MAT292 - Fall 2020

Term Test 2 - October 22, 2020

Time allotted: 90 minutes
Total marks: 50

Aids permitted: see "OK list"

Full Name: SOLUTIONS
Last First

Student Number:

Email: @mail.utoronto.ca

Do not forget to fill in the integrity statement on the second page!

For the entirety of this test, A will always denote a real 2×2 matrix with constant entries.

- In the first section, only answers and sometimes brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 9 pages (including this title page). Make sure you have all of them.
- You can use pages 8–9 for rough work or to complete a question (**Mark clearly**).
- Make sure to follow this timeline:
  - 9:05 am – Start test.
  - 10:35 am – Stop writing test, fill in integrity page (second page).
    You MUST stop writing the test at 10:35 am.
    The last 25 minutes are for submission, not for test writing.
  - 11:00 am – Upload deadline.
    No extensions will be given.

Question	Q1-4	Q5	Q6	Q7	Q8	**Total**
Marks	10	8	12	12	8	**50**

HAVE FUN!

This is the ONLY page you can fill in *after* 10:35 am.

If you don't complete and sign this page, you will receive a grade of zero for the entire test.

We at U of T want you to feel proud of what you accomplish as a student. Please respect all of the hard work you're doing this term by making sure that the work you do is your own.

We don't expect you to score perfectly on the assessments and there will be some things that you may not know. Using an unauthorized resource or asking someone else for the answer robs you of the chance later to feel proud of how well you did because you'll know that it wasn't really your work that got you there.

Success in university isn't about getting a certain mark, it's about becoming the very best person you can by enriching yourself with knowledge, strengthening yourself with skills, and building a healthy self-esteem based on how much you've grown and achieved. No one assessment captures that but your conscience will stay with you forever.

Make yourself and your loved ones proud of the student that you are by conducting yourself honestly at all times. Hold each other accountable to these standards.

In submitting this assessment ...	Short sentences
... I confirm that my conduct regarding this test adheres to the Code of Behaviour on Academic Matters.	I know the Code.
... I confirm that I have not acted in such a way that would constitute cheating, misrepresentation, or unfairness, including but not limited to, using unauthorized aids and assistance, impersonating another person, and committing plagiarism.	I didn't cheat.
... I confirm that the work I am submitting in my name is the work of no one but myself.	This is only my work.
... I confirm that all pages have been handwritten by myself.	I wrote all pages.
... I confirm that I have not received help from others, whether directly or indirectly.	I didn't receive help.
... I confirm that I have not provided help to others, whether directly or indirectly.	I didn't provide help.
... I confirm that I have only used the aids marked as "OK" on the list.	I only used "OK" aids.
... I am aware that not disclosing another student's misconduct despite my knowledge is an academic offence.	I know I must report cheating.

In this box, handwrite the sequence of short sentences (starting with "I know the Code. I didn't cheat...").

Your student number

Your signature

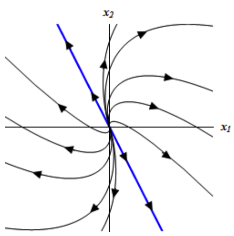
Submission date

SECTION I Provide the final answer. Justify briefly when asked. **(18 marks)**

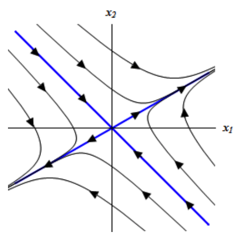
1. **(4 marks)** Consider two-dimensional systems of the form $\vec{x}\,'(t) = A\,\vec{x}(t)$ and the following setups of eigenvalues for the matrix A.

   P: $\lambda_1 = 1, \lambda_2 = -1$ **Q:** $\lambda_1 = 1 + i, \lambda_2 = 1 - i$ **R:** $\lambda_1 = i, \lambda_2 = -i$ **S:** $\lambda_1 = \lambda_2 = 1$

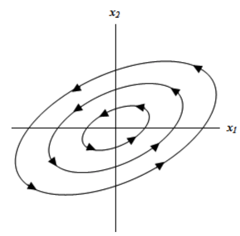
   Below each phase plot below, **write the letter** of the matching setup.



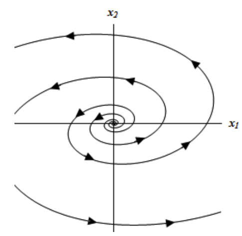
S



P

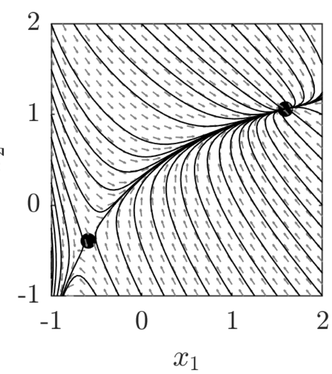


R



Q

2. **(1 mark)** Why can the phase plot on the right *not* result from a linear, autonomous system of differential equations for $x_1(t)$ and $x_2(t)$?
   Solution: This phase plot cannot result from a linear, autonomous system since it has exactly two equilibria (linear, autonomous systems have either one or an infinite number of equilibria). Also, the trajectories approach/depart the equilibria along curves (these would be lines for a linear, autonomous system).



3. **(3 marks)** Consider the IVP $y' = e^{-t} - y,\ y(0) = y_0$ with solution $y(t) = te^{-t} + y_0e^{-t}$. For which value(s) of y_0 does Euler's method with a stepsize $\Delta t = 1$ make no error on the first step? *Hint: Write down the value from Euler's method and compare with the value from the true solution.*
   Solution:
   Euler's method: $y_1 = y_0 + \Delta t f(t_0, y_0) = y_0 + 1(e^{-0} - y_0) = 1$
   true solution: $y(\Delta t) = 1e^{-1} + y_0e^{-1} = e^{-1}(1 + y_0)$
   Equating these (so that there is no error) gives $1 = e^{-1}(1 + y_0)$ with unique solution $y_0 = e - 1$.

$y_0 = e - 1$

4. (2 marks) The following Matlab implementation of the improved Euler method is flawed. Circle the line number of the line containing the error. Briefly explain what is wrong.

```
1  f = @(t,y) y+5*(t-1);
2  t0 = 0; tf = 2; y0 = 1;
3  N = 100;
4  dt = (tf-t0)/N;
5  t = zeros(1,N); t(1) = t0;
6  y = zeros(1,N); y(1) = y0;
7  for n = 1:N
8      t(n+1) = t(n) + dt;
9      k1 = f(t(n),y(n));
10     u = y(n) + dt*k1/2;
11     k2 = f(t(n+1),u);
12     y(n+1) = y(n) + dt*k1/2 + dt*k2/2;
13 end
```

Solution: Line 10 contains the mistake. A step of size $\Delta t/2$ has been used give the value for `u`, but that should be a step size of Δt, i.e. the corrected line is

`u = y(n) + dt*k1;`

On lines 5 and 6, there should be an allocation of size N+1 instead of N for the solution storage, but that does not affect the computed result.

5. (8 marks) For each of the following statements, decide if it is true or false. Then justify your choice.

Remember: A statement is only true if it is always true. If a statement only works in special cases/under certain circumstances, it is false.

(a) If a solution $y(t)$ to a first-order linear ODE exists, then $y(t)$ is defined for all time $t \in \mathbb{R}$. ○ TRUE ⊗ FALSE

Solution: For example, $y' = 1/t$ is a first-order linear ODE, but the solution $y = \log t$ is defined only for $t > 0$.

(b) If $e^{(2+i)t}\begin{bmatrix}1\\ i\end{bmatrix}$ is a solution to $\vec{x}\,' = A\vec{x}$, for real A, then $e^{(2-i)t}\begin{bmatrix}1\\ -i\end{bmatrix}$ is also a solution. ⊗ TRUE ○ FALSE

Solution: Taking the complex conjugate $\overline{\vec{x}}' = \overline{A\vec{x}} = \overline{A}\overline{\vec{x}}$ or $\overline{\vec{x}}' = A\overline{\vec{x}}$ since $\overline{A} = A$ showing that the complex conjugate of a solution is a solution.

(c) Every first order ODE is separable or linear. ○ TRUE ⊗ FALSE

Solution: For example, $y' = (t+y)^2 = t^2 + 2ty + y^2$ is a first order ODE that is neither separable nor linear.

(d) The slope field of $y' = e^y$ has no horizontal lines. ⊗ TRUE ○ FALSE

Solution: Since $e^y > 0$ for all y, y' is never zero and therefore the slope field contains no horizontal lines.

SECTION II **Justify** your answers. **(32 marks)**

6. Consider a two-dimensional system of ODEs $\vec{x}\,'(t) = A\,\vec{x}(t)$ with solutions $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. **(12 marks)**

Does an A with the following properties exist? Make a choice. Then give an example/explain why not.

(a) A matrix A for which zero is an eigenvalue such that the system has no stable equilibria.

⊗ such an A exists (give an example)

○ such an A does NOT exist (explain why)

Solution: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has eigenvalues $1, 0$ and the general solution $\vec{x} = \begin{bmatrix} c_1 e^t \\ c_2 \end{bmatrix}$ shows that all equilibria $\begin{bmatrix} 0 \\ c_2 \end{bmatrix}$ are unstable.

(b) A matrix A such that any solution with $\vec{x}(0) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has the property $x_1(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$.

○ such an A exists (give an example)

⊗ such an A does NOT exist (explain why)

Solution: Any A for which some solutions have a first component that goes to $-\infty$ will also have some solutions with a first component that goes to $+\infty$.

(c) A matrix A with eigenvalues $\lambda_1 < 0 < \lambda_2$ such that the system has a stable equilibrium.

○ such an A exists (give an example)

⊗ such an A does NOT exist (explain why)

Solution: The presence of a positive eigenvalue $\lambda_2 > 0$ means that the origin $\vec{0}$ is unstable.
A saddle is not stable.

(d) A matrix A of the form $A = \begin{bmatrix} 1 & 1 \\ d & -3 \end{bmatrix}$ with complex eigenvalues s.t. the system has a stable equilibrium.

⊗ such an A exists (give an example)

○ such an A does NOT exist (explain why)

Solution: The eigenvalues of A satisfy $(\lambda - 1)(3 + \lambda) - d = 0$ with solutions $\lambda = -1 \pm \sqrt{4 + d}$. If $d < -4$ then the eigenvalues will be complex and $\vec{0}$ will be stable (since the real part of the eigenvalues are negative).

(e) A matrix A such that the orbit of any solution with $\vec{x}(0) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ follows a straight line.

⊗ such an A exists (give an example)

○ such an A does NOT exist (explain why)

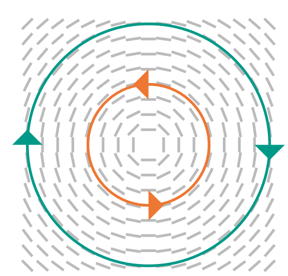
Solution: All solutions not containing $\vec{0}$ for $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are straight lines.

(f) A matrix A for which the system's phase plot looks like the one on the right.

○ such an A exists (give an example)

⊗ such an A does NOT exist (explain why)

Solution: The arrows indicating the direction for increasing t are incompatible. Centers are either clockwise or counter-clockwise and cannot mix both directions.



7. We would like to model the population of zeebrills $x(t)$, a prey species, and the population of tigerfants $y(t)$, a predatory species, in the savannah of Wakanda. Time t is measured in weeks. **(12 marks)**

All parameters in this question are positive.

Through extensive studies, the following effects have been observed:

- At any given time, half the tigerfants are female and half the zeebrills are female.
- A female tigerfant has k babies per week. A female zeebrill has l babies per week.
- Every tigerfant - male or female - kills m zeebrills per week.
- Every week, n tigerfants die of starvation.
- The savannah of Wakanda is well-known for its lush greenery. That's why every week, p zeebrills and q tigerfants join the savannah.

(a) **(9 marks)** Find a system of ODEs governing the zeebrill and tigerfant population in Wakanda. **Make sure to explain your system.**

Solution: The female half of zeebrills $x/2$ have l babies per week, resulting in a term $l/2x$ in the expression for x', the change in the zeebrill population. Likewise, there is a term $h/2y$ in the expression for y'. Each tigerfant kills m zeebrills per week, which is represented as a term $-my$ in the expression for x'. The negative sign means that there is a decrease in the zeebrill population. The starvation of tigerfants results in the inhomogeneous term $-n$ in the y' expression. The immigration to Wakanda is represented by the inhomogeneous terms $+p$ and $+q$ in the expressions for x' and y' respectively. Putting all of this together:

$$
\begin{aligned}
x' &= \frac{l}{2}x - my + p \\
y' &= \frac{h}{2}y + q - n
\end{aligned}
$$

System: $$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} l/2 & -m \\ 0 & h/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} p \\ q - n \end{bmatrix}$$

(b) **(3 marks)** An empirical study observed the following:

- In week 5, there were 2000 tigerfants and the tigerfant population was decreasing.
- In week 10, there were 2000 tigerfants and the tigerfant population was increasing.

Is the following argument correct? Explain. *"The system is autonomous, which means its behaviour doesn't depend on the specific time at which we observe it. At 2000 tigerfants, there can be only one solution due to the Existence-Uniqueness theorem. So either the tigerfant population is increasing at* $y = 2000$ *or it is decreasing at* $y = 2000$. *There must be an error in the study."*

Solution: The existence-uniqueness theorem applies to this problem, but the entire state, both the number of tigerfants and the number zeebrills, must be part of the initial condition. The tigerfant population can be increasing for $y = 2000$ and also decreasing for $y = 2000$ so long as the zeebrill population size is different in the two situations. Therefore, there is not necessarily an error in the study. See page 8 for more discussion.

8. Consider the IVP, $y' = f(y) = ry(1-y) - \frac{y^2}{1+y^2}$, $y(0) = 0.1$ with $r = 1.176595921817640$. **(8 marks)**
The following table shows the estimated values for $y(5)$ using two different numerical methods and various number of steps N. Also shown is the total number of times f was evaluated for each method.

	Method A		**Method B**	
N	$y(5)$ estimate	# of f evaluations	$y(5)$ estimate	# of f evaluations
5	0.617142496590494	10	0.603055919631019	20
10	0.607926841213706	20	0.600229689407806	40
20	0.603838322400591	40	0.600020698810837	80
40	0.601892653147372	80	0.600002143108742	160
80	0.600940306273229	160	0.600000250053907	320
160	0.600468728298439	320	0.600000038802442	640

(a) **(1 mark)** What is the most plausible value for $y(5)$?

$y(5) = 0.6$

Solution: The estimates for $y(5)$ seem to be approaching 0.6 as N is increased.

(b) **(2 marks)** What is the order of Method A? Make the most plausible choice. Then explain.

Order: 1

Solution: Between each row the value of N is doubled. We therefore expect the error to decrease between rows by a factor of $\left(\frac{1}{2}\right)^p$ for an order p method. The ratios of the errors (from the bottom up) are $\log_2(0.000940306273229/0.000468728298439) \approx 1.0$, $\log_2(0.001892653147372/0.000940306273229) \approx 1.0$, $\log_2(0.003838322400591/0.001892653147372) \approx 1.0$ This shows that the method A is most plausibly first order.

(c) **(2 marks)** What is the order of Method B? Make the most plausible choice. Then explain.

Order: 3

Solution: Between each row the value of N is doubled. We therefore expect the error to decrease between rows by a factor of $\left(\frac{1}{2}\right)^p$ for an order p method. The ratios of the errors (from the bottom up) are $\log_2(0.000000250053907/0.000000038802442) \approx 2.7$, $\log_2(0.000002143108742/0.000000250053907) \approx 3.1$, $\log_2(0.000020698810837/0.000002143108742) \approx 3.3$. This shows that the method B is most plausibly third order.

(d) **(3 marks)** Discuss the advantages and disadvantages of the two methods in terms of accuracy and computational cost (the number of evaluations of f).
Solution: For each number of steps, method B makes twice the number of function evaluations (and therefore requires roughly twice the computational work) as method A. However, method B is a lot more accurate than method A. For each **equal number of function evaluations** $20, 40, \ldots, 320$, the error for method A is 2.6, 17, 91, 440, 1900 times larger than the error for method B. Method A should not be entirely disregarded however. If low error tolerances, i.e. a solution of 0.60 ± 10^{-2}, are acceptable then method A might be faster.

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Looking at the ODE system, we can see that y' depends on y, but not x. Assuming that the ODE is a correct model for the populations sizes of zeebrillas and tigerfants, then you may conclude that there was an error in the study. In general, though, even for an autonomous system, y' would depend also on x, and in that case the study could have been correct.

The essence of the existence-uniqueness theorem is that, for an autonomous system, given any values for x and y, there is exactly one solution passing through that point, regardless of the time variable. Had the populations of both zeebrillas and tigerfants been equal at 5 and 10 weeks, this would imply that the solution had an exact time periodicity of 5 weeks, and therefore not only the values of the populations but also their rates of change would have to be the same at 5 and 10 weeks, contradicting the results in the study. However, the study did not measure the number of zeebrillas.

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