MAT292 - Fall 2017

Term Test 1 - October 23, 2017

Time allotted: 100 minutes | Aids permitted: None

Total marks: 60

Full Name: ______________________________________
Last | First

Student Number: ______________________________________

Email: ______________________________ @mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 11–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

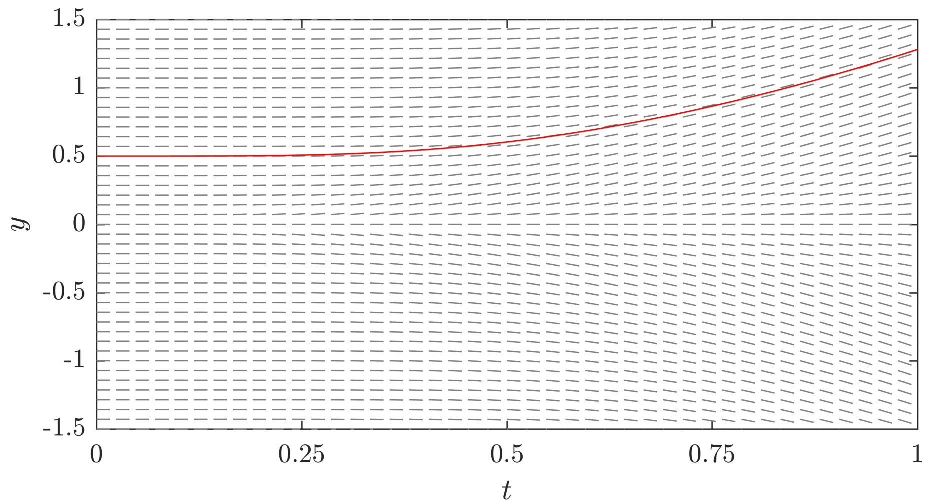
HAVE FUN!

SECTION I No explanation is necessary. (10 marks)

1. **(2 marks)** What is a solution to the initial value problem $\frac{dy}{dt} = 5y,\ y(1) = 1$?

   $y(t) =$ $e^{5(t-1)}$

2. **(2 marks)** The direction field for a differential equation is given below. Sketch a solution y such that $y(0) = 0.5$.



3. **(1 mark)** Is the differential equation corresponding to the direction field above, an automonous differential equation? Answer 'yes' or 'no'. no

4. **(2 marks)** Find A and ϕ so that $y(t) = A\sin(t + \phi)$ is a solution to $\frac{dy}{dt} + y = \sin(t)$.
   Hint: $\sin(\theta) + \cos(\theta) = \sqrt{2}\sin(\theta + \pi/4)$. $A =$ $\sqrt{2}/2$, $\phi =$ $-\pi/4 + 2\pi n, n$ integer

5. **(3 marks)** For the automonous differential equation $y' = (1 - y^2)y^2$, label the following three equilibriums solutions as stable, unstable, or semi-stable:

   $y(t) = -1$ unstable
   $y(t) = 0$ semi-stable
   $y(t) = 1$ stable

SECTION II **Justify** your answers. **(50 marks)**

6. Find all equilibrium solutions to **(5 marks)**

$$\frac{dy}{dt} = \sin\left(\frac{\pi}{y}\right)$$

and classify them as stable, semi-stable, or unstable.

Solution: Solving $\sin(\pi/y) = 0$ gives equilibrium solutions of $y = 1/n$ for any non-zero integer n. Since $\frac{d}{dy}\sin(\pi/y) = -\frac{\pi\cos(\pi/y)}{y^2}$, which has the value of $\pi(-1)^{n+1}n^2$ at the equilibrium $y = 1/n$. Thus, for even n the derivative is negative and the equilibrium is stable and for odd n the derivative is positive and the equilibrium is unstable.

7. We seek the solution $y(t)$ of some differential equation. **(5 marks)**

We run a numerical method (such as Euler, improved Euler, or Runge-Kutta) several times, each time with a different step size Δt, obtaining the following approximations for $y(1)$:

$\Delta t =$	0.08	0.04	0.02	0.01	0.005
$y(1) \approx$	1.6395	1.1602	1.0399	1.0100	1.0025

Guess the (integer) order of the numerical method that we used for this problem. Justify your answer.

Solution: It looks like the approximations of $y(1)$ are approaching 1, so we will take that as the true value. Each reduction in Δt is by a factor of 2. The ratios of the errors in successive approximations of $y(1)$ are $0.6395/0.1604 \approx 64/16 = 4, 0.1602/0.0399 \approx 16/4 = 4, 0.0399/0.0100 \approx 4/1 = 4, 0.0100/0.0025 \approx 100/25 = 4$. Therefore, the method is likely second order.

An alternative to solve this problem is to obtain a least-squares fit of $y(1) = y_{\text{true}} + C(\Delta t)^p$ to the data in the table. You can't realistically do this during a test, but the least-squares fit gives $y_{\text{true}} = 0.9999, C = 99.21, p = 1.997$, which agrees with the previous result.

8. Solve the following initial value problem for $t > 0$: **(10 marks)**

$$\frac{dy}{dt} = \frac{2\ln t}{t} y + \exp\left((\ln t)^2\right), \quad y(1) = 1.$$

Solution: An integrating factor for this first-order, linear differential equation is

$$\mu(t) = \exp\left(-\int \frac{2\ln(t)}{t}\, dt\right) = \exp\left(-(\ln(t))^2\right).$$

Therefore,

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = 1$$

and

$$y(t) = \frac{t + C}{\mu(t)} = t\exp\left((\ln(t))^2\right)$$

after selecting the constant of integration $C = 0$ so that $y(1) = 1$.

9. Consider the differential equation $\frac{dy}{dt} = \frac{\pi}{2}\sqrt{1-y^2}$ for $y \in [-1, 1]$. **(10 marks)**

(a) **(1 mark)** Find any equilibrium solutions.

Solution: $\frac{dy}{dt} = 0 \implies y = \pm 1$, so the equilibrium solutions are $y = -1$ and $y = 1$.

(b) **(3 marks)** Find a solution with the initial condition $y(0) = 0$. You may use the fact that $\frac{d}{dy}\sin^{-1}(y) = \frac{1}{\sqrt{1-y^2}}$. *Hint: Be careful! The derivative of any solution is always non-negative.*

Solution: This is a separable differential equation giving $\frac{\pi}{2}t + C = \int \frac{1}{\sqrt{1-y^2}}\, dy = \sin^{-1}(y)$ with $C = 0$ so that $y(0) = 0$. Thus, $y(t) = \sin(\frac{\pi}{2}t)$. However, the inverse of $\sin^{-1}$ has a restricted domain of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so the solution we found is only valid for $t \in [-1, 1]$. At $t = \pm 1$, the equilibrium solutions are hit and thus a solution is

$$y(t) = \begin{cases} -1, & t \leqslant -1 \\ \sin(\frac{\pi}{2}t), & -1 < t < 1 \\ 1, & t \geqslant 1 \end{cases}.$$

(c) **(2 marks)** Does there exist a unique solution to the differential equation for initial conditions $y(t_\mathrm{s}) = y_\mathrm{s}$ where $t_\mathrm{s} \in (-\infty, \infty)$ and $y_\mathrm{s} \in (-1, 1)$? What if $y_\mathrm{s} = 1$ or $y_\mathrm{s} = -1$? Justify your answer.

Solution: $\frac{d}{dy}\frac{\pi}{2}\sqrt{1-y^2} = -\frac{\pi}{2}\frac{y}{\sqrt{1-y^2}}$, which is continuous for $y \in (-1, 1)$. Since the right-hand side of the differential equation is continuous in t (it is constant in t), the existence/uniqueness theorem tells use that there is a unique solution for any initial condition $y(t_\mathrm{s}) = y_\mathrm{s}$ where $t_\mathrm{s} \in (-\infty, \infty)$ and $y_\mathrm{s} \in (-1, 1)$. Since the right-hand side is not differentible when $y = \pm 1$, the theorem does not tell us anything. However, we know that at least the equilibrium solutions pass through these initial conditions.

(d) **(2 marks)** If the solution found in part **(b)** is labelled $y_0(t)$, show that $y_c(t) = y_0(t-c)$ is a solution for any value of c. *Hint: This is a consequence of the differential equation being autonomous. You don't need to have solved part* **(b)** *to answer this quesion.*

Solution: Check that $y_c(t)$ satisfies the automonous differential equation $\frac{dy}{dt} = f(y)$:

$$\frac{d}{dt}\left(y_c(t)\right) = \frac{d}{dt}\left(y_0(t-c)\right) = f(y_0(t-c)) = f(y_c(t)).$$

Any translation of a solution to an automonous differential equation is also a solution.

(e) **(2 marks)** Write down all of the solutions that satisfy $y(0) = -1$.

Solution: The equilibrium solution $y = -1$ is a solution, as well as $y_c(t)$ for all $c \geqslant 1$.

10. An out-of-control nuclear reactor has temperature $H(t)$ described by **(10 marks)**

$$\frac{dH}{dt} = \frac{1}{3}kH^4,$$

for some constant $k > 0$. You record the temperature at $t = 1$ and $t = 2$. Let T be the time such that H approaches infinity as t approaches T from below, i.e. $\lim_{t\to T^-} H(t) = \infty$.

(a) **(3 marks)** Express k and T as functions of $H(1)$ and $H(2)$.

Solution: The equation is seperable giving $kt + C = \int 3H^{-4}dH = -H^{-3}$ or $H^{-3} = -kt - C$. We see that the solution approaches infinity as $t \to -C/k$, so $C = -kT$. From the known values $H(1)$ and $H(2)$, we get a system of equations $H(1)^{-3} = k(T-1), H(2)^{-3} = k(T-2)$ for k and T. Subtract the equations to determine $k = H(1)^{-3} - H(2)^{-3}$ and use either equation to find $T = \frac{2H(1)^{-3}-H(2)^{-3}}{H(1)^{-3}-H(2)^{-3}}$.

(b) **(1 mark)** For $H(1) = 1$ and $H(2) = 2$, give numerical values for k and T.

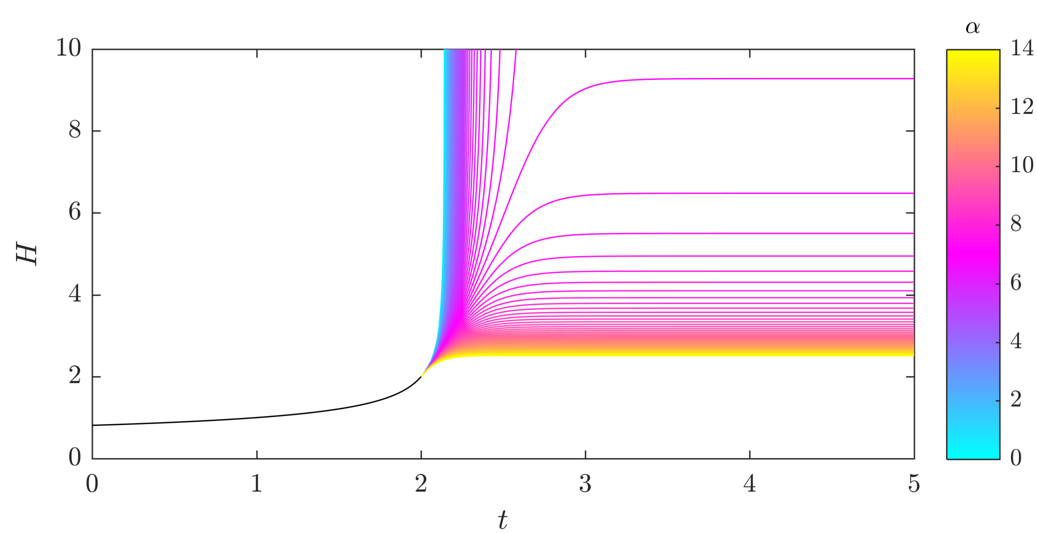
Solution: Substituting the values gives $k = 7/8$ and $T = 15/7$.

(c) **(3 marks)** The emergency control system can intervene at $t = 2$ by dumping water on the reactor which will instantaneously reduce the reactor temperature. What would the value of T become if $H(2) = 1$? Use the value of $k = \frac{7}{8}$. How low must the temperature be reduced to avoid H becoming infinite?

Solution: If $H(2)$ is 1 then we have the same initial condition at time 2 instead of at time 1 in the original situation. Since the differential equation is automonous, time has simply been shifted by one unit. So the new time at which $H \to \infty$ is $T = 1 + 15/7 = 22/7$. The finite-time blow-up is inevitable unless H is reduced to zero which would give the equilibrium solution $H = 0$.

(d) **(3 marks)** A second intervention used instead of the method from **(c)** is to slowly insert the control rods causing k to decrease as a function of time, $k(t) = \frac{7}{8}\exp(-\alpha(t-2))$ for some $\alpha > 0$. Find a formula for T as a function of α. How large does α need to be so that H will never become infinite? Use the initial condition $H(2) = 2$.

Solution: The differential equation is still seperable giving $-H^{-3} = \int \frac{7}{8}\exp(-\alpha(t-2))dt = C - \frac{7}{8}\exp(-\alpha(t-2))/\alpha$. Therefore, $H^{-3} = \frac{1}{8} + \frac{7}{8\alpha}(\exp(-\alpha(t-2)) - 1)$ with the constant of integration selected so that $H(2) = 2$. The blow-up time T satisfies $\frac{1}{8} + \frac{7}{8\alpha}(\exp(-\alpha(T-2)) - 1) = 0$ giving $T = 2 + \frac{1}{\alpha}\ln(\frac{7}{7-\alpha})$. We see that H will not blow-up if $\alpha > 7$.



Plotted above is $H(t)$ for 100 values of $\alpha \in [0, 14]$, denoted with color.

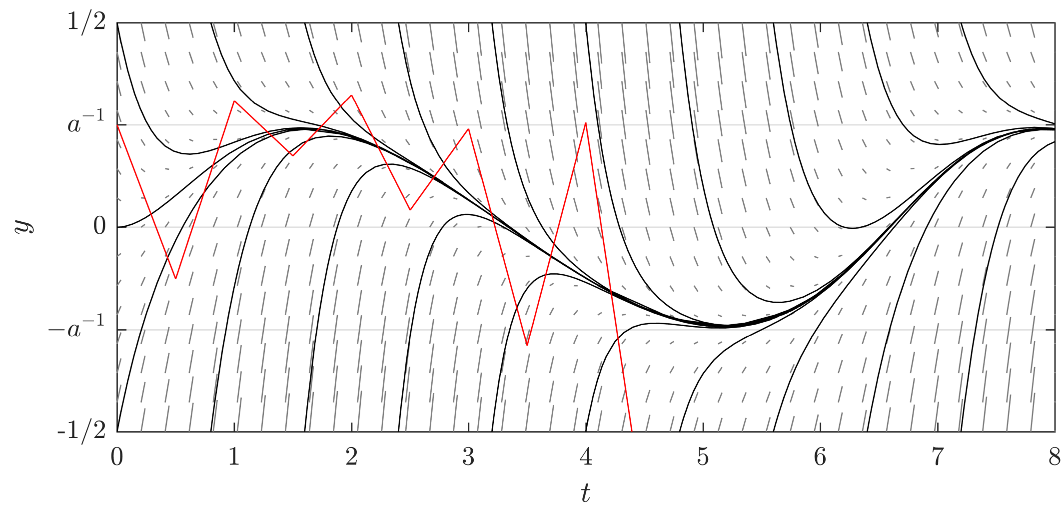
Continued...

11. Consider the differential equation $\frac{dy}{dt} = -ay + \sin(y+t)$ with $a > 0$. **(10 marks)**

(a) **(2 marks)** Write the formula for Euler's method that updates the approximate solution value y_n at time t_n to y_{n+1} at time $t_n + \Delta t$.

Solution: $y_{n+1} = y_n + \Delta t\,[-ay_n + \sin(y_n + t_n)]$.

(b) **(2 marks)** Solutions to the differential equation can only get so large: $-1/a \leqslant y(t) \leqslant 1/a$ for large t. However, if Δt is too large, then the approximations from Euler's method will grow without bound. This undesirable situation is known as numerical instability. Without computing, sketch on the direction field below a numerical solution using Euler's method starting from $y(0) = 1/4 = 1/a$. Pick Δt large enough to demonstrate numerical instability. The parameter is $a = 4$ and several solutions have been plotted over the direction field. The length of each line seqment is proportional to $|dy/dt|$.



The red line represents Euler's method with $\Delta t = 0.5$.

(c) **(3 marks)** Let's create a numerical method that doesn't have this instability problem. As an approximation, assume that the non-linear term is held constant at its value from the beginning of the timestep, $\sin(y+t) \approx \sin(y_n + t_n)$. Use the method of integrating factors to solve the initial value problem

$$\frac{dv}{dt} = -av + \sin(y_n + t_n)$$
$$v(t_n) = y_n.$$

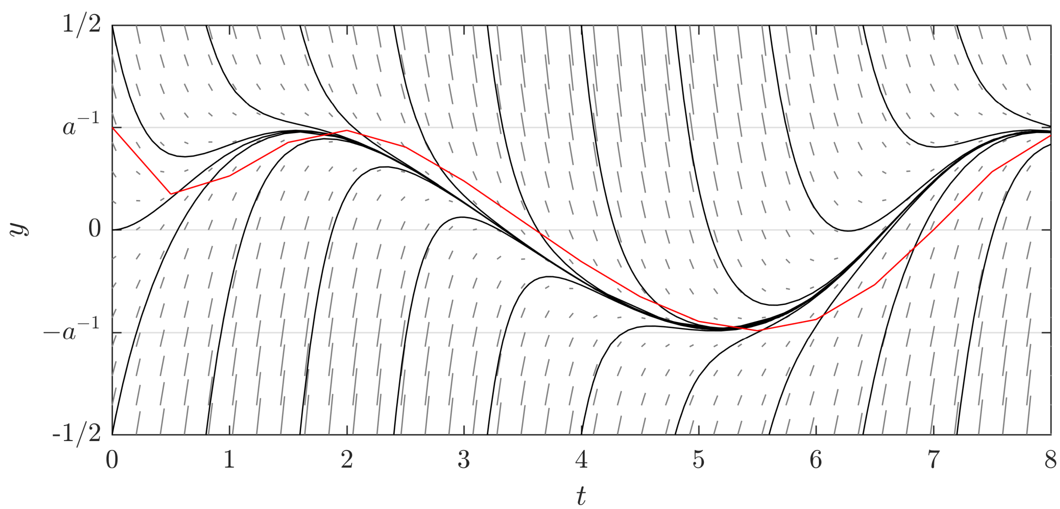
Hint: Note that $\sin(y_n + t_n)$ *is a constant - just call it* S*.*

Solution: $\frac{d}{dt}\left(e^{at}v\right) = Se^{at}$
$v(t) = \frac{S}{a} + (y_n - \frac{S}{a})e^{-a(t-t_n)} = \frac{\sin(y_n+t_n)}{a} + \left(y_n - \frac{\sin(y_n+t_n)}{a}\right)e^{-a(t-t_n)}$

(d) **(1 mark)** Write an update formula using $y_{n+1} = v(t_n + \Delta t)$ using your solution from part **(c)** .

Solution: $v(t_n + \Delta t) = \frac{\sin(y_n+t_n)}{a} + \left(y_n - \frac{\sin(y_n+t_n)}{a}\right)e^{-a\Delta t}$. The update formula is thus

$$y_{n+1} = \frac{\sin(y_n + t_n)}{a}\left(1 - e^{-a\Delta t}\right) + y_n e^{-a\Delta t}.$$



The red line represents the new method with $\Delta t = 0.5$.

(e) **(2 marks)** The new method does not have the numerical instability of Euler's method. Using the update formula from part **(d)** , show that if $|y_n| \leqslant 1/a$ then $|y_{n+1}| \leqslant 1/a$ for the new method.

Since $0 < e^{-a\Delta t} < 1$ for $\Delta t > 0$, y_{n+1} is a weighed average of y_n and $\sin(y_n + t_n)/a$ and must be between those two values. So if $|y_n| \leqslant 1/a$ and using $|\sin(y_n + t_n)/a| \leqslant 1/a$, $|y_{n+1}| \leqslant 1/a$.

Page for scratch work or for clearly-labelled overflow from previous pages

Continued...

Page for scratch work or for clearly-labelled overflow from previous pages