MAT292 - Fall 2017

Term Test 2 - November 16, 2017

Time allotted: 100 minutes

Aids permitted: None

Total marks: 60

Full Name: ______________________________

Last First

Student Number: ______________________________

Email: ______________________________ @mail.utoronto.ca

<u>**Instructions**</u>

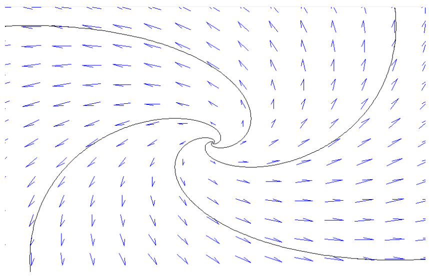
- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test is **double-sided**. Make sure you don't skip any problems.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 11–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

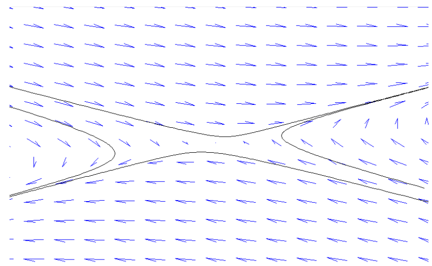
HAVE FUN!

SECTION I No explanation is necessary. **(15 marks)**

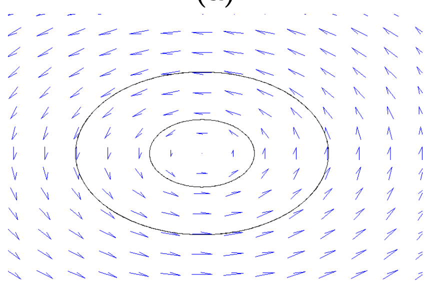
For questions 1–4, please match the differential equations with the phase portraits and circle the correct option about stability.



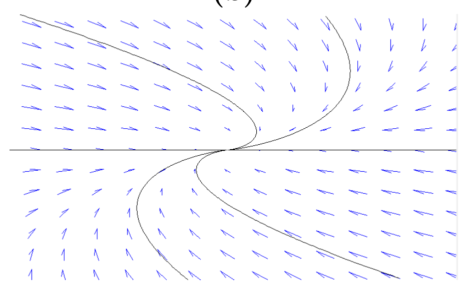
(a)



(b)



(c)



(d)

1. **(2 marks)** $\mathbf{x}' = \begin{pmatrix} -1 & 6 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ **(b), unstable** **(a)** **(b)** **(c)** **(d)**

The equilibrium solution is **stable** / **unstable** .

2. **(2 marks)** $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x}$ **(d), stable** **(a)** **(b)** **(c)** **(d)**

The equilibrium solution is **stable** / **unstable** .

3. **(2 marks)** $\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \mathbf{x}$ **(c), stable** **(a)** **(b)** **(c)** **(d)**

The equilibrium solution is **stable** / **unstable** .

4. **(2 marks)** $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x}$ **(a), unstable** **(a)** **(b)** **(c)** **(d)**

The equilibrium solution is **stable** / **unstable** .

For questions 5–8, please fill in the blanks.

5. **(2 marks)** Suppose y_1 and y_2 are two solutions to $y''(t) - \frac{1}{t}y'(t) = 0$ for $t > 0$, and the general solution is given by $y(t) = c_1 + c_2t^2$.
If $W[y_1, y_2](1) = 1$, then $W[y_1, y_2](t) =$ t ______.

6. **(1 mark)** For which value(s) of γ are solutions to $3y''(t) + y(t) = \cos(\gamma t)$ unbounded?
$\gamma = \pm\sqrt{3}/3$ ______.

7. **(2 marks)** For which value(s) of r is x^r a solution to $x^2y''(x) + 5xy'(x) + 4y(x) = 0$?
$r = -2$ ______.

8. **(2 marks)** If $\mathbf{x}' = \begin{pmatrix} t & 1 \\ 2 & t \end{pmatrix}\mathbf{x}$, $\mathbf{x}_0 = \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\mathbf{x}_1$ is the result of applying Euler's method to numerically approximate this system with step size $h = 1$, then

$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ______.

SECTION II **Justify** your answers. **(45 marks)**

9. Consider the system $\mathbf{x}'(t) = \begin{pmatrix} 1 & 4 \\ \alpha & 1 \end{pmatrix} \mathbf{x}(t)$ for $\alpha \in \mathbb{R}$. **(5 marks)**

a) **(1 mark)** For which value(s) of α is $\mathbf{0}$ an unstable spiral point?

Eigenvalues of the matrix are given by $0 = (1-r)(1-r) - 4\alpha = r^2 - 2r + 1 - 4\alpha$.

So, $r = \dfrac{2 \pm \sqrt{4 - 4(1-4\alpha)}}{2} = 1 \pm \sqrt{4\alpha}$.

Therefore $\mathbf{0}$ is an unstable spiral point for $\alpha < 0$.

b) **(4 marks)** Find the general solution to the system for $\alpha = -1/4$, and write it in terms of real valued functions.

The eigenvalues are $r = 1 \pm \sqrt{-1} = 1 \pm i$.

We need to find an eigenvector $\mathbf{v}$ with eigenvalue $1+i$. That is, find $\mathbf{v}$ such that

$$\begin{pmatrix} -i & 4 \\ -1/4 & -i \end{pmatrix} \mathbf{v} = 0.$$

$\mathbf{v} = \begin{pmatrix} 4 \\ i \end{pmatrix}$ suffices. One complex solution can be written as

$$e^{(1+i)t} \begin{pmatrix} 4 \\ i \end{pmatrix} = e^t(\cos t + i \sin t) \begin{pmatrix} 4 \\ i \end{pmatrix} = e^t \left[\begin{pmatrix} 4\cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} 4 \sin t \\ \cos t \end{pmatrix} \right]$$

yielding two real solutions.

Thus the general solution can be written as

$$\mathbf{x}(t) = e^t \left[c_1 \begin{pmatrix} 4\cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} 4\sin t \\ \cos t \end{pmatrix} \right]$$

where $c_1, c_2 \in \mathbb{R}$.

10. Use the method of undetermined coefficient to solve the initial value problem **(10 marks)**

$$\begin{cases} y''(t) + y'(t) - 2y(t) = 5\sin(t) + 2 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

First we find the complementary solution. The characteristic equation is $0 = r^2+r-2 = (r-1)(r+2)$ so the complementary solution is $y_c(t) = c_1e^t + c_2e^{-2t}$.
Our guess for the particular solution is

$$\begin{aligned} Y(t) &= A\cos t + B\sin t + C \\ Y'(t) &= -A\sin t + B\cos t \\ Y''(t) &= -A\cos t - B\sin t \end{aligned}$$

Plugging into the differential equation,

$$\begin{gathered} -A\cos t - B\sin t - A\sin t + B\cos t - 2(A\cos t + B\sin t + C) = 5\sin t + 2 \\ (B - 3A)\cos t + (-A - 3B)\sin t - 2C = 5\sin t + 2 \end{gathered}$$

Therefore $B = 3A$, and $-A - 9A = 5$ so $A = -1/2$ and $B = -3/2$. And $C = -1$.
The solution to the differential equation is

$$y(t) = c_1e^t + c_2e^{-2t} - \frac{1}{2}\cos t - \frac{3}{2}\sin t - 1$$

To solve the initial value problem:

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 - \frac{1}{2} - 1 = c_1 + c_2 - \frac{3}{2} \\ 0 &= y'(0) = c_1 - 2c_2 - \frac{3}{2} \end{aligned}$$

Subtracting:

$$0 = c_2$$

Then $c_1 = \frac{3}{2}$. So the solution to the initial value problem is

$$y(t) = \frac{3}{2}e^t - \frac{1}{2}\cos t - \frac{3}{2}\sin t - 1$$

11. **(10 marks)**

Suppose $y_1(x)$ is a solution to the differential equation $x^2y''(x) = 2xy'(x) - 2y(x)$, for $x > 0$.

a) **(2 marks)** Let $z(x) = \dfrac{y_1(x)}{x}$. Using $y_1(x) = x \cdot z(x)$, find $y_1'(x)$ and $y_1''(x)$ in terms of x, $z'(x)$, and $z''(x)$.

$$\begin{aligned} y_1'(x) &= z(x) + xz'(x) \\ y_1''(x) &= 2z'(x) + xz''(x) \end{aligned}$$

b) **(2 marks)** Substitute the results of **a)** into the differential equation for y_1 to get a differential equation for $z(x)$.

$$\begin{aligned} x^2y_1''(x) &= 2xy_1' - 2y_1(x) \\ x^2(2z'(x) + xz''(x)) &= 2x(z(x) + xz'(x)) - 2xz(x) \end{aligned}$$

c) **(2 marks)** Find $z(x)$.

$$\begin{aligned} 2x^2z'(x) + x^3z''(x)) &= 2xz(x) + 2x^2z'(x)) - 2xz(x) = 2x^2z'(x) \\ x^2z''(x) &= 0 \\ z(x) &= ax + b \text{ for } a, b \in \mathbb{R} \end{aligned}$$

d) **(2 marks)** If $z(1) = 1$, and $z'(57) = -1$, find $y_1(x)$.

$1 = z(1) = a + b$, $-1 = z'(57) = a$ so $a = -1$ and $b = 2$, thus $z(x) = 2 - x$.

Therefore $y_1(x) = x(2 - x) = 2x - x^2$.

e) **(2 marks)** Check that $y_1(x)$ satisfies the differential equation.

$$\begin{aligned} y_1(x) &= 2x - x^2 \\ y_1'(x) &= 2 - 2x \\ y_2''(x) &= -2 \end{aligned}$$

$LHS = x^2y_1''(x) = -2x^2$, and

$RHS = 2xy_1'(x) - 2y_1(x) = 2x(2 - 2x) - 2(2x - x^2) = 4x - 4x^2 - 4x + 2x^2 = -2x^2$.

The left and right sides are equal, so y_1 solves the differential equation.

12. For a 2×2 real valued constant matrix A, consider the initial value problem, **(10 marks)**

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \qquad (\star)$$

a) **(2 marks)** Write the solution to the initial value problem $(\star)$ in terms of e^{At} if $t_0 = 0$.
$\mathbf{x}(t) = e^{At}\mathbf{x}_0$

b) **(2 marks)** Recall that $(e^{At})^{-1} = e^{-At}$. Write the solution to the initial value problem $(\star)$ using matrix exponentials if $t_0 = 6$.
$\mathbf{x}(t) = e^{At}\left(e^{6A}\right)^{-1}\mathbf{x}_0 = e^{A(t-6)}\mathbf{x}_0$

c) **(3 marks)** If $A = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$, then the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t}$ for $c_1, c_2 \in \mathbb{R}$. Find e^{At}.

Hint: $e^{At}\big|_{t=0} = I$.

e^{At} is the matrix whose columns are solutions with $e^{At}\big|_{t=0} = I$. So we need to find a solution with initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and another solution with initial condition $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The general solution $\mathbf{x}$ at $t = 0$ is $\mathbf{x}(0) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2c_2 - c_1 \\ c_1 - c_2 \end{pmatrix}$.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} 2c_2 - c_1 \\ c_1 - c_2 \end{pmatrix} \implies c_1 = c_2 \text{ and } c_2 = 1 \implies c_1 = 1$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} 2c_2 - c_1 \\ c_1 - c_2 \end{pmatrix} \implies c_1 = 2c_2 \text{ and } c_2 = 1 \implies c_1 = 2$$

Thus

$$e^{At} = \begin{pmatrix} 2e^{-t} - e^{2t} & 2e^{-t} - 2e^{2t} \\ e^{2t} - e^{-t} & 2e^{2t} - e^{-t} \end{pmatrix}$$

d) **(3 marks)** Use the method of integrating factors to find the vector function $\mathbf{v}(t)$ such that the general solution to the system of differential equations,

$$\mathbf{x}'(t) + \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} \mathbf{x}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

can be written as $\mathbf{x}(t) = e^{-At}\mathbf{v}(t) + e^{-At}\mathbf{c}$ where $\mathbf{c} \in \mathbb{R}^2$.

$$\begin{aligned}
\left(e^{At}\mathbf{x}(t)\right)' &= e^{At}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\left(e^{At}\mathbf{x}(t)\right)' &= \begin{pmatrix} 2e^{-t} - e^{2t} & 2e^{-t} - 2e^{2t} \\ e^{2t} - e^{-t} & 2e^{2t} - e^{-t} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\left(e^{At}\mathbf{x}(t)\right)' &= \begin{pmatrix} 2e^{-t} - e^{2t} \\ e^{2t} - e^{-t} \end{pmatrix} \\
e^{At}\mathbf{x}(t) &= \begin{pmatrix} -2e^{-t} - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{2t} + e^{-t} \end{pmatrix} + \mathbf{c} \\
\mathbf{x}(t) &= e^{-At}\begin{pmatrix} -2e^{-t} - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{2t} + e^{-t} \end{pmatrix} + e^{-At}\mathbf{c}
\end{aligned}$$

So $\mathbf{v}(t) = \begin{pmatrix} -2e^{-t} - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{2t} + e^{-t} \end{pmatrix}$.

13. **(10 marks)**

Lidocaine is a drug used for treating ventricular arrhythmia. When lidocaine is given to a patient, it first passes into body tissue before moving into the bloodstream where it is effective. Additionally, over time the drug is filtered out of body tissue and the bloodstream. Let

- $x(t) =$ the amount of lidocaine in the bloodstream,
- $y(t) =$ the amount of lidocaine in body tissue,

where t represents time in hours. A system of differential equations that models $x(t)$ and $y(t)$ is

$$\begin{cases} x'(t) &= -9x(t) + \alpha \cdot y(t) \\ y'(t) &= 6x(t) - \beta \cdot y(t) \end{cases} \qquad (\#)$$

where α and β are strictly positive real numbers.

a) **(1 mark)** What does the initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$ represent?

The initial condition represents the amount of lidocaine in the bloodstream and body tissue at time $t = 0$. In this case, at $t = 0$ there is no lidocaine in the bloodstream and y_0 units of lidocaine in the body tissue, it represents a dose of lidocaine before any of it passes into the bloodstream.

b) **(2 marks)** If all of the lidocaine that leaves body tissue must enter the bloodstream (it can't go anywhere else), what can you say about α and β? Justify in one or two sentences.

$\beta \cdot y(t)$ represents the rate at which lidocaine leaves body tissue. Since all of it must enter the bloodstream, the rate at which lidocaine enters the bloodstream must be equal to the rate at which lidocaine leaves the body tissue. Therefore, $\beta \cdot y(t) = \alpha \cdot y(t)$ so $\alpha = \beta$.

c) **(3 marks)** If $\alpha = \beta$, classify the critical point $\mathbf{0}$ of the system (#).

The eigenvalues are given by $0 = (-9-r)(-\beta-r)-6\alpha = r^2+(9+\beta)r+9\beta-6\alpha = r^2+(9+\beta)r+3\beta$ so $r = \dfrac{(-9-\beta) \pm \sqrt{(9+\beta)^2 - 12\beta}}{2}$ Since $\sqrt{(9+\beta)^2 - 12\beta} < 9 + \beta$, both eigenvalues will be negative and distinct so the $\mathbf{0}$ will be a stable node.

d) **(1 mark)** What is the physical interpretation of your classification of the critical point from part **c**? Does it make sense?

Since the critical point is a stable node, all the solutions will go to $\mathbf{0}$ as $t \to \infty$, physically this means that lidocaine levels will go to 0 in body tissue and the bloodstream. This makes sense as we expect lidocaine levels to decrease to 0 as the drug is processed by the body.

e) **(3 marks)** Ideally, we would have a treatment plan where $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for some optimized constant lidocaine levels x_1, and y_1. A researcher has developed a new treatment that they claim will keep $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ close to $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for all $t > 0$. The new treatment, with a sinusodially driven bloodstream lidocaine, is modelled by the differential equations

$$\begin{cases} (x(t) - x_1)' &= -(y(t) - y_1) + \cos(t) \\ (y(t) - y_1)' &= \quad (x(t) - x_1) \end{cases}$$

Will $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ remain close to $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for all $t > 0$ under this treatment plan?

Hint: Let $\tilde{x}(t) = x(t) - x_1$ and $\tilde{y}(t) = y(t) - y_1$, then find a second order differential equation for $\tilde{y}(t)$.

$$\tilde{y}'' = \tilde{x}' = -\tilde{y} + \cos t$$
$$\tilde{y}'' + \tilde{y} = \cos t$$

$0 = r^2 + 1 \implies r = \pm i$ so the solution to the homogeneous equation is $\tilde{y}(t) = c_1 \cos t + c_2 \sin t$. Since $\cos t$ is in resonance with the general solution, a solution to the nonhomogeneous equation will have a term of the form $t \sin t$ and/or $t \cos t$. In either case, the solution will be unbounded as $t \to \infty$, so it will not remain close to $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

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