University of Toronto, Faculty of Applied Science and Engineering

MAT292H1F - Ordinary Differential Equations

Final Exam - December 12, 2017

Examiners: A. Stinchcombe and C. Sinnamon

Time allotted: 150 minutes

Aids permitted: None

Total marks: 90

Full Name: ______________________________
Last First

Student Number: ______________________________

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test is **double-sided**. Make sure you don't skip any problems.
- This test contains 18 pages, including this title page and a formula sheet. Make sure you have all of them.
- You can use pages 15–16 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 15–16.
- No calculators, cellphones, or any other electronic gadgets are allowed.
- You may detach the formula sheet. Work on the formula sheet will NOT be graded.

SECTION I No explanation is necessary. (25 marks)

For questions 1–5, please fill in the blanks.

1. **(2 marks)** If $y(t)$ satisfies $y' = e^y(y-1)(y-3)$ with $y(12) = \frac{3}{2}$, then

$\lim_{t\to\infty} y(t) = \underline{1 \qquad\qquad\qquad\qquad}$.

2. **(2 marks)** The general solution to $\mathbf{x}'(t) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \boldsymbol{x}(t)$ is $\boldsymbol{x}(t) = \underline{c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} \qquad\qquad}$.

3. **(3 marks)** The general solution to $\frac{d^6y}{dt^6} - 4\frac{d^2y}{dt^2} = 0$ is

$y(t) = \underline{c_1 + c_2 t + c_3 e^{\sqrt{2}t} + c_4 e^{-\sqrt{2}t} + c_5 \cos(\sqrt{2}t) + c_6 \sin(\sqrt{2}t) \qquad\qquad}$.

4. **(2 marks)** Recall that $\sinh(at) = \frac{e^{at} - e^{-at}}{2}$. After simplifying as much as possible,

$\mathcal{L}\{\sinh(at)\}(s) = \underline{\frac{s}{s^2+a^2} \qquad\qquad}$.

5. **(2 marks)** For what value(s) of λ does there exist a non-zero solution $\phi : [0,\pi] \to \mathbb{R}$ to the boundary value problem $\begin{cases} \frac{\partial^2 \phi}{\partial x^2}(x) = -\lambda \phi(x) \\ \phi'(0) = \phi'(\pi) = 0 \end{cases}$?

$\lambda = \underline{n^2, \text{ for } n = 0, 1, 2, \ldots \qquad\qquad}$.

For questions 6–12, circle **True** or **False**.

6. (2 marks) $\boxed{\textbf{True}}$ **False**

There exists a solution to $\begin{cases} y'(t) = 2ty^2(t) \\ y(0) = 1 \end{cases}$ for all $t \in (-1, 1)$.

7. (2 marks) **True** $\boxed{\textbf{False}}$

If $\det A = 0$, then the general solution to $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$ is
$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

8. (2 marks) **True** $\boxed{\textbf{False}}$

If $f, g : \mathbb{R} \to \mathbb{R}$ then $W[f, g](t)$ is either never zero or always zero.

9. (2 marks) **True** $\boxed{\textbf{False}}$

If $y(t)$ solves $y''(t) + y'(t) + y(t) = A\sin(t)$, $A > 0$, then $|y(t)| \leqslant A$ for all t.

10. (2 marks) $\boxed{\textbf{True}}$ **False**

The solution to $y''(t) + 2y'(t) + 2y(t) = 7e^{-t}\sin(t)$ can be written in the form
$y(t) = e^{-t}\left((c_1 + c_2 t)\cos(t) + (c_3 + c_4 t)\sin(t)\right)$.

11. (2 marks) $\boxed{\textbf{True}}$ **False**

$\mathcal{L}\{e^{7t}\cos^2(12t) + 5t^4\}(s)$ is defined for all $s > 10$.

12. (2 marks) **True** $\boxed{\textbf{False}}$

If $Y(s) = \mathcal{L}\{y(t)\}(s)$, and $y'(t) = y^2(t) - 1$ then $sY(s) - y(0) = Y^2(s) - \frac{1}{s}$.

SECTION II **Justify** your answers. **(65 marks)**

13. **(4 marks)**

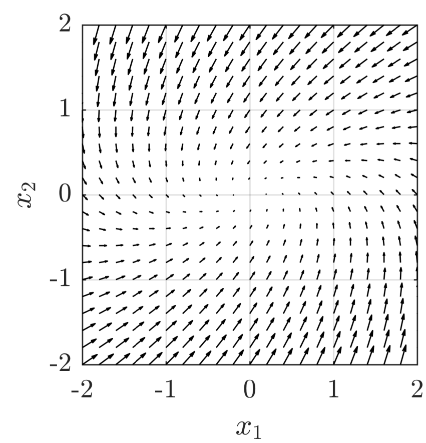
Consider the differential equation $y'(t) = f(y(t))$ where every derivative of f is continuous. Explain why $y_1(t) = \frac{1}{2} - \cos(t)$, and $y_2(t) = \cos(t) - \frac{1}{2}$ cannot both be solutions to the differential equation.

Solution: y_1 and y_2 intersect when $\frac{1}{2} - \cos(t) = \cos(t) - \frac{1}{2}$, that is, when $\cos(t) = \frac{1}{2}$, for example, when $t = \frac{\pi}{3}$. Since f is continuous with respect to t (it is constant), and $\frac{df}{dy}$ is continuous (given), there exists a unique solution to the differential equation near any point. In particular there exists a unique solution near the point $(\frac{\pi}{3}, 0)$. Since y_1 and y_2 both pass through this point, they cannot both be solutions to the differential equation or there would be a contradiction to the uniqueness of solutions.

14. **(3 marks)**

Consider the differential equation $\boldsymbol{x}' = A\boldsymbol{x}$, where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and A is a real 2x2 matrix with direction field shown (*right*).

Explain why none of the following can be an eigenvalue of A: -2, $1 - i$, $5i$.



Solution: The direction field depicts a stable spiral point. This occurs when the eigenvalues are $\alpha \pm i\beta$ with $\alpha < 0$, and $\beta \neq 0$. Since none of -2, $1+i$, or $5i$ have this form, they cannot be eigenvalues.

15. **(10 marks)**

Let t represent time in minutes. Let $N_1(t)$ represent the number of 'Bismuth 214' atoms at time t, $N_2(t)$ represent the number of 'Polonium 210' atoms at time t, and $N_3(t)$ represent the number of 'Lead 206' atoms at time t. By the process of radioactive decay 'Bismuth 214' decays into 'Polonium 210', and 'Polonium 210' decays into 'Lead 206', which does not decay. This decay chain is modeled by the system of differential equations,

$$\frac{dN_1(t)}{dt} = -20N_1(t)$$
$$\frac{dN_2(t)}{dt} = -N_2(t) + 20N_1(t)$$
$$\frac{dN_3(t)}{dt} = N_2(t)$$

a) **(3 marks)** Half-life is the time required for a quantity to reduce to half its initial value. What is the half-life of 'Bismuth 214' as predicted by the given model?

Solution: $N_1(t) = N_0 e^{-20t}$, let T be the halflife. Then $\frac{1}{2} = e^{-20T}$ so $T = \frac{-1}{20}\ln(\frac{1}{2}) = \frac{\ln(2)}{20}$.

b) **(3 marks)** If $N_1(0) = 19$, and $N_2(0) = 0$, find $N_2(t)$.

Solution: $N_2'(t) = -N_2 + 20 \cdot 19e^{-20t}$, $N_2 e^t = -20e^{-19t} + Ce^{-t}$, $N_2 = -20e^{-20t} + Ce^{-t}$.
$0 = N_2(0) = -20 + C \implies C = 20$, so $N_2(t) = -20e^{-20t} + 20e^{-t}$.

c) (2 marks) If $N_1(0) = 19$, $N_2(0) = 0$, and $N_3(0) = 0$, find $N_3(t)$.

Solution: $N_3'(t) = -20e^{-20t} + 20e^{-t}$, so $N_3(t) = e^{-20t} - 20e^{-t} + K$.
$0 = N_3(0) = 1 - 20 + K \implies K = 19$, $N_3(t) = e^{-20t} - 20e^{-t} + 19$.

d) (2 marks) This model is fairly accurate when applied to systems with large numbers of atoms, but has an important flaw that shows up when dealing with small numbers of atoms. What is this flaw? In the real world, how many 'Polonium 210' atoms would you expect to have at time 10 if you had 19 'Bismuth 214' atoms, and no 'Polonium 210' atoms at time 0?

Solution: The flaw is that it approximates a discrete quantity, the number of atoms, by a continuous value $N_i(t)$. Since $N_2(10) = -20e^{-200} + 20e^{-10}$, which is very close to zero, I would expect to have zero 'Polonium 210' atoms at time 10.

16. **(6 marks)**

a) **(3 marks)** Solve the initial value problem $y'(t) = y^2(t),\ y(0) = 1$. Express your solution as a power series in t.

Solution: The equation is separable, $\int \frac{dy}{y^2} = \int dt, \quad -y^{-1} = t - 1$. Therefore, $y = \frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \ldots$.

b) **(3 marks)** Compute three Picard iterations, $y_{n+1} = 1 + \displaystyle\int_0^t y_n^2(\tau) d\tau$, starting with $y_0(t) = 1$. For y_3, you may drop any terms with powers of t greater than 4. Extrapolate from the three iterations you computed to guess how many terms in the power series for y_n will exactly match those of the true solution found in part **a)**.

Solution:

$$y_1(t) = 1 + \int_0^t 1^2 d\tau = 1 + t$$
$$y_2(t) = 1 + \int_0^t (1+\tau)^2\, d\tau = 1 + t + t^2 + \frac{1}{3}t^3$$
$$y_3(t) = 1 + \int_0^t \left(1 + \tau + \tau^2 + \frac{1}{3}\tau^3\right)^2 d\tau = 1 + t + t^2 + t^3 + \frac{2}{3}t^4 + O(t^5)$$

$y_n(t)$ matches $n+1$ terms.

17. Find the general solution to **(5 marks)**

$$y''(t) - y(t) = e^t$$

using the variation of parameters formula.

Solution: $r^2 - 1 = 0 \implies r = \pm 1$ so $y_c(t) = c_1 e^t + c_2 e^{-t}$.
$W[e^t, e^{-t}](t) = \det \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} = -2.$

$$\begin{aligned} Y(t) &= -y_1(t) \int \frac{y_2(t)g(t)}{W[y_1, y_2](t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)} dt \\ &= -e^t \int \frac{e^{-t}e^t}{-2} dt + e^{-t} \int \frac{e^t e^t}{-2} dt \\ &= e^t \frac{t}{2} + e^{-t}(\frac{-1}{4} e^{2t}) \end{aligned}$$

So $y(t) = c_1 e^t + c_2 e^{-t} + e^t \frac{t}{2} + e^{-t}(\frac{-1}{4} e^{2t}) = \tilde{c}_1 e^t + c_2 e^{-t} + \dfrac{te^t}{2}$.

18. Consider the linear differential equation **(5 marks)**

$$\frac{d\boldsymbol{x}}{dt}(t) = A\boldsymbol{x}(t)$$

for $t \in [0,1]$ with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

a) **(2 marks)** Using Euler's method with N equal timesteps of size $h = 1/N$, what is the update formula for the approximate solution values, $\boldsymbol{x}_n$?

Solution: $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + hA\boldsymbol{x}_n = (I + hA)\boldsymbol{x}_n$.

b) **(3 marks)** Show that the approximate solution $\boldsymbol{x}_N \approx \boldsymbol{x}(1)$ approaches the true value $e^A\boldsymbol{x}_0$ as $N \to \infty$.

Solution: Solving the recursive formula gives $\boldsymbol{x}_n = (I + hA)^n\boldsymbol{x}_0$. At the end of the interval, $\boldsymbol{x}(1) \approx (I + hA)^N\boldsymbol{x}_0 = (I + \frac{1}{N}A)^N\boldsymbol{x}_0 \to e^A\boldsymbol{x}_0$ as $N \to \infty$ according to the limit from calculus defining the exponential function.

19. Consider the initial value problem **(8 marks)**

$$y''(t) + 4\pi^2 y(t) = \sum_{n=0}^{\infty} \delta(t-n), \quad y(0) = 0, \quad y'(0) = 0.$$

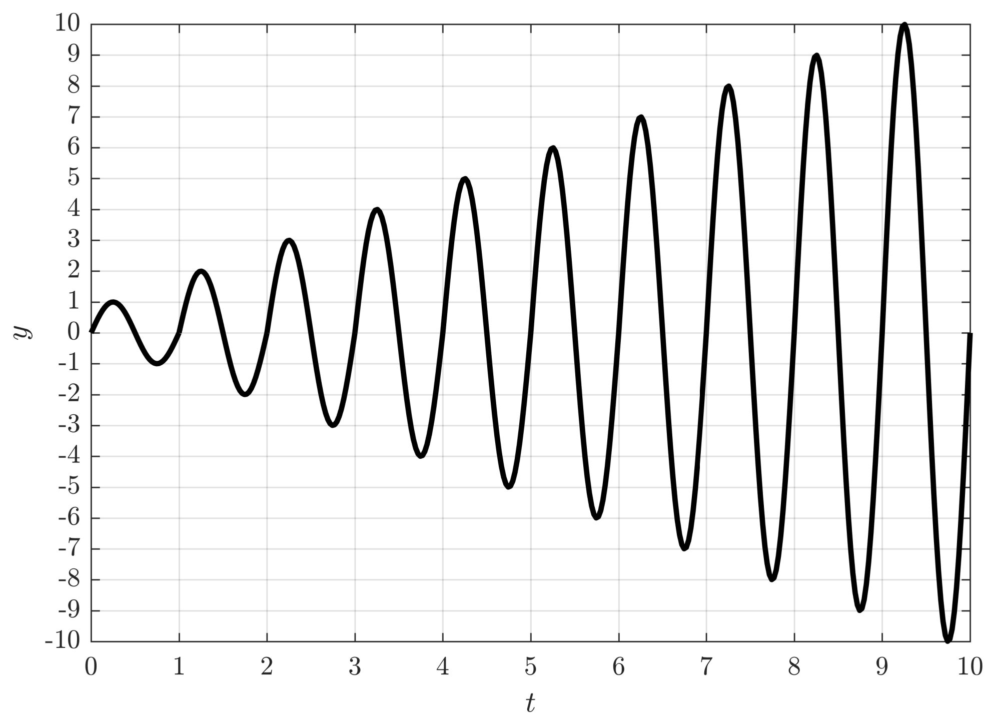
a) **(3 marks)** Find $Y(s) = \mathcal{L}\{y(t)\}(s)$. Do not evaluate the infinite series.

Solution: $s^2Y(s) + 4\pi^2 Y(s) = \sum_{n=0}^{\infty} e^{-ns}$, $Y(s) = \dfrac{\sum_{n=0}^{\infty} e^{-ns}}{s^2 + 4\pi^2}$.

b) **(3 marks)** Find $y(t) = \mathcal{L}^{-1}\{Y(s)\}(t)$ by inverting the series term by term using linearity of the Laplace transform. Do not evaluate the infinite series or worry about convergence.

Solution: $y(t) = \sum_{n=0}^{\infty} \sin(2\pi(t-n))u_n(t) = \sum_{n=0}^{\infty} \sin(2\pi t)u_n(t) = \sin(2\pi t)\sum_{n=0}^{\infty} u_n(t)$.

c) **(2 marks)** On the axes below, make a rough sketch of $y(t)$ for $t > 0$. In **one word**, how would you describe the behaviour exhibited by the solution to this differential equation?



In one word: Resonance.

20. **(12 marks)**

The heat equation describes the evolution of the temperature $v(x,t)$ in a rod, $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$. The temperatures at each end of the rod are held constant at $v(0,t) = a$ and $v(1,t) = b$ for some $a, b \in \mathbb{R}$.

a) **(3 marks)** Show that $u = v - [(b-a)x + a]$ satisfies the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with zero boundary values $u(0,t) = 0$ and $u(1,t) = 0$.

Solution: $u_t = v_t$ and $u_{xx} = v_{xx}$. So $u_t = v_t = v_{xx} = u_{xx}$. Therefore u solves the heat equations. For the boundary values, $u(0,t) = v(0,t) - a = a - a = 0$ and $u(1,t) = v(1,t) - b = b - b = 0$ as required.

b) **(4 marks)** Show that by taking the Laplace transform in t, $U(x,s) = \int_0^\infty e^{-st} u(x,t)dt$, the heat equation becomes $\frac{d^2U}{dx^2} - sU = -u(x,0)$, an ordinary differential equation. What are the boundary conditions for $U(x,s)$?

Solution: Substituting $\mathcal{L}\{u_{xx}\} = \int_0^\infty e^{-st} u_{xx} dt = \frac{\partial}{\partial x^2} \int_0^\infty e^{-st} dt = \frac{d^2U}{dx^2}$ and $\mathcal{L}\{u_t\} = sU - u(x,0)$ into the heat equation gives $\frac{d^2U}{dx^2} - sU = -u(x,0)$. The boundary conditions are $U(0,s) = \mathcal{L}\{u(0,t)\} = \mathcal{L}\{0\} = 0$. Likewise, $U(1,s) = 0$.

c) **(5 marks)** Solve the heat equation with zero boundary conditions ($u(0,t) = 0$ and $u(1,t) = 0$) and $u(x,0) = \sin(2\pi x)$ by first solving the ordinary differential equation from **b)** for $U(x,s)$ and then inverting the Laplace transform in s.

Hint: Remember, functions of x are constants with respect to the Laplace transform in t and the inverse Laplace transform in s.

Solution: $U'' - sU = -\sin(2\pi x)$, $U(0,s) = U(1,s) = 0$. The complementary solution is $U_c(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$. The particular solution will have the form $U_p(x,s) = A\sin(2\pi x)$ where A may be a function of s. Plugging it into the differential equation yields $(-4\pi^2 - s)A = -1 \implies A = \frac{1}{s+4\pi^2}$.
So $U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{\sin(2\pi x)}{s+4\pi^2}$. Using the boundary conditions, $c_1 = c_2 = 0$, so $U(x,s) = \frac{\sin(2\pi x)}{s+4\pi^2}$. Therefore $u(x,t) = e^{-4\pi^2 t}\sin(2\pi x)$.

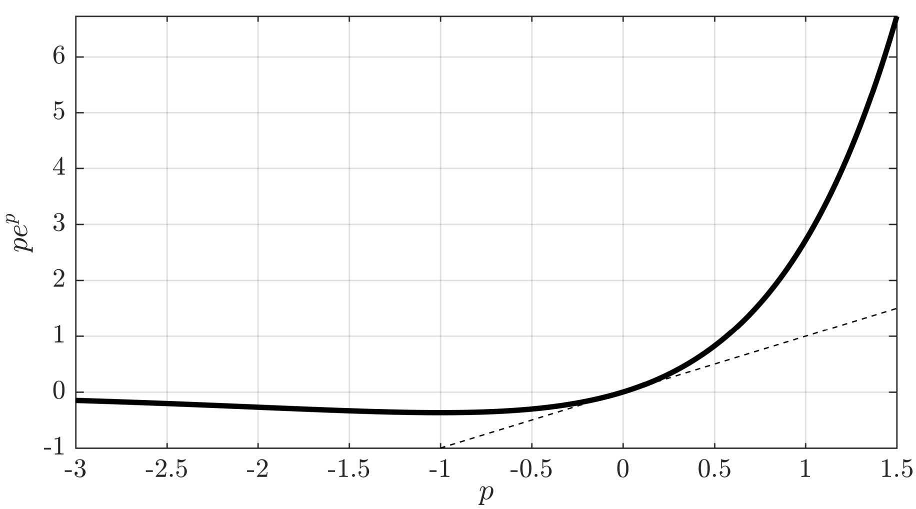
21. Consider the autonomous differential equation **(12 marks)**

$$\frac{dy}{dt}\exp\left(\frac{dy}{dt}\right) = (1-y^2)^2. \tag{$\star$}$$

(a) **(2 marks)** Find any equilibrium solutions.

Solution: $\frac{dy}{dt} = 0 \iff (1-y^2)^2 \iff y = \pm 1$, so the only equilibrium solutions are $y = 1$ and $y = -1$.

(b) **(2 marks)** It is not possible using elementary functions to write the differential equation in the form $\frac{dy}{dt} = f(y)$. Nonetheless, for each value of y there is a single value of $\frac{dy}{dt}$ given by the differential equation. Explain why this is so using the graph of pe^p versus p shown below (the dashed line is the graph of p versus p).



Solution: Since $(1-y^2)^2 \geqslant 0$, we only need pe^p to be one to one when $pe^p \geqslant 0$. Looking at the graph, we note that pe^p is strictly increasing for $pe^p \geqslant 0$, so it passes the horizontal line test and thus is one to one.

(c) **(4 marks)** Even though we can't write down an explicit form, let f be the function such that y solves $(\star)$ if and only if y solves $\frac{dy}{dt} = f(y)$. Use implicit differentiation of $f(y)e^{f(y)} = (1-y^2)^2$ to find an expression for $\frac{df}{dy}$. Apply the existence and uniqueness theorem to explain why $(\star)$ has a unique solution for any initial condition, $y(t_0) = y_0$.

Solution:

$$\begin{aligned} f(y)e^{f(y)} &= (1-y^2)^2 \\ f'(y)y'e^{f(y)} + f(y)f'(y)y'e^{f(y)} &= 2(1-y^2)(-2y)y' \\ f'(y) &= \frac{2(1-y^2)(-2y)y'}{y'e^{f(y)}(1+f(y))} \\ f'(y) &= \frac{2(1-y^2)(-2y)}{e^{f(y)}(1+f(y))} \end{aligned}$$

So $f'(y)$ is continuous except at $f(y) = -1$. However $f(y)e^{f(y)} \geqslant 0$, so $f(y) \geqslant 0$. Therefore $f'(y)$ is continuous for all y. Furthermore, f is continuous with respect to t as it is constant. So by the existence and uniqueness theorem, there exists a unique solution for any initial condition $y(t_0) = y_0$.

(d) **(2 marks)** Explain why solutions to this differential equation are always non-decreasing.

Solution: $f(y)e^{f(y)} = (1-y^2)^2 \geqslant 0$, so $f(y) \geqslant 0$. Since $y' = f(y) \geqslant 0$, y is non-decreasing.

(e) **(2 marks)** Are the equilibrium solutions in part **(a)** stable, semi-stable, or unstable? Give a carefully reasoned explanation.

Solution: The equilibrium solutions are 1 and -1. Solutions are always non-decreasing, so 1 and -1 are both semi-stable.

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FORMULA SHEET

First-Order Linear Differential Equations. $y' + p(t)y = g(t)$.

- $\mu(t) = e^{\int p(t)\,dt}$
- $y = \dfrac{1}{\mu(t)} \displaystyle\int \mu(t)g(t)\,dt + \dfrac{C}{\mu(t)}$.

Euler Method. $y' = f(t, y) \qquad y(t_0) = y_0$.

- $t_n = t_0 + n \cdot h$
- $y_{n+1} = y_n + f(t_n, y_n)h \qquad$ or $\qquad y'(t_n) = \dfrac{y_{n+1} - y_n}{h}$
- $E_n \leqslant Ch$

Improved Euler Method. $y' = f(t, y) \qquad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + k_{n,2}}{2}h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f(t_{n+1}, y_n + k_{n,1}h)$
- $E_n \leqslant Ch^2$

Runge-Kutta Method. $y' = f(t, y) \qquad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + 2k_{n,2} + 2kl_{n,3} + k_{n,4}}{6}h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f\left(t_n + \frac{h}{2}, y_n + k_{n,1}\frac{h}{2}\right)$
- $k_{n,3} = f\left(t_n + \frac{h}{2}, y_n + k_{n,2}\frac{h}{2}\right)$
- $k_{n,4} = f(t_{n+1}, y_n + k_{n,3}h)$
- $E_n \leqslant Ch^4$

Euler's Formula. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Continued...

Limits and Series.

- $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $r < 1$.

- $\displaystyle\exp(At) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$.

- $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{n}A\right)^n = e^A$.

Variation of Parameters.

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}dt$$

Laplace Transforms.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)\,e^{-st}\,dt.$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(k\,t)\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(k\,t)\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0),$$

$$\mathcal{L}\{e^{a\,t}\,f(t)\} = F(s-a), \quad \mathcal{L}\{u_a(t)\,f(t-a)\} = e^{-s\,a}\,F(s),$$

$$\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\frac{d^n}{ds^n}F(s),$$

$$\mathcal{L}\left\{\sum_{k=0}^{\infty} f(t-kT)\big(u_{kT}(t) - u_{(k+1)T}(t)\big)\right\} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}},$$

$$\mathcal{L}\{f * g\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right\} = F(s)\,G(s),$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}.$$

 The end.