for the action of the spring through the equation for potential energy. Thus a situation diagram is called for, showing the initial compression distance d and the final position of the block a distance ℓ up the ramp (Figure WG10.7). With this approach, the initial elastic potential energy of the compressed spring is converted to gravitational potential energy as the block rises and to thermal energy dissipated through friction. Although the kinetic energy changes during the process, it is the same at start and finish: zero.

Figure WG10.7



❷ **DEVISE PLAN** If we choose the block, spring, ramp, and Earth as our system, there are no external forces, and therefore no work is done on the system:

$$\Delta E = W = 0$$

$$\Delta K + \Delta U_{\text{spring}} + \Delta U^G + \Delta E_{\text{th}} = 0. \tag{1}$$

This expression will contain all our variables, and so with a little algebra we should be able to isolate ℓ, the variable we are after.

The thermal energy dissipated is the product of the magnitude of the force of kinetic friction and the distance ℓ over which the block moves:

$$\Delta E_{\text{th}} = F^k_{\text{rb}}\ell = (\mu_k F^n_{\text{rb}})\ell. \tag{2}$$

To calculate the magnitude F^n_{rb}, we note that the normal force is aligned along the y axis. We employ the vector sum of force components along this axis and the fact that along this axis $a_y = 0$:

$$\sum F_{\text{b}y} = ma_y = m(0) = 0. \tag{3}$$

Equations 1–3 should give us all the information we need to derive an expression for ℓ.

❸ **EXECUTE PLAN** Substituting an expression that contains the quantities given in the problem statement for each term in Eq. 1 gives us

$$\left(\tfrac{1}{2}mv_{\text{f}}^2 - \tfrac{1}{2}mv_{\text{i}}^2\right) + \left[\tfrac{1}{2}k(x_{\text{f}} - x_0)^2 - \tfrac{1}{2}k(x_{\text{i}} - x_0)^2\right] + (mgh_{\text{f}} - mgh_{\text{i}}) + F^k_{\text{rb}}\ell = 0$$

$$(0 - 0) + \left[\tfrac{1}{2}k(0)^2 - \tfrac{1}{2}k(-d)^2\right] + mg(h_{\text{f}} - h_{\text{i}}) + \mu_k F^n_{\text{rb}}\ell = 0$$

$$-\tfrac{1}{2}kd^2 + mg(+\ell \sin\theta) + \mu_k F^n_{\text{rb}}\ell = 0$$

$$\ell(mg\sin\theta + \mu_k F^n_{\text{rb}}) = \tfrac{1}{2}kd^2, \tag{4}$$

where x_0 is the block's position when the spring is at its relaxed length and we have used h to designate vertical height coordinates so as not to mistakenly use the tilted y coordinate. Equation 4 looks pretty close to what we are after (an expression for ℓ in terms of k, d, m, θ, and μ_k). All we need to derive is an expression for F^n_{rb}, which we obtain from Eq. 3, the vector sum of the forces along the y axis. Four forces are exerted on the block, so

$$\Sigma F_{\text{b}y} = F^k_{\text{rb}y} + F^n_{\text{rb}y} + F^c_{\text{sb}y} + F^G_{\text{Eb}y} = 0$$

$$0 + (+F^n_{\text{rb}}) + 0 + (-F^G_{\text{Eb}} \cos\theta) = 0$$

$$F^n_{\text{rb}} = F^G_{\text{Eb}} \cos\theta = mg\cos\theta.$$

Substituting this expression for F^n_{rb} into Eq. 4 gives us what we're after:

$$\ell(mg\sin\theta + \mu_k mg\cos\theta) = \tfrac{1}{2}kd^2$$

$$\ell = \frac{kd^2}{2mg(\sin\theta + \mu_k\cos\theta)}. \checkmark$$

❹ **EVALUATE RESULT** We expect that the greater the force of friction (in other words, the higher the value of μ_k), the shorter the distance ℓ the block travels, as this expression predicts. We also expect that for any fixed value of μ_k, the distance ℓ increases as θ decreases. When $\theta = 0$ (the block sits on a horizontal surface), $\sin\theta = 0$ and ℓ has a large value. Because $\sin\theta$ increases faster than $\cos\theta$ decreases, the $\theta = 0$ value of ℓ is in fact its maximum value (assuming that $\mu_k < 1$).

The larger the spring constant k or the greater the compression distance d, the farther the block should go, which is what our expression predicts. Our result behaves the way we would expect.

Note that the x components of the four forces in this problem contribute to the work done on the system. The two y components, $F^n_{\text{rb}y}$ and $F^G_{\text{Eb}y}$, do not. Their lack of contribution to the system energy is not just because they cancel each other, however. Let us consider the effect of the normal force, which is perpendicular to the direction of motion. The scalar product of two vectors is nonzero only when the vectors have a component in the same direction, so $\vec{F}^n_{\text{rb}} \cdot \Delta\vec{r}_F = 0$ for the normal force as well as for any other force that is always perpendicular to the direction of motion. This means that such forces cannot rearrange energy within the system.

Guided Problem 10.6 Moving a load

A variation of the Atwood machine (see Figure WG8.6) can be used to haul a load up an incline. Using this variation, you must design a coal cart of inertia m_c that moves up an incline set at an angle θ with the horizontal. A rope attached to the cart is thrown over a pulley at the top of the incline, and a metal block of inertia m_b is hung vertically at the other end of the rope as a counterweight. (*a*) In terms of θ and the two inertias, what is the magnitude of the acceleration of the cart if the block alone is used to draw the cart up the incline? (*b*) In terms of θ and m_c, what must the inertia of the block be if the cart is to move at constant speed?

❶ **GETTING STARTED**

1. Sketch the physical situation.
2. What variable must you determine in part *a*? Which approach is more relevant for determining it: force analysis or energy analysis? Can the result of part *a* be adapted to solve part *b*?
3. How many objects are involved in the motion? How are their motions related?
4. When you analyze forces for two or more objects, each must have its own free-body diagram and coordinate system, leading to a set of force-acceleration equations for each object.
5. Put the acceleration of each object in your diagrams, and recall that it is useful to have one coordinate axis parallel to an object's acceleration, with the axes oriented such that the motion of one object along a positive axis in its coordinate system corresponds to the motion of the other object along its corresponding positive axis.

❷ **DEVISE PLAN**

6. How does the tension in the rope segment between cart and pulley compare with the tension in the segment between block and pulley?
7. Which component, x or y, of Newton's second law applied to the cart contains the requested acceleration? Do you need to consider the other component equation?
8. Make sign and magnitude decisions for the components of vectors where possible (for example, the x component of the gravitational force).
9. Repeat your analysis for the block and then count equations and unknowns.
10. If the speed of the cart is to be constant, what is its acceleration? How can this information be used to obtain the value needed for m_b?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

11. Examine your expression for the required acceleration. Do the magnitude and sign of the acceleration depend as you expect on m_c and m_b? On θ?
12. Do your answers to questions 11 make sense in the limit $\theta = 0$? In the limit $\theta = 90°$?

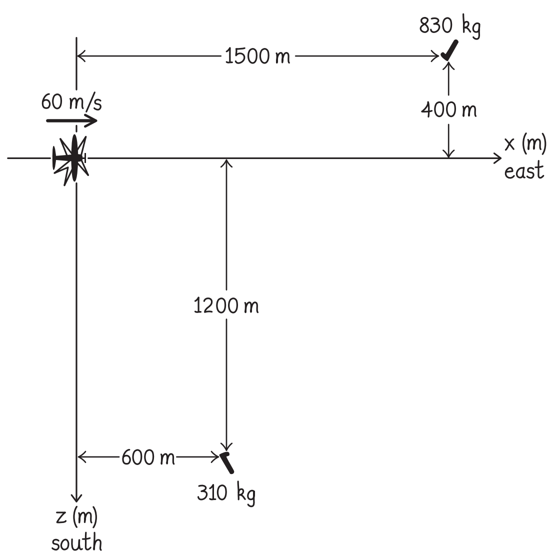
Worked Problem 10.7 Accident investigation

While on a test flight at an altitude of 2000 m, a 1500-kg unmanned drone moving eastward at 60 m/s exploded into three pieces. Satellite data show that the explosion occurred exactly 50 km east of Reno, Nevada. Beginning directly beneath the explosion site, searchers find a 310-kg piece 1200 m to the south and 600 m to the east. An 830-kg piece is found 400 m to the north and 1500 m to the east. Assume that air resistance is negligible and that all three pieces hit the ground simultaneously. What were the velocities of the three pieces immediately after the explosion, and where should investigators look for the third piece?

❶ **GETTING STARTED** We are seeking the final velocities of three pieces and the final position of one piece. The initial velocity and position of the drone are given, as well as the final positions of two of the three pieces. That still leaves a lot of unknowns! Moreover, this appears to be a three-dimensional problem, something we have not yet encountered. However, each piece is a projectile that moves in a plane (though each moves in a different plane). But it gets better because we have some very useful clues. For example, the assumption that all three pieces hit the ground at the same time will allow us to immediately solve for motion in the vertical direction. Then it will be necessary only to analyze two horizontal components of the motion by using the techniques we already have at our disposal.

Let's organize our thoughts by drawing before and after satellite views (looking down from high above) of the system (Figure WG10.8). The figure shows the drone immediately before the explosion and the known final positions of two of the pieces. The initial velocity is also indicated, and we set up coordinates with the origin directly under the explosion site, with the positive x direction to the east, the positive y direction upward (not shown), and the positive z direction to the south. We can treat the explosion like an explosive separation, and so the drone is an isolated system. Therefore the momentum of the pieces must be the same before and after the explosion.

Figure WG10.8



❷ **DEVISE PLAN** Because we assume the three pieces hit the ground simultaneously, in the absence of air resistance each must have the same final vertical component of velocity. This is easily seen from the vertical position equation for a projectile, Eq. 10.20:

$$y_f = y_a + v_{y,a}\Delta t - \tfrac{1}{2}g(\Delta t)^2.$$

The subscripts are "a" for immediately after the explosion and "f" for final (just before hitting the ground). Each piece started at the same height y_a, each reached the same final height ($y_f = 0$), and each experienced the same values of g and Δt, so the only unspecified variable is the velocity just after the explosion $v_{y,a}$. The equation thus demands the same value of vertical velocity immediately after the explosion for each piece. Moreover, this vertical velocity must be zero! Prior to the explosion, the system had zero velocity, and thus zero momentum, in the vertical direction. After the explosion, the vertical momentum must still be zero. This might be arranged by having some of the pieces ejected upward and some downward, but then they would not hit the ground at the same time. All three pieces therefore fall from rest as far as the vertical motion is concerned:

$$v_{1y,a} = v_{2y,a} = v_{3y,a} = 0.$$

This allows us to compute the time of fall, which together with the given position information and simple kinematics should give all three x and z components of velocity just after the explosion.

The location of the missing piece can be found using the momentum equation. Alternatively, we might compute it by considering that the center of mass of a system moves with the system momentum.

❸ **EXECUTE PLAN** We carry one extra digit in all of our calculations so as not to accumulate rounding error, but we report the results to two significant digits to match the precision of the given information.

The time of fall from 2000 m for each piece is identical:

$$y_{\rm f} = y_{\rm a} + v_{y,{\rm a}}\Delta t - \tfrac{1}{2}g(\Delta t)^2 = y_{\rm a} + (0)\Delta t - \tfrac{1}{2}g(\Delta t)^2$$
$$= y_{\rm a} - \tfrac{1}{2}g(\Delta t)^2$$
$$(\Delta t)^2 = \frac{y_{\rm a} - y_{\rm f}}{\frac{1}{2}g} = \frac{+2000\ \text{m}}{\frac{1}{2}(9.8\ \text{m/s}^2)}$$
$$\Delta t = 20.2\ \text{s}.$$

This leads immediately to the x and z components of velocity after the explosion of the two pieces whose final positions we know:

$$x_{1\rm f} = x_{1\rm a} + v_{1x,\rm a}\Delta t$$
$$v_{1x,\rm a} = \frac{x_{1\rm f} - x_{1\rm a}}{\Delta t} = \frac{+600\ \text{m} - 0}{20.2\ \text{s}} = +29.7\ \text{m/s} = +30\ \text{m/s}\ ✔$$
$$x_{2\rm f} = x_{2\rm a} + v_{2x,\rm a}\Delta t$$
$$v_{2x,\rm a} = \frac{x_{2\rm f} - x_{2\rm a}}{\Delta t} = \frac{+1500\ \text{m} - 0}{20.2\ \text{s}} = +74.2\ \text{m/s} = +74\ \text{m/s}\ ✔$$
$$z_{1\rm f} = z_{1\rm a} + v_{1z,\rm a}\Delta t$$
$$v_{1z,\rm a} = \frac{z_{1\rm f} - z_{1\rm a}}{\Delta t} = \frac{+1200\ \text{m} - 0}{20.2\ \text{s}} = +59.4\ \text{m/s} = +59\ \text{m/s}\ ✔$$
$$z_{2\rm f} = z_{2\rm a} + v_{2z,\rm a}\Delta t$$
$$v_{2z,\rm a} = \frac{z_{2\rm f} - z_{2\rm a}}{\Delta t} = \frac{-400\ \text{m} - 0}{20.2\ \text{s}} = -19.8\ \text{m/s} = -20\ \text{m/s}.\ ✔$$

The velocity of the missing piece can now be obtained from the momentum equation, and its location from the same kinematics employed above. The inertia of the missing piece is 1500 kg − 310 kg − 830 kg = 360 kg. We have

$$\vec{p}_{\rm i} = \vec{p}_{\rm a}$$
$$m_{\rm plane}\vec{v}_{\rm i} = m_1\vec{v}_{1\rm a} + m_2\vec{v}_{2\rm a} + m_3\vec{v}_{3\rm a}.$$

We already know that the y components of velocities are zero immediately after the explosion, so we use the x and z component equations:

$$m_{\rm plane}v_{x,\rm i} = m_1v_{1x,\rm a} + m_2v_{2x,\rm a} + m_3v_{3x,\rm a}$$
$$v_{3x,\rm a} = \frac{m_{\rm plane}v_{x,\rm i} - m_1v_{1x,\rm a} - m_2v_{2x,\rm a}}{m_3}$$
$$= \frac{(1500\ \text{kg})(+60\ \text{m/s}) - (310\ \text{kg})(29.7\ \text{m/s})}{360\ \text{kg}}$$
$$- \frac{(830\ \text{kg})(74.2\ \text{m/s})}{360\ \text{kg}}$$
$$= +53\ \text{m/s}\ ✔$$
$$m_{\rm plane}v_{z,\rm i} = m_1v_{1z,\rm a} + m_2v_{2z,\rm a} + m_3v_{3z,\rm a}$$
$$v_{3z,\rm a} = \frac{m_{\rm plane}v_{z,\rm i} - m_1v_{1z,\rm a} - m_2v_{2z,\rm a}}{m_3}$$
$$= \frac{(1500\ \text{kg})(0) - (310\ \text{kg})(59.4\ \text{m/s})}{360\ \text{kg}}$$
$$- \frac{(830\ \text{kg})(-19.8\ \text{m/s})}{360\ \text{kg}}$$
$$= -5.50\ \text{m/s}.\ ✔$$

The signs in our values for v_3 indicate that piece 3 is moving in the positive x (eastward) and negative z (northward) directions after the explosion. To finish our task—finding the location of piece 3—we apply kinematics. In the 20 s before it struck the ground, piece 3 traveled to this final resting place:

$$x_{3f} = x_{3a} + v_{3x,a}\Delta t$$
$$= 0 + (+53.3\ \text{m/s})(20.2\ \text{s})$$
$$= +1077\ \text{m} = 1.1 \times 10^3\ \text{m} ✔$$
$$z_{3f} = z_{3a} + v_{3z,a}\Delta t$$
$$= 0 + (-5.50\ \text{m/s})(20.2\ \text{s})$$
$$= -111\ \text{m} = -1.1 \times 10^2\ \text{m.} ✔$$

❹ **EVALUATE RESULT** The calculated velocity components are all comparable in magnitude to the original velocity of the drone. Piece 3 should be found at approximately 1100 m east and 110 m north of the origin (or the point of explosion). The magnitude of the displacement of piece 3 is neither huge nor tiny compared with the displacements of the other two pieces. We can check our result by recalling that the motion of the center of mass of a system is affected only by forces acting from outside the system. Thus the center of mass of the drone follows the parabolic path of a projectile launched at altitude 2000 m with eastward velocity 60 m/s:

$$x_{\text{cm,f}} = x_{\text{cm,i}} + v_{\text{i},x}\Delta t = 0 + (60\ \text{m/s})(20.2\ \text{s})$$
$$= 1212\ \text{m} = 1.2 \times 10^3\ \text{m.}$$

We can compute the location of the center of mass of the pieces using the results obtained above:

$$x_{\text{cm}} = \frac{m_1x_{1f} + m_2x_{2f} + m_3x_{3f}}{m_{\text{plane}}}$$
$$= \frac{(310\ \text{kg})(+600\ \text{m}) + (830\ \text{kg})(+1500\ \text{m})}{1500\ \text{kg}}$$
$$+ \frac{(360\ \text{kg})(+1100\ \text{m})}{1500\ \text{kg}}$$
$$= +1218\ \text{m} = 1.2 \times 10^3\ \text{m.}$$

This method can also verify that the z component of the center-of- mass position is indeed zero, if we recall that small differences in large numbers cause reduced significant digits:

$$z_{\text{cm}} = \frac{m_1z_{1f} + m_2z_{2f} + m_3z_{3f}}{m_{\text{plane}}}$$
$$= \frac{(310\ \text{kg})(+1200\ \text{m}) + (830\ \text{kg})(-400\ \text{m})}{1500\ \text{kg}}$$
$$+ \frac{(360\ \text{kg})(-111\ \text{m})}{1500\ \text{kg}}$$
$$= +0.0267\ \text{m} = 0.$$

Under the given assumptions about direction of motion and constancy of velocities, the answers we obtain are not unreasonable.

PRACTICE

Guided Problem 10.8 Hockey with a bang

Someone puts a firecracker into a hockey puck and slides it onto frictionless ice while you watch. The explosion breaks the puck into exactly two pieces, and you see the two pieces slide across the ice without rising into the air. Piece P (inertia 60 g) moves with speed 4.5 m/s at an angle of 22° south of east. Piece Q (inertia 110 g) moves with speed 1.9 m/s at an angle of 52° north of west. Compute the initial velocity of the hockey puck before the explosion. Ignore the inertia of the firecracker.

1 GETTING STARTED

1. What are you asked to determine?
2. What is an appropriate choice of system?
3. Which fundamental principle(s) would be most useful for this analysis?
4. Make a sketch showing the appropriate initial and final conditions.

2 DEVISE PLAN

5. Write the appropriate equation(s) that express the principle you chose to make the analysis.
6. Are there sufficient equations to compute the unknowns?

3 EXECUTE PLAN

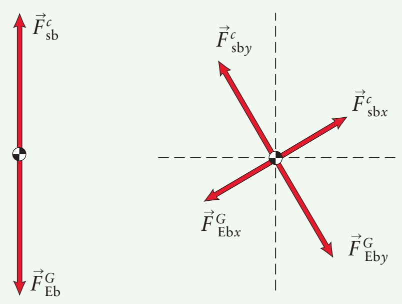
7. Insert known values and evaluate your expression(s).

4 EVALUATE RESULT

8. Is it possible that the hockey puck was at rest before the explosion?
9. Is more than one answer possible? Are your values reasonable?

Answers to Review Questions

1. The logger sees the package fall in a curved trajectory. The pilot sees it fall in a straight line directly under the plane. The plane is directly above the logger when the package lands at his feet.
2. The accelerations are the same to both observers—namely, *g*.
3. This type of motion has two vector components, one vertical and one horizontal, and we need a reference axis for each.
4. In both cases, you place the tail of the second vector at the head of the first vector, and their sum is the vector extending from the tail of the first vector to the head of the second one. The difference is that in more than one dimension the magnitude of the sum depends on the angle between the vectors.
5. Subtraction is the same as if the vectors did lie along a straight line: Reverse the direction of vector 1 and then add that reversed vector to vector 2.
6. Yes, $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. No, $\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$.
7. To add two vectors $\vec{A}$ and $\vec{B}$, draw a scale model (magnitude and direction) of vector $\vec{A}$, then draw a scale model of vector $\vec{B}$ such that its tail is placed at the head of vector $\vec{A}$. The sum is a vector drawn from the tail of $\vec{A}$ to the head of $\vec{B}$. To subtract vector $\vec{B}$ from vector $\vec{A}$, reverse the direction of vector $\vec{B}$ and then add it to vector $\vec{A}$ using the addition process. The difference is a vector drawn from the tail of $\vec{A}$ to the head of the reversed $\vec{B}$.
8. The object is not moving in a straight line during the time interval Δt, but it is moving on a curved path.
9. The component of acceleration parallel to an object's velocity changes the magnitude of the velocity (speed) but does not change its direction. The component of acceleration perpendicular to the velocity changes the direction of the velocity but does not change its magnitude.
10. The normal component of this force is perpendicular to the surface of contact, and the tangential component is parallel to this surface.
11. (*a*) (*b*)



12. With one axis parallel to the acceleration, the components of force along the other axis add to zero, simplifying the algebra.
13. All three forces are components of the contact force experienced by two surfaces in contact with each other. The normal force is the component of the contact force perpendicular to the two surfaces. The force of friction, static or kinetic, is the component of the contact force parallel to the surfaces. The direction of any normal force always opposes interpenetration of the two surfaces; the direction of any frictional force always

opposes slipping between the two surfaces. The magnitudes of the normal force and the force of static friction adjust within a range of possible values from zero to some maximum allowed value. Exceeding the maximum allowable normal force results in the failure (breaking) of one or both materials. When this happens, the surface can no longer exert a normal force and so its magnitude becomes zero. Exceeding the maximum allowable magnitude of the force of static friction results in slipping, in which case the force of static friction is replaced by a force of kinetic friction.

14. The force of static friction increases in magnitude as you push harder, so that its magnitude remains equal to the horizontal force you exert on the crate. The direction of the force of static friction opposes the direction of your push at all times.

15. Static friction rather than inertia is what needs to be overcome. The magnitude of your pushing force is smaller than the maximum magnitude of the frictional force the floor can exert on the desk.

16. The force of static friction is an elastic force and hence causes no dissipation of energy. The force of kinetic friction is not elastic and does cause energy dissipation. If this dissipation occurs at the system boundary, we have no means of knowing what fraction of the dissipated energy stays within the system and what fraction is transferred out of the system, so we cannot balance the energy equation.

17. Many answers are possible. The force of static friction between tires and road is responsible for accelerating a car, but it does no work on the car because there is zero force displacement. (Unless there is slipping, which would no longer involve static friction, the point of application of the force does not move.) When you carry a stack of books, the static friction between the top book and the one underneath it is responsible for accelerating the top book whenever you speed up or slow down. Work is done on the top book by the force of static friction because there is a nonzero force displacement.

18. Friction need not oppose motion, as in the case of the books in Review Question 17. What friction does always oppose is slipping, or *relative* motion between the surfaces in contact.

19. No, because in polar coordinates r represents a distance, not a position vector, and distances are always positive.

20. $B \sin \theta$. The sine function is the ratio of the opposite side to the hypotenuse, and the cosine is the ratio of the adjacent side to the hypotenuse. Because angles are generally given between a vector (hypotenuse) and the x axis, the cosine function is usually associated with an x component. Not in this case! (Something helpful for remembering which component is associated with $\sin \theta$ and which is associated with $\cos \theta$: Starting at the vector arrow, draw an arc across θ to the component that forms the other side of the angle. This component is the one associated with $\cos \theta$.)

21. A component vector is one of a set of vectors into which a given vector may be decomposed. Thus the sum of the component vectors equals the given vector. A vector component is the signed scalar that, together with an appropriate unit vector, describes the magnitude and direction of a component vector. The component vectors are related to the x and y components of the same vector by Equation 10.5:

$$\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{\imath} + A_y\hat{\jmath}.$$

This relationship reflects the option of considering vector addition as (scalar) addition of vector components in a chosen rectangular coordinate system, or (vector) addition of component vectors that lie along the chosen coordinate axes.

22. At the highest point, the vertical component of the velocity is 0, and so the ball's speed is given by the unchanged horizontal component: 30 m/s.

23. The path is parabolic. More specifically, $y(x)$ is a quadratic function.

24. The maximum height depends on only the vertical component of the launch velocity. However, a problem might involve a fixed launch speed with a variable angle of launch, in which case the horizontal and vertical components of the launch velocity are interdependent. The horizontal range of a projectile does depend on both components of launch velocity: the x component in order to move quickly horizontally, and the y component in order to remain in the air long enough to travel far.

25. The principles are the same: The momentum of the system is unchanged by the collision, and the coefficient of restitution determines the relative velocities of the colliding objects after the collision in terms of the relative velocities before the collision. The difference is that, in two dimensions, each momentum vector has two components, giving two equations that must be satisfied rather than one, as in one dimension. The relative velocities also have two components in two dimensions, but their relationship described by the coefficient of restitution still involves a single equation.

26. In one dimension there are two unknown final velocity components and two equations relating initial and final values. Momentum remains unchanged, and the initial and final relative velocities are related by the coefficient of restitution. In two dimensions these same principles provide three equations: two component equations to ensure that the momentum is unchanged, plus the single coefficient of restitution equation relating the initial and final relative velocities. However, in two dimensions there are in general four unknown final velocity components, two for each object. Thus in two dimensions at least one more piece of information is required for a complete solution.

27. The magnitude of one or both of the vectors might be zero, or the vectors might be perpendicular to each other, in which case the factor $\cos \phi$ in Eq. 10.33 is zero because $\phi = 90°$.

28. The work (a scalar) is the scalar product of the constant force (a vector) and the force displacement (a vector) between the two positions. It is equal to the product of the magnitudes (scalars) of these two vectors and the cosine of the angle between them.

29. The force displacement $\Delta\vec{r}_F$ cannot be used. Instead, the path along which the object moves from position 1 to position 2 must be broken up into infinitesimally small force displacements $d\vec{r}$. The work done is computed as a line integral of the scalar product of force and force displacement over the path from position 1 to position 2: $W = \int_1^2 \vec{F} \cdot d\vec{r}$.

30. The contact area is the macroscopic area over which the surfaces touch each other. The effective contact area is the actual area of microscopic contact between the two surfaces. The effective contact area is many thousands of times smaller than the contact area in everyday experience, and it is this effective contact area that increases as the normal force increases. The contact area does not increase with increasing normal force; hence, the force of friction is independent of the contact area. The available frictional force is dependent, as you would expect, on the effective contact area and hence on the normal force exerted by the surfaces on each other.

31. He has to tell you what material he had in contact with the concrete in order for the information to have any value. The coefficients of friction depend on the material of *both* surfaces in contact, not just one.

32. Kicking in the wedge increases the normal force exerted by the floor on the wedge and hence also increases the magnitude of the force of static friction exerted by the floor on the wedge.

33. Skidding involves kinetic friction because the tires and the road are sliding past each other. Rolling involves static friction because the tire and the road do not slip. The magnitude of the force of kinetic friction between any two surfaces is typically smaller than the maximum value of the magnitude of the force of static friction between the surfaces, and so it is the latter you want working for you when you stop a car.

Answers to Guided Problems

Guided Problem 10.2 (*b*) $h = \dfrac{gd^2}{2v_i^2 \cos^2\theta} - d\tan\theta$

Guided Problem 10.4 $0.88\ \text{m/s}^2$ at 11° W of N

Guided Problem 10.6 (*a*) $a = \dfrac{m_b - m_c \sin\theta}{m_b + m_c} g$; (*b*) $m_b = m_c \sin\theta$

Guided Problem 10.8 $\vec{v}_{p,i} = (0.85\ \text{m/s})\hat{\imath} + (0.37\ \text{m/s})\hat{\jmath}$

Guided Practice by Chapter

11 Motion in a Circle

Review Questions

Answers to these questions can be found at the end of this chapter.

11.1 Circular motion at constant speed

1. Give an example of (*a*) an object that has translational motion but no rotational motion, (*b*) an object that has rotational motion but no translational motion, and (*c*) an object that has both rotational motion and translational motion.
2. Is it possible for an object to have a nonzero acceleration if the object is traveling (*a*) at constant velocity and (*b*) at constant speed?
3. For an object moving along a circular path, what is the difference between the object's rotational coordinate ϑ and its polar angle θ?
4. For an object in circular motion at constant speed, describe the directions of the object's position vector (relative to the center of the circular trajectory), velocity vector, and acceleration vector at a given instant.

11.2 Forces and circular motion

5. If you are sitting in the passenger seat of a car that makes a quick left turn, your shoulders seem to lean to the right. What causes this apparent rightward motion?
6. Why shouldn't you use a rotating reference frame when analyzing forces?
7. What can you say about the vector sum of the forces exerted on an object that moves in a circle at constant speed? How does this vector sum of forces vary with speed and radius?

11.3 Rotational inertia

8. Is rotational inertia an intrinsic property of an object? Explain your answer.
9. Why does a tightrope walker carry a long pole?

11.4 Rotational kinematics

10. Does an object moving in a circle always have centripetal acceleration? Does it always have rotational acceleration? Does it always have tangential acceleration?
11. What is the mathematical expression that gives the relationship between rotational coordinate ϑ and polar angle θ?
12. Explain the distinctions among rotational acceleration, tangential acceleration, and centripetal acceleration.
13. For an object moving in circular motion, write an expression showing the relationship between (*a*) its arc length along the circle s and its rotational coordinate ϑ, (*b*) its velocity v_t and rotational velocity ω_ϑ, and (*c*) its tangential acceleration a_t and its rotational acceleration α_ϑ.
14. You are whirling a ball on the end of a string in a horizontal circle above your head. How does the ball's centripetal acceleration change if you increase the speed v so that the time interval needed to complete a revolution is halved?

11.5 Angular momentum

15. Define *rotational inertia* for a particle moving in a circle. Define *lever arm distance* for a particle moving near an axis of rotation.
16. Describe the relationship between the rotational kinetic energy of an object moving in circular motion and the object's rotational inertia.
17. How is angular momentum L related to momentum mv for an object of inertia m moving with constant velocity?
18. If both the rotational inertia I and the rotational speed ω of an object are doubled, what happens to the object's rotational kinetic energy?
19. What is the meaning of the statement: Angular momentum is conserved?

11.6 Rotational inertia of extended objects

20. Your physics book can be rotated around three mutually perpendicular axes passing symmetrically through its center. About which axis is the rotational inertia smallest? About which axis is it largest?
21. A diatomic molecule, which has a dumbbell shape, can rotate about three symmetrical axes that pass through the center of the molecule. Describe them. Rotation at a specific rotational speed about which axis gives the molecule the lowest kinetic energy?
22. State the parallel-axis theorem in words.

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The speed v of a point on the equator as Earth rotates (D, P)
2. The rotational inertia of a bowling ball about an axis tangent to its surface (A, R, X)
3. Your rotational inertia as you turn over in your sleep (V, C)
4. The angular momentum around the axle of a wheel/tire combination on your car as you cruise on the freeway (E, I, O, AA, S)
5. The angular momentum of a spinning ice skater with each arm held out to the side and parallel to the ice (G, X, N, U)
6. The speed you would need to orbit Earth in a low orbit (F, P)
7. The magnitude of the force exerted by the Sun on Earth to hold Earth in orbit (B, L, T, Z)
8. The kinetic energy associated with Earth's rotation (Z, P, D)
9. The angular momentum, about a vertical axis through your house, of a large car driving down your street (H, Y, M)
10. The kinetic energy of a spinning yo-yo (K, W, J, Q)

Hints

A. What is the inertia of a bowling ball?
B. How long a time interval is needed for Earth to make one revolution around the Sun?
C. What simple geometric shape is an appropriate model for a sleeping person?
D. What is Earth's rotational speed?
E. What is the combined inertia of the wheel and tire?
F. What is the relationship between force and acceleration for this orbit?
G. How can you model the skater's shape during her spin?
H. What is the inertia of a midsize car?
I. What is the radius of the tire?
J. How many turns are needed to rewind the yo-yo?
K. What is the yo-yo's rotational inertia?
L. What is the radius of Earth's orbit?
M. What is the perpendicular distance from the house to the car's line of motion?
N. What is the skater's rotational inertia with arms held out?
O. How can you model the combined rotational inertia of the wheel and tire?
P. What is Earth's radius?
Q. What is the final rotational speed?
R. What is the radius of a bowling ball?
S. What is the rotational speed of the tire?
T. What is the required centripetal acceleration?
U. What is the skater's initial rotational speed?
V. What is your inertia?
W. When thrown, how long a time interval does the yo-yo take to reach the end of the string?
X. What is needed in addition to the formulas in Table 11.3 in order to determine this quantity?
Y. What is a typical speed for a car moving on a city street?
Z. What is Earth's inertia?
AA. What is a typical freeway cruising speed?

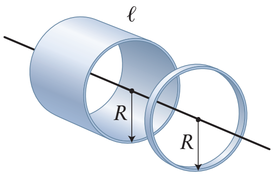
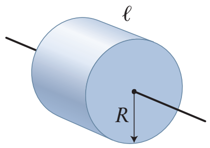
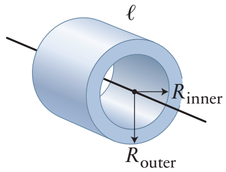
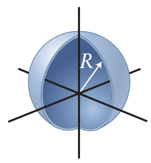
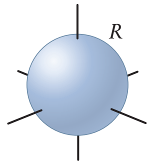
Key (all values approximate)

A. 7 kg; **B.** 1 y $= 3 \times 10^7$ s; **C.** solid cylinder of radius 0.2 m; **D.** period $=$ 24 h, so $\omega = 7 \times 10^{-5}\ \text{s}^{-1}$; **E.** 10^1 kg; **F.** from Eqs. 8.6, 8.17, and 11.16, $\sum\vec{F} = m\vec{a}$, so $mg = mv^2/r$; **G.** a solid cylinder with two thin-rod arms of inertia 4 kg held out perpendicularly; **H.** 2×10^3 kg; **I.** 0.3 m; **J.** 2×10^1 turns; **K.** $6 \times 10^{-5}\ \text{kg}\cdot\text{m}^2$ (with yo-yo modeled as solid cylinder); **L.** 2×10^{11} m; **M.** 2×10^1 m; **N.** $4\ \text{kg}\cdot\text{m}^2$; **O.** between MR^2 (cylindrical shell representing tire) and $MR^2/2$ (solid cylinder representing wheel)—say, $3MR^2/4$; **P.** 6×10^6 m; **Q.** about twice the average rotational speed, or $\omega = 5 \times 10^2\ \text{s}^{-1}$; **R.** 0.1 m; **S.** no slipping, so $\omega = v/r \approx 10^2\ \text{s}^{-1}$; **T.** $8 \times 10^{-3}\ \text{m/s}^2$; **U.** $\omega \approx 10\ \text{s}^{-1}$; **V.** 7×10^1 kg; **W.** 0.5 s; **X.** the parallel-axis theorem; **Y.** 3×10^1 mi/h; **Z.** 6×10^{24} kg; **AA.** 3×10^1 m/s

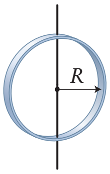
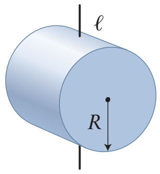
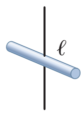
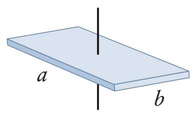
Worked and Guided Problems

Table 11.3 **Rotational inertia of uniform objects of inertia *M* about axes through their center of mass**

Rotation axes oriented so that object could roll on surface: For these axes, rotational inertia has the form cMR^2, where $c = I/MR^2$ is called the *shape factor*. The farther the object's material from the rotation axis, the larger the shape factor and hence the rotational inertia.

	thin-walled cylinder or hoop	solid cylinder	hollow-core cylinder	thin-walled hollow sphere	solid sphere
Shape					
Rotational inertia	MR^2	$\frac{1}{2}MR^2$	$\frac{1}{2}M(R^2_{\text{outer}} + R^2_{\text{inner}})$	$\frac{2}{3}MR^2$	$\frac{2}{5}MR^2$
Shape factor $c = I/MR^2$	1	$\frac{1}{2}$	$\frac{1}{2}\left[1 + \left(\frac{R_{\text{inner}}}{R_{\text{outer}}}\right)^2\right]$	$\frac{2}{3}$	$\frac{2}{5}$

Other axis orientations

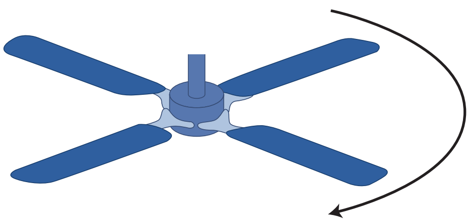
	thin-walled hoop	solid cylinder	thin rod	rectangular plate
Shape				
Rotational inertia	$\frac{1}{2}MR^2$	$\frac{1}{4}MR^2 + \frac{1}{12}M\ell^2$	$\frac{1}{12}M\ell^2$	$\frac{1}{12}M(a^2 + b^2)$

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 11.1 The fan

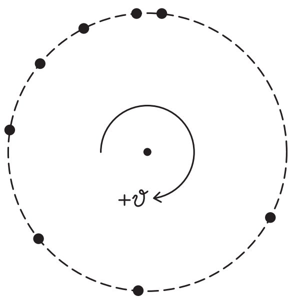
The ceiling fan in Figure WG11.1 has blades that are 0.60 m long and is spinning at 80 rev/min. It is turned off and comes to rest in 40 s. (*a*) What are the speed v and the magnitude a_c of the centripetal acceleration of the tip of any blade just before the fan is switched off? (*b*) After it is switched off, what is the fan's average rotational acceleration? (*c*) How many revolutions does the fan complete before coming to rest?

Figure WG11.1



❶ **GETTING STARTED** We are given information about initial and final rotational speeds, $\omega_i = 80$ rev/min and $\omega_f = 0$. We must obtain kinematic quantities, and this sounds like a straightforward rotational kinematics problem. We approach this problem as we would a kinematics problem involving translational motion. We begin with a motion diagram for the tip of one of the blades (Figure WG11.2). We note that the fan is slowing down, which means that the points on the circle representing the blade tip at different instants get closer together as time goes on. We have to pick a direction for the positive sense of the rotation, which we arbitrarily choose to be clockwise as we look up at the fan. This is the equivalent of choosing the direction of the x axis in a translational motion problem. With this choice for direction, the initial rotational velocity is positive. The rotational variables we need should be directly available from the equations given in the *Principles and Practice* text and the centripetal acceleration can be obtained once we have computed the speed v of the blade tip.

Figure WG11.2



❷ **DEVISE PLAN** The magnitude of the tip's centripetal acceleration is $a_c = v^2/r$, and its speed is $v = \omega r$. The magnitude of the fan's average rotational acceleration after the fan is turned off can be obtained from $\alpha_\vartheta = \Delta\omega_\vartheta/\Delta t$. The number of revolutions can be computed from this rotational motion expression from Table 11.2:

$$\Delta\vartheta = \omega_{\vartheta,i}\Delta t + \tfrac{1}{2}\alpha_\vartheta(\Delta t)^2.$$

❸ **EXECUTE PLAN**
(*a*) Before we can calculate the numerical answers, we need to convert the given initial rotational speed from revolutions per minute to inverse seconds. The initial rotational speed is the operating speed, which is constant until the fan turns off. Therefore its initial instantaneous value is equal to the average (operating) value:

$$\omega_i = |\omega_{\vartheta,i}| = \left|\frac{\Delta\vartheta}{\Delta t}\right| = \left(\frac{1}{1\text{ rad}}\right)\frac{\Delta\theta(\text{rad})}{\Delta t}$$

$$= 80\,\frac{\text{rev}}{\text{min}}\left(\frac{2\pi\text{ rad}}{\text{rev}}\right)\left(\frac{1}{1\text{ rad}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 8.4\text{ s}^{-1}.$$

Before the fan starts to slow down, the speed of the blade tip is

$$v_i = \omega_i r = (8.4\text{ s}^{-1})(0.60\text{ m}) = 5.0\text{ m/s}. \checkmark$$

The magnitude of the initial centripetal acceleration of the tip is

$$a_c = \frac{v_i^2}{r} = \frac{(5.0\text{ m/s})^2}{0.60\text{ m}} = 42\text{ m/s}^2. \checkmark$$

(*b*) From our choice of clockwise as the positive rotation sense, the initial rotational velocity is positive: $\omega_{\vartheta,i} = +\omega_i$. The average rotational acceleration during spin-down is

$$\alpha_\vartheta = \frac{\Delta\omega_\vartheta}{\Delta t} = \frac{\omega_{\vartheta,f} - \omega_{\vartheta,i}}{\Delta t} = \frac{0 - (+8.4\text{ s}^{-1})}{40\text{ s}} = -0.21\text{ s}^{-2}, \checkmark$$

with the minus sign telling us that the direction of this rotational acceleration is counterclockwise, opposite our clockwise choice for the positive sense of rotation.
(*c*) The angle the fan goes through in slowing down can be found from the change in the rotational coordinate:

$$\Delta\vartheta = \omega_{\vartheta,i}\Delta t + \tfrac{1}{2}\alpha_\vartheta(\Delta t)^2$$

$$= (+8.4\text{ s}^{-1})(40\text{ s}) + \tfrac{1}{2}(-0.21\text{ s}^{-2})(40\text{ s})^2$$

$$= 1.7 \times 10^2.$$

PRACTICE

Thus the change in angle is equivalent to 170 rad, or

$$(170 \text{ rad})\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 27 \text{ rev.} ✔$$

❹ **EVALUATE RESULT** The negative sign in our result for part *b*, -0.21 s^{-2}, indicates that the rotational acceleration is causing the rotational velocity to change in the direction opposite our chosen positive direction. Because the initial rotational velocity is positive, the negative value for rotational acceleration means the rotational velocity is becoming less positive and its magnitude is decreasing; in other words, the fan is slowing down. This is also consistent with a negative tangential acceleration of the tip.

The numerical values do not seem unusually large or small for a ceiling fan. It was necessary to treat the average rotational acceleration as if it were a constant rotational acceleration in order to use the kinematic equation for the angle the blade tip turns through as the fan slows down. This is permissible because, just as with translational motion, an average acceleration produces the same effect as a constant acceleration of the same magnitude.

Guided Problem 11.2 Swift shuttle

A space shuttle follows a circular orbit at an altitude of 300 km above the ground and has a period of 90.5 min. What are the shuttle's (*a*) rotational speed ω, (*b*) speed v, and (*c*) centripetal acceleration magnitude a_c?

❶ **GETTING STARTED**

1. How is rotational speed related to the given information?
2. Are kinematic equations sufficient to provide the relationships you need?
3. Sketch a motion diagram for the shuttle. How should the dots representing its position at various instants be spaced?

❷ **DEVISE PLAN**

4. Write an expression for the speed in terms of the rotational speed.
5. What is the numerical value of the orbit radius?
6. Write an expression for the magnitude of the centripetal acceleration.

❸ **EXECUTE PLAN**

7. Substitute numerical values for the known quantities. Watch out for units!

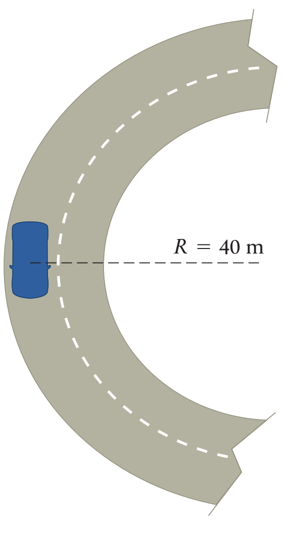
❹ **EVALUATE RESULT**

8. Is the speed reasonable for a spacecraft? How does the acceleration compare to g?

Worked Problem 11.3 Careful on that curve!

A highway has a flat section with a tight level curve of radius 40 m (Figure WG11.3). If the coefficient of static friction between tires and pavement surface is 0.45, what is the maximum speed v at which a car of inertia m can negotiate the curve safely—in other words, without the tires slipping?

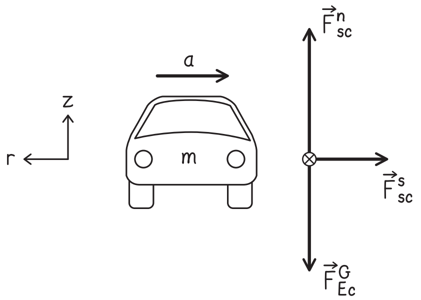
Figure WG11.3



PRACTICE

❶ **GETTING STARTED** We know from experience that if it goes too fast in the curve, the car goes off the outer edge of the road. This tells us there must be an inward force exerted on the car to provide centripetal acceleration and thus keep it moving in circular motion. In this case, that force is the frictional force. (Note that even though the tires are moving, the force is the force of *static* friction because the tires are not sliding on the road surface.) Let us use Newton's second law, keeping in mind that there is an acceleration toward the center of the curve. The first step is to produce a free-body diagram for the car and choose our coordinate directions (Figure WG11.4).

Figure WG11.4



The frictional force is always directed opposite the direction the car would slide if there were no friction, which would be outward in this case. Thus the frictional force is aimed inward, supplying the necessary centripetal component of force directed toward the center of the curve (to the right in Figure WG11.4).

❷ **DEVISE PLAN** Because the acceleration and the frictional force are along our chosen radial r axis, we begin with Newton's second law along that axis:

$$\begin{aligned} \sum F_r &= ma_r \\ F^n_{sc,r} + F^s_{sc,r} + F^G_{Ec,r} &= ma_r \\ 0 + (-F^s_{sc}) + 0 &= m(-a_c) \\ F^s_{sc} &= ma_c \qquad (1) \end{aligned}$$

The centripetal acceleration magnitude is related to the car's speed through $a_c = v^2/R$, where R is the (fixed) radius of the curve. The frictional force is related to the normal force through Eq. 10.54, $F^s_{sc} \leq \mu_s F^n_{sc}$. At the maximum speed, just before slipping, $\leq$ is replaced by $=$ and we have

$$F^s_{sc,max} = \mu_s F^n_{sc}. \qquad (2)$$

This means we have enough equations to solve for the maximum speed v if we know the normal force. Because the normal force is along the z axis, we use Newton's second law along that axis to determine its magnitude.

❸ **EXECUTE PLAN** Starting with Eq. 1 in the form

$$F^s_{sc} = ma_c = m\frac{v^2}{R},$$

we have, at maximum speed,

$$F^s_{sc,max} = m\frac{v^2_{max}}{R}.$$

From Eq. 2 we make the substitution

$$\mu_s F^n_{sc} = m\frac{v^2_{max}}{R}. \qquad (3)$$

To get the magnitude of the normal force, we use Newton's second law in the z direction:

$$\begin{aligned} \sum F_z &= ma_z \\ F^n_{sc,z} + F^s_{sc,z} + F^G_{Ec,z} &= ma_z \\ (+F^n_{sc}) + 0 + (-mg) &= m(0) \\ F^n_{sc} &= mg. \end{aligned}$$

We substitute this result into Eq. 3 to get

$$\mu_s mg = m\frac{v_{\max}^2}{R}$$

$$v_{\max} = \sqrt{\mu_s gR}$$

$$= \sqrt{(0.45)(9.8\,\text{m/s}^2)(40\,\text{m})}$$

$$= 13\,\text{m/s} = 30\,\text{mi/h.}\ \checkmark$$

❹ **EVALUATE RESULT** The most remarkable feature of the result we obtained for $v_{\max}$ is that $v_{\max}$ does not depend on the car's inertia. This means that the maximum safe speed is 30 mi/h for both a compact car and a cement truck. Does this make sense? It seems that a truck should go more slowly because more force is required to keep it moving in a circle with the same acceleration as that exerted on a compact car. Although that is true, the frictional force is proportional to the normal force, which in turn is proportional to the inertia. Therefore the inertia cancels out.

A larger coefficient of friction μ_s means you can go faster without slipping, as is also the case for a bigger (straighter) curve with larger radius R. This is exactly what $v_{\max} = \sqrt{\mu_s gR}$ says.

No assumptions were made, but we should review our decision to use the maximum available force of static friction. In order for the tires to slip, we must require an inward force greater than friction can provide, and so our use of maximum values is correct. The numerical value for the maximum speed is plausible and raises no red flag to cause us to question our numerical result.

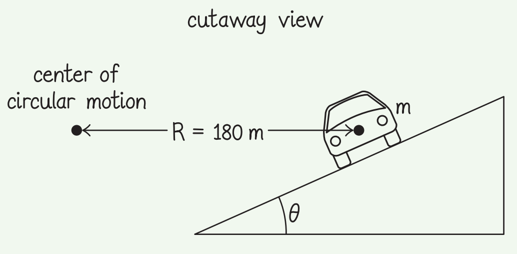
Guided Problem 11.4 It's in the bank

Highway curves are often banked to reduce a vehicle's reliance on friction when negotiating the turn: On a banked curve, there is a centripetal component of normal force acting on the vehicle. When the angle and speed are such that friction plays no role in a vehicle's motion in the curve, the nature of the road surface is immaterial and thus the posted speed limit applies in both wet and dry weather. Suppose the posted speed limit is 100 km/h on a curve of radius 180 m. At what angle θ to the horizontal should the curve be banked so that reliance on friction is not necessary?

❶ **GETTING STARTED**

1. Draw a cutaway view that shows the bank angle θ (Figure WG11.5).

Figure WG11.5



2. Do you notice the similarity to Worked Problem 11.3 and to the roller skater leaning into a curve in Example 11.4?

❷ **DEVISE PLAN**

3. What forces are exerted on the car? Remember that you want the force of static friction to be zero.
4. Draw a free-body diagram of the car as seen in the cutaway view.
5. Show the centripetal acceleration that keeps the car moving in a circle in your diagram. Be careful about the direction of this acceleration. Where is the center of the circle?
6. Which directions provide the best choice of an xy coordinate system? Recall that the acceleration should lie along one axis.
7. Count the unknowns. Does the bank angle appear in your equations?

❸ **EXECUTE PLAN**

8. From the component force equations and the relationship of centripetal acceleration magnitude to speed, perform some algebra to solve for the bank angle.
9. Get the numerical value for the angle. Don't forget units.

❹ **EVALUATE RESULT**

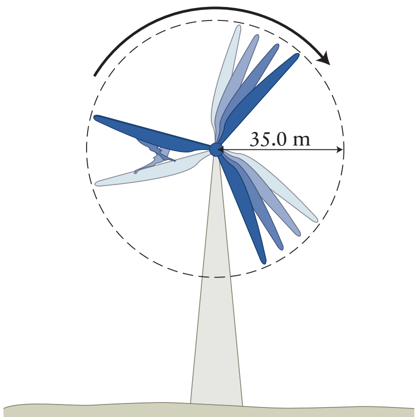
10. Examine your algebraic expression for the bank angle. How does the required angle change as the speed increases? As the radius of the curve increases? Are these variations what you expect?
11. Does the angle depend on a vehicle's inertia? Is this reasonable?
12. Consider the numerical value. Is this a reasonable angle? How steep are the angles you usually see on highways?

Worked Problem 11.5 Wind power

The commercial wind turbine in Figure WG11.6 consists of three 6400-kg blades, each 35.0 m long. It generates 1.0 MW of electrical power when it rotates with very little friction at a constant speed in the usually steady wind. However, when the wind suddenly stops, the turbine spins to a stop in 100 s. From this information, estimate the following for the situation when the wind is blowing steadily: (*a*) the rotational kinetic energy of the turbine, (*b*) its rotational speed, and (*c*) the magnitude of its angular momentum.

❶ **GETTING STARTED** When the rotational speed ω is constant, the turbine converts wind energy to electrical energy without changing its own rotational kinetic energy. When the wind stops, the turbine's rotational kinetic energy is converted to electrical energy for as long as that rotational kinetic energy lasts. It is this conversion of energy, not friction, that slows the turbine down when the wind stops. We can gain information about the constant-speed state by backtracking from the spin-down data.

Figure WG11.6



❷ **DEVISE PLAN** When the wind is steady, the turbine rotates with a constant rotational speed and has a certain amount of rotational kinetic energy. Once the wind dies, that kinetic energy is converted to electrical energy as the turbine spins down. During the spinning down, the work done by its environment on the turbine is equal to the change in the rotational kinetic energy:

$$\Delta K_{\mathrm{rot}} = W_{\mathrm{ct}}.$$

In Chapter 9 we saw that the work done by the environment on a system has the same magnitude as the work done by the system on its environment, but the two carry opposite signs. Therefore the work done by the environment on the turbine during the spin-down interval is the negative of the work W_{tc} done by the turbine on the environment during that interval. This work is the average power generated by the turbine multiplied by the time interval the turbine takes to come to rest:

$$\begin{aligned} K_{\mathrm{rot,f}} - K_{\mathrm{rot,i}} &= W_{\mathrm{ct}} \\ &= -W_{\mathrm{tc}} \\ &= -P_{\mathrm{av}}\Delta t. \end{aligned} \tag{1}$$

Knowing that the final rotational kinetic energy is zero, we can determine the initial rotational kinetic energy if we can reasonably estimate the average power during spin-down. Once we have $K_{\mathrm{rot,i}}$, we can get the initial rotational speed ω_i and the magnitude of the angular momentum from

$$K_{\mathrm{rot,i}} = \tfrac{1}{2} I_{\mathrm{t}} \omega_{\mathrm{i}}^2 \tag{2}$$

$$L_{\vartheta} = I_{\mathrm{t}} \omega_{\vartheta \mathrm{i}}. \tag{3}$$

PRACTICE

❸ **EXECUTE PLAN** (*a*) We suspect that the rate of power generation decreases as the blades slow down, but we are not given sufficient information to make this statement quantitative. So we need to make a reasonable assumption. Let us say that the average power generated during spin-down is half the constant-speed power of 1.0 MW: $P_{av} = 500$ kW. Then we can say from Eq. 1 that

$$\begin{aligned} K_{rot,f} - K_{rot,i} &= -P_{av}\Delta t \\ 0 - K_{rot,i} &= -P_{av}\Delta t \\ K_{rot,i} &= P_{av}\Delta t \\ &= (5.0 \times 10^5 \text{ J/s})(100 \text{ s}) \\ &= 5.0 \times 10^7 \text{ J} \\ &= 50 \text{ MJ}. ✔ \end{aligned}$$

(*b*) To obtain the rotational speed ω_i when the wind is blowing at a constant speed, we solve Eq. 2 for this variable:

$$\omega_i^2 = \frac{2K_{rot,i}}{I_t}.$$

First we must obtain the rotational inertia for the turbine, which can be modeled as three rods whose ends abut the turbine axle. For a rod of length ℓ and inertia m, the rotational inertia about an axis through one end of the rod is $m\ell^2/3$ (see Example 11.11). Thus the rotational inertia for the three-blade turbine is $I_t = 3(m\ell^2/3) = m\ell^2 = (6400 \text{ kg})(35 \text{ m})^2 = 7.84 \times 10^6 \text{ kg}\cdot\text{m}^2$. The steady-state rotational speed of the turbine is then

$$\begin{aligned} \omega_i &= \sqrt{\frac{2K_{rot,i}}{I_t}} \\ &= \sqrt{\frac{2(5.0 \times 10^7 \text{ J})}{7.84 \times 10^6 \text{ kg}\cdot\text{m}^2}} \\ &= 3.6 \text{ s}^{-1}. ✔ \end{aligned}$$

(*c*) For the angular momentum, Eq. 3 gives

$$\begin{aligned} |L_{\vartheta,i}| &= |I_t\omega_{\vartheta,i}| \\ &= I_t\omega_i \\ &= (7.84 \times 10^6 \text{ kg}\cdot\text{m}^2)(3.6 \text{ s}^{-1}) \\ &= 2.8 \times 10^7 \text{ kg}\cdot\text{m}^2/\text{s}. ✔ \end{aligned}$$

❹ **EVALUATE RESULT** By being careful in distinguishing between work done *by* the turbine and work done *on* it, we have gotten the signs right: We did not wind up with ω_i^2 being negative. The initial rotational speed $\omega_i = 3.6 \text{ s}^{-1}$ corresponds to a rotational period $T_i = 2\pi/\omega_i = 1.7$ s. This seems reasonable for a large commercial wind turbine and is consistent with observations if you have driven by a wind farm: The blades take a couple of seconds to complete each revolution.

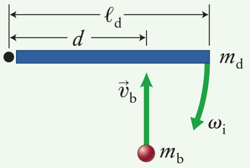
We made several approximations: that the turbine blades are rod-shaped, that the rotational inertia of the turbine hub can be ignored, and that the average power during spin-down is half the constant-speed power. This last approximation is the most uncertain, but the average power should be some fraction of the constant-speed power because it must be between the initial and final values. Instead of saying the average power is half the constant-speed value, we might have said one-third or two-thirds. Those choices are both reasonable and would affect the numerical answers but not the spirit of our solution.

The hub's rotational inertia should not be significant because the hub radius is much smaller than the length of the blades. Modeling the blades as rods introduces some error, but the alternative is a very complicated integration procedure requiring specific information about the blades' shape. The error is not greater than the uncertainty introduced by the assumption that $P_{av} = P_{constant\ speed}/2$.

Guided Problem 11.6 Hit that door

You stand in front of a door of inertia m_d and width ℓ_d as it swings toward you with rotational speed ω_i. You throw a ball of inertia m_b and speed v_b toward the door so that it strikes the door perpendicularly at a distance d from the hinged edge, as shown in the overhead view (Figure WG11.7). The ball bounces directly back toward you with one-fourth its original speed. (*a*) What is the final rotational speed ω_f of the door after the collision? (*b*) What must the speed v_b of the ball be at impact if you want the door to swing away from you after the ball hits?

Figure WG11.7



1 GETTING STARTED

1. What is the best choice of system? Is this system isolated? Closed?
2. Modify Figure WG11.7 to obtain a two-part sketch of the system before and after the collision. Because rotational quantities have signs, include a curved arrow about the rotation axis to indicate the direction you choose for the positive sense of rotation.

2 DEVISE PLAN

3. How many unknown quantities do you seek in part *a*? Are there additional unknowns for part *b*?
4. What quantity is constant in this collision? Write the algebraic expression stating the equality of the initial and final values. Make sure to account for all the objects in your system on both sides of the equation.
5. What is the most convenient choice of location for the reference axis? Select the appropriate signs for each term in your equation. Base your decisions about signs on the chosen direction of positive rotation, even for the ball moving in a straight line.

3 EXECUTE PLAN

6. Solve for the door's final rotational speed ω_f.
7. How is ω_f different if the door swings away from you instead of toward you after the ball hits?

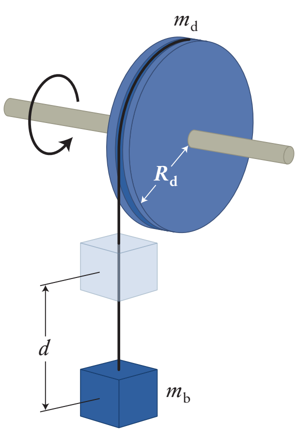
4 EVALUATE RESULT

8. Does the equation you get scale as you expect with changes in m_b, m_d, v_i, ℓ_d, and d? For instance, how does ω_f behave as the ball's inertia gets very large or very small?

Worked Problem 11.7 Just unwind

A block of inertia m_b is attached to a very light string wound about a uniform disk of inertia m_d and radius R_d, which rotates on a horizontal axle (Figure WG11.8). If the block is released from rest, what are its speed $v_{b,f}$ and acceleration a_b after it has fallen a distance d? Ignore any friction force exerted on or by the axle.

Figure WG11.8



PRACTICE

❶ **GETTING STARTED** Here we have two kinds of motion: the rotational motion of the disk about the axle and the translational motion of the block falling under the influence of gravity. We might try applying conservation of energy or angular momentum to determine the block's speed, but we will likely need Newton's second law or kinematic equations to get the block's acceleration. If we choose the disk, block, and string as our system, then the system's angular momentum about the axle does not remain constant—the block and disk both contribute increasing angular momentum in the same direction. We have no experience in dealing with possible changes in Earth's angular momentum, and so perhaps we should try conservation of energy. If we choose the disk, block, string, and Earth as our system, our system is closed.

❷ **DEVISE PLAN** We can determine the speed of the block from its kinetic energy $\frac{1}{2}mv^2$. It gains kinetic energy through the conversion of gravitational potential energy as it falls. Some of the initial potential energy must also go into the kinetic energy of the disk, however, because the rotational and translational motions are connected by the string. Fortunately, we know the relationship: Because of the connecting string, a point on the outer rim of the disk must always have the same speed v as the block. Thus from Eq. 11.10 $v_b = \omega_d R_d$. Because there is no dissipation of energy and no source energy is involved, we know that the mechanical energy $K + U$ of the system is constant:

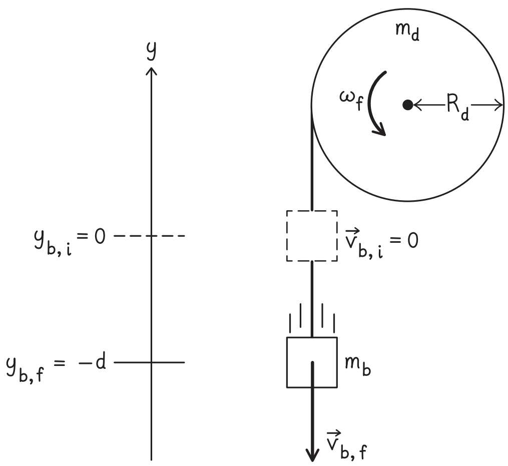
$$K_f + U_f = K_i + U_i.$$

(Remember that the absence of any subscripts denoting disk or block tells you that these variables are for the system as a whole.)

We can use energy accounting to determine the block's final speed $v_{b,f}$ after it has fallen a distance d. We can then use kinematic equations to get the block's acceleration a_b. The disk can be thought of as a very short solid cylinder, and so we can use the solid cylinder rotational inertia from Table 11.3: $I = mR^2/2$.

Because we want to use the formula for gravitational potential energy (mgy), we choose the positive y axis upward because that direction was assumed in the derivation of this equation (Eq. 7.21). We also make a diagram to represent all the information given in a convenient way (Figure WG11.9).

Figure WG11.9



For convenience, we place the origin of our y axis at the initial level of the block, making $y_{b,i} = 0$ (and thus $U_{b,i} = m_b g y_{b,i} = 0$). Because the system starts at rest, $K_i = 0$. After the block has fallen a distance d, it is at $y_{b,f} = -d$, information that we put in our diagram. We can get the block's final speed $v_{b,f}$ by using this final position $-d$ and the relationship between the block's speed and the disk's rotational speed: $v_{b,f} = v_{d,f} = R_d\omega_{d,f}$.

❸ **EXECUTE PLAN** For our system (disk, block, string, Earth), energy conservation tells us that

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}m_b v_{b,f}^2 + \frac{1}{2}I_d\omega_{d,f}^2 + m_b g y_{b,f} + m_d g y_{d,f}$$
$$= 0 + 0 + m_b g y_{b,i} + m_d g y_{d,i}. \quad (1)$$

PRACTICE

Because the vertical position of the disk does not change, $m_d g y_{d,f} = m_d g y_{d,i}$, and so we drop these two terms from the equation. Next, we eliminate the factor containing ω by modifying Eq. 11.10 to $\omega_{d,f} = v_{d,f}/R_d$ and then replacing $v_{d,f}$ with $v_{b,f}$ because any point on the disk rim has the same speed as the block. Then, with I_d replaced by $m_d R_d^2/2$, Eq. 1 becomes

$$\tfrac{1}{2}m_b v_{b,f}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}m_d R_d^2\right)\left(\frac{v_{b,f}}{R_d}\right)^2 + m_b g(-d) = m_b g(0)$$

$$\tfrac{1}{2}m_b v_{b,f}^2 + \tfrac{1}{4}m_d v_{b,f}^2 = m_b g d$$

$$v_{b,f}^2 = \frac{2m_b}{m_b + \frac{1}{2}m_d} g d$$

$$v_{b,f} = \sqrt{\frac{2m_b}{m_b + \frac{1}{2}m_d} g d}. \checkmark \tag{2}$$

Because we do not know the time interval needed for the block to travel the distance d but we do know the initial and final speeds, we can use Eq. 3.13, $v_f^2 = v_i^2 + 2a(x_f - x_i)$, for the block's motion along the y axis to get its acceleration:

$$v_{by,f}^2 = v_{by,i}^2 + 2a_{by}(y_f - y_i)$$

$$\frac{2m_b}{m_b + \frac{1}{2}m_d} g d = (0)^2 + 2a_{by}(-d - 0) = -2a_{by} d$$

$$a_{by} = -\frac{m_b}{m_b + \frac{1}{2}m_d} g. \checkmark$$

❹ **EVALUATE RESULT** If it were not connected to the disk by the string, the block would be in free fall and its speed after it had fallen a distance d would be the familiar free-fall result $v = \sqrt{2gd}$. In our case, the speed is lower because of the presence of the term $\frac{1}{2}m_d$ in the denominator of Eq. 2. We expect this lower speed because some of the gravitational potential energy the system loses as the block falls must go into increasing the rotational kinetic energy of the disk rather than all of it going into increasing the kinetic energy of the block. We also expect that if $m_d \ll m_b$, the disk should have little effect. If we set $m_d = 0$ to get the limiting case, we indeed get the expected free-fall speed.

The negative sign in our acceleration expression indicates that the acceleration is downward because the positive y axis points upward, as we expect for a falling block that starts from rest. Because not all of the system's potential energy goes into translational kinetic energy, the block cannot fall as fast as it would if there were no rotational diversion. Therefore an acceleration magnitude less than g makes sense.

Guided Problem 11.8 Globe pulling

You have a very large globe of Earth that has an inertia of 20 kg and a radius of 0.50 m. It can rotate freely about its polar axis, and you want to start it spinning by wrapping a string around its equator and then pulling on the string. You plan to attach one end of the string to the equator with a bit of museum wax, so that it will not slip until all the string is unwound. If you are restricted to exerting a continuous pulling force of magnitude 2.0 N, how much string do you need to wrap tightly around the globe to make it spin at 0.50 rev/s?

❶ **GETTING STARTED**

1. Draw a sketch of the globe and string showing how the pulling force is exerted.
2. Is your force tangential to the globe surface? Which quantity should you focus on: angular momentum or energy? Can you compute the effect of your force on the quantity you choose?
3. What must you assume about whether the string slips as it is pulled?

❷ **DEVISE PLAN**

4. Write an equation for the quantity you named in step 2. Does the rotational speed appear in this expression?
5. What is the globe's rotational inertia? (Is the globe more like a thin spherical shell or a solid sphere?) How does this variable come into play in this problem?
6. How does the string length come into your calculation? Would a tangential force applied continuously at a single point on the equator (such as a force produced by a small rocket motor attached to the globe) produce the same effect?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

Answers to Review Questions

1. Many answers possible. Some are: (*a*) falling brick, racehorse on home stretch, cruising airplane; (*b*) spinning compact disc, Ferris wheel, roulette wheel; (*c*) thrown Frisbee, propeller on outboard motor; bowling ball headed toward pins.
2. (*a*) No, because constant velocity means zero acceleration. (*b*) Yes, because a change in direction means nonzero acceleration even if there is no change in speed.
3. The rotational coordinate specifies the location of the object on the circle in terms of a unitless number that increases by 2π for each revolution of the object. The polar angle specifies the location of the object on the circle in units of radians, degrees, or revolutions, typically measured from the positive x axis to the radius vector that locates the object.
4. The position vector points from the center of the circular trajectory to the object at the chosen instant. The velocity vector is perpendicular to the position vector, tangent to the circular trajectory in the direction of the object's motion (along the tangential axis). The acceleration vector points opposite the position vector, toward the center of the circular trajectory.
5. At the beginning of the car's curving motion, your shoulders continue going straight but your legs and lower body are pulled to the left by the frictional force exerted by the car seat. Because the car is accelerating into the curve (leftward) in the Earth reference frame, the car door comes in to meet your right shoulder, an occurrence you mistake for your shoulders traveling out of the curve (rightward) in the car's reference frame.
6. Because a rotating reference frame is accelerating and therefore noninertial. From Chapter 6 we know that the methods we use in this text for monitoring energy and momentum changes are not valid in noninertial reference frames.
7. A vector sum of the forces must be directed toward the center of the circle to make the object move at a constant speed in circular motion. The vector sum of the forces provides the required centripetal acceleration, and it increases with increasing speed and decreases with increasing radius.
8. No. An intrinsic property (see Section 4.6) is one that we cannot change without changing the object. The value we calculate for a given object's rotational inertia depends on what we choose as the axis of rotation. The rotational inertia of a thin hoop, for instance, is mr^2 for rotation about an axis perpendicular to the hoop face but only $mr^2/2$ for rotation about an axis parallel to the hoop face. Same hoop, different values of I, meaning I is not an intrinsic property.
9. The rotation axis the walker is concerned about is a horizontal axis passing through her feet. Without the pole, all the material in the system (just her) is near the axis. With the pole, some of the material of the walker-pole system is far from the axis, making it more difficult for her to rotate about that axis and fall.
10. Yes. Circular motion always involves centripetal acceleration. The other two forms of acceleration may or may not be involved in the motion. If speeding up or slowing down, the object has both rotational acceleration and tangential acceleration; these are related by a factor of the radius, so they always coexist.
11. The rotational coordinate is the polar angle (expressed in units of radians) divided by one radian:

$$\vartheta = \frac{\theta\,(\text{rad})}{1\ \text{rad}}.$$

12. Rotational acceleration has units of inverse seconds squared and measures the rate at which the rotational speed ω changes with time. Tangential acceleration has units of acceleration (m/s^2) and measures the rate at which the speed v changes with time. They are related in that the magnitude of the tangential acceleration equals the product of the radius and rotational acceleration. Centripetal acceleration has units of acceleration (m/s^2) and measures the rate at which the *direction* of velocity $\vec{v}$ changes with time. Centripetal acceleration is the radial component of acceleration (toward the center of the circle), while tangential acceleration is the component of acceleration tangent to the circle.
13. These relationships are found in Table 11.1: (a) $s = r\vartheta$, (b) $v_t = r\omega_\vartheta$, (c) $a_t = r\alpha_\vartheta$.
14. If the time interval is halved, the speed v is doubled. Equation 11.15, $a_c = v^2/r$, tells you that doubling the speed quadruples the centripetal acceleration.
15. The rotational inertia of a particle is defined as the product of the particle's inertia and the square of its radius of circular motion. The lever arm distance (or lever arm) is the perpendicular distance between the line of action of the momentum of the particle and the rotation axis.
16. The rotational kinetic energy of an object is related to its rotational inertia in the same way that the kinetic energy of a particle is related to its inertia, with rotational inertia substituted for inertia and rotational speed substituted for speed:

$$\tfrac{1}{2}mv^2 \leftrightarrow \tfrac{1}{2}I\omega^2.$$

17. The angular momentum of this object is the product of its momentum and the perpendicular distance from the line of action of its momentum to the axis of rotation: $L = I\omega = mvr_\perp$ (Eqs. 11.34 and 11.36).
18. It increases by a factor of 8. The doubling of I in $K_{\text{rot}} = \frac{1}{2}I\omega^2$ causes K_{rot} to increase by a factor of 2, and the doubling of ω causes K_{rot} to increase by a factor of 4.
19. Angular momentum can be neither created nor destroyed (but it can be transferred from one object to another).
20. Smallest rotational inertia: axis running through the center of the book and parallel to its largest dimension. Largest rotational inertia: axis passing through the center of the book and perpendicular to both covers.
21. Two axes pass through the molecule's center perpendicular to the bond connecting the two atoms; the third axis runs parallel to the bond. Rotation about either perpendicular axis involves a sizable rotational inertia. Rotation around the parallel axis involves a smaller rotational inertia and therefore a smaller kinetic energy.
22. The rotational inertia of an object about any axis A is equal to the rotational inertia of the same object about an axis that runs through its center of mass and is parallel to axis A plus a term equal to the product of the object's inertia and the square of the perpendicular distance between the two axes.

Answers to Guided Problems

Guided Problem 11.2 (*a*) $\omega = 1.16 \times 10^{-3}\ \text{s}^{-1}$, (*b*) $v = 7.73 \times 10^3\ \text{m/s}$, (*c*) $a_c = 8.94\ \text{m/s}^2$

Guided Problem 11.4 $\theta_{\text{bank}} = 23.6°$

Guided Problem 11.6 (*a*) $\omega_f = \dfrac{15m_b v_b d}{4m_d \ell_d^2} - \omega_i$, (*b*) $v_b > \dfrac{4m_d \ell_d^2 \omega_i}{15m_b d}$

Guided Problem 11.8 $\ell = 8.2\ \text{m}$

PRACTICE

Guided Practice by Chapter

12 Torque

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

12.1 Torque and angular momentum

1. Can a small force produce a greater torque than a large force?
2. A friend pushes a doorknob to open the door for you, and the door swings all the way open in 2 s. On the way back, you return the favor but push not on the knob but close to the hinged edge. Although the door again opens all the way in 2 s, your friend notices your exertion and comments, "You had to exert a lot more torque doing it that way." Is he right?
3. What reference point should you choose for computing torques for a stationary object that is not mounted on a pivot, axle, or hinge?
4. Describe a case where the line of action of a force exerted on an object that can rotate either freely or about a fixed axis passes through the pivot even though the point of application of the force is not at the pivot. What is the torque in this case?

12.2 Free rotation

5. Can an object in free rotation rotate or revolve about any point you might choose?
6. Why is the concept of center of mass so useful in analyzing rotational motion?

12.3 Extended free-body diagrams

7. When you hold your left forearm parallel to the floor, what is the direction of the force exerted by the humerus bone on the elbow joint?
8. What is the main difference between a standard free-body diagram and an extended free-body diagram? Why is it necessary to use extended free-body diagrams when we study rotational motion?
9. If what you need in a problem is an extended free-body diagram, why draw a standard free-body diagram first? Isn't it possible to use an extended free-body diagram to compute vector sums of forces as well as vector sums of torques?

12.4 The vectorial nature of rotation

10. Describe the right-hand rule for determining a vector direction for the rotation of an object.
11. What is the direction of the rotational velocity vector for the second hand of a clock hanging on the wall?
12. Why are the words *clockwise* and *counterclockwise* insufficient to describe rotation in three dimensions?

12.5 Conservation of angular momentum

13. Which statement is correct: "In mechanical equilibrium, the vector sum of all the torques and forces is zero" or "In mechanical equilibrium, the vector sum of the torques is zero and the vector sum of the forces is zero."
14. Suppose a single force (one that is not canceled by any other forces) is exerted on an object. Is it possible for this single force to change both the object's momentum and its angular momentum?
15. A baton-twirler tosses her spinning baton up into the air. While it is in the air, is the baton's momentum constant? Is its angular momentum constant?
16. What is rotational impulse, and how is it related to the angular momentum law?
17. Under what circumstances is angular momentum conserved? Under what circumstances does the angular momentum of a system remain constant?

12.6 Rolling motion

18. For a wheel that is rolling without slipping, what is the relationship between the wheel's rotational speed and the speed v of its center of mass?
19. If two gears are in contact, with their teeth meshing, what do they have in common: rotational speed, tangential speed v_t of the teeth, or some other quantity?
20. What is the shape factor for a rolling object, and what does it measure?
21. An object rolls without slipping down a hill. How are the magnitudes of its rotational acceleration α_ϑ and the acceleration of its center of mass a_{cm} related?

12.7 Torque and energy

22. Explain how to compute the work done on a rotating object when you know both the sum of the torques caused by the external forces exerted on the object and the number of rotations the object makes in the time interval of interest.
23. If an object's motion is both translational and rotational, how should its kinetic energy be determined?
24. For an object that rolls without slipping, describe what effect static friction has on the work done on the object and on the object's energy.

12.8 The vector product

25. Draw a vector arrow on a sheet of paper, draw another vector arrow on a second sheet, and then lay the sheets on your desk in any arbitrarily chosen orientations. What angle does the vector representing the vector product of these two vectors make with the desktop? Does your answer depend on the orientation of the two vectors?
26. How is the magnitude of the vector product of two vectors related to the vectors?
27. What is $\vec{A} \cdot (\vec{A} \times \vec{B})$ equal to?
28. If you accidentally compute a torque using $\vec{F} \times \vec{r}$ rather than $\vec{r} \times \vec{F}$, what is wrong with your result?
29. What is the vector relationship between the angular momentum and the momentum for an isolated particle?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The magnitude of the torque caused by the force you exert to open your bedroom door (C, O)
2. The magnitude of the torque caused by the force required to accelerate a sports car from 0 to 60 mi/h in 4 s (E, R, K)
3. The maximum torque caused by the gravitational force exerted on your bicycle when you lean into a tight turn (A, V)
4. The length of a lever you need to raise the front end of a car in which the engine is in the front (B, H, N, W)
5. The gear ratio (ratio of front gear radius to rear gear radius) for the highest gear of a racing bicycle (D, I, S, Q)
6. The largest feasible gear ratio for a hand-held electric drill (G, L, U)
7. The maximum useful torque that can be delivered to each of a car's two drive wheels (Y, P, K)
8. The work required to remove and replace five lug nuts as you change a flat tire on your car (F, X)
9. The maximum torque caused by the force you exert on a diving board when you dive into a pool (Z, T, J, M)

Hints

A. What is the distance from the pivot to the center of mass?
B. What is the distance from the axis of rotation to the car's center of mass?
C. What is the lever arm distance?
D. What is the diameter of a bicycle wheel?
E. What acceleration is needed?
F. What torque through what angle is needed to "break" each nut loose?
G. What is the minimum number of teeth on a gear to allow smooth turning?
H. What is the distance from the car's axis of rotation to the point where the lever touches the car?
I. At what speed can a strong cyclist move in the highest gear?
J. What is the average force you exert on the diving board?
K. What is the radius of each wheel?
L. If each tooth is 1 mm thick, what minimum radius is needed for the smaller (driveshaft) gear?
M. What distance does the diving board extend beyond any support?
N. What is the ratio of the required lifting force to the gravitational force exerted on you by Earth?
O. What is the force magnitude?
P. What normal force (needed to support the car) is exerted on each drive wheel?
Q. What is the minimum comfortable time interval needed for the cyclist's foot to complete one cycle?
R. What is the vector sum of the forces required?
S. What is the rotational speed of the wheel?
T. What is the time interval between the instant you hit the board on your last bounce and the instant the board is at maximum depression?
U. What maximum gear radius could fit comfortably in the drill casing?
V. What is your maximum angle of lean?
W. What is the minimum distance from the point where you pivot the lever to the location where the lever touches the car?
X. What torque through what angle is needed to unscrew each loosened lug nut?
Y. What coefficient of friction can be used to move the car forward most effectively?
Z. If you jump high on your last bounce, with what speed do you hit the end of the diving board?

Key (all values approximate)

A. 1 m; **B.** axis at rear wheels, center of mass just behind front wheels, so 2 m; **C.** 0.7 m; **D.** 0.7 m; **E.** 7 m/s^2; **F.** 1×10^2 N · m through one-quarter of a turn; **G.** 1×10^1; **H.** about 3 m; **I.** 2×10^1 m/s; **J.** consider both gravitational force and momentum change divided by the time interval, about 2×10^3 N for an 80-kg person; **K.** 0.3 m; **L.** about 3 mm, allowing for spacing between teeth; **M.** less than 3 m; **N.** 14 to 1; **O.** 1×10^1 N; **P.** 4×10^3 N, or one-third of the gravitational force for a front-wheel-drive car; **Q.** 0.4 s; **R.** 0.7 times the gravitational force exerted on the car—say, 7 kN; **S.** 6×10^1 s^{-1}; **T.** 0.3 s; **U.** 3×10^1 mm; **V.** 40° away from vertical; **W.** 0.3 m; **X.** 8 N · m through 5 revolutions; **Y.** to avoid wasteful slipping, use the coefficient of static friction, which may be close to 1; **Z.** 3 m/s

Worked and Guided Problems

Procedure: Extended free-body diagrams

1. Begin by making a standard free-body diagram for the object of interest (the *system*) to determine what forces are exerted on it. Determine the direction of the acceleration of the center of mass of the object, and draw an arrow to represent this acceleration.
2. Draw a cross section of the object in the plane of rotation (that is, a plane perpendicular to the rotation axis) or, if the object is stationary, in the plane in which the forces of interest lie.
3. Choose a reference point. If the object is rotating about a hinge, pivot, or axle, choose that point. If the object is rotating freely, choose the center of mass. If the object is stationary, you can choose any reference point. Because forces exerted at the reference point cause no torque, it is most convenient to choose the point where the largest number of forces is exerted or where an unknown force is exerted. Mark the location of your reference point and choose a positive direction of rotation. Indicate the reference point in your diagram by the symbol ⦿.
4. Draw vectors to represent the forces that are exerted on the object and that lie in the plane of the drawing. Place the tail of each force vector at the point where the force is exerted on the object. Place the tail of the gravitational force exerted by Earth on the object at the object's center of mass.* Label each force.
5. Indicate the object's rotational acceleration in the diagram (for example, if the object accelerates in the positive ϑ direction, write $\alpha_\vartheta > 0$ near the rotation axis). If the rotational acceleration is zero, write $\alpha_\vartheta = 0$.

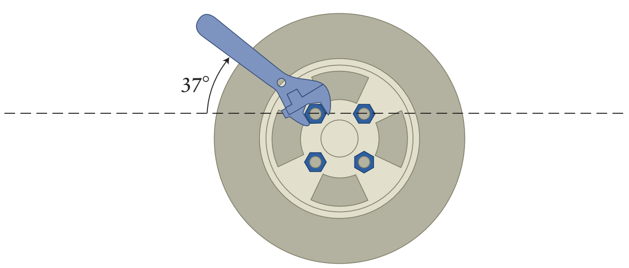
*Is the gravitational force really exerted at the center of mass? Suppose the gravitational force exerted by Earth were exerted at some other point. The force would then cause a permanent torque about the center of mass, and any object dropped from rest would begin spinning, which is not true.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 12.1 Using a wrench

Loosening the rusted lug nuts on a car tire requires 220 N · m of torque. (*a*) A 60-kg mechanic holds a wrench horizontally on one nut. The wrench is 400 mm long, and she holds it at the free end and exerts a force equal to the magnitude of the gravitational force exerted on her. If the direction of the force she exerts is straight downward, how much torque does she cause on the nut? (*b*) Suppose one nut is oriented such that the wrench makes a 37° angle with the horizontal, as in Figure WG12.1. By exerting the identical force she exerted in part *a*, can the mechanic loosen this nut with the 400-mm wrench? If not, how long a wrench must she use to loosen the nut?

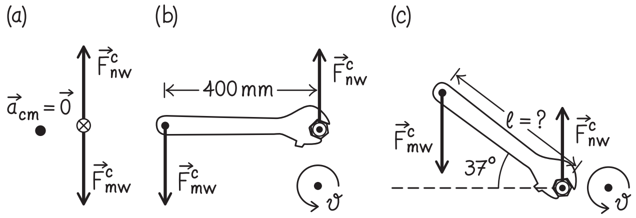
Figure WG12.1



❶ GETTING STARTED Given that this problem involves torques, we follow the procedure for drawing extended free-body diagrams at the top of this page. We begin by making a free-body diagram for the wrench (Figure WG12.2*a*). The wrench is subject to three forces: a downward force exerted by the mechanic on the wrench, an upward contact force exerted by the nut on the wrench, and a downward force of gravitation exerted by Earth on the wrench. Because the gravitational force on the wrench is negligible compared to the other two forces, we ignore this force in this problem. Next we draw extended free-body diagrams for the wrench for both parts of this problem (Figure WG12.2*b* and *c*). In both cases we choose the nut as the reference point. With that choice of reference point, only the force $\vec{F}^{c}_{mw}$ causes a torque.

PRACTICE

Figure WG12.2



❷ **DEVISE PLAN** The magnitude of the torque caused by a force exerted on an object is

$$\tau = |\vec{r} \times \vec{F}| = rF \sin \theta, \tag{1}$$

where θ is the angle between the line of action of the force and the position vector that points from the rotation axis to the point of application of the force. The magnitude of the contact force exerted by the mechanic on the wrench is the magnitude of the gravitational force exerted on her: $F^{c}_{mw} = F^{G}_{Em} = m_{m}g$. The lever arm distance $r_{\perp}$ is the wrench length ℓ because the point of application of her force is the very end of the wrench. In part *a* the angle between $\vec{r}$ and $\vec{F}^{c}_{mw}$ is 90°.

The force magnitude and direction remain the same in part *b*, but the angle θ changes. We note that the 37° angle the wrench makes with the horizontal is *not* the angle we use to calculate the torque. We want the angle between the downward force and the wrench, which is $\theta = 90° - 37°$. We can use Eq. 1 for part *b*, with $r = \ell$ in this case, if we solve for the wrench length: $\ell = \tau/(F \sin \theta) = \tau/(F^{G}_{Em} \sin \theta)$.

❸ **EXECUTE PLAN**
(*a*) The magnitude of the gravitational force exerted on the mechanic is $m_{m}g = (60 \text{ kg})(9.8 \text{ m/s}^2) = 588 \text{ N}$. The torque caused by the force exerted by the mechanic is then

$$\tau = (0.400 \text{ m})(588 \text{ N})(\sin 90°) = 235 \text{ N} \cdot \text{m}$$
$$= 2.4 \times 10^2 \text{ N} \cdot \text{m.} \checkmark$$

(*b*) With the wrench making a 37° angle with the horizontal but the force exerted vertically downward, the angle between the line of action of the force and the wrench is $\theta = 90° - 37° = 53°$. The torque her force causes now is

$$\tau = (0.400 \text{ m})(588 \text{ N})(\sin 53°) = 188 \text{ N} \cdot \text{m.}$$

This is insufficient to loosen the rusted nut. She can loosen it with a longer wrench, though. The minimum wrench length ℓ to achieve the needed 220-N · m torque is

$$\ell = \frac{\tau}{F^{G}_{Em} \sin \theta} = \frac{220 \text{ N} \cdot \text{m}}{(588 \text{ N})(\sin 53°)} = 0.47 \text{ m.} \checkmark$$

A slightly longer wrench will loosen the nut, or the mechanic can just put a piece of hollow pipe over the end of her 400-mm wrench to increase the lever arm distance a bit.

❹ **EVALUATE RESULT** Our answers are close to the needed torque given in the problem statement, 220 N · m, which gives us confidence in our results, but what other bounds can we establish to test their reasonableness? Consider the force $\vec{F}^{c}_{wm}$ exerted by the wrench on the mechanic in part *a*. By Newton's third law, this force is equal in magnitude to the force exerted by the mechanic on the wrench, but in the opposite direction. Making full use of the gravitational force exerted on the mechanic in our calculation means we assume that no support force other than $\vec{F}^{c}_{wm}$ is necessary to keep the mechanic from falling—the wrench provides all the required support. In fact, this is the greatest force she can exert on the wrench without invoking some creative arrangement. In order to exert a downward force greater than the gravitational force, the mechanic would need to be able to push upward against some nearby fixed object, perhaps with her feet. With no mention of such an effort in the problem statement, we assume she did no such contortions.

Exerting a horizontal force requires some frictional force to compensate, or a nearby object on which the mechanic can push horizontally with her feet in order to maintain a zero sum of horizontal forces exerted on herself. Coefficients of friction rarely exceed 1, so again the gravitational force is the upper limit for a typical person to exert on a wrench. In practice, the maximum force possible with a person's hands is likely to be a bit smaller than the gravitation force. Most people must stand on the wrench in order to supply a force as great as the gravitational force exerted on them by Earth.

Guided Problem 12.2 At the grindstone

A circular grindstone, radius 0.17 m, is used to sharpen a knife. The knife is pressed against the grindstone with a normal force of 20 N, and the coefficient of kinetic friction between knife and stone is 0.54. What torque must the motor driving the grindstone maintain in order to keep the stone rotating at constant speed?

❶ GETTING STARTED

1. Consider the force exerted by the knife on the grindstone. Does any component of this force cause a torque about the grindstone's axle?

❷ DEVISE PLAN

2. To maintain a constant rotational speed, what must be the vector sum of the torques caused by the forces exerted on the grindstone? What two torques are involved?
3. Does the normal force exerted by the knife on the grindstone contribute to the torque equation?

❸ EXECUTE PLAN

4. Compute the magnitude of the torque that must be canceled by the motor.

❹ EVALUATE RESULT

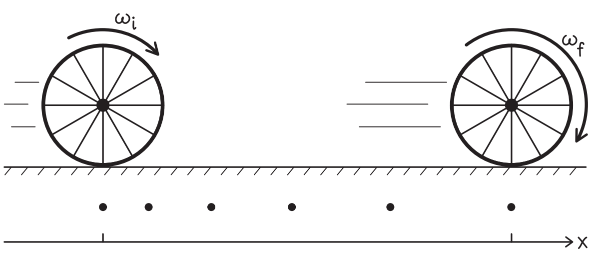
5. Examine the expression for torque used in step 4. What happens to the torque as the normal force changes? As the coefficient of friction changes? Are these two trends reasonable?
6. Consider your numerical result. Could a motor supply this amount of torque? Could a person supply this amount of torque with a foot pedal? (If you have trouble answering these two questions, review the Developing a Feel questions in this chapter.)

Worked Problem 12.3 Accelerating bicycle

A bicyclist initially traveling at a steady 3.0 m/s accelerates at a constant 1.1 m/s^2. The time interval over which the acceleration takes place is long enough for the wheels to make 20 rotations. If the wheels have a radius of 0.30 m, what is their rotational speed after the acceleration stops?

❶ GETTING STARTED The appropriate diagram for kinematics problems is a motion diagram (Figure WG12.3). The bicyclist moves horizontally while the wheel both rotates and moves horizontally, with increasing rotational speed as the bicyclist accelerates. This is a simple kinematics problem, but the key is to establish a relationship between the given translational quantities and the desired rotational quantities. We use the fact that for rolling without slipping, there is a relationship between the distance the center of each wheel travels and the number of rotations the wheel makes.

Figure WG12.3



❷ DEVISE PLAN The acceleration of the bicycle is the acceleration of the center of each wheel, which is equal to the tangential acceleration of a point on the rim. The relationships between translational and rotational quantities for rolling without slipping are

$$x = \vartheta R$$
$$v_{\mathrm{cm}\,x} = \omega_\vartheta R$$
$$a_{\mathrm{cm}\,x} = \alpha_\vartheta R.$$

Because the acceleration of the bicycle is constant, the rotational acceleration is also constant. The rotational speed of the wheels is described by the equation

$$\omega_{\vartheta,\mathrm{f}}^2 = \omega_{\vartheta,\mathrm{i}}^2 + 2\alpha_\vartheta(\vartheta_\mathrm{f} - \vartheta_\mathrm{i}). \tag{1}$$

Thus we have enough information to compute $\omega_{\vartheta,\mathrm{i}}$ and α_ϑ, and $\vartheta_\mathrm{f} - \vartheta_\mathrm{i}$ is the 20 rotations expressed as a change in the rotational coordinate.

PRACTICE

❸ **EXECUTE PLAN** We use Eq. 1 to determine the final rotational speed, substituting translational variables for $\omega_{\vartheta,\mathrm{i}}$ and α_ϑ:

$$\omega_\mathrm{f}^2 = \left(\frac{v_{\mathrm{cm}\,x,\mathrm{i}}}{R}\right)^2 + 2\left(\frac{a_{\mathrm{cm}\,x}}{R}\right)(\vartheta_\mathrm{f} - \vartheta_\mathrm{i}).$$

The final rotational speed is therefore

$$\omega_\mathrm{f} = \left[\left(\frac{3.0\ \mathrm{m/s}}{0.30\ \mathrm{m}}\right)^2 + 2\left(\frac{1.1\ \mathrm{m/s^2}}{0.30\ \mathrm{m}}\right)(20\ \mathrm{rev})(2\pi/\mathrm{rev})\right]^{1/2}$$

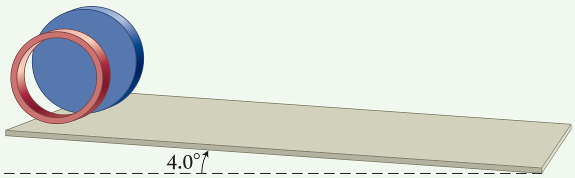
$$= 32\ \mathrm{s^{-1}}. ✔$$

❹ **EVALUATE RESULT** The algebraic equation for ω_f shows that it is greater for larger accelerations and higher initial speeds. Both of these trends are reasonable. The reasonableness of the numerical result can be checked by noting that the final speed is $v = \omega R = (32\ \mathrm{s^{-1}})(0.30\ \mathrm{m}) = 9.6\ \mathrm{m/s} = 34\ \mathrm{km/h} = 21\ \mathrm{mi/h}$, which is a reasonable speed for a cyclist.

Guided Problem 12.4 Inclined race

A solid disk and a hoop, which have different radii and different inertias, are set near the top of a ramp and then released (Figure WG12.4). The ramp is inclined at an angle of 4.0° and is 1.8 m long. (*a*) Which object reaches the bottom first? (*b*) After the first object reaches the bottom, how many seconds pass before the other object reaches the bottom?

Figure WG12.4



❶ **GETTING STARTED**

1. What types of motion are involved here?
2. Consider possible approaches to part *a*: conservation laws, other fundamental principles, or kinematics alone. Repeat for part *b*. If more than one approach is feasible, you may be able to use one method to check an answer you obtained using another method.

❷ **DEVISE PLAN**

3. Draw a diagram showing the physical quantities needed to describe the approach you have decided to use. Write a general equation based on your diagram for the acceleration and rotational acceleration of each object. Consider whether you need separate diagrams for parts *a* and *b*.
4. How does the shape factor for a hoop compare with that for a cylinder (remember that the disk is a very short cylinder)?
5. Which object experiences the greater rotational acceleration: the one with the larger shape factor or the one with the smaller shape factor?
6. For each object, write an expression that relates the acceleration to the time interval the object takes to reach the bottom. Repeat for rotational acceleration. Do you now have enough information to solve the problem?

❸ **EXECUTE PLAN**

7. Solve the equations you wrote in step 6 for the time interval each object takes to roll down the ramp.
8. Subtract your expression for the time interval for the object that reaches the bottom first from the time interval for the object that reaches the bottom second. How do you know from this difference whether you are wrong about which object reaches the bottom first?

❹ **EVALUATE RESULT**

9. Examine your expression for how the time interval needed to reach the bottom changes as you change the physical conditions. What happens as the length of the ramp increases? As the angle of inclination increases? Are these trends reasonable?
10. Consider your numerical result. Is the order of magnitude about right?

Worked Problem 12.5 Falling yo-yo

A yo-yo is basically a spool consisting of an inner small axle and two outer disks, with a string wrapped around the axle. Calculate the acceleration of a 0.130-kg yo-yo if it's allowed to drop while the free end of the string is held at a fixed distance above the floor. Each disk has a radius of 17.0 mm, which means the yo-yo radius is the same, $R_Y = 17.0$ mm, and the axle radius is $R_a = 5.00$ mm. Ignore the inertia of the axle and assume that the entire 0.130-kg inertia is for the two disks.

❶ **GETTING STARTED** We know that the yo-yo accelerates as it falls because of the gravitational force. At the same time, however, the string tends to pull the yo-yo upward. We also know that the yo-yo increases its spin rate as it falls, an increase that must be due to a vector sum of torques caused by these two opposing forces. This information implies that we have to use force and torque laws to get our acceleration.

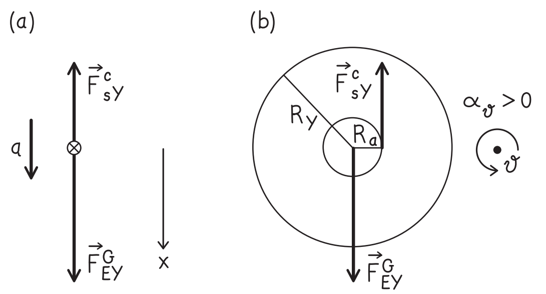
❷ **DEVISE PLAN** The torque caused by the force exerted by the string converts gravitational potential energy to kinetic energy. However, energy methods give us the yo-yo's speed after it has dropped a given distance rather than the instantaneous acceleration requested. We can get this acceleration from Newton's second law, except we do not know the value of the tension in the string. Another equation is needed.

If we assume that the string does not slip, we can relate the rotational acceleration to the desired acceleration via $a_{\text{cm}\,x} = R_a\alpha_\vartheta$ and then get the rotational acceleration from the vector sum of the external torques, $\sum\tau_{\text{ext}\,\vartheta} = I\alpha_\vartheta$ (Eq. 12.10).

Although the acceleration of the yo-yo's center of mass does not depend on the locations of the points of application of the forces, its rotational acceleration does. Thus we need both a free-body diagram and an extended free-body diagram for the yo-yo (Figure WG12.5). The only forces exerted are the downward gravitational force exerted by Earth $\vec{F}^G_{\text{EY}}$, the point of application of which is at the yo-yo's center of mass; and the upward force exerted by the string $\vec{F}^c_{\text{sY}}$, which is directed tangent to the axle.

We have to be careful about the signs of our accelerations. The easiest way to do this is to choose directions for our axes such that positive translation corresponds to positive rotation. Because we know that the yo-yo accelerates downward, we choose the positive x axis pointing downward. A positive downward translation corresponds in Figure WG12.5 to a counterclockwise rotation as the string unwinds, so we put a counterclockwise curved arrow labeled ϑ in our diagram. We note that the only force that produces a nonzero torque about the center of mass is the force exerted by the string, because the force of gravity has a zero lever arm distance about this axis.

Figure WG12.5



❸ **EXECUTE PLAN** We begin with the equation of motion based on the vector sum of forces:

$$F^c_{\text{sY}x} + F^G_{\text{EY}x} = m_Y a_{\text{cm}\,x}.$$

Substituting in the sign and magnitude for each component gives us

$$-F^c_{\text{sY}} + F^G_{\text{EY}} = m_Y a_{\text{cm}\,x}. \tag{1}$$

We suspect that $a_{\text{cm}\,x}$ must be positive, but let us see if the final result bears this out.

PRACTICE

The rotational equation of motion for the components along the axis specified by the curved arrow for our rotational coordinate system is

$$\sum \tau_{\text{ext}\,\vartheta} = \tau_{\text{sY}\,\vartheta} + \tau_{\text{EY}\,\vartheta} = I\alpha_\vartheta$$

$$+R_a F^c_{\text{sY}} + 0 = \left(\tfrac{1}{2} m_Y R_Y^2\right)\alpha_\vartheta, \tag{2}$$

where we have used the fact that the yo-yo is approximately the shape of a solid cylinder, for which the expression for rotational inertia is $I = mR^2/2$.

The relationship between the rotational acceleration and the acceleration of the center of mass is

$$a_{\text{cm}\,x} = \alpha_\vartheta R_a.$$

Note that the axle radius is the appropriate radius to use here. Solving this equation for α_ϑ, substituting the result into Eq. 2, and dividing through by R_a, we get

$$+R_a F^c_{\text{sY}} + 0 = \left(\tfrac{1}{2} m_Y R_Y^2\right)\left(\frac{a_{\text{cm}\,x}}{R_a}\right)$$

$$+F^c_{\text{sY}} = \left(\tfrac{1}{2} m_Y R_Y^2\right)\left(\frac{a_{\text{cm}\,x}}{R_a^2}\right). \tag{3}$$

Next we substitute Eq. 3 into Eq. 1, solve the result for $a_{\text{cm}\,x}$, and insert numerical values:

$$-\left(\tfrac{1}{2} m_Y R_Y^2\right)\left(\frac{a_{\text{cm}\,x}}{R_a^2}\right) + F^G_{\text{EY}} = m_Y a_{\text{cm}\,x}$$

$$F^G_{\text{EY}} = m_Y a_{\text{cm}\,x} + \left(\tfrac{1}{2} m_Y R_Y^2\right)\left(\frac{a_{\text{cm}\,x}}{R_a^2}\right)$$

$$m_Y g = m_Y\left(1 + \frac{1}{2}\frac{R_Y^2}{R_a^2}\right)a_{\text{cm}\,x}$$

$$a_{\text{cm}\,x} = \frac{g}{1 + \frac{1}{2}\frac{R_Y^2}{R_a^2}}$$

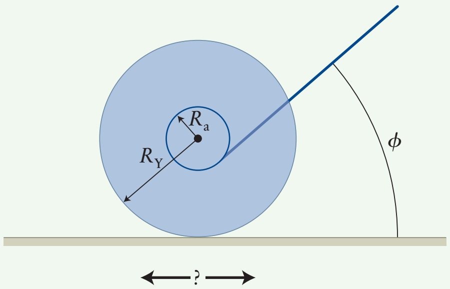
$$a_{\text{cm}\,x} = \frac{9.80\ \text{m/s}^2}{1 + \frac{1}{2}(0.0170\ \text{m})^2/(0.00500\ \text{m})^2} = 1.45\ \text{m/s}^2. \checkmark$$

❹ **EVALUATE RESULT** The expression we obtained for $a_{\text{cm}\,x}$ tells us that the acceleration of the yo-yo depends on the radius R_a of the axle and on the shape factor $c = \frac{1}{2}$ (from Table 11.3). If R_a decreases so that more of the yo-yo's inertia is concentrated at the end pieces (or if R_Y increases), the yo-yo accelerates more slowly. As we expected, our expression shows that $a_{\text{cm}\,x}$ can only be positive. The numerical value of the acceleration is reasonable: A value quite a bit smaller than the free-fall value makes sense because the force exerted by the string impedes the falling motion.

Guided Problem 12.6 Fickle yo-yo

The yo-yo in Figure WG12.6, which has inertia m_Y, outer radius R_Y, and axle radius R_a, sits on a rough horizontal surface. Depending on the angle ϕ at which the string is pulled with tension $\mathcal{T}$, the yo-yo rolls either to the left or to the right in the drawing. (*a*) For a given angle ϕ, determine expressions for the magnitude and direction of the acceleration. (*b*) What is the critical angle at which the yo-yo changes the direction in which it rolls?

Figure WG12.6



1 GETTING STARTED

1. What sort of motion results when the string is pulled at the critical angle? (Hint: What motion could be intermediate between rolling right and rolling left?)
2. What sort of diagram is appropriate for analyzing torques? Analyzing forces? How many cases must you draw separately?
3. Identify all the forces exerted on the yo-yo. What is the direction of the frictional force? (Hint: In which direction would the yo-yo slide if the surface were ice?)

2 DEVISE PLAN

4. What is a good choice for the axis about which to calculate the torques?
5. Draw both a horizontal x axis and a curved arrow for your selection of a positive sense of rotation about your chosen axis.
6. Write the second-law force and torque equations. Account for all forces and torques. Are any zero? How many unknowns do you have?
7. What is a reasonable shape factor to use for the yo-yo?
8. What condition must be met at the critical angle if the yo-yo rolls neither left nor right?
9. Derive an expression for the critical angle in terms of known quantities.

3 EXECUTE PLAN

10. Make sign and magnitude decisions for each force component and each torque component. Attempt a solution.

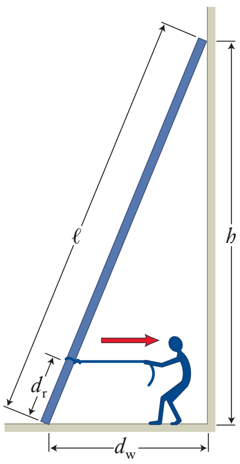
4 EVALUATE RESULT

11. Does the coefficient of friction or the magnitude of the frictional force play a role? Is this surprising? Consider possible hidden assumptions.
12. What do your equations tell you about the motion as ϕ approaches zero? As ϕ approaches 90°?
13. Obtain a spool or yo-yo and give this a try. Can you reproduce the behavior exhibited by the equations?

Worked Problem 12.7 Leaning ladder of physics

A ladder of length $\ell = 13.0$ m leans against a smooth vertical wall. The base rests on rough ground at a distance $d_w = 5.00$ m from the wall, and the top is at height $h = 12.0$ m above the ground (Figure WG12.7). The magnitude of the force of gravity exerted on the ladder is 324 N. A rope is attached to the ladder at a distance $d_r = 2.00$ m up from the base, and a man pulls horizontally on this rope with a force of magnitude 390 N. In order for the ladder not to slip as he pulls, what are (*a*) the vertical component and (*b*) the horizontal component of the force exerted by the ground on the ladder?

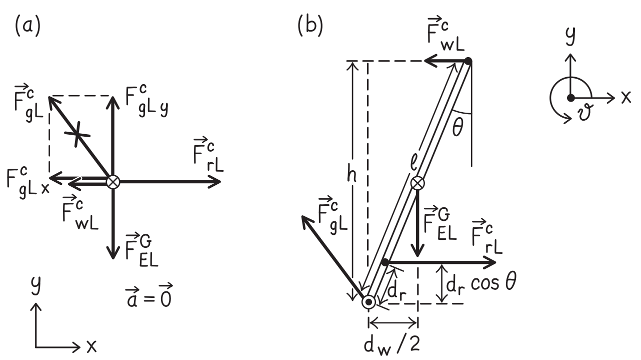
Figure WG12.7



❶ GETTING STARTED A sketch has been provided with the problem, so we can focus on which physical principles elucidate this situation. Because nothing is changing the ladder's state of motion, this is a case of mechanical equilibrium. Therefore the sum of the forces exerted on the ladder must be zero, and the sum of the torques caused by these forces must be zero. Three force components of unknown magnitude are exerted on the ladder, so we should get three component equations from our force and torque analysis.

❷ DEVISE PLAN As usual, we draw a free-body diagram and an extended free-body diagram (Figure WG12.8). Because we need to deal with components of forces and torques, we have to specify axes and a rotational direction. Let us choose the positive *x* direction to be to the right, the positive *y* direction to be up, and the pivot about which to calculate the torques to be the base of the ladder, with positive rotation counterclockwise. We put these two sign notations in our diagram.

Figure WG12.8



We know the magnitudes and directions of the gravitational force and the force exerted by the man, so we draw those to scale first in our free-body diagram. The man exerts a force to the right, and its magnitude exceeds that of the gravitational force. That means we suspect that in the absence of friction the bottom of the ladder would slide to the right. Thus we draw the contact force exerted by the ground as pointing upward and to the left. If this turns out to be wrong, our equations will provide a correction by virtue of a negative answer.

It is useful to identify a relevant angle for our analysis, and let us select the one between the top of the ladder and the wall, calling it θ. The ladder forms a 5–12–13 right triangle with the wall, and this angle satisfies the relationships $\sin\theta = \frac{5.00}{13.0}$ and $\cos\theta = \frac{12.0}{13.0}$ (making $\theta = 22.6°$).

We want to know the force exerted by the ground on the ladder. We know that because the wall is smooth we can ignore any frictional force exerted by the wall. This means the wall exerts no vertical force, only a horizontal (normal) contact force. We can therefore look at the sum of the forces exerted on the ladder in the vertical direction and not worry about any force in this direction exerted by the wall. The sum of the torques about the base of the ladder must be zero because there is no rotation. We know that the force exerted by the wall is directed to the left in the orientation shown in Figure WG12.7, and we know that the horizontal forces must balance because the ladder isn't accelerating in the *x* direction. This gives us enough equations to solve for all unknowns.

PRACTICE

3 EXECUTE PLAN

(*a*) Because the sum of the vertical components of the four forces must be zero, we write

$$\Sigma F_y = F^c_{\text{wL}y} + F^G_{\text{EL}y} + F^c_{\text{rL}y} + F^c_{\text{gL}y} = 0.$$

The vertical component of the contact force exerted on the ladder by the ground is the normal force, so we call its magnitude $F^n_{\text{gL}} = |F^c_{\text{gL}y}|$. (The horizontal component is the friction force.) Thus we get

$$\Sigma F_y = 0 + (-F^G_{\text{EL}}) + 0 + F^n_{\text{gL}} = 0$$

$$F^n_{\text{gL}} = |F^c_{\text{gL}y}| = F^G_{\text{EL}} = 324 \text{ N.} \checkmark$$

Note that the direction of $F^c_{\text{gL}y}$ must be upward because the only other vertical force (the gravitational force) is downward.

(*b*) To determine the horizontal component $F^c_{\text{gL}x}$, we need two equations because there are two unknowns in the equation for the horizontal sum of the forces: $F^c_{\text{gL}x}$ (the one we have to determine) and $F^c_{\text{wL}x}$. Thus we begin with torques. The sum of the ϑ components of the torques caused by the four forces about the rotation axis must be zero because there is no rotation. Using our generic symbol $r_\perp$ for lever arm distances, we write our first torque equation in the general form

$$\Sigma\tau_\vartheta = \pm r_{\text{wL}\perp}F^c_{\text{wL}} \pm r_{\text{EL}\perp}F^G_{\text{EL}} \pm r_{\text{rL}\perp}F^c_{\text{rL}} \pm r_{\text{gL}\perp}F^c_{\text{gL}} = 0.$$

We note that the term $r_{\text{gL}\perp}F^c_{\text{gL}}$, the torque caused by the force exerted by the ground on the ladder, must be zero because the lever arm distance for this torque is zero for a pivot at the base of the ladder.

Our chosen direction for positive rotation determines the sign of each torque. We determine these signs by looking at Figure WG12.8*b* to see which way each force tends to rotate the ladder about the pivot. Remember that counterclockwise in the figure is positive rotation:

$$\Sigma\tau_\vartheta = +hF^c_{\text{wL}} + \left(-\tfrac{1}{2}d_{\text{w}}F^G_{\text{EL}}\right) + \left[-(d_{\text{r}}\cos\theta)F^c_{\text{rL}}\right] + 0 = 0$$

$$(12.0 \text{ m})F^c_{\text{wL}} - \tfrac{1}{2}(5.00 \text{ m})(324 \text{ N})$$

$$- (2.00 \text{ m})\left(\tfrac{12.0}{13.0}\right)(390 \text{ N}) = 0.$$

This gives us the magnitude of the force exerted by the wall:

$$F^c_{\text{wL}} = \frac{\tfrac{1}{2}(5.00 \text{ m})(324 \text{ N}) + (2.00 \text{ m})\left(\tfrac{12.0}{13.0}\right)(390 \text{ N})}{12.0 \text{ m}} = 128 \text{ N.}$$

The sum of the horizontal forces is

$$\Sigma F_x = F^c_{\text{wL}x} + F^G_{\text{EL}x} + F^c_{\text{rL}x} + F^c_{\text{gL}x} = 0$$

$$(-F^c_{\text{wL}}) + 0 + (+F^c_{\text{rL}}) + F^c_{\text{gL}x} = 0,$$

where we have used the known information about the directions of the forces exerted by the wall and the rope. We now solve for the horizontal component of the force exerted on the ladder by the ground:

$$F^c_{\text{gL}x} = +F^c_{\text{wL}} - F^c_{\text{rL}} = 128 \text{ N} - 390 \text{ N} = -262 \text{ N.} \checkmark$$

4 EVALUATE RESULT As we suspected, the man pulls on the rope hard enough that the ladder would slide toward him if the ground weren't holding it back. So the ground must supply a horizontal force in the direction opposite the direction of the force exerted by the rope, which means in the negative *x* direction. Thus, the negative value for $F^c_{\text{gL}x}$ is reasonable and agrees with our free-body diagram. If the man had pulled with less force, we might have obtained a positive value. We also know that the ground needs to be pushing up on the ladder (if it were not, the ladder would fall through the ground), so our +324 N result in part *a* is reasonable. However, let's double-check our answer by calculating the torque about the ladder's center of mass, which should also be zero:

$$\Sigma\tau_\vartheta = \left[+\left(\tfrac{1}{2}\ell - d_{\text{r}}\right)\cos\theta\right]F^c_{\text{rL}} + \left[+\tfrac{1}{2}\ell\cos\theta\right]F^c_{\text{wL}}$$

$$+ \left[+\tfrac{1}{2}\ell\cos\theta\right]F^c_{\text{gL}x} + \left[-\tfrac{1}{2}\ell\sin\theta\right]F^c_{\text{gL}y}.$$

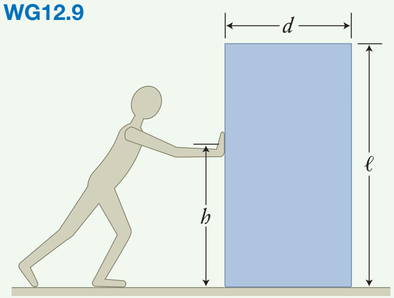
Note the absence of any contribution by the gravitational force here. Why is this so?

When we substitute numerical values, we get zero, verifying our earlier torque result. This is one of the nice things about mechanical equilibrium problems. You can pick whichever pivot is easiest to balance around to get an initial answer and then pick a different pivot to check that answer.

Guided Problem 12.8 Moving a refrigerator

Your new refrigerator, of inertia m, has been delivered and left in your garage. As shown in Figure WG12.9, it has length ℓ in vertical dimension and each side of its square base is of length d. You need to slide it across the rough garage surface to get it into your house. The coefficients of static and kinetic friction between base and garage surface are almost equal, so you approximate $\mu = \mu_s = \mu_k$. You push horizontally at a height h above the surface, exerting a force just big enough to keep the refrigerator moving. You dislike bending over, so you push at the highest possible point that will not cause the refrigerator to tip as it slides. Thus the refrigerator is always on the verge of tipping. (*a*) Where along the base of the refrigerator is the effective point of application of the normal force exerted by the garage surface on the refrigerator; that is, at what location can you picture the normal force as being concentrated? (*b*) If the refrigerator is not to tip, and if its center of mass is at its center, what is the maximum value h_{max} at which you can push?

Figure WG12.9



1 GETTING STARTED

1. What condition must be met in order for the refrigerator not to rotate?
2. How can you determine the force you push with to just keep the refrigerator moving?
3. Which force(s) tend to tip the refrigerator, and which tend to prevent it from tipping?

2 DEVISE PLAN

4. Draw a free-body diagram and an extended free-body diagram for the refrigerator. Indicate a sign for each coordinate axis (x, y, and ϑ) so that you can correctly determine the signs of the components.
5. What is the lever arm distance of the normal force $\vec{F}^n_{fr}$ exerted by the floor?
6. How does the height at which you push affect the point of application of $\vec{F}^n_{fr}$?
7. Is there enough information to solve for the value of the lever arm distance of $\vec{F}^n_{fr}$ at which the refrigerator begins to tip?
8. What condition exists just before tipping begins?

3 EXECUTE PLAN

4 EVALUATE RESULT

9. In your expression for the lever arm distance, does each term have a sign that is physically plausible?
10. Does your answer make sense if μ is reduced to zero or increased to 1.0? What if d becomes very large or very small?

Answers to Review Questions

1. Yes, because the torque magnitude depends not only on the magnitude of the force but also on the lever arm.
2. Not quite. The fact that the door moves the same distance in 2 s in both cases tells you that the rotational acceleration is the same in both cases, which means that the torque caused by your force must be the same as the torque caused by your friend's force. However, because the rotation axis runs along the hinged edge, the magnitude of your force must be greater than the magnitude of his force because the lever arm of your force is smaller than the lever arm of his force.
3. You may choose any reference point, though it is best to choose a reference point that simplifies the calculation by eliminating one or more unknowns from the torque calculation.
4. There are many examples. The normal force that the ground exerts on the rim of a bicycle tire (when the tire is vertical) is one example. The torque is zero because there is no component of the force perpendicular to the line of action. Similarly, if you pull on a door handle with a force whose line of action passes through the hinged edge, the door will not rotate.
5. No. Objects can be made to rotate or revolve about a specified point only by imposing a constraint, such as a hinge, pivot, or string. Free rotation is the absence of such constraints, so a freely rotating object will rotate about its center of mass.
6. The center of mass of an object or system moves as if all the inertia of the object or system were concentrated at that point. Hence, any free rotation occurs about an axis through the center of mass, allowing complicated motion to be expressed as two independent motions: the translational motion of the center of mass and the rotational motion around that point.
7. Primarily downward, with a small horizontal component toward the hand. One way to see this is to look at the stationary forearm. Being free to choose any pivot around which to sum torques, let's arbitrarily take the pivot to be at the hand. About this point, the torque caused by the

PRACTICE

force exerted by the biceps muscle on the forearm bone is clockwise from your perspective. To balance this clockwise torque exactly (remember that the arm is not rotating), the torque caused by the force exerted by the humerus on the forearm bone must be counterclockwise. To get this counterclockwise torque, the force causing it must be directed downward. Another viewpoint: If the biceps muscle pulls upward on the forearm, then the humerus must exert a downward force on the elbow-joint end of the forearm in order to prevent upward translation of that end.

8. An extended free-body diagram shows the location of the point of application of each force on the object in question, a necessity for computing torques. Torques are required in order to analyze rotational motion. A standard free-body diagram shows the forces as if the points of application of all of them were at a common point, which is fine for force computations.
9. It is possible to use the extended diagram for both forces and torques, but it is best to draw a standard free-body diagram first until you have gained a lot of experience. You are familiar with free-body diagrams, so just a glance at one will tell you whether the force components are capable of canceling or adding up to the required components of *ma*. The extended free-body diagram is less familiar and has the forces moved to remote locations rather than all tail to tail at one spot. If you accidentally leave one force off the extended free-body diagram, it will be much harder to tell at a glance that anything is wrong. Leaving out a force (or including an inapplicable force) leads to an erroneous result.
10. Curl the fingers of your right hand so that they follow the rotation of the object. Your thumb then points in the direction of the vector that specifies the direction of rotation.
11. The direction is into the wall, by the right-hand rule.
12. In three dimensions, rotations that appear clockwise from one perspective can appear counterclockwise from another.
13. The latter statement is correct. The former cannot be right because the units of force and torque aren't the same and so they can't be added.
14. Yes. As long as the single force is exerted, the momentum changes. As long as that force has a lever arm with respect to some pivot, the force causes a torque, which changes the angular momentum.
15. No. The momentum is not constant because a gravitational force is exerted on the baton. The baton is unconstrained and hence rotates about an axis through its center of mass. Because only a gravitational force is exerted on the baton while it is in the air, and this force is exerted straight downward on the center of mass, it causes no torque about that point. Thus the angular momentum is constant.
16. Rotational impulse represents the transfer of angular momentum between a system and its environment. It can be computed as the product of external torque and the time interval during which the torque is applied, quantified in the rotational impulse equation, $J_\vartheta = (\sum \tau_{\text{ext}\,\vartheta})\Delta t$. The angular momentum law, by analogy to the momentum law from translational motion, states that any change in the angular momentum of a system must be due to the rotational impulse caused by external torque: $\Delta L_\vartheta = J_\vartheta$.
17. Angular momentum is always conserved, which means it cannot be created or destroyed. The angular momentum of a system remains constant if the system is in rotational equilibrium. This occurs when the system is isolated or when the rotational impulse acting on the system is zero.
18. They are related by a factor of the radius of the wheel: $|v_{\text{cm}\,x}| = |R\omega_\vartheta|$.
19. When gears mesh, they turn without slipping, so the tangential speeds of the teeth of the two gears must be identical. The rotational speeds are generally not the same because they depend on the radii of the gears.
20. The shape factor is the ratio of an object's rotational inertia to the rotational inertia of a hoop of the same inertia m and radius R: $c = I/mR^2$. This factor compares the rotational inertia of an object with the rotational inertia if all the object's inertia were concentrated on its rim, so it measures the distribution of inertia in the object. This distribution is loosely referred to as "shape."
21. They are related by a factor of the object's radius: $|a_{\text{cm}\,x}| = |R\alpha_\vartheta|$.
22. The work done is the product of the sum of the torques $\sum\tau$ and the change in the object's rotational coordinate $\Delta\vartheta$. It is necessary to first convert the number of rotations to the change in the rotational coordinate.
23. The kinetic energy of an object or system that is in both translational and rotational motion is equal to the sum of its center-of-mass and rotational kinetic energies.
24. The force of static friction does no work on the object and converts some of the object's translational kinetic energy to rotational kinetic energy.
25. The vector product vector is perpendicular to the desktop, pointing either up or down. Which direction is correct is determined by the right-hand rule. Other than that, the angle is independent of the orientation of the two vectors.
26. The magnitude of the vector product of two vectors is equal to the area of the parallelogram defined by the two vectors. It is also equal to the product of the magnitudes of each vector and the sine of the smaller angle between them when they are placed tail to tail.
27. Zero. The term in parentheses is perpendicular to $\vec{A}$, and so the scalar product is zero.
28. The sign. The magnitude of your answer is correct, but the direction is opposite the correct direction.
29. The angular momentum of an isolated particle about any reference point is equal to the vector product of the position vector that locates the particle relative to the reference point and the momentum of the particle: $\vec{L} = \vec{r} \times \vec{p}$.

Answers to Guided Problems

Guided Problem 12.2 $|\vec{\tau}| = 1.8\ \text{N}\cdot\text{m}$

Guided Problem 12.4 (*a*) The disk wins; (*b*) $\Delta t = 0.4$ s

Guided Problem 12.6 (*a*) $a_x = \dfrac{\mathcal{T}}{m_Y(1 + c)}\left(\cos\phi - \dfrac{R_a}{R_Y}\right)$;

(*b*) $\phi_{\text{crit}} = \cos^{-1}\left(\dfrac{R_a}{R_Y}\right)$

Guided Problem 12.8 (*a*) beneath the edge of the base opposite you; (*b*) $h_{\max} = d/2\mu$

PRACTICE

Guided Practice by Chapter

13 Gravity

Review Questions

Answers to these questions can be found at the end of this chapter.

13.1 Universal gravity

1. Which object exerts a greater pulling force on the other: Earth or the Moon? Which object experiences greater acceleration due to this mutual pull?
2. What is the evidence for Newton's $1/r^2$ law of universal gravitation?
3. What orbital shapes are possible with Newton's $1/r^2$ law of universal gravitation?
4. The length of a year on any planet is defined as the time interval the planet takes to make one revolution around the Sun. What is the length of a year on a planet whose orbital radius is four times Earth's orbital radius?
5. What is the distinction between mass and inertia?

13.2 Gravity and angular momentum

6. Explain how, in an isolated system of two objects interacting through a central force, each object has constant angular momentum about the system's center of mass.
7. State Kepler's three laws of planetary motion.
8. In Figure 13.12*a* no force is exerted on the moving object, whereas in Figure 13.12*b* and *c*, there is (the force that keeps the object moving along a curved trajectory). Why is the conclusion about angular momentum the same in all three cases?
9. Which has greater acceleration in its orbit around Earth: the Moon or the International Space Station?

13.3 Weight

10. What is your weight measured by a spring scale as you fly through the air over a trampoline? Assume that you can keep the spring scale beneath your feet.
11. Suppose a balance is loaded with objects of equal mass on each side and then dropped in free fall. Does it remain balanced?
12. You stand on a spring scale placed on the ground and read your weight from the dial. You then take the scale into an elevator. Does the dial reading increase, decrease, or stay the same when the elevator accelerates downward as it moves upward?
13. Explain why astronauts in an orbiting space station are able to float around in the cabin.

13.4 Principle of equivalence

14. State the principle of equivalence, and explain why you must use the word *locally* in your statement.
15. Should we expect light to travel along a curved path when it passes near a massive object?

13.5 Gravitational constant

16. Gravitational forces decrease with a $1/r^2$ dependence, and yet the value of g is constant near Earth's surface. Is this a contradiction?
17. Does a planet that has a greater mass than Earth necessarily have a greater acceleration due to gravity near its surface? (Saturn, for example, has about a hundred times the mass of Earth.)

13.6 Gravitational potential energy

18. What choice causes the gravitational potential energy described by Eq. 13.14, $U(r) = -Gm_1m_2/r$, to be negative for any finite r?
19. What does it mean to say that the work done by the gravitational force exerted on an object is *path independent*?
20. Under what condition is it reasonable to approximate a $1/r$ potential energy curve by a straight line?

13.7 Celestial mechanics

21. When you throw a ball from the roof of a building, is the trajectory a parabola or an ellipse?
22. Where in the orbit of any planet in our solar system is the planet's speed smallest? Where is it greatest?
23. If $E = 0$ for a system that consists of an object orbiting a much more massive object, is the orbit bound or unbound?
24. Describe the orbital shapes available for motion under the gravitational force exerted by the Sun, and relate each to the energy of the system comprising the Sun and the orbiting object.

13.8 Gravitational force exerted by a sphere

25. What approach is used to show that the gravitational force exerted by a solid sphere on an external particle is the same as if all the mass of the sphere were concentrated at its center?
26. Can the process for computing the force exerted by a solid sphere on an external particle be extended to a particle located inside the sphere? To a hollow sphere? To a sphere of nonuniform density?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The magnitude of the gravitational force exerted on you by your partner for a tango (B, N)
2. The magnitude of the gravitational force exerted on you by the population of Earth (B, Q, D, J)
3. The magnitude of the gravitational force exerted on you by a nearby large mountain (S, I, B)
4. The radius of the orbit in which your running speed would be sufficient to keep an object moving in a circular orbit around Earth (A, V)
5. The orbital speed required for a near-surface circular orbit of the Moon (V, C, R)
6. The maximum radius for a planet of Earth's density that you could jump from without falling back (F, K, T)
7. The radius to which Earth must be compressed in order for a light beam to orbit the planet in a circular path (V, G, O)
8. The energy required to remove all the planets from the solar system (E, P, L)
9. The maximum and minimum magnitudes of the vector sum of forces you experience during a bungee jump (H, M, U)
10. The magnitude of Earth's angular momentum as it orbits the Sun (W)

Hints

A. What is a typical human running speed for long distances?
B. What is the mass of a typical human?
C. What is the mass of the Moon?
D. How is the population of Earth distributed?
E. What is the gravitational potential energy of the system after the planets are removed?
F. What is the average density of Earth?
G. What principle provides the key to analyzing the gravitational behavior of light?
H. Where in the jump does the bungee cord begin to stretch?
I. What is the distance to the center of mass of a nearby mountain?
J. What is the radius of Earth?
K. How much can you raise your center of mass on Earth by running and then jumping up?
L. How many terms should you include in computing the initial gravitational potential energy of the solar system?
M. Where in the jump does the bungee cord exert maximum force on you?
N. What typical distance separates your center of mass from that of your partner as you tango?
O. What is the required orbital speed?
P. Do you need to add or remove energy?
Q. What is the population of Earth?
R. What is the radius of the Moon?
S. What is the mass of a large mountain?
T. How can you relate jumping height to vertical speed or planet radius?
U. How is the energy stored in a bungee cord related to the distance through which it has stretched?
V. How is the speed of a small object of mass m_s orbiting a large object of mass m_ℓ related to the radius of the orbit?
W. How is angular momentum related to radius and speed of orbit?

Key (all values approximate)

A. 4 m/s; **B.** 7×10^1 kg; **C.** 7×10^{22} kg; **D.** around Earth's surface—assume a uniform spherical-shell distribution, which is a rough approximation; **E.** zero; **F.** 6×10^3 kg/m^3; **G.** the principle of equivalence; **H.** about one-third of the way down; **I.** 3×10^1 km is a reasonable choice; **J.** 6×10^6 m; **K.** 0.7 m for an average person; **L.** ideally, one term for each pair of objects (Mercury-Sun, Mercury-Venus, Mercury-Earth, and so on), but compare terms for the Sun and the three largest planets to get a sense of scale; **M.** the spring force kx is greatest where the stretch length x is greatest, at the bottom; **N.** 0.5 m; **O.** the speed of light, 3×10^8 m/s; **P.** for each planet in orbit, the gravitational potential energy of the Sun-planet system is negative, so you must add potential energy; **Q.** 7 billion; **R.** 2×10^6 m; **S.** 2×10^{13} kg; **T.** use energy methods: the initial kinetic energy is converted to gravitational potential energy during a jump; **U.** the energy stored is approximately $kx^2/2$, where x is the stretch and k is the cord's spring constant; **V.** the gravitational force is the only force in the vector sum for the small object, and the acceleration is centripetal for circular orbit, so $v_s = (Gm_\ell/r)^{1/2}$; **W.** for circular orbit, $L = rmv$

Worked and Guided Problems

Table 13.1 Solar system data (in SI units and relative to Earth)

	Mass (kg)	Mass (m_E)	Equatorial radius (m)	Equatorial radius (R_E)	Orbit[†] semimajor axis (m)	Orbit semimajor axis (a_E)	Orbit eccentricity	Orbit period (s)	Orbit period (years)
Sun	2.0×10^{30}	3.3×10^{5}	7×10^{8}	110	–	–	–	–	–
Mercury	3.30×10^{23}	0.06	2.440×10^{6}	0.38	5.79×10^{10}	0.39	0.206	7.60×10^{6}	0.24
Venus	4.87×10^{24}	0.81	6.052×10^{6}	0.95	1.082×10^{11}	0.72	0.007	1.94×10^{7}	0.62
Earth	5.97×10^{24}	1	6.378×10^{6}	1	1.496×10^{11}	1	0.017	3.16×10^{7}	1
Mars	6.42×10^{23}	0.11	3.396×10^{6}	0.53	2.279×10^{11}	1.52	0.09	5.94×10^{7}	1.88
Jupiter	1.90×10^{27}	318	7.149×10^{7}	11.2	7.783×10^{11}	5.20	0.05	3.74×10^{8}	11.86
Saturn	5.68×10^{26}	95.2	6.027×10^{7}	9.45	1.427×10^{12}	9.54	0.05	9.29×10^{8}	29.45
Uranus	8.68×10^{25}	14.5	2.556×10^{7}	4.01	2.871×10^{12}	19.2	0.05	2.65×10^{9}	84.02
Neptune	1.02×10^{26}	17.1	2.476×10^{7}	3.88	4.498×10^{12}	30.1	0.01	5.20×10^{9}	164.8
Pluto	1.31×10^{22}	0.002	1.151×10^{6}	0.18	5.906×10^{12}	39.5	0.25	7.82×10^{9}	247.9
Moon	7.3×10^{22}	0.012	1.737×10^{6}	0.27	3.84×10^{8}	0.0026	0.055	2.36×10^{6}	0.075

[†]The elliptical orbits of the planets and the Moon are specified by their *semimajor axis a* (half the major axis) and eccentricity *e*; see Figure 13.7. With the exception of Mercury and Pluto, the eccentricity is small and so the orbits are close to being circular.

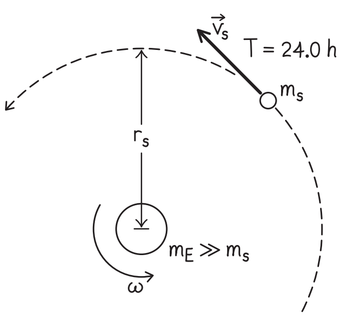
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 13.1 Reliable 24/7 communications

In order to supply 24-h reception, a communications satellite is often placed in *geosynchronous orbit,* which means the satellite always appears in the same location relative to the receiver on the ground. This requires, among other things, that the rotational speed at which the satellite orbits Earth is the same as the rotational speed at which Earth spins on its axis. How far above Earth's surface is a satellite in geosynchronous orbit? What else is required?

❶ GETTING STARTED We seek an orbit whose period is $T = 24.0 \text{ h} = 8.64 \times 10^4$ s. Determining the radius of such an orbit is important, but what else might be needed? Let's consider the goal in detail: In order for the satellite to always appear at the same location relative to a point on the ground, its orbital speed must be constant, matching the rotational speed of the Earth. This means the orbit must be circular. We draw a sketch of the system, labeling both the known mass of Earth m_E and the unknown mass of the satellite m_s (Figure WG13.1).

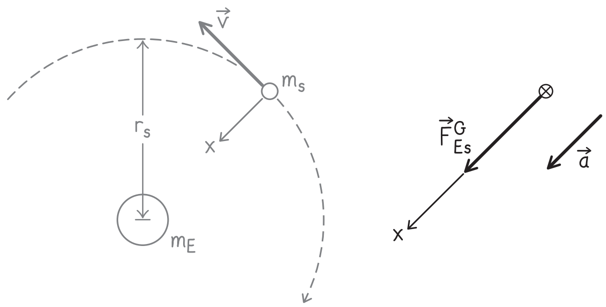
Figure WG13.1



Kepler's third law says the square of the orbital period is proportional to the cube of the distance from the center of the object exerting the gravitational force. Thus a specified period implies a certain orbit radius r. We need to obtain an expression for r in terms of our given parameters. However, does just any orbit of this radius work? The goal is for the satellite to remain fixed relative to a specific point on Earth. Points on the surface of Earth move in circles parallel to the equator as Earth rotates. Thus it seems that our circular orbit must also be parallel to the plane of the equator in order to remain at the same location relative to a point on the ground.

❷ **DEVISE PLAN** As the satellite moves in a circular orbit, it is subject to the gravitational force of Earth only. We draw a free-body diagram for the satellite (Figure WG13.2), choosing the x axis to be in the direction of this force, toward the center of the orbit. Because the orbit is circular, we can use the relationship between radial (centripetal) acceleration and rotational speed ω from Eq. 11.15: $a_c = \omega^2 r$. The satellite's rotational speed is 2π rad in 24.0 h, making $\omega = 2\pi/T$. The centripetal acceleration is due to the force of gravity exerted by Earth. Because the satellite is not near Earth's surface, we cannot set a_c equal to g. Instead, we can solve for the acceleration by force analysis, using Newton's law of gravity, $F^G_{Es} = Gm_E m_s/r^2$. The only unknown in the equation of motion (Eq. 8.8) will be r.

Figure WG13.2



❸ **EXECUTE PLAN** We begin by substituting known formulas into the component form of the equation of motion for the satellite:

$$\Sigma F_x = ma_x$$

$$F^G_{Es\,x} = m_s a_x$$

$$+F^G_{Es} = m_s(+a_c)$$

$$G\frac{m_E m_s}{r^2} = m_s(\omega^2 r)$$

$$Gm_E = \omega^2 r^3.$$

We want to solve for r in terms of T:

$$Gm_E = \left(\frac{2\pi}{T}\right)^2 r^3$$

$$r^3 = Gm_E\left(\frac{T}{2\pi}\right)^2.$$

This is Kepler's third law in mathematical form for a satellite orbiting Earth. Substituting the values given, we get

$$r^3 = (6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97\ 10^{24}\ \mathrm{kg})\left(\frac{8.64 \times 10^4\ \mathrm{s}}{2\pi}\right)^2$$

$$r = 4.22 \times 10^7\ \mathrm{m}.$$

Thus the satellite must follow a circular orbit whose plane is parallel to Earth's equator, and it must orbit at height

$$h = r - R_E = (4.22 \times 10^7\ \mathrm{m}) - (6.38 \times 10^6\ \mathrm{m})$$
$$= 3.58 \times 10^7\ \mathrm{m} = 3.6 \times 10^4\ \mathrm{km}. ✔$$

❹ **EVALUATE RESULT** This is pretty far up, considering that the International Space Station (ISS) orbits at only a few hundred kilometers above ground. On the other hand, our value for r is an order of magnitude smaller than the distance to the Moon, so the answer is at least plausible. The ISS completes an orbit in a couple of hours, while the Moon requires a month. Our geosynchronous orbit is thus bracketed by these examples in both period and radius—which gives us some confidence in our results.

Note that the mathematical form of Kepler's third law does not depend on the mass of the satellite. (Neither does the version expressed purely in words!) Any geosynchronous satellite must therefore orbit in a circular, equatorial orbit at a common height. This raises a practical issue: If many countries and companies all want geosynchronous satellites, how do we find room up there for them?

PRACTICE

Guided Problem 13.2 Disruptive spin

Objects on Earth's surface remain there because of gravitational attraction, despite the fact that objects near the equator move at about 1600 km/h because of Earth's rotation. If the Earth were to spin a lot faster, though, the gravitational force might not be strong enough to prevent objects from flying away from Earth's surface. This effect of rapid spin is especially acute for fluid objects like stars, particularly for the remnant cores of stars that have exploded in a supernova explosion. Some of these, called millisecond pulsars, rotate very rapidly (about once every millisecond) because their angular momentum remained constant as they collapsed. These stars typically have 2 times the mass of our Sun. What is the maximum radius a millisecond pulsar can have in order to avoid losing matter at its surface?

❶ GETTING STARTED

1. Describe the problem in your own words. How is the radius involved?
2. What is the shape of the path followed by a small clump of material at the surface of a millisecond pulsar? What force keeps the clump on that path rather than traveling in a straight line?
3. Draw an appropriate diagram.

❷ DEVISE PLAN

4. How is the magnitude of the relevant force related to the mass and radius of the pulsar?
5. What physics law allows you to relate this force to the rotational speed of the pulsar? How is the rotational speed related to the pulsar's rotational period?
6. Which unknown quantity do you need to determine?
7. Does your approach allow you to express the unknown quantity in terms of known ones?

❸ EXECUTE PLAN

8. Work through the algebra to solve for the unknown quantity.

❹ EVALUATE RESULT

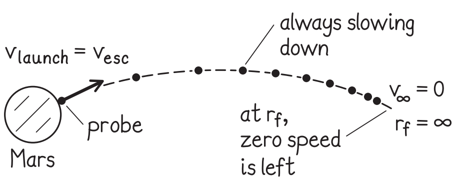
9. Did you make any assumptions?
10. Compare your answer with the range of pulsar radii available online.

Worked Problem 13.3 Escape at last

The Mars Colony wants to launch a deep-space probe, but they have no rocket engines. They decide to launch a probe with an electromagnetic cannon, which means they must launch at escape speed. Determine this speed.

❶ GETTING STARTED Let us do a quick sketch to help our thinking (Figure WG13.3). We select the Mars-probe system for analysis. In order to reach "deep space," the probe must attain a very great distance from Mars. This will require a significant amount of initial kinetic energy, which the probe must acquire during launch. After launch, the kinetic energy immediately begins to decrease, and the potential energy of the Mars-probe system increases as the separation distance increases. We assume a reference frame where Mars is fixed and only the probe moves. When the probe is far enough away (infinity, really, but practically it doesn't need to go quite this far), the kinetic energy has its minimum value, which we can take to be zero because the colonists presumably do not want to supply any more energy than needed to get the probe out there. The gravitational potential energy has its maximum value, which is also zero. (Remember that universal gravitational potential energy is *negative*.) We also assume that the Sun and other planets have a negligible influence on our system, and we ignore the rotation of Mars.

Figure WG13.3



❷ DEVISE PLAN We can use conservation of energy because the probe has all of the needed kinetic energy at the beginning, as it is shot from a cannon. As the probe travels, this kinetic energy is converted to gravitational potential energy of the Mars-probe system. We want to know the initial speed of the probe acquired at launch. The initial potential energy is the value when the probe is still near the Martian surface. The final state of the probe is zero speed at an infinite distance from Mars. The *Principles and Practice* text analyzes a similar situation in Section 13.7, leading to Eq. 13.23, so there is no need to derive this result again here. We begin with Eq. 13.23, solving this version of an energy conservation equation for $v_i = v_{esc}$ in terms of the known quantities.

❸ **EXECUTE PLAN** Let us use r_i for the initial Mars-probe radial center-to-center separation distance, $r_f = \infty$ for the final separation distance, R_M for the radius of Mars, and m_M and m_p for the two masses. We begin with Eq. 13.23:

$$E_{mech} = \tfrac{1}{2}m_p v_{esc}^2 - G\frac{m_M m_p}{R_M} = 0$$

$$\tfrac{1}{2}v_{esc}^2 - G\frac{m_M}{R_M} = 0$$

$$\tfrac{1}{2}v_{esc}^2 = G\frac{m_M}{R_M}$$

$$v_{esc} = \sqrt{2G\frac{m_M}{R_M}}$$

$$v_{esc} = \sqrt{2(6.67 \times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})\frac{6.42 \times 10^{23}\,\mathrm{kg}}{3.40 \times 10^{6}\,\mathrm{m}}}$$

$$= 5.02 \times 10^3\,\mathrm{m/s} = 5.02\,\mathrm{km/s}.\ ✔$$

Notice that this speed does not depend on the mass of the probe. A probe of any other size shot from the cannon would need the same minimum speed to break free of Mars's gravitational pull.

❹ **EVALUATE RESULT** Our algebraic expression for the escape speed is plausible because it involves the mass of Mars, the initial center-to-center radial separation distance of our two objects (which is Mars's radius), and G. We expect v_{esc} to increase with m_M because the gravitational pull increases with increasing mass. We also expect v_{esc} to decrease as the distance between the launch position and Mars's center increases because the gravitational force exerted by the planet on the probe decreases with increasing separation distance. All this is just what our result predicts.

An escape speed of 18,000 km/h is smaller than (but on the order of) the escape speed from Earth, and so the answer is not unreasonable.

We assumed that the initial Mars-probe separation distance is equal to the planet's radius. Of course, the length of the cannon may be tens of meters, but this tiny difference would have no impact on the numerical answer. We ignored the rotation of Mars, which could supply a small amount of the needed kinetic energy. We also ignored the effect of the Sun, which is fine for getting away from the surface of Mars, but we would need to account for it if the destination was another star.

Guided Problem 13.4 Spring to the stars

Suppose that, instead of using chemical rockets, NASA decided to use a compressed spring to launch a spacecraft. If the spring constant is 100,000 N/m and the mass of the spacecraft is 10,000 kg, how far must the spring be compressed in order to launch the craft to a position outside Earth's gravitational influence?

❶ **GETTING STARTED**

1. Describe the problem in your own words. Are there similarities to Worked Problem 13.3?
2. Draw a diagram showing the initial and final states. What is the spacecraft's situation in the final state?
3. How does the spacecraft gain the necessary escape speed?

❷ **DEVISE PLAN**

4. What law of physics should you invoke?
5. As the spring is compressed, is the gravitational potential energy of the Earth-spacecraft system affected? If so, can you ignore this effect?
6. What equation allows you to relate the initial and final states?

❸ **EXECUTE PLAN**

7. What is your target unknown quantity? Algebraically isolate it on one side of your equation.
8. Substitute the numerical values you know to get a numerical answer.

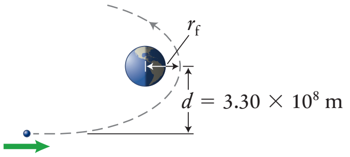
❹ **EVALUATE RESULT**

9. Is your algebraic expression for the compression plausible for how the compression changes as the spring constant and Earth's mass and radius change?
10. If you were the head of a design team, would you recommend pursuing this launch method?

Worked Problem 13.5 Close encounter

Suppose astronomers discover a large asteroid, still far away, headed for Earth. Calculations indicate that it is traveling at $v_i = 754$ m/s and that a straight-line extension of its velocity would bring it within a distance $d = 3.30 \times 10^8$ m of Earth (which is about the radius of the Moon's orbit around Earth). However, Earth's gravitational attraction will cause its trajectory to be a conic section, as Figure WG13.4 shows. (*a*) What is the value of r_f, the radius of its orbit when the asteroid is closest to Earth? (*b*) What speed v_f does the asteroid have when it gets closest to Earth? Note from Figure WG13.4 that r_f is to be measured from Earth's center, not from Earth's surface.

Figure WG13.4



1 GETTING STARTED The problem is one of celestial mechanics. We have an asteroid in orbit (it may be an unbound orbit), and we need to calculate the radius r_f of its orbit when it comes closest to Earth and its speed v_f when it is at that closest-approach position. We know both its speed and direction of motion a long way off, although we don't know its mass. Perhaps we don't need to know this mass. We assume that when the asteroid is first sighted, Earth's gravitational influence on it is negligible.

2 DEVISE PLAN For celestial mechanics problems, we know that both the energy and the angular momentum of a system of objects interacting only gravitationally are constant: $E_i = E_f$ and $\vec{L}_i = \vec{L}_f$. In Earth's reference frame, the angular momentum of the asteroid (of unknown mass m_a) and the energy of the Earth-asteroid system do not change as the asteroid moves from far away to its position of closest approach, where the radius of its orbit is r_f. We can determine the magnitude of its angular momentum at closest approach by looking at its angular-momentum magnitude when it is far from Earth. Its velocity is always tangent to its trajectory, and when it is far away we have information about the perpendicular distance of this tangent line from Earth:

$$L_i = |\vec{r}_i \times \vec{p}_i| = m_a v_i d,$$

where $\vec{p}_i$ is the asteroid's momentum when it is traveling along its straight-line course. At the position of closest approach, the direction of velocity is also perpendicular to its radius of curvature (see Example 13.5), so its angular momentum is $L_f = r_f m_a v_f$.

The energy of the Earth-asteroid system at the initial position of sighting, where the influence of Earth's gravitational pull is negligible, is purely kinetic. At the point of closest approach, however, it has both kinetic energy and (negative) gravitational potential energy. All that remains is to do the algebra to solve for r_f and v_f.

3 EXECUTE PLAN

(*a*) Using r_i and r_f for the initial and final Earth-asteroid radial center-to-center distances, we have from conservation of energy:

$$K_i + U_i^G = K_f + U_f^G$$

$$\tfrac{1}{2} m_a v_i^2 + \left(-G\frac{m_E m_a}{r_i}\right) = \tfrac{1}{2} m_a v_f^2 + \left(-G\frac{m_E m_a}{r_f}\right)$$

$$\tfrac{1}{2} v_i^2 + \left(-G\frac{m_E}{\infty}\right) = \tfrac{1}{2} v_f^2 + \left(-G\frac{m_E}{r_f}\right)$$

$$\tfrac{1}{2} v_i^2 = \tfrac{1}{2} v_f^2 - G\frac{m_E}{r_f}.$$

This equation contains the two desired quantities: the orbit radius at closest approach r_f and the speed v_f when the asteroid is at that position. Next we apply conservation of angular momentum:

$$L_i = L_f$$

$$m_a v_i d = r_f m_a v_f$$

$$v_f = \frac{v_i d}{r_f}.$$

We note that the asteroid's mass m_a cancels in all our equations. Substituting this expression for v_f into our energy equation yields

$$\tfrac{1}{2} v_i^2 = \tfrac{1}{2}\left(\frac{v_i d}{r_f}\right)^2 - G\frac{m_E}{r_f} = \frac{1}{2}\frac{(v_i d)^2}{r_f^2} - G\frac{m_E}{r_f}.$$

PRACTICE

Multiplying through by r_f^2 and then by $2/v_i^2$, we get a quadratic equation:

$$\tfrac{1}{2}v_i^2 r_f^2 = \tfrac{1}{2}(v_i d)^2 - Gm_E r_f$$

$$r_f^2 + 2\frac{Gm_E}{v_i^2}r_f - d^2 = 0$$

$$r_f = -\frac{Gm_E}{v_i^2} \pm \tfrac{1}{2}\sqrt{\left(2\frac{Gm_E}{v_i^2}\right)^2 + 4d^2}$$

$$= -\frac{Gm_E}{v_i^2} \pm \sqrt{\left(\frac{Gm_E}{v_i^2}\right)^2 + d^2}.$$

The positive result is the physical one we seek, yielding

$$r_f = 7.38 \times 10^7\,\text{m} = 7.38 \times 10^4\,\text{km}. ✔$$

(*b*) The speed at closest approach is

$$v_f = \frac{dv_i}{r_f} = \frac{(3.30 \times 10^8\ \text{m})(754\ \text{m/s})}{7.38 \times 10^7\,\text{m}}$$

$$= 3.37 \times 10^3\ \text{m/s} = 3.37\ \text{km/s}. ✔$$

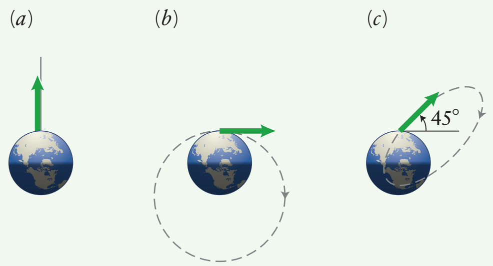
❹ **EVALUATE RESULT** Earth's radius is 6.38×10^3 km, making $r_f \approx 10R_E$, which might generate a bit of anxiety among the planet's inhabitants when one considers possible observational errors. The distance of closest approach is about 74,000 km, roughly one-fourth of the Earth-Moon distance. Fortunately, the asteroid does not hit Earth because at closest approach it is moving a lot faster than it was at first sighting ($v_f \approx 5v_i$). You also might notice that the speed at closest approach is barely greater than the escape speed for an object at this distance r_f from Earth's center, $v_{esc} = \sqrt{2Gm_E/(7.4 \times 10^7\ \text{m})} = 3.3 \times 10^3\ \text{m/s}$. A near miss indeed!

The expression for r_f shows that a greater straight-path distance d gives a greater r_f value, as we expect. Because we are not given any information about how far away the asteroid is initially, we assumed that Earth has negligible gravitational influence at that instant. Without that assumption, we would not have been able to solve the problem. In any case, if our assumption is false, the decrease in potential energy would have been (somewhat) smaller and therefore the final velocity would have been smaller.

Guided Problem 13.6 More rocket launches

Three rockets are launched from Earth as shown in Figure WG13.5. One rocket is launched vertically (*a*), one horizontally (*b*), and one at 45° with respect to the ground (*c*). All three are launched with a speed $\frac{3}{4}$ of the escape speed from Earth's surface, $v_{esc} = 1.12 \times 10^4$ m/s (from Checkpoint 13.22). Calculate the maximum distance from Earth each rocket achieves. Ignore Earth's rotation, and express your answer in terms of Earth's mass and radius.

Figure WG13.5



❶ **GETTING STARTED**

1. What are the physical quantities of interest in this problem?
2. Consider the shape of each trajectory. Figure 13.37 may aid your thinking.

❷ **DEVISE PLAN**

3. What conservation law(s) should you use? How are the maximum and minimum orbital distances related to other physical quantities?
4. Is the information about the magnitude and direction of the velocities in each of the three cases useful?
5. Which unknown quantities do you need to determine in each case?

PRACTICE

3 EXECUTE PLAN

6. Does the same *general* approach work in all three cases?
7. Work through the algebra and obtain an expression for the maximum distance in terms of Earth's mass and radius. Is it possible to use the same expression in all three cases?

4 EVALUATE RESULT

8. Does your expression (or expressions) show plausible algebraic behavior when you vary the launch condition and when you vary Earth's mass and radius?
9. Are your assumptions reasonable?

Worked Problem 13.7 Black hole

A black hole is the remnant of a massive star that, once its nuclear fuel burned out, has collapsed to essentially a point under its own gravitational attraction. The gravitational force exerted by a black hole is so great that not even light can escape from within its interior—hence the name: (black, for the absence of light). Suppose a certain black hole has a mass equal to ten times the mass of the Sun. What is the radius of closest approach for a passing light ray to avoid capture?

1 GETTING STARTED The problem asks about the motion of a light ray in the vicinity of a black hole. We might consider the gravitational force exerted by the black hole on the ray or work from energy or angular momentum conservation. Light has no mass, however, and so we will likely need to invoke the equivalence principle. Conservation principles are usually easier to use than other laws, so let's begin there. It is possible that energy alone can provide the equation we need because only one unknown is sought: a radial distance from the center of the black hole. We know the speed of light, $c = 3.0 \times 10^8$ m/s, and are given the mass of the black hole as 10 solar masses. The mass of the Sun is given in Table 13.1.

2 DEVISE PLAN In any two-object system where one object moves relative to the other, angular momentum, gravitational force, and gravitational potential energy each depend on the masses of the objects. That seems to rule out any of our usual methods for solving this problem because one of our objects has no mass. By the equivalence principle, however, we might be able to treat the light ray as if it were an object subject to the usual laws. Thus we might imagine the ray as having an initial kinetic energy to use in attempting to escape the gravitational pull of the black hole. We used energy methods in Worked Problem 13.3 to derive an expression for the escape speed that depends *not* on the mass of the "escaping" object but only on the mass of the object exerting the gravitational force (which means m_h here) and on the radial distance r separating the objects at the instant of launch: $v_{esc} = \sqrt{2Gm_h/r}$. This expression was derived for an object escaping from a planet's surface, and so we must assume that the gravitational behavior of the black hole is similar enough to that of a planet to allow us to use this relationship. We apply this approach here, with the light ray being the escaping "object" and with r being the "escape radius," the radius at which the light can just barely escape from the black hole, which is the radius of closest approach that we seek. We assume the ray obeys mechanical laws and is launched with speed c, which must be the escape speed at the radius of closest approach.

3 EXECUTE PLAN Squaring our equation for the escape speed from Worked Problem 13.3, substituting $v_{esc} = c$, and solving for r, we get

$$r = 2G\frac{m_h}{c^2}.$$

The mass of the Sun is 2.0×10^{30} kg. Putting in numbers, we arrive at

$$r = 2(6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2})\frac{10(2.0 \times 10^{30}\,\mathrm{kg})}{(3 \times 10^8\,\mathrm{m/s})^2}$$

$$= 3.0 \times 10^4\,\mathrm{m} = 30\,\mathrm{km}. ✔$$

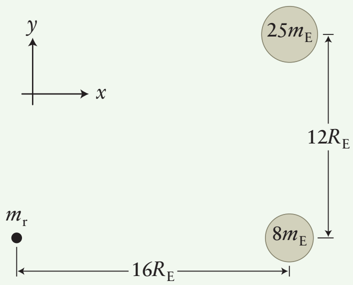
4 EVALUATE RESULT The gravitational force must be very strong to affect light significantly, and the inverse square dependence of this force (or the inverse first power dependence of the gravitational potential energy) on the radius therefore demands a very small radius. That black holes have much smaller radii than ordinary objects of the same mass is no surprise: Black holes must be very exotic or we would have noticed them in our everyday lives. In addition, light seems unaffected by gravity in ordinary life, so we would expect that a very intense gravitational pull would be required. If the mass is fixed, that leaves only the radius to adjust, and the smaller the radius, the greater the gravitational force.

Note: The assumption we made—that we can ignore the fact that light has no mass and simply apply the escape speed formula—is not technically valid. However, when this calculation is done correctly, using Einstein's general theory of relativity, our expression for the black hole radius of closest approach is correct. The value of this radius is known as the "event horizon" for the black hole because events that take place at radii smaller than this cannot be seen from the outside.

Guided Problem 13.8 Heavenly rock

A planet that has a mass 25 times that of Earth is orbited by a moon that has a mass eight times that of Earth (Figure WG13.6). The orbital radius is equal to twelve Earth radii. A large rock located near this system is sixteen Earth radii away from the moon, with the line connecting the rock and moon perpendicular to the line connecting the planet and moon. What is the magnitude of the rock's acceleration in terms of the free-fall acceleration (g) it would have on Earth?

Figure WG13.6



1 GETTING STARTED

1. Given that the question asks for acceleration, what approach should you take?
2. Does it matter that the rock's mass is not specified? Even though this mass is not given, you should give it a symbolic value m_r in any diagrams you draw and any equations you use to see how it comes in and, perhaps, cancels out.
3. How many forces are exerted on the rock? What is the direction of each force? Draw a free-body diagram for the rock.

2 DEVISE PLAN

4. What is the center-to-center distance between rock and moon? The center-to-center distance between rock and planet?
5. What is the magnitude of the gravitational force exerted by the moon on the rock? The magnitude of the gravitational force exerted by the planet on the rock?

3 EXECUTE PLAN

6. Determine the x component of the vector sum of forces exerted on the rock.
7. Determine the y component of the vector sum of forces exerted on the rock.
8. Remember that acceleration is a vector.

4 EVALUATE RESULT

9. Is your result greater or smaller than g? Given the masses and distances, is this reasonable?

Answers to Review Questions

1. They exert forces of equal magnitude because the two forces in an interaction pair are always of equal magnitude. However, the Moon has the smaller mass, and so, from $a = F/m$, its acceleration is greater.
2. Several bits of evidence are presented in this section. First, the centripetal acceleration of the Moon (required to produce its known orbital radius and speed) matches the expected $1/r^2$ strength of gravity at the Moon's orbital radius (a distance of 60 Earth radii). Second, the proportionality between period squared and radius cubed for all planets, first identified by Kepler, agrees with a $1/r^2$ force law. Finally, there is the elliptical shape of the planetary orbits, which can be derived from the inverse square law plus conservation of energy.
3. Ellipse, circle, parabola, and hyperbola are possible.
4. Because period squared is proportional to orbital radius cubed, if the radius quadruples, then the period increases by a factor of 8. Thus a "year" on the planet is equivalent to eight years on Earth.
5. Mass is the property of an object that determines the magnitude of the gravitational force it feels due to, or exerts on, another object. Inertia is the property of an object that determines its acceleration under the influence of any nonzero vector sum of forces. These are very different concepts, even though it turns out that mass and inertia have been measured to be identical to great accuracy under ordinary circumstances.
6. The torque about this axis is zero because the lever arm is always zero. This means that the angular momentum about this axis does not change.
7. (1) Planets move in elliptical orbits with the Sun at one focus.
(2) As a planet orbits the Sun, a straight line connecting the two bodies sweeps out equal areas in equal time intervals. (3) The period squared for any planet is proportional to its orbital radius cubed.
8. The force exerted in (b) and (c) is a central force and so causes zero torque. For angular momentum to change, the vector sum of torques must be nonzero.
9. The ISS. The acceleration due to gravity is independent of the orbiting object's mass and is inversely proportional to (the square of) the distance from Earth. Because the station orbits closer to Earth, its acceleration is greater.
10. Zero. During your entire trajectory, the spring scale is pulled toward the trampoline right along with you, so you do not exert any force on it.

11. Yes. The balance compares the normal forces needed to support each mass against their respective gravitational forces, but in free fall no such normal forces are needed. Thus the normal force is zero on each side; that is, it is still balanced.
12. The reading decreases because the supporting force exerted by the scale has a smaller magnitude when you are accelerating downward.
13. While the astronauts are in orbit, the gravitational force exerted on them by Earth is exactly the amount needed to supply the centripetal acceleration that keeps them in orbit. Thus there is no support force required to satisfy Newton's second law.
14. The principle of equivalence states that there is no way to distinguish locally between gravitational effects and those due to acceleration of the reference frame. The qualification *locally* must be used because over large scales the inverse square nature of gravitational forces would become apparent, causing, for example, the gravitational acceleration experienced by an object at two widely separated locations to be different.
15. Yes. According to the principle of equivalence, anything—including light—that moves close to any object that has mass must move in a curved path.
16. The gravitational force exerted by Earth on an object in free fall near the ground does indeed vary as $1/r^2$. However, as the object moves, the change in its distance r from Earth's center is negligible. Therefore the force and hence the acceleration g are constant to a very good approximation.
17. Not necessarily. The larger planet might also have a larger radius, and the acceleration due to gravity decreases inversely with the radius squared. In fact, the acceleration due to gravity on Saturn is very close to that on Earth.
18. The choice that the potential energy is zero when two objects in a system are infinitely far apart makes the potential energy at all finite distances negative.
19. For any choice of initial and final positions, the work done by the gravitational force on an object is the same regardless of the particular path over which the object moves to get from initial to final position.
20. It is reasonable over any distance that is much smaller than the separation distance r.
21. Strictly speaking, the path is an ellipse. You cannot throw an object fast enough so that its kinetic energy exceeds the gravitational potential energy of the Earth-object system. Thus the mechanical energy is negative and the path is elliptical. However, the path is well approximated by a parabola.
22. The speed is smallest when the planet is at aphelion and greatest when the planet is at perihelion.
23. It is just barely unbound; the object can "escape" to an infinite distance from the more massive object.
24. The orbit is an ellipse if the mechanical energy of the object-Sun system is negative (negative potential energy has a greater magnitude than positive kinetic energy). The circle is also possible as a special case of an ellipse with zero eccentricity. The orbit is one branch of a hyperbola if the mechanical energy of the object-Sun system is positive (positive kinetic energy has a greater magnitude than negative potential energy). If the mechanical energy is zero, the orbit is a parabola, the limiting case between an ellipse and a hyperbola.
25. The sphere is divided into concentric thin shells, and then a shell is divided into rings such that each ring lies between two planes perpendicular to the line joining the particle and the center of the sphere. The force exerted by a ring on the particle is computed, and then this is integrated to obtain the force exerted by one spherical shell. The results for each shell are then superposed to obtain the result for the full sphere.
26. The particle may be located inside the sphere; the same process yields a result of zero for the force on a particle inside a spherical shell. The sphere may be hollow because the process involves superposing the results for spherical shells. The density does not have to be uniform across the solid sphere, but it is assumed to be spherically symmetrical for each shell. Hence the density could vary with radius without affecting the result, but the result would change if the density varied with some other component of the position vector.

Answers to Guided Problems

Guided Problem 13.2 $R_{max} = 19$ km
Guided Problem 13.4 $\Delta x = 3.5$ km
Guided Problem 13.6 **vertical launch:** $r_{max} = \frac{16}{7}R_E$ or $h_{max} = \frac{9}{7}R_E$;
horizontal launch: $r_{max} = \frac{9}{7}R_E$ or $h_{max} = \frac{2}{7}R_E$;
45° launch: $r_{max} = 1.96R_E$ or $h_{max} = 0.96R_E$
Guided Problem 13.8 $a_{rock} = 0.089g$

PRACTICE

Guided Practice by Chapter

14 Special Relativity

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

14.1 Time measurements

1. What is an event?
2. Name two attributes of clocks that allow us to measure precisely the instants at which various events occur.
3. What must happen in order for an event to be observed?
4. What must you observe about the locations of two events in order to measure the proper time interval between them?
5. Describe what an observer must do to measure the instant at which an event occurs in a reference frame that consists of synchronized clocks placed at various equally spaced positions.

14.2 *Simultaneous* is a relative term

6. Where must a simultaneity detector be placed in order to determine whether two events are simultaneous? When placed at that location, how does the detector operate?
7. How does the speed of light in vacuum depend on the relative velocity of the source and the observer?
8. What can you say about the simultaneity of two events seen by two observers moving relative to each other along the line joining the events?

14.3 Space-time

9. Clock A is moving relative to identical clock B. To an observer at rest relative to B, how does the rate at which A runs compare with the rate at which B runs?
10. Cite two experiments that give direct evidence for time dilation.
11. What is the proper length of an object?
12. What is length contraction?

14.4 Matter and energy

13. What experimental evidence shows that the kinetic energy expression $K = \frac{1}{2}mv^2$ does not apply at relativistic speeds?
14. Which of the variables—inertia, mass, kinetic energy, and internal energy—are invariant, and which depend on the reference frame of the person measuring them?
15. If you change the internal energy of a system, does the system's inertia change, and if so, how?
16. What happens to an object's inertia when you decrease the object's kinetic energy?

14.5 Time dilation

17. What is the definition of the Lorentz factor, and at what speed do its effects on space-time properties become noticeable?
18. What is the relationship between the proper time interval between two events and the time interval between the events measured by an observer for whom the events occur at different locations?
19. What is the difference among *timelike, lightlike,* and *spacelike* values of the space-time interval $s^2 = (c_0\Delta t)^2 - (\Delta x)^2$ between two events?
20. How does the nature of the space-time interval between events (*timelike, lightlike,* or *spacelike*) affect the principle of causality?

14.6 Length contraction

21. What is the relationship between the proper length of an object and the length measured by an observer moving relative to the object?
22. How are time dilation and length contraction related to each other? Consider muons traveling from the top of Earth's atmosphere to the ground to support your answer.
23. What are the requirements for validity of the Lorentz transformation equations? How do they differ from the requirements for the time dilation and length contraction equations (Eqs. 14.13 and 14.28)?

14.7 Conservation of momentum

24. The momentum of a particle was defined in Chapter 4 as the product of its inertia and its velocity. What change, if any, is required so that momentum can be defined for a particle that is traveling at a relativistic speed?
25. Does the momentum of an isolated system remain constant when one or more components travel at a relativistic speed?
26. What is the relationship (*a*) between the mass m of a particle and the inertia m_v we measure as the particle moves at some speed v relative to us, and (*b*) between the particle's inertia and its momentum?

14.8 Conservation of energy

27. How does the mathematical expression for the kinetic energy of a particle moving at relativistic speeds differ from that for a particle moving at nonrelativistic speeds? Are the two expressions compatible?
28. For a system of mass m, what does the quantity mc_0^2 represent?
29. For a system of mass m, what does the quantity γmc_0^2 represent?
30. What is the difference between *conserved* and *invariant,* and how is this difference relevant when we measure energy and momentum?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The difference between 1 and the Lorentz factor for a high-speed rifle bullet (E, T)
2. The difference between the time intervals needed to complete your walking trip, as measured by you and by your friend at rest in the Earth reference frame, beginning when you depart New York City and ending when you reach Los Angeles (Each of you measures only the time intervals you spend *walking*.) (V, J, M, B)
3. The difference between the time intervals, as measured by an astronaut on board the International Space Station (ISS) and by an observer at rest in the Earth reference frame, beginning when the ISS passes directly above Los Angeles and ending when the ISS is directly above New York City (F, J, U, C, P)
4. The length by which the distance from New York City to Los Angeles is contracted when measured by an observer moving east to west relative to Earth at the speed of the ISS (F, J, U, C, P)
5. The distance traveled by a pion during its lifetime, 26 ns in the pion rest reference frame, when the particle is moving at a speed of $0.995c_0$, as measured by an observer at rest in the Earth reference frame (H, Q)
6. The extra inertia in the Earth reference frame of a midsize car traveling at highway speed compared to the same car at rest (R, D, O)
7. The amount of mass converted to energy in the Sun every second (N, A, G)
8. The ratio of the energy of a proton moving at $0.99999c_0$ in the Earth reference frame to the *kinetic* energy of a snail in the same reference frame (S, L, I, K)

Hints

A. What is the distance from the Sun to Earth?
B. What is the Lorentz factor for your walking speed in the Earth reference frame?
C. What fraction of an orbital period does the ISS need to travel from Los Angeles to New York City in the Earth reference frame?
D. What is the Lorentz factor for this speed? (You calculated this in Checkpoint 14.14.)
E. What is the speed of a high-speed bullet in the Earth reference frame?
F. At what height does the ISS orbit in the Earth reference frame?
G. How much power does the Sun generate?
H. What length of Earth moves past the pion during its lifetime in the pion reference frame?
I. What is the mass of a snail?
J. What is the distance from New York City to Los Angeles in the Earth reference frame?
K. What is the top speed of a snail in the Earth reference frame?
L. What is the Lorentz factor for this proton in the Earth reference frame?
M. What time interval (in the Earth reference frame) would you need to walk that distance nonstop?
N. What is the amount of average solar power that arrives at each square meter of Earth?
O. What is the mass of a midsize car?
P. What is the Lorentz factor for the ISS in the Earth reference frame?
Q. What is the Lorentz factor for the pion in the Earth reference frame?
R. What is a typical highway speed in the Earth reference frame?
S. What is the mass of a proton?
T. What is the Lorentz factor for this bullet in the Earth reference frame?
U. What is the period of orbit of the ISS in the Earth reference frame?
V. What is your average walking speed in the Earth reference frame?

Key (all values approximate)

A. 2×10^{11} m; **B.** $1 + 2 \times 10^{-17}$; **C.** 1×10^{-1}; **D.** $1 + 5 \times 10^{-15}$; **E.** 1×10^{3} m/s; **F.** 4×10^{2} km; **G.** 5×10^{26} W; **H.** 8 m; **I.** 10^{-2} kg; **J.** 4×10^{6} m; **K.** 10^{-3} m/s; **L.** 2×10^{2}; **M.** 2×10^{6} s; **N.** 1×10^{3} W/m^2; **O.** 1×10^{3} kg; **P.** $1 + 3 \times 10^{-10}$; **Q.** 1×10^{1}; **R.** 3×10^{1} m/s; **S.** 2×10^{-27} kg; **T.** $1 + 6 \times 10^{-12}$; **U.** 9×10^{1} min; **V.** 2×10^{0} m/s

Worked and Guided Problems

Procedure: Time and length measurements at very high speeds

At very high speeds, measurements of lengths and time intervals depend on the reference frame. Problems involving objects or observers moving at very high speeds therefore require extra care when we consider lengths and time intervals.

1. Identify each observer or reference frame with a letter and determine in which reference frame each numerical quantity given in the problem statement is measured.
2. If the problem involves any time intervals, identify the events that define that time interval. Determine in which observer's reference frame, if any, the two events occur at the same position. This observer measures the proper time interval between the events (Figure 14.26).
3. Identify any distances or lengths mentioned in the problem. Determine for which observer, if any, the object defining that distance or length remains at rest. This observer measures the proper length (Figure 14.29).
4. Use Eqs. 14.13 and 14.28 to determine the time intervals and lengths in reference frames other than the ones in which the intervals and lengths are given.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

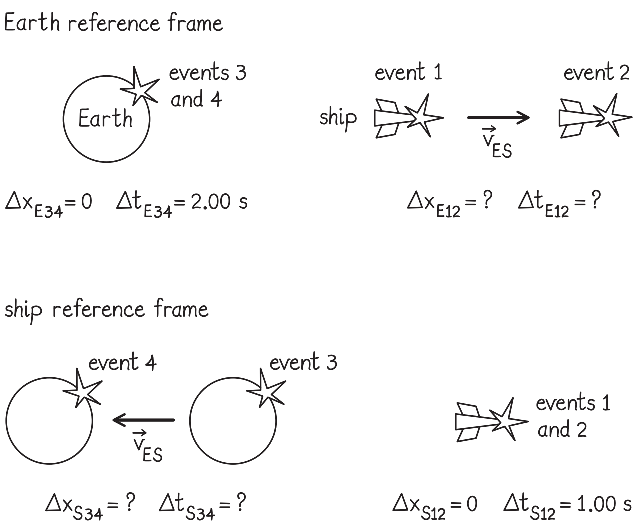
Worked Problem 14.1 Space rescue

You are playing a rescue dispatcher on Earth in a video game involving interstellar travel. In the game, a spaceship that left Earth a few months ago, accelerating toward the star Vega, has suffered an engine failure and is now drifting toward Vega. A distress signal from the ship has been received on Earth, with the signal pulses arriving 2.00 s apart. You know that the distress beacon on the disabled ship emits pulses 1.00 s apart in the ship's reference frame. (*a*) In order to give instructions to a rescue crew being sent out, you must know the speed at which the disabled ship is drifting toward Vega. What is this speed in the Earth reference frame? (*b*) The standard reply signal telling a disabled ship that help is on the way consists of pulses arriving at the ship 1.00 s apart. If the disabled ship is to receive these reply pulses at 1.00-s intervals, what interval between pulses must you use in sending the reply signal?

❶ GETTING STARTED The problem asks us to relate the time difference between the pulses that are emitted in one reference frame and the pulses that are received in another reference frame, so let us consider two pairs of events: Event 1 is the emission of a pulse by the spaceship, and event 2 is the emission of the next pulse by the ship. Event 3 is the reception of the first pulse on Earth, and event 4 is the reception of the next pulse on Earth. Events 1 and 2 occur at the same location relative to the disabled ship, so crewmembers on the ship measure the proper time interval between those two events. Events 3 and 4 occur at the same location relative to Earth, so we measure the proper time interval between them. Because the ship is moving at constant speed with respect to Earth, we can use the formula for time dilation to determine the time interval between events 1 and 2 in the Earth reference frame. We need to remember, though, that we measure the instant at which an event takes place with a clock that is at rest in our reference frame *at the location of the event*. Because events 1 and 2 take place at different locations in the Earth reference frame, the time interval that we measure between them is not the same as the time interval that we measure between events 3 and 4. These two intervals are, however, related by the speed of the ship and the speed of the signal (that is, the speed of light, c_0).

We get oriented with a sketch showing the four events of interest in both the Earth and Ship reference frames (Figure WG14.1).

Figure WG14.1



From the point of view of the crew on the disabled ship, they are at rest and Earth is moving away from them at the constant speed that we measure for the ship. Because the respective situations are symmetrical, we should be able to use similar reasoning to solve parts *a* and *b*.

❷ **DEVISE PLAN** Because the crewmembers on the disabled ship measure the proper time interval between events 1 and 2, we know that the interval as measured in the Earth reference frame is longer by the Lorentz factor, γ:

$$\Delta t_{E12} = \gamma \Delta t_{S12}. \tag{1}$$

In the Earth reference frame, the time interval it takes for the signals to reach Earth is the distance from the ship to Earth divided by the speed of light, so

$$\begin{aligned}\Delta t_{E34} &= t_{E4} - t_{E3} \\ &= (t_{E2} + x_{E2}/c_0) - (t_{E1} + x_{E1}/c_0) \\ &= \Delta t_{E12} + \Delta x_{E12}/c_0. \end{aligned} \tag{2}$$

The distance that the ship moves between events 1 and 2 is the ship's speed multiplied by the time interval:

$$\Delta x_{E12} = v_{ES} \Delta t_{E12}. \tag{3}$$

Substituting first Eq. 3 and then Eq. 1 into Eq. 2 gives

$$\begin{aligned}\Delta t_{E34} &= \Delta t_{E12} + v_{ES}\Delta t_{E12}/c_0 \\ &= \Delta t_{E12}(1 + v_{ES}/c_0) \\ &= \gamma \Delta t_{S12}(1 + v_{ES}/c_0),\end{aligned}$$

which we can simplify by using the definition of the Lorentz factor:

$$\begin{aligned}\Delta t_{E34} &= \frac{\Delta t_{S12}}{\sqrt{1 - (v_{ES}/c_0)^2}}(1 + v_{ES}/c_0) \\ &= \Delta t_{S12}\sqrt{\frac{(1 + v_{ES}/c_0)^2}{1 - (v_{ES}/c_0)^2}} \\ &= \Delta t_{S12}\sqrt{\frac{1 + v_{ES}/c_0}{1 - v_{ES}/c_0}}.\end{aligned} \tag{4}$$

Finally, we can solve Eq. 4 for v_{ES}, the speed of the ship in the Earth reference frame:

$$\begin{aligned}\Delta t_{E34}^2(1 - v_{ES}/c_0) &= \Delta t_{S12}^2(1 + v_{ES}/c_0) \\ \Delta t_{E34}^2 - \Delta t_{S12}^2 &= \Delta t_{E34}^2 v_{ES}/c_0 + \Delta t_{S12}^2 v_{ES}/c_0 \\ &= v_{ES}(\Delta t_{E34}^2/c_0 + \Delta t_{S12}^2/c_0) \\ v_{ES} &= c_0\frac{\Delta t_{E34}^2 - \Delta t_{S12}^2}{\Delta t_{E34}^2 + \Delta t_{S12}^2}.\end{aligned}$$

❸ **EXECUTE PLAN** (*a*) Substituting the time intervals measured in each reference frame, we obtain

$$v_{ES} = c_0\frac{(2.00\text{ s})^2 - (1.00\text{ s})^2}{(2.00\text{ s})^2 + (1.00\text{ s})^2} = c_0\frac{3.00}{5.00} = 0.600c_0. ✔$$

PRACTICE

(b) Because the relationship between the Earth reference frame and the ship's reference frames is symmetrical, we can use Eq. 4—but this time with E and S exchanged and labeling the two emissions on Earth as 5 and 6 and the two receptions on the ship as 7 and 8—to determine the ratio between the rate of pulses that are sent from Earth and the rate of pulses that are received on the ship:

$$\Delta t_{S78} = \Delta t_{E56}\sqrt{\frac{1 + v_{SE}/c_0}{1 - v_{SE}/c_0}}$$

$$\Delta t_{E56} = \Delta t_{S78}\sqrt{\frac{1 - v_{SE}/c_0}{1 + v_{SE}/c_0}}$$

$$\Delta t_{E56} = (1.00 \text{ s})\sqrt{\frac{1 - 0.600}{1 + 0.600}} = 0.500 \text{ s. } \checkmark$$

❹ EVALUATE RESULT The speed that we calculated in part *a* is substantial. An interstellar ship would have to move at great speed to travel between stars in a reasonable amount of time.

It makes sense that the ship's speed is relativistic because of the large factor by which the received pulse rate differs from the rate at which the pulses were emitted. It also makes sense that we got the same factor for the Earth-to-ship pulses as for the ship-to-Earth pulses because of the symmetry of their situations with respect to each other.

As a check on our math, we can compare the invariant interval between events 1 and 2 using numerical values from each reference frame:

$$s^2 = (c_0\Delta t^2) - \Delta x^2$$

For events 1 and 2 in the Earth frame we have (using Eqs. 1 and 3):

$$\begin{aligned} s_{12}^2 &= (c_0\Delta t_{E12})^2 - \Delta x_{E12}^2 = (c_0\Delta t_{E12})^2 - (v_{ES}\Delta t_{E12})^2 \\ &= (c_0\gamma\Delta t_{S12})^2 - (v_{ES}\gamma\Delta t_{S12})^2 \\ &= [(1.25c_0 \text{ m/s})(1.00 \text{ s})]^2 - [(0.600c_0 \text{ m/s})(1.25)(1.00 \text{ s})]^2 \\ &= c_0^2 \text{ m}^2 \end{aligned}$$

In the Ship frame

$$\begin{aligned} s_{12}^2 &= (c_0\Delta t_{S12})^2 - \Delta x_{S12}^2 \\ &= [(c_0 \text{ m/s})(1.00 \text{ s})] - 0 \\ &= c_0^2 \text{ m}^2 \end{aligned}$$

This is reassuring.

We are reminded that, when we consider objects that move at relativistic speeds, what we measure for distant events is not what we actually see when we look at them: When an object is moving at a significant fraction of the speed of light, we have to account for the finite time interval that it takes for light to travel from the object to our eyes (or another detector) because we define our measurements as being made locally to the events. In this situation, with the disabled spaceship moving away from Earth, the time interval between the pulses that we receive on Earth is even longer than the time-dilated interval that we measure for the emission of those pulses from the ship. If the ship had been approaching Earth, the time interval between the pulses that we received on Earth would have been shorter than the time interval that we measured for their emission from the ship. (The sign of Δx in Eq. 2 would be opposite, giving us the inverse ratio in Eq. 4.)

Guided Problem 14.2 Relativistic fly-by

Spaceship A makes a fly-by of space station B at very close distance. The duty officer on B determines that, during the fly-by, the length of A was 50.0 m in the reference frame of B. Records show that the measured length of A was 80.0 m when A was in dry-dock on B. (*a*) What does the duty officer on B measure for A's speed during the fly-by? (*b*) The crew on B measures the length of B to be 200 m. What length does an observer on A measure for the length of B during the fly-by?

❶ GETTING STARTED

1. Which reference frames should you compare?
2. For each reference frame you use, make a sketch of the objects involved and the information you know about them.

❷ DEVISE PLAN

3. How is the speed you must determine related to the two lengths given for the spaceship?
4. Is there any reference frame in which the station length you must determine is a proper length?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

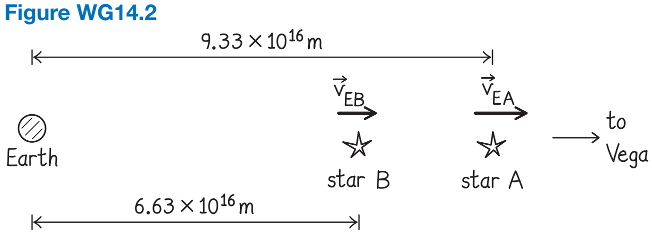
5. Are your proper lengths greater than their corresponding nonproper lengths?

Worked Problem 14.3 Galactic fireworks

A fast *nova* (from the Latin, *nova stella*, "new star") occurs when a white dwarf star experiences a runaway nuclear reaction, which increases its brightness dramatically during a short interval of time. The enhanced brightness then persists for about a month. Suppose that bright light from "new star" A first appears to observers at MacDonald Observatory on Earth at 11:00 p.m. on July 12. Measurements indicate that star A is moving almost directly away from Earth with a relative speed of 6.0×10^4 m/s and is first observed approximately 9.33×10^{16} m from Earth in the direction of Vega. While still studying star A, observers see another "new star," star B, also in the direction of Vega. Bright light from star B first arrives at the observatory at 2:00 a.m. on July 30, and measurements indicate that star B is moving almost directly away from Earth with a relative speed of 4.0×10^4 m/s and is first observed approximately 6.63×10^{16} m from Earth. (*a*) Determine the space-time interval between these two events: star A goes nova and star B goes nova. (*b*) Describe an inertial reference frame in which these two events are simultaneous, or argue that such a reference frame does not exist.

❶ **GETTING STARTED** The information we have is from the Earth reference frame, so we begin with a picture of the situation in that reference frame (Figure WG14.2). It is convenient to choose an x axis along the straight line that contains both stars and Earth. We arbitrarily choose the positive direction to be toward Vega (toward both new stars). We can compute the light travel time from each star from the given distance. It may be necessary to adjust for the stellar motion, but we have speed and direction information in any case. This should allow us to compute both the x component of displacement between the two events and the time interval between the two events in the Earth reference frame. The value of the space-time interval is the same in all inertial reference frames. It can be computed directly from these Δx and Δt values in the Earth reference frame. Depending on the sign of this result, we should be able to determine whether there is a reference frame in which the events are simultaneous. If there is, such a reference frame could be imagined as moving past Earth at some relative velocity in the x direction—that is, toward one new star and away from the other.

Figure WG14.2



❷ **DEVISE PLAN** For part *a*, the space-time interval between two events can be computed in any reference frame using Eq. 14.18:

$$s^2 = (c_0 \Delta t)^2 - (\Delta x)^2.$$

We have enough information to compute the displacement and time interval between the two specified events, but we must decide whether the velocities of the two stars relative to Earth should be taken into account. Both relative velocities are of order of magnitude 10^5 m/s, and the time interval between observations of "first" light from the two stars is of the order 10^6 s. This means that star A might have moved as much as $d = vt = (10^5 \text{ m/s})(10^6 \text{ s}) = 10^{11}$ m before star B was first observed. However, we are given stellar distance information to three significant digits, and both distances are of the order 10^{17} m. Thus any consideration of additional distance due to stellar motion is several digits beyond the precision of our data. In addition, relativistic effects depend on the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{(1 - v^2/c_0^2)}}.$$

For the given relative velocities of these stars, the difference between γ and 1 is on the order of 10^{-8}, which is well beyond the precision of our distance information and, assuming we know the measured times to within 1 minute, also several digits beyond the precision of our time data. Thus a simple calculation of light travel time is appropriate in determining the space-time interval.

For part *b*, the space-time interval can be positive, negative, or zero. If it is positive, the interval is timelike, and it is possible to establish a definite order of events. Thus one of the two events occurs first in all inertial reference frames, and there is no inertial reference frame in which the two events are simultaneous. If the space-time interval is negative, the interval is spacelike. In this case, there is a reference frame in which the events are simultaneous, and we can obtain it by using a simultaneity detector—that is, by devising a reference frame that receives light from each event at an instant when it is equidistant in space between the two events. If the space-time interval is zero, it is lightlike in all inertial reference frames, so a ray of light could travel from one event to the other. Because the speed of light is the same in all reference frames, the events cannot be simultaneous unless they occur at exactly the same place.

PRACTICE

3 EXECUTE PLAN

(*a*) The time interval between the two events must be computed using the travel time of light. We observe star A at the location where its light originated—that is, where the star was when it "went nova." Light would require a time interval of

$$\Delta t_{\text{EA}} = \frac{\Delta x_{\text{EA}}}{c_0}$$

to travel from star A to Earth.

Similarly, the travel time for light from the second event to reach Earth is

$$\Delta t_{\text{EB}} = \frac{\Delta x_{\text{EB}}}{c_0}.$$

However, these time intervals overlap because star B does not wait for light from star A to reach Earth before "going nova." Thus the time interval between the two events is the difference in the two light travel times plus the delay in reception of the signal from star B compared to star A. The result is a time interval, in the Earth reference frame, between the two events of

$$\begin{aligned}
\Delta t_{\text{E}} &= \Delta t_{\text{EA}} - \Delta t_{\text{EB}} + \Delta t_{\text{E detection}} \\
&= \frac{\Delta x_{\text{EA}}}{c_0} - \frac{\Delta x_{\text{EA}}}{c_0} + \Delta t_{\text{E detection}} \\
&= \frac{9.33 \times 10^{16}\ \text{m}}{3.00 \times 10^{8}\ \text{m/s}} - \frac{6.63 \times 10^{16}\ \text{m}}{3.00 \times 10^{8}\ \text{m/s}} \\
&\quad + \left[17\ \text{days}\left(\frac{24\ \text{h}}{1\ \text{day}}\right) + 3\ \text{h}\right]\left(\frac{3600\ \text{s}}{1\ \text{h}}\right) \\
&= (3.11 \times 10^{8}\ \text{s}) - (2.21 \times 10^{8}\ \text{s}) + (1.48 \times 10^{6}\ \text{s}) \\
&= 9.15 \times 10^{7}\ \text{s}.
\end{aligned}$$

The displacement between the two events is, to the precision available, simply the difference in their apparent positions when they went nova:

$$\begin{aligned}
\Delta x_{\text{EA}} &= 9.33 \times 10^{16}\ \text{m} \\
\Delta x_{\text{EB}} &= 6.63 \times 10^{16}\ \text{m} \\
\Delta x_{\text{E}} &= \Delta x_{\text{EA}} - \Delta x_{\text{EB}} = (9.33 \times 10^{16}\ \text{m}) - (6.63 \times 10^{16}\ \text{m}) \\
&= 2.70 \times 10^{16}\ \text{m}.
\end{aligned}$$

The space-time interval is then

$$\begin{aligned}
s^2 &= (c_0 \Delta t_{\text{E}})^2 - (\Delta x_{\text{E}})^2 \\
&= [(3.00 \times 10^{8}\ \text{m/s})(9.15 \times 10^{7}\ \text{s})]^2 - (2.70 \times 10^{16}\ \text{m})^2 \\
&= 2.4 \times 10^{31}\ \text{m}^2.
\end{aligned}$$

(*b*) The invariant space-time interval is positive, or timelike, so it must be possible to order the events in time. Indeed, star A must go nova first; otherwise, the light from star B, which is closer to Earth, would arrive first. This is true in any inertial reference frame. It is therefore not possible to construct an inertial reference frame in which the events are simultaneous.

4 EVALUATE RESULT Distances between stars are typically very large. The stars in this problem are at distances on the order of 10^{17} m, which is about the distance light travels in 10 years.

For part *a*, the space-time interval is less than $(c_0 \Delta t_{\text{E}})^2$, which is consistent for events that do not occur at the same location. The time required for light to travel between stars A and B is slightly smaller than the time interval we computed between the events, so in principle a signal from one star could have triggered the other to explode. This is also consistent with a timelike interval between events.

For part *b*, we know that events separated by a timelike space-time interval in one reference frame are also separated by a timelike interval in any other inertial reference frame; hence, they cannot be simultaneous in any reference frame.

Guided Problem 14.4 Lorentz would love it

Observers Carol and Diane move at constant velocity relative to each other along a straight line. They agree to align the positive x axis of their respective reference frames along this line. They set their respective clocks to zero just as the origins of the two x axes coincide. Diane then observes event 1 at $x_{D1} = +400$ m, $t_{D1} = 1.00 \times 10^{-6}$ s, and event 2 at $x_{D2} = +900$ m, $t_{D2} = 5.00 \times 10^{-7}$ s. Carol observes these two events as occurring simultaneously. (*a*) What is the velocity $\vec{v}_{DC}$ of Carol relative to Diane? (*b*) At what instant does Carol observe the events?

1 GETTING STARTED

1. For each reference frame, make a sketch of the objects involved and the information you know about them.
2. Is any length a proper length in one of the reference frames? Is any time interval a proper time interval in one of the reference frames?

2 DEVISE PLAN

3. What general transformation equation relates time and space coordinates in different reference frames?
4. How can you express Carol's observations of the time coordinates of events 1 and 2 in terms of Diane's observations of the same events?
5. How can you express the difference in time between events 1 and 2 as measured by Carol in terms of Diane's coordinates?
6. Can the Lorentz factor that relates Carol's reference frame to Diane's reference frame be zero?

3 EXECUTE PLAN

7. Can you rearrange the expression for the time difference as measured by Carol to get Carol's velocity as measured by Diane?

4 EVALUATE RESULT

8. Can you check your result for consistency by using the space-time interval between events?

Worked Problem 14.5 To the stars!

The first interstellar exploration spaceship passes remote science post P and has already turned off its engines to coast at a constant cruising speed for its journey beyond the star Alpha Centauri. The scientists on post P measure the ship's speed to be $0.600c_0$. Because Alpha Centauri is known to be 4.00×10^{16} m distant from post P, according to measurements made on post P, the scientists have some time to plan a celebration of humans moving past Alpha Centauri. Cleverly, they schedule the festivities so that astronauts aboard the ship will observe (through their telescopes) the beginning of the scientists' celebration at the same instant the ship speeds past Alpha Centauri. (*a*) What time interval is required for the trip according to spaceship clocks? (*b*) How much time will have elapsed on post P between the instant the ship passes their post and the start of the celebration?

1 GETTING STARTED Because we need to compare information from post P and the ship, we need to analyze the problem from the perspective of both the scientists at post P and the astronauts on the ship. We define two reference frames: reference frame P attached to the post and reference frame S attached to the spaceship. Thus the ship's speed relative to the post is $v_{PS} = 0.600c_0$. We are told that in reference frame P, the distance to Alpha Centauri is 4.00×10^{16} m. We define event 0 to occur when the ship passes the post, event 1 to be when the ship reaches Alpha Centauri, event 2 to occur when the party starts at post P, and event 3 to be when the astronauts on the ship as it passes Alpha Centauri observe the party starting. It is useful to note that, in order for the astronauts to observe the party starting, the light from that event must travel from post P to the ship. We can think of this light representing the start of the party as a signal sent (event 2) and received (event 3). Table WG14.1 organizes the information we know about these events. For convenience, we choose the origin of both time and position in both reference frames at event 0, with the positive x axis pointing toward Alpha Centauri in reference frame P and toward post P in reference frame S. This allows all x components to be positive for each event in each reference frame.

Table WG14.1 Events for Worked Problem 14.5

Event	t_S	x_S	t_P	x_P
0: ship passes post P	$t_{S0} = 0$	$x_{S0} = 0$	$t_{P0} = 0$	$x_{P0} = 0$
1: ship arrives at AC	t_{S1}	$x_{S1} = 0$	t_{P1}	$x_{P1} = +4.00 \times 10^{16}$ m
2: signal that party starts sent from post P	t_{S2}	x_{S2}	t_{P2}	$x_{P2} = 0$
3: signal that party starts received by ship at AC	$t_{S3} = t_{S1}$	$x_{S3} = 0$	t_{P3}	$x_{P3} = x_{P1} = 4.00 \times 10^{16}$ m

2 DEVISE PLAN We can glean significant information about the trip from the time dilation equation (Eq. 14.13). In order to determine the time interval for the trip according to scientists at post P ($\Delta t_{P01} = t_{P1} - t_{P0}$), we divide the distance traveled by the spaceship as measured in reference frame P by the ship's speed. Because in reference frame S events 0 and 1 both take place at the location of the ship, the time interval between these two events as measured by the ship's clocks ($\Delta t_{S01} = t_{S1} - t_{S0}$) is a proper time interval in reference frame S.

Therefore we can use Eq. 14.13 to relate Δt_{S01} and Δt_{P01}, using the appropriate Lorentz factor given by Eq. 14.6. The v in the Lorentz factor is equal to the speed of the ship relative to the monitoring post: $v = v_{PS} = 0.600c_0$.

Because events 1 and 3 are simultaneous in reference frame S, we know that $\Delta t_{S03} = \Delta t_{S01}$. Because event 2 takes place at the post, we know that Δt_{P02} is a proper time interval in reference frame P. Thus we can add some information to Table WG14.1 by using Eq. 14.13. Sorting events 2 and 3 requires something more general, however. We use the invariant (reference-frame-independent) speed of light plus the invariant space-time interval s^2 to compare information between reference frames and then calculate the desired temporal information.

❸ **EXECUTE PLAN** The time interval Δt_{P01} needed for the spaceship to reach Alpha Centauri as measured by clocks at the monitoring post is

$$\Delta t_{P01} = \frac{d_{P01}}{v_{PS}} = \frac{4.00 \times 10^{16}\ \text{m}}{0.600(3.00 \times 10^{8}\ \text{m/s})} = 2.22 \times 10^{8}\ \text{s}.$$

We then use Eq. 14.6 to determine the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - v_{PS}^2/c_0^2}} = \frac{1}{\sqrt{1 - (0.600)^2}} = \frac{1}{0.800} = 1.25.$$

(a) Now we can use Eq. 14.13 to determine what clocks on the spaceship measure for the proper time interval needed for the ship to travel from the monitoring post to Alpha Centauri:

$$\Delta t_{S01} = \frac{\Delta t_{P01}}{\gamma_{PS}} = \frac{2.22 \times 10^{8}\ \text{s}}{1.25} = 1.78 \times 10^{8}\ \text{s}. \checkmark$$

Both Δt_{S01} and Δt_{P01} begin at $t = 0$, and so we now have full information about events 0 and 1 in Table WG14.1:

$$t_{S1} = 1.78 \times 10^{8}\ \text{s}, \quad t_{P1} = 2.22 \times 10^{8}\ \text{s}.$$

Note that we also have most of the information about event 3, missing only a value for t_{P3}. In reference frame S, events 1 and 3 are simultaneous and occur at the same position. This means that in this reference frame the space-time interval s^2 between events 1 and 3 is zero. Thus these two events are separated by a lightlike space-time interval. Events that are simultaneous *and* occur at the same location in one reference frame must also be simultaneous and occur at some common location in *any* inertial reference frame. This is not immediately obvious because it might be possible to calculate a number of different time and space intervals between events 1 and 3 that, when combined, give a space-time interval $s^2 = 0$.

Let us try it: Suppose scientists at the monitoring post see the spaceship reach Alpha Centauri (event 1) at some instant and then see the light signal indicating the start of the party received at the star (event 3) at a different instant. Then the signal must be received at a position that makes the space-time interval between events 3 and 1 equal to zero, just as in reference frame S. However, that would mean that the ship travels from the position of event 3 to the position of event 1 at the speed of light because only then does $s^2 = 0 = (c_0\Delta t)^2 - (\Delta x)^2$. This is not possible! The result is that the scientists must also see the signal as being received at Alpha Centauri just as the ship reaches the star. Thus we add to our table of values:

$$t_{P3} = t_{P1} = 2.22 \times 10^{8}\ \text{s}\,.$$

(b) Now we must determine what scientists at the post measure for the time interval between the instant the ship passes the post (event 0) and the instant the scientists begin their party (event 2). The scientists know that both the ship and the light signal indicating the start of the party must arrive at Alpha Centauri at instant $t_{P3} = t_{P1} = 2.22 \times 10^{8}$ s. They also know the speed at which that signal travels and the distance it travels. Thus from the instant the signal is sent out (event 2) to the instant it is received at Alpha Centauri (event 3) is a time interval of

$$\Delta t_{P23} = t_{P3} - t_{P2} = \frac{d_{P23}}{c_0} = \frac{d_{P01}}{c_0} = \frac{4.00 \times 10^{16}\ \text{m}}{3.00 \times 10^{8}\ \text{m/s}}$$
$$= 1.33 \times 10^{8}\ \text{s}.$$

We know t_{P3} and so

$$t_{P2} = t_{P3} - \Delta t_{P23} = (2.22 \times 10^{8}\ \text{s}) - (1.33 \times 10^{8}\ \text{s})$$
$$= 0.89 \times 10^{8}\ \text{s}.$$

We have set $t_{P0} = 0$, so by the clocks at post P the party must start

$$\Delta t_{P02} = t_{P2} - t_{P0} = t_{P2} - 0 = 8.9 \times 10^7 \text{ s} \checkmark$$

after the ship passes.

4 EVALUATE RESULT The light signal indicating the start of the party travels at speed c_0, and the spaceship travels at 60% of that value. If the two must travel the same distance in reference frame P and the signal moves 1.67 times faster than the ship in that reference frame, it seems reasonable that the post scientists should start their party after about 30% to 40% of the ship travel time interval has elapsed. The astronauts, of course, measure a different value for the instant at which the party signal is sent. Because the time interval Δt_{P02} is proper in reference frame P, we can use Eq. 14.13 to calculate the time interval in reference frame S:

$$\Delta t_{S02} = \gamma \Delta t_{P02} = 1.25(8.9 \times 10^7 \text{ s}) = 1.1 \times 10^8 \text{ s}.$$

In reference frame P, the space-time interval between events 0 and 2 is

$$s_{02}^2 = (c_0 \Delta t_{P02})^2 - 0 = 7.1 \times 10^{32} \text{ m}^2.$$

The space-time interval must have the same value in reference frame S, allowing us to compute the displacement of post P for the ship when the signal was sent in reference frame S:

$$s_{02}^2 = 7.1 \times 10^{32} \text{ m}^2 = (c_0 \Delta t_{S02})^2 - (\Delta x_{S02})^2$$

$$\Delta x_{S02} = \sqrt{[c_0(1.1 \times 10^8 \text{ s})]^2 - (7.1 \times 10^{32} \text{ m}^2)} = 1.9 \times 10^{16} \text{ m}.$$

Thus in reference frame S the party signal travels this distance to arrive at the ship (at Alpha Centauri) for event 3. The time interval the astronauts measure for how long the signal takes to travel this distance is

$$\Delta t_{S23} = \frac{1.9 \times 10^{16} \text{ m}}{3.00 \times 10^8 \text{ m/s}} = 6.3 \times 10^7 \text{ s},$$

so that the time interval from passing post P to the arrival of the party signal is

$$\Delta t_{S03} = \Delta t_{S02} + \Delta t_{S23} = (1.1 \times 10^8 \text{ s}) + (6.3 \times 10^7 \text{ s})$$
$$= 1.7 \times 10^8 \text{ s}.$$

The results are all consistent within the two significant digits allowed by the subtraction we used in calculating t_{P2}.

Guided Problem 14.6 Space-time interval

An unstable high-energy particle enters a detector and, before decaying, leaves a track 1.05 mm long. The speed of the particle relative to the detector is $0.992c_0$. Show that the value of the space-time interval s^2 for the interval from the instant the particle enters the detector to the instant the particle decays is the same when computed in a reference frame D at rest relative to the detector and in a reference frame P at rest relative to the particle.

1 GETTING STARTED

1. Identify the two events that bracket the desired space-time interval.
2. Which time or space intervals given in the problem statement are useful in computing the desired space-time interval? In which reference frame(s)?

2 DEVISE PLAN

3. How can you determine the time interval in reference frame D?
4. Is Eq. 14.18 relevant?
5. Is a proper time interval involved in either reference frame? What about a proper length? How might either of these allow you to obtain information in the second reference frame?

3 EXECUTE PLAN

4 EVALUATE RESULT

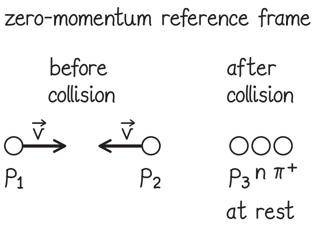
6. Do your answers for the space-time interval in the two reference frames agree?

Worked Problem 14.7 Pion production

Protons p_1 and p_2, initially moving at the same speed in the Earth reference frame, collide head-on, and the products of the collision are a proton p_3, a neutron n, and a pion π^+ that are at rest in the Earth reference frame. The masses of these particles, in terms of the proton mass m_p, are: neutron mass $m_n = 1.0014\, m_p$ and pion mass $m_{\pi^+} = 0.1488 m_p$. In a reference frame in which proton p_2 is at rest, what is the ratio of the kinetic energy of proton p_1 to its internal energy?

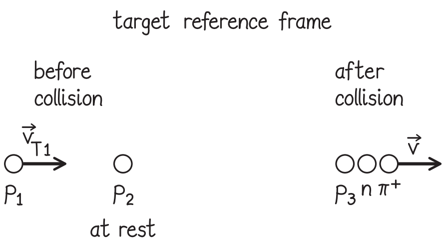
❶ **GETTING STARTED** We are given a description of two protons p_1 and p_2 colliding to create a proton p_3, a neutron, and a pion, and we are told the mass ratios of the products: $m_n/m_p = 1.0014$ and $m_{\pi^+}/m_p = 0.1488$. Our task is to determine the value of the ratio $K_{p1}/E_{\text{int},p1}$ in a reference frame in which p_2 is at rest. We begin by making a sketch of the collision in the Earth reference frame, showing an unknown common speed v for p_1 and p_2 (Figure WG14.3). Note that the Earth reference frame is the zero-momentum reference frame because the protons are moving with opposite velocities and hence the sum of their momenta is zero. We choose to label this reference frame Z because this label will remind us of important momentum information.

Figure WG14.3



We must also consider the collision from a reference frame in which p_2 is at rest. To keep things simple, let's call this the target reference frame T (because in this reference frame p_2 behaves like a stationary target waiting to be hit by p_1). In reference frame T, the three product particles are all moving to the right at speed v (Figure WG14.4).

Figure WG14.4



❷ **DEVISE PLAN** We can consider the system (which initially is the protons p_1 and p_2 and finally is the three product particles p_3, n, and π^+) as being both isolated and closed. Therefore the system's initial energy equals its final energy, and its initial momentum equals its final momentum. We know that the energy of a particle is the sum of its kinetic energy and internal energy (Eq. 5.21) and that its internal energy is $E_{\text{int}} = mc_0^2$ (Eq. 14.54).

Determining E_{int} for p_1. We can set the initial energy equal to the final energy in both reference frames. This gives us two equations in five quantities: $K_{Z,i}$, $E_{\text{int},i}$, $E_{\text{int},f}$, $K_{T,i}$, and $K_{T,f}$. We can express the internal energies in terms of c_0^2 and the masses m_p, m_n, and m_{π^+}. We can also express the internal energies in terms of the internal energy of a proton, $E_{\text{int},p}$.

Determining K for p_1: We know that in the zero-momentum reference frame Z protons p_1 and p_2 are moving with speed v, and so we can calculate $K_{Z,i}$ using Eq. 14.51 and the Lorentz factor γ_{Zv} (Eq. 14.6). In order to calculate γ_{Zv}, we use Eqs. 14.53 and 14.54 to calculate the initial and final energies of p_1 and p_2 in reference frame Z and then set the initial energy equal to the final energy in order to determine the inertia m_{vp} of p_1 and p_2. We can then use Eq. 14.41 to determine γ_{Zv} for p_1 and p_2. We know that in reference frame T the three product particles are moving with speed v, and so we can calculate $K_{T,f}$ using the same Lorentz factor γ_{Zv}. We can then solve for the kinetic energy $K_{T,i}$ of p_1 in reference frame T.

❸ **EXECUTE PLAN** In the zero-momentum reference frame, the three product particles are created at rest. Therefore none of the energy initially contained in p_1 and p_2 goes into kinetic energy in the product particles. All of the initial energy of the system thus is converted to internal energy in the product particles. Setting the system initial energy (kinetic energy of p_1 and p_2, internal energy in p_1 and p_2) equal to the system final energy (internal energy only, in p_3, n, π^+) yields

$$E_Z = K_{Z,i} + E_{\text{int},i} = E_{\text{int},f},$$

which tells us that the initial kinetic energy is equal to the increase in internal energy:

$$K_{Z,i} = E_{\text{int},f} - E_{\text{int},i}. \tag{1}$$

In reference frame T, the initial momentum is nonzero, which means that the final momentum must also be nonzero. Thus the proton, neutron, and pion are moving at speed v, and the kinetic energy $K_{T,f}$ is also nonzero. Setting the initial energy equal to the final energy yields

$$E_T = K_{T,i} + E_{int,i} = K_{T,f} + E_{int,f},$$

and so we have an expression for the initial kinetic energy in the target reference frame T:

$$K_{T,i} = (E_{int,f} - E_{int,i}) + K_{T,f}. \tag{2}$$

This equation tells us that in reference frame T, the initial kinetic energy is equal to the increase in internal energy plus the final kinetic energy of the three product particles.

Substituting Eq. 1 into Eq. 2 tells us that the initial kinetic energy in reference frame T is

$$K_{T,i} = K_{Z,i} + K_{T,f}. \tag{3}$$

In the zero-momentum reference frame, we set the initial energies of p_1 and p_2 (in the form $m_v c_0^2$ from Eq. 14.53) equal to the final internal energy of the product particles (in the form mc_0^2 from Eq. 14.54) to determine the inertia m_{vp} of p_1 and p_2:

$$2m_{vp}c_0^2 = 2\gamma_{Zv}m_p c_0^2 = (m_p + m_n + m_{\pi^+})c_0^2.$$

Now we use Eq. 14.41 to determine the Lorentz factor γ_{Zv} associated with p_1 and p_2:

$$\begin{aligned} m_v &= \gamma_{Zv} m \\ \gamma_{Zv} &= \frac{m_v}{m} = \frac{m_p + m_n + m_{\pi^+}}{2m_p} = \frac{1 + (m_n/m_p) + (m_{\pi^+}/m_p)}{2} \\ &= \frac{1 + 1.0014 + 0.1488}{2} = 1.0751. \end{aligned}$$

Because p_1 and p_2 move at speed v in the zero-momentum reference frame, the incoming kinetic energy is (Eq. 14.51)

$$K_{Z,i} = 2(\gamma_{Zv} - 1)m_p c_0^2 = 0.1502 m_p c_0^2.$$

Because the three particles are moving with speed v in reference frame T, we can use the same Lorentz factor to determine the final kinetic energy in this reference frame:

$$\begin{aligned} K_{T,f} &= (\gamma_{Zv} - 1)m_p c_0^2 + (\gamma_{Zv} - 1)m_n c_0^2 + (\gamma_{Zv} - 1)m_{\pi^+}c_0^2 \\ &= (0.0751)(m_p + m_n + m_{\pi^+})c_0^2 \\ &= (0.0751)(1 + 1.0014 + 0.1488)m_p c_0^2 \\ &= 0.162 m_p c_0^2. \end{aligned}$$

We now substitute these values for $K_{Z,i}$ and $K_{T,f}$ into Eq. 3 to determine the initial kinetic energy measured in reference frame T. Because we have specified that p_2 is at rest in this reference frame, this initial kinetic energy all belongs to p_1:

$$\begin{aligned} K_{T,i} = K_{p_1} &= K_{Z,i} + K_{T,f} \\ &= 0.1502 m_p c_0^2 + 0.162 m_p c_0^2 = 0.312 m_p c_0^2. \end{aligned}$$

From Eq. 14.54 we know that $m_p c_0^2$ is the internal energy of p_1. Thus the ratio we are asked for, $K_{p_1}/E_{int,p_1}$ in a reference frame in which p_2 is at rest, is

$$\frac{K_{p_1}}{E_{int,p_1}} = \frac{K_{T,i}}{m_p c_0^2} = 0.312. ✔$$

❹ **EVALUATE RESULT** We see that $K_{T,i} > 2K_{Zi}$. If we ignore relativistic effects, the speed of p_1 in the target reference frame is $2v$, and so the kinetic energies in the two reference frames are $K_{T,i} = \frac{1}{2}m(2v)^2 = 2mv^2$ and $K_{Z,i} = 2(\frac{1}{2}mv^2) = mv^2$. Thus $K_{T,i} = 2K_{Z,i}$. We would expect that the relativistic increase of inertia with speed should provide extra initial kinetic energy in reference frame T, so our result makes sense.

We can also evaluate our result by using an alternative method to calculate the ratio. First we calculate the speed v of p_1 and p_2 in the zero-momentum reference frame from our calculation for γ_{Zv}:

$$\gamma_{Zv} = \frac{1}{\sqrt{1-(v/c_0)^2}}$$

$$\gamma_{Zv}^2 = \frac{1}{1-(v/c_0)^2}$$

$$1-(v/c_0)^2 = \frac{1}{\gamma_{Zv}^2}$$

$$(v/c_0)^2 = 1-\frac{1}{\gamma_{Zv}^2}$$

$$v = \left(\sqrt{1-\frac{1}{\gamma_{Zv}^2}}\right)c_0 = \left(\sqrt{1-\frac{1}{(1.0751)^2}}\right)c_0 = 0.367c_0.$$

In order to determine the initial kinetic energy of p_1 in the target reference frame, we need to know the x component of the velocity $v_{\mathrm{T}x,\mathrm{i}}$ of p_1 (we choose the positive x direction to be the direction in which p_1 moves). We use Eq. 14.33 with $v_{\mathrm{ZT}x} = -v$ and $v_{\mathrm{Z}x,\mathrm{i}} = v$:

$$v_{\mathrm{T}x,\mathrm{i}} = \frac{v_{\mathrm{Z}x,\mathrm{i}} - v_{\mathrm{ZT}x}}{1-\dfrac{v_{\mathrm{ZT}x}}{c_0}\dfrac{v_{\mathrm{Z}x,\mathrm{i}}}{c_0}} = \frac{v+v}{1+\dfrac{v^2}{c_0^2}} = \frac{2(0.367)c_0}{1+(0.367)^2} = 0.647c_0.$$

Because the kinetic energy of p_1 in reference frame T depends on the Lorentz factor associated with its motion (Eq. 14.51), we first calculate that factor:

$$\gamma_{\mathrm{T}v} = \frac{1}{\sqrt{1-\left(\dfrac{v_{\mathrm{T}x,\mathrm{i}}}{c_0}\right)^2}} = \frac{1}{\sqrt{1-(0.647)^2}} = 1.311.$$

The kinetic energy of p_1 in the target reference frame is thus

$$K_{\mathrm{T,i}} = (\gamma_{\mathrm{T}v} - 1)m_{\mathrm{p}}c_0^2 = 0.311m_{\mathrm{p}}c_0^2$$

$$\frac{K_{\mathrm{T,i}}}{m_{\mathrm{p}}c_0^2} = 0.311,$$

in agreement with our result above.

Guided Problem 14.8 Particle production

A pion π^- of mass m_{π^-} collides with a proton p of mass m_{p} that is at rest relative to an inertial reference frame R. The pion has the minimum kinetic energy necessary such that the collision destroys the proton and pion and produces two particles: a neutral kaon K of mass m_{K} and a lambda particle Λ of mass m_Λ. Derive an expression that gives, in reference frame R, the kinetic energy of the pion in terms of c_0 and the masses of the four particles.

1 GETTING STARTED

1. If possible, choose an isolated and closed system.
2. Decide how many reference frames you must analyze, and sketch the initial and final states in each reference frame.
3. In the zero-momentum reference frame, at the minimum kinetic energy necessary to produce the kaon and lambda particle, are these two particles at rest or moving? Is the reference frame R the zero-momentum reference frame?
4. Assign symbols for the velocity of the kaon and lambda particle in the reference frame R in which the proton is at rest. It is generally good to include symbols in your sketches for any quantities that you may need for the energy equations and the momentum equations.

2 DEVISE PLAN

5. Can you write invariant expressions for the energy and momentum for the pion and proton in reference frame R?
6. Is there a reference frame in which the magnitude of the momentum of the kaon equals that of the lambda particle? What is the invariant expression for the energy and momentum of the kaon-lambda particle combination?

3 EXECUTE PLAN

7. Determine the kinetic energy of the pion in the reference frame R in which the proton is at rest.

4 EVALUATE RESULT

8. In the reference frame in which the proton is at rest, do you expect the sum of the energy of the pion and the internal energy of the proton to be greater than, equal to, or less than the sum of the internal energies of the kaon and lambda particle?

Answers to Review Questions

1. An event is something that happens at a specific location at a specific instant.
2. Clocks measure equal known time intervals, and they can be synchronized so that their readings agree.
3. A signal of some sort must travel from the event to the observer.
4. You must see the two events as occurring at the same location. If you see them taking place at different locations, you cannot measure the proper time interval.
5. The observer must note, at the instant the event occurs, the reading on the clock nearest the event location.
6. The detector must be placed equidistant from the two locations at which the events take place. If the signals from the two events reach it at the same instant, the detector activates; if the signals arrive at different instants, the detector stays unchanged.
7. The speed is always $c_0 = 3.00 \times 10^8$ m/s, independent of the relative velocity of the source and the observer.
8. If the relative velocity of the observers is sufficiently large, they do not both report the two events as being simultaneous.
9. The observer sees time passing more slowly (time *dilates*) on clock A.
10. Time dilation has been confirmed by measurements on atomic clocks flown in airplanes around the world and by observations of cosmic-ray muons.
11. An object's proper length is its length measured by an observer who is at rest relative to the object.
12. Length contraction is the phenomenon whereby an observer moving relative to an object measures the object's length in the direction of motion as being shorter than the object's proper length.
13. This expression solved for v says that quadrupling K doubles v, but experiments on objects initially moving at relativistic speeds measure an increase in speed much less than the predicted doubling.
14. Mass and internal energy are invariant; inertia and kinetic energy depend on the measurer's reference frame.
15. The inertia changes in the same direction as the internal energy: Increasing the internal energy increases the inertia, and decreasing the internal energy decreases the inertia.
16. The inertia decreases.
17. The Lorentz factor is $\gamma = 1/\sqrt{1-(v^2/c_0^2)}$ (Eq. 14.6). Its effects become noticeable at speeds greater than $0.1c_0$.
18. The time interval the observer measures is equal to the product of the Lorentz factor and the proper time interval (Eq. 14.13.)
19. The space-time interval is spacelike if less than zero, timelike if greater than zero, and lightlike if zero. When the interval between the events is timelike, there is an inertial reference frame in which the events occur at the same location but at different instants and are hence separated by only a time interval. When the interval is spacelike, there is an inertial reference frame in which the events occur at different locations but at the same instant and hence are separated by only a distance in space. When the interval is lightlike, the two events are separated spatially by exactly the distance light travels in the time interval between the events.
20. To distinguish cause and effect, it is necessary to determine the sequence of events. For a pair of events that have a timelike space-time interval ($s^2 > 0$), we can establish a reference frame in which we can study both events at the same position, and so we can determine their sequence with certainty and establish their causal relationship. For a pair of events that have a spacelike space-time interval ($s^2 < 0$), we cannot uniquely determine the time order of events and therefore cannot determine any causal relationship. However, for two events to be causally related, they must be able to interact physically. When $s^2 < 0$, the events occur in such rapid succession that a light signal cannot travel from one event to the other in the time interval between them. Because no information can travel faster than the speed of light, such events cannot interact and so cannot be causally related. Only events that have a timelike space-time interval can be causally related, and for such events the time sequence can be uniquely determined.
21. The length the observer measures is equal to the proper length divided by the Lorentz factor (Eq. 14.28).
22. Time dilation and length contraction are the same effect seen by different observers. Muons can travel the proper distance through the atmosphere to reach Earth's surface because according to an observer at the surface their lifetime is lengthened by time dilation and according to an observer traveling with the muons their proper lifetime is sufficient for them cover the contracted length of the atmosphere.
23. The Lorentz transformation equations allow observers in any two inertial reference frames to compare measurements, whereas the length contraction and time dilation equations allow comparisons only between specific reference frames in which a proper time interval or proper length cannot be measured.
24. The momentum for a particle traveling at a relativistic speed is still defined as the product of its inertia and its velocity.
25. Yes. The system momentum remains constant because even with components moving at relativistic speeds, the definition of momentum is still the product of each component's inertia and its velocity.
26. (*a*) The inertia m_v is equal to the product of the particle's mass and the particle's Lorentz factor: $m_v = \gamma m$ (Eq. 14.41). (*b*) The momentum is the product of the particle's inertia and its velocity: $\vec{p} = \gamma m\vec{v} = m_v\vec{v}$.
27. The nonrelativistic form of kinetic energy is $K = \frac{1}{2}mv^2$. The relativistic expression is $K = (\gamma - 1)mc_0^2 = m_v c_0^2 - mc_0^2$ (Eq. 14.51). The two expressions are compatible; the latter reduces to the former at speeds much smaller than the speed of light in vacuum.
28. The quantity mc_0^2 represents the internal energy of the system (Eq. 14.54).
29. The quantity γmc_0^2 represents the energy of the system (Eq. 14.53), which equals the sum of the kinetic and internal energies (Eq. 14.52).
30. A conserved quantity is one that cannot be created or destroyed, but observers in different reference frames can measure different values for a given conserved quantity. An invariant quantity is one that has the same value for all observers, regardless of any difference in the reference frames of the observers. Because they are conserved quantities, the energy and momentum of an isolated, closed system have constant values for a given observer but can have a different set of values for an observer measuring from a different reference frame. In order to obtain, for a given system, a parameter value that is the same for all observers in all reference frames, the invariant combination $E^2 - (c_0p)^2 = (mc_0^2)^2$ must be used.

Answers to Guided Problems

Guided Problem 14.2 (*a*) $0.781c_0$; (*b*) 125 m

Guided Problem 14.4 (*a*) $\vec{v}_{DC} = -0.300c_0\hat{\imath}$; (*b*) $t_{C1} = t_{C2} = 1.47 \times 10^{-6}$ s

Guided Problem 14.6 $s^2 = 1.79 \times 10^{-8}$ m^2

Guided Problem 14.8 $K_{\pi^-} = \dfrac{(m_K + m_\Lambda)^2c_0^4 - m_{\pi^-}^2c_0^4 - m_p^2c_0^4}{2m_pc_0^2} - m_{\pi^-}c_0^2$

PRACTICE

Guided Practice by Chapter

15 Periodic Motion

Review Questions

Answers to these questions can be found at the end of this chapter.

15.1 Periodic motion and energy

1. What is required in order that the motion of an object be called periodic?
2. The terms *amplitude* and *frequency* are often applied to periodic motion. What is the meaning of each?
3. It requires 5.00 s for an oscillator to complete four cycles. What is the period T? What is the frequency f?
4. In periodic motion, where is the kinetic energy of the moving object greatest? Where is the magnitude of its potential energy greatest? Where is each of these variables zero?

15.2 Simple harmonic motion

5. A playground swing moves in simple harmonic motion. If you don't pump your legs on the swing, the amplitude of your motion decreases. What happens to the period?
6. Explain why the restoring force exerted on an object in simple harmonic motion must be a linear function of the object's displacement.
7. What is a turning point in simple harmonic motion?
8. How is simple harmonic motion related to circular motion at constant speed?

15.3 Fourier's theorem

9. State Fourier's theorem and explain why it is important in the study of periodic motion.
10. How can the fundamental frequency of a Fourier series be determined from the time dependence of the original periodic function?
11. What is meant by the *spectrum* of a periodic function?
12. What is Fourier analysis, and what is Fourier synthesis?

15.4 Restoring forces in simple harmonic motion

13. Which of the three types of equilibrium—stable, unstable, neutral—allow periodic motion?
14. Two pendulums have identical shape and length, but one of them has twice the mass of the other. Which has the greater period, if either?
15. What common feature of all restoring forces is crucial for ensuring simple harmonic motion for small displacements from equilibrium?
16. A child and an adult are on adjacent identical swings at the playground. Is the adult able to swing in synchrony with the child?

15.5 Energy of a simple harmonic oscillator

17. Is there anything actually going around in a circle as a spring oscillates? If not, what purpose does the reference circle serve?
18. Describe the simple harmonic oscillator equation in words.
19. If you know the initial position of an oscillator, what else do you need to know in order to determine the initial phase of the oscillation?
20. How do we distinguish frequency f from angular frequency ω if both have units of $(\text{time})^{-1}$?

15.6 Simple harmonic motion and springs

21. If you know the mass of an object hanging from a spring in an oscillating system, what else do you need to know to determine the period of the motion?
22. You stop your car to pick up a member of your car pool. After she gets in, does the angular frequency ω of oscillation due to the car's suspension increase, decrease, or stay the same?
23. Given an object suspended by a spring, which of these variables of the motion can you control by varying the initial conditions: period, amplitude, energy of the system, frequency, phase, maximum velocity, maximum acceleration?
24. How does the frequency of a vertically oscillating block-spring system compare with the frequency of an identical block-spring system oscillating horizontally?

15.7 Restoring torques

25. Two pendulums have identical shape and mass, but pendulum B is twice as long as pendulum A. Which has the greater period, if either?
26. The pendulum on a grandfather clock consists of a heavy disk fastened to a rod in such a way that the position of the disk along the rod is adjustable. If the clock runs slow, how should you adjust the pendulum?
27. What approximation is necessary in order for a pendulum to execute simple harmonic motion?
28. What are the units of the torsional constant in Eq. 15.25, $\tau_\vartheta = -\kappa(\vartheta - \vartheta_0)$?

15.8 Damped oscillations

29. Does an overdamped oscillator oscillate?
30. Describe the changes in the periodic motion of an oscillator as the damping coefficient is gradually increased from $b = 0$ to $b = 0.5m\omega_0$.
31. During a time interval of one time constant, by what factor does the mechanical energy of an oscillating system decrease?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The energy involved in the oscillation of water sloshing in a bathtub (E, J, P, Z)
2. The energy involved in the oscillation of a child on a playground swing (O, A, F, U)
3. The maximum frequency at which you can continuously shake a bowling ball horizontally at chest height for 1 minute, fully extending your arms during each cycle (N, G, C, K, Q)
4. The length of a grandfather-clock pendulum for which the period of oscillation is 2 s (O, W)
5. The angular frequency ω of your leg oscillating as a pendulum (D, I, M)
6. The angular frequency ω of each piston in a four-cylinder gasoline engine as you drive your car, and the initial phase difference $\Delta\phi_i$ between each of the four oscillating pistons (R, X, L, T)
7. The maximum restoring torque caused by the forces exerted on a child on a playground swing (J, A, F, U)
8. The damping coefficient for a shock absorber on a midsize car (S, Y, B, H, V)

Hints

A. What is the maximum angle of oscillation?
B. What spring constant is needed for each wheel?
C. How many times during each cycle must you provide the full oscillation energy to move or stop the bowling ball?
D. What are the mass and length of your leg?
E. What maximum height above the equilibrium level does the sloshing water reach?
F. What is the mass of the child plus the swing seat? (You can ignore the mass of the chains supporting the swing.)
G. What energy is required for these pushups?
H. What is the frequency of the vertical oscillation of your car?
I. What is the rotational inertia of your leg pivoted at your hip?
J. What type of restoring force is involved?
K. What is the mass of a bowling ball?
L. Does each piston fire simultaneously?
M. What is the length from the pivot to your leg's center of mass?
N. How many pushups can you do in 1 minute?
O. Is it reasonable to approximate this as a simple pendulum?
P. How is the water's energy related to the maximum height the water reaches?
Q. What is the amplitude of oscillation?
R. What is a typical cruising-speed angular frequency ω, in revolutions per minute, for the crankshaft (the shaft to which the pistons are attached)?
S. What is the mass of a midsize car?
T. What is the initial phase between two pistons adjacent in the firing order?
U. What is the length of the pendulum?
V. How many oscillations occur before the shock absorber damps out the motion?
W. How is period related to length for a simple pendulum?
X. What is the angular frequency ω of each piston?
Y. Through what distance does a fender deflect when you sit on it?
Z. At maximum height, where is the center of mass of the volume of water above the equilibrium level?

Key (all values approximate)

A. 45°; **B.** 2×10^4 N/m; **C.** four; **D.** 1×10^1 kg, 1 m; **E.** 0.2 m; **F.** 3×10^1 kg; **G.** $30mgh$, where $h \approx 0.3$ m, $m \approx 7 \times 10^1$ kg, gives 6×10^3 J; **H.** 1 Hz; **I.** 3 kg · m² because it is roughly a uniform rod; **J.** the gravitational force exerted by Earth; **K.** 7 kg; **L.** no, they fire one at a time; **M.** 0.4 m; **N.** if you are fit, 3×10^1; **O.** yes; **P.** at maximum height, there is no kinetic energy; **Q.** 0.3 m; **R.** 2×10^3 rev/min; **S.** 1×10^3 kg; **T.** π rad; **U.** 3 m; **V.** a well-adjusted system allows less than one full oscillation; **W.** $T = 2\pi\sqrt{\ell/g}$; **X.** the same as for the crankshaft, $2 \times 10^2\ \mathrm{s}^{-1}$; **Y.** 5×10^1 mm; **Z.** this volume has a triangular shape, so its center of mass is at about one-third the maximum height

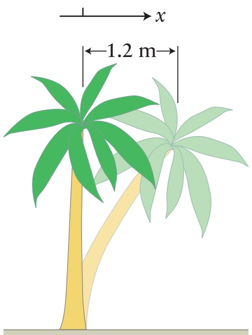
Worked and Guided Problems

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 15.1 Swayin' in the breeze

A treetop sways back and forth with a period of 12 s, and you estimate the amplitude of the motion to be 1.2 m (Figure WG15.1). You begin timing at some instant after the treetop passes through the upright position, and after you have been watching this motion for 36 s, the treetop is 0.60 m to the left of the upright position. (*a*) Write the equation for the position of the treetop as a function of time. (*b*) What is the maximum speed of the treetop? (*c*) At what instants does the treetop have this speed? (*d*) What is the maximum magnitude of the treetop's acceleration? (*e*) What is the magnitude of its acceleration at the instant $t = 36$ s?

Figure WG15.1



❶ **GETTING STARTED** This treetop is executing periodic motion, and we assume the motion can be described with a sine function. We are given several bits of information from which to construct the position as a function of time. Two of these bits are the period $T = 12$ s and the amplitude $A = 1.2$ m. We also know that the treetop is 0.60 m to the left of vertical at $t = 36$ s. Parts *b*–*e* all require the general equation of motion, and so once we have that, we're set.

❷ **DEVISE PLAN** We begin with the general equation for position in simple harmonic motion, Eq. 15.6:

$$x(t) = A\sin(\omega t + \phi_i).$$

With the data we have, we can determine the amplitude A, angular frequency ω, and initial phase ϕ_i. Then we can obtain the maximum speed and the acceleration magnitude from $v_x = dx/dt$ and $a_x = dv_x/dt$.

❸ **EXECUTE PLAN** (*a*) The angular frequency is $\omega = 2\pi/T = 2\pi/(12\text{ s}) = (\pi/6)\text{ s}^{-1}$. (Generally you should leave π in expressions because it often either cancels or produces an easily evaluated trig function.) The initial phase ϕ_i can be found by using the information we have about the position at 36 s:

$$x(t = 36\text{ s}) = -0.60\text{ m} = (1.2\text{ m})\sin\left[\left(\tfrac{1}{6}\pi\text{ s}^{-1}\right)(36\text{ s}) + \phi_i\right]$$

$$-0.50 = \sin(6\pi + \phi_i) = \sin(\phi_i)$$

$$\phi_i = \sin^{-1}(-0.50) = -\tfrac{1}{6}\pi.$$

With this value for ϕ_i, $x(t) = A\sin(\omega t + \phi_i)$ for this particular motion becomes

$$x(t) = (1.2\text{ m})\sin\left[\left(\tfrac{1}{6}\pi\text{ s}^{-1}\right)t - \tfrac{1}{6}\pi\right]. \checkmark$$

(*b*) The velocity is given by Eq. 15.7:

$$v_x(t) = \frac{dx}{dt} = \frac{d}{dt}A\sin(\omega t + \phi_i) = \omega A\cos(\omega t + \phi_i)$$

$$= \left(\tfrac{1}{6}\pi\text{ s}^{-1}\right)(1.2\text{ m})\cos\left[\left(\tfrac{1}{6}\pi\text{ s}^{-1}\right)t - \tfrac{1}{6}\pi\right]$$

$$= (0.20\pi\text{ m/s})\cos\left[\left(\tfrac{1}{6}\pi\text{ s}^{-1}\right)t - \tfrac{1}{6}\pi\right].$$

PRACTICE

The maximum speed occurs when the cosine function is 1, which means that $v_{\max} = 0.20\pi \text{ m/s} = 0.63 \text{ m/s}$. ✔

(c) The tree has this maximum speed whenever it passes through the position $x = 0$, which is when the tree is vertical. This occurs when the cosine function is ± 1. Solving first for the positive values, we have

$$1 = \cos\left[\left(\tfrac{1}{6}\pi \text{ s}^{-1}\right)t - \tfrac{1}{6}\pi\right]$$

$$\Rightarrow \left(\tfrac{1}{6}\pi \text{ s}^{-1}\right)t_n - \tfrac{1}{6}\pi = 2n\pi, \qquad n = 0,1,2,\ldots$$

$$t_n - \tfrac{1}{6}\pi\left(\frac{6}{\pi}\text{ s}\right) = 2n\pi\left(\frac{6}{\pi}\text{ s}\right), \quad n = 0,1,2,\ldots$$

$$t_n = (1 + 12n)\text{ s}, \quad n = 0,1,2,\ldots$$

$$t = 1 \text{ s},\ 13 \text{ s},\ 25 \text{ s}, \ldots. ✔$$

We repeat the calculation for the negative values:

$$-1 = \cos\left[\left(\tfrac{1}{6}\pi \text{ s}^{-1}\right)t - \tfrac{1}{6}\pi\right]$$

$$\Rightarrow \left(\tfrac{1}{6}\pi \text{ s}^{-1}\right)t_n - \tfrac{1}{6}\pi = n\pi, \qquad n = 1,3,5,\ldots$$

$$t_n - \tfrac{1}{6}\pi\left(\frac{6}{\pi}\text{ s}\right) = n\pi\left(\frac{6}{\pi}\text{ s}\right), \quad n = 1,3,5,\ldots$$

$$t_n = (1 + 6n)\text{ s}, \quad n = 1,3,5,\ldots$$

$$t = 7 \text{ s},\ 19 \text{ s},\ 31 \text{ s}, \ldots. ✔$$

(d) The acceleration is given by Eq. 15.8:

$$a_x(t) = \frac{dv_x}{dt} = \frac{d}{dt}[\omega A\cos(\omega t + \phi_i)] = -\omega^2 A \sin(\omega t + \phi_i)$$

$$= -(0.33\text{ m/s}^2)\sin\left[\left(\frac{\pi}{6}\text{ s}^{-1}\right)t - \frac{\pi}{6}\right].$$

The maximum acceleration occurs when the sine function is 1, which makes $a_{\max} = 0.33 \text{ m/s}^2$. ✔

(e) At $t = 36$ s, the acceleration magnitude is $a_x(36 \text{ s}) = 0.16 \text{ m/s}^2$. ✔

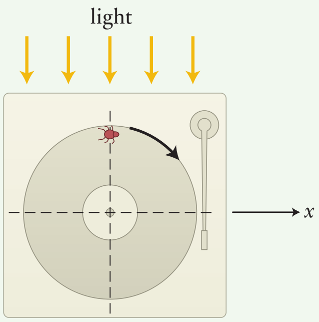
❹ **EVALUATE RESULT** Given that the amplitude is not very large relative to the tree height, the acceleration maximum indicates gentle swaying in the breeze rather than a hurricane. The value we obtained for the maximum speed confirms this. The answers are the right order of magnitude for a swaying tree.

PRACTICE

Guided Problem 15.2 Archaic music medium

A long-playing phonograph record is 12.0 in. (305 mm) in diameter and rotates at $33\frac{1}{3}$ rev/min. Suppose a bug is at the edge of one such record, as in Figure WG15.2, at instant $t = 0$. Imagine that a distant light source casts a shadow of the bug along an x axis that runs tangent to the turntable at the rim position opposite the bug's initial position. (*a*) Write an equation that describes the motion of the shadow as a function of time. (*b*) What is the shadow's maximum speed along the x axis? (*c*) What is the maximum magnitude of its acceleration along the x axis?

Figure WG15.2



1 GETTING STARTED

1. What does it mean to have the bug's motion projected onto the x axis? Make sketches showing the shadow's location at several instants as the turntable and bug rotate.
2. How can the properties of a right triangle and the sine and cosine functions help you in this problem?
3. Describe the relationship between circular motion and oscillation.

2 DEVISE PLAN

4. What unknown quantities do you need to determine?
5. What equations allow you to express the unknown quantities in terms of known ones?
6. Write the generic equation for part *a*. Do you have to take derivatives of an expression?

3 EXECUTE PLAN

7. Manipulate your equations to obtain an expression for the position of the shadow along the x axis as a function of time.
8. How do you get the shadow's speed and the magnitude of its acceleration from the expression you found in step 7?
9. Which trigonometric function must be a maximum when the speed is a maximum, and what is that function's maximum value?
10. Which trigonometric function must be a maximum when the acceleration magnitude is a maximum, and what is that function's maximum value?

4 EVALUATE RESULT

11. Are the maximum speed and acceleration of the shadow of the right order of magnitude? Compare these values with the speed and maximum acceleration of the bug.

Worked Problem 15.3 Keeping lunar time

(*a*) What is the period of a simple pendulum of length 0.50 m on the Moon? (*b*) How does this period compare with the period of the same pendulum on Earth?

1 GETTING STARTED We have to think about the factors that affect the period and about how these factors change when we move from Earth to the Moon. From videos of astronauts on the Moon, we are aware that things drop more slowly there, which implies that the free-fall acceleration is smaller there than on Earth.

2 DEVISE PLAN From Example 15.6, we know that the angular frequency of a simple pendulum is $\omega = \sqrt{g/\ell}$ and its period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}.$$

The derivation of this relationship in the *Principles and Practice* text does not require that the pendulum be oscillating on Earth, and so the result is general. Because Equation 13.4 in the generic form $g = Gm/R^2$ tells us the free-fall acceleration g at the surface of *any* celestial body of radius R, we can apply it to the Moon:

$$g_M = G\frac{m_M}{R_M^2}.$$

PRACTICE

We get the mass and radius of the Moon from Table 13.1.

❸ EXECUTE PLAN (*a*) The magnitude of G is the same no matter where we are (that's why Newton's gravitational law is called the law of *universal* gravitation), so Eq. 13.4 gives

$$g_M = (6.674 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})\frac{7.3 \times 10^{22}\ \mathrm{kg}}{(1.74 \times 10^6\ \mathrm{m})^2} = 1.6\ \mathrm{m/s^2}.$$

The period of a 0.50-m lunar pendulum is therefore

$$T_M = 2\pi\sqrt{\frac{0.50\ \mathrm{m}}{1.6\ \mathrm{m/s^2}}} = 3.5\ \mathrm{s}. \checkmark$$

(*b*) We can use the general expression for period and our knowledge of the lunar and earthly free-fall accelerations to get the desired ratio of periods without having to calculate the pendulum's period on Earth:

$$\frac{T_M}{T_E} = \sqrt{\frac{g_E}{g_M}} = \sqrt{\frac{9.8\ \mathrm{m/s^2}}{1.6\ \mathrm{m/s^2}}} = 2.5. \checkmark$$

❹ EVALUATE RESULT The longer period on the Moon makes sense because, as we found in part *a*, the gravitational acceleration on the Moon, $1.6\ \mathrm{m/s^2}$, is about one-sixth that on Earth.

Guided Problem 15.4 In sync

A block of mass m is hung from a vertical spring, stretching the spring a distance h beyond its relaxed length. The block is pulled down a bit and released, resulting in a vertical oscillation. The block is then removed from the spring and used as the bob on a simple pendulum. If the period of the oscillating pendulum is the same as the period of the oscillating block-spring system, what is the length of the pendulum?

❶ GETTING STARTED

1. How are the motions of the spring and pendulum analogous to each other?
2. Sketch both oscillating systems at an arbitrary instant, labeling key variables.
3. What assumptions do you need to make in order to solve this problem?

❷ DEVISE PLAN

4. What general expressions give you the periods of the two systems?
5. What unknown quantities must you determine before setting the two periods equal? What information can help you determine these quantities? For example, how can you calculate the spring constant k? Does calculating this constant help?

❸ EXECUTE PLAN

6. Work through the algebra and solve for the pendulum length.

❹ EVALUATE RESULT

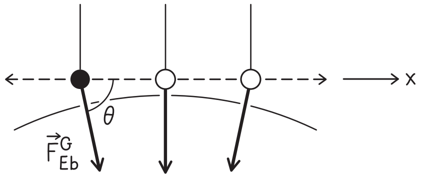
7. Is the answer plausible, that is, does your expression behave as you expect it to with variations in m and h?

Worked Problem 15.5 A very tall grandfather clock

Suppose you could construct a pendulum consisting of a very long string (nearly infinitely long) and a bob that swings just above Earth's surface with an amplitude that is small compared to Earth's diameter. What would the period of its oscillation be?

❶ GETTING STARTED The problem statement describes a simple pendulum. From Worked Problem 15.3, we know that the period of a pendulum increases with the square root of its length: $T = 2\pi\sqrt{\ell/g}$. This relationship implies that the period of our very long pendulum should be really long, increasing without limit to infinity as the length increases. So, what is at issue? Why do we bother with this problem if the solution is that easy? To answer this question, we need to derive an expression for the period. Let us start with a sketch showing how this infinite pendulum might differ from a normal pendulum (Figure WG15.3).

Figure WG15.3



PRACTICE

We note that an infinite string length means that the string is always vertical and the arc described by the bob is very nearly a straight line. We choose this line of motion as the x axis, with the origin at the equilibrium point and the positive direction to the right in the figure. Next we notice that the angle of the gravitational force exerted on the pendulum bob changes direction, always pointing toward Earth's center rather than straight down in the figure. From this changing direction, we conclude that although gravity provides a restoring force that pulls the bob toward the center of the motion, it is not the same as the constant gravitational force used in the analysis of a normal pendulum.

Because this is a hypothetical ("Suppose you could…") problem, we assume that practical considerations of pendulum construction and air drag should be ignored.

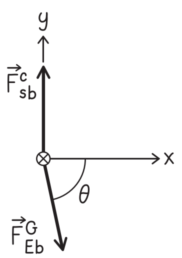
❷ **DEVISE PLAN** Because the pendulum's motion is nearly linear, we will use an analysis similar to that in Example 15.8 to derive the angular frequency, beginning with Newton's second law, $\Sigma\vec{F} = m\vec{a}$, and casting it in a form that looks like a harmonic oscillator equation (Eq. 15.9):

$$a_x = \frac{d^2x}{dt^2} = -\omega^2 x. \tag{1}$$

We use a linear variable x to describe the position of the bob along its essentially straight-line horizontal path (straight and horizontal because we are taking our infinitely long string to be always vertical).

❸ **EXECUTE PLAN** From the free-body diagram (Figure WG15.4), we see that two forces are exerted on the bob: the tensile force exerted by the string and the force of gravity exerted by Earth. Because the pendulum is near Earth's surface at all times, we can use g for the acceleration magnitude.

Figure WG15.4



The horizontal component of the gravitational force, which points to the right in Figure WG15.4, is unbalanced and so produces an acceleration, causing the oscillations. The positive x direction is, for consistency with our earlier choice, to the right in Figure WG15.4. Thus the x component of force is

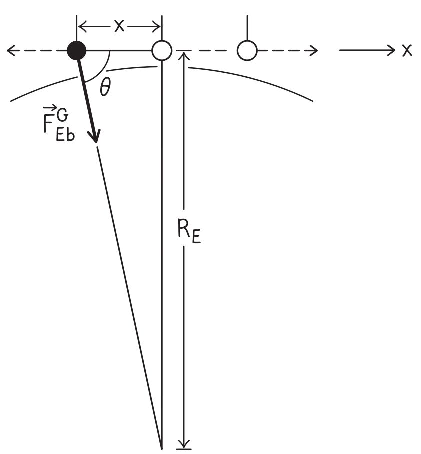
$$F^G_{\text{Eb}x} = ma_x = mg\cos\theta. \tag{2}$$

The gravitational-force vector in Figure WG15.4 is directed from the pendulum bob to the center of Earth. We can construct a right triangle (Figure WG15.5) that contains this same angle θ using the distance from the bob to Earth's center, the displacement x, and the radius of Earth. Note that the height above Earth's surface of the pendulum bob at the midpoint of its motion is negligible compared to the radius of Earth. Using this triangle, we can rewrite $\cos\theta$ in terms of the bob's displacement x and Earth's radius:

$$\cos\theta = \frac{-x}{\sqrt{R_E^2 + x^2}} \approx \frac{-x}{R_E}.$$

Using Eq. 2 in the form $a_x = g\cos\theta$ and then writing the acceleration a_x as the second derivative of position with respect to time give

Figure WG15.5



PRACTICE

$$\frac{d^2x}{dt^2} = g\cos\theta = -\frac{g}{R_E}x, \tag{3}$$

our desired equation of motion for the oscillator.

We obtain the period of oscillation by comparing Eqs. 1 and 3:

$$a_x = -\omega^2 x$$

$$\omega^2 = -\frac{a_x}{x} = -\left(\frac{-gx}{R_E}\right)\left(\frac{1}{x}\right) = \frac{g}{R_E}.$$

Equation 15.1, $\omega = 2\pi/T$, gives us

$$\omega^2 = \frac{g}{R_E} = \left(\frac{2\pi}{T}\right)^2, \quad \text{and so} \quad T = 2\pi\sqrt{\frac{R_E}{g}}.$$

Notice that the period does not depend on the mass of the bob. Putting in numbers, we get

$$T = 2\pi\sqrt{\frac{6.38 \times 10^6\ \text{m}}{9.8\ \text{m/s}^2}} = 5070\ \text{s} = 84\ \text{min.} ✔$$

❹ **EVALUATE RESULT** This is a large oscillation period, but long pendulums have large periods. Even if we didn't work through the calculation, it would be surprising that the period is not infinite, as the result of Example 15.6, $T = 2\pi(\ell/g)^{1/2}$, implies it should be. The derivation of that equation assumed that the gravitational force always points vertically downward, whereas in this case we used the fact that the gravitational force is directed radially toward the center of Earth.

We assumed that the magnitude of the gravitational acceleration and the string direction do not change over the path of the swing, which is consistent with the requirement that the amplitude of the motion be small compared to the diameter of Earth. These assumptions do not undermine the main element of the derivation: The direction of the gravitational force changes even though we have assumed its magnitude does not.

Guided Problem 15.6 One deep hole

In another hypothetical exercise, suppose you could dig a tunnel that begins where you are standing now, passes through Earth's center, and comes out on the other side of the globe. What would happen to a rock you drop into this tunnel? How long would it take the rock to return to you? (For simplicity, assume Earth does not rotate and ignore air drag.)

❶ **GETTING STARTED**

1. Describe the physical situation and posed task in your own words. Draw a sketch consistent with your description.
2. What do you expect the rock to do after you release it?

❷ **DEVISE PLAN**

3. Recall from Section 13.8 that a solid sphere exerts a gravitational force as if all the mass of the sphere were concentrated at its center.
4. To determine the magnitude of the gravitational force exerted by Earth on the rock after it has been dropped into the tunnel and is a radial distance $r < R_E$ from Earth's center, imagine the portion of Earth that has radius r (at the instant you are analyzing, your rock is on the surface of this imaginary sphere).
5. Refer to Checkpoint 13.23 if necessary.
6. What is the rock's acceleration as it passes through Earth's center?
7. Write the equation of motion for the rock as it passes a specific position x in its motion through the tunnel. Can you make this look like a simple harmonic oscillator equation?
8. What is the average density of Earth? Add information to your sketch as needed to help you visualize your approach.

❸ **EXECUTE PLAN**

9. Write an equation for the position of the rock as a function of time.
10. What is the angular frequency ω of the oscillation? What is the period T?

❹ **EVALUATE RESULT**

11. Compare your period with that of the "infinite" pendulum of Worked Problem 15.5. What insight do you gain from this comparison?

Worked Problem 15.7 A damping influence

You decide to test the effect of a viscous liquid on the motion of a simple pendulum consisting of a thin rod 9.0 m long, a 2.0-kg spherical bob, and pivot bearings at the end of the rod opposite the bob. You place the pendulum in a huge vat of maple syrup. The rod is so thin that it experiences very little drag, so essentially the entire drag force is exerted on the bob. You start the pendulum swinging with a small amplitude, and 10 s later the motion is indiscernible. (*a*) Derive an equation that shows how the angle the pendulum makes with the vertical depends on time. (*b*) What is the approximate period of oscillation of this pendulum? Ignore any friction in the bearings.

❶ **GETTING STARTED** This problem involves damped oscillations but for a pendulum rather than the block-spring system examined in Section 15.8. A drag force exerted by the syrup opposes the motion of the pendulum. We do not know anything about the amplitude of the motion except that after 10 s the motion is undetectable. So we have to use our judgment here. Damped oscillations die out exponentially, which means that they never truly reach zero amplitude (the motion just becomes too small to notice). A reasonable guess is that the final amplitude is 1% of the initial value.

❷ **DEVISE PLAN** To make use of the solution worked out in Section 15.8, we need to derive an equation in the form of Eq. 15.36,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

and make the appropriate mathematical correspondence with our rotational quantities (for instance, x becomes ϑ). As with the block-spring system of Section 15.8, we assume that the drag force exerted by the syrup on the pendulum is proportional to the pendulum's velocity: $\vec{F}^{d}_{sp} = -b\vec{v}_p$ (Eq. 15.34). This force produces a torque on the pendulum in addition to the gravitational torque produced by Earth. Therefore we can use Eq. 15.24, $\Sigma\tau_\vartheta = I\alpha_\vartheta$, in the form

$$I\frac{d^2\vartheta}{dt^2} = \tau_{\mathrm{Ep}\,\vartheta} + \tau_{\mathrm{sp}\,\vartheta}. \tag{1}$$

We use Eq. 15.31, $\tau_\vartheta = -(m\ell_{\mathrm{cm}}g)\vartheta$, to simplify the $\tau_{\mathrm{Ep}\,\vartheta}$ term. The torque $\tau_{\mathrm{sp}\,\vartheta}$ is proportional to the pendulum's translational speed dx/dt and therefore to its rotational speed $d\vartheta/dt$. This information should give us the derivatives we need in the equation of motion for the damped oscillator in part *a*.

In order to get the period of the motion, we need to know the number of oscillations in a given time interval, which means we need to know the angular frequency ω_d of the damped motion. Because damping is present, the angular frequency ω_d is given by Eq. 15.38. Once we have ω_d, we obtain the period of the motion from the relationship $T_d \approx 2\pi/\omega_d$.

❸ **EXECUTE PLAN**
(*a*) Let ℓ be the length of the rod and m the mass of the pendulum. The torque caused by the gravitational force is given by Eq. 15.31, as noted above. The torque caused by the drag force exerted by the syrup is

$$\begin{aligned}\tau_{\mathrm{sp}\,\vartheta} &= F^{d}_{\mathrm{sp}\perp}\ell = -(bv_{\mathrm{p}\perp})\ell = -[b(\ell\omega_\vartheta)]\ell \\ &= -b\ell^2\frac{d\vartheta}{dt},\end{aligned}$$

where we have used Eq. 15.34 to substitute for the factor $F^{d}_{\mathrm{sp}\perp}$ and the relationship between speed and rotational speed for motion along a circular arc, $v_{\mathrm{p}\perp} = \ell\omega_\vartheta$, to substitute for the factor $v_{\mathrm{p}\perp}$. Inserting these values into Eq. 1 yields

$$I\frac{d^2\vartheta}{dt^2} = [-(m\ell g)\vartheta] + \left[-b\ell^2\frac{d\vartheta}{dt}\right]$$

$$I\frac{d^2\vartheta}{dt^2} + (b\ell^2)\frac{d\vartheta}{dt} + (m\ell g)\vartheta = 0.$$

This result has the same form as Eq. 15.36,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

PRACTICE

with the substitutions I for m, ϑ for x, $b\ell^2$ for b, and $m\ell g$ for k. We infer from Eq. 15.37 that the equation for the rotational position as a function of time for the damped pendulum is

$$\vartheta(t) = Ae^{-b\ell^2 t/2I}\sin(\omega_d t + \phi_i).$$

(Recall from Section 15.5 that ϕ_i is the initial phase of the motion.) The rotational inertia for a simple pendulum is $I = m\ell^2$. Substituting this rotational inertia into our rotational position equation yields

$$\vartheta(t) = Ae^{-b\ell^2 t/2(m\ell^2)}\sin(\omega_d t + \phi_i),$$

where

$$\omega_d = \sqrt{\frac{m\ell g}{m\ell^2} - \left(\frac{b\ell^2}{2m\ell^2}\right)^2}$$

$$\vartheta(t) = Ae^{-bt/2m}\sin(\omega_d t + \phi_i),$$

where

$$\omega_d = \sqrt{\frac{g}{\ell} - \left(\frac{b}{2m}\right)^2}. ✔ \quad (2)$$

As with the block-spring case, the angular frequency ω is modified by a term, $(b/2m)^2$, that depends on the damping, telling us that the oscillations decrease exponentially. The ratio $b/2m$ determines the damping.

(*b*) We assumed that the oscillation amplitude decreases to 1% of its initial value in 10 s: $(0.010)A = Ae^{-b(10\text{ s})/2m}$. Solving for $b/2m$ yields

$$\ln 0.010 = -\frac{b}{2m}(10\text{ s})$$

$$\frac{b}{2m} = \frac{4.61}{10\text{ s}} = 0.46\text{ s}^{-1}.$$

The estimated angular frequency and period are therefore

$$\omega_d = \sqrt{\frac{9.8\text{ m/s}^2}{9.0\text{ m}} - (0.46\text{ s}^{-1})^2}$$

$$= \sqrt{(1.09 - 0.21)\text{ s}^{-2}} = 0.94\text{ s}^{-1}$$

$$T_d \approx \frac{2\pi}{0.94\text{ s}^{-1}} = 6.7\text{ s}. ✔$$

❹ **EVALUATE RESULT** If we remove the pendulum from the syrup, we have an undamped simple pendulum, for which the angular frequency is $\omega = \sqrt{g/\ell}$, which is what our expression for ω_d reduces to for $b = 0$.

We can gain further confidence by seeing whether our algebraic expression for ω_d behaves the way we expect as we change the value of the damping coefficient b. If b increases, we expect ω_d to decrease because the pendulum must overcome a greater resistive force as it swings. We see that the term $b/2m$ is subtracted in the square-root term in Eq. 2, causing ω_d to decrease with increasing b.

Note that the pendulum's angular frequency without syrup would be $\omega = 1.04\text{ s}^{-1}$, so under our assumption the syrup has a 10% effect on the angular frequency and period. Our decision to make 1% of the original amplitude the state of undetectable motion is of course arbitrary. What if we chose 0.10% instead? The value of $b/2m$ becomes 0.69 s^{-1}, which makes $\omega_d = 0.78\text{ s}^{-1}$, and now we are closer to a 25% effect of damping on the angular frequency. The size of the effect clearly depends on our assumption. However, under this revised assumption, the period would be about 8 s, so the sensitivity is not dramatic.

One last thing to consider is why we must say that the period is only approximately equal to $2\pi/\omega_d$ instead of using the equality $T = 2\pi/\omega$ (from Eq. 15.1, $\omega = 2\pi/T$). The reason we must use the approximation is that the motion is not strictly periodic because of the exponential time dependence of the amplitude. The motion does not simply repeat itself, which makes the definition of *period* a bit ambiguous. Thus $T_d \approx 2\pi/\omega_d$ is the best estimate we can make.

Guided Problem 15.8 Prevent oscillation

A typical engineering design element in systems susceptible to unwanted oscillation is *critical damping*. The idea is that a disturbance displaces some moving part, but rather than oscillating around its equilibrium position, the part returns as quickly as possible to its equilibrium position without overshooting it. This requires that the damped angular frequency be zero: $\omega_d = 0$. (*a*) For the pendulum in Worked Problem 15.7, what must be the value of the damping coefficient b if critical damping is desired? (*b*) When the system is critically damped, how long does it take for the amplitude to decrease to 1% of its initial value?

1 GETTING STARTED

1. What is the physical significance of the damping coefficient b?
2. Should the value of $b/2m$ be greater or smaller than it is in Worked Problem 15.7?

2 DEVISE PLAN

3. Is the condition for critical damping given in the problem statement, $\omega_d = 0$, useful in determining the desired value for part *a*?
4. What equation or expression can be used to solve part *b*? Can you save a bit of derivation by using results from Worked Problem 15.7?

3 EXECUTE PLAN

5. Solve algebraically for the desired quantities, doing part *a* first and then part *b*. Are there any other unknown quantities in your expression that must be eliminated?
6. Substitute known values and compute your answers.

4 EVALUATE RESULT

7. How does your algebraic expression for part *b* behave if the value of b is twice the critical value? If it is half the critical value?

Answers to Review Questions

1. Motion is periodic if the motion repeats itself at regular time intervals.
2. The amplitude is the magnitude of the moving object's maximum displacement from the equilibrium position. The frequency is the number of cycles of the repetitive motion completed during each second.
3. The period is the time interval needed to complete one cycle: $(5.00\text{ s})/(4\text{ cycles}) = 1.25\text{ s/cycle}$; $T = 1.25$ s. The frequency is the reciprocal of the period, $f = 1/T = 0.800$ Hz.
4. The kinetic energy is greatest at the equilibrium position and zero at the extremes of the motion. The magnitude of the potential energy is greatest at the extremes of the motion and zero at the equilibrium position.
5. As long as the amplitude is not too great, the period does not change because simple harmonic motion is isochronous.
6. All simple harmonic motion is isochronous. The acceleration for isochronous motion is directly proportional to the object's displacement from its equilibrium position. The restoring force is proportional to the object's acceleration by Newton's second law and hence directly proportional to the object's displacement. Direct proportionality guarantees a linear relationship.
7. A turning point is a point at which the object turns around—that is, changes its direction of motion. These points are the extrema of the motion, the points farthest from the equilibrium position.
8. Circular motion is two dimensional, and hence the position of an object in circular motion can be described using two orthogonal components. Either component may be expressed as a sinusoidal function of time. Any motion that can be expressed as a sinusoidal function of time is simple harmonic motion. Thus simple harmonic motion has the same functional behavior as one component of constant-speed circular motion.
9. Any periodic function, no matter how complex, can be written as a sum of sinusoidal simple harmonic functions of frequency $f_n = n/T$, where $n \geq 1$ is an integer and T is the period of the motion. This means that any periodic function can be treated as a superposition of simple harmonic motions, and so we need to understand only simple harmonic motion in order to deal with any periodic motion.
10. The fundamental frequency is the same as the frequency of the original periodic function.
11. The spectrum of a periodic function is a plot of the square of the amplitudes of each harmonic frequency in the Fourier series versus frequency.
12. Fourier analysis is the breaking down of a function into a Fourier series (a sum of sinusoidal functions with frequencies that are integer multiples of the fundamental frequency). Fourier synthesis is the inverse of Fourier analysis: the construction of periodic functions by adding together sinusoidal functions with frequencies that are integer multiples of some fundamental frequency.
13. Stable equilibrium allows periodic motion.
14. They have identical periods. Both the restoring force and the rotational inertia are proportional to m, and so the dependence on m cancels.
15. For sufficiently small displacements away from the equilibrium position, restoring forces are always linearly proportional to the displacement.
16. The period of the swing does not depend on the mass of the pendulum bob—in this case, the rider. Thus the two stay in synchrony, as your common experience tells you.
17. No, even the angular frequency ω associated with simple harmonic motion is merely a convenience that exploits the mathematical similarity between simple harmonic motion and circular motion at constant speed. (That is why we don't call it *rotational velocity* here, although it is mathematically the same.) The reference circle helps you relate simple harmonic motion to circular motion, providing a visual interpretation for angular frequency, phase, and amplitude.
18. The variable x describes simple harmonic motion if (and only if) the second derivative of x with respect to time is equal to minus a constant (the square of the angular frequency) times the variable x.

19. You need to know the amplitude A because $\sin \phi_i = x_i/A$ (from Eq. 15.6, with $t = 0$).
20. Angular frequency is given in units of s^{-1}, just as for circular motion, while frequency is given in units of Hz (Hertz).
21. You need to know the spring constant k because $T = 2\pi\sqrt{m/k}$ (from Eqs. 15.1 and Eq. 15.22).
22. It decreases. After another person gets into the car, the mass of the load compressing the springs increases, and angular frequency ω is inversely proportional to the square root of that mass (Eq. 15.22).
23. You can control all except period and frequency, which are determined by the design (mass and spring constant) of the oscillating system.
24. When the friction in the horizontal system is negligibly small, the frequencies are identical.
25. B. Equation 15.33 shows the ratio ℓ/I in the expression for the angular frequency: $\omega = \sqrt{m\ell g/I}$. Length ℓ is double for B, but I for this pendulum is greater by a factor of 4 because it is proportional to the length squared (Equation 11.30; $I = mr^2$). Hence ω is smaller for B. Because period is inversely proportional to angular frequency ($\omega = 2\pi/T$), T is greater for this pendulum.
26. Because the clock is losing time, you need to increase the angular frequency ω of the oscillation. You should therefore shorten the length of the pendulum by moving the disk higher up on the rod to decrease the rotational inertia.
27. The small-angle approximation is required. The equation of motion for a pendulum equates the gravitational torque, which is proportional to $\sin\vartheta$, to the inertial term (which is proportional to the second derivative with respect to time of the angular position ϑ). The simple harmonic oscillator equation requires that the variable involved in the second derivative also be present as a linear term (first power only) in the restoring term. This is possible only if we replace $\sin\vartheta$ with ϑ. This is called the "small-angle approximation" because it is valid for only angular positions much less than 1, or angles much less than 1 radian.
28. In Eq. 15.25, the rotational displacement is unitless, so the torsional constant has same units as torque, newton-meters.
29. No. The overdamped condition occurs when the damping is so great that the system returns to equilibrium without oscillating.
30. The amplitude of the oscillations dies out more and more rapidly as the damping coefficient increases. When b reaches its greatest value, the system makes only a few complete oscillations before the motion becomes too small to measure. Also the angular frequency ω decreases as the damping increases, but this effect amounts to no more than a 5% change.
31. The energy decreases by a factor of e, or 2.718, so that approximately 37% of the energy at the beginning of the time interval remains at the end.

Answers to Guided Problems

Guided Problem 15.2 (*a*) $x(t) = (152\text{ mm})\sin[(3.49\text{ s}^{-1})t]$; (b) $v_{max} = 532\text{ mm/s}$; (c) $a_{max} = 1.86\text{ m/s}^2$

Guided Problem 15.4 $\ell = h$

Guided Problem 15.6 The rock begins to oscillate and it falls all the way through Earth, then returns. The time interval needed for the rock to return to you is one period, $T = 84$ min.

Guided Problem 15.8 (*a*) $b = 2m\omega$; (*b*) $t = 0.73T = 4.4$ s

Guided Practice by Chapter

16 Waves in One Dimension

Review Questions

Answers to these questions can be found at the end of this chapter.

16.1 Representing waves graphically

1. What physical quantities are transported by a wave?
2. Classify each wave as longitudinal or transverse: the wave created when a whip is snapped, the wave represented by cars slowing down and speeding up in a traffic jam on a highway, water waves spreading out from the location where a stone is thrown into a pond, the wave created when a guitar string is plucked.
3. What is the difference between the wave speed c of a wave traveling along a rope and the speed v of a small segment of that rope?
4. For a given wave pulse, is it true that the wave function (a function of time) always looks the same as the displacement curve (a function of space)?

16.2 Wave propagation

5. You are sitting at the beach watching the waves. What is the difference between their frequency and their speed?
6. If the hand in Figure 16.5 initially stretches the spring leftward rather than compresses it as shown, what happens to the wave speed?
7. Are waves always periodic? Are they always harmonic?
8. If you shake a rope at one end, slowly at first and then faster and faster, what changes about the wave you create in the rope? What remains the same?

16.3 Superposition of waves

9. In what sense is the constructive or destructive interference caused by two wave pulses a temporary phenomenon?
10. How are the values of kinetic energy and potential energy carried by any wave pulse related to each other?
11. When superposing waves, how are the wave functions combined?

16.4 Boundary effects

12. (*a*) How does a pulse reflected from a fixed boundary differ from the incident pulse? (*b*) How does a pulse reflected from a free boundary differ from the incident pulse?
13. Describe the graphical method for determining the shape of the reflected pulse when a wave pulse traveling along a string arrives at a boundary.

16.5 Wave functions

14. List all the variables required to describe a traveling harmonic wave.
15. What is the definition of *wave number,* and what is its purpose?
16. What is the direction of travel of the wave represented by the time-dependent wave function $D(x, t) = 3\sin(5x - 2t)$?

16.6 Standing waves

17. What is an *antinode*?
18. What is the distance between a node and an adjacent antinode in a standing wave of wavelength λ?
19. Is the displacement of the medium in which a standing wave exists ever zero at an antinode?

16.7 Wave speed

20. An astronaut takes his ukulele out on a space walk. When he strums the strings in space, is the wave speed on each string higher than, lower than, or the same as on Earth?
21. A vibrating string is clamped at both ends, with one of the clamps being a tension-adjustment screw. By what factor must you change the tension in the string to double its frequency of vibration without changing the wavelength?
22. What is the *linear mass density* of a string or rope?

16.8 Energy transport in waves

23. Describe in words the relationship between power and wave properties expressed in Eq. 16.42.
24. When you shake the end of a string to produce a wave, what fraction of the energy you transfer to the string becomes kinetic energy associated with the motion of the string? What fraction becomes potential energy associated with the stretching of the string?

16.9 The wave equation

25. How does a partial derivative differ from an ordinary derivative?
26. A wave pulse moves along a stretched string. Describe the relationship among the speed at which the pulse moves, the pulse curvature, and the acceleration of small segments of the string.

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The longest wavelength possible for a standing wave on a guitar string (B, N)
2. The speed of transverse waves on a guitar string (A, R)
3. The speed of transverse vibrations in a telephone wire strung between two poles (C, I, L, T)
4. The frequency of a radio wave whose wavelength equals your height (D, J)
5. The longest wavelength possible for a standing wave around the circumference of a helium atom (O, S)
6. The wave number for a wave whose wavelength equals the circumference of Earth's orbit about the Sun (F, P)
7. The power involved in the vibration of a guitar string (A, H, M, R)
8. The ratio of the linear mass density of the thickest string on a guitar to the linear mass density of the thinnest string (K, Q, U, E, G)

Hints

A. What is the linear mass density of a typical guitar string?
B. What is a typical length for a guitar string?
C. What is the distance between the telephone poles?
D. At what speed do radio waves travel?
E. How does the linear mass density vary with wave speed?
F. What is the radius of Earth's orbit?
G. How do the two wave speeds compare?
H. What is the amplitude of the vibration?
I. What is the maximum vertical sag in the wire?
J. What is the relationship among wavelength, frequency, and speed?
K. What is the fundamental frequency played on each string?
L. What angle does the tensile force exerted on the wire make with the horizontal?
M. What is a typical frequency for a vibrating guitar string?
N. What is the node/antinode pattern on the string when it is vibrating?
O. What is the radius of a helium atom?
P. What is the relationship between wave number and wavelength?
Q. Do the two strings have equal tension?
R. What is a typical guitar string tension?
S. What is the node/antinode pattern in a standing wave that has the longest wavelength possible?
T. Is it possible to compute the vibration speed without knowing the mass of the wire?
U. How do the wavelengths of the fundamental (single-antinode) vibrations compare on the two strings?

Key (all values approximate)

A. 2×10^{-3} kg/m; **B.** 0.7 m; **C.** 2×10^1 m; **D.** the speed of light, 3×10^8 m/s; **E.** $\mu = \mathcal{T}/c^2$; **F.** 2×10^{11} m; **G.** because the wavelengths are equal, c is proportional to f; **H.** 3 mm; **I.** 0.5 m; **J.** $\lambda f = c$; **K.** 1×10^2 Hz on the thickest string, 4×10^2 Hz on the thinnest string; **L.** 5×10^{-2} rad (sag height divided by half the wire length); **M.** 3×10^2 Hz; **N.** for the fundamental vibration, one antinode at the center and a node at each end; **O.** 1×10^{-10} m; **P.** $k = 2\pi/\lambda$; **Q.** yes, to a first approximation; **R.** 7×10^1 N; **S.** two nodes and two antinodes; **T.** yes, to a first approximation, mass appears in both numerator and denominator and so cancels; **U.** two strings of the same length, with the same fundamental node/antinode pattern, have identical wavelengths;

Worked and Guided Problems

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

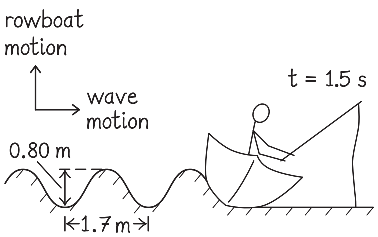
Worked Problem 16.1 Bobbing up and down

A passing speedboat sends out a wake that rocks a rowboat in which a physics student calmly fishes. She notices that the wake has a trough-to-peak height of 0.80 m and that the wave rocks the rowboat with a period of 1.5 s. The distance between crests is 1.7 m. What are (*a*) the wave speed, (*b*) the wave function, and (*c*) the maximum vertical speed of the rowboat?

PRACTICE

❶ **GETTING STARTED** Our physics student has measured several parameters of the wave, and we need to calculate others. We begin with a sketch to help us visualize our ideas and approach (Figure WG16.1). We need to calculate three things: the wave speed, which is the speed at which the wave moves across the water surface, the wave function that describes the wave, and the speed at which the rowboat bobs up and down. In calculating this last value, we assume the boat rides lightly on the surface of the waves as opposed to riding low in the water; with this assumption the amplitude and speed of the boat's motion will match those of the water beneath the boat. For simplicity, we assume that the wave approximates a sine wave in determining the wave function.

Figure WG16.1



❷ **DEVISE PLAN** The wave speed can be found from the period and wavelength, $c = \lambda/T$ (Eq. 16.9). The rowboat's vertical speed is the speed of a water particle at the boat's location moving up and down in simple harmonic motion as the wave passes the location. Because the boat follows the vertical displacement of the water, the displacement curve for the boat should match that of the water. Because the motion of the water at a particular value of x along the wave is perpendicular to the direction along which the wave propagates (which we take to be the x axis), we take the displacement of the water surface (and the rowboat) to be along a y axis, making the wave function

$$y(x, t) = D(x, t) = A\sin(kx - \omega t).$$

Because the boat motion is up and down along the y axis, we can obtain the y component of the boat's velocity by taking the partial derivative of the boat's y coordinate with respect to time:

$$v_y = \frac{\partial y}{\partial t}.$$

We determine the maximum value this function can have to obtain the boat's maximum bobbing speed.

❸ **EXECUTE PLAN** (*a*) The wave speed is

$$c = \frac{\lambda}{T} = \frac{1.7\ \text{m}}{1.5\ \text{s}} = 1.1\ \text{m/s}. ✔$$

(*b*) The peak-to-trough height is twice the wave amplitude, which gives $A = \frac{1}{2}(0.80\ \text{m}) = 0.40\ \text{m}$. We know from Eqs. 16.7 and Eq. 16.11 that $k = 2\pi/\lambda$ and $\omega = 2\pi f = 2\pi/T$. Consequently, the wave function is

$$\begin{aligned} y(x, t) &= A\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) \\ &= (0.40\ \text{m})\sin\left(\frac{2\pi}{1.7\ \text{m}}x - \frac{2\pi}{1.5\ \text{s}}t\right) \\ &= (0.40\ \text{m})\sin[(3.7\ \text{m}^{-1})x - (4.2\ \text{s}^{-1})t]. ✔ \end{aligned}$$

(*c*) Because the wave is sinusoidal, the x component of the boat's vertical velocity is

$$v_y(x, t) = \frac{\partial}{\partial t}A\sin(kx - \omega t) = -\omega A\cos(kx - \omega t).$$

No matter where the boat is located along the x axis, there is some instant t_{max} at which the bobbing speed reaches a maximum. This instant occurs when $|\cos(kx - \omega t_{max})| = 1$, and at that instant the bobbing speed is ωA:

$$v_{max} = \omega A(1) = \frac{2\pi}{1.5\ \text{s}}(0.40\ \text{m}) = 1.7\ \text{m/s}. ✔$$

❹ EVALUATE RESULT If you think about water waves moving across the surface of a lake, a wave speed of 1.1 m/s and a bobbing speed of 1.7 m/s both seem reasonable. We assumed that the boat moves with the water, so our results for the boat are based on the wave function, which describes how a water particle at position x moves along the y axis as the wave passes. The assumption seems reasonable for a small boat.

We also assumed that the wave caused by the wake of the boat is sinusoidal. The greatest possible bobbing speed for waves of any shape is given by assuming the boat is in free fall as it moves from a crest to a trough, which means that

$$v_{\max}^2 < 2gh = 2(9.8\ \mathrm{m/s^2})\,(0.80\ \mathrm{m}) = 16\ \mathrm{m^2/s^2}$$

$$v_{\max} < 4.0\ \mathrm{m/s}.$$

The value we obtained, 1.7 m/s, appears to be in the right ballpark.

Guided Problem 16.2 Human wave

A "people wave" is created by the crowd in a stadium when each person stands up just as one neighbor sits down and then sits down just as the other neighbor stands. A sophisticated crowd produces a "people wave" with each person approximately executing simple harmonic motion as they repeatedly stand and sit. (*a*) Estimate the frequency, wavelength, wave speed, and amplitude of such a wave, and use these values to construct a wave function that describes the wave. (*b*) What is the maximum speed of each person in the wave as she or he stands up and then sits down?

❶ GETTING STARTED

1. Describe the problem in your own words.
2. Assuming a sinusoidal wave, what is a generic expression for the wave function?

❷ DEVISE PLAN

3. Which of the asked-for values can you confidently estimate? Which ones do you have more uncertainty about? You might want to start with amplitude and wavelength.
4. In addition to amplitude and wavelength, what other quantities do you need in order to specify the wave function?
5. How can you determine the stand–sit speed of the wave "particles" (the people)?
6. How can you determine the maximum speed of each person?

❸ EXECUTE PLAN

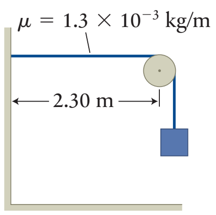
❹ EVALUATE RESULT

7. Are your assumptions and estimates plausible? What errors are introduced by the uncertainty of your estimates?

Worked Problem 16.3 Musical wire

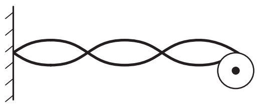
One end of a wire is attached to a wall, and the other end is strung over a pulley and attached to a block (Figure WG16.2). The distance from wall to pulley is 2.30 m, and the wire has a linear mass density of 1.3×10^{-3} kg/m. Plucking the horizontal section of the wire in specific ways can produce standing waves that contain specific numbers of antinodes: The first harmonic has one antinode, the second harmonic has two antinodes, and so on. What should the block's mass be in order that the third-harmonic standing wave (containing three antinodes) vibrates at frequency $f = 550$ Hz?

Figure WG16.2



❶ **GETTING STARTED** The tensile force exerted by the block on the wire provides the tension in the wire, and so we need to determine this force and then use the relationship between the tensile force and the gravitational force on the block to obtain the block mass needed to create the standing wave asked for. The number of antinodes is related to how many wavelengths of the traveling waves fit into the horizontal section of the wire. The third harmonic has three antinodes and is illustrated in Figure WG16.3. Thus we want to calculate what tension is required to create, along the horizontal section of the wire, a three-antinode standing wave vibrating at 550 Hz. We assume that the block moves negligibly when the wire is vibrating.

Figure WG16.3



❷ **DEVISE PLAN** We need to relate the frequency f, wavelength λ, and tensile force. Because the block is supported in equilibrium by the string, the tension (magnitude of tensile force) is $\mathcal{T} = |\vec{F}^c_{sb}| = |\vec{F}^G_{Eb}| = m_b g$. Equations 16.10 and 16.30 permit us to relate the tension to wavelength and frequency, and we can determine λ from Eq. 16.21. For the third harmonic, there are three antinodes and four nodes. The first node is at the wall; the fourth node (the one for which $n = 3$ in Eq. 16.21) occurs at $x = \ell$, where ℓ is the length of wire between the wall and the pulley. Knowing λ, we can solve Eq. 16.30 for the tension $\mathcal{T}$ and then equate $\mathcal{T}$ with the tensile force F exerted by the block to determine m_b.

❸ **EXECUTE PLAN** Substituting ℓ for x and 3 for n in Eq. 16.21, we get $\ell = \frac{3}{2}\lambda$. This tells us that the third harmonic has wavelength $\lambda = 2\ell/3$. Solving Eq. 16.30 for the tension gives us

$$c = \sqrt{\frac{\mathcal{T}}{\mu}}$$

$$\mathcal{T} = \mu c^2,$$

and from Eq. 16.10 we can rewrite this as $\mathcal{T} = \mu(\lambda f)^2$. Because the tension is created by the tensile force exerted by (or on) the block, we have

$$m_b g = \mathcal{T} = \mu(\lambda f)^2 = \mu(\tfrac{2}{3}\ell f)^2$$

$$m_b = \frac{4\mu(\ell f)^2}{9g}$$

$$= \frac{4(0.00130\ \text{kg/m})[(2.30\ \text{m})(550\ \text{Hz})]^2}{9(9.8\ \text{m/s}^2)} = 94\ \text{kg.}\ ✔$$

❹ **EVALUATE RESULT** Our result shows that we need quite a massive block to obtain the desired third harmonic. We know one has to increase the tension in a string to obtain a higher pitch (higher frequency) from it. Our algebraic expression indeed shows that the block's mass must be increased to increase the frequency f. We would also expect the required mass to be inversely proportional to g because if the gravitational pull were stronger, a block of smaller mass would be sufficient to create the required tension.

Our assumption that the block does not move is likely to be accurate given the mass of the block. We ignored the mass of the hanging portion of the wire in our calculation, but that seems reasonable considering the very small value for μ. The mass of the 2.30-m horizontal section is only 0.003 kg, so even a relatively long section of string has negligible mass relative to that of the block.

PRACTICE

Guided Problem 16.4 Pitch-perfect guitar

A guitarist breaks the highest string on her guitar. If a new string is a steel wire of mass density $7.8 \times 10^3\ \text{kg/m}^3$ and radius 0.120 mm, what tension must the string be under if it is to produce the frequency 330 Hz as its fundamental (first harmonic, one antinode) frequency? The distance between the wire's two attachment points on the guitar is 640 mm.

❶ **GETTING STARTED**

1. Sketch the shape of the string oscillating at its fundamental frequency.
2. What variables affect the frequency?

❷ **DEVISE PLAN**

3. What equation relates the wave's frequency and wavelength to the tension?
4. How can you use the information that there is only one antinode in the standing wave to determine the wavelength of the traveling waves that interfere to produce this standing wave?
5. How can you determine the mass of the wire from the mass density of steel? How can you determine the wire's linear mass density from this?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

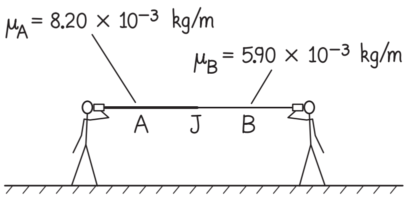
6. Is your computed value of tension the right order of magnitude for the tension in a guitar string?

Worked Problem 16.5 Basic telephone service

Two children playing telephone use some string to connect two paper cups. To make the separation large enough, they use two strings of different linear mass densities connected at the center. String A has a linear mass density of 8.20×10^{-3} kg/m, and string B has a linear mass density of 5.90×10^{-3} kg/m. Consider a traveling wave that originates on string A, passes the junction at the center, and continues along string B. What is the ratio of the wavelength of this wave on string B to its wavelength on string A?

❶ **GETTING STARTED** We start with a sketch (Figure WG16.4), labeling the junction at the center by the letter J. We are asked to determine how the wavelength changes as a wave traveling from A to B passes the junction. The wavelength is related to the frequency and speed of the wave. Because the two strings have different linear mass densities, waves travel at different speeds in the two strings. We need to determine out how this change in speed affects the wavelength. We might also have to consider the tension in the strings because the tension affects the wave speed.

Figure WG16.4



❷ **DEVISE PLAN** The wave speed along a taut string is given by Eqs. 16.10 and 16.30, $\lambda f = c = \sqrt{\mathcal{T}/\mu}$, where $\mathcal{T}$ is the tension in the strings and μ is its linear mass density. Neither the tension nor the frequency f in the two strings is specified. However, we can reasonably argue that the tension is the same in the two strings by considering what would happen if it were not. If the tension on the left were greater, then a free-body diagram of the junction would show a greater tensile force to the left than to the right, causing the junction to accelerate to the left. This would reduce the tension on the left and increase it on the right, ultimately leading to equilibrium at a new junction position.

To determine the frequency, we consider the vertical motion of the junction. The frequency at which the junction moves up and down is equal to the frequency of the incoming wave from A, which means that the junction end of string A produces the wave that goes into B with the same frequency. Therefore the frequency of the wave *must* be constant throughout the joined strings. However, the wavelength does change because the wave speed changes due to the difference in linear mass densities.

We can solve Eq. 16.30 for the tension $\mathcal{T}$ in each segment, set the two tensions equal to each other, and use Eq. 16.10 to replace each c by its equivalent λf. The f values cancel, leaving us with the ratio of the wavelengths.

❸ **EXECUTE PLAN** Solving Eq. 16.30 for the tension and then substituting λf for c from Eq. 16.10 give

$$\mathcal{T} = \mu(\lambda f)^2.$$

Setting the tensions equal in the two strings, we have

$$\mathcal{T}_A = \mathcal{T}_B$$

$$\mu_A \lambda_A^2 f_A^2 = \mu_B \lambda_B^2 f_B^2.$$

Canceling the frequencies because they are the same in both segments gives us

$$\frac{\lambda_B}{\lambda_A} = \sqrt{\frac{\mu_A}{\mu_B}} = \sqrt{\frac{8.20 \times 10^{-3}\ \text{kg/m}}{5.90 \times 10^{-3}\ \text{kg/m}}} = 1.18. ✔$$

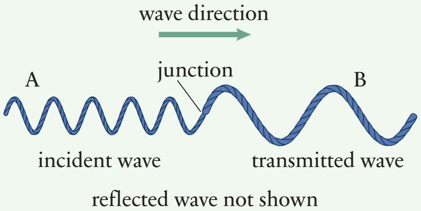
❹ **EVALUATE RESULT** The relationship $c = \sqrt{\mathcal{T}/\mu}$ tells us that the string that has the smaller linear mass density generates the greater wave speed c. Thus in one period of the motion the leading edge of the traveling wave moves a greater horizontal distance in the thinner string, which is B. This means that the distance between two adjacent crests is greater in B than in A; in other words, the wavelength is longer in B, which is consistent with our calculation. The 18% difference we obtained is reasonable: The mass densities differ by more than 30%, and we need to take the square root of these mass densities to obtain the wavelength.

Our assumption that the tension is the same in the two strings would be strictly valid only if no gravity were present. The gravitational force exerted by Earth causes both strings to sag, and the tension in the two strings must adjust differently to this sagging because the strings have different masses. However, the gravitational forces exerted on the strings are much smaller than the typical tension in a string telephone, and so our assumption should have a negligible effect on our answer.

Guided Problem 16.6 At the junction

In Figure WG16.5, string B has twice the linear mass density of string A. When an incident wave described by the function $y_I(x, t) = A_I \sin(k_A x - \omega t)$ initially traveling left to right through string A hits the junction, some of the wave is reflected and the rest is transmitted into string B. What fraction of the incident wave power is reflected and what fraction is transmitted? (For simplicity, Figure WG16.5 does not show the reflected wave.)

Figure WG16.5



❶ GETTING STARTED

1. Does the boundary at the junction behave as fixed or free?
2. The incident wave $y_I(x, t)$ is given. The reflected wave $y_R(x, t)$ must have an amplitude $A_R \neq A_I$, but what is the expression for its wave function? In writing your expression, consider direction of travel, inverted or upright amplitude, and so on.
3. The transmitted wave $y_T(x, t)$ has an amplitude $A_T \neq A_I$. What is an appropriate form for its wave function? Remember that the wave number k_B, the number of wavelengths in 2π m of string B, may not equal k_A.)

❷ DEVISE PLAN

4. Wave power is proportional to the square of the wave amplitude by Eq. 16.42, $P_{av} = \frac{1}{2}\mu A^2\omega^2 c$.
5. Where do the waves have the greater speed; in A or in B? Why?
6. Based on what you learned from the solution to Worked Problem 16.5, what is the relationship between the frequencies f_A and f_B (and thus the relationship between ω_A and ω_B)? What is the relationship between the tensions $\mathcal{T}_A$ and $\mathcal{T}_B$?
7. What is the relationship between the wavelengths λ_A and λ_B? Between the wave numbers k_A and k_B?
8. The wave function and its slope (derivative) must be continuous everywhere, including at the junction, where superposition gives $y_I(x_J, t) + y_R(x_J, t) = y_T(x_J, t)$ and $\frac{\partial}{\partial x}[y_I(x, t) + y_R(x, t)]_{x=x_J} = \frac{\partial}{\partial x}[y_T(x, t)]_{x=x_J}$.

❸ EXECUTE PLAN

9. Take these derivatives to infer what the ratios A_R/A_I and A_T/A_I must be in order to ensure continuity.
10. The algebra is simplified if you choose the origin at the junction so that $x_J = 0$.

❹ EVALUATE RESULT

11. Does the sum of the transmitted and reflected power equal the incident power? If not, why not?

Worked Problem 16.7 A pulse on a rope

A certain transverse wave pulse traveling along a rope is described by the time-dependent wave function $f(x, t) = Ae^{-(kx-\omega t)^2}$, with wave number $k = 2\pi\ \text{m}^{-1}$ and angular frequency $\omega = 2\pi\ \text{s}^{-1}$. (*a*) Sketch the time-independent wave function at $t = 0$ and $t = 5.00$ s. (*b*) Sketch the pulse as a function of time (the displacement curve) at $x = 0$ and $x = 5.00$ m. (*c*) Show that the function given above satisfies the wave equation. (*d*) What is the wave speed of the pulse?

❶ GETTING STARTED We are given an equation for a particular wave pulse. We need to sketch the pulse at various positions and instants and then show that the given expression is a solution to the wave equation. Then we need to determine the wave speed of the pulse.

❷ DEVISE PLAN To make the sketches, we can make tables of values at various positions for the wave function of part *a* and various instants for the displacement curve of part *b*, and then plot the data. A graphing calculator would make this task easier, but we shall make one table just to get the idea.

Then we need to show that the given expression satisfies the wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

To do that, we can take appropriate derivatives of the given function, insert calculated values on both sides of the wave equation, and check that the values we get for the two sides are equal to each other.

The wave speed required for part *d* is given by the value of *c* in the wave equation.

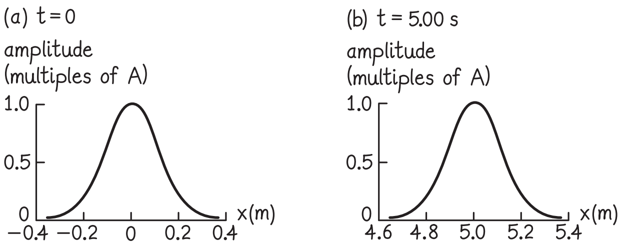
PRACTICE

❸ **EXECUTE PLAN** (a) First let us make a table of the wave function for the $t = 0$ case. Because the value of the amplitude A is not given, we divide $f(x)$ by A and compute only the exponential function.

x (m)	$f(x)/A$
−0.500	5.17×10^{-5}
−0.400	1.81×10^{-3}
−0.300	2.86×10^{-2}
−0.200	2.06×10^{-1}
−0.100	6.74×10^{-1}
0	1.00×10^{0}
0.100	6.74×10^{-1}
0.200	2.06×10^{-1}
0.300	2.86×10^{-2}
0.400	1.81×10^{-3}
0.500	5.17×10^{-5}

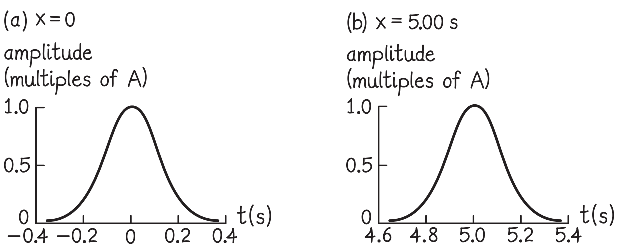
A similar calculation can be done for $t = 5.00$ s. The resulting plots for the pulse positions at these two instants are shown in Figure WG16.6. This exponential pulse is just a bump that travels to the right. In part a, $t = 0$, the peak is at $x = 0$, and in part b, $t = 5.00$ s, the peak has moved on to $x = 5.00$ m.

Figure WG16.6



(b) The displacement curves $f(x = 0, t)$ and $f(x = 5.00 \text{ m}, t)$ at the two positions given in the problem statement yield the graphs of Figure WG16.7a and b, where we have plotted $f(t)/A$ as a function of t. ✔

Figure WG16.7



(c) Now, to confirm that the expression given for the pulse is a solution to the wave equation, we have

$$\frac{\partial^2}{\partial x^2} A e^{-(kx-\omega t)^2} \stackrel{?}{=} \frac{1}{c^2}\frac{\partial^2}{\partial t^2} A e^{-(kx-\omega t)^2}$$

$$-\frac{\partial}{\partial x} 2k(kx - \omega t) A e^{-(kx-\omega t)^2}$$

$$\stackrel{?}{=} \frac{1}{c^2}\frac{\partial}{\partial t} 2\omega(kx - \omega t) A e^{-(kx-\omega t)^2}$$

$$[-2k(kx - \omega t)]^2 A e^{-(kx-\omega t)^2} - 2k^2 A e^{-(kx-\omega t)^2}$$

$$\stackrel{?}{=} \frac{1}{c^2}\{[2\omega(kx - \omega t)]^2 A e^{-(kx-\omega t)^2} - 2\omega^2 A e^{-(kx-\omega t)^2}\}.$$

Eliminating the exponential factor gives

$$[-2k(kx - \omega t)]^2 - 2k^2 \stackrel{?}{=} \frac{1}{c^2}\{[2\omega(kx - \omega t)]^2 - 2\omega^2\}$$

$$2k^2[2(kx - \omega t)^2 - 1] \stackrel{?}{=} \frac{1}{c^2}2\omega^2[2(kx - \omega t)^2 - 1]$$

$$2k^2 \stackrel{?}{=} \frac{1}{c^2}2\omega^2.$$

This equation is true only if

$$c = \frac{\omega}{k},$$

which we know to be true from Eq. 16.11, $kc = \omega$. Our pulse function is indeed a solution to the wave equation. ✔

(*d*) The wave speed of the pulse is

$$c = \frac{\omega}{k} = \frac{2\pi \text{ s}^{-1}}{2\pi \text{ m}^{-1}} = 1.00 \text{ m/s}. ✔$$

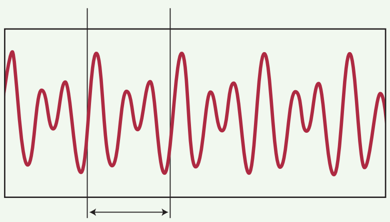
The values of ω and k are known to many significant digits, but we choose to report three because the other data are known to three significant digits.

❹ **EVALUATE RESULT** We've demonstrated that this type of pulse is a solution to the wave equation. The speed is reasonable, if a bit slow. Also, the shape looks reasonable for a pulse on a rope.

Guided Problem 16.8 Superposed waves

Most of the waves in our everyday world, such as radio waves from a station's transmitter or sound waves from a plucked or bowed violin string, are not simple sine or cosine waves but rather a superposition of many different harmonic waves, as shown in Figure WG16.8 As an example, consider the wave function $f(x, t) = \sin(kx - \omega t) + (1.5)\sin(2kx - 2\omega t)$. You are informed that $k = 8.00 \text{ m}^{-1}$ and that the frequency at which the pattern repeats is $f = 440$ Hz. (*a*) Sketch the wave as a function of time at $x = 0$ for three cycles on a suitable graph. (*b*) Show that this wave function is a solution to the wave equation, $(\partial^2 f/\partial x^2) = (1/c^2)(\partial^2 f/\partial t^2)$. (*c*) What is the speed of this wave?

Figure WG16.8



❶ **GETTING STARTED**

1. You'll need numerical values in order to complete parts *a* and *c*. List the values you know and those you need to determine.
2. What must be the value of ω to get the interference pattern to repeat at the specified frequency? What time interval elapses before the pattern starts to repeat itself?
3. For a fixed value of x, how many seconds would it take for the first term in the wave function to go through one cycle? How many seconds for the second term to go through one cycle?

❷ **DEVISE PLAN**

4. Think through a simple consistent procedure to determine the displacement curve at the specified value of x for three cycles. How do you determine how closely spaced the time values along your horizontal axis should be?
5. The procedure in Worked Problem 16.7 can be used to see whether the function satisfies the wave equation, but how can the wave speed be related to the quantities you know?

❸ **EXECUTE PLAN**

6. Plot your data.
7. Write the wave equation and take the required partial derivatives of the wave function in its symbolic form (without the numerical values you have inferred for ω). Is it possible for the equation to balance? What does your answer to this question imply about the wave speed?

❹ **EVALUATE RESULT**

Answers to Review Questions

1. A wave transports energy and momentum but no matter, which means that for mechanical waves, each portion of the medium is at its original location after the wave passes through.
2. The traffic jam is longitudinal, and the other three are transverse.
3. The wave speed is the speed at which the wave moves along the rope. The speed of any small segment of the rope is the magnitude of the velocity at which that segment gets displaced from its equilibrium position as the wave passes through.
4. No, unless the pulse is symmetrical. In general, the displacement curve is the mirror image of the wave function.
5. The frequency is the number of wave crests that arrive each second, in units of hertz (Hz). The wave speed is how fast a wave travels in meters per second (m/s).
6. It remains the same. See Checkpoint 16.5.
7. Waves are not always periodic. A periodic wave is created only when the motion that causes the wave is repeated at regular intervals. A wave pulse is generally not periodic. Neither are waves always harmonic. A harmonic wave is created only when the motion that causes the wave is harmonic (that is, the motion is the particular type of periodic motion that yields a sine curve).
8. Assuming that you move your hand only vertically, the horizontal component of force you exert on the rope as you shake it remains constant and equal to the tension. The wave speed is therefore unaffected by whether you move your hand rapidly or slowly. Assuming that you move your hand up and down by the same distance each time, the amplitude of each pulse you create remains the same. Thus only the period, frequency, and wavelength of the wave change.
9. The interference is temporary in the sense that it occurs at only those instants and places where the pulses overlap. After the pulses move apart, they again have their original size and shape, and the interference no longer exists.
10. Wave pulses always carry equal amounts of kinetic and potential energy.
11. The wave functions are added algebraically at each point of overlap.
12. (*a*) The reflected pulse is inverted and left–right reversed. (*b*) The reflected pulse is left–right reversed but not inverted.
13. At each instant that the shape is desired, sketch a scale diagram of the wave pulse approaching the boundary and a reversed pulse (traveling in the opposite direction at the same speed) designed to arrive at the boundary at the same instant as the actual pulse. If the string beyond the boundary has a greater linear mass density (or if the boundary is a fixed end), the reversed pulse is drawn inverted. If the string beyond the boundary has a smaller linear mass density (or the boundary is a free end), the reversed pulse is drawn upright. Now superimpose the two pulses where they overlap by adding the displacements of each pulse at common positions. The result is an outline of the shape of the pulse at the chosen instant.
14. A complete description requires amplitude A; plus period T, frequency f, or angular frequency ω; plus wavelength λ, wave number k, or wave speed c; plus direction of motion.
15. The wave number of a harmonic wave is the number of wavelengths of the wave that fit in a length of 2π meters. The wave number is a conversion constant that is used to change distances in meters into unitless angles in the arguments of the sinusoidal functions that are used to describe harmonic waves.
16. The wave travels in the positive x direction. The relative sign of the position and time terms gives you this information.
17. An antinode is any point in a standing wave where the amplitude of the displacement attains its greatest value. These points occur halfway between nodes (positions of zero amplitude).
18. They are separated by a distance $\lambda/4$.
19. Yes. The antinodes cycle through all values of medium displacement, including zero.
20. The same. The wave speed depends on the tension and linear mass density of the strings, neither of which has changed.
21. You need to double the wave speed. This means quadrupling the tension, assuming that the string does not stretch (change its linear mass density).
22. Linear mass density is the mass per unit length of a string or rope, measured in units of kg/m.
23. The average power required to generate a wave is directly proportional to the linear mass density of the string and the wave speed and to the square of the amplitude and angular frequency of the wave.
24. Half of the energy transferred becomes kinetic energy and half potential energy.
25. A partial derivative is used for functions that contain more than one variable and is evaluated as an ordinary derivative with respect to one variable while all other variables are held constant.
26. The curvature of any location on the pulse increases as the acceleration of the string segments in the vicinity increases and decreases as the wave speed increases.

Answers to Guided Problems

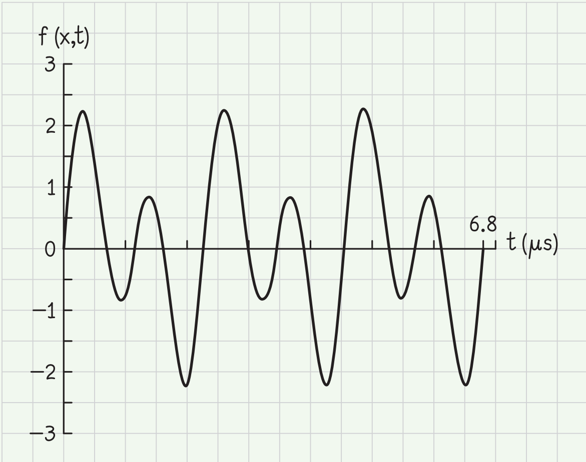
Guided Problem 16.2 (*a*) A range of values is possible; for example, $f = 1/3$ Hz, $\lambda = 12$ m, $c = 4$ m/s, and $A = 1$ m give $f(x, t) = (1\text{ m})\sin(\pi x/6 - 2\pi t/3 + \phi)$. (*b*) These values result in a vertical speed of 2 m/s for each participant.

Guided Problem 16.4 $\mathcal{T} = 63$ N

Guided Problem 16.6 $\dfrac{A_R}{A_I} = \dfrac{k_A - k_B}{k_A + k_B}$ and $\dfrac{A_T}{A_I} = \dfrac{2k_A}{k_A + k_B}$, so 97.1% of the incident power is transmitted and 2.9% is reflected.

Guided Problem 16.8 (*a*) Figure GPS16.8

Figure GPS16.8



(*c*) $c = \dfrac{\omega}{k} = \dfrac{2\omega}{2k} = \dfrac{2\pi(440\ \text{Hz})}{8.00\ \text{m}^{-1}} = 346\ \text{m/s}$

Guided Practice by Chapter

17 Waves in Two and Three Dimensions

Review Questions

Answers to these questions can be found at the end of this chapter.

17.1 Wavefronts

1. Name two ways in which two- and three-dimensional waves differ from one-dimensional waves.
2. Which of the following factors plays a role in how much a wave's amplitude decreases as the wave travels away from its source: dissipation of the wave's energy, dimensionality of the wave, destructive interference by waves created by other sources?
3. Consider a sound wave, created at one end of a hollow pipe, that moves through the medium filling the pipe. How does the amplitude change as the wave travels away from the sound source inside the pipe? Ignore energy dissipation.

17.2 Sound

4. Are sound waves in air longitudinal or transverse?
5. What is the range of frequencies audible to humans who have normal hearing?
6. Describe a demonstration that shows that air is elastic.
7. In a sound wave, how do the locations of maximum and minimum medium displacement compare with the locations of maximum and minimum medium density?

17.3 Interference

8. Is the sound created by two sound waves passing a given position P always louder than the sound at P created by either wave alone?
9. Suppose your seat at a poorly designed concert hall is at a *dead spot,* which is a position where destructive interference destroys the sound coming from the stage. Is it a dead spot for all musical notes?
10. Picture two identical coherent wave sources placed side by side and sending out waves that interfere with each other, creating a moiré pattern. If the distance between the two sources is increased, does the number of nodal lines in the pattern increase, decrease, or remain the same?
11. Can destructive interference occur when waves emanating from three or more sources overlap?

17.4 Diffraction

12. What is Huygens' principle?
13. Why is it important that a bat use high-frequency waves to locate insects sonically?
14. Describe the relative extent of spreading of sound waves after they pass through a gap in a barrier. Consider three possibilities: The wavelength is (*a*) much smaller than the gap width, (*b*) comparable to the gap width, and (*c*) much greater than the gap width.

17.5 Intensity

15. A point source emits sound waves uniformly in all directions. How does the sound intensity I vary with distance from the source?
16. Increasing your distance from a point source of spherical waves by a factor of 10 reduces the intensity level β by how many decibels?
17. Explain why the units of intensity I differ for a ripple on the surface of a pond and for a sound wave from an emergency alert siren.

17.6 Beats

18. How is the beat frequency created by two interfering waves related to the frequencies of the waves?
19. One way to tune a piano is to strike a tuning fork (which emits only one specific frequency), then immediately strike the piano key for the frequency being sounded by the fork, and listen for beats. In making an adjustment, a piano tuner working this way causes the beat frequency to increase slightly. Is she going in the right direction with that adjustment?

17.7 Doppler effect

20. A train approaches as you wait at a crossing. Is the whistle frequency you hear higher than, lower than, or the same as the frequency you would hear if the train were stationary?
21. Which produces a greater frequency ratio f_o/f_s: a wave source approaching a stationary observer at a speed of $0.250c$ or an observer approaching a stationary source at a speed of $0.250c$?

17.8 Shock waves

22. Does the angle of a shock wave depend on the frequency of the sound emitted?
23. A boat travels through the water. If its speed increases, what happens to the angle of the bow wave it creates in the water?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The wavelength of the sound wave traveling in air when you whistle (M, T)
2. The number of wavelengths of the musical tone A above middle C (440 Hz) in a standing wave that fills an auditorium (H, M, Q)
3. The width of the smallest object that can be located using sound in the range of human hearing (P, B, F)
4. The greatest distance at which you can just hear a large fireworks shell explode (A, C, I, R, S, U)
5. The power output of the warning horn on a freight train (G, J, N, R, E)
6. The maximum distance over which you can hear a train horn (E, R, L)
7. The ratio of the wavelength of a sound on your vocal cords to the wavelength of the same sound in air (D, K, O)

Hints

A. How much energy is released by a commercial fireworks shell?
B. Which end of the range of human hearing is more useful for detecting a very small object?
C. During what time interval is the energy released?
D. What is the length of a human vocal cord?
E. What is the intensity level β near the horn?
F. What is the frequency range of human hearing?
G. At what distance must the horn be heard?
H. What is the greatest dimension of an auditorium?
I. What fraction of the energy released goes into sound?
J. What must the intensity level β be inside an automobile approaching the tracks in order for the driver to hear the horn?
K. What is a typical frequency for the human voice?
L. What is the minimum intensity level β needed to hear this sound in an open field?
M. What is the speed of sound in air?
N. What must the intensity level β be outside the car?
O. What is the wavelength in air for this typical voice frequency?
P. What is the relationship between the width of the object and the wavelength of the sound wave being used to detect it?
Q. What is the wavelength for this tone?
R. How does the intensity I drop off with distance?
S. What fraction of the sound energy is absorbed or dissipated over great distances?
T. What is the frequency of sound emitted when you whistle?
U. What background noise level is appropriate?

Key (all values approximate)

A. 2 J; **B.** the smallest object requires the smallest possible wavelength for the sound wave, which means the highest possible frequency in the human-hearing range; **C.** 5 ms; **D.** 2×10^1 mm; **E.** 120 dB 1 m from horn; **F.** 2×10^1 Hz to 2×10^4 Hz; **G.** 1×10^2 m for a fast train, which may travel this distance in a few seconds; **H.** 6×10^1 m; **I.** taking into account energy to create light and kinetic energy of fragments—say, 1/2; **J.** above the intensity level of conversation or radio loudness—say, 65 dB; **K.** 2×10^2 Hz; **L.** 30 dB; **M.** 3×10^2 m/s; **N.** because the windows may be closed, allow a good margin—say, 80 dB; **O.** 2 m; **P.** wavelengths longer than the object's width diffract around the object instead of being reflected back to the detector; **Q.** 0.7 m; **R.** as $1/r^2$, because the wavefronts are spherical; **S.** say, another 1/2; **T.** 1 kHz; **U.** for a best case, 30 dB

Worked and Guided Problems

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

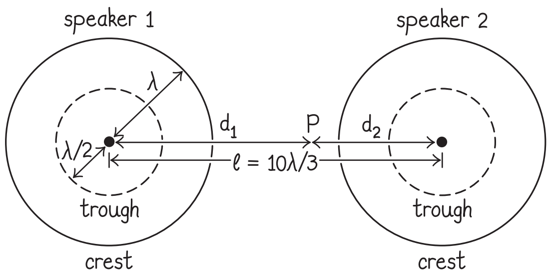
Worked Problem 17.1 Minimal sound

Two small, spherical speakers simultaneously emit a note of wavelength λ. The speakers are a distance $10\lambda/3$ apart. If a microphone is moved along the line between their centers, at what positions does it detect (*a*) an unusually loud sound and (*b*) an unusually faint sound?

PRACTICE

❶ **GETTING STARTED** We know that the closer the microphone is to a speaker, the louder the sound it detects because the intensity I of the sound waves is greatest near the source. However, the question says *unusually loud,* which hints that we are being asked about the effects of constructive and destructive interference. We are asked for the positions where the microphone records very loud and very faint sounds, and so we draw a diagram that shows what we know and specifies a suitable reference position against which to measure distances. We choose this reference position to be the center of the left speaker, which we call speaker 1 (Figure WG17.1).

Figure WG17.1



❷ **DEVISE PLAN** Consider the arbitrary position labeled P in Figure WG17.1, which is on the line that connects the speaker centers. Constructive interference occurs at P when waves from the two speakers arrive at P in phase. To achieve this, the phase difference at P must be a multiple of one cycle: 0, $\pm 2\pi$, $\pm 4\pi$, ... Because there is one wavelength in each cycle, the number of wavelengths in distances d_1 and d_2 must differ by an integral number of wavelengths: 0, $\pm\lambda$, $\pm 2\lambda$, ... So, for constructive interference,

$$\text{constructive interference: } \frac{d_1 - d_2}{\lambda} = n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$d_1 - d_2 = n\lambda.$$

Destructive interference occurs in locations where the two waves are out of phase by $180° = \pi$ rad. Using the same argument as above, we can say

$$\text{destructive interference: } \frac{d_1 - d_2}{\lambda} = n + \tfrac{1}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$d_1 - d_2 = (n + \tfrac{1}{2})\lambda.$$

We have to make sure that our values for d_1 and d_2 remain between the two speakers, which requires that $d_1 + d_2 = \ell$ and that $0 < d_1 < \ell$ and $0 < d_2 < \ell$.

❸ **EXECUTE PLAN**

(*a*) Because $d_2 = \ell - d_1$, we have for constructive interference

$$d_1 - d_2 = d_1 - (\ell - d_1) = 2d_1 - \ell = 2d_1 - \tfrac{10}{3}\lambda = n\lambda$$

$$d_{1,n}^{\text{con}} = \tfrac{1}{2}(n + \tfrac{10}{3})\lambda, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now we can substitute the first few values of n to obtain all the constructive-interference positions between the speakers:

$$d_{1,0} = \tfrac{1}{2}(0 + \tfrac{10}{3})\lambda = \tfrac{5}{3}\lambda$$

$$d_{1,-1} = \tfrac{1}{2}(-1 + \tfrac{10}{3})\lambda = \tfrac{7}{6}\lambda$$

$$d_{1,+1} = \tfrac{1}{2}(+1 + \tfrac{10}{3})\lambda = \tfrac{13}{6}\lambda$$

$$d_{1,-2} = \tfrac{1}{2}(-2 + \tfrac{10}{3})\lambda = \tfrac{2}{3}\lambda$$

$$d_{1,+2} = \tfrac{1}{2}(+2 + \tfrac{10}{3})\lambda = \tfrac{8}{3}\lambda$$

$$d_{1,-3} = \tfrac{1}{2}(-3 + \tfrac{10}{3})\lambda = \tfrac{1}{6}\lambda$$

$$d_{1,+3} = \tfrac{1}{2}(+3 + \tfrac{10}{3})\lambda = \tfrac{19}{6}\lambda$$

$$d_{1,-4} = \tfrac{1}{2}(-4 + \tfrac{10}{3})\lambda = -\tfrac{1}{3}\lambda$$

$$d_{1,+4} = \tfrac{1}{2}(+4 + \tfrac{10}{3})\lambda = \tfrac{11}{3}\lambda.$$

PRACTICE

These last two distances are not between the speakers, which are a distance $10\lambda/3 = 20\lambda/6$ apart, and so we discard them. Thus the positions between the speakers where the microphone detects unusually loud sounds are at these distances from speaker 1:

$$d_1^{\text{con}} = \tfrac{1}{6}\lambda, \tfrac{2}{3}\lambda, \tfrac{7}{6}\lambda, \tfrac{5}{3}\lambda, \tfrac{13}{6}\lambda, \tfrac{8}{3}\lambda, \tfrac{19}{6}\lambda. \checkmark$$

(b) Using the same procedure, we discover that the distances from speaker 1 that produce destructive interference satisfy the expression

$$2d_1 - \tfrac{10}{3}\lambda = (n + \tfrac{1}{2})\lambda$$

$$d_{1,n}^{\text{des}} = \tfrac{1}{2}(n + \tfrac{23}{6})\lambda, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$d_{1,0} = \tfrac{1}{2}(0 + \tfrac{23}{6})\lambda = \tfrac{23}{12}\lambda$$

$$d_{1,-1} = \tfrac{1}{2}(-1 + \tfrac{23}{6})\lambda = \tfrac{17}{12}\lambda$$

$$d_{1,+1} = \tfrac{1}{2}(+1 + \tfrac{23}{6})\lambda = \tfrac{29}{12}\lambda$$

$$d_{1,-2} = \tfrac{1}{2}(-2 + \tfrac{23}{6})\lambda = \tfrac{11}{12}\lambda$$

$$d_{1,+2} = \tfrac{1}{2}(+2 + \tfrac{23}{6})\lambda = \tfrac{35}{12}\lambda$$

$$d_{1,-3} = \tfrac{1}{2}(-3 + \tfrac{23}{6})\lambda = \tfrac{5}{12}\lambda$$

$$d_{1,+3} = \tfrac{1}{2}(+3 + \tfrac{23}{6})\lambda = \tfrac{41}{12}\lambda$$

$$d_{1,-4} = \tfrac{1}{2}(-4 + \tfrac{23}{6})\lambda = -\tfrac{1}{12}\lambda$$

$$d_{1,+4} = \tfrac{1}{2}(+4 + \tfrac{23}{6})\lambda = \tfrac{47}{12}\lambda.$$

Again we want only positions that lie between the speakers, which are $10\lambda/3 = 40\lambda/12$ apart. The last three values we just calculated are not on this center-to-center line, and so our destructive-interference positions are

$$d_1^{\text{des}} = \tfrac{5}{12}\lambda, \tfrac{11}{12}\lambda, \tfrac{17}{12}\lambda, \tfrac{23}{12}\lambda, \tfrac{29}{12}\lambda, \tfrac{35}{12}\lambda. \checkmark$$

4 EVALUATE RESULT We expect constructive interference halfway between the speakers because the waves from the two speakers travel the same distance to reach this position. This distance is $\tfrac{1}{2}(\tfrac{10}{3}\lambda) = \tfrac{5}{3}\lambda = d_{1,0}^{\text{con}}$, and we see that this value does appear in our constructive-interference list.

We expect the spacings between constructive and destructive positions to be related to the wavelength in some simple way. In other words, the same type of interference should occur as the difference $d_1 - d_2$ changes by one wavelength. This is easiest to see with the constructive-interference positions. The midpoint between the speakers is a position of constructive interference. As we move from this midpoint toward speaker 2, d_1 increases and d_2 decreases. If d_1 increases by $\lambda/2$, d_2 decreases by $\lambda/2$, and the difference $d_1 - d_2$ changes by one wavelength. Thus positions of constructive interference should be a distance $\lambda/2$ apart, positions of destructive interference should be a distance $\lambda/2$ apart, and this is just what we see.

We expect destructive interference halfway between each pair of constructive-interference positions, which is just what we see—for example,

$$\tfrac{1}{2}[d_{1,+1}^{\text{con}} + d_{1,0}^{\text{con}}] = \tfrac{1}{2}\left[\tfrac{13}{6}\lambda + \tfrac{5}{3}\lambda\right] = \tfrac{23}{12}\lambda = d_{1,0}^{\text{des}}.$$

We made one assumption that can affect the accuracy of our answers. We assumed that the amplitude of the waves is constant along the line that joins the speakers, and this is not true. The closer you are to one speaker, the higher the amplitude of the wave coming from it and the lower the amplitude of the wave coming from the other speaker. Thus, positions where the phase difference is $\lambda/2$ are not positions where the crest of one wave cancels the trough of the other. This deviation from the ideal case causes both the constructive and the destructive interference to be less than complete. Nevertheless these are positions of unusually loud or faint sound.

Guided Problem 17.2 Dripping disturbance

A dripping faucet creates waves on the surface of a still pool of water in a stoppered sink. The drops enter the water at position P, and the crest-to-trough height of the waves 10 mm from P is 7.0 mm. (*a*) What is the amplitude of the waves 150 mm from P? (*b*) By what factor does the wave intensity 150 mm from P differ from the wave intensity 10 mm from P?

1 GETTING STARTED

1. Describe what happens to the waves as they spread out.
2. What is the difference between a water wave spreading out in two dimensions and a sound wave spreading out in three dimensions?
3. Equation 17.1, $I = P/A$, is for three-dimensional waves. How does the denominator of the fraction change when you write the intensity equation for a two-dimensional wave?

2 DEVISE PLAN

4. What unknown quantities do you need to determine?
5. What principles or methods can help you obtain these quantities?
6. How is energy related to a quantity you've studied in this chapter?
7. Which equations allow you to express the unknown quantities in terms of known quantities?

3 EXECUTE PLAN

4 EVALUATE RESULT

8. Are these results consistent with your experience with such water waves?

Worked Problem 17.3 Bottle music

When you blow across the mouth of a bottle, you can cause a standing wave in the air in the bottle, which makes a musical sound. Blowing harder can give a higher musical pitch, but that takes practice. Assume for any liquid height in the bottle that you can reliably obtain only the fundamental (first harmonic) and the next higher harmonic frequency by blowing. You have a beverage bottle 0.23 m in length, with an original liquid height of 0.20 m (before you drink any of the beverage). Given that the speed of sound in air is 343 m/s, what range of frequencies can you obtain as you slowly sip the liquid in the bottle?

1 GETTING STARTED Sound waves are longitudinal waves, but standing waves in the bottle should form only for certain wavelengths, as for transverse standing waves on a string (see Section 16.6). We must determine what standing wave patterns form in the air above the liquid in the bottle. The top surface of the liquid in the bottle (or the bottom of the bottle when the contents are consumed) acts as a fixed end because sound waves are longitudinal, and the displacement of air molecules is greatly reduced at the liquid or glass surface. Therefore we expect a node at this location. The top end of the bottle is open, and displacement of air molecules is not constrained at this end of the bottle. This corresponds to a free end for reflection, resulting in a displacement antinode near the top of the bottle. Now we just need to figure out how many wavelengths fit in the bottle to match these boundary conditions. The smallest number we obtain corresponds to the fundamental frequency.

2 DEVISE PLAN We need to determine what fraction of a wavelength satisfies our boundary conditions. From the allowed wavelengths and the known speed of sound in air, we can then use the relationship $c = \lambda f$ to compute the frequencies.

One node in a standing wave pattern is separated from the next node by half a wavelength, so to get from one node to an adjacent antinode is a quarter of a wavelength. The fundamental pattern requires one-quarter wavelength to fit into the open space in the bottle. For each shorter wavelength that satisfies the same conditions, we need to add another half wavelength, adding an additional node and antinode so that there is still a node at the bottom and an antinode at the top of each pattern. Like the standing waves on a string, the patterns differ from each other by one-half wavelength. Unlike the standing waves on strings, the higher harmonics do not simply occur for every integer multiple of the half wavelength. Instead, the harmonics occur for odd integer multiples of one-quarter wavelength:

$$\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots .$$

Only the first two patterns are relevant for this problem. If the length of air in the bottle is ℓ, each pattern must fit into that length. The first two patterns must satisfy

$$\ell = \frac{\lambda}{4} \quad \text{and} \quad \ell = \frac{3\lambda}{4}$$

$$\lambda = 4\ell \quad \text{and} \quad \lambda = \frac{4\ell}{3}.$$

The resulting frequencies are

$$f = \frac{c}{\lambda} = \frac{c}{4\ell} \quad \text{and} \quad \frac{3c}{4\ell}.$$

PRACTICE

We must determine these frequencies for the range of lengths available.

❸ **EXECUTE PLAN** The minimum length of air in the bottle occurs before we take the first sip: It is the difference between the height of the bottle and the original height of the liquid. For this case we obtain frequencies

$$f_1 = \frac{c}{4\ell} = \frac{343 \text{ m/s}}{4(3.0 \times 10^{-2} \text{ m})} = 2.9 \times 10^3 \text{ Hz}$$

$$f_3 = \frac{3c}{4\ell} = 3f_1 = 8.6 \times 10^3 \text{ Hz.}$$

Any longer length ℓ will result in smaller frequencies, so the maximum frequency we can attain is just above 8500 Hz. The lower frequency limit will occur for the greatest value of the air column—that is, when the bottle is empty:

$$f_1 = \frac{c}{4\ell} = \frac{343 \text{ m/s}}{4(0.23 \text{ m})} = 3.7 \times 10^2 \text{ Hz}$$

$$f_3 = 3f_1 = 1.1 \times 10^3 \text{ Hz.}$$

The lowest possible frequency for these conditions is thus just over 370 Hz, giving a range of

$$370 \text{ Hz} < f < 8600 \text{ Hz.} \checkmark$$

❹ **EVALUATE RESULT** Many of us have produced tones by blowing across the mouth of a bottle, so it is reasonable that we obtain frequencies that can be heard. Both ends of this frequency range are well within the range of human hearing, although the upper limit is a pretty high note, perhaps reminiscent of a piccolo. Experience also tells us that longer bottles produce deeper notes, in agreement with our results.

Guided Problem 17.4 Bottle orchestra

Preparing to watch a ballgame on television, you raid the refrigerator for two 0.23-m-tall bottles of your favorite beverage for yourself and your roommate. After a few minutes the game is boring, so your roommate begins blowing across the top of his bottle. That looks like fun, so you decide to try it, too. Each of you has a bottle with a slightly different level of liquid, but the depth of liquid in each bottle is roughly 10^2 mm. What is the minimum difference in the heights of the liquid in the two bottles that will allow you to hear two different musical notes when you and your roommate "play" your bottles simultaneously at their fundamental frequencies?

❶ **GETTING STARTED**

1. If the two bottles produce notes whose frequencies differ by just a few Hz, what pattern of sound do you expect to hear?
2. What is the lowest frequency note that humans can hear?
3. How is the fundamental frequency related to the depth of the liquid in the bottle?

❷ **DEVISE PLAN**

4. Given the height of the bottle and the depth of the liquid, can you determine the frequency of the sound that forms a fundamental standing wave in the air in the bottle?
5. Examine Eq. 17.11, the immediate precursor to the beat equation (Eq. 17.12). Is there any difference in principle between the sine and cosine terms? Figure 17.27, the superposed displacement curves of two waves of slightly different frequencies, may help you answer.
6. What minimum difference in frequencies is needed so that either the sine or cosine term in Eq. 17.11 might represent a note that can be detected by the human ear?
7. Relate this minimum frequency difference to the minimum difference in the depths of liquid in the two bottles.

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

8. Is your result plausible? What are the two frequencies that can be heard if the average depth of liquid is 100 mm and the difference in the depths is your computed value?
9. You might try this experimentally, but realize that you may not have perfect hearing.

Worked Problem 17.5 Sharp cruising

Driving down a quiet street on a summer day with the car windows down, you see a convertible approaching from the opposite direction, with the driver playing her car stereo loudly. Having a well-trained musician's ear, you notice that the song you hear is in a key that is one semitone (the interval between adjacent notes on a piano) above the key in which the song is typically played. You also know that to shift any note by one semitone (from C to C-sharp, for instance, or from E-flat to E), the frequencies must be related by a factor of about 1.06. Assuming that both drivers are traveling at about the same speed, what is this speed?

❶ **GETTING STARTED** You know that if you are moving toward a stationary sound source, the frequency you hear is higher than the frequency you would hear if you were standing still. The same is true if a moving source approaches you, so both motion effects should contribute to a higher frequency. Thus we need to determine the common speed that gives us $f_o/f_s = 1.06$. The use of the word *about* in the problem statement should cause us to reduce the number of significant digits in our answer from three to two.

❷ **DEVISE PLAN** For a moving observer who is approaching a moving source, the change in frequency is given by Eq. 17.21:

$$\frac{f_o}{f_s} = \frac{c \pm v_o}{c \pm v_s}.$$

Under the assumption that the speeds of the two cars are equal, we can substitute $v = v_o = v_s$. We also must decide which signs to use, so we will pick the signs that produce a higher observed frequency due to motion of both the source and observer. Then, using the known speed of sound in air, we can solve for the single unknown v.

❸ **EXECUTE PLAN**

$$\frac{f_o}{f_s} = 1.06 = \frac{c + v}{c - v}$$

$$1.06(c - v) = c + v$$

$$0.06c = 2.06v$$

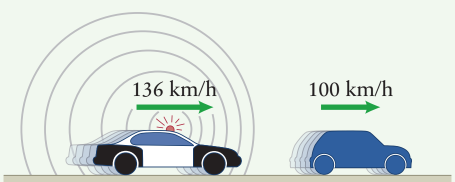
$$v = \frac{0.06c}{2.06} = \frac{0.06(343 \text{ m/s})}{2.06} = 10 \text{ m/s.} \checkmark$$

❹ **EVALUATE RESULT** The positive sign for v is consistent with our choices for signs in the numerator and denominator; a negative value for v would indicate some error in our thinking. This speed is reasonable for cars that are approaching each other on a quiet, therefore likely suburban, street (10 m/s = 22 mi/h).

Guided Problem 17.6 Cut to the chase

A motorist driving at 100 km/h is overtaken by a police car, siren blaring, chasing down a speeder (Figure WG17.2). The police car is moving at 136 km/h. What frequency does the motorist hear for the siren (*a*) before and (*b*) after he is passed if the frequency in the police car's reference frame is 1526 Hz?

Figure WG17.2



❶ **GETTING STARTED**

1. You have a sketch of the situation before the police car overtakes the motorist, so draw a sketch showing the situation after the police car has passed the motorist.
2. Can part *a* and part *b* be solved separately?
3. Keep in mind that both cars are moving relative to the air.

❷ **DEVISE PLAN**

4. What is the motional relationship between source and observer before the police car passes the motorist? After the police car passes?
5. Which Doppler equation is appropriate for each part of the motion?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

6. Are your answers plausible? Does the frequency change the way you expect it to (that is, compare your answers with the general trend of frequency shifts when the source is approaching or receding)?

PRACTICE

Worked Problem 17.7 Leaving a wake

A speedboat travels across the water of a shallow lake at 55 km/h, creating behind it a wake that spreads out from the line that bisects the boat along the boat's direction of travel. You estimate that the wake moves 1.0 m perpendicular to the bisector for every 3.0 m of forward motion of the boat. (*a*) How fast do the surface waves travel on the water? (*b*) At what water equivalent of a Mach number is the boat traveling?

❶ **GETTING STARTED** A wake is a shock wave that forms because the speed of the boat on the water surface is greater than the speed of the waves on the water surface. We're asked to determine (*a*) what this wave speed is and (*b*) how it compares with the speed of the boat.

❷ **DEVISE PLAN** Equation 17.22 relates the angle of a shock wave to the speed of the waves that create the shock wave and to the source speed:

$$\sin\theta = \frac{c}{v_s}.$$

We should be able to calculate the angle θ of the shock wave from the given distance information, and we know the boat speed v_s, so we can determine the wave speed c.

Mach numbers relate object speed to wave speed in a given medium. Once we have the wave speed, therefore, we can determine the "water Mach number" for the boat.

❸ **EXECUTE PLAN**
(*a*) The wake moves 1 m sideways for each 3-m forward motion of the boat, so the angle of the wake to the bisector of the boat should be given by

$$\tan\theta = \frac{1.0\ \text{m}}{3.0\ \text{m}} = 0.33 \qquad \theta = 18.4°.$$

Solving Eq. 17.22 for the wave speed gives

$$c = v_s \sin\theta = (55\ \text{km/h})\sin 18.4° = 17.4\ \text{km/h} = 4.8\ \text{m/s}. ✔$$

(*b*) Mach number is defined as the ratio of the speed of an object in a given medium to the speed of waves in that medium. The water Mach number is therefore

$$\frac{v_s}{c} = \frac{55\ \text{km/h}}{17.4\ \text{km/h}} = 3.2. ✔$$

❹ **EVALUATE RESULT** The answer to part *a* indicates that water waves spread across the surface at about 15 ft/s, a value that is not unreasonable. Part *b* shows that it is much easier to break the "water barrier" with a speedboat than to break the "sound barrier" with a jet airplane!

Guided Problem 17.8 Shocked, I say—shocked

A high-speed jet is flying across the Atlantic Ocean at an altitude of 15 km. It passes over a ship, and 34 s after it passes directly overhead, a sonic boom is heard by the ship's passengers. At what Mach number and speed is the jet flying?

❶ **GETTING STARTED**
1. Start with a diagram of the situation showing the relevant distances and the shock wave angle θ.
2. Do you need to make any assumptions?

❷ **DEVISE PLAN**
3. What information can help you determine the required unknowns?
4. How is the angle of the shock wave related to the speed of the jet?
5. Which equations allow you to express the unknown quantities in terms of known ones?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**
6. Can a jet travel this fast? If not, go back and check your work.

Answers to Review Questions

1. (1) The amplitude of two- and three-dimensional waves decreases as they propagate, even in the absence of energy dissipation. (2) Some interference effects that are not observed in one-dimensional waves occur in two- and three-dimensional waves.
2. Dissipation of the wave's energy leads to a decrease in amplitude as the wave travels, but even without dissipation, the dimensionality of the wave is important. Amplitude decreases as $1/\sqrt{r}$ in a two-dimensional wave and as $1/r$ in a three-dimensional wave.
3. The pipe circumference and thus its cross-sectional area are unchanged all along the pipe. Therefore the wave's amplitude remains constant because there is no change in the area over which the wave's energy is spread as the wave propagates from one end of the pipe to the other.
4. Sound waves in air are longitudinal.
5. The range is 20 Hz to 20 kHz.
6. Squeeze a plastic soda bottle that contains only air and has its cap screwed on tightly. You can decrease the volume only slightly because the compressional force you exert on the walls is transmitted to the air molecules inside. The air compresses elastically, exerting a force on the walls in the opposite direction.
7. Maximum and minimum medium densities occur at locations where the medium displacement is zero. Therefore the locations of maximum and minimum medium density are one-quarter wavelength away from (90° out of phase with) the locations of maximum and minimum displacement.
8. No. The sound is louder only if at P the interference between the two waves is constructive. If it is destructive, the sound when the two waves pass P is softer than the sound of either wave passing P alone.
9. No. A given position in the hall is dead for only certain frequencies because the relative phase of two waves coming to that position is determined by the ratio $\Delta r/\lambda$. Thus destructive interference occurs for only certain frequencies.
10. The number of nodal lines increases.
11. Yes. Any number of sources can be arranged such that wave maxima and minima overlap at certain locations. For example, a set of closely spaced collinear point sources creates constructive interference along a line perpendicular to the sources and destructive interference in other directions.
12. Huygens' principle is the model that considers wavefronts as a collection of closely spaced, coherent point sources, each of which creates new coherent waves that travel in all directions but, when superposed, cancel in all directions other than forward. The result can be used to sketch the shapes of additional forward-propagating wavefronts.
13. High frequencies correspond to short wavelengths, and wavelengths comparable to the size of the insect are needed for reliable location by reflected waves.
14. The amount of spreading depends on the size of the wavelength relative to the gap width: The longer the wavelength, the greater the extent to which the waves spread out. Thus in (*a*) very little spreading occurs, in (*b*) the spreading is quite noticeable, and in (*c*) the waves spread out in all directions, producing a pattern similar to that of a point source.
15. If we ignore dissipation, the intensity varies inversely with the square of distance from the source: $I \propto 1/r^2$ (Eq. 17.2). This is because the energy of the waves is constant regardless of the distance from the source, but the area over which this energy is distributed increases as the distance from the source increases.
16. The intensity I decreases by a factor of $(10)^2 = 100$ because of its $1/r^2$ dependence: $I = P/4\pi r^2$ (Eq. 17.2). Equation 17.5 for intensity level, $\beta = (10\text{ dB})\log(I/I_{\text{th}})$, tells you that there is a 10-dB drop in intensity level β for each power-of-ten reduction in intensity I, and so the drop in intensity level is 20 dB.
17. In the case of ripples on a pond, as each ripple expands, the energy is spread over the circumference of a circle (formula $2\pi r$). This gives intensity units of W/m. An emergency warning siren is typically mounted on a tall pole, which allows the sound energy to spread out in all directions. In this case the energy carried by each wavefront is spread across the surface of a sphere (formula $4\pi r^2$). The intensity units are therefore W/m^2.
18. The beat frequency is the difference between the two wave frequencies.
19. No. As the tuning fork frequency and the piano key frequency approach equality, which is the tuner's aim, the beat frequency decreases.
20. The frequency is higher because of the Doppler effect for a moving source.
21. The ratio is greater for source approaching observer. Here's why. The source moving toward the stationary observer means that the observed frequency is higher than the emitted frequency and thus $f_o/f_s > 1$. Because $v_o = 0$, Eq. 17.21 becomes $f_o/f_s = c/(c \pm v_s)$. In order to have the fraction be greater than 1, you use a minus sign in the denominator: $f_o/f_s = c/(c - 0.250c) = 1.33$.

    The observer moving toward the stationary source means again $f_o > f_s$ and thus again $f_o/f_s > 1$. Because in this case $v_s = 0$, Eq. 17.21 becomes $f_o/f_s = (c \pm v_o)/c$. In order to have the fraction be greater than 1, you use a plus sign in the numerator: $f_o/f_s = (c + 0.250c)/c = 1.25$.
22. No. Equation 17.22, $\sin\theta = c/v_s$, contains no information about frequency.
23. As v_s in Eq. 17.22, $\sin\theta = c/v_s$, increases, $\sin\theta$ decreases, which means that θ decreases and the bow wave narrows.

Answers to Guided Problems

Guided Problem 17.2 (*a*) 0.90 mm; (*b*) $\frac{1}{15}$
Guided Problem 17.4 about 4 mm
Guided Problem 17.6 (*a*) 1.58×10^3 Hz; (*b*) 1.49×10^3 Hz
Guided Problem 17.8 Mach number 1.6 or 5.5×10^2 m/s

PRACTICE

Guided Practice by Chapter

18 Fluids

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

18.1 Forces in a fluid

1. What are the three basic types of stress? Describe the forces involved in each type.
2. What is the difference in how nonviscous liquids and viscous liquids react to shear stress? Give an example of each type of liquid.
3. Describe the difference in how gases and liquids respond to bulk stress.
4. What is the relationship between force and pressure? How are the two different from each other?
5. What effect does Earth's gravitational force have on the pressure in a fluid at rest in a container?
6. How does a pressure change applied to the top of a column of liquid held in a closed container affect the pressure throughout the liquid?

18.2 Buoyancy

7. What is the relationship between Earth's gravitational force and the buoyant force exerted by a fluid on a fully or partially submerged object?
8. What determines whether an object sinks or floats in a liquid?
9. How can an object composed of a material that typically sinks in water be made to float on water?

18.3 Fluid flow

10. How do the properties of streamlines differ for laminar and turbulent flow?
11. What is *viscosity*?
12. What factors determine whether the flow of fluid past an object is laminar or turbulent?
13. What does it mean to streamline an object, and what benefit is there to doing so?
14. What does it tell you if the streamlines for a fluid in laminar flow are closer together in one part of the flow and farther apart in another part of the flow?
15. What is the connection between flow speed and pressure in a moving fluid when the flow is laminar?

18.4 Surface effects

16. Describe the interaction between two particles of fluid as a function of their separation distance.
17. Describe the difference between the forces experienced by particles in the interior of a liquid and the forces experienced by particles at the liquid surface that creates the effect called surface tension.
18. How does the relationship between surface tension in a liquid surface and surface shape differ from the relationship between tension in an elastic membrane enclosing a fluid and membrane shape?
19. What factor determines whether or not a liquid wets a solid surface with which it is in contact? (Ignore the presence of any surrounding gas.)
20. What determines whether the meniscus formed by a liquid in a capillary tube is concave or convex?

18.5 Pressure and gravity

21. What are three effects that contribute to the pressure in all fluids? What is a fourth effect that contributes to the pressure only in liquids?
22. Approximately what is the difference in air pressure between your feet and your head when you're standing at sea level?
23. What is hydrostatic pressure, and how does it vary in a liquid at rest?
24. How does an object's average mass density determine whether the object floats, sinks, or hangs motionless when placed in a fluid?

18.6 Working with pressure

25. What is *gauge pressure,* and what is its relationship to atmospheric pressure?
26. Describe the main components of a hydraulic system and how the system can multiply a force magnitude.

18.7 Bernoulli's equation

27. For laminar flow of a nonviscous liquid through a tube in which the cross-sectional area at point 1 is different from the cross-sectional area at point 2, how does the mass of liquid flowing past point 1 at a given instant differ from the mass of liquid flowing past point 2 at that instant?
28. An aqueduct of constant diameter is at some points along its length very close to the ground and at other points far above the ground. If the flow of water in the duct is laminar, how does the water pressure at some location 1 along the duct length compare with the water pressure at some other location 2 along the length?
29. A horizontal aqueduct is constructed of pipes that have different diameters. If the water flow in the duct is laminar, how does the water pressure in a small-diameter portion compare with the water pressure in a large-diameter portion?

18.8 Viscosity and surface tension

30. What is a velocity gradient in a fluid?
31. How does the temperature dependence of viscosity in liquids compare with the temperature dependence of viscosity in gases?
32. For any fluid moving through a tube, what does Poiseuille's law tell us about the relationship among viscosity, tube size, and volume flow rate?
33. How is surface tension related to the energy associated with the surface of a liquid?
34. How does pressure differ across a curved fluid surface that is under surface tension?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The mass of the atmosphere [C, N, X]
2. The difference between your mass density and the mass density of water [A, G, Y]
3. The pressure at the lowest point in the ocean [U, F, Y]
4. The decrease in air pressure as you move from ground level to the top of a skyscraper [J, V]
5. The magnitude of the force required to separate two 0.50-m-diameter rigid hemispheres when the air between them has been pumped out [T, I, O]
6. The magnitude of the force required to open a passenger airplane emergency door at cruising altitude [D, Q]
7. The upward force exerted on the roof of a single-family house by a hurricane-force wind blowing over it [S, E, L]
8. The magnitude of the horizontal force exerted on a passenger car traveling at highway speed when a truck traveling in the opposite direction passes close by [P, B]
9. The volume flow rate of water moving through a garden hose 30 m long [W, Z, K, H]
10. The maximum mass of a 70-mm wire that can be supported by a soap film [R, M]

Hints

A. When you float vertically in a pool with only your head above water, what fraction of your body volume remains unsubmerged?
B. What is the area of the side of the car?
C. What is the surface area of Earth?
D. What is the pressure difference between the air inside and outside the airplane?
E. What is the difference between the air pressure just inside and just outside the roof?
F. How does pressure vary with depth?
G. What is the relationship between the mass of your whole body (submerged portion plus above-water portion) and the mass of the water displaced by the submerged portion?
H. What is the hose radius?
I. What is the relationship between the magnitude of the force exerted on the sphere surface and the pressure difference from Hint T?
J. What is the height of a skyscraper?
K. What pressure difference (inside versus outside the hose) is needed to squirt the water to the appropriate height?
L. What is the area of the roof?
M. What fraction of the surface tension of water is appropriate for soap film?
N. What is atmospheric pressure at sea level?
O. What is the equatorial cross-sectional area of the sphere?
P. What is the difference between the speed of the air passing the driver's side and the speed of the air passing the passenger side?
Q. What is the area of the door?
R. What forces must cancel?
S. What is the wind speed?
T. What is the pressure difference between the inside and outside of the sphere?
U. How deep below sea level is the bottom of the Mariana Trench, which is the deepest recorded part of the ocean?
V. What is the mass density of air?
W. What water viscosity value is appropriate in this situation?
X. How is the force exerted by the atmosphere on all of Earth's surface related to the atmosphere's mass?
Y. What is the mass density of water?
Z. To what height will the water rise if the hose end is held vertically?

Key (all values approximate)

A. 8×10^{-2}; **B.** 3 m^2; **C.** 5×10^{14} m^2; **D.** 5×10^4 Pa; **E.** 8×10^2 Pa; **F.** Eq. 18.17: $P = P_{atm} + \rho gh$; **G.** these two masses are equal; **H.** 1×10^1 mm; **I.** $F = (P_{in} - P_{out})\pi R^2$, where R is the sphere radius; **J.** 3×10^2 m; **K.** 0.1 atm or 1×10^4 Pa; **L.** 2×10^2 m^2; **M.** $\frac{1}{3}$ of 7×10^{-2} N/m; **N.** 1×10^5 Pa; **O.** 2×10^{-1} m^2; **P.** 3×10^1 m/s; **Q.** 1 m^2; **R.** gravitational and surface tension forces; **S.** 4×10^1 m/s; **T.** 1×10^5 Pa; **U.** 1×10^1 km; **V.** 1 kg/m^3; **W.** 1×10^{-3} Pa • s, the value at 20°C; **X.** that force is the atmosphere's mass times the acceleration due to gravity; **Y.** 1×10^3 kg/m^3; **Z.** 1 m or less without a nozzle

Worked and Guided Problems

Procedure: Working with pressure in liquids at rest

The branch of physics that deals with pressure in a liquid at rest is called *hydrostatics*. The pressure in a liquid at rest is determined by gravity and by what happens at the boundary of the liquid. To determine the pressure in such liquids:

1. Begin by making a sketch showing all the **boundaries** and identifying all the factors that affect pressure: pistons, gases at surfaces open to the atmosphere, and so on. Note the known vertical heights of liquid surfaces, the areas of these surfaces, the surface areas of pistons, and the liquid mass densities.
2. Determine the **pressure at each surface.** The pressure at a liquid surface open to the air is equal to atmospheric pressure P_{atm}. The pressure at a liquid surface bordering a vacuum is zero: $P = 0$. The pressure at a liquid surface open to a gas other than the atmosphere is equal to the pressure in the gas: $P = P_{\text{gas}}$. The pressure at a liquid surface that is in contact with a solid, such as a piston, is $P = F^c_{\text{sl}}/A$, where F^c_{sl} is the magnitude of the force exerted by the solid on the liquid and A is the area over which that force is exerted.
3. Use **horizontal planes.** The pressure is the same at all points on a horizontal plane in a connected liquid. The pressure difference between two horizontal planes 1 and 2 is given by $P_1 = P_2 + \rho g d$ (Eq. 18.7), where d is the vertical distance between the horizontal planes and 1 is below 2.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 18.1 Mars balloon

It has been suggested that we could explore the surface of Mars by sending an inflated balloon fitted with an instrument package to hover above the surface. To see if this suggestion is feasible, calculate the radius needed in order for such a balloon (supporting only itself and the gas filling it—no instruments) to hover just above the Martian surface, where the mass density of the atmosphere is 0.020 kg/m^3. Assume the balloon is filled with hydrogen gas (at Martian atmospheric pressure, $\rho_{\text{hy}} = 9.0 \times 10^{-4} \text{ kg/m}^3$) and is made of plastic film that has a mass density of $\rho_{\text{film}} = 1.2 \times 10^3 \text{ kg/m}^3$ and is 0.30 mm thick.

❶ GETTING STARTED We can think of the Martian atmosphere as a fluid that supports the gas-filled balloon by exerting a buoyant force on it. For the balloon to hover, the forces exerted on it must balance, which means that the magnitude of the buoyant force must be equal to the magnitude of the gravitational force exerted by Mars on the balloon and on the hydrogen gas.

❷ DEVISE PLAN The magnitude of the buoyant force exerted by the atmosphere on the balloon is equal to the magnitude of the gravitational force exerted by Mars on a volume of its atmosphere that is equal to the volume of the balloon. This buoyant force magnitude is given by Eq. 18.12: $F^b_{\text{atm,bal}} = \rho_{\text{atm}} V_{\text{disp}} g = \rho_{\text{atm}} V_{\text{bal}} g$, where g is the acceleration due to gravity on Mars. This force magnitude must be equal to the magnitude of the gravitational force exerted on the hydrogen-filled balloon, which is

$$F^G_{\text{planet,bal}} = m_{\text{bal}} g + \rho_{\text{hy}} V_{\text{bal}} g = (\rho_{\text{film}} V_{\text{film}} + \rho_{\text{hy}} V_{\text{bal}}) g.$$

Note the different subscripts on V: V_{film} is the volume not of the inflated balloon but rather of the plastic film of which the balloon is made. We assume that the balloon is a sphere of radius R that has walls of thickness $t \ll R$. The volume of the film is therefore the surface area of the sphere times the film thickness: $V_{\text{film}} = 4\pi R^2 t$. The volume of the atmosphere displaced by the balloon is equal to the balloon volume, $V_{\text{bal}} = \frac{4}{3}\pi R^3$.

❸ EXECUTE PLAN Equating the buoyant and gravitational forces exerted on the balloon gives

$$F^b_{\text{atm,bal}} = F^G_{\text{planet,bal}}$$

$$\rho_{\text{atm}} V_{\text{bal}} g = (\rho_{\text{film}} V_{\text{film}} + \rho_{\text{hy}} V_{\text{bal}}) g$$

$$(\rho_{\text{atm}} - \rho_{\text{hy}})\left(\tfrac{4}{3}\pi R^3\right) = \rho_{\text{film}}(4\pi R^2 t).$$

Solving for R gives

$$R = \frac{3\rho_{\text{film}} t}{\rho_{\text{atm}} - \rho_{\text{hy}}}$$

$$= \frac{3(1.2 \times 10^3 \text{ kg/m}^3)(0.00030 \text{ m})}{(0.020 \text{ kg/m}^3) - (9.0 \times 10^{-4} \text{ kg/m}^3)} = 57 \text{ m. } \checkmark$$

❹ EVALUATE RESULT A radius of 57 m means that the balloon diameter is about 110 m, 10% longer than a football field. This is big, but given the low mass density of the Martian atmosphere not unreasonable. In addition, the surface of Mars is mostly flat, so a large balloon may be practical for such a project. Of course, to carry an instrument package the balloon would have to be somewhat larger, depending on the mass of the instruments.

Note that we do not need to know the value of g because this variable cancels out in our algebra.

Guided Problem 18.2 Trial balloon

Before sending the balloon discussed in Worked Problem 18.1 to Mars, it would be necessary to test it on Earth. At sea level, the mass density of air is 1.286 kg/m^3 and that of hydrogen gas is 0.090 kg/m^3. (The testers might need to use helium, however, because of the danger posed by the oxygen in our atmosphere; at sea level the mass density of helium is 0.179 kg/m^3.) What must the radius of such a test balloon be if it is to be able to hover above Earth's surface? Use information as needed from Worked Problem 18.1.

❶ GETTING STARTED

1. What forces must balance in order for the balloon to hover?
2. How can you determine each of these forces?

❷ DEVISE PLAN

3. Which parts of the approach used in Worked Problem 18.1 are applicable?
4. Which values require replacement?

❸ EXECUTE PLAN

5. Repeat the calculation used in Worked Problem 18.1 for the balloon on Earth, using hydrogen as the inflating gas.
6. Using helium for safety, repeat the calculation.

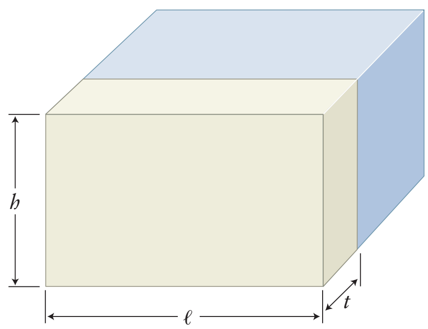
❹ EVALUATE RESULT

7. Is your answer consistent with the value in Worked Problem 18.1 for the balloon's radius?
8. Is your result surprising? Is the use of helium feasible for the Earth test?

Worked Problem 18.3 Dam the canyon

You wish to design a dam that remains upright and in place when holding back a 30-m depth of water. The opening is a canyon with a rectangular cross section that is 50 m across and 30 m top to bottom. You do not want to rely on braces or connections with the canyon walls to support the dam, and so that leaves only normal and frictional forces to keep the dam in place. The normal and frictional forces exerted on the dam base will greatly exceed those exerted on the dam sides, so ignore these forces on the dam sides. The most straightforward solution seems to be a rectangular concrete block of height $h = 30$ m, horizontal length $\ell = 50$ m, and thickness t (Figure WG18.1). What minimum thickness of high-density concrete (mass density 5200 kg/m^3) is required for this design? Assume a coefficient of static friction of $\mu_s = 0.60$ between ground and concrete.

Figure WG18.1

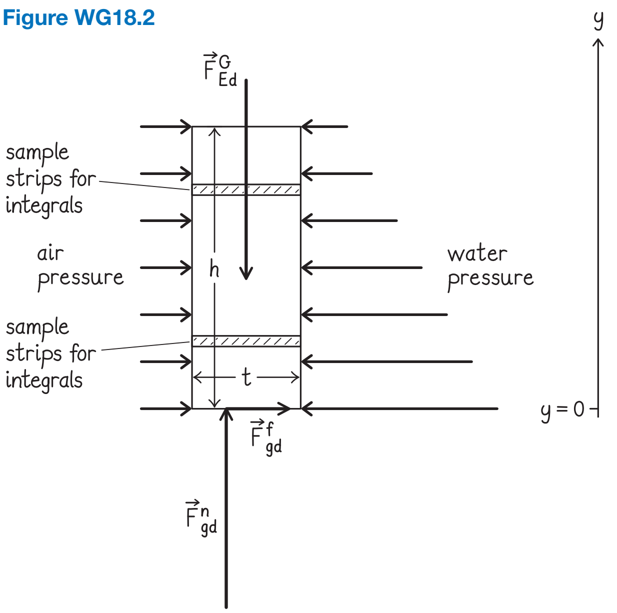


❶ GETTING STARTED Example 18.6 obtained an expression for the magnitude of the force exerted by water on a dam. The difference is that this time we need the vector sum of all forces exerted on the dam, which means we must include the force exerted by the air on the front of the dam. The mass of the dam determines the normal force, and the normal force determines the frictional force, which must balance the vector sum of the horizontal forces exerted on the dam by the water and air. Once we have an expression involving the dam mass, we can use it in combination with the concrete density to determine the volume of the dam. Because we know height and length, we can use the volume expression to calculate the required minimum thickness.

PRACTICE

There is a catch, however: The dam might not simply be pushed horizontally by the vector sum of the forces; it could tip over instead! Thus we need to consider torques as well as forces. The side-view extended free-body diagram in Figure WG18.2 shows the forces exerted on the dam by the air and water as "distributed loads." That is, these forces are represented not by single arrows, but by a distribution of arrows intended to illustrate that they are applied across entire dam surfaces. In order to sum such forces (or the torques they cause) it will be necessary to mentally divide the dam into a large number of thin horizontal strips, each small enough that these pressure forces do not vary appreciably across its (infinitessimal) height. Two such strips are illustrated in Figure WG18.2, but our calculations will rely on many (an infinite number of) strips that span the dam from bottom to top. Note that the magnitude of the force exerted by the air is the same for every strip but the magnitude of the force exerted by the water increases with depth.

Figure WG18.2



2 DEVISE PLAN The contact force exerted by the water on each horizontal strip is

$$dF^c_{ws} = P(y)dA = P(y)\ell dy,$$

where $P(y)$ is the pressure at height y and $dA = \ell dy$ is the area of a horizontal strip of length ℓ and height dy. We calculate the force exerted by the water on the dam by integrating this expression from height 0 at the base to height h at the top (Example 18.6):

$$F^c_{wd} = \ell h\left(P_{atm} + \tfrac{1}{2}\rho_w gh\right).$$

The force exerted by the air on the dam is easier to compute because atmospheric pressure does not vary noticeably over the 30-m height:

$$F^c_{ad} = P_{atm}A = P_{atm}\ell h.$$

The forces exerted by the water and air oppose each other, and so their vector sum is

$$F^c_{wd} - F^c_{ad} = \ell h(P_{atm} + \tfrac{1}{2}\rho_w gh) - \ell h P_{atm} = \tfrac{1}{2}\rho_w \ell g h^2. \quad (1)$$

This horizontal force must be balanced by the frictional force exerted by the ground on the dam, which is

$$F^f_{gd} = \mu_s F^n_{gd} = \mu_s mg = \mu_s \rho_{concrete} V_d g.$$

We now have an expression that contains $\rho_{concrete}$ and V_d, so we can compute the minimum volume of the dam, which will lead to a minimum thickness.

The thickness we obtain via force analysis may not be sufficient to prevent rotation, though, so we must check the torques caused by the forces. Suppose the dam were to tip over. If it were to rotate, it would rotate about the bottom edge of the side opposite the water, and so we choose that edge as our axis of rotation. If the dam were just about to rotate, the normal force exerted by the ground, which is usually distributed uniformly across the bottom face, would concentrate at the rotation edge, the last line of contact with the ground. In this just-about-to-rotate case, the line of action of both the normal force and the frictional force would pass through the rotation axis and thus contribute zero torque. Thus, to keep the dam from rotating, the torque caused by the gravitational force exerted on the just-about-to-rotate dam would have to cancel the torque caused by the forces exerted by the water and the air.

To compute the contribution the gravitational force makes to the torque, we must calculate the distance from the dam's center of mass, which is the point of application of this force, to the rotation axis, so that we know the lever arm for the gravitational force. To compute the contribution the forces exerted by the air and water make to the torque, we must integrate to sum the torque contributions from all horizontal strips:

$$d\vec{\tau} = \vec{r} \times d\vec{F}.$$

Whichever computation—force or torque—yields the greater minimum thickness value is our desired result because both translational equilibrium and rotational equilibrium are required.

❸ **EXECUTE PLAN** The gravitational force exerted on the dam is

$$F^G_{\text{Ed}} = mg = \rho_{\text{concrete}} V_{\text{d}} g = \rho_{\text{concrete}} \ell thg. \tag{2}$$

The magnitude of the normal force equals this gravitational force, so the frictional force is

$$F^{\text{f}}_{\text{gd}} = \mu_s F^n_{\text{gd}} = \mu_s \rho_{\text{concrete}} \ell thg.$$

For stability against translational motion, this frictional force must balance the two contact forces due to the water pressure and air pressure:

$$F^{\text{f}}_{\text{gd}} = F^{\text{c}}_{\text{wd}} - F^{\text{c}}_{\text{ad}}$$

$$\mu_s \rho_{\text{concrete}} \ell thg = \tfrac{1}{2} \rho_{\text{w}} \ell g h^2,$$

where the term on the right is from Eq. 1. Solving for the dam thickness, we get

$$t = \frac{\rho_{\text{w}} h}{2\mu_s \rho_{\text{concrete}}}. \tag{3}$$

Now consider torques about our chosen rotation axis, the bottom edge on the air side of the dam. At the instant the dam is about to rotate, the torque caused by the gravitational force tends to rotate the dam clockwise in Figure WG18.2, and the force exerted by the water on the base of the dam tends to rotate the dam counterclockwise. These two terms must cancel to prevent rotation. Beginning with the gravitational force, notice that the lever arm for this vertical force is horizontal and is just one-half the thickness of the dam. Using our Eq. 2 expression for F^G_{Ed} thus yields

$$\tau_{\text{Ed}} = F^G_{\text{Ed}}(\tfrac{1}{2}t) = \tfrac{1}{2}\rho_{\text{concrete}} t^2 \ell hg.$$

The lever arm for the force exerted by the water is vertical, but its length varies from one horizontal strip to the next. At any height y, the force exerted by the water is the vector sum of the force exerted by the water and the opposing force exerted by the air at that height. The combined pressure at y is $P_{\text{atm}} + \rho_{\text{w}} gy - P_{\text{atm}} = \rho_{\text{w}} gy$. The force exerted by the water at y is thus $dF^{\text{c}}_{\text{wd}} = PA = (\rho_{\text{w}} gy)(\ell\, dy)$. For a strip at any height y, the torque is thus

$$d\tau_{\text{wd}} = y dF^{\text{c}}_{\text{wd}} = y(\rho_{\text{w}} gy)(\ell\, dy) = \rho_{\text{w}} g y^2 \ell\, dy.$$

Performing the integration from bottom to top gives

$$\tau_{\text{wd}} = \int_0^h \rho_{\text{w}} g y^2 \ell\, dy = \rho_{\text{w}} g \ell \int_0^h y^2 dy$$

$$= \tfrac{1}{3}\rho_{\text{w}} g \ell h^3.$$

Setting these two torque magnitudes equal allows us to solve for the minimum thickness needed to prevent rotation:

$$\tau_{\text{Ed}} = \tau_{\text{wd}}$$

$$\tfrac{1}{2}\rho_{\text{concrete}} t^2 \ell hg = \tfrac{1}{3}\rho_{\text{w}} g \ell h^3$$

$$t^2 = \frac{2\rho_{\text{w}} h^2}{3\rho_{\text{concrete}}}$$

$$t = h\sqrt{\frac{2\rho_{\text{w}}}{3\rho_{\text{concrete}}}}. \tag{4}$$

Determining which thickness value is greater requires a numerical comparison. Equation 3 gives

$$t_{\text{translation}} = \frac{h}{2\mu_s}\frac{\rho_{\text{w}}}{\rho_{\text{concrete}}} = \frac{30\text{ m}}{2(0.60)}\frac{1000\text{ kg/m}^3}{5200\text{ kg/m}^3} = 4.8\text{ m},$$

and from Eq. 4 we get

$$t_{\text{rotation}} = h\sqrt{\frac{2\rho_{\text{w}}}{3\rho_{\text{concrete}}}} = (30\text{ m})\sqrt{\frac{2(1000\text{ kg/m}^3)}{3(5200\text{ kg/m}^3)}} = 11\text{ m.} ✔$$

The minimum thickness needed to keep the dam in place—neither translating nor rotating—is 11 m.

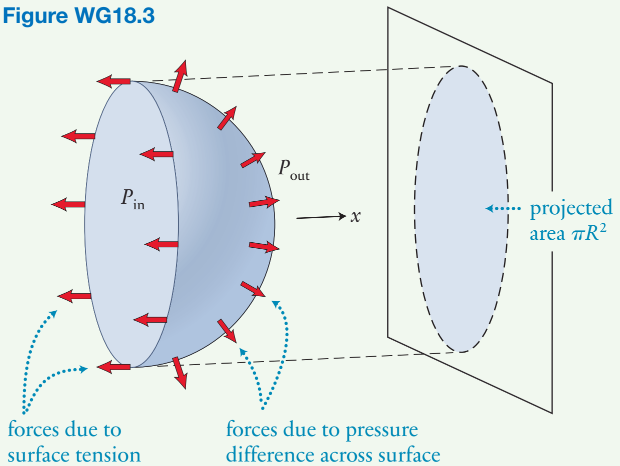
❹ **EVALUATE RESULT** The required minimum thickness is less than half the height, so the dam really would have the shape of a rectangular block. We know from experience that tipping over a tall, thin object is easier than tipping over a short, thick one, however, and so it is reasonable that the minimum thickness, though small enough to give us confidence in modeling the dam as having a rectangular cross section, is a good fraction of the height.

Most dams are not shaped like rectangular blocks because other shapes are more efficient (require less concrete). Shapes that are thicker at the base and narrower at the top, and arched shapes that allow significant support from surrounding materials (so that the dam itself need not bear the entire effort of holding back the water) are typically used. The wider base makes a greater lever arm for the gravitational force, enhancing the torque this force causes without much effect on the torque due to pressure.

Guided Problem 18.4 Under pressure

Show, by integration, that the force due to the pressure difference $P_{\text{in}} - P_{\text{out}}$ across the hemispherical surface of a liquid drop (Figure WG18.3) is equal to the pressure difference times the equatorial cross-sectional area.

Figure WG18.3



❶ **GETTING STARTED**

1. Examine the arrows illustrating the magnitude and direction of the forces exerted on the hemisphere at representative locations.
2. Why isn't the difference $P_{\text{in}} - P_{\text{out}}$ a function of location on the hemispheric surface? Have you had to make an assumption?
3. Consider symmetry in establishing whether certain force components cancel.

❷ **DEVISE PLAN**

4. Which choice of area elements is best? Hint: Consider slicing the hemisphere into thin vertical rings, each centered on the central horizontal axis of the hemisphere. Each ring has radius $R\sin\theta$ and width (tangent to the hemisphere) $R\,d\theta$, where R is the hemisphere radius and θ is the angle from the central axis to the ring. (This angle varies from 0 for the tiny ring closest to the central axis to 90° for the ring with the maximum circular cross section.)
5. Consider the horizontal and vertical components of the forces due to air pressure on one such ring. Which of these components cancel?
6. Express the force component along the horizontal symmetry axis as a function of the variables P_{in}, P_{out}, R, and θ. Call this the x component of the force due to the air pressure.

❸ **EXECUTE PLAN**

7. Have you found that the only remaining force component exerted on a ring is $dF_x = (P_{\text{in}} - P_{\text{out}})\cos\theta\,dA = (P_{\text{in}} - P_{\text{out}})\cos\theta(2\pi R\sin\theta R\,d\theta)$?
8. Integrate over the hemisphere, from $\theta = 0$ to $\theta = \pi/2$. Why isn't this integration from $\theta = 0$ to $\theta = \pi$?

❹ **EVALUATE RESULT**

9. Is the force you computed consistent with the Laplace equation for a sphere and with what you know about surface tension for water?

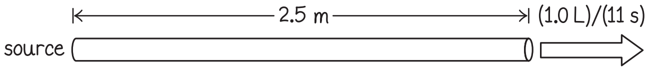
PRACTICE

Worked Problem 18.5 Plumber's dream

In a household water line, the pipes typically have an inside diameter of 15 mm. With the cold water tap turned on, it takes 11 s to fill a 1.0-L jar. The viscosity of water at 10 °C is 1.307×10^{-3} Pa · s. What is the change in water pressure as water at this temperature flows through a 2.5-m length of pipe? What is the greatest speed attained by the water?

1 GETTING STARTED We are given the flow rate for water at 10 °C moving through a certain length of pipe, the water's viscosity, and the pipe diameter and length. We must use this information to determine how much the pressure in the water changes along the given pipe length and what maximum speed the water has. We begin with a sketch showing this information (Figure WG18.4).

Figure WG18.4



The speed of a viscous fluid is not the same everywhere in a pipe, but depends on the distance from the center and is a maximum at the center (see Figure 18.57).

2 DEVISE PLAN. The time interval given for filling the 1.0-L jar tells us the flow rate Q of the water. Once we have Q, we can use Poiseuille's law, Eq. 18.47, to calculate the pressure difference $P_1 - P_2$ along the 2.5-m length of the pipe. From Eq. 18.44 we know that the speed of the water any radial distance r from the center of a pipe of radius R is $v_x = [(P_1 - P_2)/4\eta\ell](R^2 - r^2)$ and that the speed has its maximum when $r = 0$ (that is, the speed is greatest at the center of the pipe).

3 EXECUTE PLAN. Our first task is to convert the non-SI unit liters to the SI volume unit meters^3:

$$1.0\text{ L} \times \frac{1 \times 10^{-3}\text{ m}^3}{1\text{ L}} = 1.0 \times 10^{-3}\text{ m}^3.$$

The water flow rate is

$$Q = \frac{V}{\Delta t} = \frac{1.0 \times 10^{-3}\text{ m}^3}{11\text{ s}} = 9.09 \times 10^{-5}\text{ m}^3/\text{s}.$$

Solving Poiseuille's law (Eq. 18.47) for the pressure difference gives

$$\begin{aligned} P_1 - P_2 &= \frac{8\eta\ell Q}{\pi R^4} \\ &= \frac{8(1.307 \times 10^{-3}\text{ Pa}\cdot\text{s})(2.5\text{ m})(9.09 \times 10^{-5}\text{ m}^3/\text{s})}{\pi(7.5 \times 10^{-3}\text{ m})^4} \\ &= 0.24 \times 10^3\text{ Pa.}\ \checkmark \end{aligned}$$

To calculate the maximum speed, we set r equal to 0 in Eq. 18.44:

$$\begin{aligned} v_{\max} &= \frac{P_1 - P_2}{4\eta\ell}(R^2 - 0^2) \\ &= \frac{240\text{ Pa}}{4(1.307 \times 10^{-3}\text{ Pa}\cdot\text{s})(2.5\text{ m})}(7.5 \times 10^{-3}\text{ m})^2 \\ &= 1.0\text{ m/s.}\ \checkmark \end{aligned}$$

4 EVALUATE RESULT A pressure difference of 240 Pa is about 0.0024 atm, which is quite small and not significant in typical household pipes. A maximum water speed of 1.0 m/s seems reasonable as we observe water coming out of a home faucet.

PRACTICE

Guided Problem 18.6 Devastating flood

The 1889 Johnstown Flood in Pennsylvania was one of the most devastating in American history. Heavy rains caused a breach in a dam holding back a reservoir, releasing a wall of water that killed more than 2200 people. Researchers have calculated that the breach measured 13 m high by 90 m across and that the peak water discharge rate was greater than 8500 m^3/s. Determine the maximum speed of the water as it burst out of the breach and the maximum rate at which mass and energy gushed from the reservoir, with the given quantities accurate to two significant digits.

❶ GETTING STARTED

1. How are the maximum *speed,* maximum *mass* flow rate, and maximum *energy* flow rate related to the known maximum *volume* flow rate at which water left the reservoir?
2. Can the water be treated as incompressible? Nonviscous?

❷ DEVISE PLAN

3. There are lots of flow rates. Label them Q_{vol} for volume flow rate, Q_{mass} for mass flow rate, and Q_{energy} for energy flow rate.
4. Is Eq. 18.26, $Q = Av$, useful for determining the maximum water speed?
5. The water mass is $\rho_w V$, but how is V related to Q_{vol}?
6. Is the energy carried by the water as it moves through the breach its kinetic energy?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

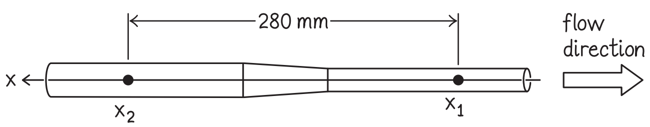
7. Is your maximum water speed reasonable? Remember that a large volume of water can pass through a huge opening in a short time interval, even at a relatively slow speed.
8. Compare the mass and energy flow rates with similar rates for something more familiar, such as the motion of cars on a freeway.

Worked Problem 18.7 Tapered flow

The inside diameter of a horizontal pipe narrows from 15 mm to 10 mm, and the pipe carries water that moves at 750 mm/s in the wide part of the pipe. Ignoring the very small viscosity of water, what is the difference between the water pressure at a position in the wide part and the water pressure at a position in the narrow part if the two positions are 280 mm apart? At which position is the water pressure higher?

❶ GETTING STARTED Figure WG18.5 shows the horizontal pipe with our two positions of interest marked x_1 in the narrow part and x_2 in the wide part, with the two separated by 280 mm. For fluid-flow problems, the first thing to consider is whether or not the flow is laminar because knowing this lets us know whether or not Bernoulli's equation and/or one of the continuity equations can be used. Because this is a nonviscous, incompressible fluid, laminar flow is likely at low speeds, so we can use these equations. Given that we are seeking a pressure difference and have diameter and speed information, using Bernoulli's equation looks like a good approach. Because our fluid is nonviscous and incompressible, we can use the Eq. 18.24 version of the continuity equation.

Figure WG18.5



❷ DEVISE PLAN The Eq. 18.24 version of the continuity equation relates the speeds at the two positions. Using d for the diameters, we have

$$A_1 v_1 = A_2 v_2$$

$$\pi\left(\frac{d_1}{2}\right)^2 v_1 = \pi\left(\frac{d_2}{2}\right)^2 v_2$$

$$v_1 d_1^2 = v_2 d_2^2. \quad (1)$$

The heights of these two positions are identical because the tube is horizontal, so Bernoulli's equation with $y_1 = y_2$ becomes

$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2$$

$$P_2 - P_1 = \tfrac{1}{2}\rho(v_1^2 - v_2^2). \quad (2)$$

We need to determine only the pressure difference $\Delta P = P_2 - P_1$ between the two positions, not the absolute pressure at each position. Thus we have two equations with two unknowns, ΔP and v_1, and so we can determine ΔP.

PRACTICE

❸ **EXECUTE PLAN** Equation 1 solved for our unknown v_1 gives

$$v_1 = v_2 \frac{d_2^2}{d_1^2}$$

$$v_1^2 = v_2^2 \left(\frac{d_2}{d_1}\right)^4.$$

Next we use this expression for v_1^2 in Eq. 2 to get the pressure difference:

$$\Delta P = P_2 - P_1 = \tfrac{1}{2}\rho(v_1^2 - v_2^2) = \tfrac{1}{2}\rho\left[v_2^2\left(\frac{d_2}{d_1}\right)^4 - v_2^2\right]$$

$$= \tfrac{1}{2}\rho v_2^2\left[\left(\frac{d_2}{d_1}\right)^4 - 1\right]$$

$$= \tfrac{1}{2}(1000\ \text{kg/m}^3)(7.50 \times 10^{-1}\ \text{m/s})^2\left[\left(\frac{15\ \text{mm}}{10\ \text{mm}}\right)^4 - 1\right]$$

$$= 1.1 \times 10^3\ \text{Pa}. ✔$$

That this result is positive means $P_2 > P_1$; that is, the pressure is greater in the wide part of the pipe. ✔

❹ **EVALUATE RESULT** The pressure difference is about 1% of atmospheric pressure, great enough to measure with either a water or mercury manometer but not great enough to deform the pipe walls or to make us question our assumption of incompressibility.

Equation 1 tells us that v_1 is 1.7 m/s. Thus the speeds in both parts of the pipe are on the order of 1 m/s and thus not fast enough to cause concern about laminar flow. We expect the speed to increase and the pressure to be reduced in the narrow part, and this is what we obtain. This is reasonable because to speed up the flow in the narrow portion, we need a force in the same direction as the motion, which is consistent with greater pressure in the wide portion.

Note that we had no reason to use the given information about the distance separating positions 1 and 2. Laminar flow is the same at all positions that have a common pipe diameter, so as long as each point is well away from the transition from one diameter to the next, the precise positions are not relevant.

Guided Problem 18.8 Reoriented flow

The pipe in Figure WG18.5 is rotated 90° counterclockwise so that it is vertical, with the wide part above the narrow part. Assuming that the speed in the wide part remains the same as in Worked Problem 18.7, what is the pressure difference between positions in the two parts now, and which position is at the greater pressure?

❶ **GETTING STARTED**

1. Sketch the situation.
2. Will the approach of Worked Problem 18.7 work in this case?

❷ **DEVISE PLAN**

3. Are there differences between Worked Problem 18.7 and this problem other than $y_1 \neq y_2$?
4. Write Bernoulli's equation and the appropriate form of the continuity equation.
5. Do you have sufficient information to solve the problem?

❸ **EXECUTE PLAN**

6. Insert the given numerical values and obtain a result.

❹ **EVALUATE RESULT**

7. Is your result reasonable? Is it unexpected?

Answers to Review Questions

1. The three types are tensile/compressive stress, bulk stress, and shear stress. Tensile stress is caused by equal-magnitude forces exerted in opposite directions on an object in such a way as to elongate the object along the line of action of the forces; compressive stress is caused by equal-magnitude forces exerted in opposite directions in such a way as to shorten the object along the line of action of the forces. Bulk stress is the result of forces that are exerted equally on all surfaces of an object and cause the object's volume to decrease. Shear stress is caused by forces exerted in opposite directions on opposite surfaces of an object, causing the object's shape to "shear" along the line of action of the forces.
2. Nonviscous liquids cannot support shear stress and so remain unchanged when such stress is exerted; examples include water, gasoline, ethanol, and mercury. Viscous liquids can sustain shear stress, which means that the shape of a given volume of such a liquid changes when subject to shear stress; examples include jelly, tar, molasses, and honey.
3. A gas that is experiencing bulk stress compresses, which means the volume of the sample decreases and its mass density increases. A liquid is essentially incompressible and so does not undergo any change in volume or mass density when subject to bulk stress.

4. For a given force magnitude, pressure is defined as the ratio of the magnitude of the force exerted by the fluid on a surface to the area of that surface, $P = F/A$. Two principal differences are: (1) Force is a vector and pressure is a scalar, and (2) because pressure depends on the surface area over which a force is exerted, two forces of the same magnitude can cause very different pressures if those forces are exerted over different surface areas.
5. The gravitational force exerted by Earth on the fluid causes the pressure in the fluid to be greatest at the bottom of the container and to decrease with height in the container.
6. The pressure change is transmitted equally to all locations throughout the liquid. This is Pascal's principle.
7. The magnitude of the buoyant force on the object is equal to the magnitude of the gravitational force exerted by Earth on the volume of fluid displaced by the object.
8. If the magnitude of the force of gravity exerted on the object is greater than the buoyant force exerted on it, the object sinks. Otherwise, it floats.
9. The object must be designed in such a way that the magnitude of the gravitational force exerted on the object is smaller than the magnitude of the gravitational force exerted on the volume of water displaced by the object. Such a design is usually achieved by making all or part of the object hollow.
10. In laminar flow, streamlines maintain their shape and relative positions; in turbulent flow, they change erratically.
11. Viscosity is the property of a fluid that measures the fluid's resistance to shear stress. Viscous fluids have high viscosity and tend not to flow easily even when experiencing shear stress. Nonviscous fluids have low or zero viscosity and flow easily.
12. The factors are the flow speed, the shape of the object, and the viscosity of the fluid.
13. Streamlining an object means adjusting its shape to maintain streamline flow around the object. This minimizes energy dissipation when the object moves at high speeds relative to the fluid.
14. The different spacings tell you that the fluid's speed is different in the two parts. The speed is greatest where the streamlines are closest together and smallest where they are farthest apart.
15. The pressure in the fluid is greatest where the speed is smallest and smallest where the speed is greatest.
16. When the particles are far apart, as in a gas, they exert no appreciable force on each other. When the separation distance is equal to a few particle diameters, which is typical of particles in a liquid, the particles exert cohesive forces on each other. At separation distances smaller than one particle diameter, as happens when gas particles collide with each other, the particles exert repulsive forces on each other.
17. Each interior particle has neighbors in all directions, which means that the vector sum of the cohesive forces exerted by neighbors on the particle in question is zero. Each surface particle has no liquid particles above it, so the vector sum of the cohesive forces exerted on it by all its neighbors is nonzero. The resultant force on the surface particle is directed inward and causes the surface particles to act like a membrane that is under tension.
18. The tension in the elastic membrane increases as the radius of curvature increases, whereas the surface tension in a liquid surface is independent of the radius of curvature of the surface.
19. The determining factor is the contact angle between liquid and surface. If the contact angle is smaller than 90°, the liquid wets the solid surface; if the contact angle is greater than 90°, the liquid does not wet the surface.
20. If the liquid is able to wet the tube walls (water in a glass tube, say), the meniscus is concave as viewed from above. If the liquid does not wet the walls (mercury in a glass tube, say), the meniscus is convex as viewed from above.
21. The gravitational force exerted on the fluid, the motion of the fluid, and collisions between fluid particles (or between particles and container walls) contribute to the pressure in all fluids. Surface tension contributes to the pressure only in liquids.
22. At sea level, the change in atmospheric pressure with height is 12 Pa/m, which means the difference for a person 2 m tall is about 24 Pa.
23. Hydrostatic pressure is that part of the pressure at any location in a liquid that results from the gravitational force. In a stationary liquid, with constant mass density, hydrostatic pressure decreases linearly with height in the liquid.
24. The object floats if its average mass density ρ_{av} is smaller than the mass density ρ_f of the fluid, hangs motionless if ρ_{av} is the same as ρ_f, and sinks if ρ_{av} is greater than ρ_f.
25. Gauge pressure is the pressure shown on the indicator of a pressure gauge. When the gauge is used to measure the pressure inside any container, the gauge pressure tells you the pressure in excess of atmospheric pressure. The gauge is itself under atmospheric pressure, and so the pressure it records is any pressure above atmospheric pressure. The true pressure in the fluid is thus the recorded value plus 1×10^5 Pa of atmospheric pressure.
26. A hydraulic system consists of two or more liquid-filled cylinders connected to one another, each cylinder fitted with a piston. The pressure created in the liquid when a force is exerted on one piston is transmitted equally to all parts of the liquid, which means that a force is exerted on all the other pistons. Force is multiplied in the system if the surface area of the piston on which the initial force is exerted is smaller than the surface area of the other pistons. Because $F = PA$ and P is the same everywhere in the liquid, the magnitude of the force exerted by the larger-surface-area pistons is greater than the magnitude of the initial force exerted.
27. There is no difference in the masses because the mass of liquid passing *any* point along the tube length is the same as the mass passing all other points, as expressed by the continuity equation (Eq. 18.23).
28. The pressure is lower at whichever location is higher (Eq. 18.36).
29. The pressure is lower in the small-diameter portion because the flow speed here is higher than the flow speed in the large-diameter portion (Eqs. 18.24 and 18.36).
30. A velocity gradient is a measure of the rate at which one fluid layer moves relative to an adjacent layer or, expressed another way, a measure of how quickly the fluid velocity changes with distance perpendicular to the flow direction.
31. In general, the viscosity of liquids decreases as temperature increases, and the viscosity of gases increases as temperature increases.
32. The law says that the volume flow rate is directly proportional to the fourth power of the tube radius and inversely proportional to the viscosity of the fluid: $Q = (\pi R^4/8\eta\ell)(P_1 - P_2)$.
33. Increasing the surface area means that work is done by surface tension forces. Increasing the surface area also means bringing more particles from the interior of the liquid to its surface, which reduces the number of bonds between particles. The work done by the surface tension forces equals the energy required to break these bonds.
34. The pressure is greater inside (on the concave side of the surface); the difference is proportional to the surface tension and inversely proportional to the radius of curvature. Quantitatively, this is given by Laplace's law: $P_{in} = (\gamma/R) + P_{out}$ for a cylindrical surface, and $P_{in} = (2\gamma/R) + P_{out}$ for a spherical surface, where γ is the surface tension of the liquid and R is the radius of curvature of the surface.

Answers to Guided Problems

Guided Problem 18.2 $R = 0.98$ m for helium

Guided Problem 18.4 This is a proof that $F = (P_{in} - P_{out})\pi R^2$.

Guided Problem 18.6 $v_{max} = 7.3$ m/s, $Q_{mass} = 8.5 \times 10^6$ kg/s, $Q_{energy} = 2.3 \times 10^8$ J/s

Guided Problem 18.8 $P_2 - P_1 = -1.6 \times 10^3$ Pa, greater pressure at bottom (narrow) end

PRACTICE

Guided Practice by Chapter

19 Entropy

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

19.1 States

1. What is the distinction between the kinetic energy of a swinging pendulum and the kinetic energy of all the molecules in the air surrounding the pendulum?
2. What is Brownian motion?
3. For a system consisting of a large number of objects, what is the difference between the system's macrostates and its basic states?
4. What does it mean to say that energy is quantized?
5. What is the recurrence time of a system?

19.2 Equipartition of energy

6. How is the energy of a system shared among the system's randomly interacting parts?
7. In a system made up of N particles, how is the average energy of a given particle related to the thermal energy E_{th} of the system?

19.3 Equipartition of space

8. With a container divided by porous walls into five equal-sized compartments, how many basic states are possible in a three-particle system if the system's macrostate is two particles confined to compartment 1?
9. How do the particles in a sample of gas distribute themselves in the volume occupied by the sample, and what is this phenomenon called?

19.4 Evolution toward the most probable macrostate

10. For a given system, what distinguishes the equilibrium state from all other possible macrostates for the system?
11. What is the underlying reason for irreversibility?
12. What tendency of natural systems gives us our sense of time moving in one direction only?
13. State the second law of thermodynamics.
14. Is the entropy of Earth necessarily increasing?

19.5 Dependence of entropy on volume

15. What are the conditions that define an ideal gas?
16. What is the statistical definition of entropy?
17. What is the entropy law?
18. Does the second law of thermodynamics apply to all systems in the universe?
19. How does a change in the volume of a closed system containing an ideal gas initially at equilibrium affect the entropy of the gas?
20. What is the relationship between the individual entropies of two systems and the entropy of the system formed by the combination of the two original systems?

19.6 Dependence of entropy on energy

21. What is the root-mean-square speed of the particles in a sample of gas?
22. How does the entropy of a sample of a monatomic ideal gas containing N particles change as the thermal energy of the gas changes?
23. Define the absolute temperature of an ideal gas in terms of its entropy and thermal energy.
24. If two ideal gases, A and B, are in thermal equilibrium, what is the relationship between their absolute temperatures?

19.7 Properties of a monatomic ideal gas

25. What two atomic parameters determine the pressure in a monatomic ideal gas?
26. State the ideal gas law in words.
27. Under what conditions is the ideal gas law a good description of the behavior of real gases?
28. Which two parameters of a monatomic ideal gas at equilibrium determine the root-mean-square speed of the gas atoms, and what is the equation expressing this speed?

19.8 Entropy of a monatomic ideal gas

29. For a monatomic ideal gas, which three macroscopic parameters determine how the entropy of the system changes as the gas changes from one equilibrium state to a different equilibrium state?
30. How does the change in entropy as a gas changes from one equilibrium state to another depend on how that change takes place?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking. For simplicity, you can treat the air as a monatomic ideal gas in any situations involving air.

1. The number of basic states of one page of text in this book [C, Q]
2. The time interval needed for one person typing letters randomly to produce the first ten letters of Shakespeare's *Hamlet* [E, W]
3. The population of Manhattan, New York, if Earth's population were equipartitioned by land area [F, T, X, I]
4. The entropy change if all the gas particles in the air in your bedroom moved so as to occupy only the top half of the room [M, S]
5. The entropy change when a helium balloon deflates in your bedroom overnight [B, H, M, Y]
6. The entropy increase of the gas particles in the air in your car tire as it heats up while you drive (assume that air is a monatomic ideal gas) [A, G, O, Z]
7. Root-mean-square speed of the gas particles in air at room temperature [K, V]
8. Root-mean-square speed of atoms in a sample of helium gas at a temperature just above the boiling point of helium [J, P]
9. The thermal energy of the atmosphere [N, V, D, R]
10. The entropy change of the atmosphere over the United States from 2:00 p.m. to 2:00 a.m. [L, N, V, D, R, U, Z]

Hints

A. What is the temperature change?
B. What is the volume of a balloon?
C. How many characters are on a page?
D. How many gas particles are there in the atmosphere?
E. What is the probability of having a random ten-letter sequence match a specific ten-letter sequence?
F. What is Earth's population?
G. What is the volume of the air in your car tires?
H. How many helium atoms are in the balloon?
I. What is the area of Manhattan?
J. What is the boiling temperature of helium?
K. What is an average room temperature?
L. What is the temperature change?
M. What is the volume of a typical bedroom?
N. What is the mass of Earth's atmosphere?
O. How many gas particles are in your tires?
P. What is the mass of a helium atom?
Q. How many basic states does one character have?
R. What is the average temperature of the atmosphere?
S. How many gas particles are contained in the volume of air in the room?
T. What is the surface area of Earth?
U. What fraction of Earth's atmosphere is over the United States?
V. What is the mass of one gas particle?
W. How many letters per second do you type?
X. What fraction of Earth's surface is land?
Y. What is the change in temperature?
Z. What is the change in volume of air?

Key (all values approximate)

A. 1×10^1 K; **B.** 3×10^{-3} m^3; **C.** 4×10^3 characters/page (each position on a page is one possible "compartment" for each character); **D.** 1×10^{44} particles; **E.** 1 in $26^{10} = 7 \times 10^{-15}$; **F.** 7×10^9 people; **G.** 2×10^{-2} m^3; **H.** 7×10^{22} atoms; **I.** 6×10^7 m^2; **J.** 4 K; **K.** 3×10^2 K; **L.** decrease of 1×10^1 K; **M.** 3×10^1 m^3; **N.** 5×10^{18} kg; **O.** 1×10^{24}; **P.** 7×10^{-27} kg; **Q.** 1×10^2 basic states (each basic state possible is one of the 26 letters of the alphabet, upper and lower case, plus numerals, symbols, and punctuation); **R.** lower than 3×10^2 K; **S.** 7×10^{26} particles; **T.** 5×10^{14} m^2; **U.** 2×10^{-2}; **V.** air is mainly nitrogen gas (N_2) of molecular mass 5×10^{-26} kg; **W.** 4 letters/s; **X.** 0.3; **Y.** assume no change in temperature. **Z.** assume no change in volume

PRACTICE

Worked and Guided Problems

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 19.1 Counting basic states

A closed wooden box has three identical holes—1, 2, and 3—of diameter d_{hole} cut into one side. Five indistinguishable balls, each of diameter $d_{ball} < d_{hole}$, are placed inside the box, and the box is shaken until all five have fallen out through one of the holes. What is the probability of two of the balls (and only two) falling through hole 1?

❶ GETTING STARTED We are told that a box containing five balls is shaken until all the balls fall out through one of three holes—1, 2, and 3—and we are asked to determine the probability of two of the balls exiting through hole 1. Thus we imagine gathering up the balls each time the box empties, placing them back in the box, shaking until the box is again empty, and repeating this action many times. Each emptying of the box is one *event*, and our task is to determine the fraction of this large number of repetitions that has two of the balls falling through hole 1.

Let N_1 be the number of balls that fall through hole 1 in any given event, N_2 be the number that fall through hole 2 in that event, and N_3 be the number that fall through hole 3. We identify a macrostate of the system as being the number of balls that fall through hole 1, and there are six possible macrostates: $N_1 = 0, 1, 2, 3, 4, 5$. In order to determine the probability of the macrostate $N_1 = 2$, we need to count the basic states possible for each of the six macrostates.

❷ DEVISE PLAN We can count the number of basic states associated with each of the six macrostates $N_1 = 0, N_1 = 1, N_1 = 2, N_1 = 3, N_1 = 4, N_1 = 5$. We can then count the number of basic states Ω_{tot} associated with the system and the number of basic states $\Omega(N_1 = 2)$ associated with the macrostate $N_1 = 2$ (two balls falling through hole 1). The probability of two balls falling through hole 1 is then the fraction $\Omega(N_1 = 2)/\Omega_{tot}$.

❸ EXECUTE PLAN The macrostate $N_1 = 0$ (no balls through hole 1) has six basic states, $\Omega(N_1 = 0) = 6$:

$$(0, 5, 0),\ (0, 4, 1), (0, 3, 2), (0, 2, 3), (0, 1, 4), (0, 0, 5),$$

where each triplet representing a basic state is of the form (N_1, N_2, N_3). The macrostate $N_1 = 1$ has five basic states, $\Omega(N_1 = 1) = 5$:

$$(1, 4, 0),\ (1, 3, 1), (1, 2, 2), (1, 1, 3), (1, 0, 4),$$

the macrostate $N_1 = 2$ has four, $\Omega(N_1 = 2) = 4$:

$$(2, 3, 0),\ (2, 2, 1), (2, 1, 2), (2, 0, 3),$$

the macrostate $N_1 = 3$ has three, $\Omega(N_1 = 3) = 3$:

$$(3, 2, 0),\ (3, 1, 1), (3, 0, 2),$$

the macrostate $N_1 = 4$ has two, $\Omega(N_1 = 4) = 2$:

$$(4, 1, 0),\ (4, 0, 1),$$

and the macrostate $N_1 = 5$ has one, $\Omega(N_1 = 5) = 1$:

$$(5, 0, 0).$$

The number of basic states available to the system is then

$$\Omega_{tot} = \sum_{i=0}^{i=5} \Omega(i) = 6 + 5 + 4 + 3 + 2 + 1 = 21.$$

The probability of two balls exiting through hole 1 is the fraction

$$\frac{\Omega(N_1 = 2)}{\Omega_{tot}} = \tfrac{4}{21}. ✔$$

❹ **EVALUATE RESULT** We assumed that there is an equal probability for each of the five balls to pass through each of the three holes, which is the only sensible assumption based on the information given. We do not know the size of the holes compared to the size of the balls, except that the holes must be large enough for a ball to pass through. Depending on the hole size, each hole is at least partially blocked for a short time interval as a ball passes through it, but we have no way to take this into account without further information.

Our numerical result indicates a probability just over 19% that two balls will pass through hole 1. There are six different numbers of balls that might fall through hole 1 (0, 1, 2, 3, 4, or 5), so we might naively expect the probability that exactly two balls will pass through is one out of six, or just under 17%. However, our analysis shows that each of these six outcomes is not equally probable, so we should not expect this to exactly match our computed result. Our list of macrostates also allows us to compute the average number of balls falling through hole 1 by summing the product of the number of balls through hole 1 and the probability of the corresponding macrostate:

$$0\left(\tfrac{6}{21}\right) + 1\left(\tfrac{5}{21}\right) + 2\left(\tfrac{4}{21}\right) + 3\left(\tfrac{3}{21}\right) + 4\left(\tfrac{2}{21}\right) + 5\left(\tfrac{1}{21}\right) = \left(\tfrac{35}{21}\right) = \tfrac{5}{3}.$$

This makes sense because there are five balls that can fall through any one of three holes, and so the average number of balls falling through any hole should be $5/3$. Two balls falling through hole 1 is thus more than we would expect on average, and so the probability of this happening should be smaller than average. As a rough upper bound, consider that roughly one-third of the balls should pass through each hole. Because 2 is greater than $5/3$, we expect the probability for two balls to pass through hole 1 to be smaller than 33%. Because our result lies between the lower bound of 17% and the upper bound of 33%, it is not unreasonable.

As a final check, we could sum the probabilities for all possible outcomes ($N_1 = 0$ through 5) to check that they add to unity, as all five balls must pass through one of the holes (you may want to verify this yourself).

Guided Problem 19.2 Shared energy

Consider a system consisting of three particles—1, 2, and 3—that share six units of energy. What is the probability that particle 3 has two units of energy?

❶ **GETTING STARTED**

1. Are the particles distinguishable or indistinguishable?
2. Is each energy unit distinguishable or indistinguishable?
3. How should you define your basic states?
4. How might you define your macrostates?

❷ **DEVISE PLAN**

5. How many basic states do you have for each macrostate?
6. How do you calculate the probability that particle 3 has two units of energy?

❸ **EXECUTE PLAN**

7. Describe the basic states associated with each macrostate.

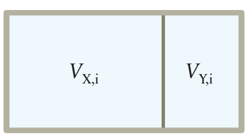
❹ **EVALUATE RESULT**

8. On average, how many energy units do you expect each particle to have?
9. Does your counting of basic states agree with what you expect?

Worked Problem 19.3 Volume and entropy changes

Figure WG19.1 shows a closed system consisting of two monatomic ideal gases that are in thermal equilibrium. A partition that is free to move horizontally keeps the gases separated in compartments X and Y but lets them exchange energy through collisions with the partition. Compartment X has four times as many atoms as compartment Y. The initial number of atoms per unit volume in compartment X is twice the number of atoms per unit volume in compartment Y. The change in entropy of the system as it evolves from this initial macrostate to the equilibrium state is 2.631×10^{23}. Determine the number of gas atoms in compartments X and Y.

Figure WG19.1



❶ **GETTING STARTED** We are trying to determine the number of gas atoms N_X in compartment X and the number of gas atoms N_Y in compartment Y. We are not given the initial volumes, $V_{X,i}$ and $V_{Y,i}$, or the final volumes, $V_{X,f}$ and $V_{Y,f}$. We know that the volume V of the system is fixed; $V = V_{X,i} + V_{Y,i} = V_{X,f} + V_{Y,f}$. From the initial information given in the problem statement, we know that $N_X = 4N_Y$. Because the gas atoms do not cross through the partition, this relationship holds while the partition moves. We also know that the initial gas atoms per unit volume (number density) in the two compartments are related by $N_X/V_{X,i} = 2N_Y/V_{Y,i}$ of each compartment.

❷ **DEVISE PLAN** Because the gases are in thermal equilibrium and the system is closed, we know that the entropy changes in each compartment are given by Eq. 19.8, $\Delta S = N\ln(V_f/V_i)$. Because entropy is an extensive quantity, the change in entropy of the system is the sum of the changes in the two compartments: $\Delta S = \Delta S_X + \Delta S_Y$ (Eq. 19.10). We know that when the new equilibrium state is reached, the number density of the gas atoms is equal on the two sides of the partition, $N_X/V_{X,f} = N_Y/V_{Y,f}$ (Eq. 19.20). We can use this condition along with the given information relating the number of atoms in each compartment and the number density in each compartment to calculate the volume ratios $V_{X,f}/V_{X,i}$ and $V_{Y,f}/V_{Y,i}$. This allows us to solve Eq. 19.10 for the number of gas atoms in compartment Y. We can then use the given information that $N_X = 4N_Y$ to obtain N_X.

❸ **EXECUTE PLAN** We can combine the given information that $N_X = 4N_Y$ and $N_X/V_{X,i} = 2N_Y/V_{Y,i}$ in order to determine a relationship between the initial volume $V_{Y,i}$ of compartment Y and the initial volume $V_{X,i}$ of compartment X:

$$V_{Y,i} = 2V_{X,i}\frac{N_Y}{N_X} = \frac{V_{X,i}}{2}. \tag{1}$$

We can substitute Eq. 1 into $V = V_{X,i} + V_{Y,i}$ to obtain

$$V = \tfrac{3}{2}V_{X,i}. \tag{2}$$

When the new equilibrium state is reached, the number of gas atoms per volume is the same in compartments X and Y (Eq. 19.20):

$$\frac{N_X}{V_{X,f}} = \frac{N_Y}{V_{Y,f}}. \tag{3}$$

We can combine $N_X = 4N_Y$ and Eq. 3, to obtain a relationship between the final volume $V_{Y,f}$ of compartment Y and the final volume $V_{X,f}$ of compartment X:

$$V_{Y,f} = V_{X,f}\frac{N_Y}{N_X} = \frac{V_{X,f}}{4}. \tag{4}$$

We can substitute Eq. 4 into $V = V_{X,f} + V_{Y,f}$ to obtain

$$V = \tfrac{5}{4}V_{X,f}. \tag{5}$$

Next we combine Eqs. 5 and 2 to determine the ratio of the final and initial volumes of compartment X:

$$\tfrac{5}{4}V_{X,f} = \tfrac{3}{2}V_{X,i} \Rightarrow \frac{V_{X,f}}{V_{X,i}} = \tfrac{6}{5}. \tag{6}$$

We now divide Eq. 4 by Eq. 1 and then use Eq. 6 to determine the ratio of the final and initial volumes of compartment Y:

$$\frac{V_{Y,f}}{V_{Y,i}} = \frac{1}{2}\frac{V_{X,f}}{V_{X,i}} = \tfrac{3}{5}. \tag{7}$$

Equations 19.10 and 19.8 provide the change in entropy of the system:

$$\Delta S = \Delta S_X + \Delta S_Y = N_Y\left(4\ln\left(\frac{V_{X,f}}{V_{X,i}}\right) + \ln\left(\frac{V_{Y,f}}{V_{Y,i}}\right)\right). \tag{8}$$

We now use Eqs. 6 and 7 and our given value for the change in entropy in Eq. 8 to solve for the number of particles in compartment Y:

$$N_Y = \frac{\Delta S}{\left(4\ln\left(\frac{V_{X,f}}{V_{X,i}}\right) + \ln\left(\frac{V_{Y,f}}{V_{Y,i}}\right)\right)} = \frac{2.631 \times 10^{23}}{(4\ln(\frac{6}{5}) + \ln(\frac{3}{5}))}$$

$$= 1.204 \times 10^{24}. \checkmark$$

The number of particles in compartment X is then $N_X = 4N_Y = 4.817 \times 10^{24}$. ✔

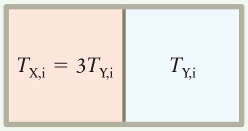
PRACTICE

❹ **EVALUATE RESULT** We know that 1 mole corresponds to 6.022×10^{23} gas atoms, so in compartment Y we have 2 moles of gas and in compartment X we have 8 moles of gas. We are not given the initial temperature, pressure, and volume in either compartment, but we know that these quantities must be related by the ideal gas law: $P = Nk_BT/V$ (Eq. 19.51). There is no mention in the problem statement of special heating or cooling apparatus or of a container able to tolerate multiple atmospheres of pressure. At room temperature and 1 atm, 1 mole occupies a volume just exceeding 24 L, which would require a container volume in the range of 250 L, or about one-quarter m^3. Such an apparatus would fit easily on a laboratory table. This lends some credence to our numerical result.

Guided Problem 19.4 Thermal equilibrium

Figure WG19.2 shows a closed system consisting of two monatomic ideal gases separated by a fixed partition that keeps the gases apart but lets them exchange thermal energy. Compartment X contains 6.0×10^{23} atoms, and compartment Y contains 1.5×10^{24} atoms. Initially, the absolute temperature of the gas in compartment X is three times the absolute temperature of the gas in compartment Y. What is the change in entropy of the system as it evolves from this initial macrostate to the equilibrium state?

Figure WG19.2



❶ **GETTING STARTED**

1. What changes and what remains the same as the system evolves from the initial state to the equilibrium state?
2. How is the change in entropy of the system related to the changes in entropy of the two gases?

❷ **DEVISE PLAN**

3. How is temperature related to the thermal energy per particle?
4. What fraction of thermal energy resides in each compartment at equilibrium?
5. How is the thermal energy per particle related to the entropy changes in each compartment?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

6. Which compartment experiences a decrease in temperature, and which an increase?
7. Is the entropy change of the system positive or negative?

Worked Problem 19.5 Cooling an ideal gas

A sample of a monatomic ideal gas containing 6.02×10^{23} atoms occupies a volume of 2.24×10^{-2} m^3 at a pressure of 1.01×10^5 N/m^2. With the volume held constant, the gas is cooled until the atoms are moving at a root-mean-square speed of 4.02×10^2 m/s. If the mass of each atom is 6.646×10^{-27} kg, what is the change in entropy of the gas?

❶ **GETTING STARTED** We are given the number of gas atoms in a sample and their mass, and we are told that the sample is cooled at constant volume. We are also given the value of this volume, the initial pressure, and the final speed of the atoms. Our task is to determine the entropy change for this cooling, but we know from Eq. 19.56, $\Delta S = \frac{3}{2}N\ln(T_f/T_i)$, that in order to do that we need to determine the initial and final temperatures, which are not given.

❷ **DEVISE PLAN** We can use the ideal gas law, Eq. 19.51, to determine the initial temperature of the gas and Eq. 19.53 to determine its final temperature. We then can use Eq. 19.56 to determine the entropy change the gas undergoes as it cools.

❸ **EXECUTE PLAN** For the initial temperature, we write Eq. 19.51 in the form

$$T_i = \frac{P_iV_i}{Nk_B} = \frac{(1.01 \times 10^5\ \mathrm{N/m^2})(2.24 \times 10^{-2}\ \mathrm{m^3})}{(6.02 \times 10^{23})(1.38 \times 10^{-23}\ \mathrm{J/K})} = 272\ \mathrm{K}.$$

For the final temperature, we write Eq. 19.53 in the form

$$T_f = \frac{m(v_{\mathrm{rms,f}})^2}{3k_B}$$

$$= \frac{(6.646 \times 10^{-27}\ \mathrm{kg})(4.02 \times 10^2\ \mathrm{m/s})^2}{(3)(1.38 \times 10^{-23}\ \mathrm{J/K})} = 25.9\ \mathrm{K}.$$

Now Eq. 19.56 gives us the change in entropy of the gas:

$$\Delta S = \tfrac{3}{2} N \ln\left(\frac{T_f}{T_i}\right)$$

$$= \tfrac{3}{2}(6.02 \times 10^{23}) \ln\left(\tfrac{25.9}{272}\right) = -2.12 \times 10^{24}. ✔$$

❹ EVALUATE RESULT A negative entropy change means that the number of basic states available to the atoms in the final macrostate is smaller than the number of basic states available in the initial macrostate. As the gas cools, the root-mean-square speed of the atoms decreases, and so the number of basic states decreases, in agreement with our answer.

Guided Problem 19.6 Heating an ideal gas

A sample of a monatomic ideal gas containing 9.03×10^{23} atoms occupies a constant volume of 4.48×10^{-2} m^3 and is initially at a pressure of 1.01×10^5 N/m^2. The mass of each atom is 6.646×10^{-27} kg. If the gas is heated until the change in entropy is 1.415×10^{23}, what is the final root-mean-square speed of the atoms?

❶ GETTING STARTED

1. What quantities do you need to know in order to calculate $v_{\text{rms,f}}$?
2. What quantities are you given?

❷ DEVISE PLAN

3. How can you determine the initial temperature of the gas?
4. How can you determine its final temperature?
5. How can you use this information to determine $v_{\text{rms,f}}$?

❸ EXECUTE PLAN

6. Calculate the initial absolute temperature.
7. Determine the final absolute temperature using the fact that $e^{\ln(T_f/T_i)} = T_f/T_i$.
8. Calculate the value of $v_{\text{rms,f}}$.

❹ EVALUATE RESULT

9. How does this speed compare with the escape speed, 1.1×10^4 m/s, from Earth?
10. Do you expect the number of basic states to increase or decrease?

Worked Problem 19.7 Expansion and entropy change

A sample of a monatomic ideal gas containing 6.00×10^{23} atoms in a volume of 8.00×10^{-3} m^3 is initially in thermal equilibrium with its surroundings at 300 K. The gas is made to expand until the final volume is 6.40×10^{-2} m^3 and the final temperature is 50 K. What is the change in entropy of the gas?

❶ GETTING STARTED We are given the number of gas atoms in a sample, the initial and final volumes, and the initial and final absolute temperatures. Our task is to determine the change in entropy of the gas as its volume and temperature change.

❷ DEVISE PLAN Both the volume and the temperature of this gas change during the expansion process. We have all the information we need in order to use Eq. 19.61 to calculate the entropy change.

❸ EXECUTE PLAN

$$\Delta S = \tfrac{3}{2} N \ln\left(\frac{T_f}{T_i}\right) + N \ln\left(\frac{V_f}{V_i}\right)$$

$$= \tfrac{3}{2} N \ln\left(\frac{50\text{ K}}{300\text{ K}}\right) + N \ln\left(\frac{6.40 \times 10^{-2}\text{ m}^3}{8.00 \times 10^{-3}\text{ m}^3}\right) = -3.6 \times 10^{23}. ✔$$

❹ EVALUATE RESULT The negative value for ΔS means that the entropy of the system decreases as the system goes from its initial macrostate to its final macrostate. This makes sense because the system is not closed so entropy does not have to increase. The negative change in entropy results from the greater decrease in entropy due to the energy leaving the system compared to the positive increase in entropy due to the volume expansion.

PRACTICE

Guided Problem 19.8 Cooling and entropy change

What is the entropy change for a sample of a monatomic ideal gas that contains 1.20×10^{24} atoms when the sample is cooled at constant pressure from 400 K to 300 K?

1 GETTING STARTED

1. What quantities do you need to know in order to calculate the entropy change?
2. What quantities are you given?

2 DEVISE PLAN

3. Does the volume of the gas increase, decrease, or remain the same?
4. You know neither V_f nor V_i, and so how can you use the ideal gas law to remove these variables from your calculation?
5. How can you then calculate the change in entropy?

3 EXECUTE PLAN

6. Determine the relationship between the temperature change and the volume change.
7. Compute the entropy change.

4 EVALUATE RESULT

8. Does a negative entropy change make sense for this situation?

Answers to Review Questions

1. The pendulum's kinetic energy is coherent mechanical energy because all the particles of the pendulum swing together. The kinetic energy of the molecules in the air is incoherent thermal energy because the motion of these molecules is random.
2. Brownian motion is the zigzag motion of a small particle suspended in a fluid. The particle is bombarded by the surrounding molecules of the fluid. Because the motion of the fluid molecules is random, the bombardment is usually not equal on opposite sides of the particle, which causes the particle to move randomly in the fluid.
3. The system's macrostates are specified by the large-scale properties of the system, such as pressure, volume, and temperature. The system's basic states are specified by descriptions of the individual objects that make up the system, such as the position and velocity of each particle in a sample of gas.
4. That energy is quantized means that it comes in very small, discrete, indivisible units. Energy transfers take place through exchanges of these units rather than in a continuous stream.
5. The recurrence time is the average time interval needed to cycle randomly through all the possible basic states of the system.
6. Each part tends to have an equal share of the system's energy. This is called the equipartition of energy.
7. The average energy of each particle is E_{th}/N.
8. Four basic states are possible, with the third particle in any one of compartments 2 through 5: 2, 1, 0, 0, 0; 2, 0, 1, 0, 0; 2, 0, 0, 1, 0; 2, 0, 0, 0, 1.
9. The particles tend to be distributed uniformly throughout the volume. This is called the equipartition of space.
10. It is the most probable macrostate.
11. Over time, a system tends to evolve from less probable macrostates to the most probable macrostate. Once the most probable macrostate is attained, if the system is disturbed into any subsequent less probable macrostate, it will tend to move back toward the most probable macrostate.
12. Our sense of the direction of time comes from the tendency of all naturally occurring processes to evolve irreversibly toward equilibrium.
13. A closed system always evolves toward the macrostate that has the maximum number of basic states and at this point is in equilibrium.
14. Because Earth is not a closed system, Earth's entropy is not necessarily increasing.
15. The conditions are a large number of particles that occupy a negligible fraction of the volume of the container holding the gas and interact with one another and with the container walls only during collisions that randomize both their energy distribution and their spatial distribution.
16. The entropy of any system is the natural logarithm of the number of basic states available to the system.
17. The entropy law is the mathematical version of the second law of thermodynamics, describing how a closed system not in equilibrium always evolves in the direction of increasing entropy ($\Delta S > 0$) until it reaches equilibrium and its entropy no longer changes ($\Delta S = 0$).
18. No. It applies only to closed systems (energy is constant). The law says that a closed system's entropy either increases ($\Delta S > 0$) or stays unchanged ($\Delta S = 0$), but if the system is not closed, the entropy of the system can increase, decrease, or stay the same.
19. The entropy change is proportional to the natural logarithm of the ratio of the final and initial volumes, with the proportionality constant being N, the number of gas particles: $\Delta S = N \ln(V_f/V_i)$ (Eq. 19.8).
20. The entropy of the combined system equals the sum of the entropies of the two initial systems.
21. The root-mean-square speed is the square root of the average of the squares of the speeds of all the particles: $v_{rms} = \sqrt{(v^2)_{av}}$ (Eq. 19.21).
22. The entropy change is proportional to the natural logarithm of the ratio of the final and initial thermal energies, with the proportionality constant being $3N/2$: $\Delta S = \frac{3}{2} N \ln(E_{th,f}/E_{th,i})$ (Eq. 19.33).
23. The absolute temperature equals the inverse of the product of the Boltzmann constant and the derivative of the entropy of the gas with respect to its thermal energy: $1/T = k_B(dS/dE_{th})$ (Eq. 19.38).
24. Being in thermal equilibrium means that the two gases have the same absolute temperature.
25. The atomic parameters number density N/V of the atoms and their average kinetic energy $\frac{1}{2}mv_{rms}^2$ determine the pressure in the gas.
26. For an ideal gas in equilibrium, the pressure P in the gas equals the number density N/V of the gas particles times the absolute temperature T times the Boltzmann constant k_B.
27. The ideal gas law describes the behavior of real gases when the pressure in the gas is low and the gas temperature is high.

28. The parameters are the absolute temperature of the gas and the mass of one gas atom. The equation is $v_{rms} = \sqrt{3k_BT/m}$ (Eq. 19.53).
29. The entropy change depends on the number of atoms in the gas, on any change in absolute temperature, and on any change in volume.
30. The entropy is a function of the equilibrium state only, so changes in it are independent of the process involved.

Answers to Guided Problems

Guided Problem 19.2 $\dfrac{\Omega(2)}{\Omega_{tot}} = \frac{5}{28}$

Guided Problem 19.4 $\Delta S = +4.4 \times 10^{23}$

Guided Problem 19.6 $v_{rms,f} = 1.58 \times 10^3\ \mathrm{m/s}$

Guided Problem 19.8 $\Delta S = -8.6 \times 10^{23}$

Guided Practice by Chapter

20 Energy Transferred Thermally

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

20.1 Thermal interactions

1. What is the symbol for energy transferred thermally into or out of a system, and when is this quantity positive?
2. What are the two ways to transfer energy to a system, and what is the distinction between them?
3. What is an adiabatic process?
4. What is a quasistatic process?
5. How does the entropy of a system change when the only energy transferred to the system is transferred thermally?
6. How does the entropy of a system change when it undergoes a quasistatic adiabatic process?

20.2 Temperature measurement

7. What is the basis of most methods used to measure temperature?
8. What is the definition of the triple point of water? What is its value on the Kelvin temperature scale?
9. How is the triple point of water used in calibrating an ideal gas thermometer? How is the absolute temperature of a gas determined by an ideal gas thermometer?
10. What happens to the thermal energy and entropy of a system at absolute zero?

20.3 Heat capacity

11. What is the heat capacity of a system?
12. What is a degree of freedom? What determines how many degrees of freedom a gas particle has?
13. How many degrees of freedom does a monatomic gas atom have? How many does a diatomic gas molecule have?
14. How is thermal energy distributed among the degrees of freedom of a gas particle in thermal equilibrium?
15. How can you determine whether a particular degree of freedom contributes to the heat capacity per particle in a gas sample?

20.4 *PV* diagrams and processes

16. For a sample of an ideal gas that contains a fixed number of particles, what does a point on the PV diagram for the gas represent?
17. Define the terms *isotherm* and *isentrope*.
18. How is a quasistatic process represented on a PV diagram?
19. When an ideal gas is transformed from one equilibrium state to another, which of the following quantities are independent of the path taken from the initial state to the final state: (*a*) change in entropy, (*b*) change in thermal energy, (*c*) work done on the gas, (*d*) quantity of energy transferred thermally?

20.5 Change in energy and work

20. Under what conditions does the energy law, $\Delta E = W + Q$, reduce to an expression that deals with only ΔE_{th}?
21. On which variables does the work done on an ideal gas depend?
22. What does the minus sign in Eq. 20.8 indicate?

20.6 Isochoric and isentropic ideal gas processes

23. According to the $\Delta E_{\text{th}} = W + Q$ form of the energy law (Eq. 20.2), what is the change in the thermal energy of an ideal gas equal to in an isochoric process and in an isentropic process?
24. Which three macroscopic properties of an ideal gas must you know in order to calculate the change in the thermal energy of the gas for any process the gas might undergo?

20.7 Isobaric and isothermal ideal gas processes

25. What is the relationship between the heat capacities per particle for an ideal gas at constant volume and at constant pressure?
26. For an isothermal process carried out on an ideal gas, what is the relationship between the quantity of energy transferred thermally and the work done on the gas?

20.8 Entropy change in ideal gas processes

27. For an isentropic process carried out on an ideal gas, what is the relationship between the gas pressure and volume? Between the gas pressure and temperature? Between the gas volume and temperature?

20.9 Entropy change in nonideal gas processes

28. What is the intensive quantity related to heat capacity, and how is it defined?
29. What are the units of heat capacity? Of specific heat capacity?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. Work done quasistatically to inflate all four flat tires on your car [C, O, G]
2. The number of strokes of a bicycle pump required to fully inflate two high-pressure bicycle tires that are initially completely deflated [K, T, X]
3. The time interval needed for a microwave oven to bring 0.3 L of water from room temperature to the boiling point [S, A, V, BB]
4. Of the energy used to heat an aluminum teakettle full of water on the stove, the fraction of the energy used to raise the temperature of the kettle [H, P]
5. The temperature of an empty iron pot after being left on a hot stove for 2 min (ignoring energy transferred thermally from the pot to the surroundings and the mass of the heating element) [E, M, Z]
6. The volume of a container that holds 4 moles of an ideal gas at standard temperature and pressure (STP), defined as $T = 0\ ^\circ\text{C} = 273.15\ \text{K}$ and $P = 1\ \text{atm} = 1.013 \times 10^5\ \text{Pa}$ [AA]
7. The work done adiabatically in compressing a tennis ball to 80% of its original volume [W, Q, Y, F]
8. The temperature increase in the fuel-air mixture in a car engine cylinder when 10^3 J are added isochorically as the fuel ignites (ignoring the mass of the gasoline burned and assuming all the energy is transferred to the air in the cylinder) [Q, Y, L]
9. The entropy change in the air inside a fully inflated 4-L Mylar balloon taken from indoors to outdoors on a very hot day, assuming the balloon does not change its volume [Y, B, R]
10. The entropy change in the air inside a floating, fully inflated 4-L Mylar balloon when it is pushed 2 m under the surface of a swimming pool [U, B, I]
11. The energy needed to heat the air in a hot-air balloon so that it can lift an additional two people (ignoring any energy transferred from the air in the balloon to the outside air) [D, J, N, Q, Y]

Hints

A. What is the temperature change?
B. How many gas particles are there in the balloon?
C. Which macroscopic properties of the air inside the tires change as you inflate the tires?
D. What is the volume of a hot-air balloon?
E. What is the mass of the pot?
F. What is the change in the temperature of the tennis ball?
G. What is the pressure in a properly inflated car tire?
H. What are the masses of the water and the kettle?
I. What is the change in the pressure?
J. What additional buoyant force is needed to lift the payload?
K. What is the volume of an inflated bicycle tire?
L. What is the maximum volume of the cylinder?
M. What is the power delivered by a stovetop heating element?
N. What temperature change is required?
O. What are the volumes of an inflated and a flat car tire?
P. What are the specific heat capacities of the water and the teakettle?
Q. How many moles are in 2 cubic meters of air at 1 atm and 20 °C?
R. What are the initial and final temperatures, T_i and T_f?
S. What is the mass of this volume of water?
T. What is the pressure in an inflated high-pressure bicycle tire?
U. Which type of process is this: isothermal, isochoric, isobaric, or isentropic?
V. What is the power delivered by the oven?
W. What is the diameter of a tennis ball?
X. What volume of air does a typical bicycle pump deliver on each stroke?
Y. How many degrees of freedom does an air particle have?
Z. What is the specific heat capacity of iron?
AA. How many particles are there in one mole?
BB. What is the specific heat capacity of water?

Key (all values are approximate.)

A. 8×10^1 K; **B.** at STP about 1/6 mol, or 1×10^{23}; **C.** volume (thus not isochoric), pressure (thus not isobaric), number of gas particles, temperature (thus not isothermal); **D.** $2 \times 10^3\ \text{m}^3$; **E.** 1 kg; **F.** 3×10^1 K; **G.** gauge pressure 2 atm, absolute pressure 3 atm $= 3 \times 10^5$ Pa; **H.** 4 kg for the water, 0.2 kg for the kettle; **I.** 2×10^4 Pa; **J.** 1×10^3 N; **K.** $2 \times 10^{-3}\ \text{m}^3$; **L.** $5 \times 10^{-4}\ \text{m}^3$; **M.** 2 kW; **N.** about 3% per person, or 2×10^1 K for two people; **O.** inflated $2 \times 10^{-2}\ \text{m}^3$, flat 20% smaller; **P.** 4×10^3 J/K · kg for the water, 9×10^2 J/K · kg for the kettle; **Q.** 8×10^1 mol; **R.** $T_i = 2 \times 10^1$ °C, $T_f = 4 \times 10^1$ °C; **S.** 0.3 kg; **T.** gauge pressure 6 atm, absolute pressure 7 atm $= 7 \times 10^5$ Pa; **U.** isothermal; **V.** 1×10^3 W; **W.** 0.06 m; **X.** $2 \times 10^{-4}\ \text{m}^3$; **Y.** air is essentially all diatomic N_2 and O_2 particles, each of which has five degrees of freedom, $d = 5$; **Z.** 4×10^2 J/K · kg; **AA.** 6.02×10^{23} particles/ mole; **BB.** 4181 J/(K · kg)

Worked and Guided Problems

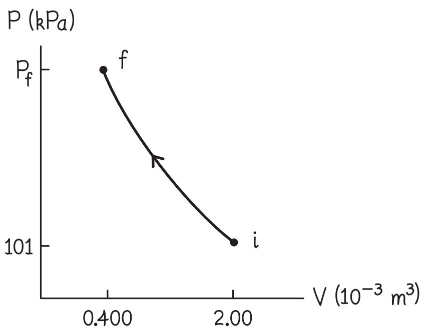
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 20.1 Compressed gas

A sample of a monatomic ideal gas is in a thermally insulated cylinder fitted with a piston that can change the cylinder volume. Initially, the cylinder volume is $V_i = 2.00 \times 10^{-3}$ m^3, the gas pressure is 1.01×10^5 Pa, and the temperature is 290 K. The piston then moves until the cylinder volume is $0.200V_i$. If the compression is quasistatic and isentropic, how much work is done on the gas?

❶ **GETTING STARTED** We begin by making a PV diagram for the process (Figure WG20.1). The process is a quasistatic isentropic compression, so in our diagram we draw the path going upward (from lower pressure to higher pressure) and leftward (from greater volume to smaller volume) along an isentrope.

Figure WG20.1



❷ **DEVISE PLAN** We know there is no energy transferred thermally in an isentropic process (Eq. 20.17: $Q = 0$). Therefore the work done on the gas during the compression increases the thermal energy (Eq. 20.18; $\Delta E_{th} = W$). The change in the thermal energy for any ideal gas process is independent of the path taken from initial state to final state and is proportional to the change in temperature (Eq. 20.15). Thus we need to determine the initial and final temperatures of the gas, the number of gas particles in the sample, and the heat capacity per particle C_V. (Remember that even though Eq. 20.15 is written in terms of C_V, it applies to all processes involving ideal gases, not just isochoric ones.) We know the initial temperature, pressure, and volume, and therefore we can use Eq. 19.51, the ideal gas law, $P = (N/V)k_BT$, to determine the number of particles. We know the initial and final volumes, so we can use Eq. 20.46 for an isentropic process to obtain the final pressure. Because the gas is monatomic, we know the number of degrees of freedom and can calculate C_V with Eq. 20.14. We then use the ideal gas law to calculate T_f and Eq. 20.19, $\Delta E_{th} = W = NC_V\Delta T$, to calculate the work done on the gas.

❸ **EXECUTE PLAN** From Eq. 19.51, the number of gas particles in the sample is

$$N = \frac{P_i V_i}{k_B T_i} = \frac{(1.01 \times 10^5 \text{ N/m}^2)(2.00 \times 10^{-3} \text{ m}^3)}{(1.38 \times 10^{-23} \text{ J/K})(290 \text{ K})} = 5.05 \times 10^{22}.$$

We now apply Eq. 20.46 to calculate the final pressure. Because the gas is monatomic and so has three degrees of freedom, the exponent in this equation is $\gamma = 1 + \frac{2}{d} = \frac{5}{3}$. The final volume is $0.200(2.00 \times 10^{-3} \text{ m}^3) = 0.400 \times 10^{-3}$ m^3, making the final pressure

$$P_f = P_i\left(\frac{V_i}{V_f}\right)^{5/3} = (1.01 \times 10^5 \text{ N/m}^2)\left(\frac{2.00 \times 10^{-3} \text{ m}^3}{0.400 \times 10^{-3} \text{ m}^3}\right)^{5/3}$$
$$= 1.48 \times 10^6 \text{ N/m}^2.$$

We then apply the ideal gas law to obtain the final temperature:

$$T_f = \frac{P_f V_f}{k_B N} = \frac{(1.48 \times 10^6 \text{ N/m}^2)(0.400 \times 10^{-3} \text{ m}^3)}{(1.38 \times 10^{-23} \text{ J/K})(5.05 \times 10^{22})} = 849 \text{ K}.$$

Being monatomic, each gas particle has three degrees of freedom and so $d = 3$. Therefore $C_V = (\frac{3}{2})k_B = 2.07 \times 10^{-23}$ J/K (Eq. 20.14), and the work done on the gas is, from Eq. 20.19,

$$W = E_{th} = NC_V\Delta T$$
$$= (5.05 \times 10^{22})(2.07 \times 10^{-23} \text{ J/K})(849 \text{ K} - 290 \text{ K})$$
$$= 584 \text{ J.} ✔$$

PRACTICE

❹ **EVALUATE RESULT** From our PV diagram, we can place upper and lower limits on the work done on the gas. The work done on the gas for an isobaric process is $W = -P\Delta V$. The change in volume is $\Delta V = -1.60 \times 10^{-3}$ m^3. If the compression were isobaric at the final pressure, $P_f = 1.48 \times 10^6$ N/m^2, then the work done on the gas would be

$$W_1 = -P_f\Delta V = -(1.48 \times 10^6\ \text{N/m}^2)(-1.60 \times 10^{-3}\ \text{m}^3)$$
$$= 2.37 \times 10^3\ \text{J}.$$

If the compression were isobaric at the initial pressure, $P_i = 1.01 \times 10^5$ N/m^2 = 101 kPa, then the work done on the gas would be

$$W_2 = -P_i\Delta V = -(1.01 \times 10^5\ \text{N/m}^2)(-1.60 \times 10^{-3}\ \text{m}^3)$$
$$= 1.62 \times 10^2\ \text{J}.$$

If the path were a diagonal straight line, then the work done would be one-half the sum of these two, $(\frac{1}{2})(W_1 + W_2) = 1.27 \times 10^3$ J. We expect the area under the isentrope to be smaller than the area under a straight line, so our answer is in the right range. The sign of the work should be positive because during compression work is done on the gas, in agreement with our answer.

Guided Problem 20.2 Work during compression

A cylinder that contains a sample of a monatomic ideal gas is fitted with a piston that can change the cylinder volume. Initially the volume is 6.00×10^{-3} m^3, the gas pressure is 1.01×10^5 Pa, and the temperature is 250 K. The gas undergoes a quasistatic isentropic compression until the cylinder volume is 3.00×10^{-3} m^3 and the gas pressure is 3.21×10^5 Pa. How much work is done on the gas sample during the compression?

❶ **GETTING STARTED**

1. Which type of process is this: isobaric, isothermal, isentropic, or isochoric? What characterizes that process?
2. Make a PV diagram.
3. Which quantities for your initial and final states are known, and which are unknown?

❷ **DEVISE PLAN**

4. For an isentropic process, what quantities do you need in order to determine the work done on the gas sample?
5. How does the fact that the gas is monatomic help you?
6. How can you use the ideal gas law to determine the relevant unknown quantities?

❸ **EXECUTE PLAN**

7. Solve for the final temperature.
8. Solve for number of particles in the sample.
9. Solve for the work done on the gas.

❹ **EVALUATE RESULT**

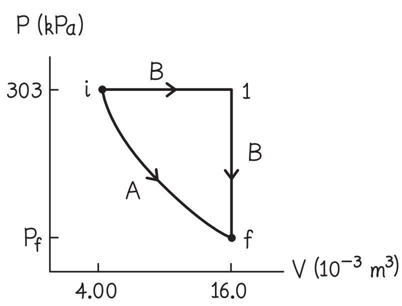
10. Does your value for the amount of work done agree with your expectation for an isentropic compression?
11. Does the sign of the work agree with what you expect for a compression?

Worked Problem 20.3 Two expansions

Consider two processes for taking a 0.500-mol sample of hydrogen gas (H_2, diatomic particles) from an initial state i to a final state f. The gas behaves ideally, and in both processes the initial volume and pressure are 4.00×10^{-3} m^3 and 3.03×10^5 N/m^2 = 303 kPa. In process A, the sample undergoes a quasistatic isothermal expansion to a volume of 1.60×10^{-2} m^3 and pressure P_f. In process B, the gas undergoes an isobaric expansion to a volume of 1.60×10^{-2} m^3 followed by an isochoric pressure decrease to P_f, with both processes quasistatic. What is the difference in the quantities of energy transferred thermally to the gas in the two processes?

❶ **GETTING STARTED** We begin by making a PV diagram showing both processes (Figure WG20.2). Process A is an isothermal expansion, and so we draw its path from initial state i to final state f downward (decreasing pressure) along an isotherm from P_i to P_f and curving to the right (increasing volume) from V_i to V_f. The path for process B has two legs. We represent the isobaric expansion by a horizontal line from initial state i rightward to intermediate state 1 at $V_1 = V_f$. The isochoric pressure decrease we represent by a vertical line from state 1 downward to final state f.

Figure WG20.2



PRACTICE

2 DEVISE PLAN We need to determine Q for two processes, one isothermal and the other an isobaric change followed by an isochoric change. Equation 20.31, $Q = Nk_BT\ln(V_f/V_i)$, is for isothermal changes; Eq. 20.21, $Q = NC_P(T_f - T_i)$, is for isobaric changes; and Eq. 20.12, $Q = (d/2)Nk_B(T_f - T_i)$, is for isochoric changes. We can use Avogadro's number to get N from the known number of moles of gas. For the heat capacities, we know that diatomic particles have five degrees of freedom. Therefore we know $C_V = \frac{5}{2}k_B$ (Eq. 20.14) and $C_P = \frac{5}{2}k_B + k_B = \frac{7}{2}k_B$ (Eq. 20.25).

For process A we know the initial and final volumes we need for Eq. 20.31, and we can calculate the constant temperature T by using initial values in the ideal gas law (Eq. 19.51). This gives us what we need to use Eq. 20.31 to calculate the energy Q_A transferred thermally to the gas.

For the isobaric expansion leg of process B, we know C_P and can use the ideal gas law to calculate the temperature at intermediate state 1. Using $T_1 - T_i$ for the temperature factor in Eq. 20.21, we get $Q_{B,\text{isobaric}}$. For the isochoric pressure decrease, we know that $T_f = T_i$ because process A is isothermal and the two processes have the same initial and final states. Thus we know the temperature difference $T_f - T_1$ needed in Eq. 20.12 and can determine $Q_{B,\text{isochoric}}$.

Our final step is to determine the difference in the quantities of energy transferred thermally to the gas in the two processes.

3 EXECUTE PLAN We use the ideal gas law to calculate the initial temperature after we calculate the number of particles in our sample:

$$0.500\text{ mol} \times \frac{6.02 \times 10^{23}\text{ particles}}{1\text{ mol}} = 3.01 \times 10^{23}\text{ particles}$$

$$T_i = \frac{P_iV_i}{Nk_B} = \frac{(3.03 \times 10^5\text{ N/m}^2)(4.00 \times 10^{-3}\text{ m}^3)}{(3.01 \times 10^{23})(1.38 \times 10^{-23}\text{ J/K})} = 292\text{ K}.$$

We next use Eq. 20.31 for process A:

$$\begin{aligned} Q_A &= Nk_BT_i\ln\left(\frac{V_f}{V_i}\right) \\ &= (3.01 \times 10^{23})(1.38 \times 10^{-23})(292\text{ K})\ln\left(\frac{1.60 \times 10^{-2}\text{ m}^3}{4.00 \times 10^{-3}\text{ m}^3}\right) \\ &= 1.68 \times 10^3\text{ J}. \end{aligned}$$

For the isobaric leg of process B, we know that $P_1 = P_i$ and thus we can get T_1 from the ideal gas law in the form

$$\begin{aligned} T_1 = \frac{P_iV_1}{Nk_B} &= \frac{(3.03 \times 10^5\text{ N/m}^2)(1.60 \times 10^{-2}\text{ m}^3)}{(3.01 \times 10^{23})(1.38 \times 10^{-23}\text{ J/K})} \\ &= 1.17 \times 10^3\text{ K}. \end{aligned}$$

Using this temperature in Eq. 20.21 and remembering that for our diatomic H_2 particles $C_P = 7k_B/2$, we get Q for this leg of process B:

$$\begin{aligned} Q_{B,\text{isobaric}} &= NC_P(T_1 - T_i) \\ &= (3.01 \times 10^{23})\left[\tfrac{7}{2}(1.38 \times 10^{-23}\text{ J/K})\right] \\ &\quad \times (1.17 \times 10^3\text{ K} - 292\text{ K}) \\ &= 1.28 \times 10^4\text{ J}. \end{aligned}$$

For the isochoric compression in process B, Eq. 20.12 is $Q_{B,\text{isochoric}} = (\frac{d}{2})Nk_B(T_f - T_1)$. Because $T_f = T_i$, we can rewrite this as

$$\begin{aligned} Q_{B,\text{isochoric}} &= \tfrac{5}{2}(3.01 \times 10^{23})(1.38 \times 10^{-23}\text{ J/K}) \\ &\quad \times (292\text{ K} - 1170\text{ K}) \\ &= -9.12 \times 10^3\text{ J}. \end{aligned}$$

The algebraic sum of $Q_{B,\text{isobaric}}$ and $Q_{B,\text{isochoric}}$ gives us the quantity of energy added thermally to the sample in process B:

$$Q_B = 1.28 \times 10^4\text{ J} + (-9.12 \times 10^3\text{ J}) = 3.68 \times 10^3\text{ J}.$$

The difference in the energies transferred thermally to the gas in the two processes is

$$Q_B - Q_A = (3.68 \times 10^3\text{ J}) - (1.68 \times 10^3\text{ J}) = 2.00 \times 10^3\text{ J}. ✔$$

PRACTICE

❹ EVALUATE RESULT From our PV diagram, the area under the curve for process B is larger than the area under the curve for process A. Because $V_f > V_i$, the work done on the gas is negative, so we expect that $W_B - W_A < 0$. Therefore $Q_B - Q_A > 0$, in agreement with our calculation. The work done on the gas for the isobaric leg in process B is

$$W_B = -P_i \Delta V = -(3.03 \times 10^5 \text{ N/m}^2)(12.0 \times 10^{-3} \text{ m}^3)$$
$$= -3.64 \times 10^3 \text{ J}.$$

The initial and final states are at the same temperature, so $P_f = P_i V_i / V_f = P_i/4$. Therefore the work done on the gas during an isobaric expansion at the final pressure would be

$$W_1 = -P_f \Delta V = -P_i \Delta V/4 = -9.1 \times 10^2 \text{ J}.$$

The difference in the energies transferred thermally to the gas in the two processes should have a magnitude near the average of these; $\frac{1}{2}(W_B + W_1) = -2270$ J. A difference of 2000 J for these two processes is therefore reasonable.

Guided Problem 20.4 Two more expansions

Consider two processes for taking a 0.200-mol sample of helium gas (He, monatomic particles) from an initial state i to a final state f. The gas behaves ideally, and in both processes the initial volume and pressure are 8.00×10^{-3} m^3 and 9.00×10^4 N/m^2. In process A, the gas undergoes a quasistatic isentropic expansion to a volume of 1.20×10^{-2} m^3. In process B, the gas undergoes an isochoric pressure decrease followed by an isobaric expansion to a volume of 1.20×10^{-2} m^3, both quasistatically. What is the difference in the amounts of work done on the gas during the two processes?

❶ GETTING STARTED

1. Draw a PV diagram showing both processes. Are there any intermediate states you can identify in process B?
2. What are your target quantities, and what principles can you use to determine them?
3. What is the heat capacity ratio for a sample of helium gas?
4. Which quantities do you know? Which quantities are unknown for the initial and final states in both processes? Which are unknown for the intermediate state in process B?

❷ DEVISE PLAN

5. Which quantities must you know to determine the work done in process A? Which must you know to determine the work done in each part of process B?
6. How can you use the fact that process A is isentropic to determine some of your target quantities?

❸ EXECUTE PLAN

7. Determine the number of gas particles in the sample and then the initial temperature.
8. Use the fact that process A is isentropic to determine the final temperature and pressure.
9. Determine the remaining quantities you seek.

❹ EVALUATE RESULT

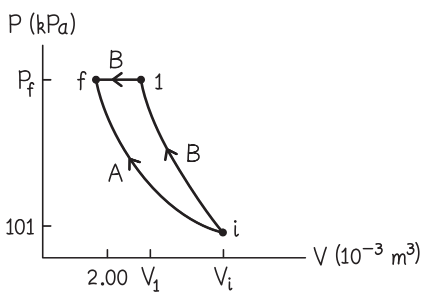
10. Based on your PV diagram, do you expect the difference in work done to be positive, zero, or negative? Does your result agree with your expectation?

Worked Problem 20.5 Two-step versus one-step compression

Consider two processes for taking a 0.200-mol sample of helium gas (He, monatomic particles) from an initial state i to a final state f. The gas behaves ideally, and in both processes the initial temperature and pressure are 150 K and 1.01×10^5 N/m^2 = 101 kPa. In process A, the gas undergoes a quasistatic isothermal compression to a volume of 2.00×10^{-3} m^3. In process B, the gas undergoes a quasistatic isentropic compression followed by a quasistatic isobaric compression to $V_f = 2.00 \times 10^{-3}$ m^3. What are (*a*) the difference $|Q_A - Q_B|$ in the quantity of energy transferred thermally to the gas in the two processes, (*b*) the difference $|W_A - W_B|$ in the amount of work done on the gas in the two processes, and (*c*) the entropy change resulting from either process?

❶ GETTING STARTED We begin by drawing a PV diagram showing both processes (Figure WG20.3). Process A is an isothermal compression, so we show its path running along an isotherm leftward (decreasing volume) and upward (increasing pressure) from initial state i to final state f. The path for process B has two legs. The isentropic compression we represent by a curve running leftward along an isentrope from initial state i to intermediate state 1 at volume V_1 and pressure $P_1 = P_f$. The isobaric compression we represent by a horizontal line running leftward from intermediate state 1 to final state f.

Figure WG20.3



PRACTICE

2 DEVISE PLAN We need to (*a*) compare the energy transferred thermally, Q, for two processes that take a gas sample from an initial state to a final state, (*b*) compare the work done on the sample in the two cases, and (*c*) determine how the entropy of the sample changes in going from the initial to final state. For parts *a* and *b*, we know that, in a system going from one macrostate to another, the change in the system's thermal energy, ΔE_{th}, is independent of path and so is the same for the two processes. Process A is isothermal, so $T_f = T_i = 150$ K. Hence Eq. 20.15 tells us that $\Delta E_{th} = NC_V\Delta T = 0$ for both processes. Therefore the Eq. 20.2 form of the energy law requires that $\Delta E_{th} = 0 = W_A + Q_A = W_B + Q_B$. From this we conclude that $|Q_A - Q_B| = |W_A - W_B|$; that is, our answer to part *b* is going to be identical to our answer to part *a*.

We can use Eq. 20.31, $Q = -W = Nk_BT\ln(V_f/V_i)$, to calculate Q_A because we can obtain V_i from the ideal gas law. For the isentropic leg of process B, we know that the energy transferred thermally to get to intermediate state 1 is $Q = 0$. Thus the only quantity we need to calculate in order to know Q_B is the value of Q for the isobaric leg, and for this we can use Eq. 20.21, $Q = NC_P\Delta T$. We know C_P from Eq. 20.25, but we need to calculate T_1 in order to know ΔT. For T_1, we can use Eq. 20.47, $P_i^{\frac{1}{\gamma}-1}T_i = P_f^{\frac{1}{\gamma}-1}T_f$, but to do so we need a value for P_f, and for that we turn to the ideal gas law once again.

The last step in solving part *a* is to determine the difference $|Q_A - Q_B|$. For part *b*, $|Q_A - Q_B| = |W_A - W_B|$, as noted above. In part *c*, we can use Eq. 20.35, $\Delta S = N\ln(V_f/V_i)$, to obtain the entropy change, which is the same for the two processes because both are changes from the same macrostate i to the same macrostate f.

3 EXECUTE PLAN (*a*) We begin with the ideal gas law (Eq. 19.51) to determine the volume V_i. Avogadro's number tells us that for this sample of helium particles, $N = (0.200\text{ mol})(6.02 \times 10^{23}\text{ particles/mol}) = 1.20 \times 10^{23}$ particles. Thus

$$\begin{aligned} V_i &= \frac{Nk_BT_i}{P_i} \\ &= \frac{(1.20 \times 10^{23})(1.38 \times 10^{-23}\text{ J/K})(150\text{ K})}{1.01 \times 10^5\text{ N/m}^2} \\ &= 2.46 \times 10^{-3}\text{ m}^3. \end{aligned}$$

We then use Eq. 20.31 to determine the quantity of energy transferred thermally in process A:

$$\begin{aligned} Q_A &= Nk_BT_i\ln\left(\frac{V_f}{V_i}\right) \\ &= (1.20 \times 10^{23})(1.38 \times 10^{-23}\text{ J/K}) \\ &\quad \times (150\text{ K})\ln\left(\frac{2.00 \times 10^{-3}\text{ m}^3}{2.46 \times 10^{-3}\text{ m}^3}\right) \\ &= -51.4\text{ J}. \end{aligned}$$

The plan for Q_B is to apply Eq. 20.21, $Q = NC_P\Delta T$, to the isobar from intermediate state 1 to final state f. We know that $T_f = 150$ K, but we don't know T_1, which is the value of T_i we need in $\Delta T = T_f - T_i$ along the isobar. To calculate T_1, we begin by solving Eq. 19.51 for P_f, remembering that $T_f = T_i = 150$ K for both processes:

$$\begin{aligned} P_f &= \frac{Nk_BT_f}{V_f} \\ &= \frac{(1.20 \times 10^{23})(1.38 \times 10^{-23}\text{ J/K})(150\text{ K})}{2.00 \times 10^{-3}\text{ m}^3} \\ &= 1.24 \times 10^5\text{ N/m}^2 = 124\text{ kPa}. \end{aligned}$$

Because this leg of process B is isobaric, we know that $P_f = P_1$.

Now we turn to the isentropic leg of process B. Here the quantity we are after, T_1, is what we must use for T_f in any equations we use, and P_1 is what we must use for P_f. We begin by solving Eq. 20.47 for T_f, which is now our T_1:

$$T_1 = \left(\frac{P_i}{P_1}\right)^{(\frac{1}{\gamma}-1)} T_i.$$

Because $\gamma = \frac{5}{3}$, the exponent $(1/\gamma) - 1$ is $-\frac{2}{5}$, and so

$$T_1 = \left(\frac{P_i}{P_1}\right)^{-2/5}(T_i) = \left(\frac{1.01 \times 10^5\text{ N/m}^2}{1.24 \times 10^5\text{ N/m}^2}\right)^{-2/5}(150\text{ K}) = 163\text{ K}.$$

Now we are ready for Eq. 20.21. For monatomic helium particles with their three degrees of freedom, $C_P = (\frac{5}{2})k_B$, and thus Eq. 20.21 is

$$\begin{aligned} Q_B &= (1.20 \times 10^{23})(\tfrac{5}{2})(1.38 \times 10^{-23}\text{ J/K})(150\text{ K} - 163\text{ K}) \\ &= -53.8\text{ J}. \end{aligned}$$

PRACTICE

Therefore the difference in the quantities of energy transferred thermally to the gas in the two processes is

$$|Q_A - Q_B| = |(-51.4\text{ J}) - (-53.8\text{ J})| = 2.4\text{ J.} ✔$$

(*b*) As noted in Devise Plan, the quantity $|Q_A - Q_B|$ is equal to the difference in the work done on the gas for the two processes, $|W_A - W_B| = 2.4$ J. ✔

(*c*) Equation 20.35 gives us the change in entropy as the gas goes from state i to state f via either process:

$$\Delta S = N \ln\left(\frac{V_f}{V_i}\right)$$

$$= (1.20 \times 10^{23}) \ln\left(\frac{2.00 \times 10^{-3}\text{ m}^3}{2.46 \times 10^{-3}\text{ m}^3}\right) = -2.48 \times 10^{22}. ✔$$

❹ **EVALUATE RESULT** By comparing the areas under the paths in the *PV* diagram, we expect that the difference in the work done on the gas in the two processes is approximately $W_B - W_A = -(\frac{1}{3})(P_f - P_i)\Delta V$. From our calculations, the change in volume is $\Delta V = V_f - V_i = -0.46 \times 10^{-3}$ m, and the change in pressure is $P_f - P_i = 0.23 \times 10^5\text{ N/m}^2$. Therefore the difference in the work done on the gas is approximately

$$W_B - W_A = -(\tfrac{1}{3})(0.23 \times 10^5\text{ N/m}^2)(-0.46 \times 10^{-3}\text{ m})$$

$$= 3.5\text{ J},$$

which is in reasonable agreement with our calculation in part *a*. That our calculated entropy change in part *c* is negative makes sense because the gas volume decreases and the particles have fewer basic states available in the smaller volume. This same entropy change must occur in process B, and so calculating that entropy change is a further check on our result in part *c*. Here we need to calculate only the change for the isobaric leg, which is given by Eq. 20.41:

$$\Delta S = \frac{NC_P}{k_B} \ln\left(\frac{T_f}{T_1}\right)$$

$$= \frac{(1.20 \times 10^{23})(\frac{5}{2}k_B)}{k_B} \ln\left(\frac{150\text{ K}}{163\text{ K}}\right) = -2.49 \times 10^{22},$$

in agreement with the ΔS value we calculated with Eq. 20.35.

Guided Problem 20.6 Two-step versus one-step entropy change

Consider two processes for taking a 0.600-mol sample of hydrogen gas (H_2, diatomic particles) from an initial state i to a final state f. The gas behaves ideally, and in both processes the initial volume and pressure are $6.00 \times 10^{-3}\text{ m}^3$ and $2.00 \times 10^5\text{ N/m}^2$. In process A, the gas undergoes a quasistatic isentropic expansion to a final volume of $2.00 \times 10^{-2}\text{ m}^3$. In process B, the gas undergoes a quasistatic isothermal expansion followed by a quasistatic isochoric pressure decrease. What is the entropy change for process A and for each leg of process B?

❶ **GETTING STARTED**

1. Make a *PV* diagram showing both processes. Are there any intermediate states that you can identify in process B?
2. What is the heat capacity ratio for H_2?

❷ **DEVISE PLAN**

3. In both legs of process B, on what quantities does the change in entropy depend?
4. Which quantities do you know and which are unknown for the initial, intermediate, and final states of process B? Which of these are your target quantities?
5. How can you use the fact that process A is isentropic to determine some of your target quantities?
6. How can the ideal gas law help you determine some of your target quantities?

❸ **EXECUTE PLAN**

7. Determine the number of gas particles in the sample and then ΔS for process A.
8. Determine the initial temperature of the gas and then the final temperature and pressure.
9. Determine ΔS for each leg of process B.

❹ **EVALUATE RESULT**

10. Do you expect a positive, zero, or negative value of ΔS for the isothermal leg of process B? For the isochoric leg?
11. Does your result for process B agree with your expectation?
12. Does the algebraic sum of your entropy changes for process B match your result for process A?

Worked Problem 20.7 Cool drink of water

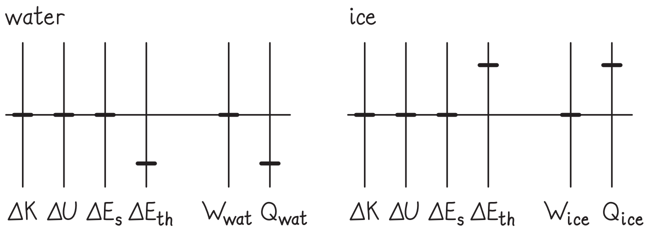
Three 0.0100-kg ice cubes initially at $-10.0\ °C$ are placed in 0.250 kg of water initially at 25.0 °C. Once all the ice melts, what is the temperature of the liquid in the glass? The specific heat capacity for ice is $2.090 \times 10^3\ J/K \cdot kg$, that for liquid water is $4.181 \times 10^3\ J/K \cdot kg$, and for ice the specific transformation energy for melting is $3.34 \times 10^5\ J/kg$.

❶ **GETTING STARTED** We must determine the final temperature T_f for a system that starts out as 0.250 kg of liquid water plus 0.0300 kg of ice and ends up as 0.280 kg of liquid water. We assume that the water and ice form a closed system, which means no energy is thermally transferred either to or from the mixture in the glass. Because the temperatures are given in degrees Celsius, we note that $-10\ °C = 263\ K$, $0\ °C = 273\ K$, and $25\ °C = 298\ K$.

To melt the ice, energy is thermally transferred out of the 0.250 kg of water, lowering the temperature of the water to T_f. This energy lost by the water is thermally transferred to the ice, first raising its temperature to 0 °C, then melting the cubes while their temperature stays at 0 °C, and then raising the temperature of the 0.0300 kg of liquid water formed from the cubes to T_f.

We draw two energy diagrams. One for the 0.250 kg that is liquid water throughout the process and one for the 0.0300 kg that is initially ice (Figure WG20.4). In the water diagram, $Q_{water} < 0$ because energy is thermally transferred out of the water. In the ice diagram. $Q_{ice} > 0$ because energy is thermally transferred to the cubes.

Figure WG20.4



❷ **DEVISE PLAN** The change in temperature for the 0.250 kg of water is $\Delta T_{water} = T_f - T_{water,i} = T_f - 298\ K$. We know the specific heat capacity of water, $c_{water} = 4.181 \times 10^3\ J/K \cdot kg$, and we know the mass of the water, which means we can use Eq. 20.49 to calculate Q_{water}, the quantity of energy transferred thermally out of the water to cause this temperature change ΔT_{water}. The quantity Q_{ice} of energy thermally transferred to the ice cubes to increase their temperature to T_f has three components. First the temperature of the cubes is raised from 263 K to 273 K, so that for this increase we can say $\Delta T_{ice,1} = 10.0\ K$. Knowing the specific heat capacity of ice, $c_{ice} = 2.090 \times 10^3\ J/K \cdot kg$, and the mass of the ice, we can use Eq. 20.49 to calculate the first component of Q_{ice}. Next the cubes melt to 0.0300 kg of liquid water while the temperature remains constant at 273 K. Knowing the specific transformation energy needed to melt ice to liquid water, $L_m = 3.34 \times 10^5\ J/kg$, and the mass of the ice, we can use Eq. 20.55 to calculate the second component of Q_{ice}. Finally the temperature of the 0.0300 kg of liquid water formed from the cubes is raised to T_f, and we express this temperature change as $\Delta T_{ice,2} = T_f - 273\ K$. To determine this third component of Q_{ice}, we use Eq. 20.49 again with the specific heat capacity of water, but this time with a mass of 0.0300 kg, to calculate the thermal energy transferred for this temperature change. Because the system is closed, $Q_{ice} = -Q_{water}$, and we use this condition to obtain T_f.

❸ **EXECUTE PLAN** Equation 20.49 gives an expression for the energy thermally transferred out of the water:

$$Q_{water} = m_{water} c_{water} \Delta T_{water}.$$

Next we use Eqs. 20.49 and 20.55 to write an expression for the three components of Q_{ice}:

$$Q_{ice} = m_{ice} c_{ice} \Delta T_{ice,1} + m_{ice} L_m + m_{ice} c_{water} \Delta T_{ice,2}.$$

Note that in the final term we must use the specific heat capacity for liquid water because now the 0.0300 kg of ice is liquid. Setting $Q_{ice} = -Q_{water}$ yields

$$m_{ice} c_{ice} \Delta T_{ice,1} + m_{ice} L_m + m_{ice} c_{water} \Delta T_{ice,2} = -m_{water} c_{water} \Delta T_{water}.$$

To have this equation contain our target quantity, T_f, we substitute for the delta factors:

$$m_{ice}c_{ice}(\Delta T_{ice,1}) + m_{ice}L_m + m_{ice}c_{water}(T_f - T_{melt})$$
$$= -m_{water}c_{water}(T_f - T_{water,i})$$

$$T_f = \frac{m_{water}c_{water}T_{water,i} + m_{ice}c_{water}T_{melt}}{(m_{water} + m_{ice})c_{water}} - \frac{m_{ice}c_{ice}(\Delta T_{ice,1}) + m_{ice}L_m}{(m_{water} + m_{ice})c_{water}}$$

$$= \frac{(0.250\ \text{kg})(4.181 \times 10^3\ \text{J/K}\cdot\text{kg})(298\ \text{K})}{(0.280\ \text{kg})(4.181 \times 10^3\ \text{J/K}\cdot\text{kg})} + \frac{(0.0300\ \text{kg})(4.181 \times 10^3\ \text{J/K}\cdot\text{kg})(273\ \text{K})}{(0.280\ \text{kg})(4.181 \times 10^3\ \text{J/K}\cdot\text{kg})}$$
$$- \frac{(0.0300\ \text{kg})(2.090 \times 10^3\ \text{J/K}\cdot\text{kg})(10.0\ \text{K})}{(0.280\ \text{kg})(4.181 \times 10^3\ \text{J/K}\cdot\text{kg})} - \frac{(0.0300\ \text{kg})(3.34 \times 10^5\ \text{J/kg})}{(0.280\ \text{kg})(4.181 \times 10^3\ \text{J/K}\cdot\text{kg})}$$

$$T_{sys,f} = 286\ \text{K} = 13.0\ ^\circ\text{C.} \checkmark$$

❹ **EVALUATE RESULT** Our result is lower than the initial temperature of the water and higher than the initial temperature of the ice, in line with our expectations. Because the mass of the water is about eight times the mass of the ice, we would expect T_f to be close to the initial water temperature—say, about 20 °C. However, once we take into account the specific transformation energy for melting the ice, we expect T_f to be lower than this, and so our 13 °C result is reasonable.

Guided Problem 20.8 Hot and cool water

Sample A is 1.50 kg of water initially at 100 °C. Sample B is 2.00 kg of water initially at 20.0 °C. When the two samples are brought into thermal contact with each other and allowed to reach equilibrium, what is the change in entropy $\Delta S_A + \Delta S_B$?

❶ **GETTING STARTED**

1. Define a closed system.
2. List the unknown quantities and decide which are your target variables.
3. Are there any quantities you may need to look up?

❷ **DEVISE PLAN**

4. Draw any relevant energy diagrams.
5. Which quantities do you need to know in order to determine the entropy change?
6. What principle do you need to use to determine the quantity of energy transferred thermally from one sample to the other?
7. Write an expression that contains the final equilibrium temperature as one term.

❸ **EXECUTE PLAN**

8. Calculate the final equilibrium temperature.
9. Use this temperature to determine the entropy changes for the two samples.

❹ **EVALUATE RESULT**

10. Do you expect the entropy to increase, stay the same, or decrease?
11. Does your result agree with your expectation?

Answers to Review Questions

1. The symbol is Q, and it is positive when energy is transferred into the system.
2. Energy can be transferred to a system by heating it or by doing work on it. Heating transfers energy by means of a thermal interaction between system and surroundings, as when a flame (surroundings) transfers some of its thermal energy to a volume of water (system). Doing work on the system transfers energy by means of a mechanical interaction, as when a system's gravitational potential energy is increased by moving a boulder up a hill.
3. An adiabatic process is one in which no energy is transferred thermally.
4. A quasistatic process is one carried out in such a way that the system remains close to equilibrium at every instant during the process.
5. The entropy increases because thermally transferring energy to the system increases the system's temperature, and entropy increases as temperature increases.
6. The entropy does not change.
7. Some physical properties of matter change as the temperature changes.
8. The triple point of water is the unique pressure-temperature combination (610 Pa, 0.01 °C) at which ice, liquid water, and water vapor coexist in equilibrium. The temperature at which this occurs is defined as 273.16 K on the Kelvin scale.
9. The vessel holding the gas is put into water that is at the triple-point temperature, $T_{tp} = 273.16$ K, and the height h_{tp} above the reference level is noted for the mercury level in the unattached arm. You then draw a straight line on a plot of height measured vs. temperature that passes through the origin (0, 0) and (h_{tp}, T_{tp}). For any other absolute temperature, measure the height h and then determine the corresponding absolute temperature T, such that the point (h, T) lies on the straight line.
10. At absolute zero, the system has only one basic state available, which means its entropy is zero, and its thermal energy is a minimum, which means $T = 0$ is the lowest possible temperature for anything.
11. The heat capacity of a system is the ratio of the quantity of energy transferred thermally into the system to the change in the absolute temperature caused by the added energy.
12. A degree of freedom is an independent way a particle can store thermal energy. The number of degrees of freedom for a particle is equal to the number of ways the particle can translate, rotate, and vibrate.
13. An atom has three degrees of freedom, one for each direction (x, y, z) in which it translates in space. At moderate temperatures, a diatomic molecule has five, three associated with the x, y, z directions in which it translates and two associated with its rotation about the two axes that do not run through the centers of both atoms.
14. The energy is distributed such that each degree of freedom has a portion of the particle's thermal energy equal to the *equipartition energy share*, $k_B T/2$.
15. The number of degrees of freedom a particle has is equal to the number of ways the particle can translate, rotate, and vibrate. Each motion requires a specific quantum (amount) of energy added to the particle. Thus you can compare the equipartition share of energy at a particular temperature, $k_B T/2$, with the quantum of energy required for a given motion. If $k_B T/2$ is less than the required quantum of energy, the motion cannot take place and that degree of freedom does not contribute to the heat capacity per particle of gas.
16. Each point represents an equilibrium state that has a definite temperature, thermal energy, and entropy (as well as a definite pressure and volume).
17. An isotherm is a curve in a PV diagram whose points represent, for a fixed amount of ideal gas, all the equilibrium states that have the same temperature. An isentrope is a curve on the diagram whose points represent all the equilibrium states that have the same entropy.
18. Because a quasistatic process is carried out in such a way that the system passes from one equilibrium state to another, it is represented by a continuous path that runs from the point representing the initial state on the diagram to a second point representing the final state.
19. (*a*) and (*b*), because both depend on only the equilibrium state, which means changes in these quantities depend on only the initial and final states, not on the path taken. (*c*) is path-dependent because it equals an area under a curve in the PV diagram for the process and that area depends on the path taken from initial to final state. (*d*) is path-dependent because it is the difference between ΔE_{th} (path-independent) and W (path-dependent).
20. In a system in which there are no changes in kinetic, potential, and source energies, $\Delta K = \Delta U = \Delta E_s = 0$, and the energy law, $\Delta E = W + Q$, reduces to $\Delta E_{th} = W + Q$ (Eq. 20.2).
21. The work done depends on the gas pressure and on the change in the gas volume: $W = -\int_{V_i}^{V_f} P\,dV$ (Eq. 20.8).
22. The minus sign indicates that if the gas volume increases, the work done on the gas by its surroundings is negative. If the volume decreases, the work done on the gas is positive.
23. In an isochoric process, the work done on the gas is zero, so the change in thermal energy equals the quantity of energy thermally transferred to the gas ($\Delta E_{th} = Q$; Eq. 20.11). In an isentropic process, no energy is transferred thermally ($Q = 0$), so the change in thermal energy equals the work done on the gas ($\Delta E_{th} = W$; Eq. 20.18).
24. You must know the number of gas particles N, the constant-volume heat capacity per particle C_V of the gas, and the change in temperature resulting from the process: $\Delta E_{th} = NC_V \Delta T$ (Eq. 20.15).
25. The heat capacity per particle at constant pressure is greater than that at constant volume by an amount equal to Boltzmann's constant: $C_P = C_V + k_B$ (Eq. 20.24).
26. Because the change in the thermal energy of the gas is zero for an isothermal process, the Eq. 20.2 form of the energy law, $\Delta E_{th} = W + Q$, requires that the quantity of energy transferred thermally equal the negative of the work done on the gas: $Q = -W$ (Eq. 20.28).
27. The pressure-volume relationship is that the product of the pressure and the volume raised to the power $\gamma = C_P/C_V$ is constant, which means $P_i V_i^\gamma = P_f V_f^\gamma$ (Eq. 20.46). The pressure-temperature relationship is $(P_i^{\frac{1}{\gamma}-1})T_i = (P_f^{\frac{1}{\gamma}-1})T_f$ (Eq. 20.47). The volume-temperature relationship is $T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$ (Eq. 20.44).
28. The related intensive quantity is specific heat capacity. For any given substance, the specific heat capacity is the quantity of energy transferred thermally divided by the mass of the system and by the temperature change.
29. The units of heat capacity are joules per kelvin; those of specific heat capacity are joules per kelvin per kilogram.

Answers to Guided Problems

Guided problem 20.2 536 J
Guided problem 20.4 $W_A - W_B = -73$ J
Guided problem 20.6 $\Delta S_A = 0$, $\Delta S_{B,isothermal} = +4.35 \times 10^{23}$, $\Delta S_{B,isochoric} = -4.35 \times 10^{23}$
Guided problem 20.8 $\Delta S_A + \Delta S_B = +6.7 \times 10^{24}$

Guided Practice by Chapter

21 Degradation of Energy

Review Questions

Answers to these questions can be found at the end of this chapter.

21.1 Converting energy

1. Which type of energy is associated with the complementary state changes that occur (*a*) when a gas burns and (*b*) when a ball drops from a rooftop to ground level?
2. How is the work done on a system defined?
3. Describe the energy transfers associated with the operations of a car engine and a refrigerator.
4. What is a steady device, and why is such a device the best choice for any everyday situation where energy must be continuously converted from one form to another?
5. What constraints are imposed by the entropy law (Eq. 19.5, $\Delta S > 0$ for a closed system moving toward equilibrium) in (*a*) a steady device designed to convert mechanical energy to thermal energy, as in Figure 21.8*a*, and (*b*) a steady device designed to convert thermal energy to mechanical energy, as in Figure 21.8*b*?

21.2 Quality of energy

6. What is the entropy gradient dS/dE equal to? What does it tell you about the relationship between a thermal reservoir's entropy and temperature when energy is added to or taken from the reservoir?
7. What is the entropy cost of an energy change?
8. Describe the energy transfer that takes place when two thermal reservoirs at different temperatures (the environment) are connected by a copper rod (the system). Is the energy transfer reversible or irreversible?
9. For a given amount of energy change, what is the relationship between energy quality and dS/dE? What is energy quality a measure of?
10. What is the energy quality associated with a mechanical input of energy?

21.3 Heat engines and heat pumps

11. What is a heat engine? Describe how a car engine operates as a heat engine.
12. Describe how entropy changes and energy transfers are illustrated in an entropy diagram.
13. How is the efficiency of a heat engine defined?
14. What is a heat pump?
15. Describe the difference in how a heat pump operates as an air-conditioner and as a heating unit for a building.
16. Define the terms *coefficient of performance of cooling* and *coefficient of performance of heating.*

21.4 Thermodynamic cycles

17. On the *PV* diagram that represents the working substance in a heat engine or heat pump, why is the cycle direction clockwise for a heat engine and counterclockwise for a heat pump?
18. Describe the steps of a Carnot cycle.
19. What properties of the Carnot cycle make it unsuitable for operating everyday devices?
20. Describe the steps of a Brayton cycle.
21. What everyday devices run on Brayton cycles?

21.5 Entropy constraints on energy transfers

22. Describe how Eq. 21.5, $\Delta S_{env} \geq 0$, restricts cyclic devices designed to transfer energy thermally from one thermal reservoir to another.
23. How can an entropy diagram be used to compute the change in entropy associated with an energy transfer for a steady process?

21.6 Heat engine performance

24. For a reversible heat engine, what is the relationship between Q_{in}, the thermal energy taken in from the higher-temperature reservoir, and Q_{out}, the thermal energy discarded to the lower-temperature reservoir, and the temperatures of the reservoirs?
25. What restrictions does the entropy law impose on the efficiency of heat engines? Consider both reversible and irreversible engines in answering.
26. In the energy diagram that represents a heat pump used as a heater, do the variables Q_{in} and T_{in} belong to the higher-temperature reservoir or the lower-temperature reservoir? What about in the energy diagram for a heat pump used as an air-conditioner?
27. How is the coefficient of performance of heating for a reversible heat pump related to the thermal energy output to the higher-temperature reservoir and the thermal energy input from the lower-temperature reservoir? How is its maximum $\text{COP}_{heating}$ defined?
28. What is the relationship between the coefficient of performance of heating for a heat pump and its coefficient of performance of cooling?
29. Does the maximum efficiency of any heat engine depend on what is used as the working substance?

21.7 Carnot cycle

30. When determining the efficiency $-W/Q_{in}$ of a heat engine running a Carnot cycle, why must you be concerned only with the work done during the two isothermal steps? Why doesn't the work done during the isentropic steps factor into the efficiency calculation?
31. How does the efficiency of an engine that operates according to a Carnot cycle compare to the maximum efficiency of any heat engine?

21.8 Brayton cycle

32. What are two advantages of the Brayton cycle over the Carnot cycle?
33. What is the relationship between the efficiency of a Brayton cycle for a jet engine and the various temperatures in the cycle?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The change in entropy for system plus environment when 1 GJ of thermal energy leaks out of a house on a cold winter day [G, O]
2. The change in entropy per second for a 1-kW baseboard heater that uses hot water as its working substance [G, K]
3. The maximum efficiency for a heat engine that converts the thermal energy of ocean water to usable energy by using warm surface water as the higher-temperature reservoir and cold deep water as the lower-temperature reservoir [I, D, N]
4. The input water flow (in kilograms per second) needed to generate 1 GW of power from the heat engine of Developing a Feel 3 [I, D, M, U, H]
5. The maximum coefficient of performance of cooling for a heat pump used to cool a house on a hot summer day [G, D, T, R]
6. The volume of outside air needed per day by a heat pump to heat a house on a cold winter day, assuming the maximum amount by which the outside air can be cooled is 7 K [G, O,V, F, A, E, J]
7. The volume of groundwater needed per month by a heat pump to heat a house in winter when the groundwater is the lower-temperature reservoir for the pump, assuming the maximum amount by which the water can be cooled is 10 K [G, C, V, F, A, H, M, Q]
8. The maximum coefficient of performance of cooling for a refrigerator that maintains helium in the liquid state [G, P, R]
9. The combustion pressure required to obtain an efficiency of 0.67 in a jet engine on a commercial airliner at cruising altitude [B, L, S]

Hints

A. How much energy is needed to heat a house on a cold winter day?
B. Which thermodynamic cycle is involved?
C. What is the temperature difference?
D. What is the temperature difference?
E. What is the specific heat capacity of air?
F. On a cold day, for what fraction of the day is the heating system in operation?
G. What is the indoor temperature?
H. What is the specific heat capacity of water?
I. What is the temperature of the surface water?
J. What is the mass density of air under these conditions?
K. What is the temperature difference?
L. What is the input pressure at cruising altitude?
M. How is Q_{in} related to the mass of water needed?
N. How are Q_{in} and Q_{out} related to the temperatures of the two reservoirs?
O. What is the temperature difference?
P. What is the highest temperature at which helium remains liquid?
Q. What is the mass density of water?
R. How is the maximum coefficient of performance of cooling related to cycle temperatures?
S. What is the heat capacity ratio for the working substance?
T. With a heat pump, T_{in} is for which reservoir?
U. How is maximum efficiency (from Developing a Feel 3) related to Q_{in} and Q_{out}?
V. What power is supplied by a typical home's central heating system?

Key (all values approximate)

A. 1 GJ; **B.** the Brayton cycle; **C.** 1×10^1 K; **D.** 2×10^1 K; **E.** 1 kJ/K · kg; **F.** as much as 1/3 of the day; **G.** 3×10^2 K; **H.** 4 kJ/K · kg; **I.** 3×10^2 K; **J.** 1 kg/m^3; **K.** 6×10^1 K; **L.** about 30% of the pressure at sea level and so, say, 3×10^4 Pa; **M.** $Q_{in} = m_{water}c_{water}\Delta T$; **N.** for an ideal engine, $Q_{out}/Q_{in} = T_{out}/T_{in}$; **O.** 3×10^1 K; **P.** 4 K; **Q.** 1×10^3 kg/m^3; **R.** $\text{COP}_{\text{cooling, max}} = Q_{in}/W = T_{in}/(T_{out} - T_{in})$; **S.** air, so $\gamma = \frac{7}{5}$; **T.** the lower-temperature reservoir; **U.** $Q_{in} = -W/\eta$ and $W = Q_{out} - Q_{in}$; **V.** 3×10^4 J/s

PRACTICE

Worked and Guided Problems

Procedure: Drawing entropy diagrams for a steady device

1. Draw an entropy gradient axis pointing right. Add a short vertical axis to indicate the origin of the axis. Label the axis $1/k_B T$ to remind yourself of the relationship between entropy gradient and temperature.
2. Mark the positions along the axis of the thermal inputs and outputs of energy of the device. Transfers at higher temperature are on the left, transfers at lower temperature on the right. Use downward pointing arrows for inputs, upward pointing arrows for outputs, and label these Q_{in} and Q_{out}.
3. Mechanical transfers of energy are placed at the origin of the axis. Use downward pointing arrow when the work done on the system is positive and an upward pointing arrow when the work done on the system is negative. Label this arrow by W and indicate whether $W > 0$ or $W < 0$.
4. Draw rectangles to represent each energy transfer. Make the height of each rectangle proportional to the amount of energy transferred (but see point 5 below). Use an arrowhead to indicate the direction of each transfer.
5. For the operation of the device to be permitted by the entropy law, the combined area of the rectangles in which the arrows point right must be larger than the combined area of the rectangles in which the arrows point left.

Procedure: Computing entropy changes from entropy diagrams

To determine the entropy change in the environment from an entropy diagram, compute the area of the rectangles representing the energy transfers as follows.

1. Determine the length of the horizontal part of each rectangle by subtracting the position of the left end of the rectangle on the $1/k_B T$ scale from the position of the right end on the scale. The position of mechanical energy is always at the zero point on this scale. Verify that the result is a positive quantity. (If it isn't, you made a mistake somewhere.)
2. Express the height of each rectangle in terms of Q_{in}, Q_{out}, and W. Remember that the height must be a positive quantity; Q_{in} and Q_{out} are always positive, but W can be positive or negative. Avoid using the input or output where two rectangles meet; if necessary you can use Eq. 21.2, $W = Q_{out} - Q_{in}$, to express the height in terms of the quantities that are given in the problem.
3. Determine the sign of the entropy change. If the rectangle points left (energy upgrade), the entropy change is negative. If the rectangle points right (energy degradation), the entropy change is positive.
4. The entropy change for each rectangle is given by the area of the rectangle (the length multiplied by the height computed in steps 1 and 2), preceded by the algebraic sign determined in step 3.

$$\Delta S = (\text{sign}) \times (\text{length}) \times (\text{width}).$$

5. The entropy change for the process is the sum of the individual entropy changes you have calculated.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

PRACTICE

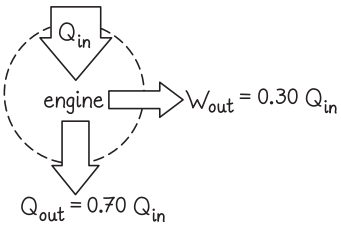
Worked Problem 21.1 Auto energy conversion

In an automobile moving at 95 km/h, the engine burns gasoline and delivers mechanical energy at a rate of 2.7×10^4 J/s with an efficiency of 0.30. What volume of gasoline is burned per second? For gasoline, the specific transformation energy for combustion is 4.4×10^7 J/kg and the mass density is 7.2×10^2 kg/m^3.

❶ GETTING STARTED We have an automobile engine burning gasoline, which means releasing the chemical energy of the gasoline, and this energy is converted to the mechanical energy that moves the vehicle. We are told the rate at which the mechanical energy is delivered and the engine efficiency, and we must determine how much gasoline is burned each second to sustain that rate of energy delivery.

This is a problem in which a quantity Q_{in} of thermal energy is transferred from the gasoline into the engine. The mechanical output of energy W_{out} of the engine is 30% of Q_{in} and the remaining 70% goes to the environment as Q_{out}. We know that the device is a steady device, so the change in energy of the device is zero. Figure WG 21.1 is an energy input-output diagram with $W_{out} = 0.30\ Q_{in}$ and $Q_{out} = 0.70\ Q_{in}$.

Figure WG21.1



❷ **DEVISE PLAN** We know that the rate at which mechanical energy is delivered, 2.7×10^4 J/s, is only 30% of the rate of energy Q_{in} released from the burning gasoline. We can use this information in Eq. 21.21 to determine Q_{in}. By differentiating Eq. 21.21 with respect to time, we get the thermal energy input per second from the burning gasoline. Next we can determine the mass of gasoline burned per second by dividing the thermal energy input per second by the specific transformation energy for combustion. Dividing the mass of gasoline burned per second by the mass density of the gasoline gives us the volume burned per second.

❸ **EXECUTE PLAN** Because the engine does work on the environment, the work done on the engine is negative, which makes the rate at which work is done $dW/dt = -2.7 \times 10^4$ J/s. Differentiating Eq. 21.21 gives us the thermal energy input per second. Thus we have

$$\frac{dQ_{in}}{dt} = -\frac{1}{\eta}\frac{dW}{dt} = -\frac{1}{0.30}(-2.7 \times 10^4\ \text{J/s}) = 9.0 \times 10^4\ \text{J/s}.$$

We now divide this rate of energy input by the specific transformation energy for combustion to determine the mass of gasoline burned per second:

$$\frac{dm_{gas}}{dt} = \frac{1}{L}\frac{dQ_{in}}{dt} = \frac{9.0 \times 10^4\ \text{J/s}}{4.4 \times 10^7\ \text{J/kg}} = 2.0 \times 10^{-3}\ \text{kg/s}.$$

We now convert this to the volume of gasoline burned per second:

$$\frac{dV_{gas}}{dt} = \frac{1}{\rho}\frac{dm_{gas}}{dt} = \frac{2.0 \times 10^{-3}\ \text{kg/s}}{7.2 \times 10^2\ \text{kg/m}^3} = 2.8 \times 10^{-6}\ \text{m}^3/\text{s}. \checkmark$$

❹ **EVALUATE RESULT** We can check the reasonableness of our result by converting the volume of gasoline burned per second to what is colloquially called *mileage*, the number of miles a vehicle travels per gallon of gasoline burned. The volume of gasoline burned each hour is

$$(2.8 \times 10^{-6}\ \text{m}^3/\text{s})\left(\frac{3600\ \text{s}}{1\ \text{h}}\right) = 1.0 \times 10^{-2}\ \text{m}^3/\text{h}.$$

To convert to gallons per hour, we reason that 1 gal is 4 qt and 1 qt is about the same volume as 1 L, which is equal to 1×10^{-3} m³. Thus

$$(1.0 \times 10^{-2}\ \text{m}^3/\text{h})\left(\frac{1\ \text{L}}{1 \times 10^{-3}\ \text{m}^3}\right)\left(\frac{1\ \text{gal}}{4\ \text{L}}\right) = 2.5\ \text{gal/h}.$$

The automobile is moving at 95 km/h, which is 60 mi/h, so the "mileage" is

$$\frac{60\ \text{mi/h}}{2.5\ \text{gal/h}} = 24\ \text{mi/gal},$$

which is a reasonable miles-per-gallon figure for automobiles.

PRACTICE

Guided Problem 21.2 Geothermal power

A geothermal power plant that has a power output of 10 MW extracts thermal energy from Earth's interior via a shaft drilled to a depth where the temperature is 600 °C and discards thermal energy at Earth's surface into a large mass of ice at −5 °C. What is the minimum possible value for the rate of thermal energy output?

GETTING STARTED

1. What assumptions do you need to make about the power plant?

DEVISE PLAN

2. How can you use the temperature difference between the two reservoirs to determine the plant's maximum efficiency?
3. For a power output of 10 MW, what is the minimum rate of thermal energy input?
4. How can you apply the energy law, $\Delta E = W + Q$, to this steady device to determine the rate of thermal energy output?

EXECUTE PLAN

5. Calculate the maximum efficiency.
6. Calculate the minimum thermal energy input rate.
7. Calculate the minimal possible rate of thermal energy output.

EVALUATE RESULT

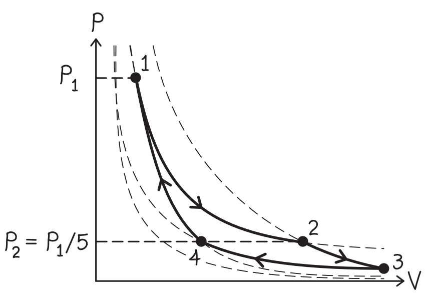
8. Is the thermal output of energy greater or smaller than the thermal input of energy?

Worked Problem 21.3 Carnot cycle

Consider a reversible heat engine that operates on the Carnot cycle. The working substance is air, which consists primarily of diatomic nitrogen molecules. During the isothermal expansion, the pressure of the air decreases by a factor of 5. During the isentropic expansion, the work done by the engine on the environment is 3.50×10^2 J. How much work does the engine do on the environment during one cycle?

❶ GETTING STARTED We assume that the working substance is an ideal gas. Because the molecules are diatomic, they have $d = 5$ degrees of freedom. We begin by drawing a PV diagram for this cycle (Figure WG21.2), identifying the beginning of the isothermal expansion as state 1 and the end of the process as state 2. The isoentropic expansion begins at state 2 and ends at state 3. The isothermal compression begins at state 3 and ends at state 4. The isentropic compression begins at state 4 and ends at state 1. We are given that $P_1/P_2 = 5$, and we show this information on our diagram. We are also given that for the isentropic expansion (process $2 \rightarrow 3$), $W = -3.50 \times 10^2$ J.

Figure WG21.2



❷ DEVISE PLAN We can use Eq. 21.34 to determine the work done W_{cycle} by the environment on the engine. We need to know the number of molecules N, the temperature difference between the two reservoirs $T_{in} - T_{out}$, and the volume ratio V_2/V_1. We are given W for the isentropic expansion and we know from Eq. 20.19 that $W = NC_V(T_{out} - T_{in})$. We also know the number of degrees of freedom of these molecules so we can calculate $C_V = (\frac{d}{2})k_B$. Therefore we can use Eq. 20.19 to determine the product $Nk_B(T_{out} - T_{in})$. For the isothermal expansion (process $1 \rightarrow 2$), we can use the ideal gas law, $P = Nk_BT/V$, to determine the ratio V_2/V_1. We can now determine W_{cycle}. The work done by the engine on the environment is then $-W_{cycle}$.

❸ EXECUTE PLAN Because $d = 5$, the heat capacity per particle at constant volume is $C_V = (\frac{d}{2})k_B = (\frac{5}{2})k_B$. To determine the product of the number of molecules and the temperature difference between the two reservoirs, we rewrite Eq. 20.19 as

$$Nk_B(T_{in} - T_{out}) = -\frac{W}{(d/2)} = -\frac{-3.50 \times 10^2\ \text{J}}{(5/2)}$$
$$= 1.40 \times 10^2\ \text{J}.$$

We now apply the ideal gas law to states 1 and 2. Because process $1 \rightarrow 2$ is an isothermal expansion, $P_1V_1 = P_2V_2$. Therefore the volume ratio is $V_2/V_1 = P_1/P_2 = 5$. We now use Eq. 21.34 to determine the work done W_{cycle} by the environment on the engine:

$$W_{\text{cycle}} = -Nk_B(T_{\text{in}} - T_{\text{out}})\ln\left(\frac{V_2}{V_1}\right)$$

$$= -(1.40 \times 10^2 \text{ J})\ln(5) = -2.25 \times 10^2 \text{ J.} ✔$$

❹ **EVALUATE RESULT** The cycle is clockwise, so the work done on the system is negative, in agreement with our result. The work done by the engine on the environment is smaller in magnitude than the the work done on the engine during the isothermal expansion because the area inside the cycle on the *PV* diagram is smaller than the area under the *PV* curve corresponding to the isothermal expansion.

Guided Problem 21.4 Brayton cycle

Consider a jet engine operating on the Brayton cycle. The working substance is air, which consists primarily of diatomic nitrogen molecules. The air is drawn in at atmospheric pressure, 1.01×10^5 Pa, at 250 K. During compression this pressure is increased to 28.3×10^5 Pa, and the combustion raises the temperature to 1500 K. How much work per mole of air does the engine do on the environment during one cycle?

❶ **GETTING STARTED**

1. Draw a *PV* diagram for this cycle. (Hint: See Figure 21.30.)
2. Label your *PV* diagram with known values of pressure and temperature.

❷ **DEVISE PLAN**

3. What quantities do you need to know in order to determine the work done by the engine?
4. How can you obtain the efficiency from the given information?
5. How can you determine Q_{in} per mole?
6. How is the number of particles related to the number of moles?
7. How can you determine the work done per mole by the engine on the environment from knowledge of the efficiency and the given information about the isentropic compression?

❸ **EXECUTE PLAN**

8. Calculate the pressure ratio.
9. How many degrees of freedom do the gas particles in air have at these temperatures?
10. Calculate the efficiency of the engine.
11. How much thermal energy per mole is transferred during the isobaric expansion?
12. How much work per mole does the engine do on the environment?

❹ **EVALUATE RESULT**

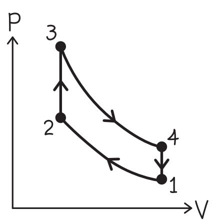
13. Does your answer agree with Eq. 21.37?

Worked Problem 21.5 Otto Cycle

In a model gasoline engine, the working substance, which is the air-gasoline mixture, can be treated approximately with the thermodynamic cycle known as the Otto cycle. In the first process ($1 \rightarrow 2$), the gas, initially at 300 K, undergoes an isentropic compression (the engine's compression stroke) during which its temperature rises to 671 K. The next process ($2 \rightarrow 3$) is an isochoric pressure increase, and the gas temperature reaches 1574 K as the gasoline burns. Next ($3 \rightarrow 4$), the gas undergoes an isentropic expansion (the engine's power stroke) during which its temperature drops to 809 K. Finally ($4 \rightarrow 1$), the gas experiences an isochoric pressure decrease during which its temperature drops to its initial value. What is the efficiency of this model engine?

❶ **GETTING STARTED** We assume that the working substance is an ideal gas. We begin by drawing a *PV* diagram for this cycle (Figure WG21.3), identifying the beginning of each process as states 1, 2, 3 and 4. We are given that $T_1 = 300$ K, $T_2 = 671$ K, $T_3 = 1574$ K, and $T_4 = 809$ K.

Figure WG21.3



No energy is transferred thermally during the isentropic processes ($1 \rightarrow 2$ and $3 \rightarrow 4$). We denote the energy transferred thermally during the isochoric pressure increase ($2 \rightarrow 3$) by $Q_{2\rightarrow 3} = Q_{in} > 0$ and the energy transferred thermally during the isochoric pressure decrease ($4 \rightarrow 1$) by $Q_{4\rightarrow 1} = -Q_{out} < 0$.

❷ **DEVISE PLAN** We can use either Eq. 20.12, $Q = \frac{d}{2}Nk_B\,\Delta T$, or Eq. 20.13, $C_V = Q/N\,\Delta T$, to calculate the quantity of energy transferred thermally during each isochoric process. The energy transferred thermally during the cycle is the sum $Q_{2\rightarrow 3} + Q_{4\rightarrow 1} = Q_{in} - Q_{out}$. We can then use Eq. 21.22 to calculate the efficiency.

❸ **EXECUTE PLAN** We choose to use Eq. 20.13. For the first isochoric process ($2 \rightarrow 3$) we have $Q_{2\rightarrow 3} = NC_V(T_3 - T_2)$, and for the second ($4 \rightarrow 1$) we have $Q_{4\rightarrow 1} = NC_V(T_1 - T_4) < 0$.

So the thermal efficiency is, from Eq. 21.22,

$$\eta = 1 - \left(\frac{Q_{out}}{Q_{in}}\right) = 1 - \left(\frac{-Q_{4\rightarrow 1}}{Q_{2\rightarrow 3}}\right) = 1 - \left(\frac{-NC_V(T_1 - T_4)}{NC_V(T_3 - T_2)}\right)$$

$$= 1 + \left(\frac{(T_1 - T_4)}{(T_3 - T_2)}\right) = 1 + \frac{300\text{ K} - 809\text{ K}}{1574\text{ K} - 671\text{ K}} = 0.436. \checkmark$$

❹ **EVALUATE RESULT** We expect the efficiency to be less than 1 for any heat engine. The only other familiar cycle is the Carnot cycle, which produces the maximum efficiency for a given temperature range. This cycle requires two constant-temperature processes, which we do not have as part of the Otto cycle. However, if we average the temperatures for the thermal input of energy for process ($2 \rightarrow 3$) as $T_{in} = (T_3 + T_2)/2 = 1123$ K and the thermal output of energy for process ($4 \rightarrow 1$) as $T_{out} = (T_4 + T_1)/2 = 555$ K, then we can compare our result to the maximum efficiency of an engine undergoing a Carnot cycle between these average temperatures. This is given by Eq. 21.23:

$$\eta_{max} = 1 - \left(\frac{T_{out}}{T_{in}}\right) = 1 - \left(\tfrac{555}{1123}\right) = 0.506.$$

This efficiency is slightly greater than our calculated efficiency for the Otto process, which gives us confidence that our calculation for the Otto cycle is reasonable.

Guided Problem 21.6 Diesel Cycle

In a diesel engine, the working substance is mostly air, a mixture of oxygen (O_2) and nitrogen gas (N_2). The air is compressed, and only then is a small amount of fuel injected at the "top" of the compression stroke. The high pressure and temperature ignite the fuel and cause rapid combustion, creating various by-products that we will ignore. The mostly diatomic working substance undergoes the following cycle: ($1 \rightarrow 2$) The gas, initially at 300 K, experiences an isentropic compression during which its temperature rises to 889 K. ($2 \rightarrow 3$) This process is an isobaric expansion during which the fuel is injected. The gas reaches 1778 K as the fuel rapidly burns. ($3 \rightarrow 4$) The gas then undergoes an isentropic expansion during which its temperature drops to 884 K. ($4 \rightarrow 1$) The final step is an isochoric pressure decrease during which the gas temperature drops to its initial value. What is the efficiency of this diesel engine?

❶ **GETTING STARTED**

1. Make a *PV* diagram for the diesel cycle.
2. During which processes is energy transferred thermally either into or out of the working substance?

❷ **DEVISE PLAN**

3. Which equations describe the thermal energy transfers made during the cycle? Which table in *Principles and Practice* might come in handy?
4. How many degrees of freedom do the molecules of the working substance have?
5. What are the heat capacities per particle at constant volume and constant pressure?
6. Which quantity is Q_{out} for the cycle? Which quantity is Q_{in}?
7. How do you determine the work for the cycle?
8. How do you determine the engine's efficiency?

❸ **EXECUTE PLAN**

9. How much energy is transferred thermally into the gas?
10. How much energy is transferred thermally out of the gas?
11. Calculate the engine's efficiency.

❹ **EVALUATE RESULT**

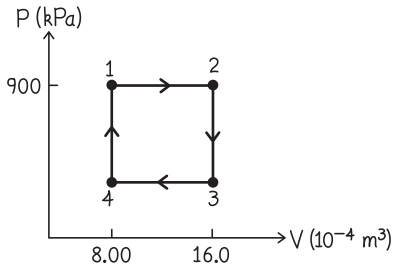
12. Compare your calculated efficiency with that of an engine operating on the Carnot cycle between 300 K and 1778 K. Are the relative efficiencies what you expect?

Worked Problem 21.7 Cyclic Process I

The working substance in an engine is 0.300 mol of nitrogen gas. Initially, in state 1, the gas volume is $V_1 = 8.00 \times 10^{-4}\ \text{m}^3$, and the pressure is $P_1 = 900\ \text{kPa} = 9.00 \times 10^5\ \text{N/m}^2$. The gas undergoes a cycle that consists of four processes: $(1 \rightarrow 2)$ an isobaric expansion until the volume doubles, $(2 \rightarrow 3)$ an isochoric pressure decrease until the temperature is the same as the initial temperature, $(3 \rightarrow 4)$ an isobaric compression until the volume is back at its initial value, and $(4 \rightarrow 1)$ an isochoric pressure increase until the pressure is what it was in state 1. Calculate Q, $W_{\text{on gas}}$, ΔE_{th}, and ΔS for each process and for the cycle.

❶ **GETTING STARTED** We are told that the working substance is 0.300 mol of nitrogen molecules. Because the molecules are diatomic, they have $d = 5$ degrees of freedom at typical engine temperatures. We draw a PV diagram for the cycle (Figure WG21.4) and add our labels, showing state 1 as the initial state positioned at $P_1 = 9.00 \times 10^5\ \text{N/m}^2$, $V_1 = 8.00 \times 10^{-4}\ \text{m}^3$. We are told that $V_2 = 2V_1$, $T_3 = T_1$, and $V_4 = V_1$. We know that $V_2 = V_3$ and $V_4 = V_1$ because processes $2 \rightarrow 3$ and $4 \rightarrow 1$ are isochoric. We know that $P_1 = P_2$ and $P_3 = P_4$ because processes $1 \rightarrow 2$ and $3 \rightarrow 4$ are isobaric. Thus the only quantities in the ideal gas law we must determine are P_3, T_1, T_2, and T_4.

Figure WG21.4



❷ **DEVISE PLAN** We know the number of moles of the working substance and so can determine the number of molecules N. We know the number of degrees of freedom of these molecules and so can calculate C_V and C_P. We can use the ideal gas law, $P = Nk_BT/V$, to calculate the values of T_1, T_2, T_4, and P_3. We then can use the summary table on page 376 to determine Q, W, ΔE_{th}, and ΔS for each process. We know that $\Delta E_{\text{th}} = 0$ for a cycle, and therefore by the energy law, $W_{\text{cycle}} = -Q_{\text{cycle}}$. We can sum the four W values to determine W_{cycle}.

❸ **EXECUTE PLAN** From Avogadro's constant, we know that the number of molecules is

$$N = (0.300\ \text{mol})\left(\frac{6.022 \times 10^{23}}{1\ \text{mol}}\right) = 1.81 \times 10^{23}.$$

Because $d = 5$, we have $C_V = \left(\frac{5}{2}\right)k_B$ (Eq. 20.14) and $C_P = \left(\frac{7}{2}\right)k_B$ (Eq. 20.25). We now apply the ideal gas law to the various states to determine T_1, T_2, T_4, and P_3:

State 1: $T_1 = \dfrac{P_1V_1}{Nk_B} = \dfrac{(9.00 \times 10^5\ \text{N/m}^2)(8.00 \times 10^{-4}\ \text{m}^3)}{(1.81 \times 10^{23})(1.38 \times 10^{-23}\ \text{J/K})} = 288\ \text{K}.$

State 2: We know that $V_2 = 2V_1$. Therefore

$$T_2 = \frac{P_2V_2}{Nk_B} = \frac{P_1 2V_1}{Nk_B} = 2T_1 = 576\ \text{K}.$$

State 3: We know that $V_3 = V_2 = 2V_1$ and $T_3 = T_1$. Therefore

$$P_3 = \frac{Nk_BT_3}{V_3} = \frac{Nk_BT_1}{2V_1} = \frac{P_1}{2} = 4.50 \times 10^5\ \text{N/m}^2.$$

State 4: We know that $V_4 = V_1$ and $P_4 = P_3 = P_1/2$. Therefore

$$T_4 = \frac{P_4V_4}{Nk_B} = \frac{P_3V_1}{Nk_B} = \frac{P_1V_1}{2Nk_B} = \frac{T_1}{2} = 144\ \text{K}.$$

PRACTICE

We now use the results from the summary table on page 376 to calculate Q, W, ΔE_{th}, and ΔS for each process:

Isobaric process $1 \rightarrow 2$: We know that $\Delta T_{12} = T_2 - T_1 = 2T_1 - T_1 = T_1$, and so

$$W_{1\rightarrow 2} = -Nk_B \Delta T_{12} = -Nk_B T_1 = -719 \text{ J}$$

$$Q_{1\rightarrow 2} = NC_P \Delta T_{12} = NC_P T_1 = 2.52 \times 10^3 \text{ J}$$

$$\Delta E_{th,1\rightarrow 2} = NC_V \Delta T_{12} = NC_V T_1 = 1.80 \times 10^3 \text{ J}$$

$$\Delta S_{1\rightarrow 2} = \frac{NC_P}{k_B} \ln\left(\frac{T_2}{T_1}\right) = (1.81 \times 10^{23})\left(\tfrac{7}{2}\right)(\ln 2) = 4.39 \times 10^{23}.$$

Isochoric process $2 \rightarrow 3$: We know that $\Delta T_{23} = T_3 - T_2 = T_1 - 2T_1 = -T_1$, so

$$W_{2\rightarrow 3} = 0$$

$$Q_{2\rightarrow 3} = NC_V \Delta T_{23} = -NC_V T_1 = -1.80 \times 10^3 \text{ J}$$

$$\Delta E_{th,2\rightarrow 3} = NC_V \Delta T_{23} = -NC_V T_1 = -1.80 \times 10^3 \text{ J}$$

$$\Delta S_{2\rightarrow 3} = \frac{NC_V}{k_B} \ln\left(\frac{T_3}{T_2}\right) = (1.81 \times 10^{23})\left(\tfrac{5}{2}\right)(\ln 0.5)$$

$$= -3.14 \times 10^{23}.$$

Isobaric process $3 \rightarrow 4$: With $\Delta T_{34} = T_4 - T_3 = \frac{T_1}{2} - T_1 = -\frac{T_1}{2}$, we have

$$W_{3\rightarrow 4} = -Nk_B \Delta T_{34} = Nk_B \frac{T_1}{2} = 360 \text{ J}$$

$$Q_{3\rightarrow 4} = NC_P \Delta T_{34} = -NC_P\left(\frac{T_1}{2}\right) = -1.26 \times 10^3 \text{ J}$$

$$\Delta E_{th,3\rightarrow 4} = NC_V \Delta T_{34} = -NC_V\left(\frac{T_1}{2}\right) = -899 \text{ J}$$

$$\Delta S_{3\rightarrow 4} = \frac{NC_P}{k_B} \ln\left(\frac{T_4}{T_3}\right) = -\Delta S_{12} = -4.39 \times 10^{23}.$$

Isochoric process $4 \rightarrow 1$: With $\Delta T_{41} = T_1 - T_4 = T_1 - \frac{T_1}{2} = \frac{T_1}{2}$, we have

$$W_{4\rightarrow 1} = 0$$

$$Q_{4\rightarrow 1} = NC_V \Delta T_{41} = NC_V\left(\frac{T_1}{2}\right) = 899 \text{ J}$$

$$\Delta E_{th,4\rightarrow 1} = NC_V \Delta T_{41} = NC_V\left(\frac{T_1}{2}\right) = 899 \text{ J}$$

$$\Delta S_{4\rightarrow 1} = \frac{NC_V}{k_B} \ln\left(\frac{T_4}{T_1}\right) = -\Delta S_{23} = 3.14 \times 10^{23}.$$

The cycle: The work done by the environment on the gas during the cycle is

$$W_{cycle} = W_{1\rightarrow 2} + W_{2\rightarrow 3} + W_{3\rightarrow 4} + W_{4\rightarrow 1} = -360 \text{ J}.$$

The energy change over the cycle must be zero, and that is just what we obtain:

$$\Delta E_{th,cycle} = \Delta E_{th,1\rightarrow 2} + \Delta E_{th,2\rightarrow 3} + \Delta E_{th,3\rightarrow 4} + \Delta E_{th,4\rightarrow 1} = 0.$$

Because $\Delta E_{th,cycle} = 0$, the energy transferred thermally from the environment to the gas during the cycle is $Q_{cycle} = -W_{cycle} = 360$ J. Finally, the entropy change over the cycle is zero:

$$\Delta S_{cycle} = \Delta S_{1\rightarrow 2} + \Delta S_{2\rightarrow 3} + \Delta S_{3\rightarrow 4} + \Delta S_{4\rightarrow 1} = 0. ✔$$

❹ **EVALUATE RESULT** We assumed that nitrogen has five degrees of freedom at typical engine temperatures, but 144 K is low enough to call that assumption into question. We can check this using the values given in Example 20.3, which show that nitrogen molecules rotate even at temperatures of this magnitude.

PRACTICE

The fact that the cycle direction is clockwise in Figure WG21.4 means that the work done on the gas is negative. We know that the magnitude of the work done per cycle is the area enclosed by the curves for the four processes. The negative of that area is

$$\begin{aligned} W_{\text{cycle, on engine}} &= -(P_1 - P_4)(V_2 - V_1) \\ &= -\left(P_1 - \frac{P_1}{2}\right)(2V_1 - V_1) = -\frac{P_1V_1}{2} \\ &= -\frac{(9.0 \times 10^5\ \text{N/m}^2)(8.0 \times 10^{-4}\ \text{m}^3)}{2} = -360\ \text{J}, \end{aligned}$$

in agreement with our result.

The work done during the cycle by the engine on the environment is positive:

$$W_{\text{cycle, on envir}} = -(W_{1\to2} + W_{2\to3} + W_{3\to4} + W_{4\to1}) = 360\ \text{J},$$

which is what we expect for a clockwise cycle.

Guided Problem 21.8 Cyclic Process II

The working substance in an engine is 3.00×10^{23} helium atoms. Initially, in state 1, the gas volume is $V_1 = 1.50 \times 10^{-3}\ \text{m}^3$ and the pressure is $P_1 = 1.00 \times 10^6\ \text{N/m}^2$. The gas undergoes a cycle that consists of four processes: $(1 \to 2)$ an isothermal expansion, $(2 \to 3)$ an isobaric compression until the volume is $V_3 = 2.00 \times 10^{-3}\ \text{m}^3$ and the pressure is $2.00 \times 10^5\ \text{N/m}^2$, $(3 \to 4)$ an isothermal compression until the volume is $V_4 = V_1$, and $(4 \to 1)$ an isochoric pressure increase until it returns to its initial state. Calculate $W_{\text{on gas}}$, Q, ΔE_{th}, and ΔS for each process and for the cycle.

1 GETTING STARTED

1. How many degrees of freedom do the gas particles have?
2. Draw a PV diagram for the cycle and label the four states 1, 2, 3, and 4, including all known P and V values.
3. List the information you know about the relationships between the various P values, between the various T values, and between the various V values.

2 DEVISE PLAN

4. What are C_V and C_P for helium atoms?
5. What information do you need to determine in order to calculate Q, W, ΔE_{th}, and ΔS for each process? Is there a useful table?
6. How can you use the ideal gas law to determine that information?
7. Which of the quantities W, Q, ΔE_{th}, and ΔS is zero for the cycle?

3 EXECUTE PLAN

8. Calculate C_V and C_P.
9. Calculate the temperatures $T_1 = T_2$ and $T_3 = T_4$ during the isothermal expansion and compression and the volume V_2 after the expansion.
10. From the summary table on page 376, calculate W, Q, ΔE_{th}, and ΔS for each process.
11. Calculate W, Q, ΔE_{th}, and ΔS for the cycle.

4 EVALUATE RESULT

12. Do you expect W_{cycle} to be positive or negative?

Answers to Review Questions

1. (*a*) When a gas burns, one state change involves the chemical bonds in the gas particles, associated with the chemical potential energy locked in the bonds, and the complementary state change is in the gas temperature, associated with the particles' thermal energy. (*b*) When a ball drops to the ground, one state change is the ball's speed, associated with its kinetic energy, and the complementary state change is in the configuration of the ball-Earth system, associated with the system's gravitational potential energy.
2. The work done on a system is the difference between the mechanical input of energy and the mechanical output of energy.
3. In a car engine, chemical energy from burned fuel is thermally transferred into the engine. Energy is mechanically transferred out of the engine as the motion of the engine does work through the linkage to produce circular motion in the wheels. Energy is also transferred thermally out of the engine via the exhaust and as the temperature of the air around the engine increases. In a refrigerator, energy is transferred mechanically into the system in the form of work done on the refrigerator by its compressor. Energy is also transferred thermally into the refrigerator from room-temperature items placed inside the refrigerator. Energy is thermally transferred out of the refrigerator as the thermal energy of the air surrounding the refrigerator increases.
4. A steady device is a device that converts energy from one form to another in such a way that either the initial state of the device never changes as the energy conversion takes place or, if the initial state does change, the change is cyclic and the steady device periodically returns to its initial

state. In a nonsteady device, some property of the device such as temperature or pressure continues to change to the point where the device no longer operates.

5. (*a*) No restrictions are imposed when the device converts mechanical energy to thermal energy because the conversion to thermal energy increases the entropy of the environment, obeying the entropy law. (*b*) Converting thermal energy to mechanical energy decreases the entropy of the environment and so violates the entropy law; such a device does not exist.
6. The entropy gradient is $dS/dE = 1/(k_B T)$. It says that the change in a reservoir's entropy dS when a given quantity of thermal energy dE is transferred to or from the reservoir is inversely proportional to the reservoir temperature. For a given quantity of energy, the magnitude of the entropy change when that quantity is removed from a higher-temperature reservoir is smaller than the entropy change when that quantity is added to a lower-temperature reservoir.
7. The entropy cost is a measure of the change in entropy caused by a given change in energy; hence it is determined by the entropy gradient at the temperature at which the energy change takes place, i.e., energy is transferred into or out of the system.
8. The rod thermally transfers energy from the higher-temperature reservoir to the lower-temperature one. The entropy of the system (rod) may increase or decrease for a short time interval while the system reaches a steady state; this depends on the initial state (temperature) of the rod. However, after a steady state is reached, the entropy of the rod will neither increase nor decrease, and the rod acts as a steady device to transfer energy. The transfer is irreversible because the ΔS for a given ΔE is inversely proportional to temperature. The amount by which the transferred energy increases the entropy, ΔS_L, of the lower-temperature reservoir is greater than the amount by which it decreases the entropy, ΔS_H, of the higher-temperature reservoir; thus the entropy change for the environment is positive, making the conversion irreversible.
9. Energy quality is a measure of how much useful work can be obtained from a given form of energy. Energy change is generally associated with an entropy change, and the entropy change of the environment for a steady device must be positive. The quality of the energy is inversely proportional to the energy gradient. High-quality energy has a small energy gradient. High-quality energy input has a small associated entropy change and so will require a small corresponding thermal energy output to the environment in order for the entropy of the environment to increase. Hence, most of the energy input will result in a mechanical output of energy. Low-quality energy input requires a greater thermal energy output to ensure that the entropy of the environment increases; hence it will have a smaller associated mechanical output of energy. Therefore the energy quality is a measure of how much useful work can be obtained from a given amount of energy change.
10. The entropy gradient associated with mechanical energy input is zero; therefore a mechanical input of energy is the highest-quality energy input.
11. A heat engine is a steady device that takes in thermal energy from a higher-temperature reservoir, converts part of this energy to mechanical energy, and transfers the rest to a lower-temperature reservoir. A car engine takes in thermal energy from burning fuel, converts part of that energy to the mechanical energy that moves the car, and transfers the remainder to the environment.
12. An entropy diagram consists of a horizontal axis representing the entropy gradient, $dS/dE = 1/k_B T$, and a vertical axis representing energy. Rectangles drawn in this coordinate system represent entropy changes, with the product of the two axes giving the magnitude of the entropy change for a given rectangle. Vertical arrows represent energy transfers into (downward arrows) and out of (upward arrows) the system. The locations of these arrows along the horizontal axis indicate at what entropy gradient in a process the energy transfers occur. A left-pointing arrowhead within an entropy-change rectangle indicates that entropy in the environment decreases; hence the energy is upgraded in that process. A right-pointing arrowhead within an entropy-change rectangle indicates that entropy in the environment increases; hence the energy is downgraded in that process.
13. The efficiency of a heat engine is the ratio of the quantity of work done by the engine on its environment ($-W$) to the quantity of energy Q_{in} transferred thermally to the engine: $-W/Q_{in}$.
14. A heat pump is a steady device that degrades mechanical energy in order to thermally transfer energy from a lower-temperature reservoir to a higher-temperature reservoir.
15. There is no difference. In both cases, the pump transfers thermal energy from a lower-temperature reservoir to a higher-temperature reservoir. In summer, the lower-temperature reservoir is a room and the higher-temperature reservoir is the outdoors. In winter, the lower-temperature reservoir is the outdoors and the higher-temperature reservoir is the room being heated.
16. The coefficient of performance of cooling, a measure of how efficient a heat pump is when used for cooling, is the ratio of the quantity of energy transferred thermally into the pump to the quantity of work done on the pump: Q_{in}/W. The coefficient of performance of heating, a measure of how efficient a heat pump is when used for heating, is the ratio of the quantity of energy transferred thermally out of the pump to the quantity of work done on the pump. Q_{out}/W.
17. A heat engine does positive work on the environment; hence the work done on the working substance (system) must be negative. An expansion process on a *PV* diagram represents negative work done on the system, and a compression process represents positive work done on the system. In order for a cyclic process to result in negative work done on the system, the expansion curve must be above the compression curve corresponding to a clockwise cycle. Therefore the path on a *PV* diagram for a heat engine must be clockwise. A heat pump does positive work on the system, so the expansion curve must lie below the compression curve, corresponding to a counterclockwise cycle.
18. A working substance sequentially undergoes an isothermal expansion (energy transferred to substance from reservoir at T_{in}), an isentropic expansion (substance temperature drops to that of reservoir at T_{out}), an isothermal compression (energy transferred from substance to reservoir at T_{out}), and an isentropic compression back to its initial state (substance temperature rises to T_{in}).
19. The engine does only a small amount of work on the environment in each cycle, which means little mechanical energy is produced, and the two isothermal steps take place very slowly, which makes the engine's power (energy delivered per unit of time) too small to be practicable.
20. A working substance sequentially undergoes an isobaric expansion, an isentropic expansion, an isobaric compression, and an isentropic compression back to the initial state.
21. Gas turbines and jet engines run on cycles that are a good approximation of a forward Brayton cycle. Refrigerators and air-conditioners run on cycles that are a good approximation of a reverse Brayton cycle.
22. It is impossible to construct a cyclic device whose only interaction with its environment is to transfer energy thermally from a lower-temperature reservoir to a higher-temperature reservoir.
23. An entropy diagram is a rectangle drawn so that the height of the rectangle represents the amount of energy transferred and the length of the rectangle represents the difference in values of $(1/k_B T)$ for the process. The product (area) thus gives the magnitude of the entropy change of the environment for the process, by Eq. 21.10. The sign of the entropy change is given by the direction of the process: If the rectangle points to the

right, energy is degraded and the entropy change of the environment is positive. If the rectangle points to the left, energy is upgraded and the entropy change is negative.

24. Q_{out} is directly proportional to the temperature of the lower-temperature reservoir and inversely proportional to the temperature of the higher-temperature reservoir: $Q_{out} = Q_{in}(T_{out}/T_{in})$ (from Eq. 21.20).
25. A heat engine upgrades some portion of Q_{in} to mechanical energy in the form of work done by the engine on the environment. This upgrading means the environment's entropy decreases. The entropy law, $\Delta S \geq 0$, requires that some Q_{in} must be degraded to compensate for this decrease in S_{env}. The degraded portion is Q_{out}. Because $Q_{out} < Q_{in}$, the efficiency $\eta = 1 - (Q_{out}/Q_{in})$ of any heat engine, reversible or irreversible, is always less than 1. Because the efficiency of irreversible heat engines is always less than that of reversible engines, the maximum efficiency, $\eta_{max} = (T_{in} - T_{out})/T_{in}$ (Eq. 21.23), is applicable to reversible engines only. In practice, no heat engine can operate at maximum efficiency because it is impossible to eliminate all dissipation of energy in the engine.
26. In both a heat pump and an air-conditioner, the thermal energy input comes from the lower-temperature reservoir; therefore Q_{in} and T_{in} belong to the lower-temperature reservoir.
27. The coefficient of performance is the quantity Q_{out} of energy transferred thermally to a higher-temperature reservoir divided by the work done on the heat pump. Because the work done on the heat pump is equal to the difference between Q_{out} and the quantity Q_{in} of energy transferred thermally from a lower-temperature reservoir, Eq. 21.25, $COP_{heating} = Q_{out}/(Q_{out} - Q_{in})$. The maximum value is the temperature T_{out} of the higher-temperature reservoir divided by the difference between T_{out} and the absolute temperature T_{in} of the lower-temperature reservoir: Eq. 21.26, $COP_{heating,\,max} = T_{out}/(T_{out} - T_{in})$.
28. The coefficient of performance of cooling equals the coefficient of performance of heating minus 1 (Eq. 21.28).
29. No. The maximum efficiency of a heat engine is independent of the working substance.
30. The isentropic work does not contribute to W because one isentropic work value contains the difference $(T_{out} - T_{in})$ and the other contains the difference $(T_{in} - T_{out})$, causing these two terms to cancel each other. The work done during the isothermal steps does not cancel.
31. The efficiency of an engine that operates on a Carnot cycle is equal to the maximum efficiency of any heat engine.
32. The Brayton cycle produces more mechanical energy than the Carnot cycle, and because the Brayton cycle involves no isothermal steps, the rate at which it produces that energy—the power output—is greater than the rate at which mechanical energy is produced in the Carnot cycle.
33. The cycle efficiency equals 1 minus the ratio of the absolute temperature T_4 of the gas just as it begins isentropic compression to the absolute temperature T_1 of the gas at the end of the compression (Eq. 21.44).

Answers to Guided Problems

Guided Problem 21.2 $Q_{out,min} = 4.4$ MW

Guided Problem 21.4 $W_{environment} = 1.52 \times 10^4$ J/mol

Guided Problem 21.6 $\eta = 0.605$

Guided Problem 21.8

$W_{12} = -2.41 \times 10^3$ J, $Q_{12} = +2.41 \times 10^3$ J,
$\Delta E_{th,12} = 0$, $\Delta S_{12} = +4.82 \times 10^{23}$

$W_{23} = +1.10 \times 10^3$ J, $Q_{23} = -2.75 \times 10^3$ J,
$\Delta E_{th,23} = -1.65 \times 10^3$ J, $\Delta S_{23} = -9.91 \times 10^{23}$

$W_{34} = +115$ J, $Q_{34} = -115$ J,
$\Delta E_{th,34} = 0$, $\Delta S_{34} = -8.63 \times 10^{22}$

$W_{41} = 0$, $Q_{41} = +1.65 \times 10^3$ J,
$\Delta E_{th,41} = +1.65 \times 10^3$ J, $\Delta S_{41} = +5.94 \times 10^{23}$

$W_{cycle} = -1.20 \times 10^3$ J, $Q_{cycle} = +1.20 \times 10^3$ J,
$\Delta E_{cycle} = 0$, $\Delta S_{cycle} = 0$

Guided Practice by Chapter

22 Electric Interactions

Review Questions

Answers to these questions can be found at the end of this chapter.

22.1 Static electricity

1. Blow up a balloon, tie it closed, hold it next to your head, and let go. What is the interaction responsible for the balloon's fall? This interaction is between which two objects? Now rub the balloon against your hair until it sticks against your head when you let go. What is the interaction responsible for the balloon's behavior? This interaction is between which two objects?
2. Name two differences between the gravitational interaction and the electric interaction.

22.2 Electrical charge

3. What experimental evidence tells us there is more than one type of electrical charge?
4. You rub a balloon on a friend's hair, and the balloon clings to her head. You say her hair put charge on the balloon, and she says the balloon put charge on her hair. Which of you is right?

22.3 Mobility of charge carriers

5. How much charge can you produce *within* any closed system by any process?
6. You scuff your feet on the carpet and then a spark leaps from your fingertip when you touch a metal doorknob. What does this observation tell you about whether your body is an electrical conductor or an electrical insulator?
7. Classify each of the following as an electrical insulator or conductor: paper, paper clip, seawater, car tire, air.
8. Breaking a piece of wood involves breaking chemical bonds. Why doesn't this usually leave the resultant pieces charged?

22.4 Charge polarization

9. Describe the process by which an electrically neutral object can be attracted by an electrically charged rod.
10. Suppose early researchers in electricity had assigned electrons a positive charge and protons a negative charge. What would be different in the electric phenomena you have studied up to this point?
11. How is charge polarization similar in electrical insulators and conductors? How is it different?
12. Negatively charged object A attracts object B. (*a*) Which state or states are possible for object B: positively charged, neutral, negatively charged? (*b*) Negatively charged object C repels object D. Which state or states are possible for object D: positively charged, neutral, negatively charged?

22.5 Coulomb's law

13. Use Newton's third law and the experimental observation that the magnitude F^E_{AB} of the electric force exerted by a charged object A on a charged object B is proportional to the charge on B to argue that the magnitude of the electric force between the two *must* be proportional to the *product* of the charges (as opposed to, say, proportional to the *sum* of the charges or to one of the charges only).
14. If you double the charge on *each* of two objects, how much must you increase the distance between the objects to restore the force exerted on them to its original value?
15. In a carbon atom, where six electrons are located around a nucleus that contains six protons (and usually six neutrons), which force magnitude is larger, if either: that of the force exerted by an electron on the nucleus or that of the force exerted by the nucleus on an electron?

22.6 Force exerted by distributions of charge carriers

16. Consider some arrangement of charge carriers 1, 2, and 3. What is wrong with the expression

$$\Sigma\vec{F}_1 = k\frac{q_2q_1}{r_{21}^2}\hat{r}_{21} + k\frac{q_3q_1}{r_{31}^2}\hat{r}_{31} + k\frac{q_3q_2}{r_{32}^2}\hat{r}_{32}$$

for the force exerted on carrier 1 by carriers 2 and 3?

17. Charged particle 1, carrying charge $+q$, is fixed at the origin of an x axis. Charged particle 2, carrying charge $-2q$, is fixed at $x = +2.0$ m. Where on the x axis should charged particle 3, carrying charge $+3q$, be placed so that the vector sum of forces exerted on particle 3 is zero: (*a*) $0 < x < 2.0$ m, (*b*) $2.0 \text{ m} < x < +\infty$, (*c*) $-\infty < x < 0$?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The minimum magnitude for the electric force exerted on a tiny bit of paper lifted by attraction to a charged comb (C, O)
2. The magnitude of the electric force between a proton and an electron in an atom (R, S)
3. The magnitude of the electric force between a sodium ion and a neighboring chloride ion in a sodium chloride molecule (D, V, S)
4. The number of protons in a 1-liter bottle of cola (E, N, H, W)
5. The number of electrons in Earth (Y, T, M, J)
6. The magnitude of the electric force between you and a friend standing 10 m apart if each of you had 1% more electrons than protons (Z, H, K, W, A)
7. The percentage of Earth's electrons that, if transferred to the Moon, would create an electrical attraction equal to the gravitational attraction of these bodies (G, L, Y, U, results of number 5 above)
8. The amount of like charge on each of two identical pith balls hung by two 80-mm strings from a common point, if the two balls are at their equilibrium positions when the angle formed by the two strings is 40° (X, Q, I)
9. The maximum amount of like charge you can put equally on two pith balls you are holding without feeling their force of repulsion (B, F, P)
10. The charge-to-mass ratio q/m for two identical, isolated particles that would remain at rest if released from rest at any separation (X, G, U)

Hints

A. What is the magnitude of charge carried by 1% of a person's electrons?
B. What minimum gravitational force can you detect with your fingers?
C. To lift any object, what force must be overcome?
D. What is the separation distance between two neighboring ions in the crystal?
E. What is the main ingredient in cola?
F. How can you minimize the magnitude of the electric force between the pith balls?
G. If you equate the gravitational force and electric force expressions, what cancels?
H. What is the mass of a molecule of water?
I. What must the magnitude of the electric force be?
J. How many electrons are there for each proton in Earth?
K. What is the mass of a person?
L. What is the mass of the Moon?
M. What is the mass of a proton?
N. What is the mass of 1 liter of cola?
O. What is the mass of a tiny bit of paper?
P. What maximum distance can you create between your hands?
Q. What is the mass of a pith ball?
R. What is a typical separation distance between a proton and an electron in an atom?
S. What is the magnitude of the charge carried by an electron or a proton?
T. What fraction of this mass is due to protons?
U. What is the ratio of gravitational to electrical constants in the force expressions?
V. What charge does each ion carry?
W. How many protons are there in one molecule of water?
X. What is the vector sum of the forces acting on each object?
Y. What is the mass of Earth?
Z. What is the main ingredient in the human body?

Key (all values approximate)

A. 4×10^7 C; **B.** about $1/10$ of the gravitational force exerted by Earth on a penny, which has a mass of 2×10^{-3} kg—thus 2×10^{-3} N; **C.** gravitational force; **D.** 2×10^{-10} m; **E.** water; **F.** hold them as far apart as possible; **G.** the separation distance; **H.** two H atoms plus one O atom, 3×10^{-26} kg; **I.** 4×10^{-4} N; **J.** one; **K.** 70 kg; **L.** 7×10^{22} kg; **M.** 2×10^{-27} kg; **N.** 1 kg; **O.** 10^{-5} kg; **P.** 2 m; **Q.** 1×10^{-4} kg; **R.** 5×10^{-11} m; **S.** 1.6×10^{-19} C, the elementary charge; **T.** half, because atoms have about equal numbers of protons and neutrons, and these two particles have nearly identical masses; **U.** with both constants in SI units, $G/k = 7 \times 10^{-21}\ C^2/kg^2$; **V.** Na^+ and Cl^-, each with an elementary charge; **W.** ten; **X.** zero; **Y.** 6×10^{24} kg; **Z.** water

PRACTICE

Worked and Guided Problems

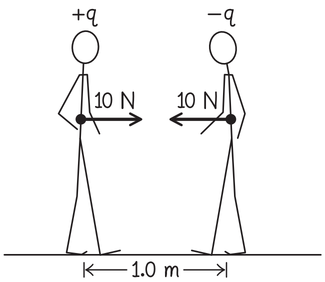
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 22.1 Mutual attraction

We often feel what is called a "physical attraction" to other people. If this were literally true and due to the electric force between two persons, one of them would have to carry surplus positive charge and the other would have to carry surplus negative charge. Suppose you and a friend stand 1 m apart and feel an attractive electric force of 10 N between you. Estimate the fraction q/q_{body} for each of you, where q is the surplus charge of a given type in either body and q_{body} is all the charge of that type in the body.

❶ GETTING STARTED We begin with a sketch of the physical situation (Figure WG22.1).

Figure WG22.1



We first need to determine the amount of surplus charge q on each body that produces 10 N of force between the bodies, and to do this we must make some assumptions about where on each body that charge is located. We then need to estimate the number of charged particles of a given type in each body. These particles are the protons and electrons in the molecules of the body. Multiplying the number of particles by the elementary charge gives the charge q_{body} of that type in each body, and so we have the fraction we are asked to calculate. This solution will require many estimates.

❷ DEVISE PLAN We know the force magnitude and the separation distance between you and your friend. Let's assume you both have the same mass and equal amounts of surplus charge (though of opposite signs). Because it would be difficult to calculate the force if the charge were uniformly distributed throughout each body, we assume that all the charge in each body is concentrated at the body's center, with a center-to-center distance of 1 m. The human body is mostly water, H_2O (see Developing a Feel 6), and so dividing body mass by the mass of a water molecule gives us the number of water molecules in each body. Multiplying that number by the number of protons or electrons in each molecule gives us the charge of each type in the body.

❸ EXECUTE PLAN We use Coulomb's law to solve for the magnitude of the surplus charge q needed to produce the 10-N force:

$$F = k\frac{|(+q)(-q)|}{r^2}$$

$$q = \sqrt{\frac{Fr^2}{k}} = \sqrt{\frac{(10\ \text{N})(1\ \text{m})^2}{9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2}} = 3.3 \times 10^{-5}\ \text{C}.$$

To determine the change q_{body} of a given type in each body, let's use 60 kg as a reasonable value for your mass. Each H_2O molecule contains ten protons and eight neutrons, for a mass of 3×10^{-26} kg, making the number of water molecules in your body

$$\frac{60\ \text{kg}}{3 \times 10^{-26}\ \text{kg}/\text{H}_2\text{O molecule}} = 2 \times 10^{27}\ \text{H}_2\text{O molecules in body.}$$

Each of these molecules contains ten protons (and ten electrons), so the charge q_{body} of a given type (positive or negative) in the body is

$$q_{\text{body}} \approx (2 \times 10^{27}\ \text{molecules})(10\ \text{elementary charges/molecule})$$
$$\times (1.6 \times 10^{-19}\ \text{C/elementary charge}) \approx 3 \times 10^9\ \text{C}.$$

Thus the fraction of surplus charge is

$$\frac{q}{q_{\text{body}}} \approx \frac{3.3 \times 10^{-5}\ \text{C}}{3 \times 10^9\ \text{C}} \approx 1 \times 10^{-14}. ✔$$

PRACTICE

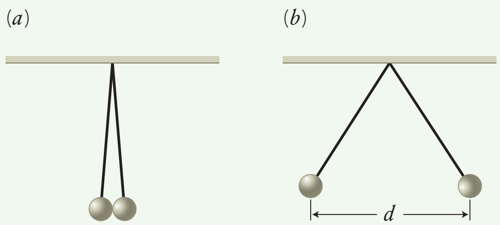
This is approximately one surplus electron (or proton) for every 10 trillion (10^{13}) water molecules!

❹ **EVALUATE RESULT** It is hard to estimate whether or not this answer is reasonable because most people do not have a good feel for the magnitudes of various electric forces in our world. In reality, of course, attractive electric forces are not felt between humans. The actual fraction of surplus charge in your body is essentially zero under normal conditions. However, you can observe some consequences of surplus body charge when you rub your feet on a carpet, transferring charge to your body. You might get a trillion or so surplus elementary charges in this case, for a surplus charge fraction of about 10^{-16}. What gives you a shock when you then touch a metal doorknob is this surplus charge leaving your body.

Guided Problem 22.2 Electroscope

In Figure WG22.2, two identical 0.017-kg spheres are attached to nonconducting threads affixed to the ceiling. Initially, the spheres touch each other (*a*), but then an equal amount of charge is put on each (by touching them, for instance, with a charged rod not shown in the drawing) (*b*). If each thread is 120 mm long and the spheres come to rest at a separation $d = 93$ mm, what is the electrical charge on each sphere?

Figure WG22.2



❶ **GETTING STARTED**

1. In what sense is this problem similar to Worked Problem 22.1? In what sense is it different?
2. Why do the spheres push apart but then come to an equilibrium configuration? What forces produce this motion?

❷ **DEVISE PLAN**

3. Draw the free-body diagram for each sphere. List any assumptions you make.
4. What quantity do you need to know to decompose the forces into components?
5. Choose a coordinate system, and then decompose the forces into x and y components.

❸ **EXECUTE PLAN**

6. Solve for the desired electrical charge q. Avoid solving for intermediate quantities if possible.

❹ **EVALUATE RESULT**

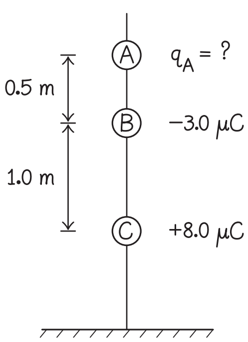
7. Think about how q should depend on the separation distance d. Does your algebraic answer reflect this expectation?

Worked Problem 22.3 Levitation

One possible way of levitating an object might be to use the forces associated with charged objects. For example, suppose you have two charged spheres fixed 0.50 m apart on a vertical pole. The lower sphere carries a fixed charge of $-3.0\ \mu$C, and the upper one carries a charge that can be adjusted. A 30-g sphere carrying a charge of $+8.0\ \mu$C can move freely on the pole below the other two. You want this sphere to levitate (float) 1.0 m below the -3.0-μC sphere. What should the charge on the upper sphere be adjusted to in order to achieve this feat?

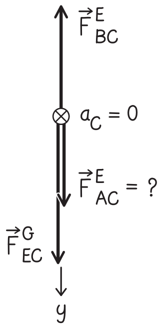
❶ **GETTING STARTED** As usual, we begin with a sketch of the situation (Figure WG22.3). Let's label the upper fixed-position sphere A, the other fixed-position sphere B, and the movable sphere C. Let's use q_A for the unknown adjustable charge on sphere A. In order for sphere C to float at the specified position, we need to balance the electric and gravitational forces exerted on this sphere so that their vector sum is zero. We assume we're near Earth, so that g is a constant 9.8 m/s^2. The gravitational forces between C and B and between C and A are much smaller than the electric force between the pairs (you should verify this), and so the only forces we need to consider are the gravitational force exerted by Earth on C, the attractive upward electric force exerted by B on C, and the electric force exerted by A on C.

Figure WG22.3



❷ **DEVISE PLAN** We want the vector sum of forces exerted on C to be zero. A free-body diagram is always helpful in force problems (Figure WG22.4). Because all the action is along a straight vertical line, we choose the reference axis to be a downward-pointing y axis. The gravitational force $\vec{F}^{G}_{EC}$ exerted on C is downward. Because the charges on B and C have opposite signs, these two particles attract each other, making the direction of the force $\vec{F}^{E}_{BC}$ upward. However, we don't know whether to make the force $\vec{F}^{E}_{AC}$ exerted by A on C attractive or repulsive (through the sign of q_A) because we do not yet know the magnitudes $\vec{F}^{G}_{EC}$ and $\vec{F}^{E}_{BC}$. In Figure WG22.4, we draw the vector arrow for $\vec{F}^{E}_{AC}$ in the direction of our y axis, but we don't assume that its component along that axis is positive. It may point in the opposite direction, depending on the sign we calculate for q_A.

Figure WG22.4



❸ **EXECUTE PLAN** For stable levitation ($a_C = 0$), we must have

$$\Sigma F_y = F^{G}_{ECy} + F^{E}_{BCy} + F^{E}_{ACy} = +F^{G}_{EC} + (-F^{E}_{BC}) \pm F^{E}_{AC} = 0.$$

We allow the final term to be either positive or negative because these are the only two options for the direction of $\vec{F}^{E}_{ACy}$. We can verify this by using the mathematical expression for the electric force (Eq. 22.7):

$$\vec{F}^{E}_{ACy} = k\frac{q_A q_C}{r^2_{AC}}(\hat{r}_{AC})_y.$$

Remembering that $\hat{r}_{AC}$ is the unit vector pointing from sphere A to sphere C, we see that $(\hat{r}_{AC})_y = +1$ in this coordinate system. The sign of the final term is thus determined by the sign of the unknown charge q_A. Thus

$$\Sigma F_y = +m_C g + \left[-k\frac{|q_B q_C|}{r^2_{BC}}\right] + \left[k\frac{q_A q_C}{r^2_{AC}}\right] = 0.$$

Solving for q_A and then substituting numerical values, we find

$$\begin{aligned} q_A &= \frac{r^2_{AC}}{kq_C}\left(k\frac{|q_B q_C|}{r^2_{BC}} - m_C g\right) = \frac{r^2_{AC}}{r^2_{BC}}|q_B| - \frac{m_C g r^2_{AC}}{kq_C} \\ &= \frac{(1.5\text{ m})^2}{(1.0\text{ m})^2}|-3.0 \times 10^{-6}\text{ C}| \\ &\quad - \frac{(0.030\text{ kg})(9.8\text{ m/s}^2)(1.5\text{ m})^2}{(9.0 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)(8.0 \times 10^{-6}\text{ C})} \\ &= -2.4 \times 10^{-6}\text{ C.} \checkmark \end{aligned}$$

This result shows us that we must put a negative charge on sphere A. This charge produces a force that is directed upward (attracting positively charged sphere C) and helps sphere B counteract the downward gravitational force exerted by Earth on C.

❹ **EVALUATE RESULT** Do we expect q_A to be negative, which makes $\vec{F}^{E}_{AC}$ attractive (aimed upward)? If we make rough estimates of our known force magnitudes, we see that $F^{G}_{EC} \approx 3 \times 10^{-1}$ N and $F^{E}_{BC} \approx 2 \times 10^{-1}$ N. Thus Earth pulls C down more than sphere B pulls it up, which means sphere A needs to supply an upward force to make the vector sum of the forces exerted on C be zero. With its negative $m_C g$ term, our expression for q_A correctly predicts the negative sign for q_A.

Is this system in stable or unstable equilibrium (see Section 15.4)? Does your answer to that question depend on the sign of q_A? As a further check of your understanding of the problem, re-derive the expression for q_A for the case where the charges on B and C have the same sign.

PRACTICE

Guided Problem 22.4 Electron orbit

In the classical model of the hydrogen atom, a single electron orbits the single proton of the hydrogen's nucleus at a radius of 0.053 nm. (*a*) How fast is the electron moving? (*b*) How long does it take to complete one orbit?

1 GETTING STARTED

1. Make a sketch showing the electron going around a circle and a proton fixed at the center of the circle.
2. Is the assumption that the proton is fixed justified?
3. What force or forces are exerted on the electron?

2 DEVISE PLAN

4. Draw a free-body diagram for the electron.
5. Do you see any analogy between this problem and the topics discussed in Chapter 13?
6. What is the relationship between the electron's speed and the period of the orbit?

3 EXECUTE PLAN

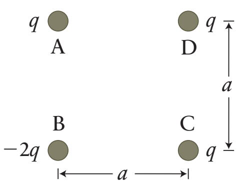
4 EVALUATE RESULT

7. How does the electron's speed compare with the speed of light?

Worked Problem 22.5 Charge square

Four charged particles are arranged in a square as shown in Figure WG22.5. If $q = +3.9 \times 10^{-6}$ C and $a = 6.9$ mm, what is the vector sum of the forces exerted on particle D by the other three particles?

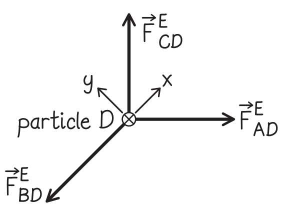
Figure WG22.5



1 GETTING STARTED This is an application of Coulomb's law, Eq. 22.7. We know all the charge magnitudes and the distances r_{AD} and r_{CD}, and we can use trigonometry to obtain the distance r_{BD}.

2 DEVISE PLAN The first step is to draw Figure WG22.6, a free-body diagram of particle D. We can simplify the problem by choosing our axes to exploit symmetry, as shown. With the axes aligned this way, the force $\vec{F}^E_{BD}$ has no *y* component and the components F^E_{CDy} and F^E_{ADy} cancel each other. Thus all we need to calculate are the *x* components of the forces, which we get from Eq. 22.7.

Figure WG22.6



3 EXECUTE PLAN The expression for the vector sum of the *x* components is

$$\begin{aligned}
\Sigma F_{Dx} &= F^E_{ADx} + F^E_{BDx} + F^E_{CDx} \\
&= F^E_{AD}\cos 45^\circ + (-F^E_{BD}) + F^E_{CD}\cos 45^\circ \\
&= 2F^E_{AD}\cos 45^\circ + (-F^E_{BD}) \\
&= 2\left(k\frac{qq}{a^2}\right)\frac{\sqrt{2}}{2} - k\frac{(2q)q}{(\sqrt{2}a)^2} \\
&= k\frac{q^2}{a^2}(\sqrt{2} - 1) \\
&= (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(3.9 \times 10^{-6}\ \text{C})^2}{(0.0069\ \text{m})^2}(\sqrt{2} - 1) \\
&= 1.19 \times 10^3\ \text{N} = 1.2\ \text{kN},
\end{aligned}$$

which means the vector sum of the forces exerted on D is a force of magnitude 1.2 kN directed along the positive x axis of our coordinate system. ✔

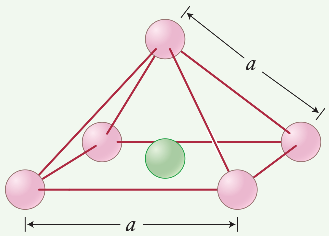
❹ **EVALUATE RESULT** Because two of the forces are repulsive and one is attractive, we expect to see terms of different signs in the algebraic answer. Reassuringly, we see that a *difference* of two terms is computed: the first term due to the two repulsive forces $\vec{F}^{E}_{AD}$ and $\vec{F}^{E}_{CD}$ and the second term due to the attractive force $\vec{F}^{E}_{BD}$. The computed force magnitude does seem large. However, the particles carry a sizable charge (remember, 1 C is a lot of charge) and they are close to one another.

Symmetry is a very powerful tool in approaching physics problems, and you should take advantage of it whenever possible.

Guided Problem 22.6 Charge pyramid

Six ions are locked in a square pyramidal structure as shown in Figure WG22.7. The length of each side of the pyramid is $a = 0.13$ nm, and the sixth (shown in green) ion is at the center of the base of the pyramid. Calculate the vector sum of the electric forces exerted on the top ion if each of the six ions is missing one electron.

Figure WG22.7



❶ **GETTING STARTED**

1. What causes the electric force experienced by the top ion?
2. Look for any symmetry you can exploit to make the problem simpler. Remember that this is a three-dimensional situation.
3. What is the amount of surplus charge on each ion?

❷ **DEVISE PLAN**

4. Which equations allow you to express the electric force experienced by the top ion in terms of known quantities?
5. Can the problem be reduced from three dimensions to two by an appropriate choice of coordinate axes?

❸ **EXECUTE PLAN**

6. Write an algebraic expression for the electric force you are seeking, and then substitute known values to get a numerical result.

❹ **EVALUATE RESULT**

7. Compare your numerical answer to the electric force between two protons a distance a apart.

Answers to Review Questions

1. The gravitational interaction between the balloon and Earth makes the balloon fall. After charging, the electric interaction between your hair and the balloon dominates. (Friction between your hair and the balloon is the source of the force that opposes the gravitational interaction between the balloon and Earth.)
2. The electric interaction is much stronger than the gravitational interaction. The electric interaction can be attractive or repulsive, while the gravitational interaction is only attractive.
3. A charged object can interact in different ways with a second charged object, either repelling or attracting that object. This means there are different possibilities for the type of charge on the second object.
4. Without further information, it's impossible to say who is right. Positive charge carriers moved from an object A to an object B or negative charge carriers moved from B to A have the same effect. However, because we know that plastic combs become negatively charged when rubbed through hair, and the hair acquires positive charge, we could speculate that the hair transfers electrons to the balloon.
5. Zero. Charge can only be moved from one place to another or created or destroyed in equal positive and negative portions. Because charge is a conserved quantity, the amount of charge in a closed system is constant.
6. The surplus charge causing the spark entered your body at your feet but interacted with the knob at your fingers. That the charge migrated tells you that your body is a fair electrical conductor.
7. Paper is an electrical insulator; to see this for yourself, insert a slip of paper between two batteries in a flashlight and turn on the switch. A paper clip is metal and thus an electrical conductor. Seawater is an electrical conductor because of the presence of dissolved ions. Tires are rubber and so electrical insulators, just as the rubber insulation on electrical power cords is. Air is usually an electrical insulator, but as with most other electrical

insulators, a sufficient electric force exerted on a volume of air can cause a breakdown, or spark, and then charge carriers can flow through the air, making it an electrical conductor in the breakdown region.

8. Breaking wood leaves two pieces of similar material. There's no reason the symmetrical result (same material on both sides of the break) should leave a positive charge on one piece and a negative charge on the other. It is usually in the breaking of bonds in *dissimilar* materials that a surplus of charge is left behind on the two materials.

9. The charged rod attracts opposite charges and repels like charges on the neutral object, causing the object to become polarized. An oppositely charged region forms near the rod, while a like-charged region forms farther away. The proximity of the oppositely charged region causes the object to be attracted (weakly) to the rod.

10. There would be no difference. Opposite charges would still be opposite and attract, and like charges would still be like and repel.

11. Polarization is accomplished by bringing an already charged item near the object to be polarized. In each case the polarization effect creates regions of surplus charge in the object, with the region of charge opposite to those on the charged item forming closest to the charged item. In a conductor, some electrons are free to move throughout the material, and it is these mobile electrons that move from one part of the object to another as the object becomes polarized. In an insulator, electrons are not free to move through the material, but rather the electron cloud in each atom shifts slightly toward or away from the charged item (depending on the sign of its surplus charge). The effect of many superposed atomic cloud shifts creates thin regions of surplus charge on the surface of the insulating object.

12. (*a*) Positively charged or neutral; (*b*) negatively charged only.

13. The magnitude F^E_{AB} is proportional to the charge on B, experimentally. Likewise, the magnitude F^E_{BA} is proportional to the charge on A. Because Newton's third law tells you that these two forces must be equal in magnitude, they must both be proportional to both the charge on A and the charge on B. The only way to make this happen is for the product $q_\text{A}q_\text{B}$ to appear in the expression for the force. Note that the sum $q_\text{A} + q_\text{B}$ does not reproduce the proportionality behavior; doubling q_B does not double the sum.

14. Doubling both charges without changing the distance would increase the force exerted on each by a factor of 4. Doubling the distance would compensate for this factor of 4, because the force is inversely proportional to the square of the distance.

15. The magnitude of the force exerted on either partner in the interaction pair is identical, as is always the case with any interaction.

16. The last term does not involve charge carrier 1 and therefore has nothing to do with the force exerted on 1. This term should be omitted from the expression.

17. Because particles 1 and 2 carry charges of opposite types, the situation requires that particle 3 not be placed between 1 and 2 in order that the directions of the two forces oppose each other. To obtain equal magnitudes for the two forces, particle 3 must be placed closer to the particle with smaller charge (particle 1). Thus the answer is (*c*).

Answers to Guided Problems

Guided Problem 22.2 Let α be the angle that each thread makes with respect to vertical. Then

$$Q = d\sqrt{\frac{mg\tan\alpha}{k}} = (0.093\text{ m})\sqrt{\frac{(0.017\text{ kg})(9.8\text{ m/s}^2)(0.4203)}{9.0\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2}}$$

$$= 2.6\times10^{-7}\text{C},$$

where $\alpha = \sin^{-1}\left(\frac{d}{2\ell}\right) = \sin^{-1}\left(\frac{93\text{ mm}}{2\times120\text{ mm}}\right) = 22.80°$, and $\tan\alpha = 0.4203$.

Guided Problem 22.4

$$(a)\; v = e\sqrt{\frac{k}{mR}}$$

$$= (1.6\times10^{-19}\text{ C})\sqrt{\frac{9.0\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2}{(9.1\times10^{-31}\text{ kg})(5.3\times10^{-11}\text{ m})}}$$

$$= 2.2\times10^6\text{ m/s};$$

$$(b)\; T = \frac{2\pi R}{v} = \frac{(2\pi)(5.3\times10^{-11}\text{ m})}{2.2\times10^6\text{ m/s}} = 1.5\times10^{-16}\text{ s}.$$

Guided Problem 22.6 The force on the top ion points upward, away from the central base ion, and has magnitude

$$F = ke^2\left[\frac{1}{h^2} + \frac{4}{a^2\sqrt{2}}\right] = 6.6\times10^{-8}\text{ N, where } h = \frac{a}{\sqrt{2}}.$$

PRACTICE

Guided Practice by Chapter

23 The Electric Field

Review Questions

Answers to these questions can be found at the end of this chapter.

23.1 The field model

1. What is the physics definition of *field*?
2. What are the two major reasons for introducing the field concept?
3. When two charged Ping-Pong balls, A and B, are held a small distance apart, which ball is the source of the electric field that acts on ball B?
4. What is the difference between a scalar field and a vector field? Give an example of each type.
5. What is a vector field diagram, and how is it useful?

23.2 Electric field diagrams

6. How can a charged test particle be used to determine the electric field at a point P caused by a charged source object S located nearby?
7. A positively charged styrofoam pellet is placed in the electric field surrounding a charged sphere. If the charge on the pellet is changed from positive to negative, what happens to the magnitude and direction of the sphere's electric field at the pellet's location?
8. A negatively charged test particle is placed at position P in an electric field, and it experiences an electric force directed to the west. What is the direction of the electric field at point P?

23.3 Superposition of electric fields

9. Balloon 1 is charged and taped to a board, and a pushpin is inserted at a different location on the board. The electric field at the pushpin due to balloon 1 is measured to have magnitude E_1. Balloon 1 is removed and the process is repeated with balloon 2, taped at a third location. The electric field at the pin due to balloon 2 is measured to have magnitude E_2. If both balloons are present, is the magnitude of the electric field at the pin equal to $E_1 + E_2$?

23.4 Electric fields and forces

10. An electron initially moving horizontally near Earth's surface enters a uniform electric field and is deflected upward. What can you say about the direction of the electric field (assuming no other interaction)? What can you say about the direction of the electric field if the electron is deflected downward?
11. A uniform electric field is directed to the right. Sketch the trajectory of a proton and an electron launched straight up in this field.
12. Is it possible for a dipole that is initially moving in a straight line to be deflected by a uniform electric field?
13. An electron is traveling in circular motion at constant speed due to the effect of an electric field. Is it possible for the electric field to be uniform?

23.5 Electric field of a charged particle

14. If the charge on particle 2 in *Principles and Practice* Figure 23.23 were doubled, what aspects of the electric field at point P would change: the magnitude, the direction, or both?
15. A delicate instrument is two paces away from a highly charged metallic sphere. If you want to reduce the magnitude of the electric field at the instrument to 1% of its present value, how many paces away from its present position must you move the instrument?

23.6 Dipole field

16. Which is greater, the magnitude of the electric field a distance r from a small sphere carrying charge q or the magnitude of the electric field a distance r from a small electric dipole of dipole moment qd, where d is the dipole separation and $r \gg d$?
17. What is the definition of *dipole moment*?
18. An electric dipole is fixed at the origin of a coordinate system, and an electric field detector can be moved anywhere along the surface of an imaginary sphere of radius R that is centered on the origin. The radius of this sphere greatly exceeds the dipole separation, $R \gg d$. As the detector is moved over the spherical surface, what is the ratio of the largest and smallest magnitudes of the electric field it detects, $E_{\text{maximum}}/E_{\text{minimum}}$?

23.7 Electric fields of continuous charge distributions

19. A hemispheric shell made of a material that is an electrical insulator has a radius R and a charge q distributed uniformly over it. It is placed in the $z \geq 0$ region of an xyz coordinate system, centered about the z axis, with its base resting on the xy plane. Write the expression for the charge dq on an infinitesimally thin ring portion of the shell of width $Rd\theta$, where θ is the angle from the z axis to the ring.
20. What is the distinguishing radial dependence of the electric field (*a*) around a charged particle (charge distributed in no dimensions), (*b*) around a long uniformly charged wire (charge distributed in one dimension), and (*c*) around a large uniformly charged sheet (charge distributed in two dimensions)?

23.8 Dipoles in electric fields

21. What are the SI units of polarizability?
22. The magnitude of the electric field due to a permanent dipole decreases as the cube of distance r from the dipole: $E \propto 1/r^3$. The electric force exerted by the dipole on a charged particle placed in the electric field of the dipole is also proportional to $1/r^3$: $F^E_{\text{dp}} \propto 1/r^3$. How does the force exerted by the charged particle on the dipole depend on r?
23. What are the similarities and differences between permanent and induced dipoles?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The electric field magnitude 0.1 nm from a proton (P, B)
2. The electric field magnitude 1 m from a ball of 10^6 electrons (I)
3. The charge-to-mass ratio for a sodium (Na^+) ion (G, P)
4. The charge-to-mass ratio for either of two identical particles whose gravitational and electric interactions cancel each other (E, R)
5. The volume charge density of a hydrogen nucleus (F, K, P)
6. The surface charge density on a 200-mm length of a freshly pulled strip of transparent tape (D, H, M, Q, U)
7. The linear charge density on a rubbed plastic rod that is 300 mm long (C, H, L, Q, T)
8. The electric field strength 0.1 m from the long axis of the plastic rod of Question 7 (Answer to Question 7, A)
9. The magnitude and direction of the uniform electric field required to "float" a proton near Earth's surface (N, P, S)
10. The electric dipole moment of a water molecule (J, O)

Hints

A. Is this location close or far enough from the rod to use an approximation for the magnitude of $\vec{E}$?
B. What is 0.1 nm in meters?
C. What is the mass of the rod?
D. What is the mass of the tape strip?
E. If you equate the gravitational and electric force expressions, what factor cancels?
F. What is the radius of a hydrogen nucleus?
G. What is the mass of a sodium ion?
H. What fraction of this mass is due to protons?
I. What is the charge on the ball?
J. How should you model the charge distribution?
K. What is the formula for the volume of a sphere?
L. What is the number of electrons in the rod?
M. What is the number of electrons in the strip?
N. What is the mass of a proton?
O. What is the separation of the centers of charge?
P. What is the charge of a proton in coulombs?
Q. What fraction of the electrons is redistributed during charging?
R. What is the ratio of the gravitational constant to the proportionality constant in Coulomb's law?
S. If the electric field points upward, what is the direction of the electric force it exerts on the proton?
T. What is the length along which the surplus charge is distributed?
U. What is the surface area of the strip?

Key (all values approximate)

A. No, which means you must use the expression in the Chapter Summary: $E_x = kq/x(\ell^2/4 + x^2)^{1/2}$; **B.** 1×10^{-10} m; **C.** 0.1 kg; **D.** 1×10^{-4} kg; **E.** the factor for the distance between the particles; **F.** 1×10^{-15} m; **G.** 4×10^{-26} kg; **H.** half, assuming the atoms contain equal numbers of protons and neutrons; **I.** -2×10^{-13} C; **J.** there is an imbalance of about one elementary charge, so the dipole consists of one "surplus" proton centered between the H atoms and one surplus electron centered on the O atom; **K.** $V = 4\pi r^3/3$; **L.** 3×10^{25}; **M.** same as the number of protons, 3×10^{22}; **N.** 2×10^{-27} kg; **O.** 5×10^{-11} m, because of the bonding angle of 105° between the H atoms; **P.** $+2 \times 10^{-19}$ C; **Q.** about one in 10^{12} (see Section 22.3); **R.** $G/k \approx 7 \times 10^{-21}$, with both constants in SI units; **S.** upward; **T.** 0.2 m, or two-thirds the length of the rod; **U.** 0.002 m^2

Worked and Guided Problems

Procedure: Calculating the electric field of continuous charge distributions by integration

To calculate the electric field of a continuous charge distribution, you need to evaluate the integral in Eq. 23.15. The following steps will help you evaluate the integral.

1. Begin by making a sketch of the charge distribution. Mentally divide the distribution into small segments. Indicate one such segment that carries a charge dq_s in your drawing.
2. Choose a coordinate system that allows you to express the position of the segment in terms of a minimum number of coordinates (x, y, z, r, or θ). These coordinates are the integration variables. For example, use a radial coordinate system for a charge distribution with radial symmetry. Unless the problem specifies otherwise, let the origin be at the center of the object.
3. Draw a vector showing the electric field caused by the segment at the point of interest. Examine how the components of this vector change as you vary the position of the segment along the charge distribution. Some components may cancel, which greatly simplifies the calculation. If you can determine the direction of the resulting electric field, you may need to calculate only one component. Otherwise express $\hat{r}_{sP}$ in terms of your integration variable(s) and evaluate the integrals for each component of the field separately.
4. Determine whether the charge distribution is one-dimensional (a straight or curved wire), two-dimensional (a flat or curved surface), or three-dimensional (any bulk object). Express dq_s in terms of the corresponding charge density of the object and the integration variable(s).
5. Express the factor $1/r_{sP}^2$, where r_{sP} is the distance between dq_s and the point of interest, in terms of the integration variable(s).

At this point you can substitute your expressions for $dq_s$and $1/r_{sP}^2$ into Eq. 23.15 and carry out the integral (or component integrals), using what you determined about the direction of the electric field (or substituting your expression for $\hat{r}_{sP}$).

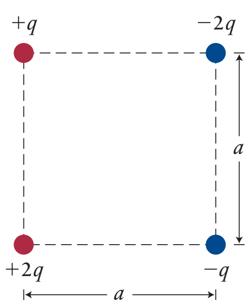
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 23.1 Charge square

Four charged particles are arranged in a square as shown in Figure WG23.1, with $q = 3.9 \times 10^{-4}$ C and $a = 6.9$ mm. Determine the magnitude and direction of the electric field at the center of the square.

PRACTICE

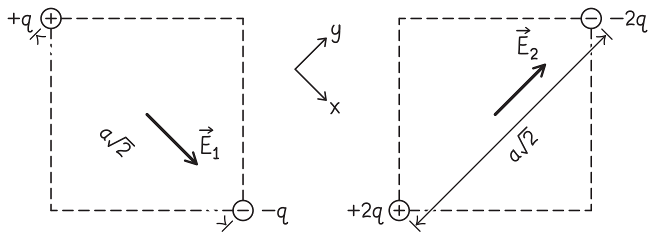
Figure WG23.1



❶ GETTING STARTED We use the superposition principle, expressed in Eq. 23.5, on the individual electric fields created by the four particles to determine the electric field at the center.

❷ DEVISE PLAN We could calculate the electric field due to each particle and then add the components to get the field at the square's center. However, we can simplify the problem by exploiting the symmetry of the configuration. One way to use that symmetry is to separate the four particles into two pairs: $+q/-q$ having separation distance $a\sqrt{2}$ and contributing an electric field $\vec{E}_1$ to the center and $+2q/-2q$ having separation distance $a\sqrt{2}$ and contributing an electric field $\vec{E}_2$ to the center (Figure WG23.2).

Figure WG23.2



Each of the two fields is produced by a pair of charged particles carrying charges of equal magnitude but opposite signs, with the fields pointing away from the positively charged particle and toward the negatively charged particle. Note in Figure WG23.2 that we have aligned the axes of our coordinate system in a way that facilitates the calculation of components. Because we seek the electric field at the center of the square, we know that the magnitudes E_1 and E_2 are related by a factor of 2 (that is, $2E_1 = E_2$) due to the similarities of distance and charge. All we have to do is add the electric field vectors $\vec{E}_1$ and $\vec{E}_2$ to obtain the electric field at the center.

❸ **EXECUTE PLAN** For the $+q/-q$ pair, the electric field at the square's center is

$$\vec{E}_1 = k\frac{q}{r_{+q}^2}\hat{r}_{+q} + k\frac{(-q)}{r_{-q}^2}\hat{r}_{-q}$$

$$= k\frac{q}{(\frac{1}{2}a\sqrt{2})^2}\hat{\imath} + k\frac{(-q)}{(\frac{1}{2}a\sqrt{2})^2}(-\hat{\imath})$$

$$= 4k\frac{q}{a^2}\hat{\imath}.$$

For the $+2q/-2q$ pair, we note that $\vec{E}_2$ is produced along the y direction by particles that carry charges that are twice as large as in the $+q/-q$ pair. We can therefore infer that

$$\vec{E}_2 = 8k\frac{q}{a^2}\hat{\jmath},$$

which means that the electric field at the center of the square is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = 4k\frac{q}{a^2}\hat{\imath} + 8k\frac{q}{a^2}\hat{\jmath}.$$

The magnitude of this electric field is

$$E = \sqrt{E_x^2 + E_y^2} = \sqrt{\left(\frac{4kq}{a^2}\right)^2 + \left(\frac{8kq}{a^2}\right)^2} = 4\sqrt{5}k\frac{q}{a^2},$$

and its direction is

$$\theta = \tan^{-1}\frac{E_y}{E_x} = \tan^{-1}\frac{8kq/a^2}{4kq/a^2} = \tan^{-1} 2 = 63°.$$

Substituting values yields

$$\vec{E} = \frac{4\sqrt{5}(9.0 \times 10^9\ \mathrm{N\cdot m^2/C^2})(3.9 \times 10^{-4}\ \mathrm{C})}{(0.0069\ \mathrm{m})^2}$$

$$= 6.6 \times 10^{11}\ \mathrm{N/C},$$

at 63° above the positive x axis as oriented in Figure WG23.2. ✔

❹ **EVALUATE RESULT** The electric field we obtained is large, but the charge on these particles is large and the separation distance is small. The direction makes sense because it points approximately along the line that connects the two particles carrying a charge $\pm 2q$.

PRACTICE

Guided Problem 23.2 Charge triangle

Three particles form an equilateral triangle with side length a. Two of the particles carry a positive charge $+q$, and the third particle carries a charge $-2q$. What are the magnitude and direction of the electric field at the center of the triangle?

❶ GETTING STARTED

1. Draw a diagram showing the particles, charges, and distances.
2. Locate the center of the triangle.
3. What causes the electric field at the center? Account for all sources that contribute to this field.
4. Is there any symmetry you can use to simplify your work?

❷ DEVISE PLAN

5. Must you work with vectors to solve this problem? You know that particles carrying the same type of charge repel each other and particles carrying opposite types of charge attract each other. Be sure to take this aspect of the interaction into account in your diagram.

❸ EXECUTE PLAN

6. What are the relevant distances to the center of the triangle?

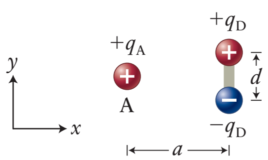
❹ EVALUATE RESULT

7. Does your answer behave in a physically plausible way when the side lengths or charge magnitudes are varied?

Worked Problem 23.3 Torqued dipole

Figure WG23.3 shows an electric dipole located near a fixed charged particle A that carries charge $+q_A$. The dipole charge is q_D. (*a*) What torque does the electric force exerted by A create about the dipole's midpoint? (*b*) Sketch a graph showing how the torque caused by $\vec{F}^E_{AD}$ changes as the dipole separation d increases while the distance a between particle A and the dipole remains constant.

Figure WG23.3



❶ GETTING STARTED The positive pole of the dipole is repelled by A, and the negative pole is attracted to A. This combination of attraction and repulsion causes a torque about the dipole midpoint that tends to rotate the dipole clockwise. The right-hand rule tells us that this torque is directed into the page. The two electric forces exerted on the dipole are not parallel to each other because the electric field created by A is directed radially outward.

❷ DEVISE PLAN One way to approach this problem is by brute force: Calculate the electric force exerted by A on one pole of the dipole, determine the torque caused by it about the dipole midpoint, do the same for the other pole, and then add the torques to get their vector sum. If we think of the dipole and particle A as a system, however, all the interactions are internal: The system is isolated. This means the vector sum of torques on the system about a chosen reference point is zero. The dipole midpoint is the reference point specified in the problem statement, so we choose it as the origin of our coordinate system. The torque on the dipole is the negative of the torque on particle A. Because *Principles and Practice* derives a formula for the magnitude, at any position, of the electric field created by a dipole (Eq. 23.8, which using our symbols for variables is $E = -kq_Dd/[a^2 + (d/2)^2]^{3/2}$), we can solve this problem.

❸ EXECUTE PLAN To keep track of the dipole variables in our calculations, we label them with subscript D.

(*a*) From Eq. 23.1 we know that the electric force exerted by the dipole on particle A is

$$\vec{F}^E_{DA} = q_A\vec{E}_D. \tag{1}$$

From Eq. 23.8, the electric field surrounding the dipole at the position of particle A (on the x axis) is

$$\vec{E}_D = -k\frac{q_Dd}{[a^2 + (d/2)^2]^{3/2}}\hat{\jmath}. \tag{2}$$

The torque on A about the midpoint of the dipole is

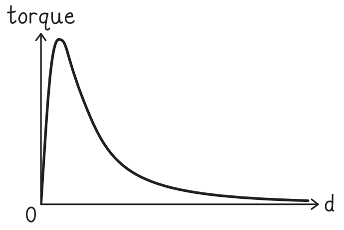
$$\vec{\tau}_{DA} = \vec{r}_{DA} \times \vec{F}^E_{DA} = (-a\hat{\imath}) \times q_A\left(-k\frac{q_Dd}{[a^2 + (d/2)^2]^{3/2}}\hat{\jmath}\right).$$

By remembering that $\hat{\imath} \times \hat{\jmath} = \hat{k}$ (out of the page) and that the torque on the particle is equal and opposite to the torque on the dipole, we obtain the torque that the force F^E_{AD} causes on the dipole:

$$\vec{\tau}_{\text{AD}} = -\vec{\tau}_{\text{DA}} = -k\frac{q_{\text{A}}q_{\text{D}}ad}{[a^2 + (d/2)^2]^{3/2}}\hat{k}. \checkmark \tag{3}$$

(*b*) Figure WG23.4 shows the magnitude of this torque as a function of d, with a held fixed. At $d = 0$, the torque is zero. For small values of d ($d < a$), the torque increases almost linearly as d increases because the denominator is nearly equal to a^3. As the dipole separation approaches $d = \sqrt{2}a$, the torque reaches a maximum value. For larger values of d, the denominator is dominated by the d term and is nearly equal to d^3. This means that the torque decreases almost quadratically as d increases in this region, as illustrated in Figure WG23.4.

Figure WG23.4



❹ **EVALUATE RESULT** Equation 3 says that if $q_{\text{A}} > 0$, the torque on the dipole is in the negative z direction—that is, into the page. This information combined with the right-hand rule of Section 12.4 means that the dipole should rotate clockwise. This is what we expect because a positively charged particle A causes the top of the dipole to rotate away from A and pulls the bottom of the dipole toward A.

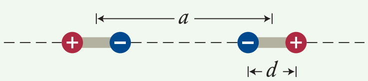
We could also check the result by using the brute force method mentioned above. The direction of the torque for $q_{\text{A}} < 0$ is also as we expect, twisting the dipole counterclockwise. The electric force exerted on the particle is in the downward y direction, as we see by substituting Eq. 2 into Eq. 1. In what direction is the electric force exerted by the particle on the dipole? Does this result make sense? We have seen from Eq. 3 that as the dipole separation d gets large, the torque magnitude decreases. Why is this?

Figure WG23.4 shows that there is a maximum torque. Why is this? Think about what happens to the lever arm of the torque as the dipole separation decreases.

Guided Problem 23.4 Interacting dipoles

The centers of the two dipoles in Figure WG23.5 are separated from each other by a center-to-center distance a. Both dipoles have a dipole charge q_{p} and a dipole separation d. What are the magnitude and direction of the electric force exerted by one dipole on the other?

Figure WG23.5



❶ **GETTING STARTED**

1. For a dipole aligned along a y axis, Eq. 23.11,

$$E_y = k\frac{q_{\text{p}}}{[y - (d/2)]^2} - k\frac{q_{\text{p}}}{[y + (d/2)]^2},$$

gives the magnitude of the electric field at any position on the y axis (the dipole's axis). How can you use this equation to determine the electric forces exerted by one dipole on the two poles of the other dipole? (Hint: A useful approach is to model one dipole as a set of source particles for an electric field and the other as a set of two oppositely charged test particles in that field. You need to identify the source particles and test particles so that you do not double count.)

❷ **DEVISE PLAN**

2. Can you take advantage of any symmetry in the problem?
3. How can you keep track of the various relevant distances in your diagram? Be careful not to use the same symbol to represent two different things.

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

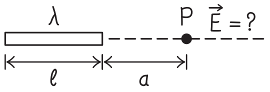
4. Does the electric force in your result point in the direction you expect? Consider the relative sizes of the various terms in your result.
5. What happens to the force as the center-to-center distance a from one dipole to the other becomes much greater than the dipole separation d in each dipole?

Worked Problem 23.5 Electric field created by a straight charged rod

A charged rod of length ℓ has a uniform linear charge density λ. What are the magnitude and direction of the electric field at a point P that is a distance a from one end of the rod and lies along the rod's long axis?

❶ GETTING STARTED First we make a diagram displaying the given information (Figure WG23.6). This problem is similar to examples in *Principles and Practice* for a continuous charge distribution. However, in this problem we are not given the charge q carried by the rod; all we know is the charge per unit length. Nonetheless, we should be able to use similar techniques to obtain the electric field.

Figure WG23.6



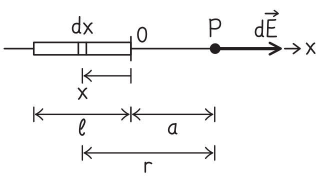
❷ DEVISE PLAN We can determine the electric field $\vec{E}$ created by a continuous charge distribution by dividing the rod into a large number of small segments and then summing their contributions $d\vec{E}$ to the electric field:

$$\vec{E} = \int d\vec{E} = \int k\frac{dq}{r^2}\hat{r}_{\text{P}},$$

where dq is the charge on any segment and r is the distance from that segment of the rod to P. To keep track of the contributions from the various segments we will draw a diagram. From symmetry, we know the electric field points along the rod's long axis.

❸ EXECUTE PLAN We draw a diagram that shows one rod segment dx and all the variables needed for our calculation (Figure WG23.7).

Figure WG23.7



For convenience, we choose the x axis to be along the rod's axis and locate the origin of this axis at the right end of the rod. If we choose the positive x axis to the right, $\hat{r}_{\text{P}} = \hat{\imath}$ and $d\vec{E} = \hat{\imath}\,dE$ is the electric field created by segment dx centered at position x. Remembering that the coordinate x represents a signed quantity (the position), we see from Figure WG23.7 that the distance r from any segment to P is $a - x$ in our coordinate system. The charge dq on the segment is $dq = \lambda dx$. The electric field at P is thus

$$\vec{E} = \hat{\imath}\int_{-\ell}^{0} k\frac{\lambda dx}{(a-x)^2} = k\frac{\lambda}{a-x}\bigg|_{-\ell}^{0}\hat{\imath}$$

$$= k\left[\frac{\lambda}{a} - \frac{\lambda}{a-(-\ell)}\right]\hat{\imath}.$$

$$= k\lambda\left[\frac{1}{a} - \frac{1}{a+\ell}\right]\hat{\imath}. ✔$$

❹ EVALUATE RESULT We first want to make sure our expression gives the correct direction for the electric field. If the charge on the rod is positive, the electric field at P should point to the right because that is the direction a small positively charged test particle would move. This is in the positive x direction for our coordinate system. For positive λ, the factor in front of $\hat{\imath}$ is indeed positive, meaning the component of the electric field in the x direction is positive, as expected.

If the rod length ℓ is much greater than the distance a from the rod end to P, we can assume ℓ is effectively infinite. As ℓ approaches infinity, the term $1/(a+\ell)$ inside the square brackets drops out and the electric field magnitude becomes $E = k\lambda/a$; that is, the field magnitude for positions very close to the rod is independent of ℓ.

As we move P closer and closer to the right end of the rod (a goes to zero), the electric field becomes infinite. At the right end of the rod, P is presumably right on top of a charged particle (zero separation), so the electric field should become very large because of the $1/r^2$ dependence of Coulomb's law. Note also that as ℓ approaches zero, the electric field becomes zero—which is what we expect because there is no charge on a rod of zero length.

Guided Problem 23.6 Electric field created by a curved charged rod

A thin semicircular rod of radius R carries a charge q uniformly distributed along its length. What are the magnitude and direction of the electric field at the center of the arc formed by the rod?

1 GETTING STARTED

1. Draw a sketch with the rod as a half circle and the given quantities placed appropriately.
2. How do you deal with the effects of a continuous charge distribution when the direction from the point of interest to one segment of the charge distribution is different from the direction from the point of interest to another segment?

2 DEVISE PLAN

3. How are you going to determine the contributions to the electric field from different segments of the rod?
4. What is the most natural variable to use in doing the integration required in this circular arc problem? How can an angular measure be related to the lengths of various rod segments?
5. What simplifications, if any, does the symmetry of the half circle allow you to make? Think in terms of the components of the vectors at the position of interest (center of the arc).

3 EXECUTE PLAN

4 EVALUATE RESULT

6. Is the direction of the electric field you obtained plausible?
7. Does your expression behave in a physically plausible way when you vary either R or q?

Worked Problem 23.7 Dipole corrections

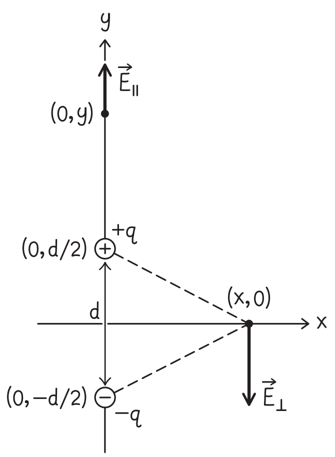
In any direction far away from a dipole, the electric field surrounding the dipole decreases approximately as the inverse cube of distance. For a dipole with dipole charge q_p, dipole separation d, and dipole moment magnitude $p = q_p d$, what is the first level of correction to this inverse-cube approximation (*a*) at any position on the dipole axis that is an intermediate distance (neither very close nor very far) from either end of the dipole and (*b*) at any position on the perpendicular bisector of the dipole axis that is an intermediate distance from the dipole?

1 GETTING STARTED Figure WG23.8 shows the dipole along with the information given in the problem statement and labels for the electric fields at an arbitrary location along the dipole axis ($\vec{E}_{\parallel}$) and at an arbitrary location on the perpendicular bisector ($\vec{E}_{\perp}$). Section 23.6 derives the expression for the nonzero components of the electric fields along the dipole axis (Eq. 23.12) and perpendicular to it (Eq. 23.8):

$$E_{\parallel} = E_y = k\frac{q_p}{y^2}\left[\left(1 - \frac{d}{2y}\right)^{-2} - \left(1 + \frac{d}{2y}\right)^{-2}\right]$$

$$E_{\perp} = E_y = -k\frac{q_p d}{[x^2 + (d/2)^2]^{3/2}}$$

Figure WG23.8



(Remember that E_x is zero in both cases.) We then saw how these equations yield the approximations $E_{\parallel} = E_y \approx 2kq_p d/y^3$ (Eq. 23.13) and $E_{\perp} = E_y \approx -kq_p d/x^3$ (Eq. 23.10) when the distance from the dipole is much greater than the dipole separation d ($y \gg d/2$ and $x \gg d/2$). We just have to make the approximations more accurate, keeping in mind that the corrections for the two cases may differ slightly from each other because the expression for $E_{\perp} = E_y$ in the x direction is different from the expression for $E_{\parallel} = E_y$ in the y direction.

PRACTICE

❷ **DEVISE PLAN** The approximations $E_{\|} = E_y \approx 2kq_{\mathrm{p}}d/y^3$ and $E_{\perp} = E_y \approx -kq_{\mathrm{p}}d/x^3$ were obtained using only the first-order term in a binomial expansion. To make the approximations more accurate, we need to take the next term in the expansion into account. The expansion is $(1 + z)^n = 1 + nz/1! + n(n-1)z^2/2! + n(n-1)(n-2)z^3/3! + n(n-1)(n-2)(n-3)z^4/4! + \cdots$. We hope to get by with just the terms through z^2 because the long-range expressions used only $1 + nz$, but we'll go to the z^3 term in case there is an exact cancellation.

❸ **EXECUTE PLAN** (*a*) Consider first the electric field $\vec{E}_{\|}$ along the dipole axis, given by Eq. 23.12. Noting that $n = -2$ in this equation and using the expansion of $(1 + z)^n$ to the z^3 term with $z = d/2y$, we have

$$\begin{aligned}\left(1 \pm \frac{d}{2y}\right)^{-2} &\approx 1 \pm (-2)\frac{d}{2y} + \frac{(-2)(-2-1)}{2}\left(\frac{d}{2y}\right)^2 \\ &\pm \frac{(-2)(-3)(-4)}{6}\left(\frac{d}{2y}\right)^3 \\ &= 1 \mp \frac{d}{y} + \tfrac{3}{4}\frac{d^2}{y^2} \mp \tfrac{1}{2}\frac{d^3}{y^3}.\end{aligned}$$

Substituting this expression into Eq. 23.12, we see that the first-level correction for intermediate distances is

$$\begin{aligned}E_{\|} = E_y &\approx k\frac{q_{\mathrm{p}}}{y^2}\left[\left(1 + \frac{d}{y} + \tfrac{3}{4}\frac{d^2}{y^2} + \tfrac{1}{2}\frac{d^3}{y^3}\right)\right. \\ &\left.- \left(1 - \frac{d}{y} + \tfrac{3}{4}\frac{d^2}{y^2} - \tfrac{1}{2}\frac{d^3}{y^3}\right)\right] = k\frac{q_{\mathrm{p}}}{y^2}\left(2\frac{d}{y} + \frac{d^3}{y^3}\right) \\ &= 2k\frac{q_{\mathrm{p}}d}{y^3}\left(1 + \frac{d^2}{2y^2}\right). ✔\end{aligned}$$

This result is similar to Eq. 23.13, $E_{\|y} \approx 2kq_{\mathrm{p}}d/y^3$, except that now we have the additional factor $[1 + d^2/2y^2]$. It is good that we kept terms through $z^3 = (d/2y)^3$ because the z^2 term cancels in the difference.

(*b*) We carry out a similar calculation for intermediate distances from the dipole along the perpendicular bisector. We start by writing Eq. 23.8 in the form

$$E_{\perp} = E_y = -\frac{kq_{\mathrm{p}}d}{[x^2 + (d/2)^2]^{3/2}} = -\frac{kq_{\mathrm{p}}d}{x^3}\left(1 + \tfrac{1}{4}\frac{d^2}{x^2}\right)^{-3/2}.$$

Using the same expansion procedure we used to obtain a first-level correction for $E_{\|}$, we see that the first-level correction for this direction is

$$\begin{aligned}E_{\perp} = E_y &\approx -\frac{kq_{\mathrm{p}}d}{x^3}\left[1 + (-\tfrac{3}{2})\tfrac{1}{4}\frac{d^2}{x^2}\right] \\ &= -\frac{kq_{\mathrm{p}}d}{x^3}\left(1 - \tfrac{3}{8}\frac{d^2}{x^2}\right). ✔\end{aligned}$$

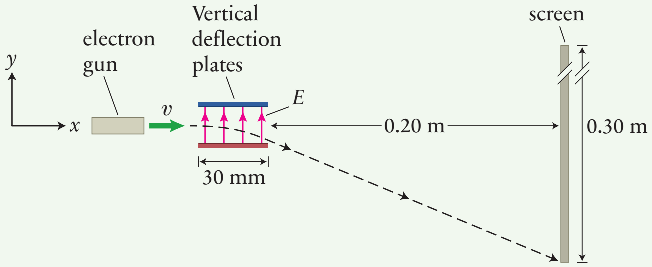
It is gratifying to note that in both cases the correction term ($1 + d^2/2y^2$ for $E_{\|}$, $1 - 3d^2/8x^2$ for $E_{\perp}$) is on the order of $(d/y)^2$ or $(d/x)^2$. We expect that these two cases should bracket the actual correction for arbitrary directions at intermediate distances, because the electric field contribution from each charged particle is spherically symmetrical, suggesting significant dependence on distance but not so much on direction.

❹ **EVALUATE RESULT** If we let either x or y get very large relative to d (that is, at locations very far from the dipole), then $d/x \to 0$ and $d/y \to 0$. At these locations, our expression for $E_{\|}$ becomes Eq. 23.13 and our expression for $E_{\perp}$ becomes Eq. 23.10. The plus sign in front of the $(d^2/2y^2)$ term in the correction for $(E_{\|})_y$ is plausible. To see why, imagine holding a positively charged test particle at a location on the dipole axis far beyond the positive end—for instance, at location $(0, y)$ in Figure WG23.8—and then moving the test particle closer and closer to the dipole. As you do this, the magnitude of the repulsive force exerted by the dipole's positive end on the test particle increases more rapidly than does the magnitude of the attractive force exerted by the more distant negative end.

Guided Problem 23.8 Cathode ray tube

A cathode ray tube uses an electric field to steer electrons to various locations on a screen. In Figure WG23.9, an electron fired from the "electron gun" passes between two parallel plates, and the uniform vertical electric field between the plates deflects the electron's trajectory in the y direction. As the electric field magnitude is varied, the electron can be directed to hit the screen anywhere vertically from top to bottom. (A separate set of plates controls the horizontal position.) If the plates are 30 mm long and located 0.20 m behind a screen that is 0.30 m high, what electric field magnitude is required to deflect an electron to the bottom of the screen if the electron has an initial horizontal velocity of 3.0×10^5 m/s?

Figure WG23.9



1 GETTING STARTED

1. What physical interactions control the electron's trajectory? What simplifications and assumptions can or must you make?
2. What happens to its motion as the electron travels from gun to screen, and why?

2 DEVISE PLAN

3. How can you determine the electron's trajectory? Remember that you have done trajectory calculations before with the gravitational force as the dominant one. Is there a similarity with the electric force in this case?
4. Does the electron experience a constant force? Does the answer to this question depend on whether the electron is between the plates or in the region from the plates to the screen? Whether or not the answer depends on the electron's position, remember that the trajectory must be continuous at the boundary between the regions.

3 EXECUTE PLAN

4 EVALUATE RESULT

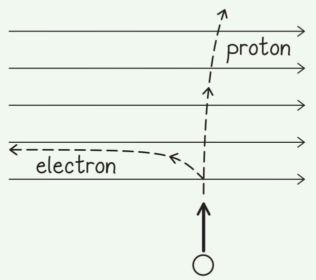
Answers to Review Questions

1. A field is a set of values, each value associated with a position in space surrounding one or more field sources.
2. It is impossible to deal with the interactions of moving charged particles without the field concept, and it is often easier to deal with fields than with distributions of charge.
3. Ball A is the source of the electric field that acts on B; the electric field created by B does not act on B.
4. A scalar field, such as the temperature at any position in the space of interest, specifies a scalar value (magnitude only) at each position. A vector field, such as the gravitational field, specifies a vector value (magnitude and direction) at each position.
5. A vector field diagram is a map that uses small vector arrows to represent, at selected positions in space, the field created by a field source. Such a diagram is useful in illustrating how the magnitude and direction of the field vary from one location to another in the field.
6. Place the charged test particle at point P and measure the electric force exerted on the particle by the source. Divide this electric force by the charge of the test particle to obtain the electric field of source S at point P.
7. Nothing. The pellet contributes nothing to the electric field created by the sphere. Because nothing has changed with the source of the field (the sphere), the field the pellet experiences doesn't change.
8. The electric field is defined to be in the direction that a positive test particle would experience an electric force, which is opposite the force on the negative test particle in this question. Hence the electric field is directed toward the east.
9. No, because the two electric field contributions must be added as vectors. The magnitude of the electric field at the pushpin can be anywhere from $|E_1 - E_2|$ to $|E_1 + E_2|$.
10. A negatively charged particle placed in a uniform electric field is accelerated in the direction opposite the direction of $\vec{E}$, and so the upward acceleration of the electron tells you that $\vec{E}$ must have a component that is directed vertically downward. (There might also be a component parallel to the initial motion of the electron. This would change the electron's speed but not its direction.) If the electron is accelerated downward, $\vec{E}$ must have a component that is directed vertically upward.

PRACTICE

11. See Figure RQA23.11.

Figure RQA23.11



12. No, because the forces exerted on the two charged regions of the dipole are equal in magnitude but opposite in direction, making the vector sum of forces zero.
13. No. The acceleration is not constant (it is centripetal and always changing direction). Therefore the electric force exerted on the electron must be nonconstant, implying the electric field is not uniform.
14. Both. The contribution E_2 makes to the electric field at P would increase, affecting both the x and y components of the electric field (and not in the same proportion).
15. The field magnitude decreases in proportion to the square of the distance between the two objects. To reduce the field by a factor of 100, therefore, you must increase the separation distance by a factor of 10, which means moving it to a position 20 paces from the sphere. Moving the instrument another 18 paces out will do the trick.
16. The magnitude of the electric field due to the sphere is proportional to $1/r^2$, while that due to the dipole is proportional to $1/r^3$. The ratio $E_{\text{dipole}}/E_{\text{sphere}}$ is thus $(qd/r^3)/(q/r^2) = d/r < 1$. That is, the dipole field is weaker at this distance.
17. The dipole moment is a vector pointing from the negative charge in an electric dipole to the positive charge. The magnitude of the dipole moment is $p = qd$, where q is the magnitude of either charge on the dipole and d is the dipole separation.
18. The ratio is 2. The magnitude of the field differs along the axis of the dipole and along the perpendicular bisector of that axis, as given by Equations 23.10 and 23.13. These two, which differ by a factor of 2, are the extreme cases at a given distance R. In the space between these axes the field smoothly varies between these extremes.
19. The radius of the annular ring is $R \sin\theta$, and its width is $R\,d\theta$. Thus the area of the ring is $2\pi(R\sin\theta)R\,d\theta = 2\pi R^2 \sin\theta\,d\theta$. Since the hemispheric half-shell has an area of $2\pi R^2$, the ratio of the ring areas to the shell area is $\sin\theta\,d\theta$. This ratio of the areas also represents the ratio of the infinitesimal charge on the ring dq to the charge on the shell q: $dq/q = \sin\theta\,d\theta$. Thus the infinitesimal charge on the ring is $dq = q\sin\theta\,d\theta$.
20. (*a*) $1/r^2$, (*b*) $1/r$, (*c*) constant.
21. The units are $C^2 \cdot m/N$.
22. Because the two forces form an interaction pair, the force magnitude F^E_{pd} must equal the magnitude F^E_{dp}; thus $F^E_{\text{pd}} \propto 1/r^3$. The electric field around the charged particle decreases as the distance squared, not the distance cubed, but the electric force exerted on the dipole decreases more rapidly than the surrounding field does. This is because the electric force exerted on the dipole is the result of a near cancellation of the electric forces exerted on the two ends of the dipole. This cancellation gets more complete as the dipole separation becomes small relative to the distance between the dipole and the particle.
23. Permanent dipoles have separation between their centers of positive and negative charge even in the absence of external electric fields, whereas charge separation for an induced dipole is created only when the object is placed into an external electric field. The charge separation in an induced dipole is proportional to the external field strength E, at least for small values of E.

Answers to Guided Problems

Guided Problem 23.2 At the center, $\vec{E}$ points toward the negatively charged particle and has magnitude $E = 9kq/a^2$. Clever trick: Due to symmetry, adding charge $-q$ to each of the three vertices will not change $\vec{E}$ at the center.

Guided Problem 23.4 In the orientation shown in the figure, the force between the two dipoles is repulsive, with magnitude

$$F = \frac{kq_p}{(a+d)^2} + \frac{kq_p}{(a-d)^2} - \frac{2kq_p}{a^2},$$

which for $d \ll a$ can be approximated as $F \approx 6kq_p d^2/a^4$.

Guided Problem 23.6 If we take the semicircular rod to be the top half of a circle, then $\vec{E}$ at the circle center points downward (for $q > 0$) and has a magnitude

$$E = \frac{kq}{\pi R^2}\int_{\phi=0}^{\pi} \sin\phi\,d\phi = \frac{2kq}{\pi R^2}.$$

Guided Problem 23.8 To accelerate the electrons downward, the electric field between the plates points upward with magnitude

$$E = \frac{mv_i^2 h}{e\ell(\ell + 2d)}$$

$$= \frac{(9.1 \times 10^{-31}\ \text{kg})\left(3.0 \times 10^5\ \dfrac{\text{m}}{\text{s}}\right)^2 (0.30\ \text{m})}{(1.6 \times 10^{-19}\ \text{C})(0.030\ \text{m})(0.030\ \text{m} + 2 \times 0.20\ \text{m})}$$

$$= 12\ \text{N/C}.$$

PRACTICE

Guided Practice by Chapter

24 Gauss's Law

Review Questions

Answers to these questions can be found at the end of this chapter.

24.1 Electric field lines

1. Describe how to draw an electric field line for a given charge distribution.
2. Explain how to determine the number of field lines to draw around a given charged object.

24.2 Field line density

3. Explain how electric field lines represent both the direction and the magnitude of the electric field at a given location in an electric field.
4. What is the definition of *field line density*?
5. Why do we measure field line density only through a surface that is perpendicular to the field lines?

24.3 Closed surfaces

6. The field line flux through a certain closed surface is zero. Does that mean there are no charged objects inside the closed surface?
7. The enclosed charge for a closed surface is zero. Does that mean there are no electric field lines crossing the surface?
8. Eight electrons are the only charged particles inside an isolated balloon, producing a field line flux through its surface. (*a*) If eight additional electrons are set outside the balloon to form the eight corners of a cube, how does the field line flux through the balloon change? (*b*) How does the flux change if the eight additional electrons are all placed at one location outside the balloon instead of being distributed at the corners of a cube?
9. Given a drawing of the electric field lines surrounding a closed surface, explain how to determine the field line flux through that surface.

24.4 Symmetry and Gaussian surfaces

10. Describe the electric fields inside and outside a uniformly charged spherical shell.
11. Describe the characteristics of planar symmetry.

24.5 Charged conducting objects

12. Under what conditions is the electric field magnitude equal to zero inside a conducting material?
13. Under what conditions is the electric field magnitude nonzero inside the cavity enclosed by a hollow conducting sphere?
14. A hollow conducting object in electrostatic equilibrium carries a surplus electric charge. Without additional information, what must be true about the distribution of this surplus charge?

24.6 Electric flux

15. What is the distinction between *field line flux* and *electric flux*? Why is electric flux the preferred variable for analyzing electric fields?
16. In the formula for electric flux, $\Phi_E = EA\cos\theta$, what does A represent? What does θ represent?
17. When working with irregular surfaces and/or nonuniform electric fields, we obtain the electric flux by integrating a scalar product of the electric field and a small segment of the area of the surface. Why is the scalar product needed?
18. In flux calculations for a closed surface, why is it important to define the area vector as pointing outward, away from the interior of the surface?

24.7 Deriving Gauss's law

19. A spherical surface encloses one electron and one proton. Is the enclosed charge positive, negative, or zero?
20. Would Gauss's law hold if the electric field due to a charged particle had a $1/r$ dependence rather than a $1/r^2$ dependence?

24.8 Applying Gauss's law

21. Inside a ball of volume V and radius R that carries a uniformly distributed charge q, you draw a spherical Gaussian surface that is concentric with the ball and has radius $r < R$. Write an expression for the charge q_{enc} enclosed by the Gaussian surface.
22. A thin spherical metal shell of radius R carries a uniformly distributed charge $+q$. It is surrounded by a concentric thin spherical metal shell of radius $2R$ that carries a uniformly distributed charge $-q$. For which of the following regions enclosed by a concentric spherical Gaussian surface of radius r is the electric field zero: $r < R$, $R < r < 2R$, $r > 2R$?
23. In sketching a Gaussian surface for an infinitely long charged rod when you need to determine the electric field at some distance r from the rod axis, it makes sense to draw a cylinder of radius r for the Gaussian surface, but what value should you use for the length of the cylindrical Gaussian surface? Must this length be infinity just as the rod length is?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The gravitational flux that enters the roof of a typical one-story house due to Earth's gravitational field (E, I, P, L)
2. The electric flux through a horizontal cake pan held 10 m directly above a small object that carries a charge of 100 μC (B, H, M)
3. The surface charge density that results when you place 50 μC of charge on a metal baking sheet (D, J)
4. The surface charge density required to "float" an electron 10^{-2} m above a plastic cutting board (C, G, K, N, A)
5. The surface charge density on the ground that is required to create an upward electric field of magnitude 100 N/C a few meters above the ground (O, F)

Hints

A. What electric field strength is required to create the electric force needed to cancel the gravitational force exerted on the electron?
B. What is the bottom surface area of a cake pan?
C. What forces must cancel in order to float the electron?
D. What is the area of a baking sheet?
E. What is the floor area of a typical one-story house?
F. Does it matter exactly how far above the ground you are?
G. What are the dimensions of a cutting board?
H. What is the magnitude of the electric field at the location of the cake pan?
I. What should you do about the tilt angle θ of the roof surface?
J. How does the charge distribute itself?
K. What does the order-of-magnitude difference between the 10-mm separation distance and the surface area of the board tell you about the board?
L. What are the gravitational quantities that correspond to these electrical quantities?
M. Is the electric field uniform over the cake pan surface?
N. What is the inertia of an electron?
O. Which geometry is more appropriate here: spherical or planar?
P. If this estimate were of electric flux, which quantities would you need?

Key (all values approximate)

A. 6×10^{-11} N/C; **B.** 0.04 m^2; **C.** the electric force exerted by the charged board and the gravitational force exerted by Earth; **D.** 0.2 m^2; **E.** 150 m^2; **F.** this value is not needed as long as you approximate Earth's surface as an infinite sheet; **G.** 0.3 m $\times$ 0.4 m; **H.** 9000 N/C; **I.** nothing—the floor area, $A \cos \theta$, includes the roof slope; **J.** approximately uniformly on both sides of the sheet; **K.** the board might as well be infinitely large at this small separation distance; **L.** $\vec{E} \leftrightarrow \vec{g}$, $\vec{A} \leftrightarrow \vec{A}$; **M.** because the pan is about 0.3 m across, the variation in the angular direction of $\vec{E}$ over the surface is diameter/distance $= 0.3\text{ m}/10\text{ m} = 0.03$ rad $\approx 2°$, small enough to consider the field uniform; **N.** 9×10^{-31} kg; **O.** planar because Earth's curvature is negligible for such a small separation distance; **P.** $\vec{E}$, $\vec{A}$

Worked and Guided Problems

Procedure: Calculating the electric field using Gauss's Law

Gauss's law allows you to calculate the electric field for charge distributions that exhibit spherical, cylindrical, or planar symmetry without having to carry out any integrations.

1. Identify the symmetry of the charge distribution. This symmetry determines the general pattern of the electric field and the type of Gaussian surface you should use (see Figure 24.27).
2. Sketch the charge distribution and the electric field by drawing a number of field lines, remembering that the field lines start on positively charged objects and end on negatively charged ones. A two-dimensional drawing should suffice.
3. Draw a Gaussian surface such that the electric field is either parallel or perpendicular (and constant) to each face of the surface. If the charge distribution divides space into distinct regions, draw a Gaussian surface in each region where you wish to calculate the electric field.
4. For each Gaussian surface determine the charge q_{enc} enclosed by the surface.
5. For each Gaussian surface calculate the electric flux Φ_E through the surface. Express the electric flux in terms of the unknown electric field E.
6. Use Gauss's law (Eq. 24.8) to relate q_{enc} and Φ_E and solve for E.

You can use the same general approach to determine the charge carried by a charge distribution given the electric field of a charge distribution exhibiting one of the three symmetries in Figure 24.27. Follow the same procedure, but in steps 4–6, express q_{enc} in terms of the unknown charge q and solve for q.

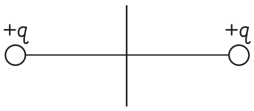
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 24.1 Pair of charged spheres

(*a*) Draw the electric field line pattern that arises from two very small spheres, each carrying a charge $+q$. (Assume that the radius of each sphere is much smaller than the distance between the spheres.) (*b*) What is the electric flux through the plane that bisects and is perpendicular to a line segment drawn between the centers of the two spheres? (c) How would the field line pattern change if the charge on each small sphere were $-q$?

❶ GETTING STARTED We have two spheres that carry equal charge, and we must show the field line pattern for the electric field they create in two cases: both spheres positively charged and both negatively charged. We must also determine the electric flux through a plane midway between the spheres. We start with the sketch of Figure WG24.1, showing our given information and the quantities we must determine.

Figure WG24.1



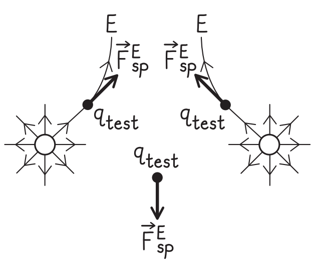
We know the relationship between the type of charge an object carries and the direction of the electric field around the object, and we know how to draw the field lines for a known electric field. We therefore need to figure out the electric field direction at several locations in the space around the spheres and then connect these points to draw the field lines. To determine the electric flux through the bisecting plane, we need to sum over the entire plane the quantity $\vec{E} \cdot d\vec{A}$, which is $E_x\,dA$ if we take the separation between the two spheres to be in the x direction. This seems as if it could be a lengthy calculation, but we hold out hope that once we start drawing field lines, we may find that symmetry simplifies the problem. Because we are told that the spheres are much smaller than their separation distance, we will treat each sphere as a particle in drawing field lines.

❷ DEVISE PLAN We obtain the direction of the electric field by determining the direction of the electric force exerted on a positively charged test particle located near the spheres. We could determine this force direction at many locations around the spheres and from these electric force vectors determine the direction of $\vec{E}$ and then draw the field lines. For a complex charge distribution, this is the only way. In this problem, however, we can use the symmetry of our two-sphere system to figure out the field line pattern.

Once we have the field line diagram, we can use Eq. 24.4 to express the electric flux through the specified plane. We can answer part *c* from our knowledge of the relationship between charge type and field line direction.

❸ EXECUTE PLAN (*a*) Because the charge on each sphere is positive, the electric field is directed radially outward from each sphere. We know that, no matter where we put it, the test particle is repelled by both spheres. We also know that wherever we place the test particle it is repelled more strongly by the sphere closer to it. In the plane midway between the spheres, the electric forces exerted on the test particle by the two spheres must be of equal magnitude because the spheres are equidistant from the plane. Thus for the test particle placed in the plane between the spheres, the force $\vec{F}^E_{\text{sp}}$, which is the vector sum of forces exerted on the test particle by the two spheres, has no x component. Drawing in a few electric force vectors gives the pattern shown in Figure WG24.2. Using the force vectors as a guide and the fact that field lines cannot cross each other, we trace out a dozen or so lines for the electric field due to the two spheres (Figure WG24.3). ✔

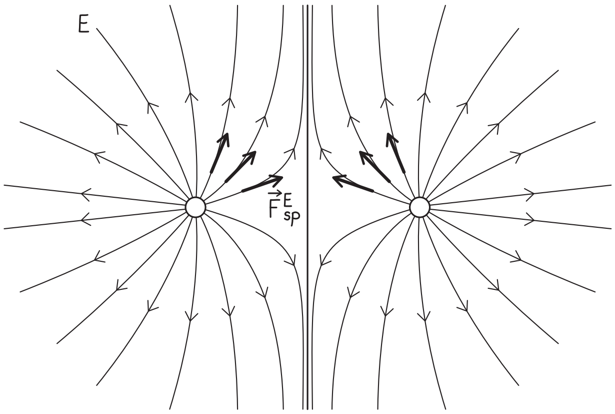
Figure WG24.2



(*b*) Notice in Figure WG24.3 that no electric field lines cross the plane between the two spheres. This tells us that the normal component E_x of the electric field is zero, and thus the electric flux through this plane is zero. ✔

(*c*) If both spheres were negatively charged, the field line pattern would look the same except the directions of the field lines would be reversed. ✔

Figure WG24.3



❹ EVALUATE RESULT We know from Section 24.1 that electric field lines between two objects that are oppositely charged run from the positively charged object to the negatively charged one. Thus, we expect that the field lines between two objects that carry the same type of charge should look as if they are pushing the objects apart, just as Figure WG24.3 shows. It makes sense that the electric flux through the bisecting plane is zero because the electric field component perpendicular to the plane is zero by symmetry: The charge distribution is unchanged when reflected about the bisecting plane.

Guided Problem 24.2 Charged square sheet

A square sheet carries a charge $+q$ uniformly distributed over its surface. Make a sketch of the sheet, and draw the electric field lines outside the sheet in the plane that contains the sheet. Based on this sketch, what is the electric flux through a plane perpendicular to the sheet and passing through its center?

❶ GETTING STARTED

1. After drawing your sketch, describe the problem in your own words. What is the problem asking you to determine? Are you asked for a qualitative or a quantitative answer?
2. What do you know about the direction of the electric field near the sheet edges?
3. What should the electric field look like at distances far from the sheet?

❷ DEVISE PLAN

4. Are there any symmetries you can use to simplify the problem?
5. What equation helps you get the electric flux? Can you already guess what the flux is?

❸ EXECUTE PLAN

6. Draw several field lines connecting the two limiting cases: close to the sheet and far away from it.

❹ EVALUATE RESULT

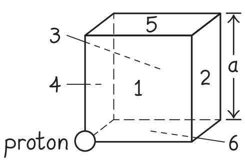
7. What can you say about the electric flux through any plane that is perpendicular to the sheet through a symmetry axis of the sheet?

Worked Problem 24.3 Flux through a cube

As you ponder your drink, you notice an ice cube floating in your glass. Because you are studying electric flux, this floating cube leads you to formulate an interesting question: What electric flux is produced though each face of an imaginary cube of side length a, at one corner of which is a single proton?

❶ GETTING STARTED We are given the size of a cube and the fact that one corner is occupied by a proton. We are asked to determine the electric flux through the cube's faces. We draw a sketch containing this information (Figure WG24.4). We treat the proton as a particle with a uniform charge distribution, which means that it creates a spherically symmetrical electric field. This means that the field line density across any face of the cube is nonuniform, with most (but not all) field lines missing the cube entirely.

Figure WG24.4



PRACTICE

❷ **DEVISE PLAN** We know the flux is given by Eq. 24.5, and because the electric field is not constant over each face of the cube, the integration may be challenging. However, we can treat each cube face separately and hope to exploit symmetry to simplify the flux calculation. We also know that the electric field due to the single proton is given by Eq. 24.15, with $q = +e$:

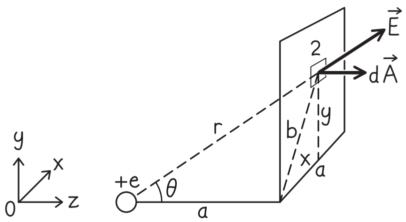
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{e}{r^2}\hat{r},$$

where q is the charge on the proton and r is the radial distance from the proton to the point at which $\vec{E}$ is evaluated. Although the electric field magnitude varies over the three faces that have the proton at their common corner (faces 1, 4, and 6 in Figure WG24.4), we know the *directions* of the field at these faces: $\vec{E}$ at face 1 is parallel to face 1, $\vec{E}$ at face 4 is parallel to face 4, and $\vec{E}$ at face 6 is parallel to face 6. Because the electric field is parallel to these three faces, we can say that for each of them $\vec{E}\cdot d\vec{A} = E\cos 90°\, dA = 0$. Moreover, the fluxes through faces 2, 3, and 5 are all the same because each vector area segment $d\vec{A}$ on one of these three faces has a corresponding area segment on the other two faces, all making the same angle with the electric field. Thus, the only thing we need to calculate is the flux through one of the three faces that have nonzero flux. Figure WG24.5, showing face 2, helps us with setting up the integral. With the xyz axes shown in Figure WG24.5, the area segment dA equals $dx\,dy$. For the right triangle that has the proton at one vertex, we have $\cos\theta = a/r$ and $r^2 = a^2 + b^2$, where b is the distance from the area segment $d\vec{A}$ to the corner of face 2 that is nearest the proton: At that corner, $x = y = 0$ and $z = a$. For the right triangle inside face 2, we have $b^2 = x^2 + y^2$. Thus $r = \sqrt{a^2 + (x^2 + y^2)}$. The flux through face 2 is

$$\Phi_E = \int \vec{E}\cdot d\vec{A} = \int E\cos\theta\, dA = \int \frac{1}{4\pi\epsilon_0}\frac{e}{r^2}\cos\theta\, dA$$

$$= \int_0^a\int_0^a \frac{1}{4\pi\epsilon_0}\frac{ea}{\sqrt{(a^2 + x^2 + y^2)^3}}dx\,dy.$$

Figure WG24.5



❸ **EXECUTE PLAN** Referring to integral tables (or using a computer program to do symbolic integration), we determine that integration over x yields

$$\Phi_E = \frac{ea}{4\pi\epsilon_0}\int_0^a\left[\frac{x}{(a^2 + y^2)\sqrt{a^2 + x^2 + y^2}}\right]_0^a dy$$

$$= \frac{ea^2}{4\pi\epsilon_0}\int_0^a \frac{dy}{(a^2 + y^2)\sqrt{2a^2 + y^2}}.$$

If you cannot find this last integral in a reference source, the integration over y is not easy, but a substitution of variables can put the integral in a form found in most integral tables. The substitutions

$$u^2 = a^2 + y^2$$

$$dy = \frac{u}{y}du$$

transform the integral into

$$\Phi_E = \frac{ea^2}{4\pi\epsilon_0}\int_a^{\sqrt{2}a}\frac{(u/y)du}{u^2\sqrt{u^2 + a^2}}$$

$$= \frac{ea^2}{4\pi\epsilon_0}\int_a^{\sqrt{2}a}\frac{du}{u\sqrt{u^2 + a^2}\sqrt{u^2 - a^2}}$$

$$= \frac{ea^2}{4\pi\epsilon_0}\int_a^{\sqrt{2}a}\frac{du}{u\sqrt{u^4 - a^4}}$$

Checking a table of integrals, we obtain

$$= \frac{ea^2}{4\pi\epsilon_0}\left[\frac{-2}{4a^2}\arcsin\frac{a^2}{u^2}\right]_a^{\sqrt{2}a}$$

$$= -\frac{e}{8\pi\epsilon_0}\left[\arcsin\left(\tfrac{1}{2}\right) - \arcsin(1)\right] = -\frac{e}{8\pi\epsilon_0}\left[\frac{\pi}{6} - \frac{\pi}{2}\right].$$

The electric flux through faces 2, 3, and 5, the three faces that do not contain the proton at one corner, is thus

$$\Phi_E = \frac{e}{24\epsilon_0}. ✔$$

As noted in Devise Plan, the electric flux through the other three faces is zero.

❹ **EVALUATE RESULT** You might say "There must be an easier way!" Sometimes there is; sometimes not. (In this case there *is* a much easier way, shown in Worked Problem 24.5.) Nonetheless, you should not be afraid to tackle integration problems like this. The important thing is to set up the physics properly, as we did in Devise Plan. Then use the math tools you know or can find to obtain an answer.

Although it might seem surprising that the flux through faces 2, 3, and 5 is independent of the cube side length a, this is to be expected when you think of the "rays" of the electric field that radiate from the proton. Doubling the size of the cube would still catch the same number of field lines on these three faces and thus not affect the flux through them.

Guided Problem 24.4 Charged rod, cubed

An infinitely long, positively charged rod whose linear charge density is λ, runs through a cube of side length a. The rod passes through the cube's center and is perpendicular to the top and bottom faces. Use integration to determine the electric flux through each face.

❶ **GETTING STARTED**

1. What information are you given? What quantity are you asked to determine? Make a sketch showing all this information.
2. In what sense is this problem similar to Worked Problem 24.3? In what sense is it different?
3. What physical symmetry can you use to simplify the problem? What does the electric field due to an infinitely long charged wire look like?

❷ **DEVISE PLAN**

4. Is the electric flux through any face zero? Why?
5. Which equations allow you to express the unknown quantities in terms of known ones? What information can help you determine the electric field at a position on, the location of, and the surface normal direction of a face of the cube that has nonzero flux?
6. Begin with the definition of electric flux. Determine the direction of the electric field at any location on each face so that you can integrate over the surface. Rewrite dA in terms of the coordinates of that location (probably x, y, z).
7. The scalar product includes the factor cos θ. Can you relate this cosine function to the coordinate variables? What about the distance from the rod to area segment dA?
8. Once you have the flux through one face, can you use symmetry or analogy to infer the flux through the other faces?

❸ **EXECUTE PLAN**

9. Perform the integration over one face that has nonzero flux and extend the result to the other faces.

❹ **EVALUATE RESULT**

10. As we will work out in detail in Guided Problem 24.6, you can cleverly calculate the answer without using integration, as a check on the brute-force answer. Using symmetry to argue that two faces have zero flux and four faces have equal fluxes, see whether the sum of the fluxes through the nonzero faces equals q_{enc}/ϵ_0, as Gauss's law requires.

Worked Problem 24.5 Revisiting flux through a cube

Use Gauss's law to determine the electric flux through each face of the cube of Worked Problem 24.3.

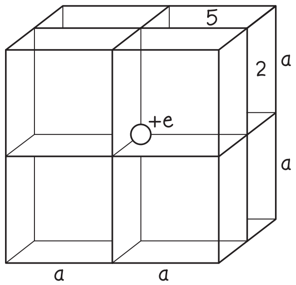
❶ **GETTING STARTED** As in Worked Problem 24.3, our given information is that the cube side length is a and the cube has an isolated proton embedded at one corner (Figure WG24.4). Our task here is to use Gauss's law, Eq. 24.8, to solve the problem of how much electric flux passes through each face of the cube. We know how to calculate the electric field surrounding the proton, and we know the locations and orientations of the faces. We also know that a cube possesses symmetries that we can exploit to simplify the problem.

❷ **DEVISE PLAN** Because the proton is at the corner of the cube, the cube does not enclose it and so cannot be our Gaussian surface. However, we can enclose the proton by attaching seven additional cubes, each of side length a, to our original cube, creating a cube of side length $2a$

PRACTICE

and having the proton at its geometric center (Figure WG24.6). This large cube is our Gaussian surface. We can apply Gauss's law to get the electric flux through any face of this surface and then use symmetry to get the flux through any face of our original cube of side length a.

Figure WG24.6



❸ **EXECUTE PLAN** From Gauss's law, we know that the flux through the large cube enclosing a single proton is e/ϵ_0. Because the proton is at the geometric center, symmetry tells us that the electric flux is the same through all six faces of the large cube. As Figure WG24.6 shows, each face of the large cube is composed of four squares, each of side length a: One face of the large cube contains four faces of our original cube. Thus, if we apply Gauss's law and symmetry, the flux through each of these smaller faces must be

$$\Phi_E = \frac{1}{(6)(4)}\frac{e}{\epsilon_0} = \frac{e}{24\epsilon_0}.$$

This is the same answer we obtained in Worked Problem 24.3 by tedious integration!

This result gives us the flux through sides 2, 3 and 5 of our original cube. The fact that the electric field is parallel to faces 1, 4, and 6 of the original cube tells us that the flux through sides 1, 4, and 6 is still zero. ✔

❹ **EVALUATE RESULT** Using Gauss's law is much easier than the method we used in Worked Problem 24.3 to do the same task. This law is extremely powerful in cases where symmetry can be exploited. It takes practice and experience to see how to construct Gaussian surfaces, but as this example shows, learning to do so is well worth the effort.

Guided Problem 24.6 Revisiting charged rod cubed

Use Gauss's law to determine the electric flux through each face of the cube of Guided Problem 24.4.

❶ **GETTING STARTED**

1. Because this problem asks you to determine the same quantity as in Guided Problem 24.4, should you draw the same sketch?
2. In what sense is this problem similar to Worked Problem 24.5? In what sense is it different?
3. What physical symmetry can you use to simplify the problem? What does the electric field due to an infinitely long charged wire look like?

❷ **DEVISE PLAN**

4. Is the electric flux through any face zero? Why?
5. Which equations allow you to express the unknown quantities in terms of known ones? What information can help you determine these unknown quantities?
6. What should you draw as your Gaussian surface? What is the amount of charge enclosed by this surface?

❸ **EXECUTE PLAN**

7. Apply Gauss's law, using the symmetry of the system.

❹ **EVALUATE RESULT**

8. Should your answer agree with your answer in Guided Problem 24.4? Does it?

Worked Problem 24.7 Nonuniformly charged cylinder

A very long, solid nonconducting cylinder of radius R has a volume charge density that is proportional to the radial distance r from the long central axis of the cylinder according to the expression $\rho(r) = \rho_0 r$, where ρ_0 is a constant with units C/m^4. (*a*) For any location near the midpoint of the cylinder's length, obtain an expression for the magnitude of the electric fields inside and outside the cylinder in terms of ρ_0, R, and r. (*b*) Sketch a graph showing the electric field magnitude as a function of the radial distance r from the cylinder's long central axis.

❶ **GETTING STARTED** We are given an equation showing how volume charge density changes inside a solid cylinder and told that the cylinder is very long. Our tasks are (*a*) to determine expressions for the magnitude of the electric field inside and outside the cylinder, in both cases working near the middle of the cylinder's length, and (*b*) to graph how the electric field changes as we move radially away from the cylinder's long central axis.

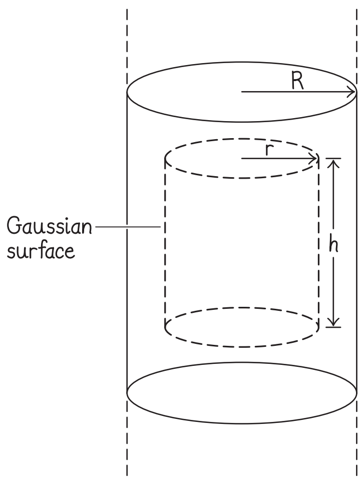
The cylinder's symmetry makes this a prime candidate for the application of Gauss's law. Because our task is to determine the magnitude of the electric field midway along the length and we're told that the cylinder is very long, we assume that we can ignore any deviation from cylindrical symmetry of the electric field near the cylinder ends. Thus we can assume that the field in the region we concentrate on is directed radially outward from the long axis of the cylinder.

❷ **DEVISE PLAN** We'll use Gauss's law to obtain expressions for the electric fields inside and outside the cylinder, making use of cylindrical symmetry. For our task in part *b*, we will take the expressions obtained in part *a*, insert numerical constants where needed, and graph E versus r, making sure that the expressions agree at $r = R$.

❸ **EXECUTE PLAN** (*a*) Because the symmetry is cylindrical, we use a cylindrical Gaussian surface and show it surrounding a portion of the charge inside the cylinder (Figure WG24.7). As usual, we use r for the Gaussian surface radius and show the surface extending some distance h along the cylinder's length. Because it is radial, the electric field is perpendicular to this Gaussian surface everywhere except on the two end caps of the surface. On the end caps, $d\vec{A}$ is perpendicular to $\vec{E}$ (because the surface is parallel to $\vec{E}$, and $d\vec{A}$ is normal to the surface) and so $\vec{E} \cdot d\vec{A} = E \cos 90° \, dA = 0$. By symmetry, the magnitude of the electric field is the same everywhere on the curved part of the Gaussian surface. Thus, we need to integrate only over the area of the curved part:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E \cos 0° \, dA = E \oint dA = E(2\pi r h).$$

Figure WG24.7



Note that the scalar product leads us to an integral over the scalar area segment dA rather than over the vectorial area segment $d\vec{A}$. Using the vectorial form would yield $\oint d\vec{A} = \vec{0}$! (You should be able to explain why.) Applying Gauss's law, we say

$$\Phi_E = E(2\pi r h) = \frac{q_{\text{enc}}}{\epsilon_0}$$

$$E = \frac{q_{\text{enc}}}{2\pi r h \epsilon_0}.$$

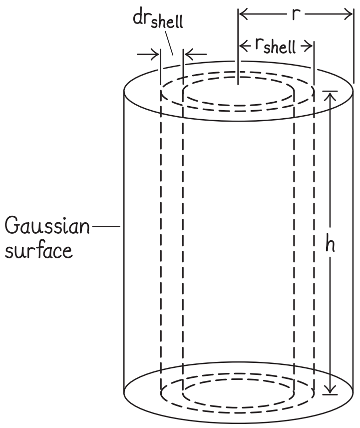
In determining the charge enclosed by our Gaussian surface, we must remember that the volume charge density $\rho = \rho(r) = \rho_0 r$ varies with the radial distance r from the cylinder's axis. We therefore divide the interior of the Gaussian surface into a series of thin-walled cylindrical shells each of radius r_{shell}, wall thickness dr_{shell}, and wall length h (Figure WG24.8). The infinitesimal amount of charge contained in the wall of each shell is $dq = \rho(r = r_{\text{shell}})dV$, where $dV = 2\pi r_{\text{shell}} h dr_{\text{shell}}$ is the volume of the wall. The charge contained within the Gaussian surface of radius r is thus

$$q_{\text{enc}} = \int dq = \int \rho(r_{\text{shell}})dV = \int_0^{r_{\text{shell}}=r} \rho_0 r_{\text{shell}} \left[2\pi r_{\text{shell}} h dr_{\text{shell}}\right]$$

$$= \frac{2\rho_0 \pi h r^3}{3}.$$

PRACTICE

Figure WG24.8



The magnitude of the electric field at radius r inside the solid cylinder ($r < R$) is thus

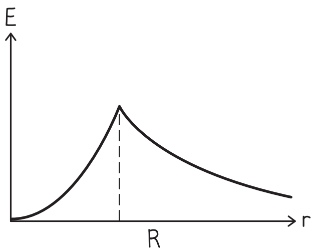
$$E_{\text{inside}} = \frac{2\rho_0\pi hr^3/3}{2\pi rh\epsilon_0} = \frac{\rho_0}{3\epsilon_0}r^2. ✔$$

Outside the cylinder, the amount of charge enclosed by the Gaussian surface is independent of the radial distance because the charge distribution ends at the surface of the cylinder. Thus we evaluate the charge enclosed at distance $r = R$, which means $q_{\text{enc}} = 2\pi h\rho_0 R^3/3$. This gives an electric field magnitude at radial distance r outside the cylinder ($r > R$) of

$$E_{\text{outside}} = \frac{2\rho_0\pi hR^3/3}{2\pi rh\epsilon_0} = \frac{\rho_0 R^3}{3\epsilon_0}\frac{1}{r}. ✔$$

(b) Plotting the magnitude E as a function of the radial distance r from the cylinder's central axis yields the graph of Figure WG24.9. The shape of the curve tells us that the field magnitude is proportional to r^2 up to the cylinder's surface ($r = R$) and then is proportional to $1/r$ for $r > R$. ✔

Figure WG24.9



❹ **EVALUATE RESULT** It makes sense that the electric field magnitude is an increasing function of r inside the solid cylinder but is a decreasing function of r outside. As we go radially outward from the cylinder's center to its surface, more and more charge is enclosed by the Gaussian surface and so the electric field becomes stronger and stronger until we reach $r = R$. Once outside the solid cylinder, the field magnitude decreases as we get farther away, just as in Exercise 24.7 for an infinitely long thin rod. We also check to make sure that our expressions for the electric fields inside the rod and outside the rod agree at the rod's boundary at $r = R$, and gratifyingly we determine that they give the same result for the limiting case of $r \to R$ from either side; in both cases, $E(r = R) = \rho_0 R^2/3\epsilon_0$.

Guided Problem 24.8 Nonuniformly charged sphere

A solid nonconducting sphere of radius R carries a charge q in its interior. The charge is distributed throughout the region $r < R$ nonuniformly, but with spherical symmetry. The volume charge density as a function of the radial distance r from the center of the solid sphere is proportional to $1/r$. Determine the electric fields for positions inside and outside the solid sphere, and then sketch a graph showing the field magnitude as a function of r.

1 GETTING STARTED

1. List the information you are given, and then describe what you must determine. Make a sketch of the situation.
2. In what sense is this problem similar to Worked Problem 24.7? In what sense is it different?
3. Do you have enough information to work the problem? Are there any assumptions you must make?

2 DEVISE PLAN

4. What physical symmetry can you use to simplify the problem? What does the electric field look like for a sphere of charge?
5. What is the volume charge density as a function of the radius, in terms of quantities given in the problem statement?
6. What is an appropriate Gaussian surface?
7. What amount of charge is enclosed in this Gaussian surface?

3 EXECUTE PLAN

8. Apply Gauss's law, using the symmetry of the system. The volume charge density is not uniform in this case, so determining the enclosed charge involves an integral. Be attentive!

4 EVALUATE RESULT

9. Does the way the electric field inside the sphere depends on r make sense? Explain why this behavior of E is reasonable.

Answers to Review Questions

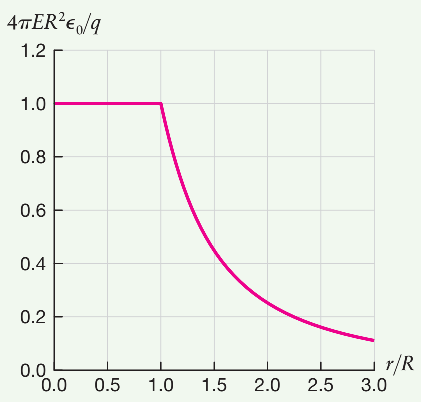
1. Imagine placing a positively charged test particle near any charged object in the distribution—call it object 1—and then imagine dragging the test particle a short distance in the direction of the electric force exerted by object 1 on the particle. Draw a directed line segment along the drag path, with the arrowhead of the segment pointing in the direction of the test particle's motion. This directed line segment is the first portion of the field line. Examine the surroundings of the test particle in its new position and then imagine dragging it a short distance in the direction of the vector sum of the electric forces it now experiences due to the charge distribution. Extend the line segment along this drag path, again showing an arrowhead pointing in the direction of the test particle's motion. Repeat until you reach a charged object in the distribution (or until you reach the edge of the area you are using for your field drawing).
2. The number is arbitrary, subject to the constraint that, in a given field line diagram, the number of lines that begin or terminate on any charged object contributing to the electric field is proportional to the magnitude of the charge on that object.
3. The direction of the electric field vector at any point is tangent to the field line at that point. The density of the electric field lines in any region represents the magnitude of the electric field in that region.
4. Field line density is the number of lines per unit area that cross a surface that is perpendicular to the electric field.
5. If the surface were parallel to the electric field, no field lines would cross the surface. At an intermediate angle θ from the perpendicular, a fraction $\cos\theta < 1$ as many field lines would cross the surface. To define field line density unambiguously, we choose the surface to be perpendicular to the field lines.
6. Not necessarily. That the flux through the surface is zero tells you only that the enclosed charge (the algebraic sum of the charges on all charged objects inside the surface) is zero. There may be several charged objects inside, whose charges sum to zero. A closed surface surrounding the one proton and one electron that make up a hydrogen atom is one example.
7. Not necessarily. If any electric field lines cross the surface, then there must be one outgoing field line for every incoming field line, so that the field line flux through the surface is zero. One example is a closed surface surrounding an electric dipole. Another example is a charged particle outside a closed surface whose enclosed charge is zero.
8. (*a*) Because the amount of charge enclosed by the balloon does not change, the field line flux through its surface does not change. (*b*) Again the field line flux does not change because the enclosed charge does not change. (However, in either case the field line *pattern* changes when the electrons are placed outside the balloon.)
9. Count the number of field lines that penetrate the surface going outward as $+1$ for each penetration; count the number of field lines that penetrate the surface going inward as -1 for each penetration. Do not count field lines that do not penetrate the surface. The algebraic sum of the inward and outward values is the field line flux.
10. The electric field in the space enclosed by a uniformly charged spherical shell is zero. The electric field at points outside a uniformly charged spherical shell is the same as the electric field due to a particle carrying an equal charge located at the center of the shell.
11. Planar symmetry is the symmetry of an infinite sheet. The appearance of the sheet remains unchanged if it is rotated by any amount about an axis perpendicular to the sheet, or translated by any amount along either of the two axes perpendicular to this perpendicular axis.
12. The electric field is zero under conditions of electrostatic equilibrium (any condition where the distribution of the charged particles in the conductor does not vary with time) because the vector sum of the electric forces exerted on a test particle within the conducting material is zero.
13. If there are no charged particles inside the cavity, then the electric field is zero inside the cavity, no matter what the charge distribution is outside the sphere. The electric field inside the cavity can be nonzero when there are charged particles inside the cavity. In either case, the electric field is zero within the conducting material of the sphere itself.

14. The surplus charge must be distributed on the surfaces of the conducting object. The surplus charge resides on the outer surface of the conducting material in the absence of other charged objects, and possibly also on the inner surface surrounding the cavity if there is a charged object in the cavity.
15. The two are proportional to each other. The field line flux has an arbitrary value, depending on how many lines are drawn to represent a certain amount of charge. The electric flux is uniquely defined by Eq. 24.4.
16. A represents the area of a flat surface through which we wish to measure the electric flux. θ represents the angle between the electric field and the surface normal. So $\cos\theta = \pm 1$ if the surface is perpendicular to the electric field (that is, if the surface normal vector is parallel or antiparallel to the electric field).
17. For each small segment of the area of the surface, we need to compute the contribution $\Phi_E = EA\cos\theta$. If we simply multiplied the magnitudes of $\vec{E}$ and $\vec{A}$ without using the scalar product, we would miss the factor $\cos\theta$. The scalar product correctly incorporates the magnitude of $\vec{E}$, the magnitude of $\vec{A}$, and the cosine of the angle between $\vec{E}$ and the surface normal.
18. If we had defined the area vector as pointing inward, we would have obtained the wrong sign for Φ_E. Just as the field line density for a closed surface enclosing a positively charged particle is defined to be positive, we want the electric flux to be positive through a closed surface enclosing a positively charged particle.
19. The enclosed charge is zero because we must compute the algebraic sum of the charges on the individual particles enclosed by the spherical surface.
20. No. Checkpoint 24.20 makes a similar point. Gauss's law is a direct consequence of the $1/r^2$ dependence of the electric field due to a charged particle. Any deviation from this behavior renders Gauss's law invalid.
21. Because the charge distribution is uniform, the ratio of the charge enclosed by the Gaussian surface to the charge enclosed by the ball is the same as the ratio of the volumes of the two spheres. Therefore $q_{\text{enc}}/q = V_{\text{enc}}/V = (4\pi r^3/3)/(4\pi R^3/3)$, or $q_{\text{enc}} = q(r/R)^3$.
22. The electric field is zero anywhere the algebraic sum of the charge inside the Gaussian surface is zero. This happens for $r < R$ and $r > 2R$.
23. It doesn't matter how long you make the cylindrical Gaussian surface because this length cancels as you do the calculation coming out of Eq. 24.8. (Such a cancellation is a feature of many types of physics problems. You have to define a variable to completely specify the calculation, but if that variable is not important for the physics—and the details of an imaginary surface certainly should not be important—that variable should cancel somewhere during your calculation. If it does not, you've probably made a mistake somewhere.)

Answers to Guided Problems

Guided Problem 24.2 The electric flux is zero because the charge distribution is symmetrical under reflection through this plane.
Guided Problem 24.4 The electric flux through the top and bottom faces is zero. The electric flux through each of the other four faces is $\lambda a/4\epsilon_0$.
Guided Problem 24.6 The electric flux through the top and bottom faces is zero. The electric flux through each of the other four faces is $\lambda a/4\epsilon_0$.
Guided Problem 24.8 For $r \leq R$, $E = q/(4\pi R^2\epsilon_0)$. For $r \geq R$, $E = q/(4\pi r^2\epsilon_0)$.

Figure WGA24.8



PRACTICE

Guided Practice by Chapter

25 Work and Energy in Electrostatics

Review Questions

Answers to these questions can be found at the end of this chapter.

25.1 Electric potential energy

1. What is electric potential energy?
2. In what way is electric potential energy more complicated than gravitational potential energy?
3. How does the electric potential energy of a dipole placed near a stationary charged object depend on the orientation of the dipole relative to the charged object?

25.2 Electrostatic work

4. What is electrostatic work?
5. When a charged particle moves in an electrostatic field, on which aspects of its path does the electrostatic work done on the particle depend? Which aspects of the path have no bearing on the amount of electrostatic work done on the particle?
6. What is electrostatic potential difference?
7. What is the distinction between the potential difference in an electric field and the potential at any point in the field?

25.3 Equipotentials

8. What are equipotential lines? Equipotential surfaces? Equipotential volumes?
9. How are equipotential lines and surfaces spatially related to electric field lines?
10. In an electric field, how does the electrostatic potential vary along a given field line?
11. What characteristic of a charged particle determines whether it tends to move from a region of higher electrostatic potential to a region of lower electrostatic potential or from a region of lower potential to one of higher potential?

25.4 Calculating work and energy in electrostatics

12. What is the mathematical expression for the electrostatic work done on a charged particle in moving the particle from one location to another in the electrostatic field created by a second charged particle?
13. What is the relationship between the electrostatic work done on a charged particle in moving it around in an electric field and the path of the motion?
14. What is the mathematical relationship between the electrostatic work done on a charged particle in moving it from point A to point B and the electrostatic potential difference between points A and B?
15. How can the electric potential energy of a system made up of two charged particles be defined when only differences in potential energy are physically meaningful?
16. What is the electric potential energy for any system of charged particles?

25.5 Potential difference

17. What is the SI unit of potential difference?
18. For (*a*) a system of charged particles and (*b*) an electric circuit, what is the common reference position used to allow us to assign a unique value of electrostatic potential to each position in the system?
19. What is the expression for determining the electrostatic potential at any given position in a system of charged particles when the potential at infinity is assigned a value of zero?
20. Is it possible to extract energy from an electrostatic field by going around a closed path?

25.6 Electrostatic potentials of continuous charge distributions

21. How can the expression for the electrostatic potential of an object that carries a continuous distribution of charge be obtained from the expression for the electrostatic potential of a system of charged particles?
22. Is there any advantage to computing the electrostatic potential of a distribution of charge rather than the electric field due to the distribution?

25.7 Obtaining the electric field from the potential

23. Can the field line pattern for the electrostatic field created by a charged object be determined from a map of the equipotentials?
24. What is the relationship between the electrostatic field and the electrostatic potential?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The electrostatic work done while assembling one proton and one electron to form a hydrogen atom (F, X, A)
2. The minimum work done by an external agent to bring two protons together to form a helium nucleus (F, M, A)
3. The potential difference needed to give 10^{-12} J of kinetic energy to a proton that is initially at rest (Q, A)
4. The electrostatic potential at a point that is 30 mm from one end of a charged glass rod and lies on the line that runs through that end perpendicular to the rod's long axis (C, R, W)
5. The surface charge density on a utility-hole cover charged with a car battery (D, V, I, N, S)
6. The potential difference between floor and ceiling needed to "float" a proton in your room (Z, K, AA, U, P)
7. The potential difference required to stop an electron whose initial speed is 1×10^6 m/s (H, Q, A)
8. The initial speed, expressed as a multiple of the speed of light c, of an alpha particle (which is a helium atom stripped of its two electrons) that approaches to within 9×10^{-15} m of the center of a gold nucleus (BB, B, G, L, Q)
9. The minimum work needed to assemble eight protons, one at each of the eight corners of a 5-mm cube (E, J, O, T, Y)

Hints

A. What is the magnitude of charge on any proton or electron?
B. What is the electrostatic potential at the particle's initial position? At its final position?
C. What is the length of the charge distribution on a typical glass rod used in the laboratory?
D. How can this be done physically, and what type of charged object results?
E. How much work is required to place the first proton at the lower left front corner?
F. What is the initial separation between the two particles?
G. What is the charge of a gold nucleus?
H. What is K_{initial}?
I. What equipotential surface is at a potential of 12 V?
J. How much work is required to place the second proton at the lower left rear corner?
K. Which of the two surfaces, floor or ceiling, must be positively charged?
L. What is the particle's final speed?
M. What is their final separation?
N. What is a typical radius of a utility-hole cover?
O. How much work is required to place the third proton at the lower right rear corner?
P. What is the distance from floor to ceiling?
Q. How are ΔV and ΔK related for the particle?
R. What is a reasonable guess for how much charge the rubbed rod can carry?
S. How does this situation relate to the charged disk in Example 25.7?
T. Do you see a trend in the number of terms?
U. What value of E is required in order to have the electric force balance this gravitational force?
V. What is the potential difference between the terminals of a car battery?
W. Where is the potential equal to zero?
X. What is their final separation?
Y. How many terms must be combined to get the minimum work required to place all eight protons?
Z. What forces must balance to float the proton?
AA. What is the magnitude of the gravitational force exerted?
BB. What is the initial separation distance?

Key (all values approximate)

A. elementary charge, $e = 1.6 \times 10^{-19}$ C; **B.** zero at initial position, $V = (1/4\pi\epsilon_0)(q/r)$ at final position; **C.** 0.2 m; **D.** place cover on nonconducting stand, attach positive battery terminal to cover, and negative battery terminal to ground, producing a charged disk; **E.** none, because there is no potential difference to move through; **F.** essentially infinite; **G.** 79 protons, thus $+79e$; **H.** 5×10^{-19} J; **I.** the surface of the cover; **J.** $W = q\Delta V = (1/4\pi\epsilon_0)(q^2/r)$, where r is cube edge length and q is proton charge; K. the floor; **L.** zero; **M.** nuclear separation distance, 2×10^{-15} m; **N.** 0.3 m; **O.** $q\Delta V = (1/4\pi\epsilon_0)(q^2/r) + (1/4\pi\epsilon_0)(q^2/\sqrt{2}r)$; **P.** 3 m; **Q.** assuming only electrostatic work, $q\Delta V + \Delta K = 0$; **R.** 1×10^1 μC; **S.** both upper and lower surfaces of the cover are charged, so there are two thin disks to superpose; **T.** each proton added requires one term for each proton already in place; **U.** $E = mg/q \approx 1 \times 10^{-7}$ V/m; **V.** 12 V; **W.** infinitely far from the rod; **X.** the atom's radius, 5×10^{-11} m; **Y.** 28, counting each pair of protons once: 12 edge terms, 12 face diagonals, and 4 body diagonals; **Z.** gravitational force exerted by Earth on proton and electric forces exerted by charged floor and ceiling on it; **AA.** 2×10^{-26} N; **BB.** infinite

Worked and Guided Problems

Procedure: Calculating the potential difference between two points in an electric field

The potential difference between two points in an electric field is given by Eq. 25.25. The following steps will help you evaluate the integral:

1. Begin by making a sketch of the electric field, indicating the points corresponding to the two points between which you wish to determine the potential difference.
2. To facilitate evaluating the scalar product $\vec{E} \cdot d\vec{\ell}$, choose a path between the two points so that $\vec{E}$ is either parallel or perpendicular to the path. If necessary, break the path into segments. If $\vec{E}$ has a constant value along the path (or a segment of the path), you can pull it out of the integral; the remaining integral is then equal to the length of the corresponding path (or the segment of the path).
3. Remember that to determine V_{AB} ("the potential difference between points A and B"), your path begins at A and ends at B. The vector $d\vec{\ell}$ therefore is tangent to the path, in the direction that leads from A to B (see also Appendix B).

At this point you can substitute the expression for the electric field and carry out the integral. Once you are done, you may want to verify the algebraic sign of the result you obtained: *negative* when a positively charged particle moves along the path in the direction of the electric field, and *positive* when it moves in the opposite direction.

Procedure: Calculating the electrostatic potential of continuous charge distributions

To calculate the electrostatic potential of a continuous charge distribution (relative to zero potential at infinity), you need to evaluate the integral in Eq. 25.34. The following steps will help you work out the integral:

1. Begin by making a sketch of the charge distribution. Mentally divide the distribution into infinitesimally small segments carrying a charge dq_s. Indicate one such segment in your drawing.
2. Choose a coordinate system that allows you to express the position of dq_s in the charge distribution in terms of a minimum number of coordinates (x, y, z, r, or θ). These coordinates are the integration variables. For example, use a radial coordinate system for a charge distribution with radial symmetry. Never place the representative segment dq_s at the origin.
3. Indicate the point at which you wish to determine the potential. Express the factor $1/r_{sP}$, where r_{sP} is the distance between dq_s and the point of interest, in terms of the integration variable(s).
4. Determine whether the charge distribution is one dimensional (a straight or curved wire), two dimensional (a flat or curved surface), or three dimensional (any bulk object). Express dq_s in terms of the corresponding charge density of the object and the integration variable(s).

At this point you can substitute your expressions for dq_s and $1/r_{sP}$ into Eq. 25.34 and work out the integral.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

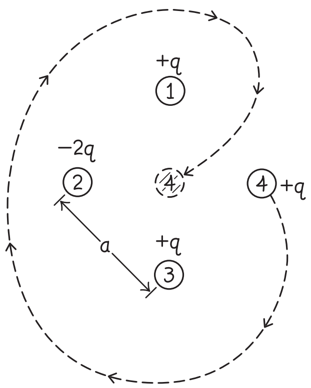
Worked Problem 25.1 Relocating a charged particle

Consider a group of charged toner particles and how one of them can be moved around in the group. Four charged particles form a square with side length $a = 6.9\ \mu\text{m}$. Particles 1, 3, and 4 carry charge $+q = 3.9 \times 10^{-15}$ C, and particle 2 carries charge $-2q$. Particles 2 and 4 are diagonally opposite each other. How much work must be done by an external agent on particle 4 to pull it out of its corner, slide it around outside the perimeter of the square past particles 3, 2, and 1, and then place it stationary at the center of the square?

PRACTICE

❶ **GETTING STARTED** We begin with a sketch of the arrangement and what we want to do with particle 4 (Figure WG25.1). We are not given much information about the external agent that causes particle 4 to move, but we do know that the change in the particle's kinetic energy is zero because it starts out with zero kinetic energy and has zero kinetic energy again after it is placed at the center of the square. Thus the work done by the external agent, the quantity we must calculate, is the negative of the electrostatic work done by the other three particles as particle 4 is moved.

Figure WG25.1



❷ **DEVISE PLAN** Because the change in kinetic energy for particle 4 is zero, we can use

$$W_{\text{on }4} = W_{\text{by agent}} + W_{\text{by }1,2,3} = \Delta K = 0.$$

Because electrostatic interactions are nondissipative, we know that the work done by particles 1, 2, and 3 on particle 4 equals the negative of the change in electric potential energy of the system. Thus we can use

$$W_{\text{by agent}} = -W_{\text{by }1,2,3} = -(-q_4\Delta V) = q_4\Delta V. \tag{1}$$

For particle 4 at any given location, the electrostatic potential at that location due to particles 1, 2, and 3 is given by

$$V = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{3}\frac{q_j}{r_j}, \tag{2}$$

where r_j is the separation distance between particle 4 and particle j (where $j = 1, 2, 3$).

We need to determine the difference between the electrostatic potential at the initial and final positions of particle 4. We can calculate this difference in potential in two ways: use Eq. 2 with either the final or initial separation distances to calculate the potentials at those positions and then substitute the values obtained into Eq. 1, or substitute Eq. 2 into Eq. 1 and sum the potential differences due to each particle separately. We choose the latter method because it allows us to consolidate common factors easily.

❸ **EXECUTE PLAN** The work done by the external agent on particle 4 is

$$W_{\text{by agent}} = q_4[V_\text{f} - V_\text{i}] = q_4\frac{1}{4\pi\epsilon_0}\left[\sum_{j=1}^{3}\frac{q_j}{r_{j,\text{f}}} - \sum_{j=1}^{3}\frac{q_j}{r_{j,\text{i}}}\right]$$

$$= \frac{q_4}{4\pi\epsilon_0}\left\{\sum_{j=1}^{3}q_j\left[\frac{1}{r_{j,\text{f}}} - \frac{1}{r_{j,\text{i}}}\right]\right\} = \frac{q_4}{4\pi\epsilon_0}\left\{q_1\left[\frac{1}{\frac{1}{2}a\sqrt{2}} - \frac{1}{a}\right]\right.$$

$$\left. + q_2\left[\frac{1}{\frac{1}{2}a\sqrt{2}} - \frac{1}{a\sqrt{2}}\right] + q_3\left[\frac{1}{\frac{1}{2}a\sqrt{2}} - \frac{1}{a}\right]\right\}.$$

PRACTICE

Removing the subscripts on q to simplify, with $q_4 = q_1 = q_2 = q$ and $q_2 = -2q$, we have

$$W_{\text{by agent}} = \frac{q}{4\pi\epsilon_0}\left[q\left(\frac{1}{\frac{1}{2}a\sqrt{2}} - \frac{1}{a}\right) + (-2q)\left(\frac{1}{\frac{1}{2}a\sqrt{2}} - \frac{1}{a\sqrt{2}}\right)\right.$$

$$\left. + q\left(\frac{1}{\frac{1}{2}a\sqrt{2}} - \frac{1}{a}\right)\right] = \frac{2q^2}{4\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{2(3.9 \times 10^{-15}\ \text{C})^2}{4\pi(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(6.9 \times 10^{-6}\ \text{m})}\left[\frac{1}{\sqrt{2}} - 1\right]$$

$$= -1.2 \times 10^{-14}\ \text{J.}\ \checkmark$$

❹ EVALUATE RESULT As a check on this result, let's try the alternative approach of calculating the potential V_f at particle 4's final position, due to particles 1, 2, and 3; then calculating V_i at particle 4's initial position, due to particles 1, 2, and 3; and directly using $W_{\text{by agent}} = q_4[V_f - V_i]$. Taking the potential at infinity to be zero, we obtain

$$V_f = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a/\sqrt{2}} + \frac{q}{a/\sqrt{2}} + \frac{(-2q)}{a/\sqrt{2}}\right) = 0,$$

and we get

$$V_i = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a} + \frac{q}{a} + \frac{(-2q)}{a\sqrt{2}}\right) = \frac{q}{4\pi\epsilon_0 a}(2 - \sqrt{2}).$$

So

$$q_4[V_f - V_i] = 0 - \frac{q^2}{4\pi\epsilon_0 a}(2 - \sqrt{2}) = -1.2 \times 10^{-14}\ \text{J},$$

just as we found above. It makes sense that $V_f = 0$ because particle 4's final position is equidistant from three charged particles whose charges sum to zero. And it makes sense that $V_i > 0$ because particle 4's initial position is closer to the two positively charged particles than to the negatively charged particle. The electrostatic work done on particle 4 as it moves from $V_i > 0$ to $V_f = 0$ (negative potential difference) is positive. Because particle 4's kinetic energy does not change, this positive electrostatic work is balanced by negative work done by the external agent. So it makes sense that our answer is negative.

Guided Problem 25.2 Forming and straightening a charged triangle

You have three charged particles that are initially very far apart from one another, and you must bring them together to form an equilateral triangle of side length ℓ. Particles A and B each carry a charge $-q$, and particle C carries a charge $4q$. How much work must you do to form the triangle? How much work must you do to rearrange the particles so that they are in a straight line with particle C in the center and a distance ℓ between adjacent particles?

❶ GETTING STARTED

1. Sketch the triangle and the line, and describe the problem in your own words. What two values are you asked to determine?

❷ DEVISE PLAN

2. How can you relate the work you need to do in forming the triangle to the electric potential energy of the configuration?
3. What kind of drawing should you make to show the process of forming the triangle? What kind to show the straightening process?
4. How many terms are in the potential-energy sum for each configuration? What are those terms?

❸ EXECUTE PLAN

5. By comparing the initial and final potential differences, compute the work done in forming the triangle. Repeat for the work done in forming the straight line.
6. Is there a shortcut for evaluating the amount of work needed to form the straight line?

❹ EVALUATE RESULT

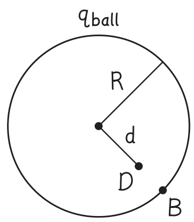
7. Which configuration do you expect to have a greater potential energy?

Worked Problem 25.3 Nonuniformly charged rubber ball

A solid rubber ball of radius R carries a charge q_{ball}, with the volume charge density increasing linearly from zero at the center to the ball's surface. What is the difference in potential between a position B on the ball's surface and any position D located a distance $d < R$ from the ball's center?

❶ **GETTING STARTED** As usual we begin with a sketch of the situation (Figure WG25.2). Note that the problem statement does not specify that positions B and D must be on the same radial line. We could just as easily have shown D on, say, the radius that runs from the center to the 12 o'clock position at the ball's surface. Because of symmetry, the electrostatic potential is the same at each point on a spherical shell concentric with the ball, however, and so we arbitrarily show B and D on the same radial line.

Figure WG25.2



The difference in potential between B and D is a line integral of the electric field from the initial position at B, where radial distance $r = R$, to the final position at D, where $r = d$. Thus we must determine $\vec{E}$. We know from symmetry that the electric field is directed radially, and we also know that the field magnitude is affected by the charge distribution, so we must determine a way to calculate that magnitude as a function of the charge distribution.

❷ **DEVISE PLAN** The difference in potential between B and D is given by Eq. 25.25, $V_{BD} = -\int_B^D \vec{E} \cdot d\vec{\ell}$. We know from Chapter 24 that we can determine the electric field at any radial distance r from the center by using the symmetry of the problem and Gauss's law (Eq. 24.8), $\Phi_E = \oint \vec{E} \cdot d\vec{A} = q_{enc}/\epsilon_0$, where q_{enc} in our case is the quantity of charge enclosed within a sphere that is concentric with the ball and has a radius $r = d$. We can get q_{enc} by noting that the volume charge density increases linearly from zero with distance x from the center, $\rho(x) = \rho_0 x$, where ρ_0 is a constant of dimension C/m^4, and using

$$q_{enc}(r) = \int \rho dV,$$

where dV is the volume of a thin spherical shell concentric with the ball. The value of ρ_0 can be determined by noting that q_{ball} is the charge carried by the ball—that is, $q_{ball} = q_{enc}(R)$.

❸ **EXECUTE PLAN** The charge enclosed by a solid sphere of arbitrary radius r is the sum of the contributions from a series of spherical shells each of radius x and thickness dx:

$$q_{enc}(r) = \int \rho dV = \int_0^r (\rho_0 x)(4\pi x^2 dx) = \pi \rho_0 r^4.$$

From this, we can express ρ_0 in terms of q_{ball}:

$$q_{enc}(R) = \pi \rho_0 R^4 = q_{ball}$$

$$\rho_0 = \frac{q_{ball}}{\pi R^4}.$$

Now we use Gauss's law (Eq. 24.8). Because the ball's symmetry requires that the electric field be radially directed, we have at any radial distance r from the ball's center

$$\oint \vec{E}(r) \cdot d\vec{A} = E(r)A(r) = \frac{q_{enc}(r)}{\epsilon_0}$$

$$E(r)(4\pi r^2) = \frac{\pi \rho_0 r^4}{\epsilon_0} = \frac{q_{ball} r^4}{\epsilon_0 R^4}$$

$$E(r) = \frac{q_{ball}}{4\pi \epsilon_0 R^4} r^2.$$

PRACTICE

The difference in potential between point B on the ball's surface ($r = R$) and point D a distance $r = d$ from the ball's center is then

$$V_d - V_R = -\int_R^d \vec{E}(r) \cdot d\vec{r} = -\int_R^d E(r)dr$$

$$= -\int_R^d \frac{q_{\text{ball}}}{4\pi\epsilon_0 R^4} r^2 dr = -\frac{q_{\text{ball}}}{4\pi\epsilon_0 R^4}\left[\tfrac{1}{3}r^3\right]_R^d$$

$$= \frac{q_{\text{ball}}}{12\pi\epsilon_0 R^4}(R^3 - d^3). ✔$$

❹ EVALUATE RESULT If q_{ball} is positive, the electric field is directed radially outward. This means that as we move from B to D, we are moving in the direction opposite the direction of the electric field, thereby increasing the electrostatic potential. Consequently, the potential difference should be positive, which is what our answer says. The opposite is true if q_{ball} is negative, which is also consistent with our equation. Our equation says that the potential difference is a linear function of q_{ball}, and we expect $V_d - V_R$ to be greater if the ball carries more charge. We note that if we locate D such that $d = R$, the potential difference is zero, as it must be.

Guided Problem 25.4 Charged plastic sphere

A quantity q_{sphere} of charge is uniformly distributed throughout a solid plastic sphere of radius R. What is the potential difference between the center and a point A located a distance r from the center when $r < R$ and when $r > R$?

❶ GETTING STARTED

1. Sketch the sphere.
2. What information is given, and what variable or variables are you asked to determine?

❷ DEVISE PLAN

3. How do you determine the potential difference between any two positions inside the sphere? What quantities do you need to know in order to determine this difference?
4. How do you determine the charge enclosed within a spherical shell of any radius?
5. How do you relate the potential at any position outside the plastic sphere to the potential at any position inside by matching expressions at a boundary of these two regions?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

6. Does your expression give the correct result for the potential at a location very far from the sphere center?

Worked Problem 25.5 Line of charge

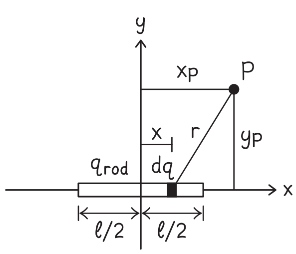
A plastic rod of length ℓ is rubbed with fur to uniformly distribute a quantity q_{rod} of surplus charge. It is then placed on a nonconducting stand, and the stand is positioned so that the rod lies along the x axis of an xy coordinate system and the rod's center is at the origin. Determine the electrostatic potential at any point P located at an arbitrary position (x_P, y_P) outside the rod.

❶ GETTING STARTED This problem is similar to Example 25.6, which determines the electrostatic potential for points along a line that runs through one end of a rod and is perpendicular to the rod's long axis. However, we're now asked to determine the potential at *any* location in the space surrounding the rod. We need to calculate the electrostatic potential that a charge distribution creates at a point P, and for this we need Eq. 25.34:

$$V = \int \frac{1}{4\pi\epsilon_0}\frac{dq}{r},$$

where r is the distance from P to an infinitesimal charge element dq in the charge distribution. We draw a diagram that shows the necessary information (Figure WG25.3).

Figure WG25.3



PRACTICE

❷ **DEVISE PLAN** We need to determine the contribution to the potential from each infinitesimal charge element dq of the rod because the distance from P to the rod is different for each element. Then we integrate all the contributions to get the potential due to all the charge distributed in the rod. We need to determine appropriate values for r and dq for each position x along the rod. The amount of charge in an infinitesimal rod length dx is $dq = \lambda dx$, where $\lambda = q_{\text{rod}}/\ell$ is the linear charge density. We then integrate from one end of the rod to the other to add up the contribution of each dq to the potential.

❸ **EXECUTE PLAN** The charge on the element dq is

$$dq = \frac{q_{\text{rod}}}{\ell}dx$$

and the distance from the element to P is

$$r = \sqrt{(x_{\text{P}} - x)^2 + y_{\text{P}}^2}.$$

Substituting gives us our expression for the potential at P:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{rod}}}{\ell}\int_{-\ell/2}^{\ell/2}\frac{dx}{\sqrt{(x_{\text{P}} - x)^2 + y_{\text{P}}^2}}.$$

By making a simple substitution, we can put the integral in a form we can look up:

$$u = x_{\text{P}} - x$$

$$du = -dx$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{rod}}}{\ell}\int_{x_{\text{P}}+\ell/2}^{x_{\text{P}}-\ell/2}\frac{-du}{\sqrt{u^2 + y_{\text{P}}^2}}.$$

Note how the limits of integration have changed to match the change of variable to u. This integral is identical to the integral in Example 25.6:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{rod}}}{\ell}\left[-\ln\left(u + \sqrt{u^2 + y_{\text{P}}^2}\right)\right]_{x_{\text{P}}+\ell/2}^{x_{\text{P}}-\ell/2}$$

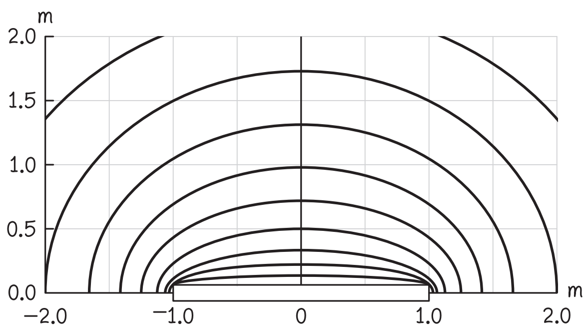
$$= \frac{1}{4\pi\epsilon_0}\frac{q_{\text{rod}}}{\ell}\Big[\ln\left(x_{\text{P}} + \ell/2 + \sqrt{(x_{\text{P}} + \ell/2)^2 + y_{\text{P}}^2}\right)$$

$$-\ln\left(x_{\text{P}} - \ell/2 + \sqrt{(x_{\text{P}} - \ell/2)^2 + y_{\text{P}}^2}\right)\Big]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_{\text{rod}}}{\ell}\ln\left(\frac{x_{\text{P}} + \ell/2 + \sqrt{(x_{\text{P}} + \ell/2)^2 + y_{\text{P}}^2}}{x_{\text{P}} - \ell/2 + \sqrt{(x_{\text{P}} - \ell/2)^2 + y_{\text{P}}^2}}\right). ✔ \quad (1)$$

We now use this equation to plot the potential due to the rod everywhere in the xy plane. A contour plot of the potential for a rod 2.0 m long is shown in Figure WG25.4.

Figure WG25.4



❹ EVALUATE RESULT If P is located on a line parallel to the y axis above or below the end of the rod, then $x_P = \ell/2$. You should check Eq. 1 for V in this special case of $x_P = \ell/2$ to confirm that we get the same expression as in Example 25.6:

$$V = \frac{q}{4\pi\epsilon_0 \ell} \ln\left(\frac{\ell + \sqrt{\ell^2 + d^2}}{d}\right),$$

with the modification that our y_P corresponds to d in the equation. Equation 1 is a complicated expression, but with it we can calculate the potential anywhere in the xy plane outside the rod. If P is very far from the rod, the rod viewed from P looks like a particle, and the potential at P calculated with Eq. 1 should be that due to a particle carrying charge q_{rod}: zero. If we let either x_P, y_P, or both become infinite in Eq. 1, the argument of the natural logarithm is unity, and $\ln(1) = 0$, yielding the expected potential of zero.

Guided Problem 25.6 Charged ring

A thin, flat ring made of a nonconducting material has inner radius R_{in} and outer radius $R_{\text{out.}}$ Its surface charge density $\sigma(a)$ decreases with distance a from the center as $\sigma(a) = \sigma_0/a$, where σ_0 is a constant with units C/m. Determine the electrostatic potential at any location along the axis that passes through the ring's center and is perpendicular to the plane of the ring.

❶ GETTING STARTED

1. Draw a sketch of the system.
2. What makes locations on this axis special for determining electrostatic potential?

❷ DEVISE PLAN

3. What expression can you use to determine the contributions from all the charge elements dq on the ring?
4. What symmetry of the ring can you exploit to establish a group of charge elements that are all at the same distance from the desired location?
5. What mathematical quantities can you use to specify this group of charge elements?
6. What is the distance between this group of charge elements and the location you are interested in?
7. What variable(s) do you integrate to account for all the charge elements on the ring, and what are the appropriate limits of integration?

❸ EXECUTE PLAN

8. Where can you obtain an analytic expression for the value of the resulting integral?

❹ EVALUATE RESULT

9. Does the expression you obtained for electrostatic potential behave properly at large distances from the ring?

Worked Problem 25.7 Electric field around a charged rod

The electric field surrounding a charged object that lacks sufficient symmetry to use Gauss's law can be determined either from Coulomb's law (as we saw in Chapter 23) or from the electrostatic potential as given by Eq. 25.40. Use the latter method to determine, for any location in space, the component of the electric field perpendicular to the rod in Worked Problem 25.5.

❶ GETTING STARTED We are given, from Worked Problem 25.5, the orientation of a charged rod of length ℓ and charge q_{rod} in an xy coordinate system and are asked to use Eq. 25.40 to obtain an expression, valid at any location in the space surrounding the rod, for the perpendicular component of the electric field due to the rod. From Figure WG25.3 we see that the y component of the electric field is the one perpendicular to the rod.

❷ DEVISE PLAN We need to calculate $E_y = -\partial V/\partial y$. We have to take the partial derivative of the potential with respect to y while holding all other variables constant; in other words, we treat the position x as a constant. At any arbitrary location (x, y) the electrostatic potential is given by

$$V(x, y) = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{rod}}}{\ell} \ln\left(\frac{x + \ell/2 + \sqrt{(x + \ell/2)^2 + y^2}}{x - \ell/2 + \sqrt{(x - \ell/2)^2 + y^2}}\right).$$

Taking the derivative of the logarithm's argument appears daunting, but we can simplify things by using the identity $\ln(a/b) = \ln a - \ln b$.

PRACTICE

❸ EXECUTE PLAN

$$E_y = -\frac{\partial V}{\partial y} = \frac{1}{4\pi\epsilon_0}\frac{q_{rod}}{\ell}\frac{\partial}{\partial y}\Big\{\ln\left[x - \ell/2 + ((x - \ell/2)^2 + y^2)^{1/2}\right]$$

$$-\ln\left[x + \ell/2 + ((x + \ell/2)^2 + y^2)^{1/2}\right]\Big\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_{rod}}{\ell}\left\{\left[\frac{\frac{1}{2}((x - \ell/2)^2 + y^2)^{-1/2}2y}{x - \ell/2 + ((x - \ell/2)^2 + y^2)^{1/2}}\right]\right.$$

$$\left.-\left[\frac{\frac{1}{2}((x + \ell/2)^2 + y^2)^{-1/2}2y}{x + \ell/2 + ((x + \ell/2)^2 + y^2)^{1/2}}\right]\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_{rod}}{\ell}y\left\{\left[\frac{((x - \ell/2)^2 + y^2)^{-1/2}}{x - \ell/2 + ((x - \ell/2)^2 + y^2)^{1/2}}\right]\right.$$

$$\left.-\left[\frac{((x + \ell/2)^2 + y^2)^{-1/2}}{x + \ell/2 + ((x + \ell/2)^2 + y^2)^{1/2}}\right]\right\}. ✔$$

This algebra is complicated but very useful because it is much easier to get the electric field component via the known potential than from superposition of electric fields as was done in Chapter 23. The expression for the x component of $\vec{E}$ is even more complicated than this expression for E_y, but the procedure for obtaining it is the same. You should try it.

❹ EVALUATE RESULT We can check to see that our expression for E_y is not unreasonable by looking at a few limiting cases. If the location we're interested in is along the x axis, E_y should be zero by symmetry. We see that setting $y = 0$ gives $E_y(x, 0) = 0$ as expected. If the location is along the perpendicular bisector of the rod ($x = 0$), the expression we get for $E_y(0, y)$ should match the expression obtained in Example 23.4 once we note that our rod is along the x axis whereas the rod in Example 23.4 is along the y axis:

$$E_y(0, y) = \frac{1}{4\pi\epsilon_0}\frac{q_{rod}}{\ell}y\left\{\left[\frac{((\ell/2)^2 + y^2)^{-1/2}}{((\ell/2)^2 + y^2)^{1/2} - \ell/2}\right]\right.$$

$$\left.-\left[\frac{((\ell/2)^2 + y^2)^{-1/2}}{((\ell/2)^2 + y^2)^{1/2} + \ell/2}\right]\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_{rod}}{\ell}y\left\{\frac{\ell((\ell/2)^2 + y^2)^{-1/2}}{y^2}\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_{rod}}{y((\ell/2)^2 + y^2)^{1/2}} = k\frac{q_{rod}}{y(\frac{1}{4}\ell^2 + y^2)^{1/2}}.$$

This indeed matches the result of Example 23.4. (Remember that from Eq. 24.7 defining the permittivity constant ϵ_0, we know $k = 1/4\pi\epsilon_0$.)

Guided Problem 25.8 Electric field due to a charged disk

A thin disk of radius R has a uniform surface charge density σ. Use Eq. 25.40 to obtain an expression for the electric field along the axis that runs through the disk center and is perpendicular to the plane of the disk. Orient the disk in the xy plane of an xyz coordinate system, with the disk center at the origin, so that the axis you work with is the z axis.

❶ GETTING STARTED

1. What are your knowns in this problem? What must you determine? Obtain the expression for the potential for this system from the Chapter Summary above.

❷ DEVISE PLAN

2. What is the expression for the potential $V(z)$ along the z axis ($x = y = 0$)?
3. What is the relationship between an electric field and the electrostatic potential at any location in the field?
4. Which component or components of the electric field are zero by symmetry?
5. Which quantities should you hold constant when you take the appropriate partial derivative?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

6. Is the result obtained the same as that derived in Example 23.6? That result expressed E_z in terms of ϵ_0 and R:

$$E_z = \frac{1}{2\epsilon_0}\sigma z\left(\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{z^2 + R^2}}\right).$$

Answers to Review Questions

1. Electric potential energy is the potential energy associated with the relative positions of objects that carry electrical charge.
2. The gravitational interaction can be only attractive, but the electric interaction can be either attractive or repulsive, depending on whether the charged objects carry charge of the same type (interaction is repulsive) or charge of opposite types (interaction is attractive). For oppositely charged objects, electric potential energy (like gravitational potential energy) decreases with decreasing separation, but for charged objects of the same sign, electric potential energy increases with decreasing separation.
3. The electric potential energy of the dipole is at its minimum when the dipole is oriented in the direction of the electric field created by the charged object and increases as the angle between the dipole orientation and the field direction increases.
4. Electrostatic work is the work done by an electrostatic field on a charged particle or charged object.
5. The amount of electrostatic work done on the particle depends on the two endpoints of the path. The work done is not affected by either path length or path shape.
6. The electrostatic potential difference between any two points A and B in an electrostatic field is the negative of the electrostatic work per unit charge done on a charged particle moved along any path from A to B.
7. Potential difference is a quantity related to *two locations* in an electric field and is numerically equivalent to the electrostatic work per unit charge done on a charged particle moved from one location to the other. Potential is a quantity that is related to any *one location* in an electric field; a value for potential can be assigned to any location in the field once we choose a reference location at which the potential in the field is zero.
8. Equipotential lines and surfaces are any lines and surfaces in an electric field along which the electrostatic potential is the same everywhere. An equipotential volume is the interior of a conducting object in electrostatic equilibrium, because the electric field magnitude in that interior is zero, making the whole interior an equipotential volume.
9. Equipotential lines and surfaces in an electric field are always perpendicular to the field lines.
10. The potential decreases as you move along the field line in the direction of the field.
11. The type of charge the particle carries. The electric force exerted on a positively charged particle is in the direction of the electric field, which means the force tends to move the particle in the direction from higher to lower electrostatic potential. The electric force exerted on a negatively charged particle is in the opposite direction and so tends to move the particle from a region of lower potential to a region of higher potential.
12. The expression, derived from a line integral of the electric force over the path of the particle's motion, is given by Eq 25.5.
13. There is no relationship. The amount of electrostatic work done is independent of the path; it depends on only the initial and final positions of the particle.
14. The electrostatic work per unit charge done on a charged particle as it moves from A to B is the negative of the potential difference between point A and point B.
15. We define the system's electric potential energy by choosing some reference configuration of the particles in which the electric potential energy is defined to be zero. Usually, the best choice for this zero-potential-energy configuration is infinite separation between the particles (which corresponds to zero electric force between them). The potential energy of the system when the separation distance is some finite value is then the difference between U^E at that separation distance and $U^E(=0)$ at infinity.
16. For a system of charged particles, the electric potential energy is the sum of the potential energies for each pair of particles, as given in Eq. 25.14 for a system of three particles.
17. The volt, which equals one joule per coulomb.
18. (*a*) The reference position at which electrostatic potential is defined to be zero is either infinity (which means infinite separation between particles) or (*b*) in dealing with electric circuits the ground.
19. Equation 25.30, $V_P = (1/4\pi\epsilon_0)\sum_n q_n/r_{nP}$, where r_{nP} is the distance between the particle whose charge is q_n and the position P at which the electrostatic potential is being measured.
20. No, because the work done by an electrostatic field is path-independent, which means the work done by the field is zero for a closed path because the beginning and end of the path are at the same location.
21. Treat each infinitesimal element of charge as a particle and integrate over the object, obtaining Eq. 25.34, $V = (1/4\pi\epsilon_0)\int(dq/r)$, where r is the distance between the infinitesimal element that carries charge dq and the position at which the potential is being measured.
22. Yes, because electrostatic potential is a scalar (which is easier to calculate than an electric field, which is a vector) and potential and field represent two different ways of expressing equivalent information.
23. Yes, because we know that the field lines are perpendicular to the equipotentials and that the direction of the field is the same as the direction of decreasing potential. The field magnitude correlates with the density of the equipotentials: The more closely spaced the equipotentials, the greater the field magnitude.
24. The component of the electrostatic field in any of the Cartesian directions x, y, z is equal to the negative of the derivative of the electrostatic potential with respect to distance in that direction (Eq. 25.40).

Answers to Guided Problems

Guided Problem 25.2 To form the triangle: $W_1 = U_i^E = -\dfrac{7q^2}{4\pi\epsilon_0\ell}$. To rearrange:

$$W_2 = U_f^E - U_i^E = -\frac{q^2}{8\pi\epsilon_0\ell}.$$

Guided Problem 25.4 The potential difference from the center of the sphere to radius r is $V_{0r} = -\dfrac{q_{\text{sphere}} r^2}{8\pi\epsilon_0 R^3}$ for $r < R$ and

$$V_{0r} = \frac{q_{\text{sphere}}}{8\pi\epsilon_0}\left(\frac{2}{r} - \frac{3}{R}\right) \text{ for } r > R.$$

Guided Problem 25.6 With the ring centered at the origin of the z axis,

$$V(z) = \frac{\sigma_0}{2\epsilon_0}\ln\left(\frac{R_{\text{out}} + \sqrt{R_{\text{out}}^2 + z^2}}{R_{\text{in}} + \sqrt{R_{\text{in}}^2 + z^2}}\right).$$

Guided Problem 25.8 $E_z = \dfrac{1}{2\epsilon_0}\sigma z\left(\dfrac{1}{\sqrt{z^2}} - \dfrac{1}{\sqrt{z^2 + R^2}}\right)$

PRACTICE

Guided Practice by Chapter

26 Charge Separation and Storage

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

26.1 Charge separation

1. What is the difference between the potential difference in a system and the system's electric potential energy?
2. When two charged objects that make up a system are separated by some distance, on what quantities does the system's electric potential energy depend?
3. In a system of charged objects separated from one another, where in space should one say that the system's electric potential energy is stored?
4. What does a Van de Graaff generator produce?

26.2 Capacitors

5. What is a capacitor?
6. Describe a parallel-plate capacitor. What geometric approximation is usually made in treating parallel-plate capacitors, and what simplification for its electrical properties results?
7. How is the amount of charge on each plate of a parallel-plate capacitor related to the potential difference between the plates?
8. For a given potential difference between the plates of a parallel-plate capacitor, how does the quantity of charge on each plate vary with plate area and separation distance?
9. In practice, what limits the quantity of charge that can be stored on the plates of a parallel-plate capacitor?

26.3 Dielectrics

10. Distinguish between the two general types of dielectrics, including a description of how each type behaves in an external electric field.
11. From a macroscopic point of view, what happens to a dielectric material when it is placed in an external uniform electric field?
12. What is the distinction between bound charge and free charge?
13. Why is the electric field magnitude in a parallel-plate capacitor with no dielectric between the plates greater than the electric field magnitude when a dielectric is inserted between the plates?
14. Does the electric field created in a polarized dielectric store less, the same, or more energy than an electric field of equal magnitude in a vacuum? Why?

26.4 Voltaic cells and batteries

15. What is a battery, and what does it do?
16. What is emf?

26.5 Capacitance

17. How is the capacitance of a capacitor defined?
18. On what properties of a capacitor does capacitance depend?
19. What is the unit of capacitance? Is its size appropriate to most practical applications in electronic devices?

26.6 Electric field energy and emf

20. How does the electric potential energy stored in a capacitor depend on the quantity of charge on each conductor?
21. How does the electric potential energy stored in a capacitor depend on the potential difference between the conductors?
22. What is energy density, and what is the algebraic relationship between the energy density of an electric field and the field magnitude?
23. How is the potential difference between the negative and positive terminals of the charge-separating device related to the emf in an ideal device and in a real-world device?

26.7 Dielectric constant

24. What is the definition of dielectric constant for a dielectric material?
25. Why is the dielectric constant of liquid water greater than that of other common materials used in capacitors, like paper or mica?
26. For a parallel-plate capacitor with a dielectric material filling the space between its plates, how is the bound charge on either dielectric surface related to the free charge on the surface of the adjacent conducting plate?
27. Why does the energy stored in an isolated charged capacitor decrease as a dielectric is inserted into the capacitor?

26.8 Gauss's law in dielectrics

28. When Gauss's law is used to calculate the electric field in a dielectric, the charge enclosed by any Gaussian surface we choose is $q_{\text{free, enc}} - q_{\text{bound, enc}}$. How can we use the law for a dielectric when we know nothing about $q_{\text{bound, enc}}$?
29. What is Gauss's law in a dielectric material, and what is its relationship to the form of Gauss's law we worked with in Chapter 24?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The maximum potential difference between two metal baking sheets separated by 100 mm of air (A, D)
2. The maximum quantity of charge that can be placed on a metal-coated basketball in air (A, J, S, Q)
3. The potential difference that causes a lightning strike (A, O)
4. The dimensions of the plates in a square, 1-F parallel-plate capacitor with air between the plates (T, U, F, K, P)
5. The capacitance of a raindrop in dry air (B, G)
6. The plate area of a 50-fF capacitor on a computer memory chip (C, N)
7. The maximum energy density achievable in an electric field in air (A, V)
8. The maximum amount of electrical energy that can be stored in an electric field that fills your physics laboratory (A, H)
9. The capacitance of the coaxial cable that connects the cable box sitting atop your television to the cable outlet in the wall (E, I, M, R)
10. The capacitance of a metal-coated softball (G, L)

Hints

A. What is the electric field at the breakdown threshold of air?
B. What is the radius of a raindrop?
C. What is between the plates?
D. How does the magnitude of the electric field between two parallel plates E relate to the magnitude of the potential difference V between them?
E. What is the length of the cable?
F. For a given capacitance, what is the relationship between plate area and plate separation distance?
G. What is the radius of the "other" sphere serving as a conductor in this "capacitor"?
H. What is the laboratory volume?
I. What are the radii of the conductors?
J. Where is the magnitude of potential greatest?
K. What is a reasonable value for the smallest gap that can be maintained between two large metal plates?
L. What is the radius of a softball?
M. What shape can you assume for the cable?
N. What is the gap width?
O. What is the length of a lightning bolt?
P. What plate area is needed in order to have 1-F capacitance when the plate separation distance is 2 mm?
Q. What is the radius of a basketball?
R. What is a reasonable dielectric constant value for the cable?
S. If you model the ball as a conducting sphere, how do you determine its capacitance?
T. Is 1 F considered a common capacitance value or an extremely high value?
U. What does having such a large capacitance suggest about plate size?
V. What is the relationship between the energy density of an electric field and the field magnitude?

Key (all values approximate)

A. 3×10^6 V/m; **B.** 0.004 m; **C.** some dielectric material, probably silicon dioxide (κ approximately 5); **D.** $V = Ed$; **E.** 3 m; **F.** as separation distance increases, plate area must increase proportionally to maintain constant capacitance, which means you want a small gap; **G.** infinite; **H.** 4×10^2 m^3; **I.** $R_{inner} = 5 \times 10^{-4}$ m, $R_{outer} = 3 \times 10^{-3}$ m; **J.** near the ball surface; **K.** 2 mm; **L.** 50 mm; **M.** ignore any bends and loops, assuming all radii of curvature are much greater than R_{outer}, so that you can treat the cable as a cylinder; **N.** 1×10^{-8} m; **O.** 2×10^3 m; **P.** 2×10^8 m^2; **Q.** 0.1 m; **R.** with plastic insulation, κ is approximately 2; **S.** imagine the ball being concentric with a sphere of infinitely large radius so that you form a capacitor in which the ball surface is one conductor, the surface of the infinite sphere is the other conductor, and the gap is filled with air; **T.** extremely high; **U.** plates must be quite large; **V.** the energy density is proportional to the square of the field magnitude, with $\epsilon_0/2$ as the proportionality constant.

Worked and Guided Problems

Procedure: Calculating the capacitance of a pair of conductors

To calculate the capacitance of a pair of conductors:

1. Let the conductors carry opposite charges of magnitude q.
2. Use Gauss's law, Coulomb's law, or direct integration to determine the electric field along a path leading from the negatively charged conductor to the positively charged conductor.
3. Calculate the electrostatic work W done on a test particle carrying a charge q_t along this path (Eq. 25.24) and determine the potential difference across the capacitor from Eq. 25.15,

$$V_{cap} = -W_{q_t}(- \to +)/q_t.$$

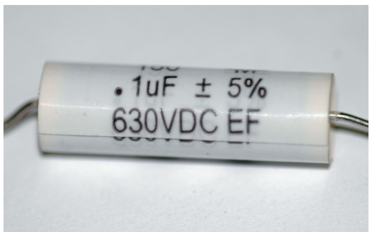
4. Use Eq. 26.1, $C \equiv q/V_{cap}$, to determine C.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 26.1 Roll-up capacitor

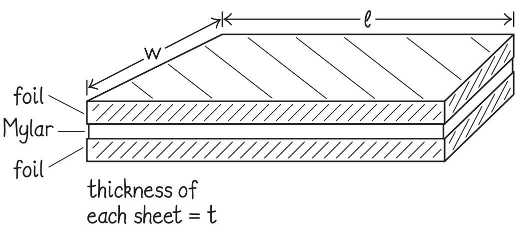
In a roll-up parallel-plate capacitor (Figure WG26.1), the plates are thin sheets of metal foil with a Mylar dielectric between them. Suppose that the foil and Mylar sheets are each 0.0500 mm thick and the capacitor is 20.00 mm tall and has a radius of 6.00 mm. Estimate the charge stored in the capacitor when there is a 25-V potential difference across the capacitor leads.

Figure WG26.1



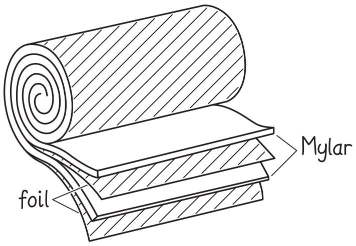
❶ GETTING STARTED We must first visualize how this capacitor is constructed. As usual, we begin with a sketch, in this case a sketch of the components—two foil sheets separated by a Mylar sheet—before the capacitor is rolled up (Figure WG26.2). We label the thickness of each sheet t. We assume that the Mylar completely fills the space between the foil sheets, so that the plate separation distance is equal to the Mylar thickness. Now we roll it up. However, there's a trick here: If we roll up the three-sheet sandwich of foil-Mylar-foil, the bottom foil touches the top foil, connecting the two plates and effectively nullifying the capacitor. To avoid this, the unrolled capacitor we start with has to be a four-sheet sandwich configured foil-Mylar-foil-Mylar (Figure WG26.3). The thickness of this sandwich is $4t$, where $t = 0.0500$ mm is the thickness of each sheet, and the radius of the rolled-up capacitor is $R = 6.00$ mm.

Figure WG26.2



PRACTICE

Figure WG26.3



2 DEVISE PLAN The radius R of the capacitor is related to the sheet thickness t and to the number of sheets in the length R. Because we are given values for the capacitor height and radius, as well as the thickness of each sheet, we should be able to unroll the sandwich and compute the capacitance of the equivalent parallel-plate capacitor. The problem asks how much charge can be stored in this capacitor when the potential difference across it is 25 V. Because capacitance is defined as the quantity of charge that can be stored for a given potential difference, calculating the capacitance should allow us to determine the quantity of stored charge.

If we unroll the capacitor, we have two foil sheets (the parallel plates) separated by a Mylar sheet of the same size. (The second Mylar sheet shown in Figure WG26.3 lies outside the capacitor and hence is irrelevant to the capacitance.) From Eq. 26.11 and the result of Example 26.2, $C = \epsilon_0 A/d$, we know that the capacitance of parallel plates with a dielectric between them is

$$C = \frac{\kappa\epsilon_0 A}{d}.$$

In this expression, d is the separation distance between the plates, and in our problem that distance is the thickness $t = 0.0500$ mm of the Mylar sheet. After we look up the dielectric constant of Mylar ($\kappa = 3.3$), what we need to do is estimate the area $A = \ell w$ of each foil sheet. When the capacitor is unrolled, the height of the capacitor is one dimension of either foil sheet—let's say the width, so that $w = 20.00$ mm. We know how long the foil and Mylar sheets are (the ℓ in $A = \ell w$), and their combined thickness determines the radius $R = 6.00$ mm of the rolled-up capacitor. So we can infer the length of the foil sheets from what we know about their thickness, the Mylar thickness, and the radius of the rolled-up capacitor.

3 EXECUTE PLAN One way to estimate the length of the unrolled foil sheets is to assume the capacitor is rolled as tightly as possible and then use the volume of the rolled-up capacitor. When you imagine starting with the four-sheet sandwich (foil-Mylar-foil-Mylar) and rolling it up, the width $w = 20.00$ mm in Figure WG26.2 becomes the height of the capacitor. The other dimension of the four sheets, ℓ, is related to the capacitor radius ($R = 6.00$ mm). When the capacitor is fully rolled up, the volume of the unrolled parallel-plate sandwich ($w \times \ell \times 4t$) must match the volume of the cylindrical capacitor ($\pi R^2 w$). Thus we have

$$V = \pi R^2 w = w\ell(4t)$$

$$\ell = \frac{\pi R^2}{4t}.$$

Another way to estimate the sheet length ℓ is to note that it requires many turns to roll up the capacitor. Each turn is a spiral ring of thickness $4t$, but because there are many turns, we can model the capacitor as if the spirals were circles. That means the capacitor is composed of a set of nested cylindrical shells, each of thickness $4t$ but with different radii, ranging from $r = 2t$ for the inner shell to $r = R - 2t$ for the outer shell. Then $R/4t$ gives the number of four-sheet shells across the capacitor radius. If we add up the circumferences of these shells, we have our ℓ. The average circumference of the shells in the roll is then $2\pi R/2$, so the length ℓ becomes

$$\ell = (\text{number of four-sheet shells}) \times (\text{average circumference})$$

$$= \left(\frac{R}{4t}\right)\left(\frac{2\pi R}{2}\right) = \frac{\pi R^2}{4t}.$$

It is reassuring that both methods give the same estimate for sheet length ℓ.

Now we obtain our expression for the capacitance of this Mylar-filled capacitor, noting that the separation distance d between "plates" is the thickness t of the Mylar between the two foil sheets:

$$C = \frac{\kappa\epsilon_0 A}{d} = \frac{\kappa\epsilon_0 \ell w}{t} = \frac{\kappa\epsilon_0\left(\dfrac{\pi R^2}{4t}\right)w}{t} = \frac{\kappa\epsilon_0 \pi R^2 w}{4t^2}. \quad (1)$$

Finally, the quantity of charge that can be stored with a 25-V potential difference across the capacitor is

$$q = CV = \frac{\kappa\epsilon_0 \pi R^2 w V}{4t^2}$$

$$= \frac{3.3(8.85 \times 10^{-12}\ \text{F/m})\pi(6.00 \times 10^{-3}\ \text{m})^2(20.0 \times 10^{-3}\ \text{m})(25\ \text{V})}{4(5.00 \times 10^{-5}\ \text{m})^2}$$

$$= 1.7 \times 10^{-7}\ \text{C.} \checkmark$$

❹ EVALUATE RESULT Our algebraic result shows that a larger capacitor (greater R or w) means that more charge can be stored for a given value of V, which makes sense. It also says that if the sheets are packed more tightly (smaller t), the quantity of charge that can be stored also increases because more foil is packed in the same volume. (However, you then need to worry about electrical breakdown of the Mylar film.) Equation 1 yields a capacitance of 6.6×10^{-9} F. Although you probably haven't developed a feel yet for capacitors, a capacitance in the nanofarad range is not unreasonable for a capacitor of this size and construction. Typical capacitors in most electric circuits have capacitances ranging from a few picofarads to several hundred microfarads. Because we know that $q = CV$ and we are given $V = 25$ V, a charge in the hundred-nanocoulomb (10^{-7} C) range makes sense for this capacitor.

Guided Problem 26.2 Home-made capacitor

Estimate the greatest capacitance you might achieve at home by rolling layers of aluminum foil and waxed paper around a number 2 pencil.

❶ GETTING STARTED

1. Are the results of Worked Problem 26.1 applicable?

❷ DEVISE PLAN

2. What is the length of an unused number 2 pencil?
3. What is the greatest length of foil or waxed paper you can (*a*) easily obtain and (*b*) handle without aid while creating the roll?
4. How are these lengths related to the capacitance?

❸ EXECUTE PLAN

5. What is a reasonable value for the dielectric constant of waxed paper?

❹ EVALUATE RESULT

6. Is your numerical value reasonable?
7. Try it and see if you are able to construct such a capacitor.

Worked Problem 26.3 Capacitance of television cable

Estimate the capacitance of a length L of the "RG-6" coaxial cable used for cable television. This type of cable consists of an inner wire of 0.50-mm radius and a concentric outer conductor of 6.8-mm diameter, with the space between the two conductors filled with polyethylene (dielectric constant $\kappa = 2.3$).

❶ GETTING STARTED We are asked to estimate the capacitance of a given length of television cable. We know that the geometry closely resembles that of Example 26.3, except that we need to include the dielectric material between the two conductors. Because doubling the length of the cable doubles the surface area of each conductor while keeping the separation unchanged, we expect the capacitance to be proportional to L. We know that putting charge $+q$ on the inner wire and charge $-q$ on the outer conductor results in an electric field pointing radially outward from the inner wire, and hence a potential difference between the two conductors. Because the space between the two conductors is filled with a dielectric material, we need to account for the effect of the dielectric material on the electric field in the region between the two conductors.

❷ DEVISE PLAN We will follow the Procedure box on page 903 to calculate the capacitance. For the path over which electrostatic work is done, we choose the straight path that goes radially inward from the inner surface of the outer conductor (radius 3.4 mm) to the outer surface of the inner conductor (radius 0.50 mm). To compute the electric field, we will use Gauss's law in matter, Eq. 26.25.

❸ EXECUTE PLAN Let $R_1 = 0.50$ mm be the radius of the inner conductor, and let $R_2 = 3.4$ mm be the radius of the outer conductor. The electric field in the region $R_1 < r < R_2$ is given by Gauss's law in matter: $\oint \kappa\vec{E} \cdot d\vec{A} = q_{\text{free, enc}}/\epsilon_0$. We choose a cylindrical Gaussian surface of radius r and length L, concentric with the two conductors. The electric field points radially outward, so $\vec{E} \cdot d\vec{A} = 0$ on the two flat ends of the cylinder. On the curved surface of the cylinder (area $A = 2\pi rL$), the electric field is constant due to cylindrical symmetry, so the

PRACTICE

Gauss's-law integral simplifies to $\kappa EA = q/\epsilon_0$, where the charge $+q$ on the inner conductor is the free charge enclosed by the Gaussian surface. The electric field for $R_1 < r < R_2$ points radially outward and has magnitude

$$E = \frac{q}{\kappa A \epsilon_0} = \frac{q}{\kappa 2\pi r L \epsilon_0}.$$

The potential difference between the capacitor's two conductors is then (using Eq. 26.25)

$$\begin{aligned} V_{21} &= -\int_{R_2}^{R_1} \vec{E} \cdot d\vec{\ell} = -\int_{R_2}^{R_1} E\, dr = -\frac{q}{2\pi\kappa\epsilon_0 L}\int_{R_2}^{R_1} \frac{dr}{r} \\ &= -\frac{q}{2\pi\kappa\epsilon_0 L}\left[\ln(r)\right]_{R_2}^{R_1} = -\frac{q}{2\pi\kappa\epsilon_0 L}\ln\left(\frac{R_1}{R_2}\right) \\ &= \frac{q}{2\pi\kappa\epsilon_0 L}\ln\left(\frac{R_2}{R_1}\right). \end{aligned}$$

Then using $q = CV$, we obtain

$$C = \frac{q}{V_{21}} = \frac{2\pi\kappa\epsilon_0 L}{\ln(R_2/R_1)}.$$

Putting in numbers, we calculate the capacitance per unit length to be

$$\begin{aligned} \frac{C}{L} &= \frac{2\pi\kappa\epsilon_0}{\ln(R_2/R_1)} = \frac{(6.283)(2.3)(8.85 \times 10^{-12}\,\mathrm{F/m})}{\ln(3.4/0.50)} \\ &= 6.7 \times 10^{-11}\,\mathrm{F/m} = 67\,\mathrm{pF/m}. ✔ \end{aligned}$$

❹ **EVALUATE RESULT** Example 26.3 found the capacitance of a coaxial cable (with air or vacuum between the inner and outer conductors) to be $C = (2\pi\epsilon_0 L)/\ln(R_2/R_1)$. Our expression is greater by a factor $\kappa = 2.3$. This makes sense, as we have seen in other examples that filling the space between a capacitor's two conductors with a nonconductor having dielectric constant κ increases the capacitance by a factor κ. We can also check online that the numerical value 67 pF/m agrees with tabulated values for the capacitance per unit length of commercial RG-6 television cable.

Guided Problem 26.4 Laptop power

The rechargeable batteries needed in laptop computers are both heavy and expensive. Moreover, the lifetime of many rechargeable batteries is relatively short—perhaps at best a few thousand charge-discharge cycles. Because of this, capacitors have been considered as one possible option to store the required energy. Capacitors are quickly charged, and because no chemical reaction is involved, they have a much longer lifetime than do batteries. Let us say your ultra-low-power laptop requires a potential difference of 8 V to run, and you'd like it to run for a minimum of 4 h using an average power of 1 W. Assume it is possible to "tap" a constant 8 V from a capacitor that has a potential difference greater than or equal to 8 V, but whose potential difference cannot exceed 48 V. (*a*) In order to use this capacitor in your laptop, what would its capacitance have to be? (*b*) If this were a parallel-plate capacitor, how large would the plate area have to be if the plates were separated by a Mylar layer 0.05 mm thick? Does this type of capacitor seem a feasible alternative to today's rechargeable batteries?

❶ **GETTING STARTED**

1. How much energy must be available for use?
2. Can the capacitor be fully discharged at the end of 4 h, or must it still be partially charged?

❷ **DEVISE PLAN**

3. Express the required capacitance in terms of initial and final potential differences and the required energy that must be supplied by the capacitor.
4. How does capacitance depend on surface area, separation distance, and dielectric constant in a parallel-plate capacitor?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

5. Work out numbers for the energy removed from the capacitor and for the energy needed to run the computer for 4 h. Do they agree?
6. Is the plate area feasible for a capacitor that must fit inside a laptop computer?
7. Modern dielectric materials and construction techniques (see, for example, Problem 79 in this chapter) allow capacitors on the order of 1 F to fit into a volume of $10^{-5}\,\mathrm{m}^3$. Would knowing this change your answer about feasibility?

Worked Problem 26.5 Earth-sized capacitor

What is the capacitance of Earth when you model it as a conducting sphere?

❶ **GETTING STARTED** It is hard to think of Earth as being part of an electronic device, in this case a gigantic capacitor, but let's accept that premise for the sake of argument. We know the definition of capacitance: the amount of charge that can be stored per unit of potential difference between the two conductors of a capacitor. If Earth is one conductor in our capacitor, what can act as the other conductor? That the potential of a spherical object is typically measured by defining infinitely far away as the position of zero potential gives us the clue we need: We should consider the other conductor to be an infinitely large spherical shell that is concentric with Earth.

❷ **DEVISE PLAN** We know from Eq. 26.1 that capacitance is given by $C = q/V_{\text{cap}}$, but in this problem q is the charge on Earth's surface, a quantity we do not know. To use Eq. 26.1, however, we also need to know the potential V_{cap} at Earth's surface. We found in Section 25.5 that the potential at some radial distance r from a charged particle is given by Eq. 25.21, $V(r) = q/4\pi\epsilon_0 r$. When we use this expression for V in Eq. 26.1, the two q factors cancel, and so our not knowing a value for q is not an obstacle here.

❸ **EXECUTE PLAN** In this problem, r in Eq. 25.21 is Earth's radius R_E. Substituting the expression for V given by Eq. 25.21 into Eq. 26.1 therefore yields

$$C_{\text{E}} = \frac{q}{(q/4\pi\epsilon_0 R_{\text{E}})} = 4\pi\epsilon_0 R_{\text{E}}$$

$$= 4\pi(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(6.4 \times 10^6\ \text{m}) = 0.71\ \text{mF.}\ \checkmark$$

❹ **EVALUATE RESULT** We see from the algebraic expression that the capacitance of Earth would increase if it had a larger radius. This increase is to be expected because the charge carriers on the surface would then be farther apart, and thus more charge could be stored on the surface. It is also gratifying to note that this is exactly the result we obtained in Example 26.4 for a spherical capacitor, $C = 4\pi\epsilon_0 R_1 R_2/(R_2 - R_1)$, when we make the outer sphere's radius infinite.

Guided Problem 26.6: Tin can capacitor

A certain capacitor consists of a solid cylindrical core surrounded by a metal shell. The shell is capped at both ends with metal covers, forming a connected metal surface with the shell but not the rod. The shell has radius R_{shell} and length L. The core is separated from the shell by a distance d and has length $L - 2d$. Assume that $d \ll R_{\text{shell}}$ and also that $d \ll L$. What is the capacitance of this capacitor?

❶ **GETTING STARTED**

1. Start by drawing a sketch of the capacitor, labeling all the given variables and indicating the quantity you must determine.
2. How is this capacitor similar to a coaxial capacitor? How does it differ from a coaxial capacitor?
3. Are the metal covers on the ends at the same potential as the metal shell?
4. What are the appropriate surface areas of the conductors?
5. Is the distance between core and shell (including covers) constant over all but a negligible portion of the capacitor?

❷ **DEVISE PLAN**

6. What is the expression for capacitance in a coaxial capacitor?
7. What distances correspond to the two radii in this expression?
8. How can you account for the effects of the end caps on the capacitance?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

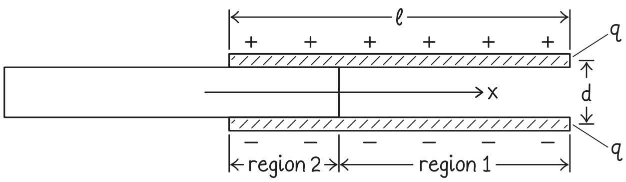
9. Pick some numerical values and compare this capacitance with that of a roll-up capacitor of similar dimensions from Worked Problem 26.1. What do you conclude?

Worked Problem 26.7 Variable capacitance

A variable capacitor can be constructed by inserting a slab of dielectric material partway between a pair of parallel plates. One such capacitor has plates of length ℓ and area A, where the plates are separated by a distance d. If the dielectric constant of the inserted material is κ, what is the capacitance as a function of how far into the space between the plates the dielectric is inserted?

❶ GETTING STARTED Figure WG26.4 shows our given information. Aligning the x axis along the plate length allows us to use x to represent the inserted length. Our drawing shows that this capacitor has two regions: region 1 with plate length $\ell - x$ and air between the plates and region 2 with plate length x and the dielectric between the plates. Because of the presence of the dielectric, we know that the electric field magnitude is not the same in the two regions. We probably need to use the general definition of capacitance, $C = q/V$, and figure out, for a given potential difference V, what surface charge is required on the plates above and below each of these two regions.

Figure WG26.4



❷ DEVISE PLAN Because there are two regions in this capacitor, we have to be careful when we use our capacitance equation. We want to know how much free charge $q_{\text{cap,free}}$ the capacitor can store for a given potential difference across the plates. We know that the potential is the same everywhere on each plate, and so the potential difference V_{cap} is the same everywhere across the plates. Because this potential difference is related to the electric field in each region by $V_{\text{cap}} = Ed$ and because d is constant, we conclude that the electric field must be the same in both regions. Because a dielectric decreases the electric field by a factor κ for a fixed density of free charge on the plates, the free-charge density must be greater by a factor κ in region 2 (dielectric-filled region) than in region 1 (air-filled region). The free charge $q_{1\text{free}}$ or $q_{q2\text{free}}$ on the plates in each region is the free-charge density (charge per unit area) in that region multiplied by the corresponding surface area. The charge $q_{\text{cap, free}}$ stored by the capacitor is the sum of $q_{1\text{free}}$ and $q_{2\text{free}}$. Then taking the ratio of $q_{\text{cap, free}}$ and V_{cap} will give us the capacitance for this variable capacitor.

❸ EXECUTE PLAN Because the plate area is A, the plate width is $w = A/\ell$. The plate area for region 1 is $A_1 = (\ell - x)w$, and the plate area for region 2 is $A_2 = xw$. There is no dielectric in region 1, and so in this region Eq. 26.12 tells us that the electric field is

$$E_1 = \frac{V_{\text{cap}}}{d} = E_{1\text{free}} = \frac{\sigma_{1\text{free}}}{\epsilon_0} = \frac{q_{1\text{free}}}{\epsilon_0 A_1},$$

which leads us to

$$q_{1\text{free}} = \frac{V_{\text{cap}}}{d}\epsilon_0 A_1. \tag{1}$$

From Eq. 26.14, the field in region 2 is

$$E_2 = \frac{V_{\text{cap}}}{d} = E_{2\text{free}} - E_{2\text{bound}} = \frac{\sigma_{2\text{free}} - \sigma_{2\text{bound}}}{\epsilon_0}$$
$$= \frac{q_{2\text{free}} - q_{2\text{bound}}}{\epsilon_0 A_2}.$$

But from Eq. 26.17 ($q_{\text{free}} - q_{\text{bound}} = q_{\text{free}}/\kappa$), we see that this last equation can be rewritten in the form

$$\frac{V_{\text{cap}}}{d} = \frac{q_{2\text{free}}}{\kappa\epsilon_0 A_2}.$$

The free charge on the plates is therefore

$$q_{\text{cap, free}} = q_{1\text{free}} + q_{2\text{free}} = \frac{V_{\text{cap}}}{d}\epsilon_0 A_1 + \frac{V_{\text{cap}}}{d}\kappa\epsilon_0 A_2$$
$$= \frac{V_{\text{cap}}}{d}\epsilon_0[(\ell - x)w + \kappa x w],$$

which means that the capacitance of our variable capacitor is given by

$$C(x) = \frac{q_{\text{cap, free}}}{V_{\text{cap}}} = \frac{\epsilon_0 w}{d}[(\ell - x) + \kappa x]$$
$$= \frac{\epsilon_0 w}{d}[\ell + (\kappa - 1)x] = \frac{\epsilon_0 A}{d}\left[1 + (\kappa - 1)\frac{x}{\ell}\right]. ✔$$

PRACTICE

❹ EVALUATE RESULT We know that dielectrics are used in capacitors to increase capacitance. Therefore $C(x)$ should increase as we insert more of the dielectric into the space between the plates (increase x). This is just what our result says because $\kappa - 1 > 0$ (remember, κ is always greater than unity). We should also check the limits $x = 0$ and $x = \ell$. With $x = 0$ (no dielectric), our equation reduces to the result from Example 26.2 for a dielectric-free parallel-plate capacitor: $C_0 = \epsilon_0 A/d$. When the capacitor is entirely filled with dielectric, $x = \ell$ and our equation becomes

$$C(\ell) = \frac{\epsilon_0 A}{d}\left[1 + (\kappa - 1)\frac{\ell}{\ell}\right] = \frac{\epsilon_0 A}{d}[1 + \kappa - 1]$$

$$= \kappa\frac{\epsilon_0 A}{d} = \kappa C_0,$$

which agrees with Eq. 26.11 giving the capacitance of a capacitor entirely filled with a dielectric.

Guided Problem 26.8 Vacuum Capacitors

Devices used in high-power radio-frequency applications require capacitors that can store large quantities of charge without electrical breakdown. Dielectric materials may be undesirable in these applications because of the materials' finite breakdown thresholds. Instead, high storage capacity is achieved with *vacuum capacitors*, which consist of conducting plates separated by a fairly large distance and sealed inside an evacuated canister so that there is vacuum between the plates rather than air. One problem with these capacitors is that the vacuum is not perfect (that is, some air molecules are present); another problem is that a sufficiently large electric field can ionize the metal plates, ripping off electrons from the plates and leading to electrical breakdown. Suppose the largest electric field that can be supported by a vacuum capacitor is 10 times greater than the breakdown threshold of air. What is the maximum potential difference that can be applied to a 1.0-nF parallel-plate vacuum capacitor with a plate separation of 1.0 mm? What is the maximum charge that can be stored without discharging?

❶ GETTING STARTED

1. How are potential difference and electric field related for a parallel-plate capacitor?
2. Where can you find the value for the maximum electric field in air and then for this capacitor?

❷ DEVISE PLAN

3. How are stored charge and potential difference related for a capacitor?
4. What dielectric constant should be used for a vacuum capacitor?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

5. Is the maximum potential difference quite large compared with potentials encountered in commonplace low-power devices?
6. Is the maximum stored charge fairly large?

Answers to Review Questions

1. Potential difference is a measure of the work per unit charge done by the electric field in moving a charged test particle (not part of the system) from one location in the system to another. The system's electric potential energy is determined by the configuration of the charged particles that make up the system.
2. The potential energy depends on the amount of charge on the objects and on their separation distance.
3. Electric potential energy is stored in the system's electric field, in any region of space where that field exists.
4. A Van de Graaff generator produces a very large separation of charge between Earth's surface ("ground") and a metal sphere attached to a nonconducting support, and hence a very large potential difference.
5. A capacitor is a pair of conducting objects separated by a nonconducting material or vacuum. Any such pair of objects stores electric potential energy when charge has been transferred from one object to the other.
6. A parallel-plate capacitor is an arrangement of two parallel conducting plates that have the same area A and are separated by a distance d. When d is small relative to the lateral dimensions of the plates, the electric field is approximately localized and uniform between the charged plates and zero outside them.
7. The magnitude of the charge on each plate is proportional to the potential difference between the plates. For a parallel-plate capacitor, this is because the potential difference equals the product of the electric field strength and the plate separation distance, and the field strength is proportional to the surface charge density on either plate.
8. The quantity of charge on a plate varies directly with plate surface area and inversely with separation distance.
9. Electrical breakdown occurs when the charge on the plates is great enough to produce an electric field sufficient to ionize the air (or other material) between the plates.
10. A dielectric is a nonconducting material. *Polar* dielectrics are made up of molecules that have a permanent dipole moment, whereas *nonpolar* dielectrics consist of molecules that do not have a dipole moment in the absence of an electric field. When either type is placed in an external

electric field, the centers of positive and negative charge in the molecules are displaced, producing an induced dipole moment. Molecules in polar dielectrics can also align themselves with the external electric field.

11. In a uniform external field, no surplus charge is induced on any volume that lies entirely inside the material, but uncompensated induced charge densities appear on the surfaces.
12. Bound charge is the surplus of charge in polarized matter due to charge carriers that are bound to atoms and cannot move freely within the bulk of the material. Free charge is the surplus of charge due to charge carriers that can move freely within the bulk of a material.
13. The direction of the electric field $\vec{E}_{\text{cap}}$ due to the plates is from the positive plate to the negative plate, but the direction of the electric field $\vec{E}_{\text{dielec}}$ due to the polarized dielectric between the plates is in the opposite direction: from the positive induced surface charge near the negative plate to the negative induced surface charge near the positive plate. The vector sum of $\vec{E}_{\text{cap}}$ and $\vec{E}_{\text{dielec}}$ gives a field magnitude for the dielectric-filled capacitor that is smaller than the magnitude of the capacitor without the dielectric.
14. The field in the dielectric stores more energy than the vacuum field. Work must be done by an external electric field on the dielectric in order to separate the positive and negative charge carriers, and this work done increases the electric potential energy stored in the dielectric's electric field, so that the amount of energy stored is greater than in a field of equal magnitude that exists in a vacuum.
15. A battery is an assembly of one or more voltaic cells that separates charge carriers by converting energy released in chemical reactions into electric potential energy.
16. Emf is the work done per unit charge by nonelectrostatic interactions in a charge-separating device (such as a battery) in order to separate positive and negative charge carriers in the device.
17. The capacitance is the ratio of the magnitude of the charge on either conductor that forms the capacitor and the magnitude of the potential difference between the conductors.
18. In the absence of a dielectric, capacitance depends on only the size and geometric properties of a capacitor—that is, on the shape of the conductors making up the capacitor and on the separation of the conductors. The capacitance also depends on the properties of the dielectric material inserted between the conductors.
19. The unit of capacitance is the farad, which is 1 C/V. Because 1 C is an enormous quantity of charge relative to the quantity of charge on conductors in electronic devices, capacitances in the range of microfarads to picofarads are more commonly used.
20. The electric potential energy is proportional to the square of the charge on each conductor: $U^E = q^2/2C$ (Eq. 26.4).
21. The electric potential energy is proportional to the square of the potential difference: $U^E = CV_{\text{cap}}^2/2$ (Eq. 26.4).
22. Energy density is the electric potential energy stored in an electric field per unit volume of the field: $u_E = U^E/\text{field volume}$. The energy density for a given field is proportional to the field magnitude squared, as given (for air or vacuum) by Eq. 26.6: $u_E = \frac{1}{2}\epsilon_0 E^2$.
23. In an ideal charge-separating device, none of the energy associated with the work done by nonelectrostatic interactions is dissipated, and the potential difference between the terminals is equal to the emf. In a nonideal device, some energy is dissipated inside the device and thus not available to be turned into electrostatic work; now the potential difference between the device's terminals is smaller than the emf. (See the discussion in the paragraphs that surround Equation 26.8.)
24. The dielectric constant for the material is the potential difference between a capacitor's conductors without any of the dielectric material between the conductors divided by the potential difference between the conductors with the dielectric material completely filling the space between them.
25. The constant for water is greater because water molecules are polar and can align themselves with the electric field, whereas the molecules in paper and mica are nonpolar. The induced polarization in liquid water is therefore much greater than that in paper or mica.
26. The sign of the bound charge on the dielectric surface is opposite the sign of the free charge on the plate, and the quantity of bound charge is smaller than the quantity of free charge. The relationship is given by Eq. 26.18: $q_{\text{bound}} = (\kappa - 1)(q_{\text{free}})/\kappa$.
27. The stored energy must decrease because as the dielectric comes near the capacitor, the capacitor does positive work on it, pulling it into the space between the conductors.
28. We use Gauss's law for determining the electric field inside a dielectric by substituting $q_{\text{free}}[(\kappa - 1)/\kappa]$ (Eq. 26.18) for q_{bound}. This substitution gives us an expression that does not contain q_{bound}. The result is that Gauss's law may be applied using only the free charge enclosed by the Gaussian surface and the electric field inside the dielectric material, Eq. 26.25.
29. Gauss's law in a dielectric material is $\oint \kappa\vec{E}\cdot d\vec{A} = q_{\text{free,enc}}/\epsilon_0$ (Eq. 26.25), where κ is the dielectric constant for the dielectric material and $q_{\text{free,enc}}$ is that portion of the free charge that is enclosed by the chosen Gaussian surface. When no dielectric is present, $\kappa = 1$ and this expression reduces to the Chapter 24 form of Gauss's law, $\oint \vec{E}\cdot d\vec{A} = q_{\text{enc}}/\epsilon_0$ (Eq. 24.8).

Answers to Guided Problems

Guided Problem 26.2 $C = \dfrac{\epsilon_0\kappa\ell w}{d} = 0.3\ \mu\text{F}.$

Guided Problem 26.4 (*a*) $C = \dfrac{2P\Delta t}{V_{\text{i}}^2 - V_{\text{f}}^2} = 13\ \text{F};$

(*b*) $A = \dfrac{Cd}{\epsilon_0\kappa} = 2.2 \times 10^7\ \text{m}^2 = (4.7\ \text{km})^2.$

Guided Problem 26.6 $C = \dfrac{\epsilon_0 A}{d} = \dfrac{2\pi\epsilon_0 R_{\text{shell}}}{d}(L + R_{\text{shell}}).$

Guided Problem 26.8 $V_{\max} = E_{\max}d = 30\ \text{kV};\ Q_{\max} = CV_{\max} = 30\ \mu\text{C}.$

Guided Practice by Chapter

27 Magnetic Interactions

Review Questions

Answers to these questions can be found at the end of this chapter.

27.1 Magnetism

1. What is a magnet? What is a magnetic material?
2. What are magnetic poles? How are the two types of magnetic poles defined?
3. How do magnetic poles interact with each other?
4. What is an elementary magnet? How does a model based on elementary magnets explain magnetization?

27.2 Magnetic fields

5. How is the concept of a magnetic field used to describe the action of a magnet?
6. What information about a magnetic field is conveyed by its magnetic field line pattern?
7. How does the magnetic field line flux through a closed surface depend on the number and strength of the elementary magnets enclosed by the surface?

27.3 Charge flow and magnetism

8. This section describes another source of magnetic fields besides a magnet. What is this source, and how is it defined?
9. Describe the magnetic field produced near a straight current-carrying wire.
10. In what direction would electrons need to flow in order to produce the same magnetic field as an equal flow of positive ions moving from left to right?
11. When electrons flow upward through a wire, how do we describe the direction of the corresponding current?
12. If you are looking at the face of a clock and see a current toward you from the center of the clock, do the magnetic field lines encircle the current in a clockwise or counterclockwise sense?
13. When a current-carrying wire is aligned perpendicular to a bar magnet, what is the relationship between the direction of the magnetic force exerted by the magnet on the wire and the direction of the magnetic field due to the magnet (call this the *external* magnetic field to distinguish it from the magnetic field due to the current in the wire)? What is the relationship between the direction of the magnetic force exerted by the magnet on the wire and the current direction?

27.4 Magnetism and relativity

14. Why should we expect any observed magnetic interaction to depend on the reference frame of the observer?
15. If the magnetic force exerted by a current-carrying wire is merely a relativistic correction to the internal electric forces in the wire, why is the magnetic force exerted by the wire so readily observable?
16. To an observer at rest relative to a current-carrying wire, how does the linear charge density of the fixed ions in the wire compare with the linear charge density of the moving electrons? How do these charge densities compare according to an observer moving along with the electrons?

27.5 Current and magnetism

17. For a current-carrying wire placed in an external magnetic field, for what orientation is the magnitude of the magnetic force exerted by the external field on the wire a maximum? For what orientation is it a minimum?
18. On what factors does the magnitude of the magnetic force exerted by an external magnetic field on a current-carrying wire depend?
19. A current through a wire is in the positive x direction in the presence of an external magnetic field that points in the positive y direction. In what direction does the magnetic force on the wire point?

27.6 Magnetic flux

20. For a flat surface located in a uniform magnetic field, how is the magnetic flux through the surface defined?
21. What is Gauss's law for magnetism, and what does it imply about magnetic sources?
22. How does the magnetic flux through one surface bounded by a loop compare with the magnetic flux through a different surface bounded by the same loop?

27.7 Moving particles in electric and magnetic fields

23. The magnetic force exerted by an external magnetic field on a current-carrying wire is a combination of what?
24. If a charged particle is moving in a magnetic field, what is the direction of the magnetic force exerted by the field on the particle?

27.8 Magnetism and electricity unified

25. Why does an observer moving along with electrons in a current-carrying wire measure the linear charge density of the fixed positive ions in the wire as being greater than the linear charge density of the electrons?
26. In the Earth reference frame, a positively charged particle moves parallel to a current-carrying wire. What is the difference between how an observer moving along with electrons in the wire interprets the force exerted by the wire on the particle and how an observer in the Earth reference frame interprets that force?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The current needed in a long straight wire to produce, 1 m away from the wire, a magnetic field magnitude equal to the magnitude of the magnetic field due to Earth (C, Q)
2. The magnitude of the flux of Earth's magnetic field through the top of your desk (C, K, F)
3. The magnitude and direction of the magnetic force exerted by Earth on an electron moving west along Earth's equator at 10^5 m/s (C, L, G)
4. The magnitude of the magnetic field contributed by a single elementary magnet in a small iron bar magnet (A, I, M, U)
5. The magnitude of a uniform magnetic field required to keep a proton moving in the field at 10^6 m/s in a circular orbit having the same radius as a basketball (L, P, T)
6. The magnitude of a uniform magnetic field needed to keep an electron moving in the field in a circular orbit if the electron is to complete 1 revolution each microsecond (D, H, L)
7. The minimum current needed to "float" a piece of 18-gauge copper wire in a horizontal magnetic field of magnitude 0.5 T (V, R, N, J, B)
8. The maximum acceleration of a copper rod that is 100 mm long has a radius of 5 mm and carries a current of 100 A in a uniform magnetic field of magnitude 1 T (O, J, S, E)

Hints

A. What is the magnitude of the magnetic field due to a small bar magnet?
B. What is the mass of a length ℓ of this wire?
C. What is the magnitude of the magnetic field near Earth's surface?
D. What is the mass of an electron?
E. What maximum magnetic force is available?
F. What is the area of an average desktop?
G. In what direction does Earth's magnetic field point along the equator?
H. How is the period related to the magnetic field strength?
I. What is the volume of a small bar magnet?
J. What is the mass density of copper?
K. What is the angle between the direction of Earth's magnetic field and the horizontal?
L. What is the magnitude of the charge carried by this particle?
M. What is the mass density of iron?
N. What is the volume of a length ℓ of this wire?
O. What is the volume of the rod?
P. What is the radius of a basketball?
Q. How are magnetic field magnitude B and current I related for a long straight wire?
R. What is the diameter of 18-gauge wire?
S. What is the mass of the rod?
T. What is the mass of a proton?
U. What is the mass of an iron atom?
V. What forces must balance to float the wire?

Key (all values approximate)

A. 0.01 T, from Table 27.1; **B.** $(7 \times 10^{-3}\,\text{kg/m})\ell$; **C.** 5×10^{-5} T; **D.** 9×10^{-31} kg; **E.** $I\ell B \sim 10$ N; **F.** 2 m^2; **G.** horizontal and north; **H.** $T = 2\pi m/qB$; **I.** 1×10^{-5} m^3; **J.** 9×10^3 kg/m^3; **K.** this varies in the continental United States from 50° to 75°, so say 60°; **L.** elementary charge, $e = 1.6 \times 10^{-19}$ C; **M.** 8×10^3 kg/m^3; **N.** $\pi r^2 \ell \approx (8 \times 10^{-7}\,\text{m}^2)\ell$; **O.** 8×10^{-6} m^3; **P.** 0.1 m; **Q.** $B = 2kI/c_0^2$; **R.** 1×10^{-3} m; **S.** 0.07 kg; **T.** 1.7×10^{-27} kg; **U.** 9×10^{-26} kg; **V.** gravitational and magnetic.

Worked and Guided Problems

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

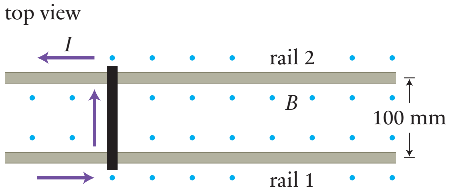
Worked Problem 27.1. Rail gun

A *rail gun* is a device used to accelerate projectiles without the use of explosives. Figure WG27.1 shows an overhead view of one such gun. A crossbar slides on two identical rails that are 100 mm apart. Charge carriers flow along one rail, into the crossbar, and then along the other rail. A uniform external magnetic field pointing up out of the page accelerates the crossbar. (*a*) In which direction is the crossbar

PRACTICE

accelerated? (*b*) What is the coefficient of kinetic friction μ_k between rails and crossbar if the current is 10 A, the magnetic field magnitude is 0.12 T, and the crossbar has a mass of 2.0 kg and moves at constant speed once it has been accelerated?

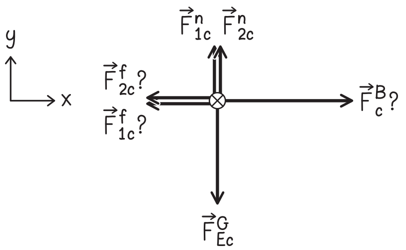
Figure WG27.1



❶ **GETTING STARTED** A current-carrying wire—or, in this case, a current-carrying crossbar—is subject to a force exerted by the external magnetic field. The force due to this field accelerates the crossbar, and our first task is to determine whether this acceleration is to the left or to the right in Figure WG27.1. (We know the acceleration cannot be up out of the page or down into the page because the magnetic force must be perpendicular to the magnetic field, which points out of the page.) Because the crossbar slides on the rails, we know it must experience a frictional force that opposes the magnetic force. We're given the magnitude and direction of the uniform magnetic field, the separation distance for the rails, values for the current and the crossbar mass, and the fact that the final crossbar speed is constant. We use this information for our second task: calculating the coefficient of kinetic friction.

❷ **DEVISE PLAN** For part *a,* the direction of the magnetic force $\vec{F}^B_{\rm c}$ exerted by the external field on the crossbar determines in which direction the crossbar is accelerated. We can determine the direction of this force from the right-hand force rule, $\vec{F}^B = I\vec{\ell} \times \vec{B}$. For part *b,* as for all problems involving forces, we begin with a free-body diagram, this one for the crossbar viewed from the side, end-on (Figure WG27.2).

Figure WG27.2



We don't yet know the direction of the magnetic force $\vec{F}^B_{\rm c}$ exerted on the crossbar, and so we arbitrarily draw that force arrow to the right and add a question mark to remind us that its direction may have to be changed. Because the direction of the frictional forces opposes the magnetic force, we show $\vec{F}^{\rm f}_{1\rm c}$ and $\vec{F}^{\rm f}_{2\rm c}$ to the left, but also add question marks. We also show the downward gravitational force and two upward normal forces, one exerted by each rail.

We can use Eq. 27.8, $\vec{F}^B_{\rm c} = I\vec{\ell} \times \vec{B}$, in its scalar form, $F^B_{\rm c} = |I|\ell B \sin\theta$ (Eq. 27.7), to get the magnetic force in terms of our given numerical values. Each frictional force exerted by a rail on the crossbar is related to the upward normal force exerted by each rail on the crossbar by Eq. 10.55: $F^{\rm f}_{2\rm c} = \mu_k F^n_{2\rm c}$ and $F^{\rm f}_{1\rm c} = \mu_k F^n_{1\rm c}$, where μ_k is the coefficient of kinetic friction.

❸ **EXECUTE PLAN** (*a*) Applying the right-hand force rule, we position our right hand above the crossbar in Figure WG27.1, with fingers pointing up in the direction of the current and the palm toward us so that the fingers can curl toward our face, in the direction of the external magnetic field. The thumb points to the right, telling us this is the direction of $\vec{F}^B_{\rm c}$. We guessed correctly! ✔

The magnetic force accelerates the crossbar to the right. The two frictional forces $\vec{F}^{\rm f}_{2\rm c}$ and $\vec{F}^{\rm f}_{1\rm c}$ therefore must be leftward. Because these frictional forces and the magnetic force oppose each other, they can cancel and allow the crossbar to move at constant velocity.

(*b*) Let us work with the *x* and *y* components of the forces, using the coordinate axes shown in Figure WG27.2. In the *x* direction, $a_x = 0$, so the vector sum of the forces exerted on the crossbar becomes

$$\Sigma F_x = ma_x$$

$$F^B_{\rm c} - (F^{\rm f}_{2\rm c} + F^{\rm f}_{1\rm c}) = 0. \qquad (1)$$

Because the rails are identical and the crossbar is placed symmetrically, $F^{\rm f}_{2\rm c} = F^{\rm f}_{1\rm c}$ and $F^n_{2\rm c} = F^n_{1\rm c}$, which means we can simplify our notation to $F^{\rm f}_{\rm rc}$ for the frictional force exerted by *each* rail on the crossbar and $F^n_{\rm rc}$ for the normal force exerted by *each* rail on the crossbar. When we make these changes in notation and substitute $I\ell B \sin\theta$ for $F^B_{\rm c}$, Eq. 1 becomes

$$I\ell B \sin\theta - 2F^{\rm f}_{\rm rc} = I\ell B \sin\theta - 2\mu_k F^n_{\rm rc} = 0.$$

Because the magnetic field and the current through the crossbar run at right angles to each other, $\theta = 90°$, $\sin 90° = 1$, and we have

$$I\ell B - 2\mu_k F^n_{\mathrm{rc}} = 0. \tag{2}$$

In the y direction, we have for the identical rails $F^n_{2\mathrm{c}} = F^n_{1\mathrm{c}} = F^n_{\mathrm{rc}}$. Newton's second law in this direction gives us

$$\sum F_y = ma_y = 0$$
$$2F^n_{\mathrm{rc}} - F^G_{\mathrm{Ec}} = 0$$
$$F^n_{rc} = mg/2. \tag{3}$$

Combining Eqs. 2 and 3 and rearranging, we obtain

$$\mu_k = \frac{I\ell B}{mg} = \frac{(10\text{ A})(0.100\text{ m})(0.12\text{ T})}{(2.0\text{ kg})(9.8\text{ m/s}^2)} = 6.1 \times 10^{-3}. ✔$$

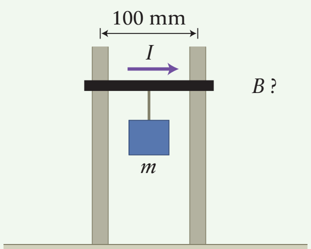
❹ **EVALUATE RESULT** That the crossbar is accelerated to the right makes sense because the magnetic force on the crossbar points to the right, as we found by applying the right-hand force rule to Figure WG27.1. Our μ_k value seems like a very small coefficient of friction, but that's exactly what you want in a rail gun because its purpose is to move objects. The idea is that at currents much greater than the 10 A we have here (1000 A, say), you can accelerate the crossbar to very high speeds.

Guided Problem 27.2 Magnetic scale

Figure WG27.3 shows a sensitive device for measuring an object's mass. The object hangs by a string from a horizontal rod through which an electric current is directed from left to right. The rod is free to slide up and down a pair of vertical columns with negligible friction, but the rod's orientation remains fixed. The entire unit is placed in an external magnetic field, and the current in the circuit is adjusted until the rod and object are motionless. Once this equilibrium has been attained, the object's mass is calculated from the magnitude of the magnetic force required to balance the gravitational force on the object.

In which direction must the external magnetic field point—into or out of the page in Figure WG27.3—in order for the scale to work? What current is required to balance the rod if the combined mass of the object and the rod is 0.157 kg and the magnetic field has magnitude 0.150 T?

Figure WG27.3



❶ **GETTING STARTED**

1. Describe the problem in your own words. What information are you given, and what quantities must you determine using the given information?
2. Draw a free-body diagram—but of the rod, the object, or something else?
3. Which concepts and principles concerning magnetism apply here?

❷ **DEVISE PLAN**

4. What is your plan for determining the magnetic field direction?
5. For a given current value, which equations allow you to express the unknown current in terms of known quantities?
6. What is the angle between the electric current in the rod and the magnetic field?

❸ **EXECUTE PLAN**

7. Determine the field direction.
8. Work through the algebra and solve for the desired unknown quantity; then substitute known values to get a numerical answer.

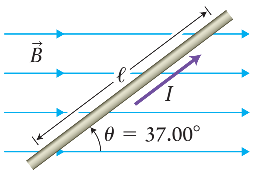
❹ **EVALUATE RESULT**

9. Does your direction for the magnetic field make sense?
10. Is your value for the current large or small? Compare it to maximum household current by noting the capacity of the circuit breakers or fuses in your home.

Worked Problem 27.3 Torque on a Wire

A straight current-carrying wire is oriented at an angle of 37.00° to a uniform external magnetic field of magnitude 0.0110 T (Figure WG27.4). The current in the wire is 10.0 A, and the portion within the field has length $\ell = 790$ mm. (*a*) What is the direction of the magnetic force exerted by the external field on the wire? (*b*) If the bottom end of the wire is fixed but the rest of the wire is free to move, what is the torque about the fixed end of the wire? In which direction does the torque cause the wire to rotate?

Figure WG27.4



❶ **GETTING STARTED** We must determine three things about a current-carrying wire placed in an external magnetic field: the direction of the force exerted by the magnetic field on the wire, the torque this force causes about one end of the wire, and the direction in which the wire tends to rotate because of this torque. For part *a*, we can determine the magnetic force direction from the right-hand force rule. For part *b*, we can use the principles we learned in Chapter 12 to determine the magnitude and direction of the torque due to the magnetic force on the wire.

❷ **DEVISE PLAN** The force $\vec{F}^B_{\rm w}$ exerted by the external field on the wire is given by Eqs. 27.8 and 27.7:

$$\vec{F}^B_{\rm w} = I\vec{\ell} \times \vec{B} \quad \text{or} \quad F^B_{\rm w} = I\ell B \sin\theta. \tag{1}$$

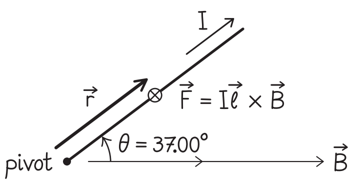
Each small wire segment of length $d\ell$ experiences the same magnitude and direction of force $d\vec{F} = Id\vec{\ell} \times \vec{B}$, so the force computed in Eq. 1 is uniformly exerted along the length of the wire. Thus, when calculating the torque, we can place the point of application of the force at the wire's midpoint—just as we take the point of application of the force of gravity to be at an object's center of mass. The torque is given by Eq. 12.1, $\tau = rF\sin\theta$, where θ is the angle between the vector $\vec{r}$ (that locates the point of application of the force relative to the rotation axis) and the line of action of the force.

❸ **EXECUTE PLAN** (*a*) Applying the right-hand force rule, we align our right hand along the current direction in Figure WG27.4 in such a way that we can curl the fingers toward the field lines representing $\vec{B}$. When we do this, our thumb points into the page, which is the direction of $\vec{F}^B_{\rm w}$. ✔

(*b*) In working with the torque in this problem, we have to be careful about notation because θ in Eq. 1 is not the same as θ in Eq. 12.1 for the torque. To keep things straight, we keep θ as the angle between the magnetic field direction and the current direction, and we change Eq. 12.1 to $\tau = rF\sin\phi$, so that ϕ is the angle between $\vec{F}^B_{\rm w}$ (into the page) and the vector $\vec{r}$ pointing from the rotation axis (the wire's fixed end) to the point of application of $\vec{F}^B_{\rm w}$ (Figure WG27.5). Using these angle symbols, Eq. 12.1 gives us

$$\begin{aligned}\tau &= rF\sin\phi = r(I\ell B\sin\theta)\sin\phi \\ &= \frac{\ell}{2}(I\ell B\sin\theta)(\sin 90°) = \tfrac{1}{2}I\ell^2 B\sin\theta \\ &= \tfrac{1}{2}(10.0\ \text{A})(0.790\ \text{m})^2(0.0110\ \text{T})(\sin 37.00°) \\ &= 0.0207\ \text{N}\cdot\text{m}. ✔\end{aligned}$$

Figure WG27.5



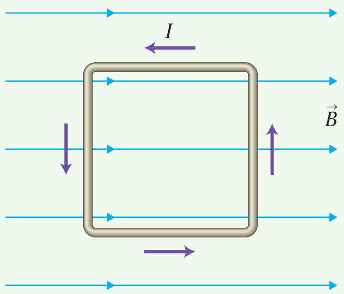
Because $\vec{F}^B_{\rm w}$ is into the page and the wire initially lies in the plane of the page, the free end of wire tends to rotate into the page. ✔

❹ **EVALUATE RESULT** Our expression above for the torque is proportional to B. This makes sense, given that a greater magnetic field results in a proportionally greater force, and thus a proportionally greater torque. The torque is zero when θ is 0 or 180°, which makes sense because there is no magnetic force on a wire that runs parallel to the magnetic field: The magnetic force (and thus the torque) is greatest when the wire is perpendicular to the field. It also makes sense that the torque is proportional to ℓ^2: one factor of ℓ for the magnetic force ($\vec{F}^B = I\vec{\ell} \times \vec{B}$) and a factor of $\ell/2$ for the lever arm in the definition of torque. We also note that newton-meters are the correct units for torque.

Guided Problem 27.4 Torque on a Current Loop

A square loop of wire carrying a 1.7-A current is oriented so that the plane of the loop is parallel to a uniform external magnetic field as shown in Figure WG27.6. Each straight segment of the loop is 10 mm long, and the magnetic field magnitude is 0.23 T. Name all of the forces exerted on the loop. Compute the vector sum of these forces. Does the loop experience a torque? If it does, determine the direction of rotation and the torque magnitude. If the loop does not experience a torque, explain why not.

Figure WG27.6



1 GETTING STARTED

1. State the problem in your own words. How do the magnetic field and current affect the loop?
2. Because forces are involved, draw a free-body diagram. Apply the right-hand force rule to determine the direction of the force on each straight segment of the loop.

2 DEVISE PLAN

3. Examine your free-body diagram to determine whether the loop experiences a torque. Use the right-hand rule for torque direction either to determine the torque direction or to conclude that there is no torque.
4. Decide which algebraic expression you can use to calculate the vector sum of the forces and, if appropriate, the torque magnitude.

3 EXECUTE PLAN

5. Substitute numbers to get numerical answers.

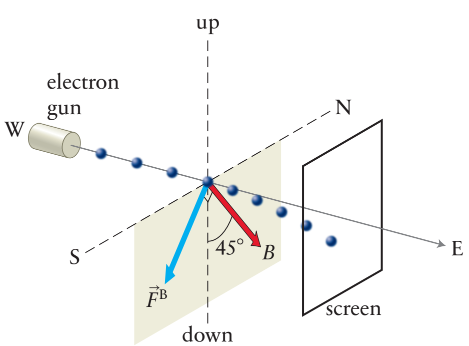
4 EVALUATE RESULT

6. Are your force and torque magnitudes reasonable? How do your calculated values compare with the values that you expected?
7. Is the torque direction what you expect?

Worked Problem 27.5. Bad picture?

Televisions manufactured during the 20th century work by shooting electrons at a screen to create the image that you see. The key component of these televisions is a *cathode ray tube* (CRT) that sprays a stream of electrons onto the screen (Figure WG27.7). The electrons are emitted from an *electron gun* located at the back of the CRT and then steered to various parts of the screen by means of variable magnetic fields. But any other magnetic field could also deflect these electrons. (*a*) Estimate the maximum deflection, due to Earth's magnetic field, of an electron aimed at the center of the screen 300 mm away from the electron gun emitting electrons that travel at 3.0×10^{7} m/s. Assume that Earth's magnetic field is directed northward, is oriented at an angle of 45° below the horizontal, and has a magnitude of 3.0×10^{-5} T. (*b*) How much, and in which direction, is the image on the screen shifted by Earth's magnetic field when the screen faces east? (*c*) What if the screen faces north?

Figure WG27.7



PRACTICE

❶ GETTING STARTED We are told that the electron beam in a television CRT might be deflected by Earth's magnetic field, and we are asked to determine the greatest possible deflection as each electron travels 300 mm through the field. We must then determine the magnitude and direction of this deflection for two possible orientations of the CRT.

❷ DEVISE PLAN The force exerted by a magnetic field on a charged particle—an electron in this case—moving through the field is given by Eq. 27.19, $\vec{F}^B = q\vec{v} \times \vec{B}$. This force is always perpendicular to the direction in which the particle moves, so as each electron is deflected, the direction of the magnetic force exerted on it changes. To simplify the problem, we will assume that the electron isn't deflected very much. We can then make the approximation that the force direction is constant and points perpendicular to the electron's initial direction. Then we can use Newton's second law and kinematics with constant acceleration to determine the electron's deflection, as we would do for projectile motion under constant gravitational acceleration. (We will check at the end whether the small-deflection approximation is justified.) First, however, we need to determine the orientation of the television that gives maximum deflection.

We assume that we can ignore the effect of gravity. Because the spatial directions in this problem are somewhat complicated, we will first write out $\vec{F}^B = q\vec{v} \times \vec{B}$ in vector components, then use the right-hand force rule as a check of our results.

❸ EXECUTE PLAN (*a*) We will define Cartesian coordinates such that $\hat{\imath}$ points east, $\hat{\jmath}$ points north, and $\hat{k}$ points up. We can check with the right-hand rule for vector products that $\hat{\imath} \times \hat{\jmath} = \hat{k}$, as the math requires. Then we can write Earth's magnetic field as $\vec{B} = (\hat{\jmath}\cos 45° - \hat{k}\sin 45°)B_0$, which points north and 45° below the horizontal plane, with magnitude $B_0 = 3.0 \times 10^{-5}$ T. The electron beam travels in the same direction as the television screen faces. Defining θ_C to be the compass direction (0° for north, 90° for east) in which the electrons travel, we can write the electron velocity as $\vec{v} = (\hat{\jmath}\cos\theta_C + \hat{\imath}\sin\theta_C)v_0$, with magnitude $v_0 = 3.0 \times 10^7$ m/s. Now we write out the components of $\vec{F}^B = q\vec{v} \times \vec{B}$:

$$F_x^B = q(v_yB_z - v_zB_y) = (-ev_0B_0)(-\cos\theta_C \sin 45°)$$

$$F_y^B = q(v_zB_x - v_xB_z) = (-ev_0B_0)(\sin\theta_C \sin 45°)$$

$$F_z^B = q(v_xB_y - v_yB_x) = (-ev_0B_0)(\sin\theta_C \cos 45°)$$

$$F^B = \left(\frac{ev_0B_0}{\sqrt{2}}\right)\sqrt{1 + \sin^2\theta_C},$$

where, in writing the magnitude F^B, we used $\cos 45° = \sin 45° = 1/\sqrt{2}$. We can see from the expression for F^B that the electrons experience a nonzero magnetic force, whichever compass direction the CRT screen faces, but that the magnitude varies with the compass direction.

To calculate how far each electron is deflected, we first compute the time interval $\Delta t = \ell/v_0$ required for the electron to travel, at speed v_0, the length $\ell = 300$ mm from the electron gun to the screen. Then we compute the deflection d due to the magnetic force $\vec{F}^B$, where we are approximating $\vec{F}^B$ to be constant and perpendicular to the length of the CRT. From Newton's second law, the electron's acceleration is $\vec{a} = \vec{F}^B/m_e$, where $m_e = 9.11 \times 10^{-31}$ kg is the electron's mass. For constant $\vec{F}^B$, the acceleration is constant and is perpendicular to the initial velocity, so we can write the deflection d due to $\vec{F}^B$ as

$$d = \tfrac{1}{2}a\Delta t^2 = \tfrac{1}{2}\left(\frac{F^B}{m_e}\right)\Delta t^2 = \tfrac{1}{2}\left(\frac{F^B}{m_e}\right)\left(\frac{\ell}{v_0}\right)^2 = \frac{F^B\ell^2}{2m_e v_0^2}$$

$$d = \frac{ev_0B_0\ell^2}{2\sqrt{2}m_e v_0^2}\sqrt{1 + \sin^2\theta_C} = \frac{eB_0\ell^2}{2\sqrt{2}m_e v_0}\sqrt{1 + \sin^2\theta_C}.$$

The maximum deflection occurs for $\sin\theta_C = \pm 1$, which occurs when $\theta_C = 90°$ (screen faces east) or $\theta_C = 270°$ (screen faces west). Substituting $\sin\theta_C = \pm 1$ and then canceling the $\sqrt{2}$ in numerator and denominator, we obtain the maximum deflection:

$$d_{max} = \frac{eB_0\ell^2}{2m_e v_0} = \frac{(1.6 \times 10^{-19}\ \text{C})(3.0 \times 10^{-5}\ \text{T})(0.300\ \text{m})^2}{(2)(9.11 \times 10^{-31}\ \text{kg})(3.0 \times 10^7\ \text{m/s})}$$

$$= 7.9\ \text{mm.}\ ✔$$

(*b*) If the screen faces east, then $\theta_C = 90°$, so $\sin\theta_C = 1$, $\cos\theta_C = 0$, and the magnitude of the deflection is 7.9 mm. The deflection is in the direction of $\vec{F}^B$, which has components $F_x^B = 0$, $F_y^B < 0$, and $F_z^B = F_y^B$. So the deflection points diagonally downward, 45° below south. From the perspective of the viewer, who is facing west, the image is shifted diagonally down and to the left, with a magnitude of 7.9 mm. ✔

(*c*) If the screen faces north, then $\theta_C = 0$, so $\sin\theta_C = 0$, $\cos\theta_C = 1$, and the magnitude of the deflection is 5.6 mm. Now $\vec{F}^B$ has components $F_x^B > 0$, $F_y^B = 0$, and $F_z^B = 0$. So the deflection points due east. From the perspective of the viewer, who is facing south, the image is shifted 5.6 mm to the left. ✔

❹ EVALUATE RESULT That seems like a pretty big deflection, one big enough to notice by turning the television to different positions. In fact, turning the screen right or left, you can actually see the picture shift by a few mm. We can check our approximation that the direction

PRACTICE

of $\vec{F}^B$ is constant by evaluating the maximum angle $\alpha = \tan^{-1}(at/v_0) = 3.0°$ (for $\sin\theta_C = \pm 1$) between the electron's initial and final velocities: 3° is a pretty small angle, justifying the approximation. You should calculate the effect of gravity on the electron to check our assumption that it was all right to ignore the gravitational force.

We can use the right-hand force rule to check the directions of the deflections in parts *b* and *c*. For the east-facing screen, the electric current is traveling west and the viewer is facing west. From the viewer's perspective, Earth's magnetic field points diagonally down and to the right, so the right-hand force rule gives a magnetic force that is diagonally down and to the left, in agreement with our calculation. For the north-facing screen, the current travels south. Earth's magnetic field points downward and toward the south-facing viewer, so the right-hand force rule gives a magnetic force that is directly to the left, again in agreement with our result.

Guided Problem 27.6 Fusion energy

Nuclear fusion, the process that powers the Sun, can be made to occur in a laboratory by superheating a gas of electrons and protons (called a plasma) to temperatures exceeding 10^6 K. When their energy is high enough, the protons combine to create helium nuclei, and energy is liberated as the protons fuse. These hot, charged particles are trapped by a "containment vessel" formed by a strong magnetic field. Alcator C-mod at the Massachusetts Institute of Technology can achieve magnetic fields greater than 8 T. Assume that the C-mod's containment vessel has a circular cross section and that the magnetic field is uniform and has a constant magnitude of 8.0 T. What is the minimum radius of the vessel so that both electrons and protons are trapped? What is the period of the orbit in which the particles travel? Assume that the average speed of the particles is about 2.0×10^7 m/s.

1 GETTING STARTED

1. Describe the problem in your own words. What two variables must you determine?
2. What type of motion is implied by "containment"?
3. Which particle, proton or electron, is harder to contain?

2 DEVISE PLAN

4. What simplification results from seeking the minimum vessel radius?
5. How is the radius related to the given information?
6. What expressions must you use to determine the orbit period?

3 EXECUTE PLAN

7. Work through the algebra and solve for the desired unknown quantities, and then substitute values you know to get numerical answers.

4 EVALUATE RESULT

8. Is your value for the vessel radius unreasonable? That is, would such a device fit in a reasonably sized laboratory room?
9. Check that a particle moving in a circular orbit at your calculated radius and period would have the speed given in the problem statement.
10. How should the ratio of the two calculated radii relate to the ratio of the proton and electron masses? Does your answer check out?

Worked Problem 27.7. Going with the flow

A proton moves parallel to a current-carrying wire, at a distance 10 mm from the long axis of the wire, with the proton's velocity equal to the average velocity of the electrons in the wire. The interaction between the wire and the proton is observed from two reference frames: Observer E is at rest in the Earth reference frame, while observer M is moving relative to the Earth reference frame. To observer E, the wire is at rest, carries a 5.0-A current, has an ion charge density $\lambda_{E,ions} = +1.60 \times 10^3$ C/m, and has an electron charge density $-\lambda_{E,ions}$, making the wire electrically neutral. (*a*) What must M's velocity be, relative to the Earth reference frame, such that M observes only an electric force (and no magnetic force) between the wire and the proton? (*b*) What magnitude does observer M measure for this electric force? (*c*) What acceleration does this force give the proton?

1 GETTING STARTED We are told that a proton moving along with the electrons in a current-carrying wire is examined by two observers: E in the Earth reference frame and M in a reference frame moving relative to E. We are also told that M sees the proton-wire interaction as purely electric and from this information must determine her speed relative to E. We must also calculate what M measures for the magnitude of the electric force exerted by the wire on the proton and the resulting acceleration of the proton.

2 DEVISE PLAN For part *a*, the only way that M can see the wire-proton interaction as purely electric is if the proton is at rest relative to M. Therefore her speed must be the same as that of the proton (and thus of the electrons), which is related to the current and the linear charge density through Eq. 27.36, $I = \lambda_{proper}v$, where $\lambda_{proper} = \lambda_{E,ions}$ is what E measures for the linear charge density of the ions in the wire.

We can then use the calculated value of v to obtain the magnitude of the electric force needed in part *b*. This electric force is given by Eq. 27.32, $\vec{F}^E_{Mwp} = q\vec{E}_M$, with the electric field magnitude measured by M given by Eq. 27.31:

$$E_M = \frac{2k\lambda_{proper}}{r}\gamma\frac{v^2}{c_0^2},$$

where $\gamma = 1/\sqrt{1-(v/c_0)^2}$

For part *c*, once we have F^E_{Mwp}, we can obtain the proton's acceleration by dividing the magnitude of this force by the proton mass.

PRACTICE

❸ EXECUTE PLAN (*a*) The speed of the electrons measured by E is, from Eq. 27.36,

$$v = \frac{I}{\lambda_{\text{proper}}} = \frac{5.0\text{ A}}{1.60 \times 10^3\text{ C/m}} = 0.0031\text{ m/s.} ✔$$

This then must be the speed at which M moves relative to the Earth reference frame.

(*b*) Using the expression for E_{M} given by Eq. 27.31, we get for the electrostatic force measured by M

$$F^E_{\text{Mwp}} = qE_{\text{M}} = q\frac{2k\lambda_{\text{proper}}}{r}\gamma\frac{v^2}{c^2}.$$

Here q is the charge on the proton, which is the elementary charge e, and the electron linear charge density is λ_{proper} as we saw in part a, so we have

$$F^E_{\text{Mwp}} = (1.6 \times 10^{-19}\text{ C})\frac{2(9.0 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)(1.60 \times 10^3\text{ C/m})}{0.010\text{ m}} \times (1.09 \times 10^{-22}) = 5.0 \times 10^{-26}\text{ N.} ✔$$

(*c*) A force of this magnitude gives the proton an acceleration that has a magnitude of

$$a = \frac{F^E_{\text{Mwp}}}{m_{\text{p}}} = \frac{5.0 \times 10^{-26}\text{ N}}{1.67 \times 10^{-27}\text{ kg}} = 30\text{ m/s}^2. ✔$$

❹ EVALUATE RESULT Notice that the speed of the electrons, 0.003 m/s, is very small. The density of electrons in the wire is so great that even a very small average speed can create a current of several amperes. The force we calculated is extremely small, as we expect because of the tiny magnitude of the elementary charge e, but the acceleration of the proton is not so small because of its very small mass: The proton's acceleration is greater than that of a freely falling particle in Earth's gravitational field.

Guided Problem 27.8. Half speed

Suppose that in the situation described in Worked Problem 27.7 the proton is moving at only half the speed of the electrons. If you are an observer at rest relative to the wire, what magnitude do you measure for the electric force exerted by the wire on the proton? Would you observe a magnetic force exerted on the proton? If so, what is the magnitude of this force?

❶ GETTING STARTED

1. What portions of Worked Problem 27.7 can you use in this problem?
2. Which equations can help you determine the two types of interaction?

❷ DEVISE PLAN

3. Which equations can help you calculate the electric and magnetic fields in this relativistic situation?
4. What important simplification should you make?

❸ EXECUTE PLAN

5. Work out the needed quantities algebraically and then substitute numbers to get a numerical answer.

❹ EVALUATE RESULT

6. How does the magnitude of the electric force exerted on the proton compare to that found in Worked Problem 27.7? Is this sensible?

Answers to Review Questions

1. A magnet is an object that attracts objects made of iron, nickel, cobalt, or certain alloys, including most types of steel. A magnetic material is one that is attracted to a magnet.
2. Magnetic poles are the places on a magnet where a magnetic material is attracted most strongly. In a magnet that is free to align in any direction, the pole that settles toward geographic north is defined to be the magnet's north pole, and the opposite pole is the south pole.
3. Two like magnetic poles repel each other; two magnetic poles of opposite types attract each other.
4. An elementary magnet is the tiny magnetic dipole corresponding to a single atom of magnetic material such as iron, nickel, or cobalt, or to an elementary particle such as an electron or proton. In the elementary-magnet model for magnetism, a large number of the elementary magnets in a magnetic material have their north poles aligned in one direction and their south poles aligned in the opposite direction. This alignment of elementary N and S poles creates the overall N and S poles of the magnet.

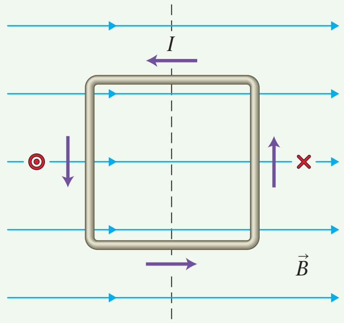
5. A magnet is surrounded by a magnetic field, which exerts a force on the poles of another magnet.
6. Magnetic field lines tell us the direction and relative magnitude of the magnetic field. The field lines point in the direction of the magnetic field, and the density of field lines is proportional to the magnetic field's magnitude. For the field created by a magnet, magnetic field lines on the outside of the magnet point from the magnet's north pole to its south pole; within the magnet, magnetic field lines point from the magnet's south pole to its north pole.
7. It doesn't; the magnetic field line flux through a closed surface is always zero, because magnetic field lines always form closed loops.
8. The magnetic field source described is moving charge carriers, which constitute a *current*, defined as the flow of charge carriers.
9. The magnetic field lines are circles, centered on the wire and in planes perpendicular to it.
10. The electrons would need to flow from right to left, because a flow of positive charge in one direction produces the same magnetic field as an equal flow of negative charge in the opposite direction.
11. Because electrons are negatively charged particles, the current direction is downward.
12. Counterclockwise, as shown using the right-hand current rule. If you point the thumb of your right hand in the direction of the current (that is, toward yourself), the fingers curl counterclockwise, indicating the direction of the magnetic field produced by that current.
13. The direction of the force exerted by the magnet on the wire is perpendicular to the direction of the external magnetic field and also perpendicular to the current direction, as determined by the right-hand force rule. When the outstretched fingers of your right hand align with the current such that they can curl toward the direction of the external magnetic field, the thumb points in the direction of the force exerted by the external field on the wire.
14. We expect this because electric currents, which both cause and interact with magnetic fields, depend on velocity, and velocity is always relative to an observer's reference frame.
15. The huge internal electric forces between the charged constituents in the wire balance to zero because there are equal amounts of positive and negative charge carriers. Because the electric forces are so huge, however, even a small relativistic correction to them is measurable, so the magnetic force is observable.
16. The observer at rest relative to the wire sees equal charge densities for the ions and the electrons, and so the wire appears electrically neutral to this observer. The observer moving with the electrons sees a greater linear charge density for the ions than for the electrons, and thus to this observer the wire is not electrically neutral.
17. The magnetic force is a maximum when the wire is perpendicular to the external field and a minimum (zero) when the wire and external field are either parallel or antiparallel.
18. It depends on the magnitude of the external magnetic field, the absolute value of the current, the length of wire in the field, and the sine of the angle between the directions of the current and the field: $F^B_w = |I|\ell B \sin\theta$ (Eq. 27.7).
19. From the right-hand force rule and Eq. 27.8, the force exerted by the field on the wire points in the positive z direction: $\hat{\imath} \times \hat{\jmath} = \hat{k}$.
20. The magnetic flux through the surface is the scalar product of the magnetic field vector and the area vector for the surface.
21. The magnetic flux through any closed surface is zero. This implies that there are no isolated magnetic monopoles (analogous to isolated charged particles).
22. The flux is the same through *all* surfaces bounded by a given loop, except for a plus or minus sign determined by the chosen direction of the normal area vector.
23. The magnetic force exerted on the wire is the vector sum of the magnetic forces exerted by the field on the moving charged particles that make up the current in the wire. The force exerted on each particle is $\vec{F}^B_p = q\vec{v} \times \vec{B}$ (Eq. 27.19).
24. The force is directed perpendicular to the magnetic field direction and perpendicular to the particle's velocity, in accordance with the right-hand force rule.
25. The ions are moving relative to the observer, so she sees the distance between ions decrease because of the length contraction due to special relativity. Decreased distance between ions means increased linear charge density for them.
26. The observer in the Earth reference frame interprets the force as a purely magnetic interaction between the particle and the current, as the wire carries no surplus charge when observed from the Earth reference frame. The observer moving along with the electrons interprets the force as a combination of electric and magnetic interactions: The electric interaction arises because the wire appears positively charged to the observer who moves along with the electrons; the magnetic interaction is now due to the motion of the positive ions as seen from the reference frame of the electrons. For the special case in which the velocity of the positively charged particle equals the velocity of the electrons, the interaction observed from the moving reference frame is purely electric.

Answers to Guided Problems

Guided Problem 27.2 $\vec{B}$ points into the page; $I = \dfrac{mg}{\ell B} = 103$ A.

Guided Problem 27.4 There is no force on the top or bottom side of the square. The force on the right side is into the plane of the figure, and the force on the left side is out of the plane of the figure, as shown in Figure WGA27.2. The vector sum of these two forces is zero. The torque tends to rotate the loop about the dashed line in the figure. The right side moves into the plane of the figure, while the left side moves out. The magnitude of the torque is $\tau = 2(\ell/2)(I\ell B) = 3.9 \times 10^{-5}\,\text{N} \cdot \text{m}$.

Figure WGA27.2



Guided Problem 27.6
For protons, $R = \dfrac{m_p v}{eB} = 0.026$ m, $T = \dfrac{2\pi m_p}{eB} = 8.2$ ns. For electrons, $R = \dfrac{m_e v}{eB} = 14\ \mu$m and $T = \dfrac{2\pi m_e}{eB} = 4.5$ ps. So the minimum vessel radius is 26 mm.

Guided Problem 27.8 In the Earth reference frame (at rest with respect to the wire), I measure no electric force exerted by the wire on the proton: $F^E_{\text{Ewp}} = 0$. But I do measure a magnetic force:

$$F^B_{\text{Ewp}} = |q| v_p \frac{2kI}{rc_0^2} = 2.5 \times 10^{-26}\ \text{N}.$$

Guided Practice by Chapter

28 Magnetic Fields of Charged Particles in Motion

Review Questions

Answers to these questions can be found at the end of this chapter.

28.1 Source of the magnetic field

1. What is the fundamental source of magnetic fields?
2. Are all magnetic forces central?
3. Is it possible to have a magnetic field without any magnetic poles?

28.2 Current loops and spin magnetism

4. What do the patterns of magnetic field lines from a current loop and from a bar magnet have in common?
5. What does the magnetic field surrounding a spinning charged particle look like?

28.3 Magnetic dipole moment and torque

6. What is the direction of the magnetic dipole moment vector used to specify the orientation of a magnetic dipole?
7. Describe the three right-hand rules used in studying magnetism.
8. Describe the magnetic interaction experienced by a current loop placed in a uniform external magnetic field.
9. What does the commutator do in an electric motor?

28.4 Ampèrian paths

10. Magnetic flux through any closed surface is zero, but electric flux need not be. How can this difference be accounted for in terms of the properties of electrostatic field lines and magnetic field lines?
11. How does the value of the line integral of the electrostatic field along a closed path encircling a charge distribution compare with the value of the line integral of the magnetic field along a closed path encircling a current-carrying wire?
12. State Ampère's law.
13. How can you determine whether a current encircled by an Ampèrian path makes a positive or negative contribution to the line integral of the magnetic field around the path?

28.5 Ampère's law

14. The proportionality constant in Ampère's law shows the relationship between what two variables associated with a current-carrying wire?
15. What kind of symmetry is displayed by a long straight wire carrying a steady current? How does this symmetry determine what you choose for the Ampèrian path when using Ampère's law to calculate the magnitude of this magnetic field?
16. How does the magnitude of the magnetic field around a long, straight current-carrying wire of radius R depend on radial distance $r > R$ from the long axis of the wire? Assuming that the current is steady and uniformly distributed within the wire, how does the field magnitude depend on the radial distance $r < R$ from the wire axis? From the point of view of Ampère's law, why is this?
17. Describe the magnetic field produced by a large flat sheet of uniformly distributed current for which the current per unit width is K. What are the magnitude and direction of the field above and below the sheet?

28.6 Solenoids and toroids

18. What is a solenoid? Describe the magnitude and direction of the magnetic field produced by a long solenoid carrying a steady current.
19. What is a toroid? Describe the magnitude and direction of the magnetic field produced by a toroid carrying a steady current.
20. Does the magnetic field inside a very long solenoid differ from that inside a toroid if the two devices carry the same current in the same number of turns? If so, how?

28.7 Magnetic fields due to currents

21. Compare and contrast the expression for the infinitesimal magnetic field at some location P near a small segment of current-carrying wire given by the Biot-Savart law (Eq. 28.12) and the expression for the electrostatic field at some location P near a source charged particle, which we obtained from Coulomb's law in Section 23.7, $d\vec{E}_s(\mathrm{P}) = k(dq_s\hat{r}_{s\mathrm{P}}/r_{s\mathrm{P}}^2)$ (Eq. 23.14).
22. What steps can we follow to calculate the magnetic force between two parallel straight wires of length ℓ, carrying constant currents I_1 and I_2 and separated by a distance d?
23. Comment on the range of applicability of the Biot-Savart law relative to the range of applicability of Ampère's law.

28.8 Magnetic field of a moving charged particle

24. What is the origin of each of the two vector products in Eq. 28.26, the expression for the electromagnetic force between two moving charged particles 1 and 2?
25. What is the numerical value of the product of the proportionality constants μ_0 and ϵ_0? How does it relate to the speed of light c_0?
26. Does the force between two moving charged particles obey Newton's third law?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The maximum magnetic field magnitude you are exposed to due to current in the electrical wiring in your house (E, K, P)
2. The straight-wire current needed to reverse the deflection of a compass needle sitting on your laboratory table (H, A, O, W)
3. The maximum magnetic field strength 10 m from a typical lightning bolt (G, R)
4. The maximum magnitude of the magnetic field you can produce in a solenoid sitting on your laboratory table when the core contains only air (D, N, Q, U)
5. The maximum magnetic force per meter between the antiparallel currents in your household wiring (P, E)
6. The electric current around the equator needed to produce Earth's magnetic field at the North Pole (B, I, M, S)
7. The magnitude of the magnetic field at the center of a Bohr-model hydrogen atom caused by the electron orbiting the nucleus (C, J, T)
8. The magnitude of the magnetic field at the North Pole that might be attributed to Earth's rotation, assuming a uniformly charged surface (F, L, I, V)

Hints

A. What is the horizontal component of Earth's magnetic field in your neighborhood?
B. What is the magnetic field magnitude at the North Pole?
C. What is the orbit radius?
D. What are the crucial variables to maximize?
E. What is the current configuration?
F. What is the magnitude of the electric field near Earth's surface?
G. How should you model the current?
H. How should the wire be oriented?
I. What is the radius of Earth?
J. What is the electron's speed?
K. How close do you come to the wiring?
L. What surface charge density could create this electric field magnitude?
M. What is the straight-line distance from a point on the equator to the North Pole?
N. What maximum current is reasonable?
O. What magnetic field strength is needed?
P. What is the maximum likely current in each conductor?
Q. What is a typical wire diameter used to carry currents of this magnitude in household or building wiring?
R. What is the peak current?
S. How can you compute the vector sum of the magnetic field magnitudes due to a large number of tiny segments of the current loop?
T. What magnitude of charge does the electron carry, and what is its inertia?
U. What is the maximum number of turns n per meter of solenoid?
V. How should you model the current?
W. How close to the wire can you place the compass?

Key (all values approximate)

A. 2×10^{-5} T; **B.** 7×10^{-5} T; **C.** 5×10^{-11} m; **D.** because $B = \mu_0 n I$, both I and n (number of windings per unit length) should be maximized; **E.** paired conductors separated by 10^{-2} m (two copper wires inside a nonconducting sheath) and carrying equal-magnitude currents in opposite directions; **F.** 100 V/m; **G.** approximately a vertical line; **H.** the compass detects horizontal magnetic fields, so orient the wire vertically; **I.** 6×10^{6} m; **J.** electrical attraction provides centripetal force, so 2×10^{6} m/s; **K.** 0.1 m if you stand near a wall; **L.** from Coulomb's law, 10^{-9} C/m^2; **M.** 9×10^{3} km; **N.** laboratory tables are powered with 20 A, 120 V circuits, so the available current will be on the order of 20 A; **O.** enough to more than cancel Earth's magnetic field, say 4×10^{-5} T; **P.** 20 A for large appliances; **Q.** About 2 mm wire diameter; **R.** 10^{5} A; **S.** by using the Biot-Savart law; **T.** $e = 1.6 \times 10^{-19}$ C, $m_e = 9.11 \times 10^{-31}$ kg; **U.** with 2-mm wire diameter, 500 turns per meter (or some small multiple of 500 turns per meter, if the wire is wound in more than one layer) are possible; **V.** as a stack of horizontal current loops of different radii (careful: current depends on radius); **W.** 0.02 m.

Worked and Guided Problems

Procedure: Calculating the magnetic field using Ampère's law

For magnetic fields with straight or circular field lines, Ampère's law allows you to calculate the magnitude of the magnetic field without having to carry out any integrations.

1. Sketch the current distribution and the magnetic field by drawing one or more field lines using the right-hand current rule. A two-dimensional drawing should suffice.
2. If the field lines form circles, the Ampèrian path should be a circle. If the field lines are straight, the path should be rectangular.
3. Position the Ampèrian path in your drawing such that the magnetic field is either perpendicular or tangent to the path and constant in magnitude. Choose the direction of the Ampèrian path so that, where it runs parallel to the magnetic field lines, it points in the same direction as the field. If the current distribution divides space into distinct regions, draw an Ampèrian path in each region where you wish to calculate the magnetic field.
4. Use the right-hand current rule to determine the direction of the magnetic field of each current encircled by the path. If this magnetic field and the Ampèrian path have the same direction, the contribution of the current to I_{enc} is positive. If they have opposite directions, the contribution is negative.
5. For each Ampèrian path, calculate the line integral of the magnetic field along the path. Express your result in terms of the unknown magnitude of the magnetic field B along the Ampèrian path.
6. Use Ampère's law (Eq. 28.1) to relate I_{enc} and the line integral of the magnetic field and solve for B. (If your calculation yields a negative value for B, then the magnetic field points in the opposite direction you assumed in step 1.)

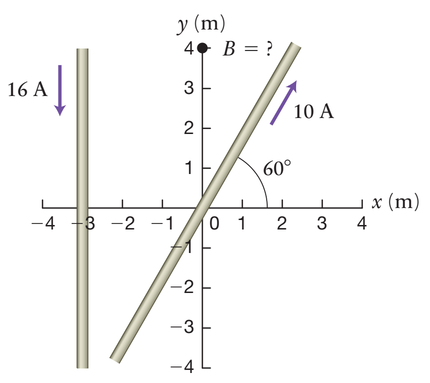
You can use the same general approach to determine the current given the magnetic field of a current distribution. Follow the same procedure, but in steps 4–6, express I_{enc} in terms of the unknown current I and solve for I.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 28.1 Electrical wiring

A hobbyist has decided to install several new electrical accessories in her houseboat. To power these devices, she lays two wires along the ceiling of her boat as indicated in Figure WG28.1. (The plane of the drawing corresponds to the boat ceiling.) The wires are designed to carry the currents indicated in the figure, but she worries that the magnetic field near the wires might be strong enough to disrupt the operation of the compass that is already mounted to the ceiling. To reassure her that there will be no such problems, determine the magnitude and direction of the magnetic field at the position $x = 0$, $y = 4.0$ m, where the compass is located.

Figure WG28.1



PRACTICE

❶ GETTING STARTED Currents generate magnetic fields. Here two current-carrying wires contribute to a magnetic field at the position indicated. We are asked to determine the magnitude and direction of this magnetic field.

❷ DEVISE PLAN We know from Example 28.3 that the magnitude of the magnetic field generated at a radial distance r from a long, straight current-carrying wire is $B = \mu_0 I/2\pi r$. Because the magnetic field is a vector, we need to determine the vector sum of the individual fields created by the two wires. We also know that near a current-carrying wire, magnetic field lines form concentric circles around the wire. We can use this information and the right-hand current rule to determine the field direction.

❸ EXECUTE PLAN Let's call the 10-A current I_1 and the corresponding magnetic field $\vec{B}_1$, and let's call the 16-A current I_2 and its magnetic field $\vec{B}_2$. Using the right-hand current rule, we point our right thumb in the direction of I_1 and notice that $\vec{B}_1$ points out of the plane of the page (i.e., points in the $+\hat{k}$ direction) at the position $x = 0$, $y = 4.0$ m, which we'll call position P. Similarly, by pointing our right thumb along I_2, we see that $\vec{B}_2$ also points out of the plane of the page at position P. The distance r_1 from the first wire to P, measured along a path perpendicular to the wire, is $r_1 = y\cos(60°) = 2.0$ m. The distance from the second wire to P is $r_2 = 3.0$ m. So the two wires' contributions to the magnetic field at point P are $\vec{B}_1 = +(\mu_0 I_1/2\pi r_1)\hat{k}$ and $\vec{B}_2 = +(\mu_0 I_2/2\pi r_2)\hat{k}$. The combined field $\vec{B}$ at point P is then

$$\vec{B} = \frac{\mu_0}{2\pi}\left(\frac{I_1}{r_1} + \frac{I_2}{r_2}\right)\hat{k} = \frac{4\pi \times 10^{-7}\ \mathrm{T\cdot m/A}}{2\pi}\left(\frac{10\ \mathrm{A}}{2.0\ \mathrm{m}} + \frac{16\ \mathrm{A}}{3.0\ \mathrm{m}}\right)\hat{k}$$

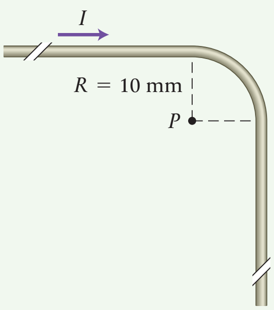
$$= +(2.1 \times 10^{-6}\ \mathrm{T})\hat{k}. ✔$$

❹ **EVALUATE RESULT** A magnitude of 2.1×10^{-6} T for the magnetic field associated with these two currents is reassuring because it is about 20 times smaller in magnitude than Earth's magnetic field, and thus unlikely to affect the compass reading. It is also reassuring that the magnetic field points up toward the sky, while the compass is designed to measure the horizontal component of Earth's magnetic field. Finally, a value smaller than Earth's magnetic field makes sense for a current of several amperes at a distance of several meters, because I know from experience that a compass is affected by everyday electrical wiring only when it is held very close to a wire.

Guided Problem 28.2 Bell wire

A student runs a long wire along one wall of her room, curves the wire at the corner, and continues to run it to a bell on the adjacent wall. The wire at the corner forms a circular arc of radius 10 mm (Figure WG28.2). What are the magnitude and direction of the magnetic field at position P if the current in the wire is 540 mA?

Figure WG28.2



❶ **GETTING STARTED**

1. Describe the problem in your own words.
2. What concept(s) apply to this situation?
3. What assumptions must you make?

❷ **DEVISE PLAN**

4. Can you break the problem into parts?
5. Which equations will help you get the magnetic field for the various parts?
6. Can the equations specific to a slightly different situation be modified for this situation?
7. Magnetic field is a vector. How can you determine the vector direction?

❸ **EXECUTE PLAN**

8. Solve for the desired unknown quantity. Substitute values you know to get a numerical answer. Be sure to determine the vector direction and include it in your answer.

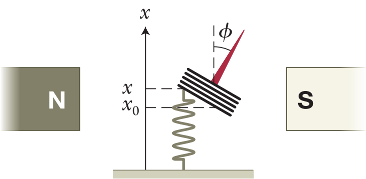
❹ **EVALUATE RESULT**

9. Does your answer behave as you expect it to as you change the radius of curvature?

Worked Problem 28.3 Galvanometer

A galvanometer is an old-fashioned device used to measure current. The simple model shown in Figure WG28.3 consists of a coil of wire, a vertical spring, a permanent magnet, and an indicator needle. A rigid rod (not shown) that runs perpendicular to the page is attached to the coil, and the coil is free to rotate about this rod axis. (The rod is also wired to allow the current to enter and leave the coil.) One end of the spring is attached to the coil, and the other end of the spring is attached to a fixed support. With no current through the device, the coil is horizontal, the spring is relaxed, and the needle points straight up. When there is a current in the coil, a torque is induced on the coil, causing it to rotate about the rigid rod, and the spring becomes either stretched or compressed. This rotation of the coil causes the needle to swing right or left, indicating the magnitude and sign of the current on a numerical scale (not shown in Figure WG28.3). As long as the angle ϕ through which the needle deflects from the vertical is not too large, the spring is stretched or compressed only vertically. (*a*) Viewed from above, what is the current direction with the coil rotated to the position shown? (*b*) Derive an algebraic expression for ϕ as a function of the current in the coil. (*c*) Calculate the current required to deflect the needle to an angle of 5.7°.

Figure WG28.3



The coil has a square cross section, with sides of length ℓ. The coil has $N = 100$ turns, the magnet produces a uniform magnetic field of magnitude 0.010 T, and the spring constant is 2.0 N/m.

❶ GETTING STARTED We are given data for a galvanometer and asked to determine (*a*) the direction of the current in the coil, (*b*) an expression for the needle-deflection angle as a function of the current, and (*c*) the value of the current when that angle is 5.7°. We know that the magnet produces a magnetic field that is directed from the north pole to the south pole—that is, left to right in Figure WG28.3. We also know that a current loop experiences a torque when placed in an external magnetic field, and we know how to compute the magnitude of such a torque.

❷ DEVISE PLAN To deduce the current direction, we can use the right-hand force rule to determine the direction of the magnetic force on each side of the coil.

To derive an expression for ϕ as a function of I, we recall that Newton's second law applied to rotational motion tells us that the rotational acceleration of an object is proportional to the sum of the torques it experiences. We have two torques in this system: one due to the spring force and one due to the magnetic force between the external magnetic field and the current in the coil. These two torques balance to give the needle zero rotational acceleration once it reaches the position on the numerical scale corresponding to the amount of current in the coil. We can compute these torques because we know that the torque an object experiences when a force is exerted on it is $\vec{\tau} = \vec{r} \times \vec{F}$ (Eq. 12.38). The spring is stretched in the view shown in Figure WG28.3, and so the spring force is exerted downward on the left side of the coil. We know from Hooke's law (Eq. 8.20) that its magnitude is $F^c = |-k(x - x_0)|$, where k is the spring constant.

Each of the four sides of the coil experiences a magnetic force. Regardless of what the current direction is, the side of the coil facing us in Figure WG28.3 and its parallel side (behind the page) must feel forces that are perpendicular to the page and pointing in opposite directions. These forces not only cancel but also create zero torque on the coil because they are directed along the axis of rotation. Thus the only magnetic torques on the coil are those caused by the forces exerted on the two sides of the coil that face the two poles of the magnet. Because Figure WG28.3 shows the needle rotated to the right through angle ϕ, we know that the force direction is upward on the coil side facing the north pole and downward on the coil side facing the south pole. The magnitude of the force exerted on any straight wire segment in the coil can be found from Eq. 28.13,

$$F^B = |I\vec{\ell} \times \vec{B}| = \ell IB;$$

where ℓ is the length of each coil side. Note that the angle in this vector product is always 90° as the coil rotates, but the angle in the vector product for the torque, $\vec{\tau} = \vec{r} \times \vec{F}$, depends on ϕ. We just need to put these pieces together to get ϕ as a function of I and then a value for I when $\phi = 5.7°$.

❸ EXECUTE PLAN (*a*) The force exerted by the stretched spring is downward, and so to keep the left side of the coil tilted upward as shown, the magnetic force exerted on this side must be upward. By the same reasoning, the right side of the coil must experience a downward magnetic force. Thus we know two of the three variables covered by the right-hand force rule—magnetic force direction and magnetic field direction—and so let's use this rule to determine the current direction. The current is either clockwise or counterclockwise as viewed from above. In the left side of the coil, a clockwise current means that, in order to make the fingers of our right hand able to sweep from I direction to $\vec{B}$ direction, we must position the hand with the fingers touching the coil's left side and pointing into the page with the palm facing to the right. Our thumb, which indicates the direction of the magnetic force on this side of the coil, points downward, but we know that this force is directed upward. Thus the current is not clockwise; it must be counterclockwise. ✔

To confirm that the current is counterclockwise as viewed from above, we apply the right-hand force rule to the right side of the coil. A counterclockwise current in this side means that when we point our fingers at this side in such a way that we can sweep them from I direction to $\vec{B}$ direction, the thumb points down for the direction of $\vec{F}^B$. Knowing that this is indeed the direction of $\vec{F}^B$ here, we have confirmed that the current in the coil is counterclockwise. ✔

(*b*) Because the needle does not accelerate once there is a steady current through the coil, the magnitude τ^B of the combined torques due to the two magnetic forces must be equal to the magnitude τ^c of the torque due to the spring force. Because these two torques must be of equal magnitude in order to balance each other, we have $\tau^B = \tau^c$:

$$\tau^B_\vartheta + \tau^c_\vartheta = \tau^B + (-\tau^c) = 0$$

$$\tau^B = \tau^c.$$

The coil is composed of a stack of N turns, and each turn consists of four segments. We know that only the segments facing the north and south magnetic poles contribute to the torque, so we must consider the magnetic force on these two segments of each turn. The torque due to the magnetic force exerted on each segment is the vector product of the force due to the magnetic field and the radius vector $\vec{r}$ pointing from the axis of rotation (the central axis of the coil) to the segment. The magnitude of this vector product is $rF^B \sin\theta = rF^B \cos\phi$, where $\theta = \frac{\pi}{2} - \phi$ is the angle between $\vec{r}$ and $\vec{F}$. Because there are N turns in the coil, there are N wire segments on each side of the coil. Because there are two sides experiencing the magnetic force that creates the torque, the torque on the coil due to the magnetic field is

$$\begin{aligned}\tau^B &= 2\left|\vec{r} \times \vec{F}^B\right| \\ &= 2\left[\tfrac{1}{2}\ell(\ell NIB)\sin\left(\tfrac{\pi}{2} - \phi\right)\right] \\ &= \ell^2 NIB\cos\phi,\end{aligned}$$

where I is the current through each segment and thus the current through the coil. The torque caused by the spring force can be approximated, for small angles, by

$$\begin{aligned}\tau^c &= \left|\vec{r} \times \vec{F}^c\right| \\ &= \tfrac{1}{2}\ell\left|k(x - x_0)\right|\sin\left(\tfrac{\pi}{2} - \phi\right) \\ &= \tfrac{1}{2}\ell\left|k(\tfrac{1}{2}\ell\sin\phi)\right|\cos\phi \\ &\approx \tfrac{1}{4}k\ell^2\sin\phi\cos\phi.\end{aligned}$$

The substitution we just used, $\Delta x = \frac{1}{2}\ell\sin\phi$, is an approximation because we have assumed that the spring stretches only vertically, whereas in reality it is also pulled slightly to the right. Nevertheless, our approximation should be fine for small angles of rotation. Setting the two torques' magnitudes equal to each other, we get for the current

$$I = \frac{k}{4NB}\sin\phi.$$

For small angles, $\sin\phi \approx \phi$ (in radians), which means that the angle of deflection is approximately linear with the current:

$$I = \frac{k}{4NB}\phi, \quad \text{or} \quad \phi = \frac{4NBI}{k}. ✔$$

(*c*) For $\phi = 5.7°$, we get

$$I = \frac{2.0\ \text{N/m}}{4(100)(0.010\ \text{T})}(5.7°)\left(\frac{1\ \text{rad}}{57.3°}\right) = 50\ \text{mA}. ✔$$

❹ **EVALUATE RESULT** In the equation we derived for I, notice how the needle-deflection angle does not depend on the size of the coil; it depends only on the number of turns N and on the strength of the magnetic field. Can you figure out why the coil size doesn't play a role? Hint: Look at how the two opposing torques—one from the spring and one from the magnetic force due to the current—depend on the dimensions of the coil.

Make sure you fully understand each vector product used in the solution and where each angle comes from. The torques caused by magnetic forces can be fairly complex because of all the angles involved.

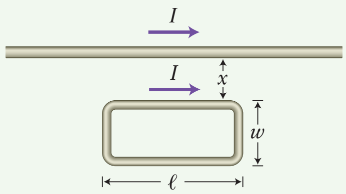
The magnitude of the current is neither excessively large nor small. Galvanometers often have a switch to allow you to measure different ranges of current while keeping the needle-deflection angle small. Moreover, a real galvanometer has a construction somewhat different from the version shown in Figure WG28.3. A spiral spring is typically used, and the magnets are designed so that the magnetic field direction is mostly radial. What would this accomplish?

This problem is a good example of how to solve a seemingly complex problem by putting several smaller pieces together. You may need to reach back to Chapter 8 to review how springs work and to Chapter 12 to review torque.

Guided Problem 28.4 Force on a rectangular current loop

A circuit board has a rectangular loop of wire next to a long straight wire located a distance $x = 0.300$ mm away from the loop (Figure WG28.4), with a 39-mA current through each element. What is the magnetic force exerted by the wire on the loop? The rectangle dimensions are $\ell = 5.7$ mm and $w = 0.90$ mm.

Figure WG28.4



1 GETTING STARTED

1. Can you break the problem into parts?
2. What simplification can you make immediately just by looking at the problem?
3. What assumptions do you need to make?

2 DEVISE PLAN

4. Along each side of the rectangle, in what direction does the magnetic field generated by the wire point? In what direction does the magnetic force exerted on that side point?
5. Are there any two sides of the rectangle whose magnetic forces cancel one another?
6. What is an expression for the magnetic force between two parallel current-carrying wires?

3 EXECUTE PLAN

7. Apply the right-hand current rule to determine the direction of the magnetic field, and the right-hand force rule to obtain the directions of the forces.
8. Obtain an algebraic expression for the vector sum of the forces exerted on the loop. Then substitute numbers to get a numerical answer.

4 EVALUATE RESULT

9. Is your value for the magnitude of the magnetic force exerted on the loop unreasonable? Do you expect this force magnitude to be large or small?
10. How does the force depend on the dimensions of the loop? If ℓ and w become so large that only one side of the rectangle is near the wire, does the force per unit length reduce to a familiar expression? Does your algebraic expression also make sense if w becomes so small that the two long sides of the rectangle lie on top of one another, with equal but oppositely directed currents?

Worked Problem 28.5 Wire with nonuniform current

A long, straight wire that has a cross-sectional radius R carries a current I, but this current is not uniformly distributed over the circular cross section of the wire. Instead, the number density of charge carriers n depends linearly on the radial distance r from the wire's central axis: $n(r) = n_R r/R$, where n_R is the number density of charge carriers at the wire surface; these charge carriers all move at the same velocity v. (*a*) Derive an expression for the magnetic field magnitude B as a function of r for $r < R$ and for $r > R$. (*b*) Plot B as a function of r.

1 GETTING STARTED We are asked to derive expressions for the magnetic field magnitude inside and outside a current-carrying wire when the current is distributed nonuniformly across the wire cross section (rather than just outside an infinitesimally thin wire, as we did in the discussion). We know that the magnetic field due to current in a wire has field lines that form closed loops around the wire. Ampère's law will be useful, but the nonuniform current requires a careful computation of the enclosed current for each value of r, the radial distance away from the wire center. This means we must break the wire cross section into a large number of small elements, with each element small enough to have a uniform current through it, and then integrate to obtain the current enclosed at any value of r. We can then draw a graph showing B as a function of r.

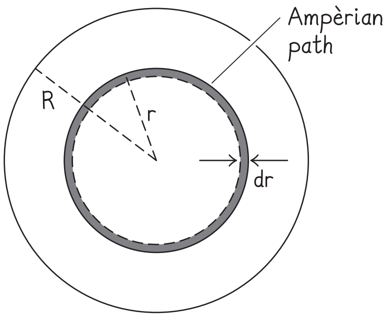
2 DEVISE PLAN Because of the cylindrical symmetry, we use Ampère's law:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}$$

for a tiny segment $d\vec{\ell}$ along some chosen Ampèrian path. The current enclosed inside any Ampèrian path we choose must be computed with an integral, but how can we set that up? From Chapter 27, we know that the current in a wire is related to the number density of charge carriers by Eq. 27.16, $I = nAqv$. Because we are dealing with an electric current, the charge q is the magnitude of the elementary charge e carried by an electron, and we are told that v does not depend on the distance r from the wire center. However, n is a function of r, so we cannot simply multiply the number density by the volume of the wire. We must imagine breaking the wire into small volume elements, but what kind? Because nothing varies with length along the wire, we can allow our elements to have any length. That is good because length along the wire is absorbed into the speed v of the charge carriers.

The cross-sectional area factor in our volume is another matter: Because n depends on r, we must break the cross-sectional area A of the wire into tiny portions, with n approximately constant over each portion. This suggests that we use very thin rings so that all portions of the cross section of a ring have the same value of r. The cross-sectional area dA of such a ring (Figure WG28.5) of radius r and thickness dr is its circumference times its thickness: $dA = 2\pi r\,dr$. This can be seen by imagining the ring being sliced through the thickness dr and unrolled to produce a rectangle of length $2\pi r$ and height dr.

Figure WG28.5



Putting all this together in applying Eq. 27.16, with the substitutions $n = n(r)$, $A = 2\pi r\,dr$, and $q = e$, we get for the current dI through any infinitesimally thin cylindrical shell of radius r:

$$dI = n(r)ev2\pi r\,dr.$$

At any radial distance r from the wire center, the current enclosed by a circular Ampèrian path can be found by integrating this expression from the wire center out to r. We should have no trouble applying Ampère's law in this case of a nonuniform current distribution, as long as we can do the integrals.

❸ **EXECUTE PLAN** (*a*) We draw a circular Ampèrian path at a radius r inside the wire (Figure WG28.5). $\vec{B}$ and $d\vec{\ell}$ are parallel all along the Ampèrian path, so $\vec{B} \cdot d\vec{\ell} = B d\ell \cos 0 = B d\ell$. Because the current depends only on the distance r from the wire center, the symmetry is not disturbed for any given path radius. Thus the magnitude of the magnetic field at all points along this circular Ampèrian path has the same value, and we can bring it outside the integral:

$$B\oint d\ell = \mu_0 I_{\text{enc}}$$

$$2\pi r B = \mu_0 I_{\text{enc}}. \quad (1)$$

We must now compute the current enclosed by integrating over a large number of infinitesimally thin cylindrical shells, each of radius r and thickness dr and carrying current dI:

$$I_{\text{enc}}(r) = \int dI = \int_0^r 2\pi evn(r)r\,dr$$

$$= 2\pi ev\int_0^r n_R\frac{r}{R}r\,dr = \frac{2\pi evn_R}{R}\int_0^r r^2 dr$$

$$= \frac{2\pi evn_R}{R}\frac{r^3}{3}, \quad (2)$$

where R is the wire radius.

We could next substitute this expression for I_{enc} into Ampère's law, but there is a great simplification to be made. We do not know values for n_R and v, but we do know that the current in the wire is I. Perhaps we can eliminate some unknowns by computing the integral all the way to the wire surface (where $r = R$), which must give us an expression that represents the wire current I:

$$I = \frac{2\pi evn_R}{R}\int_0^R r^2 dr = \frac{2\pi evn_R}{R}\frac{R^3}{3}.$$

This allows us to simplify Eq. 2 for the current enclosed at any Ampèrian path radius r to

$$I_{\text{enc}}(r) = I\frac{r^3}{R^3}.$$

Substituting this value for I_{enc} in Eq. 1 shows us that inside a wire of radius R, the magnetic field as a function of the radial distance r from the wire center is given by

$$2\pi r B = \mu_0 I (r/R)^3$$

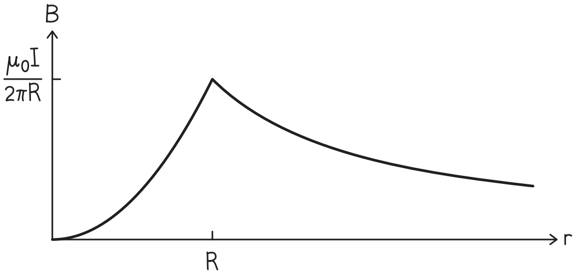
$$B = \frac{\mu_0 I}{2\pi}\frac{r^2}{R^3}.$$

Outside the wire, $r > R$, the enclosed current is just I at any path radius r, and so the expression for the magnetic field magnitude at any radial distance r from the wire center is the same as that for a thin wire:

$$B = \frac{\mu_0 I}{2\pi r}. ✔$$

(b) Figure WG28.6 is a graph of the magnetic field magnitude B as a function of the radial distance r from the wire center. ✔

Figure WG28.6



4 **EVALUATE RESULT** The problem statement tells us that n increases linearly with the radial distance r from the wire center. This nonuniform current distribution means that as we move radially away from the wire center, more and more current is enclosed by our Ampèrian path. Thus, the magnetic field magnitude increases quadratically until we reach the wire surface. Outside the wire, the current enclosed is always I, as it is for a wire carrying a uniformly distributed current, and so the magnetic field magnitude drops off as $1/r$. In this region our result agrees with what we found for an infinitesimally thin wire (see Example 28.3)

Guided Problem 28.6 Magnetic field in a coaxial cable

A coaxial cable consists of two concentric elements, an inner conducting wire and an outer conducting shell, typically separated by some dielectric material. One reason for using this type of cable is that when there is a current in it, the magnetic field due to the current is "trapped" inside the cable. Show that this is true by deriving one expression for the magnetic field magnitude in the region between the inner wire and the outer shell, and another expression for the magnetic field outside of the outer shell. Assume a uniform current and model that current as having one direction in the inner wire and the opposite direction in the outer shell. You can treat the outer shell as if its thickness were infinitesimal.

1 **GETTING STARTED**

1. How is this problem similar to Worked Problem 28.5?
2. What approach seems best to attack this problem?

2 **DEVISE PLAN**

3. Assign symbols for each radius you need. How many different radii must you use to establish the results you need?
4. What portions of Worked Problem 28.5 are relevant?

3 **EXECUTE PLAN**

5. Apply Ampère's law as many times as needed.
6. Sketch a graph of the magnetic field magnitude as a function of distance from the center of the inner wire to check the behavior inside and outside the cable.

4 **EVALUATE RESULT**

7. What is the radial dependence of the magnetic field in each region? Are your results unreasonable?
8. Do your results match what the problem statement says about the magnetic field being trapped inside the cable? If not, you probably made an error somewhere. Go back and check your work.

PRACTICE

Worked Problem 28.7 Equivalence

A charged particle moves parallel to a long, straight current-carrying wire. The particle is a perpendicular distance a away from the wire, and the direction of the particle's velocity is the same as the direction of the current in the wire. Show that F^B_{wp}, the magnitude of the magnetic force exerted by the wire on the particle, is equal to F^B_{pw}, the magnitude of the magnetic force exerted by the particle on the wire, and that these two forces are in opposite directions.

❶ **GETTING STARTED** We are told that a charged particle moves alongside a current-carrying wire, that the perpendicular distance between the particle and the wire is a, and that the particle's direction of motion is the same as the current direction. Our task is to show that the magnetic force exerted by the wire on the particle, which moves in the magnetic field created by the current in the wire, is equal in magnitude to the force exerted by the particle on the wire and that these two forces point in opposite directions. That the two forces should be equal and opposite—that is, $\vec{F}^B_{wp} = -\vec{F}^B_{pw}$—is not true in general for magnetic interactions, as discussed in Section 28.8, but we are told that it should be true in this special case.

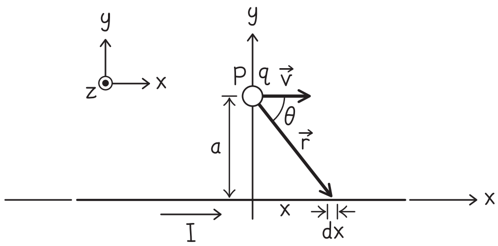
Let's spell out the concepts that will help us to solve this problem. We know that a long, straight current-carrying wire generates a magnetic field whose field lines encircle the wire, and we know that a charged particle moving through this magnetic field will experience a magnetic force that is perpendicular both to the field and to the particle's velocity. Similarly, we know that a moving charged particle generates a magnetic field and that a current-carrying wire will experience a force due to this magnetic field.

❷ **DEVISE PLAN** As we've drawn in Figure WG28.7, we let the current I in the wire run in the $+x$ direction, along the x axis. We let the position P of the charged particle be on the y axis at $x = 0$, $y = a$, $z = 0$. We call the particle's charge q, and we let the particle's velocity $\vec{v}$ point in the x direction: $\vec{v} = v\hat{\imath}$.

To evaluate the magnetic field $\vec{B}_w$ due to the wire at position P, we can use Ampère's law, Eq. 28.1, as applied in Example 28.3, where we found the magnitude of the magnetic field at a distance d from a long, straight wire to be

$$B = \frac{\mu_0 I}{2\pi d}. \tag{1}$$

Figure WG28.7



To compute the force $\vec{F}^B_{wp}$ exerted on a charged particle moving in the field of the wire, we can use Eq. 27.19:

$$\vec{F}^B_p = q\vec{v} \times \vec{B}. \tag{2}$$

To evaluate at any position (x, 0, 0) along the wire the magnetic field $\vec{B}_p$ due to the moving particle, we can use the expression for the magnetic field of a moving charged particle, Eq. 28.21, with the vector $\vec{r}$ indicated in Figure WG28.7 pointing from the moving charged particle (the source of the magnetic field) to the segment dx along the wire at $(x, 0, 0)$. Finally, to compute the force $\vec{F}^B_{pw}$ exerted on the wire, we can break the wire up into infinitesimal segments $d\vec{\ell} = dx\hat{\imath}$ along the x axis and calculate each segment's infinitesimal contribution $d\vec{F}^B_w$ to the force using Eq. 27.8:

$$d\vec{F}^B = I\, d\vec{\ell} \times \vec{B}. \tag{3}$$

We can then integrate Eq. 3 over the entire length of the wire to obtain $\vec{F}^B_{pw}$.

Finally, we can use the right-hand current rule and the right-hand force rule to check that the directions make sense for the magnetic fields and forces that we calculate.

❸ **EXECUTE PLAN** Substituting the distance a into Eq. 1 yields the magnitude $B_w = \mu_0 I/2\pi a$ at the point P where the charged particle is located. Using the right-hand current rule, we point our right thumb along the positive x axis and our fingers point out of the page at P. Because $\hat{\imath} \times \hat{\jmath} = \hat{k}$ is a vector identity, the z axis in Figure WG28.7 must point out of the page. Thus the magnetic field at point P due to the wire is

$$\vec{B}_w = \frac{\mu_0 I}{2\pi a}\hat{k}. \tag{4}$$

We obtain the magnetic force $\vec{F}^B_{wp}$ exerted on the charged particle by substituting $\vec{B}_w$ from Eq. 4 into Eq. 2:

$$\vec{F}^B_{wp} = q\vec{v} \times \left(\frac{\mu_0 I}{2\pi a}\hat{k}\right) = (qv\hat{\imath}) \times \left(\frac{\mu_0 I}{2\pi a}\hat{k}\right) = \frac{\mu_0 I q v}{2\pi a}(-\hat{\jmath}),$$

where we used the vector identity $\hat{\imath} \times \hat{k} = -\hat{\jmath}$ to see that the force points downward, in the $-y$ direction, for $q > 0$. We can check this direction using the right-hand force rule: If $q > 0$, then the vector $q\vec{v}$ points to the right, while $\vec{B}_{\rm w}$ points out of the page. Pointing the fingers of our outstretched right hand to the right and then curling them out of the page leaves our thumb pointing downward, confirming that the force on the particle points down toward the wire (along $-\hat{\jmath}$) if $q > 0$.

We evaluate the magnetic field $\vec{B}_{\rm p}$ at position $(x, 0, 0)$ along the wire due to the moving charged particle by substituting the vector $\vec{r} = x\hat{\imath} - a\hat{\jmath}$ into Eq. 28.21:

$$\vec{B}_{\rm p} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{(qv\hat{\imath}) \times \left(\dfrac{x\hat{\imath} - a\hat{\jmath}}{\sqrt{x^2 + a^2}}\right)}{(\sqrt{x^2 + a^2})^2}.$$

Remember that $\vec{r}$ points from the moving charged particle, at location $(0, a, 0)$, to the point $(x, 0, 0)$ at which $\vec{B}_{\rm p}$ is evaluated. The length of this vector is $r = \sqrt{x^2 + a^2}$, and the unit vector pointing along $\vec{r}$ is

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{\imath} - a\hat{\jmath}}{\sqrt{x^2 + a^2}}.$$

Collecting the square roots into the denominator and simplifying, we get

$$\vec{B}_{\rm p} = \frac{\mu_0 qv}{4\pi}\frac{\hat{\imath} \times (x\hat{\imath} - a\hat{\jmath})}{(x^2 + a^2)^{3/2}} = \frac{\mu_0 qva}{4\pi(x^2 + a^2)^{3/2}}(-\hat{k}),$$

where in the last step we used $\hat{\imath} \times (x\hat{\imath} - a\hat{\jmath}) = (\hat{\imath} \times x\hat{\imath}) - (\hat{\imath} \times a\hat{\jmath}) = -a\hat{k}$, because $\hat{\imath} \times \hat{\imath} = \vec{0}$ and $\hat{\imath} \times \hat{\jmath} = \hat{k}$. This magnetic field points in the $-z$ direction, which is into the plane of the page. We can check this direction by noting that a positively charged particle (assuming $q > 0$) moving to the right constitutes an electric current directed to the right. So we can point our right thumb along the charged particle's velocity vector, with our fingers wrapping around the particle, and notice that along the x axis (where the current-carrying wire is) our fingers are directed into the page, confirming the direction $-\hat{k}$.

Now we can substitute $\vec{B}_{\rm p}$ into Eq. 3 to calculate the magnetic force $d\vec{F}^B_{\rm pw}$ exerted on segment dx of the current-carrying wire. We also note that the current is in the $+x$ direction and substitute $Id\vec{\ell} = Idx\hat{\imath}$ to get

$$d\vec{F}^B_{\rm pw} = Id\vec{\ell} \times \vec{B}_{\rm p} = [\,Idx\hat{\imath}\,] \times \left[\frac{\mu_0 qva}{4\pi(x^2 + a^2)^{3/2}}(-\hat{k})\right]$$

$$= \frac{\mu_0 Iqva\,dx}{4\pi(x^2 + a^2)^{3/2}}\hat{\jmath},$$

where we used the vector identity $\hat{\imath} \times \hat{k} = -\hat{\jmath}$ to determine that the force points upward $(+\hat{\jmath})$ toward the charged particle, if $q > 0$. We can use the right-hand force rule to check that the force (for $q > 0$) points upward by pointing the outstretched fingers of our right hand to the right, along the current, and curling our fingers into the page, along $\vec{B}_{\rm p}$; when we do this, our thumb points upward. Next we integrate $d\vec{F}^B_{\rm pw}$ along the length of the wire to calculate the force $\vec{F}^B_{\rm pw}$ exerted on the wire:

$$\vec{F}^B_{\rm pw} = \frac{\mu_0 Iqv}{4\pi}\hat{\jmath}\left(\int_{x=-\infty}^{\infty}\frac{a\,dx}{(x^2 + a^2)^{3/2}}\right) = \frac{\mu_0 Iqv}{4\pi}\hat{\jmath}\left(\frac{2}{a}\right) = \frac{\mu_0 Iqv}{2\pi a}\hat{\jmath}.$$

We found that the integral in parentheses equals $2/a$ by looking it up on-line, but it turns out that by integrating over the angle θ shown in Figure WG28.7 instead of the coordinate x, we obtain an integral that is much easier to solve. Because we can see from the figure that $a/x = \tan\theta$, we can write $x = a/\tan\theta$ and differentiate to get $dx = -(a\,d\theta/\sin^2\theta)$. Then, because $a/\sqrt{x^2 + a^2} = a/r = \sin\theta$, we can write

$$\frac{a\,dx}{r^3} = -\frac{\sin\theta\,d\theta}{a}.$$

Now we can change the integration variable from x to θ and get

$$\int_{x=-\infty}^{\infty}\frac{a\,dx}{(x^2 + a^2)^{3/2}} = \int_{\theta=\pi}^{0}\left(-\frac{\sin\theta\,d\theta}{a}\right) = \frac{1}{a}\int_{\theta=0}^{\pi}\sin\theta\,d\theta = \frac{2}{a}.$$

Looking at the expressions we obtained above for $\vec{F}^B_{\rm wp}$ and $\vec{F}^B_{\rm pw}$, we can see that $\vec{F}^B_{\rm pw} = -\vec{F}^B_{\rm wp}$, as we were told in the problem statement to expect. ✔

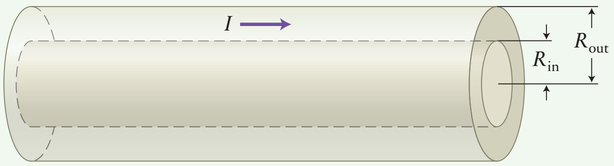
PRACTICE

❹ **EVALUATE RESULT** The force magnitude depends linearly on the current and on the speed and charge of the moving particle, as we expect. That Newton's third law applies here is reassuring, though we know that this is not necessarily to be expected in magnetic interactions. It makes sense that the particle and the wire are attracted toward one another for $q > 0$ and are repelled from one another for $q < 0$, because we can view the motion of the charged particle as an electric current, and we know that the magnetic force between two parallel currents is attractive, while the magnetic force between two antiparallel currents is repulsive.

Guided Problem 28.8 Tubular current

A very long, hollow cylindrical conductor of inner radius R_{in} and outer radius R_{out} carries current I distributed uniformly throughout its cross-sectional area (Figure WG28.8). Derive an expression for the magnitude of the magnetic field due to this current as a function of the radial distance r from the central axis in all regions: (*a*) $0 \leq r \leq R_{\text{in}}$, (*b*) $R_{\text{in}} \leq r \leq R_{\text{out}}$, (*c*) $r > R_{\text{out}}$.

Figure WG28.8



❶ **GETTING STARTED**

1. What is the symmetry of the current distribution?
2. What general law provides the easiest method to compute the magnetic field in each region?

❷ **DEVISE PLAN**

3. How many different applications of your chosen general law will be required to determine the magnetic field in each region?
4. In which region(s) will it be necessary to perform a calculation to determine the current encircled?

❸ **EXECUTE PLAN**

5. In region *b*, what fraction of the current is encircled by a circle of radius *r*?

❹ **EVALUATE RESULT**

6. How does your answer behave as *r* varies over the full range in each region? Does the result from one region match the result from the adjacent region at the boundary?

Answers to Review Questions

1. Particles in motion—either moving in a straight line or spinning about an axis—cause magnetic fields.
2. Magnetic forces are generally not central; that is, they do not point along the line joining two source elements.
3. Yes. For example, the magnetic field due to a charged particle moving in a straight line has no poles.
4. The two field line patterns are similar to each other, and both resemble the pattern of the magnetic field due to an infinitesimally small magnetic dipole.
5. The magnetic field due to a spinning charged particle is identical to the magnetic field due to an infinitesimally small magnetic dipole.
6. The direction of the magnetic dipole moment vector is the same as the direction of the magnetic field line passing through the center of the dipole. This is from S to N in a bar magnet or according to the right-hand dipole rule for a current loop.
7. The right-hand current rule gives magnetic field direction when current direction is known, and vice versa. The right-hand force rule gives magnetic force direction when current direction and magnetic field direction are known, and vice versa. The right-hand dipole rule gives magnetic dipole moment direction when current direction is known, and vice versa. These rules are illustrated in Figure 28.9.
8. The vector sum of forces exerted on the current loop is zero, but there is a nonzero vector sum of torques that tends to align the loop's magnetic dipole moment vector with the external magnetic field.
9. The commutator reverses the direction of the current through the motor's current loop every time the loop rotates one-half turn (just as the loop's magnetic dipole moment aligns with the external field), so that the magnetic torque keeps the loop always rotating in the same direction.
10. Electrostatic field lines originate and terminate on charged particles; magnetic field lines form closed loops. Therefore, the electric flux through a closed surface is proportional to the amount of charge inside, but the corresponding magnetic flux through the surface is always zero.
11. The line integral of the electrostatic field is always zero, regardless of the quantity of charge; that of the magnetic field depends on the amount of current encircled.
12. The line integral of the magnetic field along a closed path is proportional to the current encircled by the path.
13. If the direction in which the Ampèrian path encircles the current is the same as the direction of the magnetic field created by the current, the current makes a positive contribution to the line integral; otherwise, the current makes a negative contribution.
14. The constant shows the relationship between the line integral of the magnetic field around an Ampèrian path and the amount of current encircled by the path.

PRACTICE

15. The wire has cylindrical symmetry. Because of its cylindrical symmetry, the magnetic field lines are concentric circles with constant magnitude at each radius. In using Ampère's law to determine this magnitude, you should choose an Ampèrian path that coincides with one of the field lines.
16. The magnitude of the magnetic field at any point outside the wire is inversely proportional to the radial distance from the wire axis. This is true because all the current in the wire is encircled by the Ampèrian path, and the length of the path is proportional to the radial distance from the axis. At points inside the wire, the magnitude is directly proportional to the radial distance from the axis because in this case only a fraction of the current is encircled by the magnetic field line used as the Ampèrian path. How large that fraction is depends on the ratio of the area encircled by the Ampèrian path to the cross-sectional area of the wire (see Checkpoint 28.12).
17. The magnetic field is uniform on either side of the sheet; in other words, it has the same magnitude at all distances above and below the sheet, with the field magnitude given by the result of Example 28.4: $B = \frac{1}{2}\mu_0 K$, where K is the current per unit width carried by the sheet. The field direction lies in the plane of the sheet and is perpendicular to the current, as given by the right-hand current rule. The field direction is diametrically opposite on the two sides.
18. A solenoid is a tightly wound, long coil of wire used to produce a magnetic field. The field pattern is like that of a bar magnet oriented along the axis of the solenoid, and the field is much greater inside the solenoid than outside. The field is approximately uniform inside, proportional to the current and the number of turns per unit length (Eq. 28.6), and approximately zero outside.
19. A toroid is a tightly wound, donut-shaped coil of wire, as would result from bending a solenoid into a circle so that it ends meet. Its magnetic field is confined to the donut-shaped interior cavity, with magnitude inversely proportional to the distance from the center of the circle (the distance from the central axis of the donut hole), and direction in accordance with the right-hand current rule.
20. The magnetic field inside the solenoid is approximately uniform, but the field inside the toroid is not—it depends inversely on the distance from the axis about which the toroid would roll if it were a car tire.
21. Both fields are directly proportional to the strength of the source ($I\,d\vec{\ell}$ for the magnetic field, dq_s for the electrostatic field) and inversely proportional to the square of the radial separation distance, but the electrostatic field is directed along the unit vector to the field point, while the magnetic field depends on the vector product of the current segment and this unit vector.
22. First use Ampère's law (Eq. 28.1) along a circular path of radius d centered on wire 1, to determine the magnetic field at the location of wire 2 due to the current in wire 1. Then use the magnetic force law (Eq. 27.8) to calculate the force exerted on wire 2 due to the magnetic field of wire 1. The result is $F^B_{12} = \mu_0 \ell I_1 I_2/(2\pi d)$ (Eq. 28.16).
23. Ampère's law is useful only in situations where the field direction is known from symmetry and only the magnitude must be determined. The Biot-Savart law is not limited to symmetrical situations and can be used to compute the magnetic field due to a steady current circulating in any closed circuit, provided, of course, that the integral of Eq. 28.10 is amenable to analytic or numerical evaluation.
24. The vector product $\vec{v}_1 \times \hat{r}_{12}$ comes from the magnetic field produced by particle 1 at the location of particle 2, as given by the Biot-Savart law. The vector product $\vec{v}_2 \times (\vec{v}_1 \times \hat{r}_{12})$ comes from the magnetic interaction between the particles: Particle 2 is subject to the magnetic field created by particle 1.
25. $\epsilon_0\mu_0 = (8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}) = 1.11 \times 10^{-17}\ \mathrm{s^2/m^2} = 1.11 \times 10^{-17}(\mathrm{m/s})^{-2} = 1/(8.99 \times 10^{16}\ \mathrm{m^2/s^2}) = 1/(3.00 \times 10^8\ \mathrm{m/s})^2$. This is the reciprocal of the square of the speed of light.
26. No. Newton's third law requires that the forces exerted by two objects on each other be equal in magnitude and opposite in direction. The magnetic forces exerted by two moving charged particles on each other are typically neither equal in magnitude nor oriented in opposite directions, although some special cases (e.g. two charged particles moving in parallel, side by side) can satisfy $\vec{F}^B_{12} = -\vec{F}^B_{21}$. The reason is that Newton's law, $\vec{F}_{12} = -\vec{F}_{21}$, requires that the system's momentum be constant, and when fields are involved it is not enough to consider only the particles in doing a third-law analysis. Instead, the system for this analysis must include both the particles and their electric and magnetic fields.

Answers to Guided Problems

Guided Problem 28.2

$$\frac{\mu_0 I}{2\pi R} + \frac{(\mu_0 I)(\pi/2)}{4\pi R} = 1.9 \times 10^{-5}\ \mathrm{T},\ \text{pointing into the plane of the page.}$$

Guided Problem 28.4

$$F = \frac{\mu_0 I^2 \ell}{2\pi}\left(\frac{1}{x} - \frac{1}{x + w}\right) = 4.3 \times 10^{-9}\ \mathrm{N},\ \text{attracting the loop toward the wire.}$$

Guided Problem 28.6

For $R_{\text{wire}} < r < R_{\text{shell}}$, $B(r) = \dfrac{\mu_0 I}{2\pi r}$. For $r > R_{\text{shell}}$, $B(r) = 0$ because $I_{\text{enc}} = 0$.

Guided Problem 28.8

(*a*) $B = 0$. (*b*) $B = \dfrac{\mu_0 I}{2\pi r}\dfrac{(r^2 - R^2_{\text{in}})}{(R^2_{\text{out}} - R^2_{\text{in}})}$. (*c*) $B = \dfrac{\mu_0 I}{2\pi r}$.

PRACTICE

Guided Practice by Chapter

29 Changing Magnetic Fields

Review Questions

Answers to these questions can be found at the end of this chapter.

29.1 Moving conductors in magnetic fields

1. What happens when a vertical conducting rod moves east through an external magnetic field that points north?
2. What happens when a conducting loop, moving east, gradually leaves a region of zero magnetic field and enters a region of uniform, north-pointing magnetic field? Assume that the area vector of the loop points north.
3. What is an induced electric current?
4. Do the effects induced in a conducting object that is moving in an external magnetic field depend on the direction of the object's motion relative to the field direction?

29.2 Faraday's law

5. How does changing the reference frame from which you make your measurements affect your explanation of induced current in a conducting loop that is moving from a region of zero magnetic field into a region of uniform external magnetic field?
6. What is Faraday's law?
7. What is electromagnetic induction?

29.3 Electric fields accompany changing magnetic fields

8. Must a magnetic force be exerted on the charge carriers in a conducting loop in order for a current to be induced in the loop?
9. Does the nature of the force that causes electromagnetic induction depend on the choice of reference frame? If so, how? If not, why?
10. What is the difference, if any, between the field lines of the electric field that accompanies a changing magnetic field and the field lines of the electric field of a stationary charged particle?

29.4 Lenz's law

11. What does Lenz's law say about the direction of a current induced in a conducting loop?
12. What general physical principle underlies Lenz's law (that is, why would it make no sense to reverse the direction of Lenz's law)?
13. What are eddy currents?

29.5 Induced emf

14. What is induced emf?
15. When a conducting loop moves through a nonuniform external magnetic field, a current is induced as charge carriers inside the loop start moving. This is true even when the loop is oriented such that the direction of the magnetic force is perpendicular to the direction of the induced current, so the magnetic force does no work on the charge carriers. Where does the energy that causes the charge carriers to move come from?
16. What are (*a*) the quantitative statement of Faraday's law and (*b*) the significance of the negative sign in the equation?
17. Describe two differences between induced emf and potential difference.
18. How does the emf induced in a flat conducting loop that is rotating at a constant rotational speed ω in a uniform magnetic field change as the loop rotates?

29.6 Electric field accompanying a changing magnetic field

19. How is the electric field that accompanies a changing magnetic field related to the induced emf in the region of changing magnetic flux?
20. How does the direction of the electric field vary inside and outside a solenoid whose magnetic field is changing with time?
21. How does the magnitude of the electric field that accompanies the changing magnetic field described in Review Question 18 vary inside and outside the solenoid?

29.7 Inductance

22. Does changing the current in a conducting loop induce an emf in the loop?
23. What is the SI unit of inductance?
24. What is an inductor, and what purpose does it serve in electric circuits?
25. What characteristic of a conducting loop or device does inductance describe? On what properties of the conducting loop or device does inductance depend?

29.8 Magnetic energy

26. Work must be done against the induced emf in an inductor in order to change the current in it. What change in energy occurs to account for the work done?
27. How does the magnetic potential energy stored in an inductor depend on the current through the inductor? Does this energy also depend on other properties of the inductor?
28. How does the magnetic potential energy stored in the magnetic field of an inductor depend on the field magnitude? Is this relationship applicable to magnetic fields other than those of inductors?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The emf you can generate along a rotating metal clothes hanger without using magnets (P, K, G, B)
2. The maximum potential difference developed when you swing a metal curtain rod around your body (H, S, K, G)
3. The maximum current induced when you drop a small but powerful bar magnet through a metal key ring of resistance $R = 0.1\ \text{V/A}$ (R, I, M)
4. The magnetic potential energy stored in a volume of $1\ \text{m}^3$ near Earth's surface (G)
5. The maximum emf induced around the loop formed by your metal key ring when you walk under a 100-A residential power line (W, A, Q, L, F)
6. The maximum magnetic potential energy stored in a doghouse placed under a 100-A neighborhood power line (W, A, Q, C)
7. The magnitude of the electric field due to changing magnetic fields around the outside of your car when you park it under a high-tension power line leading away from a nuclear-powered electricity-generating plant (W, D, J, Q, U, F, X)
8. The magnetic field magnitude needed to have the same energy density as gasoline (N)
9. The length of wire needed to make a 1-H inductor in the form of a solenoid wrapped on a pencil (E, O, V)
10. The maximum potential difference between the two ends of a 2-m metal rod dropped from the top of a 10-story building so that it maintains a horizontal orientation as it falls (K, G, T)

Hints

A. How far above you is the power line?
B. How rapidly can you rotate the loop?
C. What is the volume of a doghouse?
D. What is the magnitude of the current carried by a power line?
E. How is inductance related to the radius and length of the pencil?
F. How does the current vary with time?
G. What is the magnetic field magnitude?
H. What is a typical length of a curtain rod?
I. Over what area is the magnetic flux changing?
J. How far above the car is the power line?
K. What is the source of the magnetic field?
L. What is the area of the conducting loop?
M. What is the smallest likely time interval for the change in flux?
N. What is the energy content of a liter of gasoline?
O. What are the length and radius of a standard pencil?
P. What is the area enclosed by the loop formed by the hanger?
Q. How can you estimate the maximum magnetic field strength?
R. What is the magnetic field strength near a pole of the magnet?
S. What is the average translational speed of points on the rod?
T. What is the maximum speed of the rod?
U. What is the "radius" of the car?
V. What is the circumference of each winding?
W. How should you model the power line?
X. How are the induced emf and electric field related for a circular path of radius R that encircles a changing magnetic flux?

Key (all values approximate)

A. 6 m; **B.** spinning at the end of a string—say, three times/s; **C.** $1\ \text{m}^3$; **D.** for a 1-GW nuclear plant, about 1×10^3 A; **E.** $L = \mu_0 N^2(\pi R^2)/\ell$; **F.** it goes from positive maximum to negative maximum and back in $\frac{1}{60}$ s; **G.** 5×10^{-5} T; **H.** 1 m; **I.** the magnet cross section—say, $10^{-4}\ \text{m}^2$; **J.** 20 m; **K.** Earth; **L.** $10^{-3}\ \text{m}^2$; **M.** the time interval during which the magnet passes through the ring, 10^{-2} s; **N.** about 3×10^7 J; **O.** $\ell = 0.2$ m, $R = 3 \times 10^{-3}$ m; **P.** $3 \times 10^{-2}\ \text{m}^2$; **Q.** from the form of Ampere's law given in Example 28.3: $B = \mu_0 I/2\pi d = (2 \times 10^{-7}\ \text{T} \cdot \text{m/A})I/d$; **R.** 1 T; **S.** 5 m/s, assuming a rotation rate slower than 2 rev/s; **T.** 2×10^1 m/s; **U.** 1 m; **V.** $2\pi R$; **W.** as a long, straight current-carrying wire; **X.** $\mathscr{E} = 2\pi RE$

PRACTICE

Worked and Guided Problems

Procedure: Calculating inductances

The inductance of a current-carrying device or current loop is a measure of the emf induced in the device or loop when current is changed. To determine the inductance of a particular device or current loop, follow these four steps.

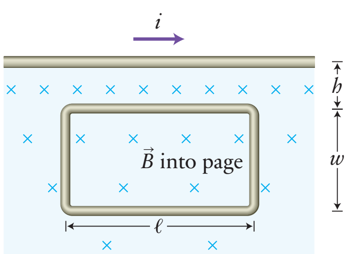
1. Derive an expression for the magnitude of the magnetic field in the current-carrying device or current loop as a function of the current. Your expression should depend only on the current I and possibly—but not necessarily— the position within the device or current loop.
2. Calculate the magnetic flux Φ_B through the device or current loop. If the expression you derived in step 1 depends on position, you will have to integrate that expression over the volume of the device or circuit. Use symmetry to simplify the integral and divide the device into segments on which B is constant.
3. Substitute the resulting expression you obtained for Φ_B into Eq. 29.21. As you take the derivative with respect to time, keep in mind that only the current varies with respect to time, so you should end up with an expression that contains the derivative dI/dt on both sides of the equal sign.
4. Solve your expression for L after eliminating dI/dt.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 29.1 Rectangular loop near a wire

The long, straight wire in Figure WG29.1 carries a current i that varies as a function of time according to $i(t) = a + bt$, where $a = 0.50$ A and $b = 4.0$ A/s during the time interval $0 < t \leq 2.0$ s. For all instants $t > 2.0$ s, the current is constant. The wire is placed a distance $h = 0.040$ m above and in the plane of a stationary rectangular loop of wire. The loop dimensions are width $w = 0.10$ m and length $\ell = 0.60$ m, and its resistance is $R = 2.8$ V/A. For the time interval $0 < t \leq 2.0$ s and for all instants $t > 2.0$ s, determine the induced emf in the loop and the magnitude and direction of the induced current in the loop.

Figure WG29.1



❶ GETTING STARTED We are given a description of a current-carrying, straight wire positioned above a rectangular loop made of conducting material, with the wire sitting in the plane defined by the loop. Our tasks are to obtain values for the emf and the current induced in the loop and to determine the direction of that induced current. We know that the emf and current are induced by a changing magnetic flux, and we know that the magnetic flux through the loop is changing because that flux is due to the magnetic field created by the current in the wire. Because that current changes during the time interval $0 < t \leq 2.0$ s, the magnetic field around the wire is also changing.

❷ DEVISE PLAN We can calculate the magnetic field created by the current in the straight wire using Ampere's law. Recall from Example 28.3 that the magnitude of this magnetic field is inversely proportional to the perpendicular distance r from the wire: $B = \mu_0 I/(2\pi r)$. Because the field lines of this magnetic field form circles around the wire, there is a magnetic flux through the rectangular loop, given by Eq. 29.5:

$$\Phi_B = \int \vec{B} \cdot d\vec{A}.$$

Because the magnetic field magnitude at any location enclosed by the loop is a function of how far from the wire that location is, we need to integrate the magnetic field over the loop area in order to calculate the magnetic flux.

PRACTICE

Once we have the changing magnetic flux, we can use Faraday's law to calculate the induced emf, which depends on a time-varying magnetic flux. Hence we need to analyze the derivative of the magnetic flux with respect to time for the interval $0 < t \leq 2.0$ s in order to calculate the emf induced in the loop:

$$\mathscr{E}_{\text{ind}} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}.$$

Our final step is to use Eq. 29.4, $I_{\text{ind}} = \mathscr{E}_{\text{ind}}/R$, to calculate the magnitude of the induced current and to use Lenz's law to establish its direction.

③ **EXECUTE PLAN** The current in the wire and hence the magnetic flux are changing only during the interval $0 < t \leq 2.0$ s. For all instants $t > 2.0$ s, the current in the wire, the magnetic field this current creates, and the magnetic flux through the loop are all constant, and therefore both the induced emf and the induced current are zero for $t > 2.0$ s. ✔

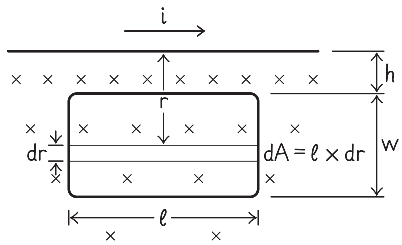
For the time interval $0 < t \leq 2.0$ s, the magnitude of the magnetic field due to a current-carrying wire is given by

$$B = \frac{\mu_0 i}{2\pi r} = \frac{\mu_0(a + bt)}{2\pi r},$$

where r is a perpendicular distance away from the wire. We begin by drawing, across the loop area, a thin strip of width dr, length ℓ, and area $dA = \ell\, dr$ (Figure WG29.2). We can consider the magnetic field to be essentially constant over the area of this infinitesimal strip. The differential magnetic flux $d\Phi_B$ through the differential area element dA is then

$$d\Phi_B = B\,dA = B\ell\,dr = \frac{\mu_0(a + bt)}{2\pi r}\ell\,dr.$$

Figure WG29.2



The magnetic flux through the loop is found by integrating this expression from $r = h$ to $r = h + w$:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \frac{\mu_0(a + bt)}{2\pi}\int_{r=h}^{r=h+w}\frac{\ell\,dr}{r}$$
$$= \frac{\mu_0(a + bt)\ell}{2\pi}\ln\left(\frac{h + w}{h}\right).$$

Applying Faraday's law gives us

$$\mathscr{E}_{\text{ind}} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\frac{\mu_0(a + bt)\ell}{2\pi}\ln(1 + w/h)\right]$$
$$= -\frac{\mu_0 b\ell}{2\pi}\ln(1 + w/h).$$

The magnitude of the induced emf is thus

$$|\mathscr{E}_{\text{ind}}| = (2 \times 10^{-7}\ \text{T}\cdot\text{m/A})(4.0\ \text{A/s})(0.60\ \text{m}) \times \ln[1 + (0.10\ \text{m}/0.040\ \text{m})]$$
$$= 6.0 \times 10^{-7}\ \text{T}\cdot\text{m}^2/\text{s}.$$

(Note that the induced emf is independent of t for the interval $0 < t \leq 2.0$ s.)

PRACTICE

Because the volt is the unit of emf, we need to check the unit in this equation before we can say we are done. Using the substitution $1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}$ gives us

$$\frac{\mathrm{T \cdot m^2}}{\mathrm{s}} = \frac{\mathrm{N \cdot m^2}}{\mathrm{A \cdot m \cdot s}} = \frac{\mathrm{N \cdot m}}{\mathrm{A \cdot s}} = \frac{\mathrm{N \cdot m}}{\mathrm{C}} = \frac{\mathrm{J}}{\mathrm{C}} = \mathrm{V}.$$

Thus the magnitude of the induced emf is

$$|\mathscr{E}_{\text{ind}}| = 6.0 \times 10^{-7}\ \mathrm{V}. ✔$$

The current $i(t)$ in the wire produces a magnetic flux that within the area of the loop is directed into the page in Figure WG29.2. The magnetic field increases as time passes, and hence the magnetic flux through the loop also increases. By Lenz's law, the direction of the current induced in the loop must be counterclockwise in order to produce an induced magnetic field out of the plane of the page to counteract the increase in flux due to the time-varying magnetic field. ✔

The magnitude of the induced current is

$$I_{\text{ind}} = \frac{|\mathscr{E}_{\text{ind}}|}{R} = \frac{6.0 \times 10^{-7}\ \mathrm{V}}{2.8\ \mathrm{V/A}} = 2.1 \times 10^{-7}\ \mathrm{A}. ✔$$

❹ **EVALUATE RESULT** Moving a magnet through the many windings of a solenoid induces a current on the order of microamperes, so we should not be surprised that the current through one turn of wire is about 0.2 μA even though the current in the wire is fairly great and the loop is close to the wire. Inductive effects are small.

Looking at the loop center gives us a way of checking our $\mathscr{E}_{\text{ind}}$ value. The distance from the wire to the center of the loop is $r = h + (w/2) = 0.090$ m, and we can use Ampere's law to obtain the magnetic field magnitude at this location at $t = 0$:

$$\begin{aligned} B(t = 0) &= \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 (a + bt)}{2\pi r} \\ &= (2 \times 10^{-7}\ \mathrm{T \cdot m/A})\left(\frac{0.50\ \mathrm{A}}{0.090\ \mathrm{m}}\right) \\ &= 1.1 \times 10^{-6}\ \mathrm{T}. \end{aligned}$$

At $t = 2.0$ s, the magnetic field at the loop center is 17 times greater than this value: $B(t = 2.0\ \mathrm{s}) = 1.9 \times 10^{-5}$ T. [The factor 17 is the ratio of the currents at these two values of t: $(8.5\ \mathrm{A})/(0.50\ \mathrm{A}) = 17$.] The average change in the magnetic field at the center is then

$$\begin{aligned} \frac{\Delta B}{\Delta t} &= \frac{(17 - 1)(1.1 \times 10^{-6}\ \mathrm{T})}{2.0\ \mathrm{s}} \\ &= 8.9 \times 10^{-6}\ \mathrm{T/s}. \end{aligned}$$

We can estimate the average time derivative of the magnetic flux by multiplying this value by the area of the loop:

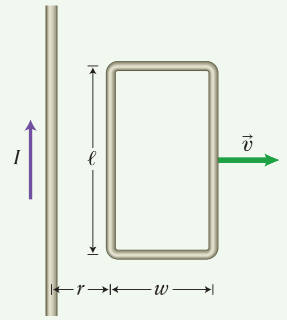
$$\begin{aligned} \frac{\Delta \Phi_B}{\Delta t} &\approx \left(\frac{\Delta B}{\Delta t}\right)(\ell w) = (8.9 \times 10^{-6}\ \mathrm{T/s})(0.60\ \mathrm{m})(0.10\ \mathrm{m}) \\ &= 5.3 \times 10^{-7}\ \mathrm{V}. \end{aligned}$$

This is close to our calculated value, $\mathscr{E}_{\text{ind}} = 6.0 \times 10^{-7}$ V, which gives us some assurance that our work is correct.

Guided Problem 29.2 Moving loop

A rectangular loop of length ℓ and width w moves at constant velocity $\vec{v}$ away from a long, straight wire carrying a current I (Figure WG29.3). The wire lies in the plane defined by the loop, and the loop's resistance is R. What are the direction and magnitude of the current induced in the loop at the instant the side nearest the wire is a perpendicular distance r away from the wire?

Figure WG29.3



1 GETTING STARTED

1. Look for similarities and differences between this problem and Worked Problem 29.1. Will a modification of the approach we used in that problem work here?
2. Which portions of the results of Worked Problem 29.1 can be used directly?

2 DEVISE PLAN

3. Determine an expression for the magnetic field at a perpendicular distance r from the wire.
4. Do you need to integrate the magnetic field over the loop in order to calculate the magnetic flux? If so, can you use the same differential area element we used in Worked Problem 29.1?
5. You will likely need to differentiate the magnetic flux through the loop with respect to t to determine the rate of change of the magnetic flux. Remember that because the loop is moving, r is a function of time.
6. Will calculating the time derivative of the magnetic flux allow you to express the induced emf in terms of the loop velocity? If so, will that result lead you to the induced current?

3 EXECUTE PLAN

7. What are the appropriate limits of integration for the flux integral?
8. Here are two possibly useful derivatives:

$$\frac{d}{dt}\ln(r+w) = \frac{1}{r+w}\frac{dr}{dt} \quad \text{and} \quad \frac{d}{dt}\ln(r) = \frac{1}{r}\frac{dr}{dt}.$$

9. Use Lenz's law to determine the direction of I_{ind}. Remember to consider not only the direction of the magnetic field through the loop but also whether the magnetic flux is *increasing* or *decreasing* over time.

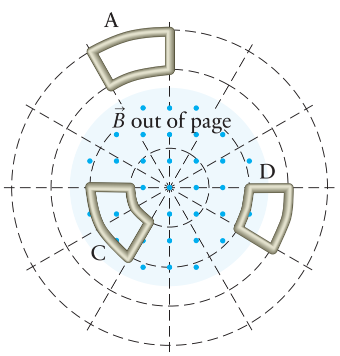
4 EVALUATE RESULT

10. Check that the current direction you found opposes the change in the magnetic flux. You can check by determining which way the magnetic force exerted by the wire on the loop points according to your direction for the induced current. The magnetic force exerted on the loop must oppose the loop's motion.
11. How does your result for emf compare with the value obtained in Worked Problem 29.1? Is this reasonable?

Worked Problem 29.3 Electric field accompanying a changing magnetic field

Figure WG29.4 shows three conducting loops, A, C, and D, positioned in a uniform magnetic field whose magnitude increases with time. The concentric circles are all equidistant from one another, at radii R, $2R$, $3R$, and $4R$, and the radial lines divide each circle into 12 equal segments. The region of nonzero magnetic field ends at $r = \frac{5}{2}R$. Obtain expressions for the magnitude of the induced emf in each of the three conducting loops A, C, and D. Then use Lenz's law to determine the direction of the induced current in each of the three loops.

Figure WG29.4



PRACTICE

❶ **GETTING STARTED** The figure shows three conducting loops, each perpendicular to a changing magnetic field. We need to determine the induced emf, induced current, and direction of the induced current in each loop. It looks as if there is a changing magnetic flux through two of the three loops, so we expect to obtain a nonzero induced emf and therefore an induced current in loops C and D. We also expect that we will be able to simplify the problem by exploiting the cylindrical symmetry of the shaded region of the figure and by using the fact that the magnetic field is uniform within this shaded region. Finally, we remember seeing a similar example of cylindrical symmetry worked out in Example 29.7.

❷ **DEVISE PLAN** Because of the cylindrical symmetry of the changing magnetic field, we know that the electric field lines that accompany this changing magnetic field must be concentric circles, such as the circles drawn in the figure.

The magnitude $E(r)$ of the electric field that accompanies the changing magnetic field can be obtained using Eq. 29.17, $\oint \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt$. We compute the line integral around each of the four circles at radii $r = R, 2R, 3R, 4R$. Because each of the circles lies along an electric field line, and the electric field magnitude is constant along any field line, we can write $|\oint \vec{E} \cdot d\vec{\ell}| = 2\pi r\, E(r)$ for the circular path at radius r, leaving us with

$$2\pi r\, E(r) = \left| -\frac{d\Phi_B}{dt} \right|. \tag{1}$$

We take the absolute value because $E(r)$ is a magnitude and therefore must be non-negative.

Because the magnetic field is uniform within the shaded region of the diagram, we can write the magnetic flux through each circle simply as the product $\Phi_B = BA$, where B is the magnetic field magnitude and A is the *shaded* area enclosed by the circle. If we call the radius of the shaded area $R_{\text{shaded}} = \frac{5}{2}R$, then for $r \leq R_{\text{shaded}}$, we have $\Phi_B = \pi r^2 B$, and for $r \geq R_{\text{shaded}}$, we have $\Phi_B = \pi R_{\text{shaded}}^2 B$.

Once we have the electric field magnitude $E(r)$, we can combine Eqs. 29.8 and 29.17 to determine the emf induced around each of the three conducting loops: $\mathscr{E}_{\text{ind}} = \oint \vec{E} \cdot d\vec{\ell}$. We must break the line integral around each conducting loop into four parts: the two radial sides and the two arc sides. Because the electric field lines are concentric circles, the line integral of the electric field for each loop is zero along the radial sides and nonzero along the arcs.

Finally, we can apply Lenz's law to determine the direction of the induced current in each loop.

❸ **EXECUTE PLAN** Combining Eq. 1 with our expressions for the magnetic flux Φ_B through each of the concentric circles, we see that for $r \leq R_{\text{shaded}}$, we have $2\pi r E = \pi r^2 |dB/dt|$, so $E(r) = (r/2)|dB/dt|$. For $r \geq R_{\text{shaded}}$, we have $2\pi r E = \pi R_{\text{shaded}}^2 |dB/dt|$, and thus $E(r) = (R_{\text{shaded}}^2/2r)|dB/dt|$. Because the magnetic field points out of the page and increases in magnitude with time, by Lenz's law the electric field must point clockwise around each circle.

Loop A is entirely outside the magnetic field, which means there is no magnetic flux through this loop and therefore no induced emf and no induced current. (Note that there is a nonzero electric field along the two arcs of loop A. However, the integral of the electric field along the inner arc must be the negative of the integral along the outer arc, so the line integral around the closed path is zero. We can see that the two canceling contributions to the emf are equal in magnitude because (for $r > R_{\text{shaded}}$) the arc length $\propto r$, and the electric field $\propto 1/r$.) ✔

Loop C lies entirely inside the magnetic field. The radius of the inner arc is R, and the radius of the outer arc is $2R$. Moving clockwise around loop C as we evaluate the line integral $\mathscr{E}_{\text{ind}} = \oint \vec{E} \cdot d\vec{\ell}$, we determine a positive contribution for the outer arc ($\vec{E}$ points in the same direction as $d\vec{\ell}$), zero for the radial segment ($\vec{E} \perp d\vec{\ell}$), a negative contribution for the inner arc ($\vec{E}$ points opposite $d\vec{\ell}$), and zero again for the second radial segment, thus completing the loop. Because each arc subtends $\frac{1}{6}$ of a circle, the arc lengths are $2\pi r/6$ for $r = 2R$ and $r = R$. Adding the four contributions gives

$$\mathscr{E}_{\text{C}} = \frac{2\pi(2R)E(2R)}{6} + 0 - \frac{2\pi R E(R)}{6} + 0.$$

Then substituting $E(r) = (r/2)(dB/dt)$ at both $r = 2R$ and $r = R$ gives us

$$\mathscr{E}_{\text{C}} = \frac{2\pi}{6}\left(\frac{4R^2}{2} - \frac{R^2}{2}\right)\frac{dB}{dt} = \frac{\pi R^2}{2}\frac{dB}{dt}$$

for the magnitude of the emf induced in loop C. Because the magnetic flux through the loop points out of the page and is increasing in magnitude, the direction of the induced current must be clockwise around loop C. ✔

We determine the induced emf for loop D in a similar manner. In this case, the outer arc, at $r = 3R$, is outside the magnetic field, while the inner arc, at $r = 2R$, is inside the magnetic field. Also, each arc subtends only $\frac{1}{12}$ of a circle, so the arc lengths are $2\pi r/12$ for $r = 3R$ and $r = 2R$. Integrating $\mathscr{E}_{\text{ind}} = \oint \vec{E} \cdot d\vec{\ell}$ clockwise over the four sides of loop D (outer arc, radial segment, inner arc, and radial segment) gives us

$$\mathscr{E}_{\text{D}} = \frac{2\pi(3R)E(3R)}{12} + 0 - \frac{2\pi(2R)E(2R)}{12} + 0.$$

PRACTICE

For the inner arc, because $2R < R_{\text{shaded}}$, we again use $E(r) = (r/2)(dB/dt)$ and obtain $E(2R) = R(dB/dt)$. For the outer arc, because $3R > R_{\text{shaded}}$, we instead use $E(r) = (R^2_{\text{shaded}}/2r)\,(dB/dt)$, so then

$$E(3R) = \frac{(5R/2)^2}{6R}\frac{dB}{dt} = \frac{25R}{24}\frac{dB}{dt}.$$

Substituting these values for the electric field, we obtain the magnitude of the induced emf in loop D:

$$\mathscr{E}_{\text{D}} = \frac{2\pi}{12}\left[(3R)\frac{25R}{24} - (2R)R\right]\frac{dB}{dt} = \frac{3\pi R^2}{16}\frac{dB}{dt}.$$

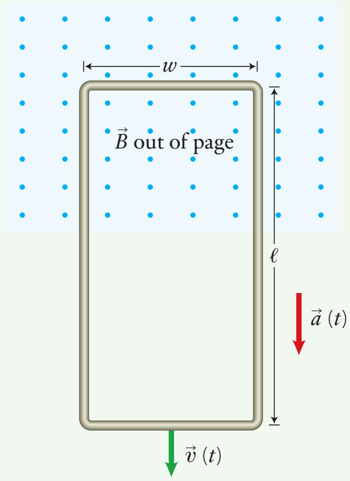
Because the magnetic flux through the loop again points out of the page and is increasing in magnitude, the induced current must be clockwise around loop D. ✔

❹ **EVALUATE RESULT** Because loop A is completely outside the magnetic field, it is reasonable that this loop has no induced emf. Loops C and D experience the changing magnetic flux to different degrees, so we expect their induced emfs to differ. Loop D, with the smaller area affected by the magnetic flux, has the smaller induced emf, as we expect.

Guided Problem 29.4 A falling loop

Under the influence of gravity, a rectangular wire loop of mass m, width w, length ℓ, and resistance R falls out of a uniform magnetic field $\vec{B}$ (Figure WG29.5). The loop is released from rest at the instant its lower edge is just barely inside the magnetic field. At the instant shown in the figure, the loop is exiting the magnetic field at speed $v(t) > 0$ (that is, the loop is moving downward). Obtain an expression for the speed of the loop as a function of time, for the time interval from the instant the loop is released to the instant just before its top edge exits the magnetic field. What happens after the top edge of the loop has left the field?

Figure WG29.5



❶ **GETTING STARTED**

1. As the loop falls, the magnetic flux through the loop changes, and that change in flux induces a current in the loop. The induced current slows the loop down (Lenz's law). A good first step is to seek a relationship between the magnetic flux through the loop at some instant t and the downward velocity of the loop. Be careful, though; at any instant, the magnetic field is zero over the lower region of the loop area and uniform and nonzero over the upper region.
2. Induced emf and induced current may be more useful here than simple magnetic flux.
3. You may need additional information to solve for the speed, so consider force analysis or energy analysis. Remember that you seek a relationship between speed and time.
4. It may be necessary to solve a differential equation to determine $v(t)$, so do not rule out acceleration as a useful quantity.

❷ **DEVISE PLAN**

5. Obtain an expression for $d\Phi_B/dt$ that involves $v(t)$.
6. Current may relate to both magnetic flux and magnetic force. What is the magnitude of the induced current in the loop at the instant shown in Figure WG29.5? (Use $I_{\text{ind}} = \mathscr{E}_{\text{ind}}/R$.)
7. Would a free-body diagram for all the vertical forces exerted on the loop be helpful? Recall that a magnetic force is exerted on the loop. Express the magnitude and direction of this force in terms of the quantities given. (Hint: Use Eq. 28.13, $\vec{F} = I_{\text{ind}}\vec{\ell} \times \vec{B}$.) Use the vertical component of this force to obtain an expression for the vertical motion of the loop.
8. You know that $a = dv/dt$, so is it possible to write a differential equation for the speed as a function of time?

PRACTICE

3 EXECUTE PLAN

9. If you get a differential equation of the form $dv/dt = \alpha - \beta v$, where $v(0) = 0$, you might want to try a solution of the form $v = (\alpha/\beta)(1 - e^{-\beta t})$.

4 EVALUATE RESULT

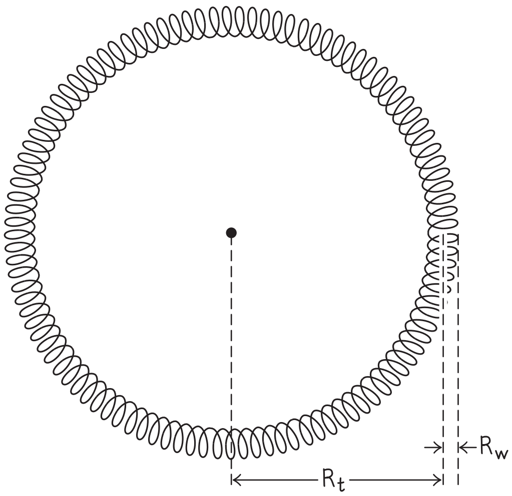
10. Taking a derivative of your expression for $v(t)$ should yield acceleration, giving you a test of the reasonableness of your result.

Worked Problem 29.5 Inductance of a toroid

Determine an expression for the inductance L of a toroid (see Section 28.6) for which the windings have a circular cross section of radius R_w and the toroid radius is $R_t \gg R_w$.

1 GETTING STARTED We begin by making a sketch of the toroid (Figure WG29.6). We know from Section 28.6 that the magnetic field is entirely contained within the cavity formed by the windings of the toroid and that the magnetic field lines are circular loops centered on the toroid's central axis (an axis perpendicular to the plane of the figure).

Figure WG29.6



2 DEVISE PLAN To calculate the inductance of the toroid, we can follow the steps in the Procedure box at the beginning of this *Practice* chapter. We first need to determine the magnetic field magnitude in terms of the current I through the toroid. We derived this expression in Section 28.6 and found that at a radial distance r from the center of a toroid that has N windings, $B = \mu_0 NI/(2\pi r)$ (Eq. 28.9). The condition $R_t \gg R_w$ means that r in Eq. 28.9 can be approximated as $r \approx R_t$, so we have $B = \mu_0 NI/(2\pi R_t)$. Having this expression for B, we can calculate the magnetic flux as a function of the current. We can then substitute this result into Eq. 29.21, $d\Phi_B/dt = L(dI/dt)$, and solve for L.

3 EXECUTE PLAN The magnetic flux through one winding is the area of one winding, πR_w^2, multiplied by the magnetic field magnitude B inside the winding. For the entire toroid, therefore, the magnetic flux is

$$\Phi_B = N(\pi R_w^2)B = \frac{\mu_0 N^2 I R_w^2}{2R_t}.$$

Substituting this expression for Φ_B into Eq. 29.21 yields

$$\frac{d}{dt}\frac{\mu_0 N^2 I R_w^2}{2R_t} = L\frac{dI}{dt}.$$

The only time-dependent factor on the left is the current, so

$$L = \frac{\mu_0 N^2 R_w^2}{2R_t}. \checkmark$$

PRACTICE

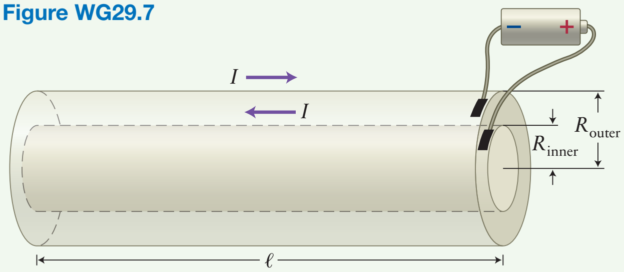
❹ **EVALUATE RESULT** Our result shows that the inductance of a toroid increases as the square of the number of windings N. This makes sense because both the magnetic field in the toroid cavity and the magnetic flux through the toroid windings increase with N. That inductance also depends on the square of the winding radius R_w makes sense because increasing R_w increases the area enclosed by each winding and therefore increases the magnetic flux by the factor R_w^2.

Finally, the inductance is inversely proportional to the radius R_t of the toroid, which makes sense because the magnetic field and therefore the magnetic flux decrease as $1/R_t$. (Note: This expression for inductance is valid only for toroids in which $R_t \gg R_w$.)

Guided Problem 29.6 Coaxial cable inductance and magnetic energy

Figure WG29.7 shows a coaxial cable that consists of two concentric long, hollow cylinders of negligible resistance. The inner cylinder has radius R_{inner}, the outer cylinder has radius R_{outer}, and the length of both cylinders is $\ell \gg R_{outer}$. There is a current I in the cable, and the current direction is from the battery into the inner cylinder and then through the outer cylinder to the battery. What is the magnetic potential energy stored in the magnetic field? What is the inductance of the cable?

Figure WG29.7



❶ **GETTING STARTED**

1. We must know the magnetic field magnitude as a function of the radius in order to determine the amount of magnetic potential energy stored. Is Ampere's law (Eq. 28.1) useful for this?
2. What is the relationship between the energy stored in the magnetic field of an inductor and the inductance?

❷ **DEVISE PLAN**

3. To obtain expressions for the magnetic field magnitude everywhere, apply Ampere's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$ (Eq. 28.1), outside the cable, where the radial distance $r > R_{outer}$; then inside the inner cylinder, where $r < R_{inner}$; and finally between the cylinders, where $R_{inner} < r < R_{outer}$.
4. For the region where the magnetic field magnitude is nonzero, use Eq. 29.29 to determine the magnetic energy density.
5. The quantity of magnetic potential energy stored can be found by integrating the magnetic energy density over the volume of space where $B \neq 0$. Is the magnetic field uniform or nonuniform in this region? You may need to choose a volume element for your integral. Based on the symmetry in the problem, the volume element for a cylindrical shell of radius r, thickness dr, and length ℓ is $dV = 2\pi r \ell \, dr$.
6. What are the appropriate limits of integration?
7. Once you have U^B, you can use it and Eq. 29.25 to determine the inductance in the cable.

❸ **EXECUTE PLAN**

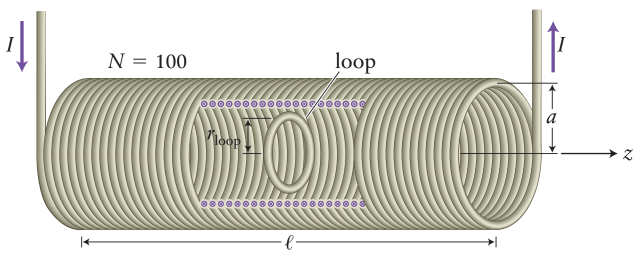
❹ **EVALUATE RESULT**

8. Consider what happens to the stored energy as the dimensions of the coaxial cable are changed.

Worked Problem 29.7 Electric fields and Faraday's law

A solenoid is made by wrapping a wire 100 times around a hollow, nonmagnetic cylinder (Figure WG29.8). The solenoid's length is $\ell = 2.5 \times 10^{-1}$ m, and its radius is $a = 3.0 \times 10^{-2}$ m. A small wire loop of radius $r_{loop} = 1.0 \times 10^{-2}$ m is placed at the midpoint of the solenoid's length and concentric with the solenoid. The loop has resistance $R = 5.0$ V/A. During the time interval $t = 0$ to $t = 2.0$ s, the current through the solenoid increases as $I(t) = bt$, where $b = 2.0 \times 10^{-1}$ A/s. After $t = 2.0$ s, the current remains constant. For $0 < t < 2.0$ s and for $t > 2.0$ s, what are (*a*) the emf induced in the loop, (*b*) the magnitude and direction of the electric field that accompanies the changing magnetic flux in the loop, and (*c*) the magnitude and direction of the current induced in the loop?

Figure WG29.8



PRACTICE

❶ **GETTING STARTED** For $t > 2.0$ s, the current is not changing, so the induced emf, electric field, and induced current are all zero. Thus we have already completed one part of our task. ✔

All we need to consider now is what happens during the interval $0 < t < 2.0$ s, when the current *is* changing. During this interval, there is a changing magnetic flux through the solenoid cavity and hence through the area enclosed by the loop. Therefore, by the Eq. 29.17 form of Faraday's law, there must be an electric field accompanying the changing magnetic flux through the loop area. This electric field is responsible for the emf that drives the induced current around the loop. We can determine the direction of the induced current by using Lenz's law or by determining the direction of the electric field. Note that the direction of the current through the solenoid is shown in Figure WG29.8.

❷ **DEVISE PLAN** In order to determine the induced emf in the loop, we must first calculate the magnetic field inside the solenoid. For this we can use Eq. 28.6, $B = \mu_0 nI$, where n is the number of windings per unit length of the solenoid:

$$B = \mu_0 nI = \mu_0 \frac{N}{\ell} I.$$

We can integrate this expression for B to determine the magnetic flux inside the loop from Eq. 29.5:

$$\Phi_B = \int \vec{B} \cdot d\vec{A},$$

where $d\vec{A}$ is the area vector perpendicular to the solenoid axis. Then the Eq. 29.8 form of Faraday's law gives us the emf induced in the loop:

$$\mathscr{E}_{\text{ind}} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(\int \vec{B} \cdot d\vec{A}\right).$$

We should then be able to obtain the electric field magnitude by using the definition of emf as the line integral of the electric field around the loop. Combining Eqs. 29.8 and 29.17, we have

$$\mathscr{E}_{\text{ind}} = \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}, \tag{1}$$

where the line integral is taken along the circle of radius r, centered on the solenoid axis, around which we wish to determine the electric field magnitude $E(r)$ due to the changing magnetic field. To determine the direction of the electric field, we infer by analogy with Checkpoint 29.12 that the electric field lines will form circles centered around the solenoid axis, in the direction required by Lenz's law.

Finally, we can use Eq. 29.4 to obtain the induced current in the loop from the emf and the resistance of the loop:

$$I_{\text{ind}} = \frac{|\mathscr{E}_{\text{ind}}|}{R}.$$

❸ **EXECUTE PLAN** (a) The magnetic field inside the solenoid is

$$\vec{B} = \mu_0 nI\,\hat{k} = \frac{\mu_0 NI}{\ell}\hat{k} = \frac{\mu_0 Nbt}{\ell}\hat{k}.$$

(From the orientation of the solenoid loops and the direction of the current in Figure WG27.8, we see that the magnetic field produced by the solenoid is in the positive z direction, so the direction of $\vec{B}$ is given by $\hat{k}$.) We choose the orientation of the area element $d\vec{A}$ in the magnetic flux integral to point in the positive $\hat{k}$ direction, parallel to the magnetic field: $d\vec{A} = dA\,\hat{k}$. The magnetic flux integral through the loop is then

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int B\,\hat{k} \cdot dA\,z = \int B\,dA.$$

Because the magnetic field is constant inside the solenoid and the area of the loop is $A = \pi r_{\text{loop}}^2$, the magnetic flux inside the loop is given by

$$\Phi_B = \int B\,dA = \frac{\mu_0 Nbt}{\ell}\int dA = \frac{\mu_0 Nbt}{\ell}\pi r_{\text{loop}}^2.$$

PRACTICE

The induced emf is then

$$\mathscr{E}_{\text{ind}} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(\frac{\mu_0 Nbt}{\ell}\right)\pi r_{\text{loop}}^2 = -\frac{\mu_0 Nb\pi r_{\text{loop}}^2}{\ell} \tag{2}$$

$$= -\frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(100)(2.0 \times 10^{-1}\,\text{A/s})\pi(1.0 \times 10^{-2}\,\text{m})^2}{(2.5 \times 10^{-1}\,\text{m})}$$

$$= -3.2 \times 10^{-8}\,\text{T}\cdot\text{m}^2/\text{s} = -3.2 \times 10^{-8}\,\text{V}. ✔$$

(*b*) We obtain the electric field that accompanies the changing magnetic flux by extracting $\vec{E}$ from the line integral of Eq. 1. For this case, that equation becomes

$$\mathscr{E}_{\text{ind}} = \oint_{\text{loop}} \vec{E}\cdot d\vec{l} = E(2\pi r_{\text{loop}}). \tag{3}$$

Note that E has the same value everywhere on the loop because the loop is centered on the symmetry axis of the solenoid, and the electric field has the same magnitude everywhere on the circle of radius r_{loop} inside the solenoid. Substituting the expression for $\mathscr{E}_{\text{ind}}$ given by Eq. 2 into Eq. 3 and taking the absolute value (to obtain the electric field magnitude E) give us

$$E(2\pi r_{\text{loop}}) = \frac{\mu_0 Nb\pi r_{\text{loop}}^2}{\ell}$$

$$E = \frac{1}{2\pi r_{\text{loop}}}\frac{\mu_0 Nb\pi r_{\text{loop}}^2}{\ell} = \frac{\mu_0 Nb r_{\text{loop}}}{2\ell}$$

$$= \frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(100)(2.0 \times 10^{-1}\,\text{A/s})(1.0 \times 10^{-2}\,\text{m})}{2(2.5 \times 10^{-1}\,\text{m})}$$

$$= 5.0 \times 10^{-7}\,\text{T}\cdot\text{m/s}. ✔$$

Because $\vec{B}$ points along the z axis and increases with time, the resulting electric field lines must loop around clockwise, as viewed from the positive z axis: This is the direction in which our fingers curl if we point our right thumb in the $-z$ direction, to oppose $d\vec{B}/dt$ (using the right-hand dipole rule). ✔

(*c*) The direction of the induced current is the same direction as the electric field. So the direction of the induced current in the loop is opposite the direction of the current in the solenoid. ✔

The magnitude of the induced current in the loop is

$$I_{\text{ind}} = \frac{|\mathscr{E}_{\text{ind}}|}{R} = \frac{\mu_0 Nb\pi r_{\text{loop}}^2}{R\,\ell} = \frac{3.2 \times 10^{-8}\,\text{V}}{5.0\,\text{V/A}}$$

$$= 6.4 \times 10^{-9}\,\text{A}. ✔$$

4 **EVALUATE RESULT** The direction of the induced current can also be determined from Lenz's law. The magnetic flux through the loop is positive (the magnetic field points in the positive z direction) and increases as the current in the solenoid increases. So, the direction of the induced current is opposite the direction of the current in the solenoid in order to oppose the change that created it.

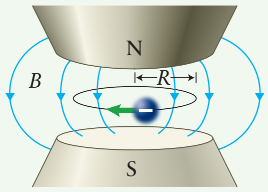
Guided Problem 29.8 Betatron

An electron of mass m and charge $q = -e$ located in a nonuniform magnetic field between the north and south poles of a magnet travels in a circular orbit of radius R (Figure WG29.9, illustrating a type of particle accelerator known as a betatron). The magnetic field magnitude is $B(r)$, where r is radial distance from an axis that is perpendicular to the plane defined by the circular orbit and passes through the center of the circle. Although B varies with r, the field is symmetrical about this axis. The magnetic field direction is perpendicular to the plane of the orbit. The average value of the field magnitude over the area bounded by the orbit is

$$B_{\text{av}} = \frac{1}{\pi R^2}\int_{\text{area}} B(r)\,dA.$$

Suppose that $B(r)$ varies with time. What relationship between dB_{av}/dt, the rate of change of the average magnetic field enclosed by the electron's orbit, and $dB(R)/dt$, the rate of change of the magnetic field at the orbit radius $r = R$, is necessary for the electron to stay in its circular orbit?

Figure WG29.9



1 GETTING STARTED

1. Assume that the electron moves in a circular orbit at constant radius R, but that the electron's speed may vary with time. (Changing the electron's speed is the purpose of a particle accelerator.) To keep the electron moving in a circle, what must be the direction of the magnetic force exerted by the magnetic field on the moving electron?
2. If the magnitude $B(R)$ of the magnetic field at the orbit radius R varies with time, what must happen to the electron's speed in order to maintain the circular motion in the orbit?
3. What relates speed to force for circular motion? Is the magnetic force exerted on the electron related to the magnetic field magnitude?

2 DEVISE PLAN

4. Write an expression for the magnetic force exerted on the electron in terms of the electron's speed v, orbit radius R, and magnetic field magnitude $B(r)$ when $r = R$.
5. What radial force (perpendicular to the particle's velocity) is needed to keep a particle traveling in circular motion at a given speed?
6. Use the relationship between force and acceleration to obtain an expression for the speed of the electron.
7. Differentiate the electron's speed with respect to time to obtain an expression for the time rate of change of the electron's speed in terms of $dB(r)/dt$, R, $-e$, and m.
8. When the magnetic field magnitude starts to increase at a rate $dB(r)/dt$, the changing magnetic flux is accompanied by an electric field. Can you express the changing magnetic flux in terms of dB_{av}/dt and R?
9. Write an expression that gives the electric field a distance R from the center of the orbit in terms of dB_{av}/dt and R. Does the electric field point in the direction of the electron's velocity or in the opposite direction?
10. Relate dv/dt to the electric field magnitude a distance R from the orbit center.
11. Determine an expression for dv/dt in terms of dB_{av}/dt, R, $-e$, and m.

3 EXECUTE PLAN

12. Compare your two expressions for dv/dt to obtain a condition that relates B_{av} and $B(r)$ and is mandatory in order for the electron to stay in its circular orbit at radius R.

4 EVALUATE RESULT

13. Consider limiting cases.

Answers to Review Questions

1. The magnetic force exerted by the external magnetic field on the charge carriers that move with the rod drives the positive carriers upward and the negative carriers downward. The result is a separation of charge and a small potential difference between the bottom and top of the rod.
2. For the portion of the loop that is in the nonzero-field region, the magnetic force exerted by the magnetic field on the charge carriers drives the positive carriers upward and the negative carriers downward. As long as part of the loop is in the nonzero-field region and part of the loop is in the zero-field region, the result is to create an electric current in the loop. When the entire loop is within the region of nonzero, uniform magnetic field, the current is no longer induced.
3. An induced electric current is the flow of charge carriers caused by electromagnetic induction, as opposed to a current that is caused by a source of potential difference such as a battery. Although the causes are different, there is no *physical* difference between induced currents and other currents.
4. Yes. The induced effects are reversed when the direction of the object's velocity is reversed, and the effects are zero when the object's velocity is parallel to the magnetic field.
5. Changing the reference frame does not affect the explanation. Whether you say the loop moves through the external magnetic field created by a stationary source or the external source moves and the loop is stationary, the result is the same: An electric current is induced in the loop as it moves between regions of differing magnetic fields.
6. Whenever the magnetic flux through a conducting loop changes, an electric current is induced in the loop.
7. Electromagnetic induction is the process by which a changing magnetic flux causes charge carriers in a conducting object to move. When the magnetic flux through a conducting loop changes, a current is induced in the loop.

PRACTICE

8. No. In the reference frame in which the loop is at rest, there is no magnetic force to create the current in the loop. Rather, the changing magnetic field is accompanied by an electric field, and this electric field is the source of an electric force exerted on the charge carriers, creating a current in the loop.
9. Yes. In a reference frame in which a conducting loop is at rest and a magnetic field source is moving, the electric field caused by the time-varying magnetic flux is the source of the electromagnetic induction. In a reference frame in which the conducting loop moves through a nonuniform external magnetic field, the magnetic force exerted on the charge carriers in the loop is the source of the electromagnetic induction.
10. The field lines of an electric field that accompanies a changing magnetic field form closed loops, whereas the field lines of the field of stationary charged particles, which originate on positively charged particles and terminate on negatively charged ones, do not.
11. The current is induced by a changing magnetic flux through the loop, and Lenz's law says that the direction of the induced current is the direction that makes the magnetic field created by the induced current oppose the changing magnetic flux through the loop.
12. The law of conservation of energy underlies Lenz's law. If the direction of an induced current were to enhance the change in the magnetic flux rather than oppose that change, the result would be an increase in the system's energy without any work being done on the system.
13. Eddy currents are electric currents induced in a conducting object by a changing external magnetic field near the object. These currents are generally not confined to a single path, as in a wire, but circulate as tiny whirlpools.
14. Induced emf is the emf (work per unit charge done in separating positive and negative charge carriers by nonelectrostatic means) associated with changing magnetic flux through a closed path.
15. The energy acquired by the charge carriers comes from the agent that causes the loop to move. If the loop is being pulled through the magnetic field, for example, that mechanical work done on the loop is transferred to the charge carriers.
16. (*a*) Faraday's law states that the emf induced around a closed path is equal to the negative of the time rate of change of magnetic flux through an area bounded by the path: $\mathscr{E}_{\text{ind}} = -d\Phi_B/dt$. (*b*) The negative sign reflects Lenz's law by making the expression say that the direction of $\mathscr{E}_{\text{ind}}$ is such that the induced current opposes the magnetic-flux change generating that current.
17. (1) Both induced emf and potential difference are related to the work done per unit charge on charged particles. In the case of potential difference, the work is done by electrostatic forces, but the work is done by nonelectrostatic forces in the case of emf. (2) Potential difference is path-independent, which means that the work done by electrostatic forces on charged particles depends only on the particles' starting and ending locations, not on the path they take to get to those locations. The work done by the emf induced by a changing magnetic field does depend on the path taken by the charged particles; that is, the work done by nonelectrostatic forces depends on the shape of the loop in which the emf is induced.
18. The induced emf oscillates sinusoidally according to the expression $\mathscr{E}_{\text{ind}} = \omega BA \cos \omega t$ (where A is the area enclosed by the loop).
19. The induced emf is the line integral of the electric field that accompanies the changing magnetic field, around the path bounding the area through which the magnetic flux is changing.
20. The direction of the electric field is tangent to the electric field lines, which form loops centered on the solenoid axis. The electric field lines lie in planes perpendicular to the solenoid axis and point in the direction dictated by Lenz's law. This is true both inside and outside the solenoid.
21. The magnitude of the electric field is proportional to the time derivative of the magnetic field strength, $E \propto (dB/dt)$. From Example 29.7 you know that inside the magnetic field region E varies directly with distance from the solenoid axis, and from Checkpoint 29.13 you know that outside the magnetic field region E varies inversely with the distance from the solenoid axis.
22. Yes. The changing current changes the magnetic flux through the loop, and Faraday's law tells you that the changing magnetic flux induces an emf to oppose the flux change.
23. The SI unit is the henry, which equals one volt-second per ampere.
24. An inductor is an electrical component that has a specified inductance, such that a given dI/dt results in a known, specified emf. Because the induced emf opposes changes in current, one use of an inductor is to even out any current fluctuations in a current-carrying loop.
25. The inductance of a conducting loop or device describes how much the magnetic flux through it changes when the current changes. The inductance depends only on the geometric properties of the loop or device—that is, on its size and shape. For example, the inductance of a solenoid depends on its length, cross-sectional area, and the number of windings: $L \propto N^2A/\ell$.
26. The work done to change the current in an inductor goes into, or comes out of, the magnetic potential energy stored in the magnetic field of the inductor. That is, a reduction in the current is accounted for by a decrease in the stored magnetic potential energy. Some external source is needed to account for the work done to increase the current, work that increases the stored magnetic potential energy.
27. The magnetic potential energy depends on the square of the current in the inductor and on the inductance: $U^B = LI^2/2$. Because inductance depends on the physical dimensions of the inductor, you can also say that the magnetic potential energy stored in an inductor depends on the square of the current and on the size and shape of the inductor; for a solenoidal inductor, for example, $U^B \propto N^2AI^2/\ell$.
28. The magnetic potential energy in an inductor is equal to the volume integral of the energy density in the magnetic field: $U^B = \int u_B dV$. The energy density is proportional to the square of the field strength, $u_B = B^2/(2\mu_0)$, which is a general relationship for any magnetic field. (If the field strength is approximately uniform over the volume of the inductor, the energy is just the product of the energy density and the volume.)

Answers to Guided Problems

Guided Problem 29.2 $I_{\text{ind}} = \dfrac{\mathscr{E}_{\text{ind}}}{R} = \dfrac{\mu_0 I\ell v}{2\pi R}\left(\dfrac{1}{r} - \dfrac{1}{r+w}\right)$. In the configuration of Figure WG29.3, the direction of the induced current is clockwise around the loop.

Guided Problem 29.4 While the top of the loop is still in the magnetic field, $v = (g\tau)(1 - e^{-t/\tau})$, with $\tau = (mR/w^2B^2)$. After the top edge of the loop has left the magnetic field, v increases with time at constant acceleration g.

Guided Problem 29.6 $U^B = \dfrac{\mu_0 \ell I^2}{4\pi}\ln\left(\dfrac{R_{\text{outer}}}{R_{\text{inner}}}\right)$, $L = \dfrac{\mu_0 \ell}{2\pi}\ln\left(\dfrac{R_{\text{outer}}}{R_{\text{inner}}}\right)$

Guided Problem 29.8 $\dfrac{dB(R)}{dt} = \dfrac{1}{2}\left(\dfrac{dB_{\text{av}}}{dt}\right)$

Guided Practice by Chapter

30 Changing Electric Fields

Review Questions

Answers to these questions can be found at the end of this chapter.

30.1 Magnetic fields accompany changing electric fields

1. What besides electric current produces a magnetic field?
2. How is the direction of the magnetic field that accompanies a changing electric field related to the direction of the changing electric field?
3. Describe the similarities and differences in the sources that produce electric and magnetic fields.
4. What is the main difference between the effects of electric fields on charged particles and the effects of magnetic fields on charged particles?
5. Describe the main similarity and the main difference in the spatial appearance of electric and magnetic field lines.

30.2 Fields of moving charged particles

6. Is the electric field surrounding a charged particle moving with constant velocity spherically symmetrical or asymmetrical? If asymmetrical, describe what the field looks like.
7. What determines how asymmetrical the electric field surrounding a charged particle moving at constant velocity is?
8. When do kinks appear in the field line pattern of the electric field that surrounds a charged particle? What two regions of a given field line does a kink join?
9. What are the two components of an electromagnetic wave?

30.3 Oscillating dipoles and antennas

10. Describe at the instant $t = T$ the electric field lines in three regions of interest near an electric dipole whose polarity changes once (from positive above negative to negative above positive) during the time interval $t = 0$ to $t = T/2$ and then remains fixed from $t = T/2$ to $t = T$.
11. In a transverse electromagnetic wave, how are the electric and magnetic fields oriented relative to each other and relative to the direction in which the wave propagates?
12. What does it mean to say that the electric and magnetic fields in an electromagnetic wave are *in phase*?
13. How is the polarization of an electromagnetic wave defined?
14. What is an antenna?
15. How is an oscillating current created in an emitting antenna?

30.4 Displacement current

16. What electrical device can be used to demonstrate that the form of Ampère's law given in Chapter 28, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$, is incomplete? Describe an Ampèrian path around the device and a surface spanned by the path that you can use to illustrate this incompleteness.
17. What is the generalized form of Ampère's law, which applies when electric fields are changing?
18. In the generalized form of Ampère's law, what does each term to the right of the equal sign represent?
19. Is displacement current really a current? If not, why is it called a current?
20. How must the displacement-current term in the generalized form of Ampère's law be modified when the capacitor being analyzed contains a dielectric between its plates?

30.5 Maxwell's equations

21. Write the four equations from the text that are collectively called Maxwell's equations, and name each one.
22. Given that all four of the laws for electric and magnetic fields were discovered before Maxwell modified one of them, why is it still fitting to refer to all of them as Maxwell's equations?
23. Describe the experimental evidence for each of Maxwell's equations.

30.6 Electromagnetic waves

24. For an electromagnetic wave pulse emanating from an accelerated charge carrier, what does the transverse (kinked) portion of the pulse look like far from the location where the acceleration occurred?
25. What is the relationship between the magnitudes of the electric and magnetic fields for an electromagnetic wave pulse traveling in vacuum?
26. What is the relationship between the magnitudes of the electric and magnetic fields for an electromagnetic wave pulse traveling through a dielectric material?
27. In terms of the electric constant ϵ_0 and the magnetic constant μ_0, what is the speed of an electromagnetic wave pulse traveling in vacuum and traveling through a dielectric?
28. For a planar electromagnetic wave, what is the directional relationship among the magnetic field, the electric field, and the wave's propagation velocity?
29. Do electromagnetic waves of different frequencies travel at different speeds in vacuum?
30. How is the speed at which an electromagnetic wave travels in vacuum related to the wave's frequency and wavelength?
31. What are the names of the major divisions of the electromagnetic spectrum?

30.7 Electromagnetic energy

32. How do the energy densities in the electric and magnetic fields of a planar electromagnetic wave in vacuum compare with each other?
33. (*a*) What is the expression for the energy density u of an electromagnetic wave in vacuum in terms of its electric and magnetic components? (*b*) How can you express the energy density if you know the electric component of the wave but do not know the magnetic component? (*c*) How can you express the energy density if you know the magnetic component but not the electric component?
34. What is the Poynting vector of an electromagnetic wave in vacuum, and what properties of the wave does this vector represent?
35. What is the intensity of an electromagnetic wave?
36. What is the relationship between the intensity of an electromagnetic wave and its energy density?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The shortest-wavelength electromagnetic wave you can generate by waving a charged rod (T, F, B)
2. The wavelength of a radio station that is called FM 104 (A, K, F, B)
3. The frequency of a radio wave if one wavelength occupies the length of a tennis court (F, B, U)
4. The maximum magnitude of the electric field 1 m from an electron that has 10^{-12} J of energy and is moving at a constant speed $v \approx c_0$ (W, G, DD, S, FF)
5. The time interval needed for a radio wave pulse emitted from ground level to travel to and then back from a communications satellite moving in a geostationary orbit at 3×10^3 m/s (A satellite in geostationary orbit moves at a translational speed equal to the product of Earth's rotational speed and the orbit's radius, so it always stays above a single location on Earth's surface.) (B, P, CC, N, H)
6. The average displacement current caused when a 1-μF capacitor is charged to 10 V in 0.1 s (J, O)
7. The root mean square (rms) value of the electric field magnitude 1 km from an AM radio station emitting antenna (E, I, X, Q, V, M)
8. The maximum magnetic field magnitude 100 mm from a 125-mW cell phone (E, Z, X, Q, BB)
9. The change in electric flux in the region between your finger and a metal doorknob when a spark jumps from your finger to the knob (Y, EE, C, R)
10. The power radiated by all the AM radio stations in the United States (M, D, L, AA)

Hints

A. What does the radio station number refer to?
B. What is the wave speed?
C. What is the electric field magnitude at which air breaks down (that is, when sparks form)?
D. How many AM stations are there in an average-size city?
E. What is the relationship between the rms values of the electric and magnetic field magnitudes?
F. What is the relationship between wavelength and frequency?
G. What is the electric field magnitude 1 m from an electron at rest?
H. What is the height above Earth's surface to geostationary orbit?
I. What is the relationship between E_{rms} and the average intensity of the wave?
J. What is the quantity of charge on each plate when the capacitor is fully charged?
K. What is the frequency unit for FM radio?
L. How many average-size cities are in the United States?
M. What power is radiated by a typical commercial radio station?
N. What is Earth's radius?
O. What is the conventional current required to charge the capacitor?
P. What is the orbital period of a geostationary orbit?
Q. What is the relationship between the average intensity of the wave and the average power of the wave source at a radial distance $r = R$ from the source?
R. What is the cross-sectional area where the electric field magnitude is greatest?
S. For a moving electron, how does the magnitude of the component of the electric field perpendicular to the line of motion 1 m from the electron compare with the magnitude of the electric field 1 m from an electron at rest?
T. At what maximum frequency can you wave the rod?
U. What is the length of a tennis court?
V. What is the relationship between P_{av} and E_{rms}?
W. What is the charge on the electron?
X. What is the area of a sphere of radius R?
Y. How should you model this situation to relate it to material covered in the text?
Z. What is the relationship between B_{rms} and the average intensity of the wave?

AA. What is a reasonable value for the number of AM stations in the United States that are outside cities?
BB. What is the relationship between P_{av} and B_{rms}?
CC. How does the period of an object in circular motion at constant speed relate to the circle's radius?
DD. Where is the electric field due to the electron's motion greatest?
EE. Is the electric field between the plates of a charged capacitor uniform or nonuniform?
FF. What is the γ factor for a 10^{-12}-J electron?

Key (all values approximate)

A. the frequency of the electromagnetic waves emitted by the station's emitting antenna; **B.** 3×10^8 m/s; **C.** 3×10^6 V/m; **D.** 30 stations; **E.** $B_{rms} = E_{rms}/c_0$; **F.** $f = c_0/\lambda$; **G.** $ke/r^2 = 1 \times 10^{-9}$ V/m ($k = 9 \times 10^9$ N·m²/C², Coulomb's law); **H.** $R - R_E \sim 4 \times 10^7$ m; **I.** $S_{av} = E_{rms}^2/c_0\mu_0$; **J.** $q_{max} = CV = 10^{-5}$ C; **K.** megahertz; **L.** at least one per state, and more in populous states—say 100 cities; **M.** 10^4 W; **N.** 6×10^6 m; **O.** $I = \Delta q/\Delta t = 10^{-4}$ A; **P.** $T = 1$ day $= 9 \times 10^4$ s; **Q.** $P_{av} = S_{av}A = S_{av}4\pi R^2$; **R.** the cross-sectional area of the smaller object, your finger—10^{-4} m²; **S.** the two magnitudes differ by the same factor that scales length contraction: $\gamma = 1/\sqrt{1 - v^2/c_0^2}$; **T.** 5 Hz; **U.** 3×10^1 m; **V.** $P_{av} = E_{rms}^2 4\pi R^2/c_0\mu_0$; **W.** -1.6×10^{-19} C; **X.** $A = 4\pi R^2$; **Y.** as a parallel-plate capacitor with your finger as one plate and the knob as the other plate; **Z.** $S_{av} = c_0 B_{rms}^2/\mu_0$; **AA.** 20 in each state = 1000 stations; **BB.** $P_{av} = c_0 B_{rms}^2 4\pi R^2/\mu_0$; **CC.** $T = 2\pi R/v$ (Eq. 11.20); **DD.** perpendicular to the line of motion; **EE.** uniform; **FF.** 1×10^1

Worked and Guided Problems

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 30.1 Magnetic from electric

The electric field of a planar sinusoidal electromagnetic wave is given by $\vec{E}(z,t) = E\sin(kz - \omega t)\hat{\imath}$, where $k = 9.0\ \mathrm{m}^{-1}$ and $E = 6.0 \times 10^2$ V/m. (*a*) What are the wavelength and angular frequency of the wave? (*b*) What are the magnitude and direction of the magnetic field associated with this electric field?

❶ **GETTING STARTED** We are given the equation that describes the electric field of an electromagnetic wave and the values for the field magnitude E and the wave number k. From the z and the minus sign in the argument of the sine function, we know that the wave is traveling in the positive z direction (Eq. 16.12). Because the wave is planar, we see from Figure 30.38 that, on each planar surface $z = a$, the electric field has a uniform value and lies along the x direction.

❷ **DEVISE PLAN** For part *a* we can determine the wavelength from Eq. 16.7, $k = 2\pi/\lambda$, and the angular frequency from Eqs. 16.11 and 16.9, $\omega = 2\pi/T$ and $\lambda = cT$, where in this case $c = c_0 = 3.0 \times 10^8$ m/s. For part *b* we use our knowledge that in a planar sinusoidal electromagnetic wave, the electric and magnetic fields are perpendicular to each other and the direction in which the wave they constitute travels is given by $\vec{E} \times \vec{B}$. We also know that the relationship between the field magnitudes is $B = E/c_0$ (Eq. 30.24) and that the fields are in phase.

❸ **EXECUTE PLAN** (*a*) From Eq. 16.7 $k = 2\pi/\lambda$, the wavelength is $\lambda = 2\pi/k = 2\pi/(9.0\ \mathrm{m}^{-1}) = 7.1 \times 10^{-2}$ m. ✔
The angular frequency of the wave is therefore

$$\omega = \frac{2\pi c_0}{\lambda} = \frac{2\pi(3.0 \times 10^8\ \mathrm{m/s})}{(2\pi/9.0)\ \mathrm{m}} = 2.7 \times 10^9\ \mathrm{s}^{-1}. ✔$$

(*b*) Consider a pair of values (z, t) such that $0 < kz - \omega t < \pi/2$. At the location and instant specified by (z, t), the sine function is positive and so the electric field points in the positive x direction. The wave travels in the positive z direction. Therefore, at (z, t), $\vec{E} \times \vec{B}$ must point in the positive z direction. We know that $\hat{\imath} \times \hat{\jmath} = \hat{k}$, so we know that the magnetic field must point in the positive y direction and be of the form

$$\vec{B}(z,t) = B\sin(kz - \omega t)\hat{\jmath}.$$

The magnitude of the magnetic field is

$$B = \frac{E}{c_0} = \frac{6.0 \times 10^2\ \mathrm{V/m}}{3.0 \times 10^8\ \mathrm{m/s}} = 2.0 \times 10^{-6}\ \mathrm{V \cdot s/m^2}.$$

PRACTICE

We know that $1\ \text{V} = 1\ \text{N}\cdot\text{m/C}$, so $1\ \text{V}\cdot\text{s/m}^2 = 1\ \text{N}\cdot\text{s/C}\cdot\text{m}$. We also know that the unit of magnetic field is the tesla, where $1\ \text{T} = 1\ \text{N}\cdot\text{s/m}\cdot\text{C}$. So the magnitude of the magnetic field is $B = 2.0 \times 10^{-6}$ T, and if we include its direction the result is

$$\vec{B}(z, t) = (2.0 \times 10^{-6}\ \text{T}) \sin[(9.0\ \text{m}^{-1})z - (2.7 \times 10^{9}\ \text{s}^{-1})t]\,\hat{\jmath}\,. ✔$$

❹ **EVALUATE RESULT** The values we calculated in part *a* lie in the range between commercial radio waves and microwaves and satisfy $c_0 = \lambda f$. In part *b* our calculated magnetic field magnitude is about 1/25 the magnitude of Earth's magnetic field near the equator. The electric field magnitude of this wave is about six times the magnitude of typical atmospheric electric fields, 100 V/m. Thus the results we obtained are not unreasonable.

Guided Problem 30.2 Electric from magnetic

The magnetic field vector of a planar electromagnetic wave is given by

$$\vec{B}(y, t) = B \cos(ky + \omega t)\hat{k},$$

where $B = 2.0 \times 10^{-5}$ T, $\omega = 3.0 \times 10^{9}\ \text{s}^{-1}$, and $\hat{k}$ is the unit vector in the *z* direction. What are (*a*) the value of the wave number *k* and (*b*) the expression for the magnitude and direction of the electric field associated with this magnetic field?

❶ **GETTING STARTED**

1. Is the approach used in Worked Problem 30.1 useful here?
2. In which direction is the wave traveling?
3. Along which axis does the direction of the magnetic field oscillate?

❷ **DEVISE PLAN**

4. How is the wave number *k* related to the angular frequency ω?
5. Along which axis does the electric field oscillate?
6. How are the magnitudes of the electric and magnetic fields related?
7. You should now be able to write an expression for the electric field.

❸ **EXECUTE PLAN**

8. Calculate the value of *k*.
9. Write an expression for the wave's electric-field component.
10. What is the magnitude of the electric field?
11. Do you have the correct units for the electric field?

❹ **EVALUATE RESULT**

12. How do your values for *E* and *B* compare with the magnitude of Earth's magnetic field near the equator and the magnitude of a typical atmospheric electric field, which is about 100 V/m?

Worked Problem 30.3 Antenna reach

A certain type of FM radio receiver requires a minimum radio signal with a root-mean-square electric field magnitude of 1.2×10^{-2} N/C to operate. How far from a radio station broadcasting its electromagnetic signal at an average power of 100 kW can this receiver be used?

❶ **GETTING STARTED** We are given the power at which a radio station broadcasts its signal, 100 kW, and we must calculate the distance from the emitting antenna to the position at which the electric-field portion of the electromagnetic waves has a root-mean-square magnitude of 1.2×10^{-2} N/C. We begin by assuming that the waves are broadcast uniformly in all directions, which allows us to model the power at any instant as being distributed on a spherical surface. Thus the distance we seek is the radius *r* of the sphere at the position where $E_{\text{rms}} = 1.2 \times 10^{-2}$ N/C.

❷ **DEVISE PLAN** We can use Eq. 30.38 to obtain the relationship between the area of any spherical surface through which the waves pass and the average power and average intensity of the waves. Thus from $A_{\text{sphere}} = 4\pi r^2$ we can obtain a value for *r*, the distance we seek. In order to use Eq. 30.38, however, we must know the average intensity of the waves, so our first step should be to calculate that value from Eq. 30.40, $S_{\text{av}} = E_{\text{rms}}B_{\text{rms}}/\mu_0$.

❸ **EXECUTE PLAN** Substituting E_{rms}/c_0 for B_{rms} in Eq. 30.40, we get for the average intensity

$$S_{\text{av}} = \frac{E_{\text{rms}}^2}{\mu_0 c_0} = \frac{(1.2 \times 10^{-2}\ \text{N/C})^2}{(4\pi \times 10^{-7}\ \text{N}\cdot\text{s}^2/\text{C}^2)(3.0 \times 10^{8}\ \text{m/s})}$$

$$= 3.8 \times 10^{-7}\ \text{N/m}\cdot\text{s}.$$

Recall that $1\ \text{W} = 1\ \text{N} \cdot \text{m/s}$, so $S_{av} = 3.8 \times 10^{-7}\ \text{W/m}^2$. The time-average version of Eq. 30.38 is

$$P_{av} = \int \vec{S}_{av} \cdot d\vec{A} = S_{av} 4\pi r^2.$$

Therefore the distance r at which $E_{rms} = 1.2 \times 10^{-2}$ N/C is

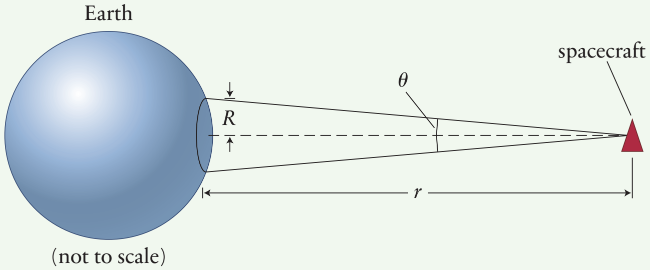
$$r = \sqrt{\frac{P_{av}}{4\pi S_{av}}} = \sqrt{\frac{1.0 \times 10^5\ \text{W}}{(4\pi)(3.8 \times 10^{-7}\ \text{W/m}^2)}} = 1.4 \times 10^5\ \text{m.} \checkmark$$

4 **EVALUATE RESULT** Our result—about 100 miles—is close to the distance at which FM stations fade out when we are driving across the country and therefore in agreement with our experience.

Guided Problem 30.4 Beam from outer space

A spacecraft 120 AU above Earth (1 AU = 1 astronomical unit = 1.5×10^8 km) has a 20-W directional dish antenna emitting at 8.0 GHz (Figure WG30.1). This emitting antenna is aimed at a 34-m-diameter receiving antenna located on Earth. Suppose you want the signal to have a root-mean-square electric field magnitude of 1.0×10^{-3} V/m at the receiving dish on Earth's surface. What must the beam width angle θ be?

Figure WG30.1



1 **GETTING STARTED**

1. How does a directional antenna differ from an antenna that emits waves equally in all directions?
2. What assumptions should you make about how the power is emitted directionally?
3. Assume the signal projects to a circular area on Earth's surface. Draw a diagram showing how the beam width is related to the radius R of that circular area and to r, the distance from the spacecraft to Earth.

2 **DEVISE PLAN**

4. How can you calculate the average intensity of the signal's electromagnetic waves?
5. What relationship can you use to determine the maximum area on Earth that the signal is directed toward in order to be detected? What relationship can you use to determine the maximum area on Earth's surface that the beam can cover and still be detectable?
6. How can you determine the maximum beam width once you know the maximum area that the signal should cover on Earth?

3 **EXECUTE PLAN**

7. Calculate the maximum intensity associated with the signal root-mean-square electric field magnitude.
8. Calculate the maximum area on Earth's surface that the beam can cover and still be detectable.
9. Calculate the radius R of the circular area on Earth's surface covered by the signal.
10. Use your values for R and r and your diagram from step 3 to obtain the beam width angle.

4 **EVALUATE RESULT**

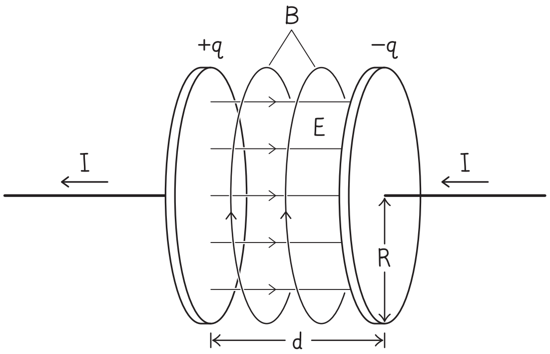
11. With the beam angle you calculated, do you think the 34-m-diameter receiving antenna can detect the signal?

Worked Problem 30.5 Discharge radiation

A capacitor consists of two circular parallel plates of radius $R = 150$ mm separated by distance $d = 20$ mm. The initial magnitude of the charge on the capacitor is $q_0 = 9.0 \times 10^{-8}$ C, and after discharge is initiated, it discharges via a steady current $I = 3.0$ mA. What is the power that crosses the closed cylindrical surface surrounding the space between the plates 2.0×10^{-5} s after discharge begins?

❶ GETTING STARTED We begin with a sketch showing the discharging capacitor and a few of the associated electric and magnetic field lines (Figure WG30.2). The closed cylindrical surface surrounding the space between the capacitor plates consists of two end faces, one coinciding with each plate, and a curved surface that joins the end faces and encloses the space between the plates. Because the distance between the plates is small relative to the radius of the plates, we assume that at any instant t, the charge $q(t)$ on the plates is uniformly distributed and the electric field $\vec{E}(t)$ is uniform between the plates (that is, we ignore edge effects). We show this uniform field in our sketch. While the capacitor is discharging, the magnitude of $\vec{E}$ is decreasing, and therefore there is a displacement current I that points in the direction of $\Delta\vec{E}$, which is opposite the direction of $\vec{E}$.

Figure WG30.2



While the electric field is changing, there is an associated magnetic field $\vec{B}(t)$ between the plates, and we know that the magnetic field lines form circular loops. We determine the direction of these loops by applying the right-hand current rule to the displacement current and add the loops to our Figure WG30.2.

At each location in the space between the plates, the electric and magnetic fields are perpendicular to each other. Because the electric field is decreasing, the energy stored in this field is also decreasing. Using our drawing and this information, we must determine, at $t = 2.0 \times 10^{-5}$ s, the power P that crosses the closed cylindrical surface defined by the capacitor volume.

❷ DEVISE PLAN In order to calculate P, we must determine the flux of the Poynting vector (Eq. 30.38). We know from Eq. 30.37 that this vector is proportional to $\vec{E} \times \vec{B}$, and so we must determine the magnitudes of the electric and magnetic fields between the plates. We can use Gauss's law (Eq. 30.10) to obtain E, which depends on the charge on the plates. Because the current is steady, we can determine the charge on the plates as a function of time. After we use Eq. 30.8 to determine B, we can calculate the flux of the Poynting vector on the closed cylindrical surface and obtain the power that crosses the surface.

❸ EXECUTE PLAN Figure WG30.3 shows a circle of radius R that is centered on and perpendicular to the line passing through the centers of the two circular plates. Consider a point Z lying on this circle. Our first task is to use Gauss's law to calculate the magnitude of $\vec{E}(t)$ at Z. We choose as our Gaussian surface the closed cylinder shown in Figure WG30.4*a*. Note that the cylinder lies partly inside the space between the capacitor plates and partly outside that space and that Z lies on the end face that is in the space between the plates, as shown in Figure WG30.4*b*. In this space, there is nonzero flux only through this end face. Thus with A_{endface} as the face area, Gauss's law, $\Phi_E = \oint \vec{E} \cdot d\vec{A} = q_{\text{enc}}/\epsilon_0$ (Eq. 24.8), becomes

$$EA_{\text{endface}} = \frac{\sigma A_{\text{endface}}}{\epsilon_0} = \frac{q(t)A_{\text{endface}}}{\pi R^2 \epsilon_0}$$

$$E(t) = \frac{q(t)}{\pi R^2 \epsilon_0}. \quad (1)$$

Figure WG30.3

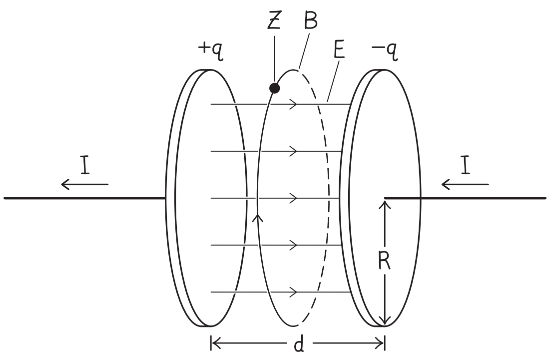
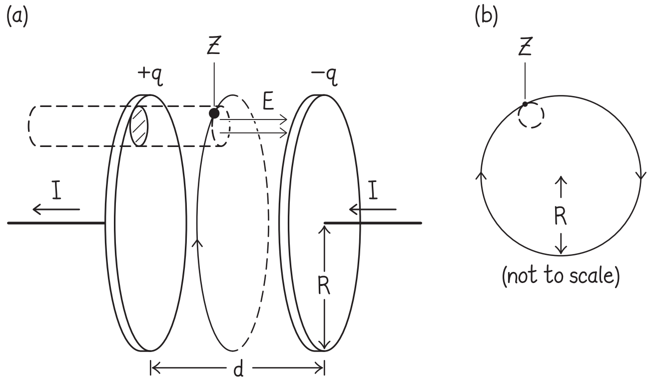


Figure WG30.4



We now use Eq. 30.8 to obtain the magnitude of $\vec{B}(t)$ at Z. We choose as our (Ampèrian) integration path the circle of radius R shown in Figure WG30.3. Because the magnitude B is constant along this integration path, the integral on the left side of Eq. 30.8 is $B(2\pi R)$. On the right side of the equation, $I_{\text{int}} = 0$ because the integration path defines an open surface, a disk of radius R, and there is no current through the disk. Because the electric field is uniform, its flux is $\Phi_E = E(\pi R^2)$, so the magnitude of the displacement current is, from Eq. 30.7,

$$I_{\text{disp}} = \epsilon_0 \frac{dE}{dt} \pi R^2.$$

Substituting Eq. 1 for E gives us

$$I_{\text{disp}} = \epsilon_0 \frac{d}{dt}\left[\frac{q(t)}{\pi R^2 \epsilon_0}\right]\pi R^2 = \frac{dq(t)}{dt}.$$

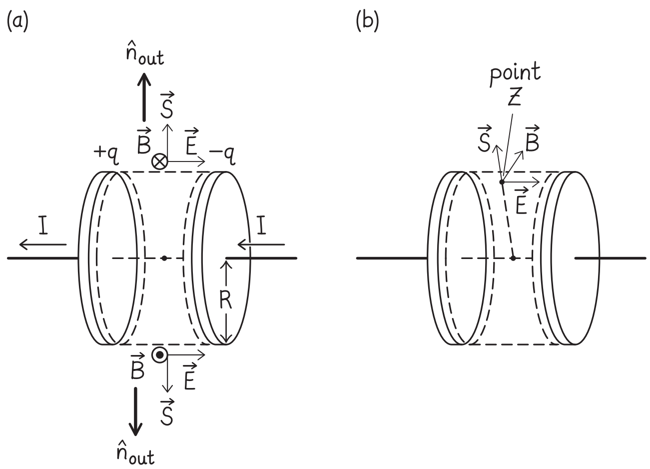
Because the current is steady and the charge is decreasing, the steady current $I = -dq(t)/dt$. Thus the magnitude of the displacement current is equal to the magnitude of the steady current. Therefore Eq. 30.8 becomes $B(2\pi R) = \mu_0 I$, and we have

$$B = \frac{\mu_0 I}{2\pi R}. \qquad (2)$$

We now apply Eq. 30.37 to calculate the magnitude and direction of the Poynting vector $\vec{S}(t)$ at point Z. Because point Z could be chosen on any circle inside the capacitor, the expression for the Poynting vector should be general. Applying the right-hand rule for the vector product $\vec{E} \times \vec{B}$ in Figure WG30.5*a*, we see that the direction of the vector product points away from the curved part of the closed cylinder—that is, radially outward at Z in Figure WG30.5*b*. This is therefore the direction of $\vec{S}$. From Eqs. 1 and 2, the magnitude is

$$S(t) = \frac{1}{\mu_0} EB = \frac{q(t)}{\pi R^2 \epsilon_0} \frac{I}{2\pi R}.$$

Figure WG30.5



PRACTICE

Note that the Poynting vector is directed radially outward at each point inside the capacitor, so energy will not cross the flat ends of the cylindrical space. The power that crosses the curved part of our closed cylindrical surface can be computed using Eq. 30.38. For a closed surface, we always choose $\hat{n}_{\text{out}}$ as the outward-pointing normal to the surface. Because $\vec{S}(t)$ points outward on all portions of our curved surface, the only thing we need to do is multiply S by the area of that curved surface, $A = 2\pi Rd$. The power that crosses the surface is then

$$P(t) = S(t)A = \frac{q(t)}{\pi R^2\epsilon_0}\frac{I}{2\pi R}2\pi Rd = \frac{q(t)I}{\pi R^2\epsilon_0}d.$$

Because the current is steady, the charge on the plates as a function of time is given by $q(t) = q_0 - It$. Therefore the power that crosses the curved part of our closed cylindrical surface as a function of time is given by

$$P(t) = \frac{(q_0 - It)I}{\pi R^2\epsilon_0}d,$$

and at $t = 2.0 \times 10^{-5}$ s the power is

$$P = \frac{[(9.0 \times 10^{-8}\text{ C}) - (3.0 \times 10^{-3}\text{ A})(2.0 \times 10^{-5}\text{ s})]}{\pi(0.150\text{ m})^2(8.85 \times 10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2)}$$

$$\times (3.0 \times 10^{-3}\text{ A})(2.0 \times 10^{-2}\text{ m}) = 2.9\text{ W}. ✔$$

4 EVALUATE RESULT In Example 30.10, a parallel-plate capacitor with circular plates of radius $R = 0.10$ m, plate separation distance $d = 0.10$ mm, and a steady charging current of 1.0 A had $(3.6 \times 10^8\text{ W/s})\Delta t$ of power crossing into the cylindrical space between the plates. For that capacitor, therefore, the power at $\Delta t = 2.0 \times 10^{-5}$ s would be 7.2×10^3 W. From Example 30.10, we know that the power is proportional to I^2 and to d and inversely proportional to R^2. Using that relationship for our capacitor, where $I_{\text{discharging}} = 3.0$ mA, $R = 150$ mm, and $d = 20$ mm, we expect a power of

$$\frac{P_{\text{discharging}}(\Delta t = 2.0 \times 10^{-5}\text{ s})}{P_{\text{charging}}(\Delta t = 2.0 \times 10^{-5}\text{ s})} =$$

$$\frac{(q_0 - I_{\text{discharging}}\Delta t)I_{\text{discharging}}}{(I_{\text{charging}}\Delta t)I_{\text{charging}}}\frac{d_{\text{discharging}}}{d_{\text{charging}}}\frac{(R_{\text{charging}})^2}{(R_{\text{discharging}})^2}$$

$$P_{\text{discharging}}(\Delta t = 2.0 \times 10^{-5}\text{ s}) =$$

$$(7.2 \times 10^3\text{ W})\frac{(3.0 \times 10^{-8}\text{ C})}{(2.0 \times 10^{-5}\text{ C})}\frac{(3.0 \times 10^{-3}\text{ A})}{(1.0\text{ A})}$$

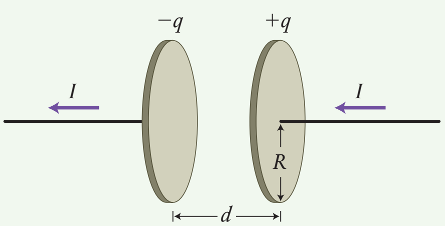
$$\times \frac{(20\text{ mm})}{(0.10\text{ mm})}\frac{(0.10\text{ m})^2}{(0.150\text{ m})^2} = 2.9\text{ W},$$

in excellent agreement with our result.

Guided Problem 30.6 Charge it!

A capacitor consists of two circular parallel plates, each with a radius $R = 200$ mm, separated by a distance $d = 15$ mm (Figure WG30.6). A steady current charges the capacitor for a time interval $\Delta t = 7.7 \times 10^{-6}$ s. During this time interval, the maximum magnitude of the magnetic field between the plates is $B_{\max} = 1.2 \times 10^{-8}$ T. What is the maximum power that crosses the closed cylindrical surface surrounding the space between the plates?

Figure WG30.6



PRACTICE

❶ GETTING STARTED

1. What quantities do you need to know in order to determine the maximum power that crosses the closed cylindrical surface that defines the space between the plates?
2. Which of these quantities are given in the problem statement?

❷ DEVISE PLAN

3. Which Maxwell equation gives you a relationship between the current and B_{max}?
4. Along what closed path does the magnetic field have its maximum magnitude?
5. Which Maxwell equation allows you to determine the maximum value of the electric field magnitude, E_{max}?
6. What is the relationship between the current and q_{max}, the maximum value of the charge on the capacitor plates?
7. How do you calculate the maximum power that crosses into the space between the plates?

❸ EXECUTE PLAN

8. Write an expression for the current.
9. Write an expression for q_{max}.
10. Write an expression for E_{max}.
11. Determine the direction of the Poynting vector, and write an expression for the maximum value of the magnitude of the Poynting vector.
12. Determine the maximum value of the power that crosses the cylindrical surface.

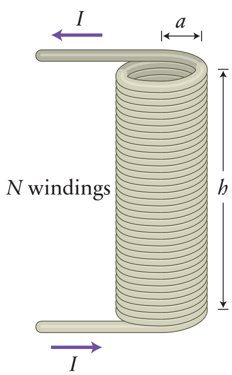
❹ EVALUATE RESULT

13. Does your value for the steady current seem reasonable?
14. Does your value for P_{max} seem reasonable?

Worked Problem 30.7 Solenoid antenna

The solenoid of Figure WG30.7 has radius $a = 0.10$ m and height $h = 0.60$ m. The current is decreasing as $I(t) = I_0 - bt$, where $I_0 = 0.40$ A and $b = 0.200$ A/s, and the current direction is as shown in the figure. If the solenoid has $N = 500$ windings, at what rate is electromagnetic energy leaving the cylindrical space inside the solenoid at $t = 1.0$ s?

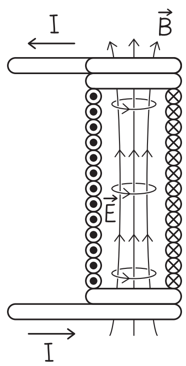
Figure WG30.7



❶ GETTING STARTED The solenoid has N/h windings per unit of height, and the current in the solenoid is decreasing. While the current is decreasing, the magnetic field inside the solenoid is decreasing. This changing magnetic field is accompanied by an electric field. Because the magnetic field is decreasing, we expect electromagnetic energy to leave the inside of the solenoid, and our task is to determine the rate at which this energy leaves.

We can ignore edge effects because the height of the solenoid is much greater than its radius. We know from Section 28.6 that the magnetic field inside the solenoid is uniform and directed along the long (vertical) axis of the solenoid. The electric field forms circular loops centered on this axis, and at each location inside the solenoid, the electric and magnetic fields are perpendicular to each other. This information is illustrated in Figure WG30.8.

Figure WG30.8



❷ **DEVISE PLAN** The rate at which electromagnetic energy leaves the cylindrical space inside the solenoid—the power—is given by the flux of the Poynting vector through the curved surface of the solenoid. Energy leaves the space inside the solenoid, and from this we know the direction of the Poynting vector—parallel to the end faces of the cylinder and directed outward, away from the long central axis.

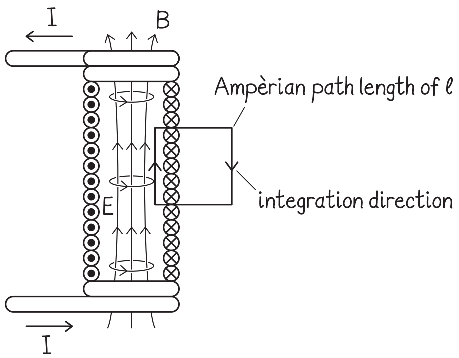
We can use Eq. 30.36 to calculate the magnitude of the Poynting vector. In order to use this equation, we must know the magnitudes E and B. We can use the Eq. 30.12 form of Faraday's law to calculate E. Because the current decreases linearly with time, the rate of change of the magnetic flux in the solenoid, $d\Phi_B/dt$, is constant and therefore the electric field magnitude is constant. Thus there is no displacement current associated with the electric field. This means Eq. 30.6, the generalized form of Ampère's law, $\oint B \cdot d\ell = \mu_0 I_{\text{int}} + \mu_0 \epsilon_0 (d\Phi_E/dt)$, reduces to Eq. 28.1, so we can use the latter to calculate the magnetic field magnitude.

To determine the power, we must integrate the Poynting vector over the cylindrical surface just surrounding the space inside the solenoid. Because the electric and magnetic fields are perpendicular to each other, tangent to and uniform over this surface, the power is the product of the Poynting vector magnitude and the area of the cylindrical surface.

❸ **EXECUTE PLAN** We begin with Ampère's law with no displacement current, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}$ (Eq. 28.1), where $d\vec{\ell}$ is a small segment of the Ampèrian path of length ℓ shown in Figure WG30.9. The direction of integration is clockwise. The magnetic field is zero outside the solenoid and points upward inside, so the left side of Eq. 28.1 is $\oint \vec{B} \cdot d\vec{\ell} = B\ell$. In the region of the solenoid we've chosen for our Ampèrian path, the current is into the plane of the page, so $I_{\text{enc}} = (N/h)\ell I$. Therefore the magnitude of the magnetic field is

$$B\ell = \frac{\mu_0 N I(t)\ell}{h} \Rightarrow B(t) = \frac{\mu_0 N (I_0 - bt)}{h}. \tag{1}$$

Figure WG30.9



Next, to obtain an expression for E, we apply the Eq. 30.12 form of Faraday's law, $\oint \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt$. This time $d\vec{\ell}$ represents a small segment of the circumference ℓ of the closed path of radius r shown in Figure WG30.10. Again we integrate clockwise (as viewed from below). The changing magnetic flux is

$$\frac{d\Phi_B}{dt} = \frac{dB}{dt}\pi r^2,$$

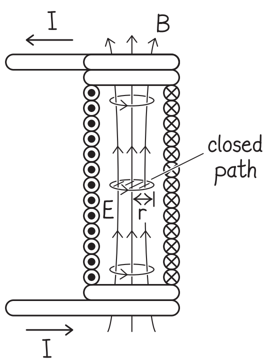
and we can use Eq. 1 to rewrite this equation in the form

$$-\frac{d\Phi_B}{dt} = -\frac{\mu_0 N \pi r^2}{h}\frac{d(I_0 - bt)}{dt} = \frac{\mu_0 N \pi r^2 b}{h}.$$

Because the electric field forms circular loops, the line integral of the electric field around this closed path is $\oint \vec{E} \cdot d\vec{\ell} = E2\pi r$. Therefore Faraday's law implies that the magnitude of the electric field is

$$E2\pi r = \frac{\mu_0 N \pi r^2 b}{h} \Rightarrow E = \frac{\mu_0 N r b}{2h}. \tag{2}$$

Figure WG30.10



Because the sign of the electric field is positive in Eq. 2, the direction of the electric field is the same as the integration direction.

Now we have all we need to determine the rate at which electromagnetic energy leaves the region inside the solenoid. We do this by calculating the flux of the Poynting vector through the cylindrical surface of the solenoid, where $r = a$. First we check the direction of the Poynting vector by determining the direction of $\vec{E} \times \vec{B}$, which points radially away from the center of the solenoid. We now use Eqs. 1 and 2 in Eq. 30.36 to calculate the magnitude of the Poynting vector:

$$S(r = a) = \frac{1}{\mu_0}EB = \frac{\mu_0 N^2 ab(I_0 - bt)}{2h^2}.$$

To determine the power at $t = 1.0$ s, we multiply S by the area of the curved surface of the cylinder, $2\pi ah$:

$$P = 2\pi ahS(r = a) = \frac{\mu_0 N^2 b(I_0 - bt)}{h}\pi a^2$$

$$= \frac{(4\pi \times 10^{-7}\ \mathrm{N \cdot s^2/C^2})(500)^2(0.200\ \mathrm{A/s})}{(0.60\ \mathrm{m})}$$

$$\times \left[0.40\ \mathrm{A} - (0.200\ \mathrm{A/s})(1.0\ \mathrm{s})\right]\pi(0.10\ \mathrm{m})^2$$

$$= 6.6 \times 10^{-4}\ \mathrm{W}. \checkmark$$

❹ **EVALUATE RESULT** This is a very small rate of energy leaving the solenoid. We have no clear standard of comparison for this value for the power. However, we expect that the electromagnetic energy leaving the cylindrical space inside the solenoid is equal to the time rate of change of the magnetic field energy stored in the space. The energy density of the magnetic field is given by Eq. 30.28, $u_B = B^2/2\mu_0$. Because the magnetic field is uniform, the energy U^B stored in it is the product of the energy density and the volume $\pi a^2 h$ of the cylindrical space:

$$U^B = \frac{B^2}{2\mu_0}\pi a^2 h = \frac{\mu_0 N^2(I_0 - bt)^2}{2h^2}\pi a^2 h.$$

The time rate of change of the stored magnetic energy is thus

$$\frac{dU^B}{dt} = \frac{d}{dt}\left[\frac{\mu_0 N^2(I_0 - bt)^2}{2h}\pi a^2\right] = -\frac{\mu_0 N^2 b(I_0 - bt)}{h}\pi a^2.$$

This expression has the same magnitude as the one we obtained for P, the rate at which electromagnetic energy leaves the solenoid, giving us confidence that our calculation is correct. The minus sign makes sense because energy is leaving the solenoid interior and hence the stored energy decreases.

Guided Problem 30.8 Coiled again!

The solenoid in Figure WG30.7 has N windings, radius a, and height $h \gg a$. The current through the windings is given by $I(t) = bt$, where b is a positive constant and has units of amperes per second. At what rate does electromagnetic energy cross the cylindrical surface defined by the windings? In which direction is this energy transfer?

❶ **GETTING STARTED**

1. How does the situation in this solenoid differ from the situation in Worked Problem 30.7?
2. Do you have enough information to determine the magnitude and direction of the electric and magnetic fields inside the solenoid? Are any assumptions needed?
3. Make a sketch of the solenoid, showing the current direction and the directions you've determined or assumed for $\vec{E}$ and $\vec{B}$.

❷ **DEVISE PLAN**

4. What is a convenient generic location to choose in order to write an expression for the magnitudes of $\vec{B}(t)$ and $\vec{E}(t)$?
5. What equation can you use to calculate the direction and magnitude of $\vec{B}(t)$?
6. What equation can you use to determine $\vec{E}(t)$?
7. How can you determine the direction and magnitude of $\vec{S}(t)$?
8. What equation can you use to determine the power?

PRACTICE

❸ EXECUTE PLAN

❹ EVALUATE RESULT

9. Write an expression for the energy stored in the magnetic field at instant t.
10. What is the time derivative of the stored magnetic energy?

Answers to Review Questions

1. A changing electric field produces a magnetic field.
2. The magnetic field forms loops around the changing electric field. The direction of the magnetic field at any location can be found using the right-hand current rule by substituting the direction of $\Delta\vec{E}$ as the direction of "current."
3. Electric fields produced by charged particles are produced both by moving charged particles and by charged particles at rest; magnetic fields produced by charged particles are produced only when the particles are moving. Fields of either kind are produced by changing fields of the other kind.
4. Electric fields exert forces on charged particles, whether the particles are moving or at rest. Magnetic fields exert forces only on moving charged particles.
5. The main similarity occurs for fields whose source is a changing counterpart field: Magnetic field lines form closed loops around a changing electric field, and electric field lines form closed loops around a changing magnetic field. The main difference occurs for fields whose source is a charged particle: Magnetic field lines form closed loops around moving charged particles, but electric field lines do not; instead, electric field lines spread outward from positively charged particles and converge on negatively charged particles, whether moving or at rest.
6. The electric field is not spherically symmetrical. It is weakest along the line of the velocity and strongest in directions perpendicular to the velocity.
7. The extent of the asymmetry is determined by the particle's speed: The faster the particle moves, the more the field surrounding it deviates from spherical symmetry.
8. Kinks appear when the charged particle is accelerated. The kink in any given electric field line joins the part of the line created by the particle before acceleration began with the part of the field line created after the particle's velocity has changed.
9. An electromagnetic wave is made up of an electric field and a magnetic field.
10. The field lines near the dipole are directed from the positive pole to the negative pole and are connected to both poles. This is true in a region of space centered on the dipole and extending in all directions by the distance that changes in the electric field could have traveled since the last change in the dipole sources: $d = cT/2$, where $T/2$ is the interval from $t = T/2$ to $t = T$. The lines far from the dipole are unconnected to the dipole sources, but they reproduce the field that existed when the dipole had its original orientation. "Far from the dipole" means far enough that electric field changes could not have traveled to this region during the time interval since the first change in the dipole sources began—that is, in the interval T between $t = T$ and $t = 0$. Between these two regions the field is not that of a dipole. Rather, the field in this region must form closed loops because there are no charged sources in the region for field lines to begin or terminate on. These loops connect the "broken ends" of the dipolar fields in the near region and in the far region. This region contains a set of loops that are "disconnected" from the dipole sources. See Figure 30.13.
11. The magnetic field is perpendicular to the electric field, and both are perpendicular to the direction in which the wave propagates.
12. *In phase* means that the two fields oscillate with the same frequency and simultaneously reach their maxima (or minima).
13. The polarization is defined to be the direction of the electric field of the wave, as seen by an observer looking along the direction of propagation.
14. An antenna is a conducting device that either emits or receives electromagnetic waves.
15. The oscillating current is created by applying across the antenna an alternating potential difference that drives charge carriers in the antenna back and forth. This oscillating motion of the carriers constitutes a current in the antenna.
16. A charging capacitor demonstrates that the Chapter 28 form of Ampère's law is incomplete. The Ampèrian path should encircle the wire leading to one of the plates, and the surface chosen to test the law should pass between the capacitor plates. The current through this surface is zero, so I_{int} for this surface is zero. There is a magnetic field between the plates, however, and so the Chapter 28 form of the law must be incomplete.
17. The generalized form of Ampère's law is $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{int}} + \mu_0\epsilon_0(d\Phi_E/dt)$.
18. The term $\mu_0 I_{\text{int}}$ represents the contribution to $\oint \vec{B} \cdot d\vec{\ell}$ of any current intercepted by a surface spanning any chosen Ampèrian path. The term $\mu_0\epsilon_0(d\Phi_E/dt)$ represents the contribution to $\oint \vec{B} \cdot d\vec{\ell}$ of any changing electric flux through the surface.
19. Displacement current is not a current in the sense of being moving charge carriers. The name is given to the factor $\epsilon_0(d\Phi_E/dt)$ in the generalized form of Ampère's law because this term is associated with a magnetic field in a region in which no charge carriers are moving. The name reminds us that we can use the right-hand *current* rule to determine the direction of the magnetic field associated with the changing electric field $\Delta\vec{E}$.
20. When a dielectric is present, the factor κ that represents the dielectric constant must be included in the displacement-current term: $\mu_0\epsilon_0\kappa(d\Phi_E/dt)$.
21. Equation 30.10, $\Phi_E = \oint \vec{E} \cdot d\vec{A} = (q_{\text{enc}}/\epsilon_0)$, is Gauss's law; Eq. 30.11, $\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0$, is Gauss's law for magnetism; Eq. 30.12, $\oint \vec{E} \cdot d\vec{\ell} = -(d\Phi_B/dt)$, is a quantitative form of Faraday's law; and Eq. 30.13, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{int}} + \mu_0\epsilon_0(d\Phi_E/dt)$, is Maxwell's generalization of Ampère's law.
22. It is fitting because Maxwell was the first to recognize that—along with conservation of charge—these four expressions completely describe all electromagnetic phenomena.
23. Equation 30.10 (Gauss's law): From the experimentally determined inverse-square relationship between the electric forces two charged particles exert on each other and the distance between the particles, and from the result that in a steady state there is no surplus charge inside a hollow charged conductor. Equation 30.11 (Gauss's law for magnetism): from the observation that magnetic monopoles do not exist, an absence that guarantees zero magnetic flux through any closed surface. Equation 30.12 (Faraday's law): from experiments on electromagnetic induction. Equation 30.13 (generalized form of Ampère's law): from measurements of the force between current-carrying wires and from observations concerning electromagnetic waves.

24. Far from where the charge carrier was accelerated, the transverse part of the pulse looks like a three-dimensional slab that extends infinitely in two dimensions and has some small finite thickness in the third dimension. The slab moves away from the charge carrier along the direction corresponding to the dimension of finite thickness.
25. The ratio E/B equals the propagation speed c_0 of the pulse.
26. The ratio E/B equals the propagation speed c of the pulse, which equals the speed of light in vacuum divided by the square root of the dielectric constant.
27. In vacuum, the speed is equal to the reciprocal of the square root of these two constants: $c_0 = 1/\sqrt{\epsilon_0\mu_0}$. In the dielectric, the speed is c_0 divided by the square root of the dielectric constant: $c = c_0/\sqrt{\kappa} = 1/\sqrt{\epsilon_0\mu_0\kappa}$.
28. The magnetic field, the electric field, and the velocity are perpendicular to one another. The direction of propagation is the same as the direction of $\vec{E} \times \vec{B}$.
29. No. In vacuum, electromagnetic waves of all frequencies travel at the same constant speed c_0.
30. The speed at which the wave travels is equal to the product of the wave's frequency and the wavelength, $f\lambda$.
31. From lowest to highest frequency, the names are radio waves, infrared, visible light, ultraviolet, x rays, and gamma rays.
32. The electric and magnetic energy densities are numerically equal to each other.
33. (*a*) $u = EB\sqrt{\epsilon_0/\mu_0}$; (*b*) $u = \epsilon_0 E^2$; (*c*) $u = B^2/\mu_0$
34. The Poynting vector is the vector product of the wave's electric field and magnetic field vectors divided by the magnetic constant. It represents the flow of energy per unit time per unit area in the wave.
35. The intensity is the magnitude of the wave's Poynting vector. For a planar electromagnetic wave, this magnitude is equal to the instantaneous electromagnetic power that crosses a unit area oriented perpendicular to the direction of $\vec{E} \times \vec{B}$.
36. The intensity of the wave equals the product of the wave's energy density and its speed: $S = uc_0$ (Eq. 30.35).

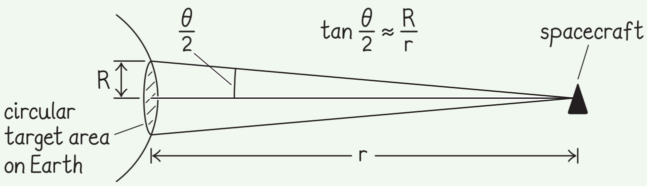
Answers to Guided Problems

Guided Problem 30.2 (*a*) $k = \dfrac{\omega}{c_0} = \dfrac{3.0 \times 10^9\ \text{s}^{-1}}{3.0 \times 10^8\ \text{m/s}} = 1.0 \times 10^1\ \text{m}^{-1}$;

(*b*) $\vec{E}(y, t) = (6.0 \times 10^3\ \text{V/m}) \cos[(1.0 \times 10^1\ \text{m}^{-1})y + (3 \times 10^9\ \text{s}^{-1})t]\hat{\imath}$

Guided Problem 30.4 $\theta = 5.4 \times 10^{-9}$ rad

Figure WGA_30.4



Guided Problem 30.6 $P_{\text{max}} = 15$ W

Guided Problem 30.8 $\dfrac{dU^B}{dt} = \dfrac{\mu_0 N^2}{h} b^2 t \pi a^2$; power crosses into the solenoid

PRACTICE

Guided Practice by Chapter

31 Electric Circuits

Review Questions

Answers to these questions can be found at the end of this chapter.

31.1 The basic circuit

1. What is the main difference between how a direct-current electric circuit is powered and how an alternating-current electric circuit is powered?
2. What is a *loop* in an electric circuit?
3. What is the *load* in an electric circuit?
4. Are the wires in an electric circuit part of the load?
5. Describe the energy conversion that takes place in a direct-current electric circuit.

31.2 Current and resistance

6. In electric circuits, to what situation does the term *steady state* refer?
7. What is the current continuity principle?
8. What does the current continuity principle say about the accumulation or depletion of charge carriers at any location in a single-loop electric circuit?
9. When a charge carrier moves from one location in a circuit to another location, how much electric potential energy is converted to some other form of energy?
10. How are circuit elements connected in a series connection?
11. What electrical property of a circuit element determines how great a potential difference must be applied across it in order to maintain a given current in it?

31.3 Junctions and multiple loops

12. In steady state, what does the current continuity principle require for (*a*) the current in any branch of a multiloop circuit and (*b*) the current into and out of a junction?
13. What electrical quantity is the same for two light bulbs that are connected in series? For two light bulbs connected in parallel?
14. What is a short in an electric circuit, and what does it do?
15. Does the orientation of elements in a circuit diagram make any difference in whether or not the diagram correctly represents the circuit? Does the length or bending of the connecting wires make any difference?

31.4 Electric fields in conductors

16. What is the electric field like inside a conductor of uniform cross section carrying a steady current?
17. On what three factors does the resistance of an electrical conductor of uniform cross section depend?

31.5 Resistance and Ohm's law

18. Describe what a metallic electrical conductor looks like at the atomic level.
19. How does the Drude model describe the average motion of free electrons in an electrical conductor in the presence of an electric field?
20. What is the conductivity of a material, and what property of the material does conductivity measure?
21. Which properties determine the conductivity of a material?
22. How is the resistance of a circuit element defined?
23. What is an ohmic material?
24. What information about resistance can be found by plotting current as a function of applied potential difference? What does such a curve look like for a circuit element made of an ohmic material?

31.6 Single-loop circuits

25. What two conditions apply to a single-loop circuit in steady state?
26. What is the equivalent resistance of a number of resistors in series?
27. What is the internal resistance of a battery, and how is it accounted for when we analyze a circuit that contains the battery?

31.7 Multiloop circuits

28. What is the equivalent resistance of a number of resistors connected in parallel?
29. Summarize the suggested strategy for analyzing multiloop circuits.

31.8 Power in electric circuits

30. What is the general expression for the rate at which electrical energy is converted to other forms of energy in a circuit element?
31. What is the expression for the rate at which energy is dissipated in a resistor in terms of the current through it? Which type of energy conversion does this dissipation represent?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The resistance of a glowing 100-W light bulb (W, E, R)
2. The resistance of a 5-W nightlight plugged into a bathroom outlet (W, E, R)
3. The height from which you must fall to gain the kinetic energy equivalent to the energy two 100-W light bulbs use in 1 h (H, A, O)
4. The length of tungsten filament in a 100-W light bulb (J, C, M)
5. The diameter of the copper wires in a 15-m heavy-duty extension cord (D, L, P, S)
6. The electric field inside the filament of a glowing 100-W light bulb (E, T)
7. The energy available for conversion in a C battery (Q, U)
8. The cost of 1 MJ (10^6 J) of C-battery energy (I, F)
9. The average rate at which electrical energy is used by the average U.S. home (N, G, V)
10. The annual cost of electricity to run the average U.S. home at \$0.15/kW-h (\$0.04/MJ) (K, B)

Hints

A. How do you determine your kinetic energy at the end of the fall?
B. How much energy does the home use in a year?
C. What is the diameter of the filament?
D. What potential difference is allowed for the cord?
E. What is the potential difference?
F. What is the cost of one C battery?
G. What is the rate of energy consumption when most items are in use?
H. How much energy do the bulbs use in 1 h?
I. How many C batteries are needed in order to have 1 MJ of energy?
J. What is the resistance of the filament?
K. How much power does the home use?
L. How much current can a heavy-duty cord carry?
M. What is the conductivity of tungsten at operating temperature?
N. How many lights and appliances are used in an average home?
O. What is your inertia?
P. What is the conductivity of copper?
Q. How long can a flashlight remain on before it dims dramatically?
R. If you opt to use $R = P/I^2$, how can you calculate current when you know the power and potential difference?
S. Describe the circuit you must analyze.
T. What is the length of the filament?
U. What is the power of a flashlight bulb?
V. During what fraction of the average day is home energy consumption near maximum?
W. What expression gives the resistance in a circuit element when you know the element's power rating?

Key (all values approximate)

A. all your gravitational potential energy *mgh* before the fall is converted to kinetic energy; **B.** 3×10^{10} J/y; **C.** 5×10^{-2} mm; **D.** wires should be almost ideal, so a few percent of household potential difference, or 5 V; **E.** 1×10^2 V, typical in household wiring; **F.** \$1 each in bulk; **G.** heating/cooling dominates, so say 5 kW; **H.** 7×10^5 J; **I.** 10 batteries, see Developing a Feel question 7; **J.** 10^2 Ω, see Developing a Feel question 1; **K.** 1 kW, see Developing a Feel question 9; **L.** 2×10^1 A; **M.** 1×10^6 A/(V · m); **N.** a dozen lights, a refrigerator, stove, TV, computer, and heating/cooling system; **O.** 6×10^1 kg; **P.** 6×10^7 A/(V · m); **Q.** 5 h; **R.** $P = IV$; **S.** two 15-m wires side by side inside a sheath made of nonconducting material, an unknown circuit element (load) plugged into the female end of the cord joins the two wires into a single-loop circuit, and the house wiring at 120 V provides the power source for the circuit; **T.** 10^{-1} m, see Developing a Feel question 4; **U.** 5 W; **V.** 1/3; **W.** either $R = P/I^2$ or $R = V^2/P$

Worked and Guided Problems

Procedure: Applying the loop rule in single-loop circuits

When applying the loop rule to a single-loop circuit consisting of resistors, batteries, and capacitors, we need to make several choices in order to calculate the current or the potential difference across each circuit element.

1. Choose a reference direction for the current in the loop. (This direction is arbitrary and may or may not be the direction of current, but don't worry, things sort themselves out in step 4.) Indicate your chosen reference direction by an arrowhead, and label the arrowhead with the symbol for the current (I).
2. Choose a direction of travel around the loop. This choice is arbitrary and separate from the choice of the reference direction for the current in step 1. (You may want to indicate the travel direction with a circular clockwise or counterclockwise arrow in the loop.)
3. Start traversing the loop in the direction chosen in step 2 from some arbitrary point on the loop. As you encounter circuit elements, each circuit element contributes a term to Eq. 31.21. Use Table 31.2 to determine the sign and value of each term. Add all terms to obtain the sum in Eq. 31.21. Make sure you traverse the loop completely.
4. Solve your expression for the desired quantity. If your solution indicates that $I < 0$, then the direction of current is opposite the reference direction you chose in step 1.

Table 31.2 Signs and values of potential differences across batteries and resistors (Figure 31.35)

Circuit element	Plus sign when traversing	Value
ideal battery	from − to +	$\mathscr{E}$
capacitor	from − to +	$q(t)/C$
resistor	opposite reference direction of current	IR

Procedure: Analyzing multiloop circuits

Here is a series of steps for calculating currents or potential differences in multiloop circuits.

1. Identify and label the junctions in the circuit.
2. Label the current in each branch of the circuit, arbitrarily assigning a direction to each current.
3. Apply the junction rule to all but one of the junctions. (The choice of which junctions to analyze is arbitrary; choose junctions that involve the quantities you are interested in calculating.)
4. Identify the loops in the circuit and apply the loop rule (see the Procedure box on page 1086) enough times to obtain a suitable number of simultaneous equations relating the unknowns in the problem. The choice of loops is arbitrary, but every branch must be in at least one of the loops. Traverse each loop in whichever direction you prefer, but be sure you traverse each loop completely and stick with the direction of travel and with the chosen directions of the currents.

There are several simplifications you can make during your analysis.

1. Multiloop circuits can sometimes be simplified by replacing parallel or series combinations of resistors by their equivalent resistances. If you can reduce the circuit to a single loop, you can solve for the current in the source. You may then need to "unsimplify" and undo the resistor simplification to calculate the current or potential difference across a particular resistor.
2. In general when solving problems, you should solve equations analytically before substituting known numerical values. When solving the simultaneous equations you obtain for multiloop circuits, however, you can often simplify the algebra if you substitute the known numerical values earlier on.

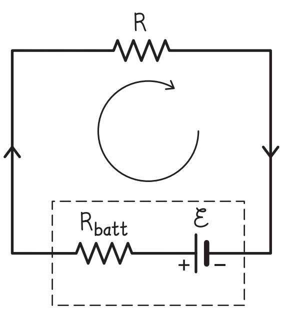
These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 31.1 Battery power

A battery of emf $\mathscr{E}$ has internal resistance R_{batt}. When it is new, the battery can maintain this emf while transporting an amount of charge q between its terminals, but then further charge transport is accompanied by a rapid reduction in the value of this emf. Suppose the battery is connected to a resistive load of resistance R. (*a*) What value of R maximizes the rate at which energy is converted in the load? (*b*) Derive an expression for the maximum rate at which energy is converted in the load. (*c*) Derive an expression for the amount of electric potential energy converted to thermal energy in the load before the battery dies.

❶ **GETTING STARTED** We are given the emf and internal resistance for a battery in a circuit that contains a load of resistance R. The battery is capable of transporting an amount of charge q before dying. Our tasks are (*a*) to determine what value of R maximizes the rate at which energy is delivered by the battery and to express (*b*) that maximum power and (*c*) the amount of energy converted in the load in terms of the given variables. Let's first draw the circuit and show the reference direction we choose for the current and the direction in which we shall travel around the circuit as we analyze the potential differences across the various circuit elements (Figure WG31.1).

Figure WG31.1



❷ **DEVISE PLAN** We first apply the loop rule (Eq. 31.21) to determine the current in the circuit. We then use Eq. 31.43 to obtain an expression for the power P, which is the rate at which energy is converted in the load. In order to calculate the value of R that maximizes P, we need to differentiate P with respect to R and then determine the value of R for which $dP/dR = 0$. Once we have this value of R, we can use Eq. 31.43 again to solve part *b*. For part *c*, we can use Eq. 31.41 to determine how much energy is converted in the load before the battery dies.

❸ **EXECUTE PLAN** (*a*) With our choices for the current reference direction and the direction in which we travel around the circuit, the loop rule takes the form

$$\mathscr{E} - IR_{batt} - IR = 0,$$

so the current is

$$I = \frac{\mathscr{E}}{R_{batt} + R}. \qquad (1)$$

The rate at which energy is converted in the load is, from Eq. 31.43,

$$P = I^2R = \left(\frac{\mathscr{E}}{R_{batt} + R}\right)^2 R.$$

Next we take the derivative of this power with respect to R and set it equal to zero:

$$\frac{dP}{dR} = \mathscr{E}^2\left[\left(\frac{1}{R_{batt} + R}\right)^2 - 2R\left(\frac{1}{R_{batt} + R}\right)^3\right] = 0.$$

This equation solved for R gives us

$$\left(\frac{1}{R_{batt} + R}\right)^2 = 2R\left(\frac{1}{R_{batt} + R}\right)^3$$

$$R_{batt} + R = 2R$$

$$R = R_{batt}. \checkmark$$

The rate at which energy is converted in the load is greatest when the load resistance is the same as the battery's internal resistance.

(*b*) Substituting R_{batt} for R in Eq. 1 gives us an expression for the current that maximizes the power:

$$I_{max} = \frac{\mathscr{E}}{R_{batt} + R} = \frac{\mathscr{E}}{2R_{batt}}.$$

PRACTICE

Thus the maximum power is

$$P_{max} = I_{max}^2 R = \left(\frac{\mathscr{E}}{2R_{batt}}\right)^2 R_{batt} = \frac{1}{4}\frac{\mathscr{E}^2}{R_{batt}}. ✔$$

(*c*) We can now use Eq. 31.41 to obtain an expression for the energy converted in the load:

$$\Delta E = -qV_{ab} = IRq = \frac{\mathscr{E}}{2R_{batt}} R_{batt} q = \frac{\mathscr{E}q}{2}. ✔$$

❹ **EVALUATE RESULT** The chemical energy converted in the battery before it dies is $\Delta E = \mathscr{E}q$, so our result tells us that half of this chemical energy is converted in the load. The other half of the chemical energy must be converted in the internal resistance of the battery! All of this chemical energy is dissipated into thermal energy due to the internal resistance of the battery or the external resistance of the load. This is reasonable for a resistive load, such as a hair dryer or toaster, but it might surprise you that by maximizing the rate of energy conversion in the load we have "wasted" 50% of the chemical energy of the battery. Returning to Eq. 1, we see that if the load resistance is considerably greater than the internal resistance of the battery, the current will be much smaller. Suppose $R = 100R_{batt}$. Then the rate of energy conversion in the load is much smaller, but Eq. 31.41 allows 99% of the converted chemical energy to appear in the load, with only 1% "wasted" in the battery. By taking a much longer time to convert energy, we use the energy stored in the battery much more efficiently.

Guided Problem 31.2 Battery-powered lift

A standard D cell battery can supply 25 mA at 1.5 V for about 300 h, and its internal resistance is about 1.0 Ω. A multispeed winch powered by such a battery is 50% efficient and is being used to lift a 60-kg object that is initially on the ground. When the winch is set to a very slow speed, the load resistance is much greater than the internal resistance of the battery. The greatest speed of the winch is limited by the maximum power supplied by the battery. (*a*) In each case, how high can the winch lift the object before the battery uses all of its energy? (*b*) What are the maximum speed at which the winch lifts the object and the minimum time interval required for the lift? Assume the battery capacity (product of current and time) is independent of the discharge rate, which is not accurate for real batteries.

❶ **GETTING STARTED**

1. Draw a circuit diagram. Which circuit element symbol should represent the winch?
2. When the winch is run at a very slow speed, how does the amount of energy dissipated due to the battery's internal resistance compare with the amount of energy converted to mechanical potential energy of the rising object?

❷ **DEVISE PLAN**

3. What relationship tells you the quantity of charge the battery delivers before it dies?
4. What relationship tells you how much energy the battery delivers before it dies?
5. How can you use energy relationships to determine the height to which the object rises before the battery dies?
6. What expression allows you to compute the rate of energy transfer to the load?
7. How do you maximize the rate at which energy is delivered to the load?
8. How do you determine the greatest speed at which the winch can lift the object?

❸ **EXECUTE PLAN**

9. Calculate the quantity of charge delivered and the energy delivered before the battery dies.
10. Calculate how high off the ground the object is lifted at very slow speed.
11. What is the greatest speed the object attains?
12. At this greatest lift rate, what is the maximum height that the winch can lift the object before the battery dies?
13. What minimum time interval is required for the object to reach this height?

❹ **EVALUATE RESULT**

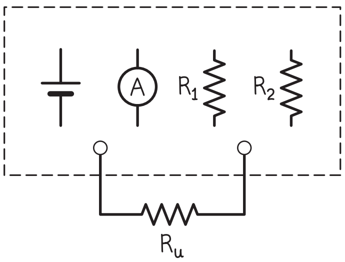
14. Are your results for the time interval required to raise the load to its maximum height at the greatest speed reasonable? Examine some limiting cases.

Worked Problem 31.3 Designing an ohmmeter

An ammeter that has a resistance of 20 Ω reads 50 μA when the needle on the meter scale is fully deflected. By combining the ammeter with a 1.5-V battery and resistors 1 and 2 that have resistances R_1 and R_2, you can convert it to an ohmmeter. When the leads of this ohmmeter are connected to each other (which is equivalent to connecting the leads across a circuit element of zero resistance), the needle on the meter scale is fully deflected. This full deflection represents both a current of 50 μA and a resistance of 0. In order to use the meter to measure the resistance of any circuit element, you must convert the meter-scale readings from amperes to ohms. Once you have done this calibration and connected the meter leads across a circuit element of unknown resistance R_u, the reading on your hand-calibrated scale indicates the resistance of the element. (*a*) If you want half-scale deflection to indicate 15 Ω, how should you connect the circuit elements to one another, and what values should you choose for R_1 and R_2? (*b*) When you align your ohm scale with the ampere scale on the meter, with the full-deflection reading $R = 0$ aligned with 50 μA on the ampere scale, which value on the ampere scale aligns with the 5.0-Ω position on your scale? With the 50-Ω position on your scale?

❶ **GETTING STARTED** We have an ammeter that has a resistance of 20 Ω, a 1.5-V battery, and resistors 1 and 2 that have resistances R_1 and R_2, and we must connect these four elements in an electric circuit in such a way that the combination acts as an ohmmeter and measures the resistance R_u of a fifth element we place in the circuit. We begin by drawing our ohmmeter as a box containing our four given elements not yet connected to one another (Figure WG31.2). We must connect the ends of our fifth element of resistance R_u to two wires that emerge from this ohmmeter "box."

Figure WG31.2



❷ **DEVISE PLAN** (*a*) To see why we need two resistors and how they must be connected, suppose at first that both R_1 and R_2 are very small and can be replaced by wires. Then we must connect only a battery, an ammeter, and the external resistance R_u. But if $R_u = 0$, then we obtain just a battery in series with an ammeter, and the current is $I_{\text{no resistors}} = \mathscr{E}_{\text{batt}}/R_A = 1.5\ \text{V}/20\ \Omega = 75\ \text{mA}$. This sounds small, but it is 1500 times the full-scale deflection current! We need a resistor to limit the current to the 50-μA scale of the meter. We can place this resistor in series with the ammeter and the battery so that the equivalent resistance of the meter + resistor combination is very great. However, we will also need to adjust the circuit so that a half-scale meter deflection corresponds to external resistance $R_u = 15\ \Omega$, without noticeably changing the portion of the circuit that secures the earlier value. Thus at least one more resistor connected in a parallel branch will be needed, and its value will need to be small (so as not to affect the earlier value very much). The numerical values can be computed using our knowledge of series and parallel resistances and the behavior of the current and potential difference for series and parallel connections.

(*b*) Having calibrated the ohmmeter, we should be able to apply the junction and loop rules to compute the scale reading for any other external resistance. If we solve for the current in the ammeter, we can determine the fraction of full-scale deflection. We can then make an appropriate mark for the corresponding scale reading in ohms.

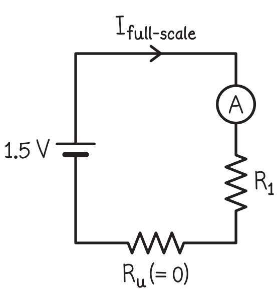
❸ **EXECUTE PLAN** (*a*) We calibrate the meter scale by using the restrictions that our new "ohmmeter" requires full-scale deflection when $R_u = 0$ and half-scale deflection when $R_u = 15\ \Omega$. To satisfy the first requirement, we connect the battery, ammeter, and resistor 1 in series inside our ohmmeter "box" and place zero resistance (a wire) for R_u (Figure WG31.3). The value of R_1 must allow full-scale deflection, so

$$I_{\text{full-scale}} = \frac{\mathscr{E}_{\text{batt}}}{R_1 + R_A}$$

$$R_1 = \frac{\mathscr{E}_{\text{batt}} - R_A I_{\text{full-scale}}}{I_{\text{full-scale}}} = \frac{1.5\ \text{V} - (20\ \Omega)(50 \times 10^{-6}\ \text{A})}{50 \times 10^{-6}\ \text{A}}$$

$$= 3.0 \times 10^4\ \Omega.$$

Figure WG31.3



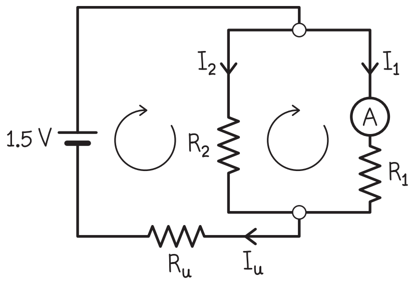
Now we add resistor 2 in a parallel branch and replace the external resistance with $R_u = 15\ \Omega$ (Figure WG31.4). We now have a two-loop circuit, and we add labels for currents and direction of travel around loops. This allows us to write the single junction equation and two loop equations:

$$I_u = I_1 + I_2 \tag{1}$$

$$+\mathscr{E}_{batt} - I_2R_2 - I_uR_u = 0 \tag{2}$$

$$+I_2R_2 - I_1R_A - I_1R_1 = 0. \tag{3}$$

Figure WG31.4



We know that the current in the ammeter for this special case is $I_1 = 0.25\ \mu A$. Thus Eq. 3 allows us to compute the potential difference across resistor 2:

$$I_2R_2 = I_1(R_A + R_1) = (25 \times 10^{-6}\ A)(20\ \Omega + 3.0 \times 10^4\ \Omega)$$
$$= 0.75\ V.$$

Next we obtain the potential difference across R_u from Eq. 2:

$$+\mathscr{E}_{batt} - I_2R_2 - I_uR_u = 0$$
$$1.5\ V - 0.75\ V - I_uR_u = 0$$
$$I_uR_u = 0.75\ V.$$

The current in the external resistor must therefore be

$$I_u = \frac{V_u}{R_u} = \frac{0.75\ V}{15\ \Omega} = 50\ mA.$$

The current in the battery must equal the current in the external resistor because these elements are connected in series. Because the current in the ammeter is known, the current in the battery can be computed from Eq. 1:

$$I_2 = I_u - I_1 = 50\ mA - 25\ \mu A = 50\ mA.$$

Notice that the current in the ammeter is entirely negligible on the scale of the current in the battery, resistor 2, and the external resistor. In fact, these three currents are identical to two significant digits. This allows us to determine the resistance R_2:

$$R_2 = \frac{V_2}{I_2} = \frac{0.75\ V}{50 \times 10^{-3}\ A} = 15\ \Omega.$$

The design of our ohmmeter is complete. ✔

(*b*) The two specific cases to be solved use the same circuit shown in Figure WG31.4. Thus we can employ Eqs. 1–3, changing only the value of R_u. In each case all resistances are known, which leaves three unknown currents to be determined from our three equations. Because we wish to solve for I_1, it is convenient to use Eq. 1 to eliminate I_u from Eq. 2, leaving us with two equations in two unknowns:

$$I_2R_2 - I_1(R_A + R_1) = 0 \tag{3}$$

$$+\mathscr{E}_{batt} - I_1R_u - I_2(R_2 + R_u) = 0. \tag{4}$$

Next we use Eq. 3 to eliminate I_2 from Eq. 4 and solve for I_1:

$$+\mathscr{E}_{batt} - I_1R_u - \frac{I_1(R_A + R_1)(R_2 + R_u)}{R_2} = 0$$

$$I_1 = \frac{\mathscr{E}_{batt}R_2}{R_2R_u + (R_A + R_1)(R_2 + R_u)}. \tag{5}$$

PRACTICE

Suppose $R_u = 5.0\ \Omega$. Then

$$I_1 = \frac{(1.5\ \text{V})(15\ \Omega)}{(15\ \Omega)(5.0\ \Omega) + (20\ \Omega + 3.0 \times 10^4\ \Omega)(15\ \Omega + 5.0\ \Omega)}$$
$$= 37\ \mu\text{A}.$$

The ammeter needle would move to 37 μA, or 75% of full-scale deflection, which is where we make the 5.0-Ω mark. ✔

Similarly, if $R_u = 50\ \Omega$, the ammeter current is

$$I_1 = \frac{(1.5\ \text{V})(15\ \Omega)}{(15\ \Omega)(50\ \Omega) + (20\ \Omega + 3.0 \times 10^4\ \Omega)(15\ \Omega + 50\ \Omega)}$$
$$= 12\ \mu\text{A}.$$

The needle is at 12 μA, or 23% of full-scale deflection, for the 50-Ω mark. ✔

❹ **EVALUATE RESULT** Our circuit is a reasonable design because we can easily find resistors that have the values we need: 15 Ω and $3.0 \times 10^4\ \Omega$. An alternative design employing the first resistor in parallel with the ammeter and the second in series with that combination and the battery would produce a first resistor with an uncomfortably small resistance that would be difficult to find and small enough to call into question our assumption that the resistance of the connecting wires can be ignored. Try it!

Our assumption of zero internal resistance in the battery is the only way to proceed. The value should be small in any case, but we are not given enough information about the battery (alkaline? rechargeable? lead-acid?) to allow us to look up a reasonable value.

The expression we obtained for the ammeter current, Eq. 5, shows that the current decreases as the external resistance R_u increases. This makes sense because the resistance scale runs backward compared to the current scale. The zero of current in Eq. 5 also corresponds to an infinitely large value of R_u, which is consistent with the nonlinear resistance scale that our numerical results require.

We should examine the effect of resistor 2 on our original calculation of full-scale deflection for zero external resistance. Solving Eq. 5 for the current in the ammeter for zero external resistance gives

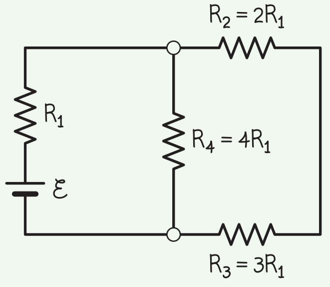
$$I_1 = \frac{(1.5\ \text{V})(15\ \Omega)}{(15\ \Omega)(0) + (20\ \Omega + 3.0 \times 10^4\ \Omega)(15\ \Omega + 0)}$$
$$= 50\ \mu\text{A}.$$

Thus the added small parallel resistance indeed had no effect on the full-scale deflection, at least to two significant digits.

Guided Problem 31.4 Four resistors

Resistors 1, 2, 3, and 4 are connected to a battery as shown in Figure WG31.5. Express the current through each resistor in terms of R_1 and the battery emf $\mathscr{E}$.

Figure WG31.5



❶ **GETTING STARTED**

1. Apply the steps in the Procedure box "Analyzing multiloop circuits." Choose a reference direction for each current and label the currents I_1, I_2, and I_4 through the three branches. Draw curved arrows to indicate the (arbitrary) direction in which you will travel around each loop.
2. Why isn't it necessary to show current I_3 in your diagram?

❷ **DEVISE PLAN**

3. How many independent equations do you need to solve the problem?
4. Is there a useful simplification for this circuit?

❸ **EXECUTE PLAN**

5. Compute equivalent resistances as needed.
6. What is the current through resistor 1?
7. What is the current through resistor 4?
8. What is the current through resistors 2 and 3?

❹ **EVALUATE RESULT**

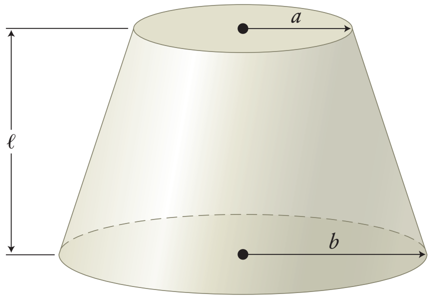
9. Check that the junction rule is satisfied.

PRACTICE

Worked Problem 31.5 Resistance of a truncated cone

Consider a resistor constructed of a material of conductivity σ that has the shape of a truncated cone of length ℓ, base radius b, and top face radius a (Figure WG31.6). The base and top face are parallel to each other. Derive an expression in terms of these variables for the resistance between the base and the top face.

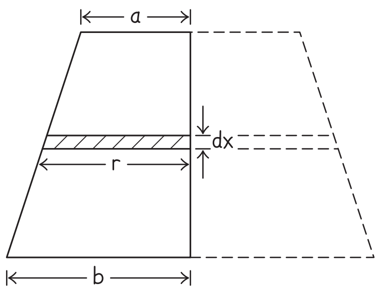
Figure WG31.6



❶ **GETTING STARTED** We know how to compute the resistance of a conductor of uniform cross section, but the cross-sectional area of this cone varies along its length. That suggests we need to integrate. We can model the cone as a stack of thin disks, oriented parallel to the base of the cone, each disk of variable radius r and thickness dx. Because the thickness is infinitesimal, each disk is essentially of uniform cross-sectional area $A = \pi r^2$. The disks are connected in series because charge carriers must flow through each disk in the order they occur in the stack.

❷ **DEVISE PLAN** Figure WG31.7 shows the truncated cone in cross section, with a representative thin disk of radius r and thickness dx. We can use Eq. 31.14 to determine the resistance dR of each disk as a function of x, the distance from the base. From the geometry of the cone, we can determine how the disk radius r varies with x and therefore how the disk area A varies with x. Because the disks are connected in series, the sum of their resistances is the resistance of the cone. To determine this sum, we can integrate the resistance $dR(x)$ from $x = 0$ to $x = \ell$.

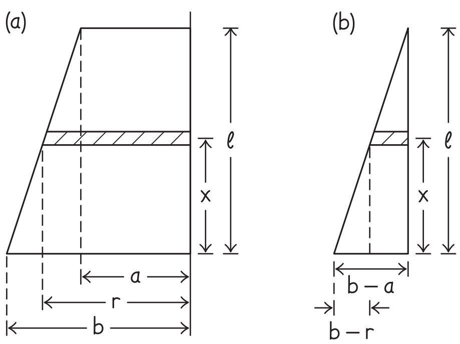
Figure WG31.7



❸ **EXECUTE PLAN** We determine how the radius r of any disk and its distance x from the base are related by working with the similar triangles shown in Figure WG31.8b. Figure WG31.8a shows their origin: In the view of the cone shown in Figure WG31.7, we draw a vertical line from the left edge of the top face to the base. From the similar triangles, r and x are related by

$$\frac{b - r}{x} = \frac{b - a}{\ell}.$$

Figure WG31.8



Rearranging this equation gives us an expression for $r(x)$:

$$r(x) = b + \frac{(a-b)}{\ell}x.$$

The area of the disk as a function of x is then

$$A(x) = \pi\left[b + \frac{(a-b)}{\ell}x\right]^2.$$

We now use Eq. 31.14 to obtain the resistance $dR(x)$ for any disk of thickness dx located a distance x from the base:

$$dR(x) = \frac{dx}{\sigma\pi\left[b + \frac{(a-b)}{\ell}x\right]^2}.$$

The resistance of the cone is therefore

$$R = \int_{x=0}^{x=\ell} \frac{dx}{\sigma\pi\left[b + \frac{(a-b)}{\ell}x\right]^2}.$$

Using the identity (see Appendix B)

$$\int \frac{dx}{(\alpha + \beta x)^2} = -\frac{1}{\beta(\alpha + \beta x)},$$

we have

$$\begin{aligned}
R &= \int_{x=0}^{x=\ell} \frac{dx}{\sigma\pi\left[b + \frac{(a-b)}{\ell}x\right]^2} \\
&= -\frac{1}{\sigma\pi}\left.\frac{1}{\left[\frac{(a-b)}{\ell}\right]\left[b + \frac{(a-b)}{\ell}x\right]}\right|_{x=0}^{x=\ell} \\
&= -\frac{1}{\sigma\pi}\left\{\frac{1}{\left[\frac{(a-b)}{\ell}\right](b + (a-b))} - \frac{1}{\left[\frac{(a-b)}{\ell}\right](b)}\right\} \\
&= -\frac{1}{\sigma\pi}\left\{\frac{(b-a)}{\left[\frac{(a-b)}{\ell}\right](ab)}\right\} = \frac{\ell}{\sigma\pi ab}.
\end{aligned}$$

✔

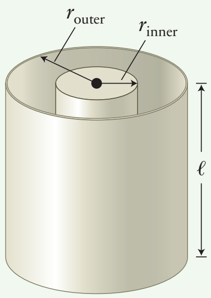
❷ **EVALUATE RESULT** Our expression involves the dimensions of the cone in a sensible fashion; that is, the resistance increases as the cone length increases and the resistance decreases as either the base or top face increases in size. If $b = a$, the cross-sectional area of the cone is a uniform $A = \pi a^2$ and our resistance expression becomes $R = \ell/\sigma\pi a^2 = \ell/\sigma A$. That this expression is identical to Eq. 31.14, the resistance for a cylindrical or rectangular resistor, gives us confidence that our answer for the resistance of the truncated cone is correct.

PRACTICE

Guided Problem 31.6 Conductivity of seawater

An oceanographer uses electrical conductivity to study how the ion concentration in seawater depends on depth. She does this by lowering into the ocean a device consisting of a metallic solid cylinder of radius $r_{inner} = 10.0$ mm concentric with a metallic cylindrical shell of radius $r_{outer} = 40.0$ mm (Figure WG31.9). The length of both the solid cylinder and the shell is $\ell = 400$ mm. The device is attached to the end of a cable, and the cable is lowered to a depth $D = 4000$ m, where the water temperature is 0.00 °C and the salinity is 3.50×10^4 mg per liter of solution. With the device at that depth, the oceanographer applies a potential difference $V_{outer} - V_{inner} = 0.500$ V between the solid cylinder and the cylindrical shell, producing an outward radial current $I = 2.93$ A. What is the conductivity of the seawater at that depth?

Figure WG31.9



1 GETTING STARTED

1. How can you model the seawater that fills the space between the solid cylinder and the cylindrical shell as a resistor?
2. Is your model a combination of thin shells connected in series or parallel?

2 DEVISE PLAN

3. How can you calculate the resistance dR of the individual shells of seawater?
4. How can you determine the contribution of all the seawater shells to get the resistance of the seawater?
5. What expression relates resistance to potential difference and current?
6. How can you calculate the seawater conductivity?

3 EXECUTE PLAN

7. Express the resistance dR of one seawater shell in terms of the shell radius r, thickness dr, and length ℓ. (Be careful: Which dimension corresponds to the length of this shell resistor?)
8. Integrate to sum the resistances of all the seawater shells.
9. Use Eq. 31.11 to relate this expression for R to the potential difference and current.
10. Use this relationship to calculate the conductivity of the seawater.

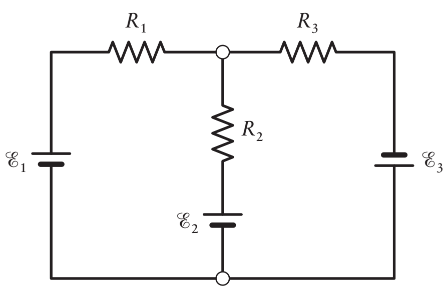
4 EVALUATE RESULT

11. Look up the value of the conductivity of seawater. Does your answer make sense based on that value?

Worked Problem 31.7 Multiple batteries

Calculate the rate at which each of the three batteries in Figure WG31.10 either delivers or absorbs (specify which in each case) energy. Assume all the batteries have zero internal resistance, and use numerical values $R_1 = R_3 = 1.0\ \Omega$, $R_2 = 2.0\ \Omega$, $\mathscr{E}_1 = 2.0$ V, and $\mathscr{E}_2 = \mathscr{E}_3 = 4.0$ V.

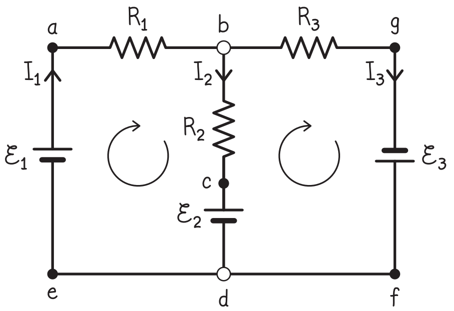
Figure WG31.10



PRACTICE

❶ **GETTING STARTED** We begin by observing that this is a multiloop circuit containing two junctions. We redraw the circuit diagram and label various locations along the wires to have a convenient way of identifying each loop we'll work with; our junctions are at b and d in the arbitrary lettering we use. Next we label the directions we choose for currents I_1, I_2, and I_3 through the three branches and add a clockwise arrow in the center of the left and right loops to show the direction in which we'll travel around each loop (Figure WG31.11).

Figure WG31.11



❷ **DEVISE PLAN** To solve this problem, we must determine the value of the current in each branch and then use our current values to calculate the power for each battery. To determine the current values, we apply the junction rule (Eq. 31.27) to one of the junctions and then the loop rule (Eq. 31.21) to as many loops as necessary. Because we have three unknown currents, we need three independent equations. With only two junctions, we know that the junction rule can give us only one independent equation, and so we must use the loop rule on two loops to get our three equations. Once we have values for I_1, I_2, and I_3, we can use Eq. 31.45 to calculate the power for each battery.

❸ **EXECUTE PLAN** The junction rule applied to junction b, with $I_{in} = I_1$ and $I_{out} = I_2 + I_3$, yields

$$I_1 = I_2 + I_3. \tag{1}$$

For loop abcdea, moving clockwise from a, we have

$$-I_1R_1 - I_2R_2 - \mathscr{E}_2 + \mathscr{E}_1 = 0. \tag{2}$$

For loop bgfdcb, moving clockwise from b, we have

$$-I_3R_3 + \mathscr{E}_3 + \mathscr{E}_2 + I_2R_2 = 0. \tag{3}$$

Now we must use these three equations to obtain expressions for I_1, I_2, and I_3. Let's begin with Eq. 2 solved for I_1:

$$I_1 = \frac{-I_2R_2 - \mathscr{E}_2 + \mathscr{E}_1}{R_1}, \tag{4}$$

and next Eq. 3 solved for I_3:

$$I_3 = \frac{\mathscr{E}_3 + \mathscr{E}_2 + I_2R_2}{R_3}. \tag{5}$$

We can now substitute these expressions for I_1 and I_2 into Eq. 1:

$$\frac{-I_2R_2 - \mathscr{E}_2 + \mathscr{E}_1}{R_1} = I_2 + \frac{\mathscr{E}_3 + \mathscr{E}_2 + I_2R_2}{R_3}.$$

Solving for I_2 and substituting given values yield

$$\begin{aligned} I_2 &= \frac{-\mathscr{E}_2(R_3 + R_1) + \mathscr{E}_1R_3 - \mathscr{E}_3R_1}{R_1R_3 + R_1R_2 + R_2R_3} \\ &= \frac{-(4.0\text{ V})(1.0\ \Omega + 1.0\ \Omega) + (2.0\text{ V})(1.0\ \Omega) - (4.0\text{ V})(1.0\ \Omega)}{(1.0\ \Omega)(1.0\ \Omega) + (1.0\ \Omega)(2.0\ \Omega) + (2.0\ \Omega)(1.0\ \Omega)} \\ &= -2.0\text{ A}. \end{aligned}$$

The minus sign means that the direction of I_2 is the opposite of the direction we chose in Figure WG31.11. Hence battery 2 is delivering energy.

We can now calculate I_1 by substituting values in Eq. 4:

$$I_1 = \frac{-(-2.0\text{ A})(2.0\ \Omega) - (4.0\text{ V}) + (2.0\text{ V})}{1.0\ \Omega} = +2.0\text{ A}.$$

PRACTICE

The positive result for this current indicates that battery 1 is delivering energy.

Equation 5 gives us

$$I_3 = \frac{(4.0\ \text{V}) + (4.0\ \text{V}) + (-2.0\ \text{A})(2.0\ \Omega)}{1.0\ \Omega} = +4.0\ \text{A},$$

and this positive value for I_3 tells us that battery 3 is delivering energy.

The rate at which energy is delivered to the circuit by battery 1 is, from Eq. 31.45,

$$P_1 = \mathscr{E}_1 I_1 = (2.0\ \text{V})(2.0\ \text{A}) = 4.0\ \text{W}, \checkmark$$

and that rate for battery 2 is

$$P_2 = \mathscr{E}_2 |I_2| = (4.0\ \text{V})(2.0\ \text{A}) = 8.0\ \text{W} \checkmark$$

Note we use the absolute value for I_2 because we have concluded that the negative value we obtained for this current meant that battery 2 is delivering energy. The power for battery 3 is

$$P_3 = \mathscr{E}_3 I_3 = (4.0\ \text{V})(4.0\ \text{A}) = 16\ \text{W}. \checkmark$$

4 **EVALUATE RESULT** We can check our results in two ways. First we check that our values for the current satisfy the junction rule:

$$I_{\text{in}} = I_1 = 2.0\ \text{A}$$

$$I_{\text{out}} = I_2 + I_3 = -2.0\ \text{A} + 4.0\ \text{A} = 2.0\ \text{A}.$$

Next we check that the rate at which energy is delivered by the batteries is equal to the rate at which energy is dissipated in the resistors. That rate for the batteries is 4.0 W + 8.0 W + 16 W = 28 W, and for the resistors it is

$$P_1 = (I_1^2)(R_1) = (2.0\ \text{A})^2(1.0\ \Omega) = 4.0\ \text{W}$$

$$P_2 = (I_2^2)(R_2) = (2.0\ \text{A})^2(2.0\ \Omega) = 8.0\ \text{W}$$

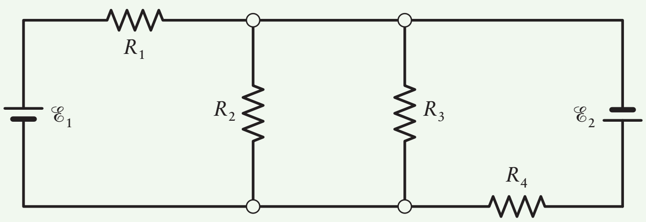
$$P_3 = (I_3^2)(R_3) = (4.0\ \text{A})^2(1.0\ \Omega) = 16\ \text{W}$$

$$4.0\ \text{W} + 8.0\ \text{W} + 16\ \text{W} = 28\ \text{W}.$$

Guided Problem 31.8 Resistor network power

Consider the circuit shown in Figure WG31.12 that consists of resistors 1, 2, 3, and 4 with values $R_1 = 2.0\ \Omega$, $R_2 = 4.0\ \Omega$, $R_3 = 4.0\ \Omega$, and $R_4 = 2.0\ \Omega$, and two batteries with emfs $\mathscr{E}_1 = 50$ V and $\mathscr{E}_2 = 20$ V. Calculate the rate at which energy is delivered to each resistor.

Figure WG31.12



1 **GETTING STARTED**

1. How can you simplify the circuit?
2. Redraw the circuit diagram, applying the steps in the "Analyzing multiloop circuits" Procedure box.

2 **DEVISE PLAN**

3. How many unknown currents must you determine?
4. Use the junction and loop rules to obtain the equations you need.
5. How do you calculate the power for each resistor?

3 **EXECUTE PLAN**

6. Apply the junction rule.
7. Apply the loop rule.
8. Solve your set of equations for the current in each branch.
9. What is the current through resistors 2 and 3 in the original circuit of Figure WG31.12?
10. Calculate the rate at which energy is delivered to each resistor.

4 **EVALUATE RESULT**

11. Check that your current values satisfy the junction rule.
12. Check that the rate at which energy is delivered by the batteries is equal to the rate at which it is dissipated in the resistors.

Answers to Review Questions

1. In a direct-current circuit, the potential difference supplied by the power source stays constant, and in an alternating-current circuit, the potential difference of the power source changes (*alternates*) over time.
2. A loop is any closed conducting path through the circuit.
3. The load is all the circuit elements connected to the power source.
4. In a nonideal circuit, yes, because some of the potential difference of the power source is converted to other forms of energy in the wires. In this chapter, however, we consider such conversions to be negligible relative to the conversion taking place in the circuit elements, and so we do not consider the wires part of the load.
5. Electric potential energy initially in charge carriers in the power source is converted to some other form of energy in the circuit load.
6. Steady state for an electric circuit means that the current in every part of the circuit is constant over time.
7. For a single-loop electric circuit in a steady state, the current is the same at all locations in the circuit.
8. The principle states that when the circuit is in steady state, charge carriers do not accumulate at any location in the circuit. For example, every charge carrier moving from the power source into the rest of the circuit is accompanied by a charge carrier moving from the rest of the circuit into the power source.
9. The amount of electric potential energy converted is equal to the potential difference between the two locations multiplied by the charge on the carrier.
10. The elements are connected such that there is only a single current path through them, so that charge carriers must flow first through one element, then the next element, and so on.
11. The resistance of the element is the determining property.
12. (*a*) The current is the same everywhere in any given branch of a multiloop circuit (the branch rule). (*b*) The number of charge carriers going into a junction equals the number exiting the junction (junction rule).
13. The electrical quantity that is the same for the bulbs connected in series is the current through them. The quantity that is the same for the bulbs connected in parallel is the potential difference across each one.
14. A short in an electric circuit is a branch of negligible resistance connected in parallel to some other element in the circuit. Essentially all the charge carriers moving through the circuit flow through the short, and essentially none flow through the element connected in parallel.
15. No. Any orientation is allowed in a diagram as long as the connections between elements are accurately portrayed. Because the wires are treated as ideal, neither the length you draw them nor whether you show them bent or straight affects the accuracy of the diagram.
16. The electric field has the same magnitude everywhere inside the conductor, and the direction of the field is parallel to the walls of the conductor (in the direction of the current).
17. The resistance of an electrical conductor of uniform cross section depends on the material the conductor is made of and the length and cross-sectional area of the conductor. The resistance is directly proportional to the length and inversely proportional to the area.
18. A metal consists of a lattice of positive ions held in fixed positions and vibrating about those positions. The ions are metal atoms that have lost one or more of their outermost electrons, and these electrons are free to move about the lattice at very high speeds, colliding occasionally with the ions.
19. At any instant, the velocity of each electron is equal to its velocity before the last collision plus the impulse per unit of mass due to the electric force exerted on the electron. The average velocity of all the electrons is then the average velocity just after their last collision plus the average impulse per unit of mass. Because an electron can move in any direction after a collision, the average velocity of the electrons just after their last collision is zero. Therefore the average velocity of the electrons at any instant is due only to the average impulse per unit of mass due to the electric force exerted on the electrons between collisions. The average impulse per unit of mass is the product of the force and the average time interval between collisions divided by the mass of the electron. This average velocity due to the impulse is the *drift velocity* of the electrons, and its direction is opposite that of the electric field.
20. The conductivity is the proportionality factor between the current density and the electric field in any conductor made of the material. It measures how well or how poorly the material conducts an electric current.
21. The conductivity depends on the number density of the charge carriers, their charge and mass, the average time interval between collisions, and temperature.
22. The resistance is the ratio of the potential difference across the element to the current through it.
23. An ohmic material is one that obeys Ohm's law, $I = V/R$ (Eq. 31.11).
24. The slope of the curve is the reciprocal of the resistance of the element. For a circuit element made of an ohmic material, the curve is a straight line, which means that the resistance is independent of the potential difference across the element.
25. Current continuity requires that the current is the same at all locations in the circuit. Energy conservation requires that the algebraic sum of the emfs and the potential differences around the loop is zero (loop rule).
26. The equivalent resistance of resistors in series is the sum of the individual resistances.
27. A battery's internal resistance accounts for the (usually small) amount of energy dissipated inside the battery and thus not available as electric potential energy outside the battery. When we analyze the circuit, a small resistor of resistance R_{batt} is added in series with the battery of emf $\mathscr{E}$.
28. The equivalent resistance of resistors connected in parallel is the reciprocal of the sum of the reciprocals of the individual resistances.
29. Identify all junctions and label the currents in each branch. Write the junction rule for all but one junction, and write the loop rule for as many loops as necessary to obtain, along with the junction rule, as many independent equations as there are unknowns to solve for. Then perform the required algebra to solve for the unknowns.
30. The rate at which energy is converted is power, and the general expression for power in a circuit element is the product of the current through the element and the magnitude of the potential difference across it.
31. The rate of energy conversion in a resistor is equal to the current squared times the resistance and represents the conversion of electric potential energy to thermal energy (Eq. 31.43).

Answers to Guided Problems

Guided Problem 31.2 (*a*) $h_{\text{slow}} = \dfrac{\Delta E_{\text{batt}}}{2\,mg} = \dfrac{(4.1 \times 10^4\ \text{J})}{2(60\ \text{kg})(9.8\ \text{m/s}^2)} = 34\ \text{m},$

$$h_{\max} = \tfrac{1}{4}\left(\frac{\Delta E_{\text{batt}}}{mg}\right) = \tfrac{1}{4}\,\frac{(4.1 \times 10^4\ \text{J})}{(60\ \text{kg})(9.8\ \text{m/s}^2)} = 17\ \text{m};$$

$$(b)\ v_{\max} = \frac{(\text{efficiency})(P_{\max})}{mg} = 4.8 \times 10^{-4}\ \text{m/s},$$

$$\Delta t_{\min} = \frac{2R_{\text{batt}}\Delta E_{\text{batt}}}{\mathscr{E}^2} = \frac{(2)(1.0\ \Omega)(4.1 \times 10^4\ \text{J})}{(1.5\ \text{V})^2}$$

$$= 3.6 \times 10^4\ \text{s} = 10\ \text{h}$$

Guided Problem 31.4 $I_1 = \dfrac{\mathscr{E}}{R_1 + R_{\text{eq}2,3,4}} = \dfrac{9}{29}\dfrac{\mathscr{E}}{R_1},$

$$I_4 = \frac{\mathscr{E} - I_1R_1}{R_4} = \frac{5}{29}\frac{\mathscr{E}}{R_1},$$

$$I_2 = \frac{\mathscr{E} - I_1R_1}{R_{\text{eq}2,3}} = \frac{4}{29}\frac{\mathscr{E}}{R_1}$$

Guided Problem 31.6 $\sigma = \dfrac{I}{\Delta V 2\pi\ell}\left(\ln\dfrac{r_{\text{outer}}}{r_{\text{inner}}}\right) = 3.23\ \text{A}/(\text{V}\cdot\text{m})$

Guided Problem 31.8 $P_{\text{R1}} = I_1^2R_1 = 8.0 \times 10^2\ \text{W}, P_{\text{R2}} = 25\ \text{W}, P_{\text{R3}} = 25\ \text{W}, P_{\text{R4}} = 4.5 \times 10^2\ \text{W}$

PRACTICE

Guided Practice by Chapter

32 Electronics

Review Questions

Answers to these questions can be found at the end of this chapter.

32.1 Alternating currents

1. What is an alternating current?
2. Describe how the current changes in a circuit that consists of a charged capacitor connected to an inductor.
3. What property specific to AC circuits must be considered when you apply the junction rule, the loop rule, or Ohm's law to such circuits?

32.2 AC circuits

4. What two quantities must you know in order to specify the time dependence of a sinusoidally varying function?
5. What is a phasor? Which phasor properties are used in analyzing oscillating quantities?
6. What do the descriptions *lead* and *lag* mean when applied to two oscillating quantities A and B that vary with the same angular frequency?
7. In a series AC circuit that contains a capacitor, an inductor, and a resistor, how is the current through each circuit element related to the potential difference across the element?

32.3 Semiconductors

8. What is a semiconductor?
9. What is the difference between an intrinsic semiconductor and an extrinsic semiconductor?
10. Describe how holes in an extrinsic semiconductor behave like positively charged particles.
11. How are doped silicon semiconductors classified, and what do the classifications mean?

32.4 Diodes, transistors, and logic gates

12. What are the two main components of a diode?
13. What is the depletion zone in a diode?
14. Describe how a depletion zone forms in a diode.
15. What happens when the *n* side of a diode is connected to a battery's positive terminal and the *p* side is connected to the negative terminal? What happens when the connections are *n* side to negative terminal and *p* side to positive terminal?
16. What is a bias potential difference, and what is its purpose in an *npn* transistor?
17. What two main functions can transistors perform in electric circuits?
18. Explain how a current is created in a field-effect transistor.
19. What is a logic gate?

32.5 Reactance

20. For a capacitor connected to an AC generator, compare the current through the capacitor and the potential difference across the capacitor in terms of their angular frequencies, amplitudes, and phase constants.
21. What is capacitive reactance in an AC circuit, and on what properties of the circuit does it depend?
22. For an inductor connected to an AC generator, compare the current through the inductor and the potential difference across the inductor in terms of their angular frequencies, amplitudes, and phase constants.
23. What is inductive reactance in an AC circuit? On what properties of the circuit does it depend?

32.6 *RC* and *RLC* series circuits

24. How can the sum of two or more quantities that vary sinusoidally at the same angular frequency be determined from the phasors representing the quantities?
25. How is the impedance of a load in an AC circuit defined?
26. Explain why the combination of a resistor and a capacitor connected in series to an AC source acts as a frequency filter.

32.7 Resonance

27. What does resonant angular frequency ω_0 refer to in a series *RLC* circuit, and which properties of the circuit determine its resonant angular frequency?
28. How does the amplitude of the current in a series *RLC* circuit vary with the angular frequency of the source?
29. Which properties of a series *RLC* circuit determine the maximum amplitude of the current?

32.8 Power in AC circuits

30. What is the advantage of using the root-mean-square value of the sinusoidally oscillating current or emf when analyzing power in an AC circuit?
31. What is the average power at a resistor in an AC circuit? What happens to the energy associated with this power?
32. What is the average power at a capacitor or an inductor in an AC circuit? Describe the instantaneous power at either element.
33. What is the power factor for a load connected to an AC source?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The maximum displacement of an electron in the electrical wire leading to a lamp in your home (E, Q, K)
2. The reactance of a 1-μF capacitor in a 60-Hz circuit (H, A)
3. The reactance of a 1-μF capacitor in a circuit that oscillates at a typical FM radio frequency (H, O)
4. The reactance of a 5-mH inductor in a circuit that oscillates at a typical FM radio frequency (C, O)
5. The capacitance needed for a high-pass filter for the 8-Ω tweeter (high-frequency speaker) in a home stereo system (F, M)
6. The inductance needed for a low-pass filter for the 8-Ω subwoofer (very-low-frequency speaker) in a home stereo system (J, D)
7. The inductance needed for resonance in an *RLC* series circuit that consists of the inductor plus a 10-μF capacitor, 1-kΩ resistor, and a 60-Hz source (L)
8. The rate at which energy is consumed in a household AC circuit for a 100-W light bulb wired in series with a 5-μF capacitor (R, S, T, I, K, A, B, P)
9. The rate at which energy is consumed by a 100-W light bulb wired in series with a 10-mH inductor in an AC household circuit (R, S, T, I, K, A, B, N)
10. The inductance of the 200-W motor running a food processor in your kitchen at maximum power if the unit has a 5-μF capacitor (K, G)

Hints

A. How does angular frequency depend on frequency?
B. What is the formula for the power factor?
C. How does inductive reactance depend on angular frequency?
D. How does the cutoff angular frequency depend on inductance?
E. What is the typical drift speed of electrons in household wiring?
F. What is a typical cutoff angular frequency for a tweeter fitted with a high-pass filter?
G. What is the requirement for maximizing the power factor?
H. How does capacitive reactance depend on angular frequency?
I. What is R for the circuit?
J. What is a typical cutoff angular frequency for a subwoofer fitted with a low-pass filter?
K. What is the frequency of household AC power?
L. What is the condition for resonance?
M. How does the cutoff angular frequency depend on capacitance?
N. What is the impedance?
O. What is a typical broadcast frequency for an FM station?
P. What is the impedance?
Q. What is the maximum time interval that an electron travels in one direction?
R. How is the rate at which the bulb consumes energy determined?
S. What is $\mathscr{E}_{max}$ for household wiring?
T. What is I for the circuit?

Key (all values approximate)

A. $\omega = 2\pi f$; **B.** $\cos\phi = R/Z$; **C.** $X_L = \omega L$; **D.** $\omega_c = R/L$; **E.** 10^{-4} m/s (from Example 31.7); **F.** $10^4\ s^{-1}$; **G.** the system angular frequency must be the resonant angular frequency, which happens when $\omega L = 1/\omega C$; **H.** $X_C = 1/\omega C$; **I.** that of the bulb, $R = \mathscr{E}^2_{max}/(100\ W) = 10^2\ \Omega$; **J.** $10^3\ s^{-1}$; **K.** 60 Hz; **L.** $\omega_0 = 1/\sqrt{LC}$; **M.** $\omega_c = 1/RC$; **N.** $1 \times 10^2\ \Omega$; **O.** 10^8 Hz; **P.** $5 \times 10^2\ \Omega$; **Q.** half the AC period; **R.** $P_{av} = \frac{1}{2}\mathscr{E}_{max}I\cos\phi$; **S.** 170 V; **T.** $I = \mathscr{E}_{max}/Z$

Worked and Guided Problems

Procedure: Analyzing AC series circuits

When analyzing AC series circuits we generally know the properties of the various circuit elements (such as R, L, C, and $\mathscr{E}$), but not the potential differences across them. To determine these, follow the procedure below.

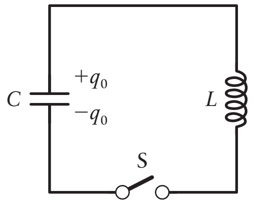
1. To develop a feel for the problem and to help you evaluate the answer, construct a phasor diagram for the circuit.
2. Determine the impedance of the load using Eq. 32.43. If there is no inductor, then ignore the term containing L; if there is no capacitor, ignore the term containing C, and so on.
3. To determine the amplitude of the current in the circuit, you can now use Eq. 32.42; to determine the phase of the current relative to the emf, use Eq. 32.44.
4. Determine the amplitude of the potential difference across any reactive element using $V = XI$, where X is the reactance of that element. For a resistor use $V = RI$.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 32.1 *LC* circuit

In the LC circuit of Figure WG32.1, $C = 80\ \mu\text{F}$. Initially, with switch S open, the capacitor is charged by an amount q_0. The switch is then closed, and 20 ms later the electric potential energy stored in the capacitor is one-fourth its initial value. What is the inductance L?

Figure WG32.1



❶ **GETTING STARTED** The potential energy in an LC circuit is stored in both the inductor and the capacitor. We are told that initially the capacitor has charge q_0, and we know there is no current through the inductor because the switch is open. We are also told that 20 ms after the switch is closed, the energy stored in the capacitor is one-fourth its initial value. Our goal is to determine the inductance of the inductor.

❷ **DEVISE PLAN** Because the energy in an LC circuit is constant, the circuit behaves like a simple harmonic oscillator, with energy that oscillates back and forth between the capacitor and the inductor. The energy stored in the capacitor is proportional to the charge squared (see Eq. 26.4), and we know that the potential difference across the capacitor oscillates sinusoidally according to $v_C = V_0 \cos \omega_0 t$ (Figure 32.4). For capacitors, we also know that $q/v_C = C$ (Eq. 26.1), so $q = Cv_C = CV_C \cos \omega_0 t$. Because the initial ($t = 0$) value of q is known to be $q(0) = q_0 = CV_C \cos(0) = CV_C$, the charge on the capacitor at any instant can be written as $q = q_0 \cos \omega_0 t$. We know that at the instant $t_1 = 20$ ms, the capacitor has only one-fourth of its initial energy. Therefore we can determine the value of ω_0 in terms of t_1. Knowing that $\omega_0 = 1/\sqrt{LC}$, we can use C and t_1 to determine the value of L.

❸ **EXECUTE PLAN** The electric potential energy stored in the capacitor at any instant t is given by Eq. 26.4

$$U_C^E(t) = \frac{q^2}{2C} = \frac{(q_0 \cos \omega_0 t)^2}{2C} = \frac{q_0^2}{2C} \cos^2 \omega_0 t.$$

Thus we can use the fact that at $t_1 = 20$ ms the energy in the capacitor is one-fourth the energy at $t = 0$ to say

$$\frac{U_C^E(t_1)}{U_C^E(0)} = \frac{\cos^2 \omega_0 t_1}{\cos^2(0)} = \frac{\cos^2 \omega_0 t_1}{1} = \tfrac{1}{4} \Rightarrow \cos \omega_0 t_1 = \tfrac{1}{2}.$$

Solving this cosine expression for $\omega_0 t_1$ yields

$$\omega_0 t_1 = \cos^{-1}(\tfrac{1}{2}) = \frac{\pi}{3} \text{rad} = 60°.$$

Therefore, with $\omega_0 = 1/\sqrt{LC}$, we obtain

$$t_1 = \frac{\pi}{3\omega_0} = \frac{\pi}{3}\sqrt{LC},$$

and the inductance is

$$L = \frac{1}{C}\left(\frac{3t_1}{\pi}\right)^2 = \frac{1}{(80 \times 10^{-6}\ \text{F})}\left[\frac{3(20 \times 10^{-3}\ \text{s})}{\pi}\right]^2$$

$$= 4.6\ \text{s}^2/\text{F} = 4.6\ \text{H.}\ ✔$$

PRACTICE

❹ **EVALUATE RESULT** Our value for the inductance, $L = 4.6$ H, is not unreasonably large compared to the inductor in Example 32.6. We can check our result by calculating the energy stored in the magnetic field of the inductor. The current in the circuit is given by

$$i = dq/dt = -\omega_0 q_0 \sin \omega_0 t.$$

The energy stored in the inductor is given by

$$\begin{aligned} U_L^B(t_1) &= \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega_0^2 q_0^2 \sin^2 \omega_0 t_1 \\ &= \tfrac{1}{2}L\frac{1}{LC}q_0^2 \sin^2 \omega_0 t_1 = \frac{q_0^2}{2C}\sin^2 \omega_0 t_1. \end{aligned}$$

When $\omega_0 t_1 = \pi/3$ rad, $\sin^2\omega_0 t_1 = \frac{3}{4}$. Therefore $U_L^B(t_1) = \frac{3}{4}q_0^2/2C$ and the energy stored in the circuit at t_1 is

$$U_L^B(t_1) + U_C^E(t_1) = \tfrac{3}{4}\frac{q_0^2}{2C} + \tfrac{1}{4}\frac{q_0^2}{2C} = \frac{q_0^2}{2C} = U_C^E(0).$$

Because there is no current through the circuit just before the switch is closed, $U_L^B(0) = 0$ and

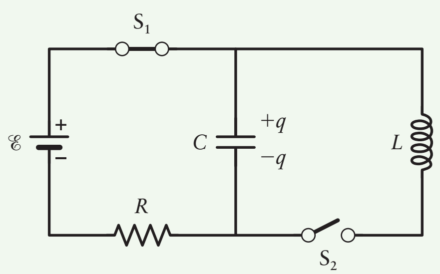
$$U_L^B(t_1) + U_C^E(t_1) = U_C^E(0) + U_L^B(0).$$

Therefore the energy is constant, as we expect because there is no resistor in the circuit.

Guided Problem 32.2 *RLC* circuit

Consider the circuit of Figure WG32.2, consisting of an inductor of inductance $L = (8.0/\pi^2) \times 10^{-3}$ H, a resistor of resistance R, a capacitor of capacitance $C = 0.50$ nF, a battery for which $\mathscr{E} = 4.0$ V, and two switches S_1 and S_2. Initially S_1 is closed and S_2 is open. After a long time interval $\Delta t \gg RC$, S_1 is opened and S_2 is closed. What is the current in the circuit at the instant t_{equal} when the magnetic potential energy stored in the inductor is equal to the electric potential energy stored in the capacitor?

Figure WG32.2



❶ **GETTING STARTED**

1. What happens at the capacitor while switch 1 is closed and switch 2 is open?
2. How do the capacitor potential difference and the current behave after switch 1 is opened and switch 2 is closed?

❷ **DEVISE PLAN**

3. How does the electric potential energy stored in the capacitor depend on the potential difference and capacitance?
4. How does the magnetic potential energy stored in the inductor depend on the current and inductance?
5. How can you use energy considerations to determine the current through the inductor at the instant t_{equal}?

❸ **EXECUTE PLAN**

6. What are the potential difference across the capacitor and the potential energy in the circuit at the instant switch 1 is opened and switch 2 is closed?
7. What is the stored energy in the inductor at the instant t_{equal}?
8. At that instant, what is the current amplitude in the circuit?

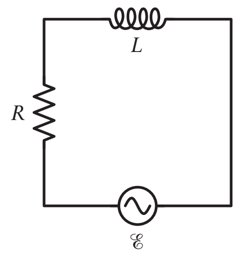
❹ **EVALUATE RESULT**

9. Is this value for the current in the range that can be easily measured with equipment from your undergraduate physics laboratory?
10. How does this value for the current magnitude I at the instant the inductor energy equals the capacitor energy compare with the maximum possible value for the current I_{max}?
11. What is I_{max}?
12. How does this value for the maximum current compare with the value you calculated?

Worked Problem 32.3 Driven *RL* circuit

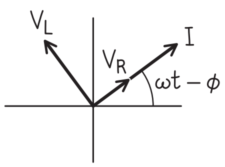
Consider the *RL* circuit of Figure WG32.3, where the AC source produces a time-varying emf given by $\mathscr{E} = \mathscr{E}_{max} \sin \omega t$. The amplitude of the emf is $\mathscr{E}_{max} = 120\sqrt{2}$ V, and the angular frequency is $\omega = 120\pi\ \mathrm{s}^{-1}$. If $R = 3.0\ \Omega$ and $L = 4.0 \times 10^{-3}$ H, what is the average power delivered by the AC source?

Figure WG32.3



❶ **GETTING STARTED** We begin by drawing a phasor diagram for the circuit showing V_R, V_L, and I (Figure WG32.4). We know that V_R and I are in phase in an AC circuit, and so we show these two phasors superimposed on each other. We also know that in an AC circuit the potential difference across the inductor leads the current by 90°, and so we draw phasor V_L at a right angle to our other two phasors. Our goal is to calculate the average power delivered by the source, which is the average rate at which the source delivers energy.

Figure WG32.4



❷ **DEVISE PLAN** We can use Eq. 32.60, $P_{av} = \frac{1}{2}\mathscr{E}_{max} I \cos \phi$, to calculate the average power delivered by the AC source, which means we must know $\mathscr{E}_{max}$, I, and $\cos \phi$. We are given the value for $\mathscr{E}_{max}$ but must determine I and $\cos \phi$. We can use Eq. 32.42 without the capacitance term (because our circuit contains no capacitors) for I, and for the power factor $\cos \phi$ we have Eq. 32.62. In order to use this relationship, however, we must know the circuit impedance Z, and we are given all the values needed to determine Z from Eq. 32.43.

❸ **EXECUTE PLAN** From Eq. 32.42, the current amplitude expressed in terms of values we know is

$$I = \frac{\mathscr{E}_{max}}{\sqrt{R^2 + (\omega L)^2}}.$$

The impedance of the circuit is, from Eq. 32.43,

$$Z = \sqrt{R^2 + (\omega L)^2},$$

and substituting this expression for Z in Eq. 32.62 gives us the power factor $\cos \phi$:

$$\cos \phi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (\omega L)^2}}.$$

Therefore the average power delivered by the AC source is, from Eq. 32.60,

$$\begin{aligned} P_{av} = \tfrac{1}{2}\mathscr{E}_{max} I \cos \phi &= \frac{1}{2}\frac{\mathscr{E}_{max}^2 R}{[R^2 + (\omega L)^2]} \\ &= \frac{1}{2}\frac{(120\sqrt{2}\ \mathrm{V})^2(3.0\ \Omega)}{[(3.0\ \Omega)^2 + (120\pi\ \mathrm{s}^{-1})^2(4.0 \times 10^{-3}\ \mathrm{H})^2]} \\ &= 3.8 \times 10^3\ \mathrm{W}. ✔ \end{aligned}$$

❹ **EVALUATE RESULT** With this large value for the average power, we expect the current amplitude to be large, so we can check that value:

$$\begin{aligned} I &= \frac{\mathscr{E}_{max}}{\sqrt{R^2 + (\omega L)^2}} \\ &= \frac{120\sqrt{2}\ \mathrm{V}}{\sqrt{(3.0\ \Omega)^2 + (120\pi\ \mathrm{s}^{-1})^2(4.0 \times 10^{-3}\ \mathrm{H})^2}} \\ &= 5.1 \times 10^1\ \mathrm{A}. \end{aligned}$$

This relatively large value for I gives us confidence that our power calculation is the right order of magnitude.

Another check for evaluating our result is to look at the phase difference, which we expect to satisfy the condition $0 < \phi < \pi/2$. From Eq. 32.44,

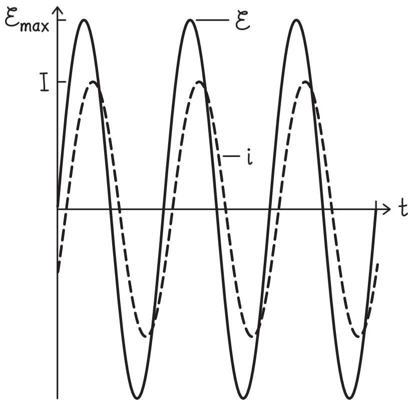
$$\phi = \tan^{-1}\frac{\omega L}{R}$$

$$= \tan^{-1}\frac{(120\pi\ \text{s}^{-1})(4.0 \times 10^{-3}\ \text{H})}{3.0\ \Omega} = 0.47\ \text{rad},$$

again giving us confidence in our result.

As a third check, we can plot on the same graph the AC source potential difference as a function of time and the current as a function of time (Figure WG32.5). The potential difference leads the current, as we expect.

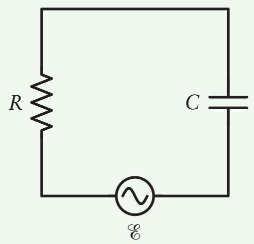
Figure WG32.5



Guided Problem 32.4 Driven *RC* circuit

In the *RC* circuit of Figure WG32.6, the emf of the AC source is given by $\mathcal{E} = \mathcal{E}_{max} \sin \omega_1 t$. The emf amplitude is $\mathcal{E}_{max} = 10\ \text{V}$, and the angular frequency is $\omega_1 = 100\ \text{s}^{-1}$. When the angular frequency is changed to a new value ω_2, the time average of the power delivered by the source decreases by a factor of 2. If $R = 3.0\ \Omega$ and $C = 5.0 \times 10^{-2}$ F, what is the value of ω_2?

Figure WG32.6



1 GETTING STARTED

1. Draw a phasor diagram for the circuit, showing V_R, V_C, and I.

2 DEVISE PLAN

2. What is the current amplitude as a function of ω?
3. What is the phase difference as a function of ω?
4. What is the time average of the power delivered by the AC source as a function of ω?
5. Write an expression showing the ratio of the average power when $\omega = \omega_1$ and the average power when $\omega = \omega_2$.
6. Solve this expression for ω_2.

3 EXECUTE PLAN

7. Determine ω_2 from the given values.

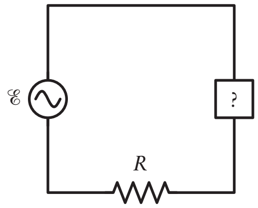
4 EVALUATE RESULT

8. Knowing that the average power decreases when the angular frequency changes from ω_1 to ω_2, do you expect the amplitude of the current to increase or decrease?
9. Does the average power vary as you expect as a function of ω?
10. Does your answer for ω_2 agree with what you expect?

Worked Problem 32.5 Unknown circuit element I

Figure WG32.7 shows a circuit containing an AC source for which the emf varies with time as $\mathscr{E} = \mathscr{E}_{max} \sin \omega t$ and a resistor of resistance R. The square with the question mark represents either an inductor or a capacitor (but not both). The amplitude of the emf is $\mathscr{E}_{max} = 100\sqrt{2}$ V, and the angular frequency is $\omega = 10\ s^{-1}$. The time-dependent current is given by $i = (10\ A)\sin(\omega t + \pi/4)$. Is the unknown element a capacitor or an inductor? What is the value of R? What is the value of the capacitance or of the inductance of the unknown element?

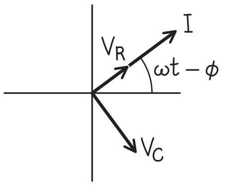
Figure WG32.7



❶ **GETTING STARTED** We know from the current expression $i = (10\ A)\sin(\omega t + \pi/4)$ that the phase difference between the AC source emf and the current is $\phi = -\pi/4$. The minus sign tells us that the current leads the AC source emf. Therefore the unknown element must be a capacitor, and we have the answer to the first question this problem asks. ✔

With this information, we can draw a phasor diagram for the circuit showing V_R, V_C, and I (Figure WG32.8).

Figure WG32.8



❷ **DEVISE PLAN** We have two unknowns, R and C, so our plan should be to obtain two independent equations containing these unknowns. We know that the unknown element is a capacitor, and we know the phase constant ϕ. Thus we can use Eq. 32.44 without the inductance term as one equation containing C and R. The current equation given in the problem statement shows the amplitude of current $I = 10$ A, and we are told the value of $\mathscr{E}_{max}$. Therefore we can use Eq. 32.42 without the inductance term as our second equation containing C and R.

❸ **EXECUTE PLAN** We know that $\phi = -\pi/4$ and $\tan(-\pi/4) = -1$. Thus Eq. 32.44 yields

$$-1 = \frac{-1/\omega C}{R}$$

$$\frac{1}{\omega C} = R. \tag{1}$$

With this result, we can eliminate $1/\omega C$ from Eq. 32.42, solve that equation for R, and use our numerical values:

$$I = \frac{\mathscr{E}_{max}}{\sqrt{R^2 + (1/\omega C)^2}} = \frac{\mathscr{E}_{max}}{\sqrt{R^2 + R^2}} = \frac{\mathscr{E}_{max}}{\sqrt{2}R}$$

$$R = \frac{\mathscr{E}_{max}}{\sqrt{2}I} = \frac{100\sqrt{2}\ \text{V}}{\sqrt{2}(10\ \text{A})} = 10\ \text{V/A} = 10\ \Omega. ✔$$

Now that we know R, we solve Eq. 1 for the capacitance:

$$C = \frac{1}{\omega R} = \frac{1}{(10\ \text{s}^{-1})(10\ \Omega)} = 1.0 \times 10^{-2}\ \text{F}. ✔$$

❹ **EVALUATE RESULT** Our value for the capacitance is somewhat large, and our value for the resistance is small but reasonable. Because $\phi = -\pi/4$, the reactance of the capacitor is equal to the resistance, the impedance of the circuit is $Z = \sqrt{R^2 + X_C^2} = \sqrt{2}R$, and therefore the amplitude of current

$$I = \frac{\mathscr{E}_{max}}{\sqrt{2}R} = \frac{100\sqrt{2}\ \text{V}}{\sqrt{2}(10\ \Omega)} = 10\ \text{A}.$$

This agrees with our result above, which gives us confidence in our calculation.

We can also check our result by comparing the average power delivered by the AC source—the average rate at which the source delivers energy to the resistor—and the power at the resistor—the average rate at which that energy is dissipated by the resistor. We know that $\cos\phi = \cos(-\pi/4) = 1/\sqrt{2}$. Hence the average power delivered by the source is, from Eq. 32.60,

$$P_{av} = \tfrac{1}{2}\mathscr{E}_{max}\frac{\mathscr{E}_{max}}{\sqrt{2}R}\frac{1}{\sqrt{2}} = \tfrac{1}{4}\frac{\mathscr{E}_{max}^2}{R}.$$

The average rate at which energy is dissipated in the resistor is, from Eq. 32.52,

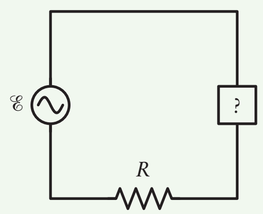
$$P_{av} = \tfrac{1}{2}I^2R = \tfrac{1}{2}\left(\frac{\mathscr{E}_{max}}{\sqrt{2}R}\right)^2 R = \tfrac{1}{4}\frac{\mathscr{E}_{max}^2}{R}.$$

We see that these two are equal, which is what we expect because the phasors V_C and I are 90° out of phase, and so no energy is dissipated in the capacitor.

Guided Problem 32.6 Unknown circuit element II

Figure WG32.9 shows a circuit containing an AC source for which the emf varies with time as $\mathscr{E} = \mathscr{E}_{max}\sin\omega t$. The emf amplitude is $\mathscr{E}_{max} = 6.0$ V, and the resistor is rated at $R = 3.0\ \Omega$. The square with the question mark represents an inductor, a capacitor, *or both*. When the angular frequency is $\omega_1 = 1.0\ \text{s}^{-1}$, the current is in phase with the source emf. When the angular frequency is $\omega_2 = 2.0\ \text{s}^{-1}$, the current is out of phase with the emf by $|\pi/4|$ rad. What is the average rate at which energy is dissipated in this circuit when the angular frequency is ω_1? Repeat for angular frequency ω_2.

Figure WG32.9



1 GETTING STARTED

1. Does the square represent an inductor, a capacitor, or both?
2. When the angular frequency is $\omega_2 = 2.0\ \text{s}^{-1}$, what reactance(s) must be accounted for?
3. Draw a phasor diagram for the circuit.

2 DEVISE PLAN

4. What is the appropriate expression relating angular frequency to the inductance and/or capacitance?
5. How can you determine whether the source emf leads or lags the current at angular frequency ω_2?
6. How can you determine the values of L and/or C?
7. How can you determine the current amplitude when the angular frequency is ω_1? ω_2?
8. How can you determine the average rate at which energy is dissipated in the resistor?
9. Is there any energy dissipated in the inductor and/or capacitor?

3 EXECUTE PLAN

10. Write the expressions relating L and/or C at both given frequencies.
11. Calculate values for L and/or C.
12. Calculate the current amplitude at ω_1 and ω_2.
13. Determine the average rate at which energy is dissipated at ω_1 and ω_2.

4 EVALUATE RESULT

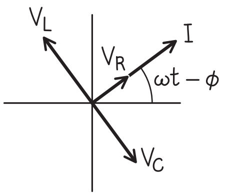
14. What is the average power delivered by the AC source at angular frequency ω_1? Did you expect that result?

Worked Problem 32.7 FM radio tuner

Suppose you want a series *RLC* circuit to tune to a radio station that broadcasts at a frequency of 89.7 MHz, but you do not want to pick up the signal from a station that broadcasts at 89.5 MHz. To achieve this, for a given input signal from your antenna, you want your resonance curve to be narrow enough to make the current in the circuit at 89.5 MHz be 100 times smaller than the current in the circuit at 89.7 MHz. You cannot avoid having a resistance of $R = 0.100\ \Omega$, and practical considerations also dictate that you use the smallest possible value for L. What values of L and C must you use?

❶ **GETTING STARTED** We begin by drawing a phasor diagram showing V_R, V_C, V_L, and I (Figure WG32.10). We first draw phasors V_R and I in phase, and then draw V_L leading I by 90° and V_C lagging I by 90°. We model the time-dependent input signal by the sinusoidal function $\mathscr{E} = \mathscr{E}_{\max} \sin \omega t$ and the time-dependent current in the circuit by $i = I \sin(\omega t - \phi)$. We want resonance to occur at $\omega_{89.7} = (2\pi)(89.7 \times 10^6\ \text{Hz})$. We also require that the current amplitude $I(\omega_{89.5})$ at $\omega_{89.5} = (2\pi)(89.5 \times 10^6\ \text{Hz})$ be one-hundredth the maximum current at resonance $I(\omega_{89.7})$: $I(\omega_{89.5}) = 0.0100 I(\omega_{89.7})$. We are given that $R = 0.100\ \Omega$.

Figure WG32.10



❷ **DEVISE PLAN** The current amplitude I is a function of ω (Eq. 32.42). We also use the resonance condition Eq. 32.47 to simplify the expression for $I(\omega_{89.7})$. We have two expressions, one for $I(\omega_{89.7})$ and one for $I(\omega_{89.5})$, and the condition that $I(\omega_{89.5}) = 0.0100 I(\omega_{89.7})$. We can thus obtain a relationship between L and C. We again use the resonance condition Eq. 32.47, which gives a second condition for L and C. We now solve our system of equations for the values of L and C.

❸ **EXECUTE PLAN** Equation 32.42 gives the current amplitude as a function of ω:

$$I = \frac{\mathscr{E}_{\max}}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}. \tag{1}$$

We know from Eq. 32.46 that at resonance, $\omega L = 1/\omega C$, which means that at our resonant angular frequency $\omega_{89.7}$, Eq. 1 reduces to

$$I(\omega_{89.7}) = \frac{\mathscr{E}_{\max}}{R}. \tag{2}$$

Once we set $\omega = \omega_{89.5}$ in Eq. 1:

$$I(\omega_{89.5}) = \frac{\mathscr{E}_{\max}}{\sqrt{R^2 + (\omega_{89.5} L - 1/\omega_{89.5} C)^2}}, \tag{3}$$

we are ready to determine values for L and C. To keep the notation simple, let's for now use h to represent the factor by which the two current amplitudes must differ: $h = 0.0100$. Thus

$$I(\omega_{89.5}) = h I(\omega_{89.7}),$$

and from Eqs. 2 and 3 we have

$$\frac{\mathscr{E}_{\max}}{\sqrt{R^2 + (\omega_{89.5} L - 1/\omega_{89.5} C)^2}} = h \frac{\mathscr{E}_{\max}}{R}.$$

Squaring both sides and eliminating $\mathscr{E}_{\max}$ yield

$$\frac{1}{R^2 + (\omega_{89.5} L - 1/\omega_{89.5} C)^2} = h^2 \frac{1}{R^2}. \tag{4}$$

A little algebraic manipulation yields the condition that

$$\frac{R^2}{R^2 + (\omega_{89.5} L - 1/\omega_{89.5} C)^2} = h^2$$

$$\Rightarrow (\omega_{89.5} L - 1/\omega_{89.5} C)^2$$

$$= \left(\frac{1}{h^2} - 1\right) R^2.$$

PRACTICE

Taking square roots, we have

$$\omega_{89.5}L - \frac{1}{\omega_{89.5}C} = -\sqrt{\frac{1}{h^2} - 1}\,R.$$

We show a minus sign on the term on the right because the circuit is below resonance, which means that $(\omega_{89.5}L - 1/\omega_{89.5}C) < 0$. Using Eq. 32.47 in the form $C = 1/L\omega_{89.7}^2$, where $\omega_{89.7}$ is the resonant angular frequency, we obtain

$$\omega_{89.5}L\left[1 - \left(\frac{\omega_{89.7}}{\omega_{89.5}}\right)^2\right] = -\sqrt{\frac{1}{h^2} - 1}\,R,$$

which yields

$$L = \frac{\sqrt{\left(\frac{1}{h^2} - 1\right)}R}{\omega_{89.5}[(\omega_{89.7}/\omega_{89.5})^2 - 1]}$$

$$= \frac{\sqrt{(1.00 \times 10^4 - 1)}(0.100\ \Omega)}{(5.62 \times 10^8\ \text{s}^{-1})[(89.7/89.5)^2 - 1]} = 3.97 \times 10^{-6}\ \text{H}. ✔$$

The capacitance is therefore

$$C = \frac{1}{L\omega_{89.7}^2} = \frac{1}{(3.97 \times 10^{-6}\ \text{H})(5.63 \times 10^8\ \text{s}^{-1})^2}$$

$$= 7.92 \times 10^{-13}\ \text{F}. ✔$$

4 EVALUATE RESULT Suppose the antenna produces a maximum signal $\mathscr{E}_{max} = 100\ \mu\text{V}$. Then at resonance, where $\phi = 0$, the time average of the signal's power is, from Eq. 32.60,

$$P_{av}(\omega_{89.7}) = \tfrac{1}{2}\mathscr{E}_{max}I(\omega_{89.7}) = \tfrac{1}{2}\frac{\mathscr{E}_{max}^2}{R}$$

$$= \frac{(1.00 \times 10^{-4}\ \text{V})^2}{2(0.100\ \Omega)} = 5.00 \times 10^{-8}\ \text{W}, \quad (5)$$

where we have substituted from Eq. 2 for $I(\omega_{89.7})$. The signal's average power at $\omega = \omega_{89.5}$ is

$$P_{av}(\omega_{89.5}) = \tfrac{1}{2}\mathscr{E}_{max}I(\omega_{89.5})\cos\phi.$$

The power factor at 89.5 MHz is given by Eq. 32.62:

$$\cos\phi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (\omega_{89.5}L - 1/\omega_{89.5}C)^2}}.$$

Therefore the average power at 89.5 MHz, when we substitute this expression for $\cos\phi$ and the expression for $I(\omega_{89.5})$ using Eq. 1, is

$$P_{av}(\omega_{89.5}) = \tfrac{1}{2}\frac{\mathscr{E}_{max}^2 R}{R^2 + (\omega_{89.5}L - 1/\omega_{89.5}C)^2}.$$

We can use Eqs. 4 and 5 to write this as

$$P_{av}(\omega_{89.5}) = \tfrac{1}{2}\frac{\mathscr{E}_{max}^2}{R}h^2 = P_{av}(\omega_{89.7})h^2$$

$$= (5.00 \times 10^{-8}\ \text{W})(1.00 \times 10^{-4}) = 5.00 \times 10^{-12}\ \text{W}.$$

So the current at 89.5 MHz is 100 times smaller than the peak current $I(\omega_0)$ at resonance, and the time average of the power of the input signal at 89.5 MHz is down by a factor of 10^4, which is a very small power value compared to the signal we want.

PRACTICE

Guided Problem 32.8 *RLC* circuit

A series *RLC* circuit with $R = 10.0\ \Omega$, $L = 400$ mH, and $C = 2.0\ \mu$F is connected to an AC source of emf $\mathscr{E} = \mathscr{E}_{max} \sin \omega t$, which has amplitude $\mathscr{E}_{max} = 100$ V. The angular frequency is $\omega = 4000\ s^{-1}$, and the time-dependent current is given by $i = I \sin(\omega t - \phi)$.

(*a*) Calculate the amplitude I of the current and the phase constant ϕ that represents the phase difference between the current and the source emf.

(*b*) What is the ratio of the potential-difference amplitude V_L across the inductor to the potential-difference amplitude V_C across the capacitor?

1 GETTING STARTED

1. Draw a phasor diagram for the circuit.

2 DEVISE PLAN

2. What quantities must you know in order to calculate the current amplitude?
3. What quantities must you know in order to calculate the phase constant?
4. How can you determine the values of the amplitudes V_C and V_L?
5. How can you determine the value of X_C you need for Eq. 32.15 and the value of X_L you need for Eq. 32.27?

3 EXECUTE PLAN

6. You have all the data you need to obtain a value for I from Eq. 32.42. However, you have surely noticed by now that many circuit problems require you to calculate ωL and $1/\omega C$, so it is usually most efficient to begin by calculating these two values and then the impedance.
7. Calculate the current amplitude.
8. Calculate the phase constant.
9. Calculate the ratio V_L/V_C.

4 EVALUATE RESULT

10. Compute the resonant angular frequency ω_0.
11. When $\omega = 4000\ s^{-1}$, is the circuit being driven at an angular frequency greater than or less than the resonant angular frequency?
12. What does this tell you about the expected current? About the expected ratio V_L to V_C?

Answers to Review Questions

1. An alternating current is a time-dependent electric current that periodically changes direction.
2. The current varies sinusoidally. Before the capacitor begins to discharge, the current is zero. As the capacitor discharges, the current increases to its maximum magnitude in one direction, then returns to zero, then increases to its maximum magnitude in the opposite direction, and then returns to zero again to end one cycle. In any nonideal circuit, the amplitude of the current oscillations decays because there is resistance in the circuit.
3. The property is that the potential differences and currents in AC circuits are time-dependent rather than constant.
4. The phase of a sinusoidal function describes how the function depends on time. The two quantities necessary to specify the phase at any instant t are ω, the angular frequency, and ϕ_i, the value of the phase at the instant $t = 0$. Then the phase at any instant t is given by $\omega t + \phi_i$.
5. A phasor is a rotating arrow whose component on a vertical axis can be used to represent oscillating quantities. The vertical component of the phasor represents the value of the oscillating quantity, the phasor length represents the quantity's amplitude, and the angle of the phasor measured counterclockwise from the horizontal axis of the reference circle at any instant represents the quantity's phase $\omega t + \phi_i$ at that instant.
6. Quantity A is said to *lead* quantity B when the maximum value of A during one oscillation cycle occurs before the maximum value of B occurs. Quantity A is said to *lag* quantity B when the maximum value of A during one cycle occurs after the maximum value of B occurs.
7. The current leads the potential difference across the capacitor by 90°, lags the potential difference across the inductor by 90°, and is in phase with the potential difference across the resistor.
8. A semiconductor is a material that has a limited supply of free charge carriers, so that its electrical conductivity is intermediate between that of conductors and that of insulators.
9. An intrinsic semiconductor is chemically pure and its conductivity too small for many practical applications (except as an electrical insulator). An extrinsic semiconductor (also called a doped semiconductor) has a precisely controlled level of impurity atoms (called dopants) introduced so that the conductivity can be adjusted to suit specific applications.
10. In a p-type semiconductor, dopant atoms have fewer valence electrons than the host lattice atoms, creating a deficit of one electron—a hole—in the surrounding covalent bonds. An electron from another covalent bond can move into the hole, as if the hole "attracted" the electron. The hole has now moved to the site previously occupied by the electron, and the process can be repeated. Because the hole moves in a direction opposite the moving electrons, the hole behaves like a positively charged particle moving between lattice sites.
11. Doped silicon semiconductors are classified as p-type (p for *positive*) or n-type (n for *negative*). A p-type semiconductor contains dopant atoms that have fewer valence electrons than the host lattice atoms and thus create holes in the semiconductor lattice; the holes behave like free positive-charge carriers. An n-type semiconductor contains dopant atoms that have more valence electrons than the host lattice atoms, providing free negative-charge carriers.
12. A diode is made up of a p-type silicon semiconductor and an n-type silicon semiconductor in contact with each other.
13. The depletion zone is a thin region containing no free charge carriers that develops at the junction between the n-type and p-type sides of the diode.
14. The depletion zone forms as electrons from the n-type side cross into the p side and fill up holes there, while holes from the p-type side cross into the n side and combine with electrons there. These electron-hole recombinations result in the accumulation of negative charge on the p-type side

of the depletion zone and accumulation of positive charge on the n-type side of the zone, with the resulting electric field directed from the n side to the p side.

15. Connecting the n side to the positive terminal and the p side to the negative terminal creates an electric field across the depletion zone directed from n to p; this external field broadens the depletion zone and maintains the condition of no charge carriers able to flow in the diode. Connecting the n side to the negative terminal and the p side to the positive terminal creates an electric field across the zone directed from p to n; when sufficiently strong, this external field shrinks and then eliminates the depletion zone with the result that charge carriers can flow through the diode. Thus, a diode allows electric current in one direction only.

16. It is a potential difference between emitter and base that shrinks the base-collector depletion zone enough to allow electrons to move from emitter to collector, thus creating a current through the transistor.

17. Transistors can function as switches or as current amplifiers.

18. First, two n-type wells are created in a layer of p-type material, then an insulating layer and a metal gate are added above the p-type region. Depletion zones form between the n- and p-type regions, preventing the flow of charge carriers between regions. When the gate receives a small positive charge, holes are driven from the p-type material beneath the gate, and the depletion zones at the boundaries of the two regions of n-type material become joined. A greater positive charge on the gate causes the depletion zone beneath it to be pushed away, creating an n-type channel connecting the two n-type wells and allowing conduction through the n-type materials.

19. A logic gate is a device that accepts two input signals, performs a logical operation on them, and emits the result as an output signal. In integrated circuits such as computer chips, logic gates are often implemented with field-effect transistors.

20. The current and the potential difference oscillate at the same angular frequency. The current amplitude I is directly proportional to the potential-difference amplitude V_C according to $I = \omega C V_C$. The current leads the potential difference by 90°, and so the phase constant of the current is $\phi = -\pi/2$ relative to $\phi = 0$ for the generator.

21. For a circuit containing a capacitor, the amplitude of the potential difference across the capacitor is proportional to the amplitude of the current through the capacitor, and the proportionality constant is the capacitive reactance of the circuit. Capacitive reactance depends on the angular frequency of the oscillation and on the capacitance: $X_C = 1/\omega C$.

22. The current and the potential difference oscillate at the same angular frequency. The current amplitude I is directly proportional to the potential-difference amplitude V_L according to $I = V_L/\omega L$. The current lags the potential difference by 90°, and so the phase constant of the current is $\phi = +\pi/2$ relative to $\phi = 0$ for the generator.

23. For a circuit containing an inductor, the amplitude of the potential difference across the inductor is proportional to the amplitude of the current through the inductor, and the proportionality constant is the inductive reactance of the circuit. Inductive reactance depends on the angular frequency of the oscillation and on the inductance: $X_L = \omega L$.

24. The sum at any given instant is determined by adding up the vertical components of the phasors of all the quantities. The result of this addition is the vertical component of the phasor of the sum, which equals the vector sum of the phasors.

25. The impedance is the ratio of the amplitude of the source emf to the amplitude of the current through the load: $Z = \mathscr{E}_{max}/I$.

26. Equation 32.34 shows that the impedance of a load consisting of a resistor and a capacitor connected in series is very great at low angular frequencies and nearly equal to the resistance at high angular frequencies. Therefore the amplitude of the potential difference across the resistor (the output) is very small if the angular frequency of the source (the input) is low but is nearly equal to the amplitude $\mathscr{E}_{max}$ of the source emf if the source angular frequency is high. This difference in impedance allows high-frequency signals to pass through the circuit but weakens low-frequency signals so that they cannot pass through the circuit—the low-frequency signals are filtered out.

27. The resonant angular frequency is the source angular frequency that results in the greatest current amplitude. This occurs when ωL is equal to $1/\omega C$, which means $\omega_0 = 1/\sqrt{LC}$. Inductance and capacitance are the circuit properties that determine the value of ω_0.

28. The current amplitude has its maximum value when the source angular frequency ω equals the circuit's resonant angular frequency ω_0 and decreases both as ω gets smaller than ω_0 and as ω gets greater than ω_0.

29. The maximum current amplitude depends on the resistance in the circuit and on the amplitude of the source emf.

30. The time average of any sinusoidally oscillating quantity is zero, and $I_{av} = 0$ and $\mathscr{E}_{av} = 0$ give no insight into how these quantities change over time. The root-mean-square value of a sinusoidally oscillating quantity is nonzero, however, and so is a more meaningful measure of the quantity.

31. The average power at a resistor is $P_{av} = I_{rms}^2 R = \frac{1}{2}I^2 R$. The energy is dissipated as thermal energy.

32. The average power at a capacitor or an inductor is zero because the potential difference across either of these circuit elements is 90° out of phase with the current through the element (i leads v_C by 90°, i lags v_L by 90°). In those parts of the oscillation cycle where i and either v_C or v_L have the same sign, the instantaneous power to the element is positive (energy is added to the element by the source); in those parts of the cycle where i and v have opposite signs, the instantaneous power is negative (energy is removed from the element by the source).

33. The power factor is the cosine of the phase difference between the source emf and the current. Because the average power to a load is $P_{av} = \frac{1}{2}\mathscr{E}_{max} I \cos\phi$, the power factor is a measure of the efficiency with which the source delivers energy to the load.

Answers to Guided Problems

Guided Problem 32.2 $I = \sqrt{\dfrac{C}{2L}}\mathscr{E} = 2.2 \times 10^{-3}$ A

Guided Problem 32.4 $\omega_2 = \sqrt{\dfrac{1}{C^2R^2 + 2/\omega_1^2}} = 6.6\ \text{s}^{-1}$

Guided Problem 32.6 The box contains a capacitor and an inductor in series, and $P_{av} = \frac{1}{2}I^2R = 6.0$ W at ω_1 and $P_{av} = 3.0$ W at ω_2.

Guided Problem 32.8 (a) $I = \dfrac{\mathscr{E}_{max}}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = 6.8 \times 10^{-2}$ A

and $\phi = \tan^{-1}\dfrac{\omega L - 1/\omega C}{R} = 1.6$ rad. (b) $\dfrac{V_L}{V_C} = \dfrac{X_L}{X_C} = 13$

PRACTICE

Guided Practice by Chapter

33 Ray Optics

PRACTICE

Review Questions

Answers to these questions can be found at the end of this chapter.

33.1 Rays

1. What has to happen in order for you to see an object?
2. What is a light ray? What is the relationship between a light ray and a light beam?
3. What is a shadow?

33.2 Absorption, transmission, and reflection

4. Describe the three possibilities for what happens to light that falls on an object.
5. What happens to most of the light that strikes an opaque object? To most of the light that strikes a translucent object?
6. State the law of reflection.
7. What is the difference between specular reflection and diffuse reflection?
8. If every ray that strikes a rough surface obeys the law of reflection, why is no image formed?
9. What is an image?
10. What is the difference between a real image and a virtual image?
11. Explain what it means to form an image of an object—for example, in a mirror. Why does an observer see the image of an object placed in front of a mirror as being located behind the mirror?
12. What do the colors of visible light correspond to?
13. How are the wavefronts and rays that represent the propagation of light drawn relative to the direction of propagation and relative to each other?

33.3 Refraction and dispersion

14. What is refraction, and why does it occur?
15. Does the wavelength change when light passes from a medium that has a high mass density into one that has a lower mass density, and if so, how?
16. For a refracting light ray, what are the names of the two angles associated with the bending, and how are their relative sizes related to the relative densities of the media?
17. What is the definition of *critical angle* for light rays undergoing refraction as they move from one medium into another?
18. Describe how total internal reflection occurs.
19. What is Fermat's principle?
20. What is dispersion, and why does it occur?

33.4 Forming images

21. A light ray traveling along a path in medium A is refracted when it enters medium B. If the ray's direction of travel is then reversed, how does its path change as the ray moves through medium B and back into medium A?
22. What are paraxial light rays?
23. For light moving from left to right through a lens, describe the paths of the three principal rays used to locate the image formed by the lens.
24. Describe the difference between the shapes of converging and diverging lens surfaces and the difference in what happens to light rays passing through them.

33.5 Snel's law

25. What is the definition of the index of refraction of a material through which light can pass?
26. How does the wavelength of light traveling in a transparent material depend on the material's index of refraction?
27. What is Snel's law?
28. How does the critical angle for total internal reflection at the boundary between two media with different mass densities depend on the indices of refraction of the media?

33.6 Thin lenses and optical instruments

29. What is the lens equation? What are the sign conventions for the distances f, o, and i in it?
30. What is the expression for determining the magnification of an image? What does the sign of the magnification tell you?
31. What is the near point of the human eye?
32. How is the angular magnification of an image formed by a lens defined?
33. What is a diopter?

33.7 Spherical mirrors

34. How does the focal length of a spherical mirror depend on the radius of curvature of the mirror?
35. Can the lens equation, $1/f = 1/o + 1/i$, be used for analyzing the image formed by a spherical mirror? If so, what are the sign conventions for the three variables?
36. What determines whether the image formed by a converging mirror is real or virtual?

33.8 Lensmaker's formula

37. What restrictions apply to the lensmaker's formula?
38. What is the sign convention for the radii of curvature in the lensmaker's formula?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The thickness of glass needed to delay light by 1 μs compared to an equal thickness of vacuum (H, A, L)
2. The thickness of air (at STP) needed to delay light by 1 μs compared to an equal thickness of vacuum (H, D, L)
3. The vertical displacement of a horizontal light ray passing through a car windshield (I, A, M, Y)
4. The angular range in water through which you can see objects in the air when looking up from underwater (J, V)
5. The angular displacement of the Sun by the atmosphere when sunlight is tangent to Earth's surface (D, Q, K, I, X)
6. The time interval during which you can see the Sun after it has dropped below the horizon (F, R)
7. The change in the eye's average focal length needed to make either distant or nearby objects visible (G, S, Z, N)
8. The radius of curvature of a funhouse mirror that makes you look very fat (U, B, O, W)
9. The radius of curvature of one side of a +5-diopter glass lens if the other side is flat (A, P, E, AA)
10. The altitude at which a large airplane casts no shadow on the ground (C, T)

Hints

A. What is the index of refraction of glass?
B. What is the object distance?
C. How large is an airplane?
D. By how much does the index of refraction of air at standard temperature and pressure exceed the index of refraction of vacuum?
E. What is the lensmaker's formula?
F. By what angle is the Sun displaced by the atmosphere when sunlight is tangent to Earth's surface?
G. What is the lens equation?
H. For any material, how are distance traveled, speed, and travel time interval related?
I. At an interface between medium 1 and medium 2, how are the incident and refracted angles related?
J. What is the critical angle?
K. If you model the atmosphere as a spherical shell of uniform mass density (hence, uniform index of refraction), what is its effective height?
L. For any material, how are the speed of light and the index of refraction related?
M. What is the windshield thickness?
N. What is the distance from the eye's lens to the image?
O. What is the relationship between magnification and o, i, and f?
P. What is the relationship between diopters and focal length?
Q. Does the index of refraction of air vary in the atmosphere?
R. How fast does Earth rotate on its axis?
S. What is the minimum distance from an object that can be clearly seen to the eye's lens?
T. What is the angular size of the Sun?
U. By how much must your image be magnified horizontally to make you look very fat?
V. What is the index of refraction of water?
W. What is the relationship between focal length and radius of curvature?
X. How can you determine the angle of refraction?
Y. What is the angle of incidence for light entering the windshield?
Z. What is the maximum distance from an object to the eye's lens?
AA. What is the radius of curvature for a flat surface?

Key (all values approximate)

A. $n = \frac{3}{2}$; **B.** 1 m; **C.** about 50 m (either length or wingspan); **D.** 3×10^{-4}; **E.** $1/f = (n-1)(1/R_1 + 1/R_2)$; **F.** 10^{-2} rad (see Developing a Feel 5); **G.** $1/f = 1/o + 1/i$; **H.** $d_{mat} = c_{mat}\Delta t_{mat}$; **I.** $n_1 \sin\theta_1 = n_2 \sin\theta_2$; **J.** $\theta_c = \sin^{-1}(n_1/n_2)$; **K.** to produce atmospheric pressure, a spherical shell of uniform mass density (1 kg/m^3) must have height $h = P_0/\rho g = 1 \times 10^4$ m; **L.** $c_{mat} = (3 \times 10^8 \text{ m/s})/n_{mat}$; **M.** 5 mm; **N.** diameter of the eyeball, 25 mm; **O.** $M = -i/o$ and $1/f = 1/o + 1/i$; **P.** $d = (1 \text{ m})/f$; **Q.** yes, the index decreases as the mass density of the atmosphere decreases nonlinearly from sea level to vacuum; **R.** $\omega = 2\pi/\text{day} = 7 \times 10^{-5} \text{ s}^{-1}$; **S.** about 0.25 m; **T.** about the same angular size as a finger held at arm's length: $\theta = 0.01 \text{ m}/1 \text{ m} = 0.01$ rad; **U.** by a factor of 3; **V.** $n = \frac{4}{3}$; **W.** $f = R/2$; **X.** sketch Earth with a refracted ray tangent to the surface, and the angle of refraction appears in a right triangle in which two of the sides are Earth's radius ($R_E = 6 \times 10^6$ m) and the radius to the top of the atmosphere ($R_E + h$); **Y.** 50°; **Z.** infinity; **AA.** $R = \infty$

PRACTICE

Worked and Guided Problems

Procedure: Simplified ray diagrams for lenses

To determine the location and orientation of an image formed by a lens, follow this procedure.

1. Draw a horizontal line representing the lens axis (the line perpendicular to the lens through its center). In the center of the diagram, draw a vertical line representing the lens. Put a + above the line to represent a converging lens or a – to represent a diverging lens.
2. Put two dots on the axis on either side of the lens to represent the foci of the lens. The dots should be equidistant from the lens.
3. Represent the object by drawing an upward pointing arrow from the axis at the appropriate relative distance to the lens. For example, if the distance from the object to the lens is twice the focal length of the lens, put the arrow twice as far from the lens as the dot you drew in step 2. The top of the arrow should be at about half the height of the lens.
4. From the top of the arrow representing the object draw two or three of the three *principal rays* listed in the following Procedure Box Principal rays for lenses.
5. The top of the image is at the point where the rays *that exit the lens* intersect (if they diverge, trace them backward to determine the point of intersection). If the intersection is on the opposite side of the lens from the object, the image is real; if it is on the same side, the image is virtual. Draw an arrow pointing from the axis to the intersection to represent the image (use a dashed arrow for a virtual image).

In general it is sufficient to draw two principal rays, but depending on the situation, some rays may be easier to draw than others. You can also use a third ray to verify that it, too, goes through the intersection. (If it doesn't, you have made a mistake.)

Procedure: Principal rays for lenses

The propagations of principal rays for converging and diverging lenses are very similar. The description below holds for rays that travel from left to right.

Converging lens

1. A ray that travels parallel to the lens axis before entering the lens goes through the right focus after exiting the lens.
2. A ray that passes through the center of the lens continues undeflected.
3. A ray that passes through the left focus travels parallel to the lens axis after exiting the lens. If the object is between the focus and the lens, this ray doesn't pass through the focus but lies on the line from the focus to the point where the ray originates.

Diverging lens

1. A ray that travels parallel to the lens axis before entering the lens continues along the line from the left focus to the point where the ray enters the lens.
2. A ray that passes through the center of the lens continues undeflected.
3. A ray that travels toward the right focus travels parallel to the lens axis after exiting the lens.

Procedure: Ray diagrams for spherical mirrors

Ray diagrams for spherical mirrors are very similar to those for lenses. The procedure below is for rays traveling from the left to the right.

1. Draw a horizontal line representing the mirror axis. In the center of the diagram, draw a circular arc representing the mirror. A converging mirror curves toward the left; a diverging mirror curves toward the right.
2. Put a dot on the axis at the center of the circular arc and label it C. Add another dot on the axis, halfway between C and the mirror. This point is the focus. Label it f.
3. Represent the object by drawing an upward pointing arrow from the axis at the appropriate relative distance to the left of the mirror. For example, if the distance from the object to a converging mirror is one-third the radius of curvature of the mirror, put the arrow a bit to the right of the focus The top of the arrow should be at about half the height of the mirror.
4. From the top of the arrow representing the object draw two or three of the following three so-called *principal rays* listed in the Procedure Box Principal rays for spherical mirrors on page 1173.
5. The top of the image is at the point where the rays that are reflected by the mirror intersect. If the intersection is on the left side of the lens, the image is real; if it is on the right, the image is virtual. Draw an arrow pointing from the axis to the intersection to represent the image (use a dashed arrow for a virtual image).

Au/Ed: Please confi
page reference.

Procedure: Principal rays for spherical mirrors

The description below holds for rays that travel from left to right.

Converging mirror

1. A ray that travels parallel to the mirror axis before reaching the mirror goes through the focus after being reflected.
2. A ray that passes through the center of the sphere on which the mirror surface lies is reflected back onto itself. If the object is between the center and the mirror, this ray doesn't pass through the center but lies on the line from the center to the point at which the ray originates.
3. A ray that passes through the focus is reflected parallel to the axis. If the object is between the focus and the mirror, this ray doesn't pass through the focus but lies on the line from the focus to the point at which the ray originates.

Diverging mirror

1. A ray that travels parallel to the mirror axis before reaching the mirror is reflected along the line that goes through the focus and the point where the ray strikes the surface.
2. A ray that passes through the center of the sphere on which the mirror surface lies is reflected back onto itself.
3. A ray whose extension passes through the focus is reflected parallel to the axis.

For both converging and diverging mirrors a ray that hits the mirror on the axis is reflected back symmetrically about the axis.

These examples involve material from this chapter but are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

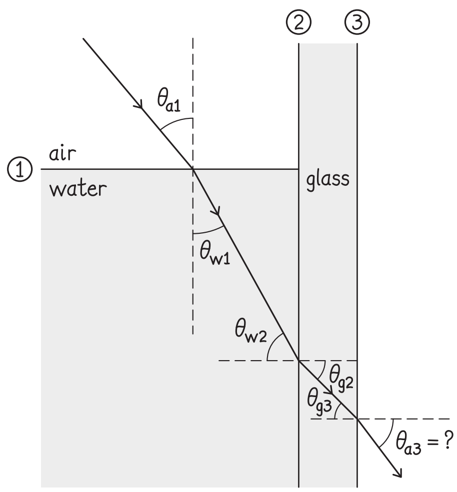
Worked Problem 33.1 Aquarium beams

You shine a laser pointer into a filled aquarium tank from above. The laser beam strikes the water surface at 40.0° from the normal and then travels to one of the glass sidewalls of the tank. Does the beam pass through that wall into the air, and if so, at what angle of refraction?

❶ GETTING STARTED This problem involves a sequence of encounters of a laser beam with interfaces between materials that have different indices of refraction. Our goal is to determine whether the beam passes through a sidewall of the tank and enters the air and, if so, at what angle of refraction. The question we must answer is: At each interface—(1) air–water, (2) water–glass, (3) glass–air—is the beam totally reflected, or does it enter the second material and become refracted? Consequently, we need to work out the change of direction of the beam at each interface.

To visualize the situation, we arbitrarily choose the right sidewall of the tank and make a sketch showing that wall, the water surface, and the laser beam (Figure WG33.1). If the beam is to pass all the way through and end up back in the air, it travels first through the air to the water surface, then through the water to the left surface of the right sidewall, and then through the glass to the right surface of this wall. To keep the diagram simple, we omit any reflected rays. We number the three interfaces and show the beam bending toward the normal to interfaces 1 and 2 as it crosses those interfaces (moving in each case from the lower-density material to the higher-density material) and bending away from the normal as it crosses interface 3 because now it moves from higher density to lower density. We label the six angles the beam makes to the interface normals θ_{a1}, θ_{w1}, θ_{w2}, θ_{g2}, θ_{g3}, and θ_{a3}.

Figure WG33.1



PRACTICE

❷ **DEVISE PLAN** We know from Table 33.1 that the index of refraction of air is smaller than that of water, which is smaller than that of glass. Thus total internal reflection—if it happens—can happen only at the glass–air interface because this is the only place the beam might move from a material of greater n value to a material of smaller n value. If total internal reflection occurs, Snel's law applied to the glass–air interface cannot be satisfied, and so the value of the critical angle is exceeded. We then obtain the indices of refraction we need in Table 33.1. Two types of glass are listed, and we arbitrarily assume the aquarium is made of flint glass. We repeatedly apply Snel's law, along with some trigonometry, to determine the angles and thereby determine whether the Snel's law condition at interface 3 can be met. If no value of θ_{a3} exists, total internal reflection occurs and the beam does not pass from glass to air; if a value exists, θ_{a3} is the exit angle.

Table 33.1 Indices of refraction for common transparent materials

Material	n (for $\lambda = 589$ nm)
Air (at standard temperature and pressure)	1.00029
Liquid water	1.33
Sugar solution (30%)	1.38
Sugar solution (80%)	1.49
Microscope cover slip glass	1.52
Sodium chloride (table salt)	1.54
Flint glass	1.65
Diamond	2.42

❸ **EXECUTE PLAN** We apply Snel's law at each interface in numerical order, beginning with interface 1: the air–water interface.

$$n_{\text{air}} \sin\theta_{\text{a1}} = n_{\text{water}} \sin\theta_{\text{w1}}.$$

We are given $\theta_{a1} = 40.0°$, and from Table 33.1 we see that $n_{\text{air}} = 1.00029$ and $n_{\text{water}} = 1.33$. Thus

$$\theta_{\text{w1}} = \sin^{-1}\left(\frac{n_{\text{a}} \sin\theta_{\text{a1}}}{n_{\text{w}}}\right) = \sin^{-1}\left(\frac{1.00029 \sin 40.0°}{1.33}\right) = 28.9°.$$

Next we note that the normals to interfaces 1 and 2 form a right triangle that has the in-water portion of the beam as its hypotenuse. Therefore $\theta_{\text{w1}} + \theta_{\text{w2}} = 90°$ and

$$\theta_{\text{w2}} = 90.0° - 28.9° = 61.1°.$$

Now we apply Eq. 33.7 to interface 2:

$$n_{\text{w}} \sin\theta_{\text{w2}} = n_{\text{g}} \sin\theta_{\text{g2}}$$

$$\theta_{\text{g2}} = \sin^{-1}\left(\frac{n_{\text{w}} \sin\theta_{\text{w2}}}{n_{\text{g}}}\right) = \sin^{-1}\left(\frac{1.33 \sin 61.1°}{1.65}\right) = 44.9°.$$

Interfaces 2 and 3 are parallel, which means $\theta_{\text{g2}} = \theta_{\text{g3}}$, allowing us to calculate θ_{a3}:

$$n_{\text{g}} \sin\theta_{\text{g3}} = n_{\text{a}} \sin\theta_{\text{a3}}$$

$$\theta_{\text{a3}} = \sin^{-1}\left(\frac{n_{\text{g}} \sin\theta_{\text{g3}}}{n_{\text{a}}}\right) = \sin^{-1}\left(\frac{1.65 \sin 44.9°}{1.00029}\right) = \sin^{-1}(1.16).$$

Because the argument of the inverse sine function exceeds 1, no angle exists that can satisfy this condition. This tells us that θ_{g3} exceeds the critical angle for the glass–air interface. The beam must undergo total internal reflection at interface 3 and thus does not exit the tank. ✔

❹ **EVALUATE RESULT** We know that total internal reflection can happen only to light traveling from a medium with a higher index of refraction into a medium with a lower index of refraction, and this is the case here, as the beam moves from glass ($n_{\text{g}} = 1.65$) to air ($n_{\text{a}} = 1.00029$). For total internal reflection to occur, $\theta_{\text{g3}} = \theta_{\text{g2}} = 44.9°$ must exceed the critical angle θ_{c} for the glass–air interface. To see whether or not $\theta_{\text{g3}} > \theta_{\text{c}}$, we can use Eq. 33.9 with glass for medium 2 and air for medium 1:

$$\sin\theta_{\text{c}} = \frac{n_{\text{a}}}{n_{\text{g}}} = \frac{1.00029}{1.65} = 0.606; \quad \theta_{\text{c}} = \sin^{-1}\left(\frac{n_{\text{a}}}{n_{\text{g}}}\right) = 37.3°.$$

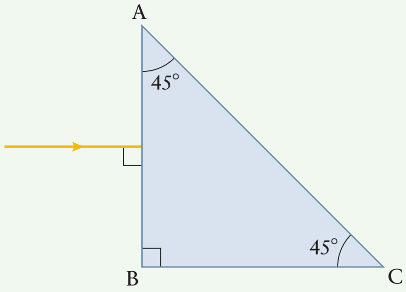
Thus $\theta_{\text{g3}} > \theta_{\text{c}}$, consistent with total internal reflection at interface 3 (the glass–air interface).

PRACTICE

Guided Problem 33.2 Prism in air or liquid

A beam of light enters a flint-glass prism as shown in Figure WG33.2, entering normal to face AB. When the prism is surrounded by air, the beam is totally internally reflected at face AC. When the prism is immersed in a clear liquid, the beam exits the prism through face AC. What minimum value of the index of refraction of the liquid permits the beam to exit?

Figure WG33.2



1 GETTING STARTED

1. In Figure WG33.2, draw the path the beam follows when the prism is surrounded by air, and label the angles.
2. What principle determines the beam's path?
3. Does the beam change direction on entering the prism?

2 DEVISE PLAN

4. How does the path of the beam *in the prism* depend on whether the prism is surrounded by air or by liquid?
5. Draw a diagram showing the beam's path when the prism is surrounded by liquid.
6. What determines the angle at which the beam strikes face AC?
7. How can you use the angle from question 6 and total internal reflection to determine the minimum index of refraction of the liquid?
8. What value do you need to look up to calculate the liquid's index of refraction?

3 EXECUTE PLAN

4 EVALUATE RESULT

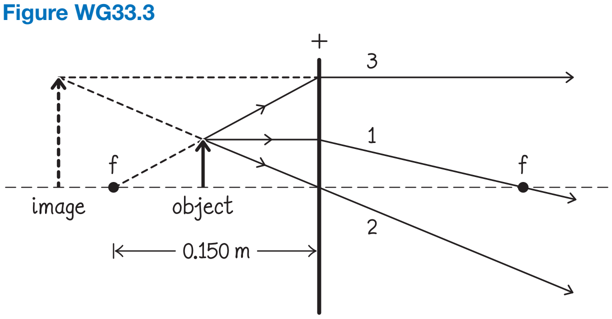
9. Is your result plausible, and why?

Worked Problem 33.3 Magnified insect

Using a magnifying lens outdoors to examine an insect, you notice that when you hold a piece of paper underneath the lens and let sunlight fall on the lens, the smallest bright spot forms when the paper is 150 mm from the lens. Where should you hold the lens to form an image of the insect that is upright and three times larger than the insect?

1 GETTING STARTED This is a problem about image formation with a magnifying lens. We need to determine the lens–object distance that produces an upright image three times larger than the object. We know that magnifying lenses are converging lenses. Because the smallest bright spot of sunlight is formed when the lens is 150 mm from the paper, we know that the sunlight is focused there, and thus the focal length f of the lens is $+0.150$ m (positive because the lens is converging). We start by drawing a simplified ray diagram (Figure WG33.3). Because our image must be upright, we position our object closer to the lens than the focal point.

Figure WG33.3



2 DEVISE PLAN We can use Eq. 33.17 to determine the image distance i in terms of the object distance o. Then we can use the lens equation (Eq. 33.16) together with our known value of f to determine o. Figure WG33.3 shows that the image is on the same side of the lens as the object and therefore the image distance i is negative. The image is upright, and therefore the image height and magnification are positive.

PRACTICE

❸ **EXECUTE PLAN** Equation 33.17 gives us $i = -Mo$. This result substituted in Eq. 33.16 yields

$$\frac{1}{f} = \frac{1}{i} + \frac{1}{o} = -\frac{1}{Mo} + \frac{1}{o} = \left(\frac{M-1}{M}\right)\frac{1}{o}.$$

Solving this expression for o yields

$$o = \left(\frac{M-1}{M}\right)f = \left(\frac{3-1}{3}\right)(0.150\ \text{m}) = 0.100\ \text{m}. ✔$$

❹ **EVALUATE RESULT** From our simplified ray diagram, we expect to obtain $o < f$, and we do. An object distance of 0.100 m is fairly short, but it is reasonable because we know from experience that we hold a magnifying glass close to the object we are viewing.

Guided Problem 33.4 Image on paper

Your friend holds a lens and a sheet of paper near a flower so that an image of the flower appears on the paper. The flower is 0.10 m away from the lens, and the image is four times the size of the flower. Which lens did your friend borrow from the physics lab to produce this image?

❶ **GETTING STARTED**

1. What quantity (value and sign) can you use to identify a lens? What algebraic symbol is used to denote this quantity?
2. Which type of lens—converging or diverging—can produce an image on a sheet of paper, or can both types of lens do so?
3. Draw a simplified ray diagram showing an enlarged image that can appear on a sheet of paper, using the type of lens you identified in question 2, and label the relevant distances. If you identified more than one type of lens, draw a simplified ray diagram for each type and identify which can produce an enlarged image.

❷ **DEVISE PLAN**

4. What equations relate image size and object size to image distance and object distance?
5. What equation relates image distance, object distance, and focal length?

❸ **EXECUTE PLAN**

❹ **EVALUATE RESULT**

6. Is your answer plausible?

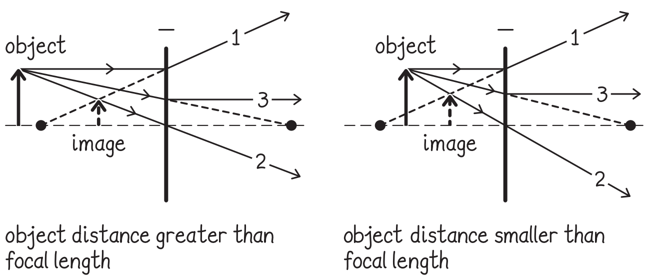
Worked Problem 33.5 Corrective lenses

Your friend's eyeglass lenses, made of a material for which the index of refraction is 1.498, have one convex surface and one concave surface. The concave surface is on the side nearer to the eye and has a radius of curvature of magnitude 71.3 mm, and the convex surface has a radius of curvature of 125 mm. If your friend looks at her computer screen at a distance of 500 mm, how far from her eyes is the image of the screen formed? Is she near-sighted (has difficulty seeing distant objects) or far-sighted (has difficulty seeing nearby objects)?

❶ **GETTING STARTED** We need to determine the image distance for eyeglass lenses used to view a computer screen (the object) at an object distance of 500 mm. We are given the index of refraction for the lens material ($n = 1.498$), the radii of curvature for the two surfaces of each lens—$R_{\text{convex}} = 125$ mm and $R_{\text{concave}} = -71.3$ mm—and the object distance $o = 500$ mm.

To determine whether your friend is near-sighted or far-sighted, we must identify the overall effect of each lens as being converging or diverging. We also need this information to draw a ray diagram. That the convex surface has a greater magnitude of radius of curvature than the concave surface means that the convex surface is flatter than the concave surface. Thus the concave surface refracts light rays more, and the lens overall is a diverging lens. We indicate this by showing a minus sign above the lens in our simplified ray diagram (Figure WG33.4). As this illustration shows, it doesn't matter, with a diverging lens, whether the object is farther or closer than the focal point. Either way, the image is virtual and appears between the lens and the object.

Figure WG33.4



❷ DEVISE PLAN We can use the lens equation (Eq. 33.16) to obtain i in terms of f and o. First, though, we must calculate f, which we can do by using the lensmaker's formula (Eq. 33.36).

❸ EXECUTE PLAN In Eq. 33.36, a convex radius of curvature is positive and a concave radius of curvature is negative. Substituting the values provided in this equation gives us

$$\frac{1}{f} = (1.498 - 1)\left(\frac{1}{125 \times 10^{-3}\,\text{m}} + \frac{1}{-71.3 \times 10^{-3}\,\text{m}}\right) = -3.00/\text{m}.$$

Now we solve Eq. 33.16 for i and calculate its value using this value for $1/f$ and our known value for o:

$$\frac{1}{f} = \frac{1}{i} + \frac{1}{o}$$

$$i = \frac{1}{\frac{1}{f} - \frac{1}{o}} = \frac{1}{-3.00/\text{m} - \frac{1}{0.500\text{ m}}} = -0.200\text{ m}.$$

A negative image distance means the image is on the same side of the lens as the object, so the image appears to be between the computer screen and your friend's eyes, 0.200 m from her eyes. ✔

From the fact that the image distance is shorter than the object distance, we infer that your friend is near-sighted, meaning she sees near objects well. She thus needs her contact lenses to bring the image of a faraway object closer to her eyes, so that the image appears where her eyes can focus on it. ✔

❹ EVALUATE RESULT The relative radii of curvature told us that the lens is overall diverging, which means the image distance must be negative, as our result is. According to the ray diagram, the image distance should be shorter than the object distance, giving us added support for the reasonableness of our result.

Guided Problem 33.6 Correcting near-sightedness

A near-sighted person's eyes cannot create focused images of distant objects on the retina because the retina is too far from the eye's lens. Near-sighted individuals see nearby objects clearly because images of closer objects form farther from the eye's lens and can reach the retina. Therefore corrective lenses for near-sightedness are designed to form images of distant objects at the person's "far point" (the farthest distance at which a person can see clearly). If a person has an eyeglass prescription of −3.00 diopters but is not wearing his glasses, what is the far point?

❶ GETTING STARTED

1. Is a −3.00-diopter lens converging or diverging? How do you know?
2. Where does this lens form an image of a distant object?
3. Draw a ray diagram showing the image of a distant object formed by this lens.
4. What quantity must you determine? How does it relate to your ray diagram?

❷ DEVISE PLAN

5. What distance in your ray diagram can you determine from the lens strength?
6. What equation can you use to relate the distances in your ray diagram?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

7. Is your answer plausible?

Worked Problem 33.7 Mirror image

You are looking at your face in a spherical mirror and see an upright image that is enlarged 1.50 times.

(*a*) Is the mirror diverging or converging? (*b*) Is the distance from you to the mirror surface longer or shorter than the mirror's focal length? (*c*) If the absolute value of the radius of curvature of the mirror is 0.96 m, where is the image?

❶ GETTING STARTED We are told that a spherical mirror produces an upright image that is 1.50 times larger than the object and asked whether the mirror is converging or diverging and where the object is located relative to the mirror's focal point. We are also given the absolute value of the mirror's radius of curvature and asked to determine the image distance.

❷ DEVISE PLAN We know that all diverging spherical mirrors form images that are smaller than the objects, so the mirror in this problem must be a converging mirror. For part b, we know that a spherical converging mirror forms an upright image only when the object distance is shorter than the mirror's focal length. To determine the image distance, we have Eq. 33.24 relating focal length f, object distance o, and image distance i. To obtain i from this relationship, however, we must know f and o. The magnification, which we know, relates o to i through Eq. 33.17, and so we have a way of expressing o in terms of i:

$$M = 1.50 = \frac{-i}{o}.$$

For f, we know that in spherical mirrors, $|f| = R/2$.

PRACTICE

3 EXECUTE PLAN

(a) This is a converging mirror, indicated by the enlarged image. ✔

(b) The fact that the image is upright tells us that the distance from the object (your face) to the mirror must be shorter than the focal length. ✔

(c) The focal length of a converging mirror is positive. Thus, for this mirror, $f = 0.96\ \text{m}/2 = +0.48\ \text{m}$. Equation 33.17 lets us express o in terms of i:

$$M = -\frac{i}{o}$$

$$o = -\frac{i}{1.50}.$$

We now have what we need to determine i:

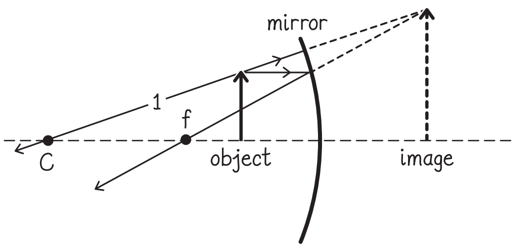
$$\frac{1}{f} = \frac{1}{i} + \frac{1}{o} = \frac{1}{i} + \left(\frac{-1.50}{i}\right) = \frac{-0.50}{i}$$

$$i = -0.50 f = -0.50(+0.48\ \text{m}) = -0.24\ \text{m}.$$

The image of your face is 0.24 m behind the mirror surface. ✔

4 EVALUATE RESULT We can use a ray diagram to check that our result is reasonable (Figure WG33.5). The figure shows that we have a virtual image, and so we expect to obtain a negative value for i. The image seems to be about 1.5 times larger than the object. Although we haven't calculated o, we can confirm that $o < f$ is correct by this reasoning: $|i| = 0.24$ m is less than $f = 0.48$ m. When an image is larger than its object, the object–mirror distance is shorter than the image–mirror distance. Thus, because $|i| < f$ and $o < |i|$, we know that $o < f$.

Figure WG33.5



Guided Problem 33.8 Reflecting flames

You want to use a spherical mirror to cast an image of a candle flame on a sheet of paper.

(a) Should you use a diverging mirror or a converging mirror? (b) If the absolute value of the radius of curvature of the mirror you use is 300 mm and you position the candle 360 mm in front of the mirror, do you see the image of the flame on the paper? If so, at what distance from the mirror must you hold the paper, and how big is the image?

1 GETTING STARTED

1. Which type of image can be viewed on a sheet of paper: real or virtual?
2. Which type of image is created by a converging mirror? By a diverging mirror?
3. Combine your answers to questions 1 and 2 to answer part a.

2 DEVISE PLAN

4. Draw a ray diagram showing the candle, the mirror, and two principal rays that allow you to locate the image. Check that you obtain the correct type of image.
5. How does the shape of the mirror determine the sign of the focal length?
6. How can you use the mirror radius of curvature and the object–mirror distance to determine where to hold the paper?

3 EXECUTE PLAN

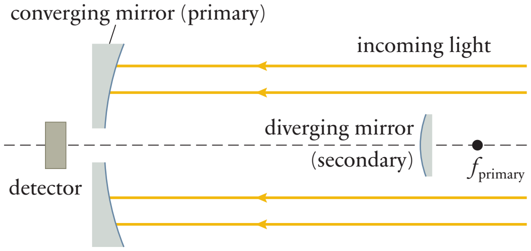
4 EVALUATE RESULT

7. Is the paper–mirror distance you calculated consistent with your answer to question 1? Is this distance reasonable?
8. Is your result for image size plausible?

Worked Problem 33.9 Reflecting telescope

A *Cassegrain telescope* (Figure WG33.6) uses a large converging spherical mirror, the *primary*, to collect light from distant stars and a small diverging spherical mirror, the *secondary*, to send the light through a gap in the primary to a detector located outside the telescope. If the primary has a focal length of 1.00 m, the secondary is mounted 0.85 m from the surface of the primary, and the final real image forms 0.12 m behind the surface of the primary, what is the focal length of the secondary?

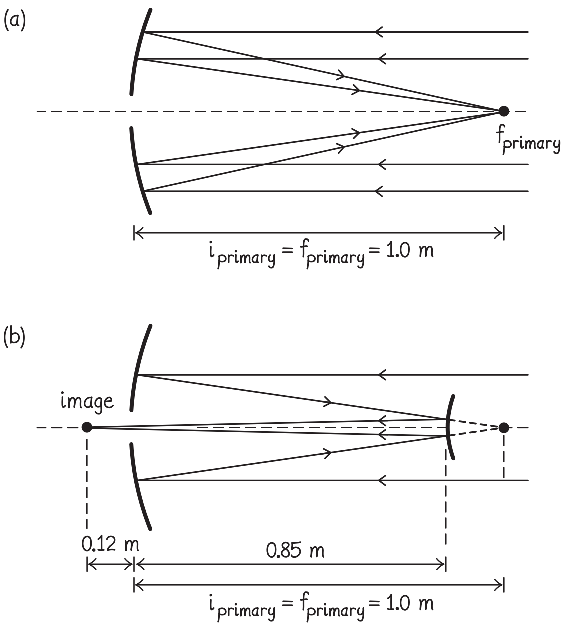
Figure WG33.6



❶ **GETTING STARTED** This is a problem about image formation by two mirrors, so we apply the principle that the image formed by the first mirror (the primary) serves as the object for the second mirror (the secondary). Because a telescope is used to observe distant objects, light from the objects enters the telescope as parallel rays. We start by drawing a ray diagram for the primary, showing these parallel rays reflected from this mirror (Figure WG33.7*a*). Parallel rays converge in the primary's focal plane, which means that the distance $i_{primary}$ from the primary to the image it forms is equal to the focal length $f_{primary}$.

Next we draw a diagram showing both mirrors (Figure WG33.7*b*). It is difficult to draw principal rays showing how light reflects from the secondary. However, the problem tells us the location of the final image—0.12 m behind the primary—and so in this diagram we show the rays reflected from the secondary as meeting at this location. We also label the other distances specified in the problem. Our task is to calculate the focal length of the secondary $f_{secondary}$, that places the final image at the proper location.

Figure WG33.7



❷ **DEVISE PLAN** The image formed by the primary is *behind* the secondary. Light does not actually reach that location, however, because the secondary is in the way. Therefore the object distance for the secondary, $o_{secondary}$, is negative. The image distance, $i_{secondary}$, is positive because the image is real. We can use the distance between the two mirrors and the distance from the final image to the primary, along with $f_{primary} = 1.00$ m, to obtain values for $o_{secondary}$ and $i_{secondary}$, and then use these distances in Eq. 33.24 to determine $f_{secondary}$.

❸ **EXECUTE PLAN** The distance from the primary to $f_{primary}$ is 1.00 m, and the distance between the mirrors is 0.85 m. Because the object for the secondary is the image formed by the primary, we have $|o_{secondary}| = 1.00\text{ m} - 0.85\text{ m} = 0.15\text{ m}$, $o_{secondary} = -0.15$ m. Also, $i_{secondary} = 0.85\text{ m} + 0.12\text{ m} = 0.97$ m. These two values in Eq. 33.24 yield

$$f_{secondary} = \left(\frac{1}{o_{secondary}} + \frac{1}{i_{secondary}}\right)^{-1}$$

$$= \left(\frac{1}{-0.15\text{ m}} + \frac{1}{0.97\text{ m}}\right)^{-1} = -0.18\text{ m.} ✔$$

PRACTICE

❹ **EVALUATE RESULT** We expect $f_{\text{secondary}}$ to be negative because the secondary is a diverging mirror. Because $|i_{\text{secondary}}| \gg |o_{\text{secondary}}|$, and thus $|1/i_{\text{secondary}}| \ll |1/o_{\text{secondary}}|$, we expect $f_{\text{secondary}}$ to be comparable to $o_{\text{secondary}}$. Our result agrees with both expectations and so seems reasonable.

Guided Problem 33.10 Double lens

You have a camera in which the lens is a combination of a diverging lens of focal length -120 mm and a converging lens of focal length 42 mm. The two are mounted 60 mm apart, and the converging lens is the one closer to the camera body. To focus the image produced on the sensor in the camera body, you move the pair of lenses closer to or farther from the sensor, keeping the distance between the lenses fixed at 60 mm. Suppose you are photographing a plant that is 400 mm tall and is located 500 mm in front of the diverging lens. When you have this object in focus, how far is the converging lens from the sensor?

❶ **GETTING STARTED**

1. What principle determines how the image forms after the light from the object passes through both lenses?
2. Draw a diagram showing the object, the two lenses, and the camera sensor. Label the important distances. If a distance is not known, label it with an appropriate symbol.
3. What quantity must you determine? How is this quantity related to the image distance for the converging lens?

❷ **DEVISE PLAN**

4. What ray diagrams should you draw?
5. What equation can you use to calculate the distance from the converging lens to the sensor?

❸ **EXECUTE PLAN**

6. Draw a diagram showing the rays for the image formed by the diverging lens.
7. Draw a ray diagram showing both lenses and the rays for the image formed by the converging lens. Remember that for a combination of two lenses, the principal rays for the second lens are not necessarily the continuation of the rays from the first lens.
8. Calculate the image distance for the converging lens.

❹ **EVALUATE RESULT**

9. Is your answer for the distance between the converging lens and the sensor plausible?

Answers to Review Questions

1. The object has to either emit light or redirect light from a source, and this light must enter your eye and be formed into an image of the object.
2. A light ray is a line drawn in a diagram to represent the direction in which light from a source travels. A ray corresponds to a very thin beam of light emitted by the source.
3. A shadow is the dark region behind an object that is facing a light source, formed because the object blocks some of the light emitted by the source, preventing it from reaching the shadow region.
4. The light can be absorbed, which means it enters the object and does not come out; transmitted, which means it passes through the object; or reflected, which means it bounces off the object surface and is redirected away from the surface.
5. Most of the light striking an opaque object is absorbed by the object, which means that light goes into the object and does not come out. Most of the light striking a translucent object is transmitted diffusely through the object, meaning the rays are redirected randomly as they pass through the object.
6. For a ray striking a smooth surface, the angle of reflection is equal to the angle of incidence, and the two angles are in the same plane.
7. Specular reflection is reflection from a smooth surface (which is a surface for which the direction of the surface normal doesn't change on the scale of the wavelength of the light). Diffuse reflection is reflection from an irregular surface in which the irregularities are comparable in size to the wavelength of the light. Because the normal to an irregular surface varies randomly, the light is reflected in all directions.
8. No image is formed because different portions of the rough surface have randomly different normal directions, so that incoming rays are randomly scattered even though, for each ray, the angle of incidence and reflection are equal.
9. An image is an optical reproduction of an object formed by light rays that travel from the object.
10. Any rays that travel from a selected point on the object and reach the image must intersect or appear to intersect at a common point on the image. A real image is formed by light rays that intersect at a point on the image. A virtual image is formed by rays that do not intersect at a point on the image, but only appear to do so when traced backward.
11. The object either emits or reflects light rays in all directions, and some of those that reach the mirror surface are reflected to the observer's eye. When the observer traces these rays backward from his eye, they appear to pass through the mirror and intersect (hence apparently originate) at a location behind the mirror, causing the image to be located there.
12. The colors correspond to the different frequencies of the waves in the visible region of the electromagnetic spectrum.
13. Wavefronts are drawn perpendicular to the direction of propagation, and rays are drawn in the direction of propagation. Thus, rays and wavefronts representing the same light are drawn perpendicular to each other.
14. Refraction is the bending of light rays as they move from a medium A into a medium B when the media have different mass densities. The rays bend because as each wavefront crosses from A into B, the speed of that portion of the wavefront that has entered B is different from the speed of the portion still in A.
15. Yes. Because $\lambda = c/f$ and frequency does not change at a boundary, wavelength λ must change as wave speed c changes. Passing into a medium of lower mass density causes c to increase, and so λ also increases.

16. The angle the unrefracted ray forms with the normal to the interface between the media is the angle of incidence; the angle the refracted ray makes with the normal to the interface is the angle of refraction. When the ray moves from a high-density medium into a low-density medium, the angle of incidence is smaller than the angle of refraction because the refracted ray bends away from the normal to the interface. When the ray moves from a low-density medium into a high-density medium, the angle of incidence is greater than the angle of refraction because the refracted ray bends toward the normal.
17. The critical angle is the angle of incidence that causes the angle of refraction to be 90°, so that the refracted rays travel along the interface between the two media.
18. When light travels from a high-density medium into a low-density medium, the angle of refraction is greater than the angle of incidence. When the angle of incidence is greater than the critical angle, the light cannot be refracted because the angle of refraction cannot exceed 90° (if it did exceed 90°, the ray would no longer be in the low-density medium). With no refraction, the interface reflects all the light back into the high-density medium.
19. The path a light ray takes in moving from one location to another is that path for which the time interval needed for the trip is a minimum.
20. In a light beam consisting of rays of different frequencies, dispersion is the spreading out of the beam as the light is refracted. The speed of light in most materials depends slightly on the frequency of the light, so the amount by which each ray in the beam is refracted depends on the ray's frequency.
21. The ray's path is not changed when the travel direction is reversed, so the ray travels along its original path back through B to the interface and then into A.
22. Paraxial light rays are rays that enter a lens near its central axis and are either parallel to that axis or oriented at a small angle to it.
23. (1) A ray that is parallel to the lens axis before entering the lens on the left is refracted to pass through the focal point on the right side of the lens, (2) a ray that passes through the lens center continues on its original path, and (3) a ray that passes through the focal point on the left side of the lens is refracted to a path parallel to the lens axis on the right. Where these three rays intersect (or appear to intersect) is the location of the image.
24. A converging lens surface curves the way the outside of a sphere curves and causes parallel light rays passing through the lens to converge at a point after they leave the lens. A diverging lens surface curves the way the inside of a sphere curves and causes parallel rays passing through the lens to diverge from one another after they leave the lens, as if they had originated from a common point on the opposite side of the lens.
25. The index of refraction is the ratio of the speed of light in vacuum to the speed of light in the material: $n_{\text{material}} = c_0/c_{\text{material}}$ (Eq. 33.1).
26. The wavelength in the material is equal to the wavelength in vacuum divided by the material's index of refraction: $\lambda_{\text{material}} = \lambda_{\text{vac}}/n_{\text{material}}$ (Eq. 33.3).
27. Snel's law describes the relationship between the angles of incidence and refraction when light traveling in a medium 1 enters a medium 2 and is refracted: The product of the sine of the angle of refraction and the index of refraction of the refractive medium 2 equals the product of the sine of the angle of incidence and the index of refraction of the incident medium 1: $n_1 \sin\theta_1 = n_2 \sin\theta_2$ (Eq. 33.7).
28. The sine of the critical angle is equal to the index of refraction of the lower-density medium divided by the index of refraction of the higher-density medium: $\sin\theta_c = n_{\text{lower density}}/n_{\text{higher density}}$ (from Eq. 33.9).
29. The lens equation relates, for any image formed by the lens, the lens focal length f to the object distance o and image distance i: $1/f = 1/o + 1/i$ (Eq. 33.16). Focal length is positive for converging lenses and negative for diverging lenses. Image distance is positive if the object and image are on opposite sides of the lens and negative if they are on the same side of the lens. Object distance is positive if the object is in front of the lens and negative if it is behind the lens, where *front* means the side on which the rays forming the image originate.
30. Magnification is the ratio of image height to object height: $M = -i/o$ (Eq. 33.17). A positive value of M indicates an upright image; a negative value of M indicates an inverted image.
31. The near point is the closest distance at which the eye can see an object in sharp focus.
32. Angular magnification is the absolute value of the ratio of the angle subtended by the image to the angle subtended by the object: $M_\theta = |\theta_i/\theta_o|$ (Eq. 33.18).
33. The diopter is the unit in which the strength of eyeglass lenses is measured. The lens strength d is given by Eq. 33.22: $d = (1\text{ m})/f$, where f is the lens focal length.
34. The focal length is half the radius of curvature: $f = R/2$ (Eq. 33.23).
35. Yes. Focal length is positive for converging mirrors and negative for diverging mirrors. Image distance is positive if the object and image are on the same side of the mirror and negative if they are on opposite sides of the mirror. Object distance is positive if the object is in front of the mirror and negative if it is behind the mirror.
36. Whether the image is real or virtual depends on where the object is located. When the object is between the focus and the mirror surface, the image forms behind the mirror and so is virtual because the rays cannot enter the space behind the mirror. When the object–mirror distance is greater than the focus–mirror distance, the image forms in front of the mirror and so is real because the rays pass through the image location.
37. The formula applies only to thin lenses and only when the rays used to calculate the radii of curvature are paraxial so that the small-angle approximation can be used.
38. The radii are positive for convex surfaces, negative for concave surfaces, and infinite for flat surfaces.

Answers to Guided Problems

Guided Problem 33.2 $n_{\text{liq}} = 1.17$

Guided Problem 33.4 A lens of focal length $f = 80$ mm

Guided Problem 33.6 0.333 m

Guided Problem 33.8 (*a*) Converging. (*b*) Yes. With the candle upright and its base on the mirror axis, place the paper 257 mm in front of the mirror and below the axis to capture the image, which is 71.4% the size of the candle.

Guided Problem 33.10 57 mm

Guided Practice by Chapter

34 Wave and Particle Optics

Review Questions

Answers to these questions can be found at the end of this chapter.

34.1 Diffraction of light

1. What is diffraction?
2. For a beam of planar waves passing through a narrow gap in a barrier located in the path of the beam, what relationship between wavelength and wavefront width determines whether the beam undergoes diffraction?

34.2 Diffraction gratings

3. What are interference fringes?
4. What does the order of an interference fringe designate?
5. When a light beam passes through two slits in a barrier, what condition determines where on a screen placed behind the barrier the bright interference fringes fall? What condition determines where on the screen the dark fringes fall?
6. Is the condition for constructive interference in light passing through a barrier containing many equally spaced slits the same as or different from the condition for constructive interference in light passing through a two-slit barrier? Is the condition for destructive interference the same or different in the two cases?
7. What is the distinction between principal and secondary maxima in an interference pattern created by light passing through a many-slit barrier?
8. What is a diffraction grating? What are the two types?

34.3 X-ray diffraction

9. What property of a crystal lattice allows the lattice to act as a diffraction grating for x rays?
10. What information does the Bragg condition convey?

34.4 Matter waves

11. What happens to a beam of electrons passing through a crystal? What does this suggest about how electrons behave?
12. What quantity measures the wave nature of a particle?
13. Why isn't it possible to observe wave behavior in macroscopic objects?

34.5 Photons

14. What is a photon, and what determines its energy?
15. The pattern on adjacent detecting screens when very-low-intensity light travels directly from a source to the screens is different from the pattern when the light passes through adjacent slits before reaching a detecting screen. Describe how this difference supports the model of light as a stream of particles.

34.6 Multiple-slit interference

16. Given that the waves leaving two slits correspond to waves from in-phase point sources, why are the waves out of phase when they arrive at a given location on a detecting screen?
17. When planar light waves pass through two side-by-side slits and then travel to a detecting screen, how does the maximum intensity in the interference pattern on the screen compare with the sum of the intensities of the waves incident on the slits?
18. What is the limiting condition for resolving two wavelengths, λ_1 and $\lambda_2 < \lambda_1$, in the dispersed spectrum formed by a diffraction grating?

34.7 Thin-film interference

19. How does thin-film interference occur?
20. When visible light passes through a transparent material, what limit on the thickness of the material determines whether or not interference occurs?
21. In thin-film interference, what two factors determine the phase difference between a light beam reflected from the front film surface and a beam reflected from the back surface?
22. Consider a ray of light traveling from medium 1 with index of refraction n_1 to medium 2 with index of refraction n_2. Describe the phase shift undergone upon reflection of the beam at the interface between media 1 and 2.

34.8 Diffraction at a single-slit barrier

23. By dividing a planar wavefront passing through a slit into suitable pairs of point sources, what feature of the interference pattern can be determined?
24. What is the expression for determining the directions to minima in the interference pattern created by light passing through a single slit?
25. What does the interference pattern created by light waves passing through a single slit look like if the slit width is smaller than the wavelength of the waves?

34.9 Circular apertures and limits of resolution

26. What does the interference pattern created by light passing through a circular aperture look like?
27. What is an Airy disk?

28. What is Rayleigh's criterion?
29. What is a diffraction limit, and how is it defined by the Airy disk of an interference pattern?

34.10 Photon energy and momentum

30. What is the photoelectric effect?
31. In an electric circuit in which the current is due to the photoelectric effect, what is the stopping potential difference, and on what property of the incident light does it depend?
32. What is the relationship between the stopping potential difference in a photoelectric circuit and the kinetic energy of the electrons ejected from the target?
33. How does the fact that the stopping potential difference for a photoelectric circuit depends on the frequency of the incident light rather than on the light intensity support a particle interpretation of light over a wave interpretation?
34. In a photoelectric circuit, what is the relationship between the current and the intensity of the incident light?
35. What is the work function of a metal?

Developing a Feel

Make an order-of-magnitude estimate of each of the following quantities. Letters in parentheses refer to hints below. Use them as needed to guide your thinking.

1. The distance between the first and second maxima 20 km away from two AM radio towers separated by 3 km, if they were to emit at the same frequency (E, L, P)
2. The angular location of the first "dark" fringe (*dark* in this case indicating no sound rather than no light) in the diffraction pattern when the sound wave created by striking the middle C key on a piano passes through a garage door (D, I, V)
3. The angular location of the first dark fringe in the diffraction pattern when a microwave signal passes through a metal doorway set in a metal wall (W, J)
4. The diameter of the image formed on the back of your eye by a point source of bright light (A, Q, U)
5. The number of photons per second emitted by a 5-mW green laser pointer (X, T)
6. The force exerted on a sheet of aluminum by the beam from a 5-mW green laser pointer (K, B)
7. The number of photons per second entering your eye from a 5-W LED lamp located 2 m away (S, F, O)
8. The wavelength of a neutron traveling at 300 m/s (M, C)
9. The minimum thickness of an oil slick lying on a puddle of water that appears red (R, H)
10. The electron speed needed in order for an electron beam to image a single virus (N, G)

Hints

A. What is the average wavelength of visible light?
B. How much momentum is carried by the laser beam in 1 s?
C. What is the neutron's momentum?
D. What is the frequency of the middle C sound wave?
E. What is the average frequency of AM radio waves?
F. What is the surface area of your pupil?
G. What wavelength must the electrons have in order to image this virus?
H. What is the index of refraction of oil?
I. What is the wavelength of this sound wave?
J. What is a typical width for a doorway?
K. How much energy is carried by the laser beam in 1 s?
L. What is the average wavelength of AM radio waves?
M. What is the neutron's mass?
N. What is a typical diameter for a virus?
O. What is the surface area of a sphere that has a radius of 2 m?
P. What is the angular separation of the two maxima?
Q. What is the focal length of the eye?
R. What is the wavelength of red light?
S. How many photons per second are emitted by the lamp?
T. What is the energy of a green photon?
U. What is the diameter of the pupil of the eye in bright light?
V. What is a typical width for a garage door?
W. What is the average wavelength of microwaves?
X. What is the wavelength of green light?

Key (all values approximate)

A. 5×10^{-7} m; **B.** 2×10^{-11} kg · m/s; **C.** 6×10^{-25} kg · m/s; **D.** 262 Hz; **E.** 1 MHz; **F.** 10^{-5} m^2; **G.** 10 nm; **H.** $n = 1.5$; **I.** 1 m; **J.** 1 m; **K.** 5 mJ; **L.** 300 m; **M.** 2×10^{-27} kg; **N.** 100 nm; **O.** 50 m^2; **P.** 0.1 rad; **Q.** 3×10^{-2} m; **R.** 7×10^{-7} m; **S.** 10^{19}; **T.** 4×10^{-19} J; **U.** 3×10^{-3} m; **V.** 3 m; **W.** 0.1 m; **X.** 5×10^{-7} m

PRACTICE

Worked and Guided Problems

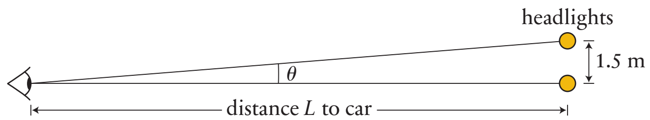
These examples involve material from this chapter, but they are not associated with any particular section. Some examples are worked out in detail; others you should work out by following the guidelines provided.

Worked Problem 34.1 Distinguishing headlights

You are driving east on a straight highway at night, with the pupils of your eyes dilated to a diameter of 6.0 mm. Far down the road, you spot a lone car traveling west toward you, and the headlights on this car are 1.5 m apart. Assuming diffraction is the only factor limiting your vision, how far away is the approaching car when you start seeing its headlights as two separate objects?

❶ **GETTING STARTED** This is a problem about distinguishing two adjacent point sources of light in the image formed by the lens in each of your eyes. We can consider just one eye, however, because the situation is the same for both eyes. To visualize the situation, we sketch it, defining L to be the distance from your eye to the headlights of the approaching car and θ the angle subtended at your eye by those headlights (Figure WG34.1).

Figure WG34.1



❷ **DEVISE PLAN** Figure WG34.1 shows that $\tan\theta = (1.5\text{ m})/L$, which means we can obtain L once we know θ. Because L is the distance at which the two headlights cease to be separate images, θ has to be the angle that satisfies Rayleigh's criterion. Equation 34.30 shows that this criterion depends on the diameter of the lens through which the light passes and on the wavelength of that light. In the eye, the pupil lies in front of the lens, which means the pupil diameter determines how much light passes through the lens. Thus we use the pupil diameter as the lens diameter. For wavelength, we choose a value near the center of the visible spectrum—$\lambda = 550$ nm—because the headlights emit visible light.

❸ **EXECUTE PLAN** Using y to represent the distance between the headlights, we have from Figure WG34.1 and the small-angle approximation

$$\theta \approx \tan\theta = \frac{y}{L}$$

Equation 34.30 says that the angle by which two objects must be separated if they are to satisfy Rayleigh's criterion is

$$\theta_r \approx \frac{y}{L} = \frac{1.22\lambda}{d}$$

and so—if diffraction is the only factor determining what you see—the distance at which you start to see the two headlights emerging from a single bright glow is

$$L = \frac{yd}{1.22\lambda} = \frac{(1.5\text{ m})(6.0\times10^{-3}\text{ m})}{1.22(550\times10^{-9}\text{ m})} = 1.3\times10^{4}\text{ m.} \checkmark$$

❹ **EVALUATE RESULT** You know that this distance, 8 miles, is much greater than the distance at which normally you can see an approaching car. (Of course, it's rare that you have a straight highway that is empty of cars over such a long distance!) In this case, the unreasonably large result suggests that diffraction is only one of the many phenomena limiting your vision. However, we can still tell that most likely the calculation is correct because the distance is on a reasonable scale—neither vastly too short (100 m or less) nor vastly too long (100 km or more).

We also notice that although we had to assume a wavelength for the light emitted by the headlights, L is inversely proportional to that wavelength. Because the range of the visible spectrum runs from 700 nm to 400 nm, the value of L we would get by choosing a different wavelength would change by at most 40%, not changing the order of magnitude of the result.

Guided Problem 34.2 Geosynchronous satellites

Television satellite dishes are approximately spherical mirrors that collect 12-GHz electromagnetic signals emitted by satellites in orbit around Earth. Each mirror serves as a circular aperture just as a lens does. How many satellites can be placed in geosynchronous orbit above the equator, at a height of 36,000 km above Earth's surface, if all of them are to send signals that can be distinguished by a satellite dish having a diameter of 0.45 m?

❶ **GETTING STARTED**

1. Draw a diagram showing two sources and the dish receiving their signals, labeling the relevant distances.
2. What condition determines whether the signals from two adjacent satellites are distinguishable? What distances does this condition depend on?

2 DEVISE PLAN

3. What quantity must you determine to solve this problem? How does this quantity relate to the distance between adjacent distinguishable satellites?
4. What information do you need to look up to work this problem?
5. How can you determine the wavelength of the electromagnetic waves broadcast by the satellites?

3 EXECUTE PLAN

4 EVALUATE RESULT

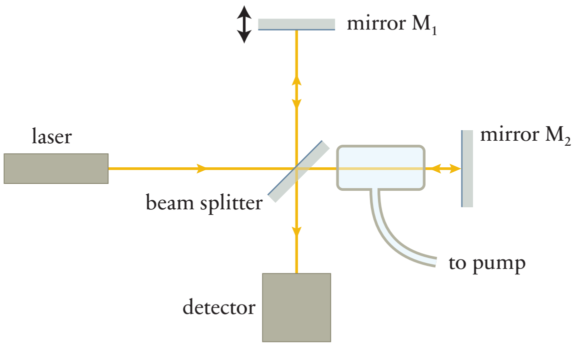
6. Is the number of satellites you obtain plausibly large enough to satisfy telecommunications needs? Small enough to be consistent with any experience you have observing satellites in the night sky?

Worked Problem 34.3 Measuring the index of refraction of air

An *interferometer* (Figure WG34.2) exploits two-source interference to measure either very short distances or differences in indices of refraction. In the device, a laser beam is directed to a *beam splitter,* which reflects half of the light in the beam to adjustable mirror M_1, which can be moved closer to or farther from the beam splitter, and transmits the other half to stationary mirror M_2. The two beams follow the paths shown in Figure WG34.2, each traveling to its mirror M_1 or M_2, back to the beam splitter, and then to a detector that measures the intensity of the combined beams. For this problem, the laser has a wavelength in vacuum of 488 nm.

A sealed cylindrical chamber that is 50.0 mm long and has glass windows at both ends is initially filled with air. The chamber is placed so that the beam traveling to M_2 passes through it, and the position of adjustable mirror M_1 is set so that the intensity at the detector is a maximum. Then the air is gradually pumped out of the chamber. While the air is being pumped out, the intensity at the detector goes to a minimum and then returns to a maximum 60.0 times. Calculate the index of refraction of air.

Figure WG34.2



1 GETTING STARTED This problem involves interference between light beams from two sources, M_1 and M_2. The amount of interference varies as the index of refraction of the contents of the chamber decreases as the air is pumped out. Changing the index of refraction changes the wavelength of the beam inside the chamber and thus changes the number of wavelengths along the path the beam travels from the splitter to M_2 and then back to the splitter.

Initially, when the chamber is filled with air, the combined intensity of the two beams at the detector is a maximum. This means that initially the beam passing through the chamber is in phase with the beam from M_1 when the two arrive at the detector. Then each intensity cycle down to a minimum and back to a maximum corresponds to decreasing the number of wavelengths in the chamber by 1 as the index of refraction decreases and consequently the wavelength gets longer. In the time interval during which the chamber goes from filled to evacuated, the minimum/maximum cycle occurs 60.0 times. We thus need to identify the index of refraction of air that causes the number of wavelengths to decrease by 60.0 as the index of refraction decreases to $n_{\text{vacuum}} = 1$.

2 DEVISE PLAN The beam traveling to M_2 passes through the chamber twice, so the distance it travels through the chamber is $2L$, where L is the chamber length. We can express the number of wavelengths N_{air} in the air-filled chamber in terms of L, the beam wavelength in vacuum λ, and the index of refraction n_{air}. Likewise, we can express the number of wavelengths N_{vacuum} in the evacuated chamber in terms of L and λ. Then we can equate the difference $N_{\text{air}} - N_{\text{vacuum}}$ to 60.0 and solve for n_{air}.

3 EXECUTE PLAN When the chamber is filled with air, the number of wavelengths in the chamber is

$$N_{\text{air}} = \frac{2L}{\lambda_{\text{air}}} = \frac{2L}{(\lambda/n_{\text{air}})} = \frac{2Ln_{\text{air}}}{\lambda},$$

and the number of wavelengths in the evacuated chamber is

$$N_{vacuum} = \frac{2L}{\lambda}.$$

Therefore

$$N_{air} - N_{vacuum} = \frac{2Ln_{air}}{\lambda} - \frac{2L}{\lambda} = \frac{2L}{\lambda}(n_{air} - 1)$$

$$n_{air} = \frac{\lambda(N_{air} - N_{vacuum})}{2L} + 1$$

$$= \frac{(488 \times 10^{-9}\ \text{m})(60.0)}{2(5.00 \times 10^{-2}\ \text{m})} + 1$$

$$= 2.93 \times 10^{-4} + 1 = 1.000293. ✔$$

Because the 1 in this calculation is an exact number and thus has an infinite number of significant digits, the number of significant digits in our value for n_{air} is determined by keeping all of the significant digits in the 2.93×10^{-4} value.

❹ **EVALUATE RESULT** This value is plausible because it is very close to but greater than 1, as it must be, and significantly smaller than the index of refraction for water (1.33) or other transparent materials. It also matches the value given in Table 33.1.

Guided Problem 34.4 Colorful soap film

As a film of soap and water evaporates and thins, the dominant reflected color changes and eventually the film disappears. (*a*) What is the last color you see as the film thins? (*b*) How thick is the film just as the last vestige of color vanishes?

❶ **GETTING STARTED**

1. What principle determines the color reflected from the film? Based on this principle, what properties of the film affect the color?
2. As the film gets thinner, does the dominant reflected wavelength increase or decrease?
3. What reasonable assumption can you make about the angle at which you view the film?
4. Draw a sketch showing the film and the paths taken by reflected light.

❷ **DEVISE PLAN**

5. How can you use what you know about interference to determine the relationship between the film thickness and the dominant color reflected at that thickness?
6. At which film surface or surfaces is there a phase shift when the light is reflected?

❸ **EXECUTE PLAN**

7. What value is reasonable for the index of refraction of the film?

❹ **EVALUATE RESULT**

8. How does your result compare with the thickness of a piece of paper? Is the ratio of the two thicknesses reasonable?
9. To evaluate the reasonableness of your result another way, estimate the volume of soap solution a child might use to blow a single bubble and the size of the resulting bubble.

Worked Problem 34.5 Overlapping rainbows

White light (400–700 nm) is incident on a diffraction grating that has 300 lines/mm. As noted in Section 34.2, the interference pattern in this case is not simply alternating bright and dark fringes but rather a sequence of red-to-violet rainbows located symmetrically to the left and right of a central white fringe. Which wavelengths in either third-order rainbow overlap the adjacent fourth-order rainbow?

❶ **GETTING STARTED** This is a problem about white light incident on a diffraction grating. To visualize the situation, we start with the interference pattern that results when monochromatic light strikes a grating (Figure WG34.3). Each wavelength produces an interference pattern consisting of narrow bright fringes separated by relatively wide dark fringes. The light in this problem is not monochromatic, however. It contains all visible wavelengths, so the interference pattern is a bright white spot at the center (where all colors undergo constructive interference) surrounded by a series of rainbows, each oriented with its violet end toward the center white spot. The orientation is this way because fringe position depends directly on wavelength. The shorter the wavelength, the closer to the midline of the interference pattern the fringe is. Because the wavelength of violet light is shorter than the wavelength of red light, in any rainbow in the interference pattern, the violet end of the rainbow is the end closer to the center. The rainbows overlap, and Figure WG34.4 shows this overlap for the first-order and second-order rainbows on either side of the central white spot.

Figure WG34.3

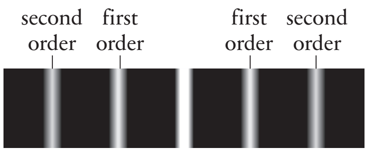
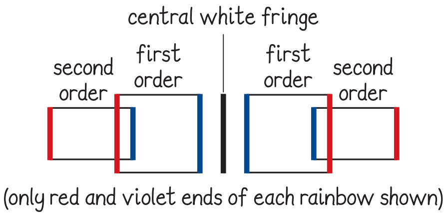


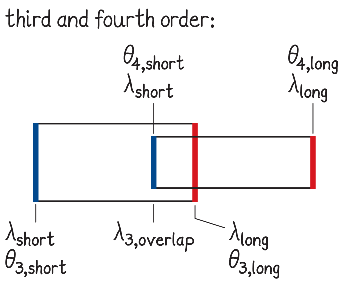
Figure WG34.4



To solve this problem, we need to find the range of angular positions of the third- and fourth-order rainbows, and identify which wavelengths fall in the same angular range.

❷ **DEVISE PLAN** Figure WG34.4 shows that the longest wavelengths (those toward the red end) in the first-order rainbow overlap the shortest wavelengths (those toward the violet end) in the second-order rainbow. The same is true for all other orders, and Figure WG34.5 shows the overlap for the third- and fourth-order rainbows to the right of the central white spot. To determine the wavelength ranges that overlap, we calculate the angular position at which the fourth-order rainbow in Figure WG34.5 begins, which is the angular position of the violet end ($\lambda_{\text{short}} = 400$ nm). Next we calculate the wavelength in the third-order rainbow that falls at that same angular position, calling this wavelength $\lambda_{3,\text{overlap}}$. The third-order rainbow ends at its red end ($\lambda_{\text{long}} = 700$ nm), so the overlap extends from $\lambda_{3,\text{overlap}}$ to λ_{long}. For all these calculations, we use the Eq. 34.16 relationship between wavelength and angular position.

Figure WG34.5



❸ **EXECUTE PLAN** The angular position of the violet end of the fourth-order rainbow is given by

$$d \sin \theta_{4,\text{short}} = 4\lambda_{\text{short}}, \tag{1}$$

where d is the grating spacing. Applying Eq. 34.16 to the wavelength $\lambda_{3,\text{overlap}}$ in the third-order rainbow where overlap begins gives us

$$d \sin \theta_{3,\text{overlap}} = 3\lambda_{3,\text{overlap}}. \tag{2}$$

Because the beginning of the overlap region and the violet end of the fourth-order rainbow occur at the same angular position, we can rewrite Eq. 2 as

$$d \sin \theta_{4,\text{short}} = 3\lambda_{3,\text{overlap}}. \tag{3}$$

Comparing Eqs. 1 and 3 tells us that $4\lambda_{\text{short}} = 3\lambda_{3,\text{overlap}}$, so

$$\lambda_{3,\text{overlap}} = \tfrac{4}{3}\lambda_{\text{short}} = \tfrac{4}{3}(400 \times 10^{-9}\text{ m}) = 533 \times 10^{-9}\text{ m}.$$

So, in the third-order rainbow, the wavelengths from 533 nm to 700 nm overlap with the fourth-order rainbow. ✔

❹ **EVALUATE RESULT** The wavelength in the third-order rainbow where overlap begins, 533 nm, is in the visible region of the electromagnetic spectrum, so our result is reasonable.

PRACTICE

Guided Problem 34.6 Two colors

Light from a red laser ($\lambda_r = 633$ nm) passes through a narrow slit located 2.00 m from a screen and creates an interference pattern on the screen. When this laser is removed and light from a blue laser ($\lambda_b = 488$ nm) passes through the slit, the $n = 1$ minima of the interference pattern move by 3.00 mm. (*a*) In which direction do the minima move: toward the center of the interference pattern or away from the center? (*b*) What is the slit width?

❶ GETTING STARTED

1. How do the angular positions of the $n = 1$ minima depend on wavelength?
2. Draw a sketch showing the pattern formed by the red light and the pattern formed by the blue light.

❷ DEVISE PLAN

3. For a single-slit interference pattern, what equation relates the angular position of the minima to the wavelength of the light and the slit width?
4. What quantity do you need to determine in order to solve this problem?
5. Does the distance by which the $n = 1$ minima move depend on only one wavelength or on both wavelengths? How can you relate this distance to the wavelength(s) and the slit width?

❸ EXECUTE PLAN

❹ EVALUATE RESULT

6. How does the slit width compare with the wavelengths of the red and blue light? To the thickness of a piece of paper?

Worked Problem 34.7 Electron and neutron diffraction

Electrons moving at 2.0×10^6 m/s pass through a double-slit apparatus, producing an interference pattern in which adjacent bright fringes are separated by 1.5 mm. (*a*) What is the bright-fringe spacing when the electrons are replaced by neutrons moving at the same speed? (*b*) Can visible light be used with this apparatus to produce the same interference pattern as the electrons or neutrons?

❶ GETTING STARTED This is a problem about the interference pattern created on a screen by particles passing through a double-slit. In this case, electrons and neutrons rather than electromagnetic waves (photons) are interfering.

❷ DEVISE PLAN For any kind of waves, the bright-fringe separation distance y is determined by the center-to-center distance d between the slits, the normal distance L from the slits to the screen, and the wavelength λ of the interfering waves: $y = L\lambda/d$ (Eq. 34.15). The same apparatus is used for the electrons and neutrons, and so d and L remain unchanged.

To solve part *a*, we use Eq. 34.15 to express the bright-fringe spacing y_n obtained with neutrons in terms of the bright-fringe spacing y_e obtained with electrons and the electron and neutron wavelengths λ_e and λ_n. We can then use the de Broglie relationship, $\lambda = h/mv$ (Section 34.4), to substitute for λ_e and λ_n and obtain a value for y_n. For part *b*, we can compare the electron and neutron wavelengths with wavelengths in the visible region of the electromagnetic spectrum.

❸ EXECUTE PLAN (*a*) Equation 34.15 gives the relationship for y_n, y_e, and the electron and neutron wavelengths:

$$y_n = \frac{L\lambda_n}{d} \qquad \frac{y_n}{\lambda_n} = \frac{L}{d}$$

$$y_e = \frac{L\lambda_e}{d} \qquad \frac{y_e}{\lambda_e} = \frac{L}{d}$$

$$\frac{y_n}{\lambda_n} = \frac{y_e}{\lambda_e}.$$

We solve this expression for y_n, substitute the de Broglie relationship for the wavelengths, and substitute values:

$$y_n = y_e\frac{\lambda_n}{\lambda_e} = y_e\frac{h/m_n v_n}{h/m_e v_e} = y_e\frac{m_e}{m_n}$$

$$= (1.5 \times 10^{-3}\ \text{m})\left(\frac{9.11 \times 10^{-31}\ \text{kg}}{1.67 \times 10^{-27}\ \text{kg}}\right)$$

$$= 8.2 \times 10^{-7}\ \text{m} = 0.82\ \mu\text{m}. \checkmark$$

PRACTICE

(*b*) The electron and neutron wavelengths are

$$\lambda_e = \frac{h}{mv} = \frac{6.626 \times 10^{-34}\ \mathrm{J \cdot s}}{(9.11 \times 10^{-31}\ \mathrm{kg})(2.0 \times 10^6\ \mathrm{m/s})}$$

$$= 3.6 \times 10^{-10}\ \mathrm{m} = 0.36\ \mathrm{nm}$$

$$\lambda_n = \frac{h}{mv} = \frac{6.626 \times 10^{-34}\ \mathrm{J \cdot s}}{(1.67 \times 10^{-27}\ \mathrm{kg})(2.0 \times 10^6\ \mathrm{m/s})}$$

$$= 2.0 \times 10^{-13}\ \mathrm{m} = 2.0 \times 10^{-4}\ \mathrm{nm}.$$

The electron wavelength is in the x-ray region of the electromagnetic spectrum, and the neutron wavelength is in the gamma-ray region. The visible spectrum extends roughly from 400 nm to 700 nm, meaning it is not possible for visible light to duplicate the interference pattern created by passing electrons or neutrons through a double slit. ✔

❹ **EVALUATE RESULT** Neutrons passing through a double slit at a given speed produce a much smaller bright-fringe spacing (by more than three orders of magnitude) than electrons passing through at the same speed. This makes sense because, with both having the same speed, the greater mass of the neutrons gives these particles greater momentum and hence shorter wavelength. Shorter wavelength results in closer bright-fringe spacing.

Because the speed in this problem is very large, the very small values we obtained for λ_e and λ_n are reasonable.

Guided Problem 34.8 Photoelectric effect with two light sources

For an experiment involving the photoelectric effect, you have two lasers for illuminating the target, and you can vary the potential difference between the target and the collector. With zero potential difference, the greatest value you measure for the kinetic energy of the electrons ejected from the target is 2.8 eV with laser 1 and 1.1 eV with laser 2. The wavelength of the light emitted by laser 2 is 50% greater than the wavelength of the light emitted by laser 1. What is the work function of the material of which the target is made?

❸ **GETTING STARTED**

1. Which conservation principle applies to this situation?
2. How is the work function related to the measured kinetic energies?

❸ **DEVISE PLAN**

3. With zero potential difference between target and collector, how is the maximum kinetic energy of the electrons measured at the collector related to the energy of the laser photons?
4. How can you relate the photon wavelength to photon energy?
5. What quantity do you need to determine in order to solve the problem?

❸ **EXECUTE PLAN**

❸ **EVALUATE RESULT**

6. Is the magnitude of your answer reasonable given the parameter values specified in the problem?

PRACTICE

Answers to Review Questions

1. Diffraction is the spreading out of a wave in the direction of wave propagation after the wave passes either through a small aperture or past the edge of a smooth barrier, such as a razor blade.
2. If the wavelength is comparable to the wavefront width, the beam diffracts after passing through the gap. If the wavefront width is much greater than the wavelength, there is no diffraction.
3. Interference fringes are the bright and dark bands formed on a screen when light passes through adjacent narrow slits in a barrier. As they travel to the screen, the beams emerging from the slits interfere with each other both constructively, producing bright fringes, and destructively, producing dark fringes.
4. The order refers to the numbering of fringes on either side of the central bright fringe, for which the order is zero. The symbol m is used for bright-fringe orders, and n is used for dark-fringe orders.
5. Bright fringes appear at any location on the screen where the difference in the slit-to-screen distance for the two slits is equal to a whole-number multiple of the wavelength. Dark fringes appear at any screen location where the difference in the slit-to-screen distance is equal to an odd multiple of $\lambda/2$.
6. The condition for constructive interference is the same in the two cases: If the path difference between adjacent slits is an integer number of wavelengths, constructive interference given by $d \sin \theta_m = \pm m\lambda$ results for N slits as for two slits. For destructive interference, the condition is not the same in the two cases. Rather than the alternating bright-dark sequence created by a two-slit barrier, the pattern created by a many-slit barrier has both strongly bright fringes called *principal maxima* and fainter bright fringes called *secondary maxima*. For a barrier containing N slits, there are $N - 1$ dark fringes between each pair of adjacent principal maxima and $N - 2$ secondary maxima between each pair of principal maxima. If N is large, the principal maxima are bright and narrow, and the minima and secondary maxima are essentially dark.
7. The principal maxima are the brightest fringes in the pattern, corresponding to constructive interference by *all* the beams created by the slits. The secondary maxima are weak bright fringes that break up the dark fringe between any two adjacent principal maxima. The secondary maxima are regions where the interference of all the beams is neither completely constructive nor completely destructive.

8. A diffraction grating is a barrier that consists of a large number of equally spaced slits or grooves. A barrier containing slits is called a transmission diffraction grating because the diffraction occurs after light passes—is transmitted—through the slits. A barrier containing grooves is called a reflective diffraction grating because the diffraction occurs after light reflects from the grooved surface of the barrier.
9. The atom-to-atom spacing in the lattice is generally of the same order of magnitude as x-ray wavelengths, so that adjacent planes of atoms act as a grating from which the x rays are diffracted.
10. For x rays diffracted by a crystal lattice, the Bragg condition defines the incidence angles θ that result in constructive interference of the rays reflected by the crystal. These angles are given by $2d\cos\theta_m = \pm m\lambda$, where d is the distance between planes in the lattice, m is any integer, and λ is the wavelength of the incident x rays.
11. The electrons are diffracted. Because diffraction is a wave phenomenon, this suggests that electrons exhibit wavelike behavior.
12. The wave nature of a particle is determined by its de Broglie wavelength $\lambda = h/p$, where $h = 6.626 \times 10^{-34}\ \mathrm{J\cdot s}$ is Planck's constant and p is the particle's momentum.
13. Observing macroscopic objects behaving like waves is not possible because the de Broglie wavelength of any macroscopic object is many orders of magnitude smaller than any length that can be measured.
14. A photon is a particle of light. The energy of a photon is proportional to the frequency of the light, with the proportionality constant being Planck's constant: $E = hf$.
15. Very-low-intensity light behaves like a beam of particles. When it is made to strike two small detecting screens placed close together, each particle of light in the beam strikes one screen or the other, and no matter for how long the experiment is run, the screens always record individual particle impacts. An interference pattern, which is characteristic of waves but not of particles, never builds up on the screens.
16. The waves are out of phase when they reach the screen because of the different distance each wave travels from its slit to the screen.
17. The maximum intensity in the interference pattern is four times the intensity of the waves incident on the slits because the intensity is proportional to the square of the electric field and the electric field is the sum of the fields of the waves from the two slits.
18. The wavelengths can be distinguished if, in the spectrum, the angular position of the mth-order principal maximum of λ_1 is greater than or equal to the angular position of the minimum adjacent to the mth-order principal maximum of λ_2.
19. Thin-film interference occurs when light reflected from the front surface of a thin film interferes with light reflected from the back surface of the film.
20. Interference occurs only if the thickness of the material is comparable to the wavelength of the light.
21. The phase difference is determined by the difference in path length of the two beams (the path of a beam reflected from the back surface is longer than the path of a beam reflected from the front surface) and by whether or not reflection from either surface inverts the wave.
22. There is a phase shift of π if the wave is inverted by the reflection, which happens when $n_2 > n_1$, and no phase shift if the wave is not inverted by the reflection, which happens when $n_1 > n_2$.
23. The directions to the dark fringes of the interference pattern can be determined.
24. The expression is $\sin\theta_n = n\lambda/a$, where θ is the angle between the direction to the minima and the original propagation direction of the light, n is any integer except zero, λ is the wavelength of the light, and a is the slit width.
25. The waves spread out in all directions once they pass through the slit, so there is no discernible interference pattern.
26. The interference pattern is a circular central maximum surrounded by alternating concentric dark fringes and secondary bright fringes.
27. An Airy disk is the circular central maximum in the interference pattern created by light passing through a circular aperture.
28. Rayleigh's criterion is a description of how close together two objects being viewed through a circular lens can be and still be distinguished from each other. The objects can be distinguished only if the distance between the centers of the two interference patterns is equal to or greater than the position of the first minimum of either pattern; this minimum separation occurs when the angular separation between the objects is at least $\theta = 1.22\lambda/d$, where d is the lens diameter.
29. A diffraction limit is a limit on how small the image of any object viewed through a lens can be; even if the object is a point, which has zero width, the image has some finite width because the light from the point diffracts as it passes through the lens. This smallest size is equal to the radius of the Airy disk.
30. The photoelectric effect is the emission of electrons from matter as a consequence of their absorption of energy from incident light with a photon energy greater than a certain minimum energy called the work function.
31. The stopping potential difference is the greatest positive value of the potential difference V_{CT} between the target and collector at which electrons ejected from the target can travel to the collector. It depends on the frequency of the incident light.
32. The stopping potential difference is proportional to the maximum kinetic energy the electrons can have: $K_{max} = eV_{stop}$ (Eq. 34.34).
33. In order to leave the target, an electron must acquire from the incident light a certain minimum amount of energy, proportional to the stopping potential difference, $K = eV_{stop}$. Because V_{stop} depends on the light's frequency rather than on its intensity, the energy delivered by the light must depend on frequency and not on intensity. In the wave model of light, the light energy is a function of intensity, and so this model does not work as an explanation of the photoelectric effect. In the particle model of light, the light consists of photons, with the energy in each photon proportional to the light frequency, agreeing with what is seen in the photoelectric effect.
34. The current is proportional to the intensity if the frequency of the light is great enough that a photon has enough energy to eject an electron from the metal surface. If the frequency of the light is too small, then there is no current, no matter how intense the incident light.
35. The work function is the minimum amount of energy necessary to free an electron from the surface of the metal.

Answers to Guided Problems

Guided Problem 34.2 110 satellites

Guided Problem 34.4 (*a*) The last color seen is violet, the color of visible light that has the shortest wavelength. (*b*) Assuming 400 nm is the shortest visible wavelength and the index of refraction of the soap film is that of water, the film thickness is 75 nm just before it becomes invisible.

Guided Problem 34.6 (*a*) The minima move toward the center of the pattern. (*b*) The slit is 0.097 mm wide.

Guided Problem 34.8 2.3 eV