Sherman-Morrison-Woodbury formula

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Overview

- Intended Outcome
- Review Sherman-Morrison-Woodbury Formula
- Fast Algorithm to solve $My = (A - uv^T)y = b$
- Summary

Intended Outcome

- To know when to use Sherman-Morrison-Woodbury formula.
- Know Sherman-Morrison-Woodbury formula step by step.

Oringinal Problem

- Known equation $Ax = b$
- Trying to solve y in $(A - UV^T)y = My = b$

Direct Solution

- Directly solve $(A - UV^T)y = My = b$
- Gaussian elimination :

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- Complexity : $\sum_{1}^{n}(n^2 + n(n-1)) \approx \frac{2n^3}{3} = O(n^3)$

Changes for Inverse Matrix

- Sherman-Morrison-Woodbury formula :

$$
\begin{aligned}
M^{-1} &= (A - UV^T)^{-1} \\
&= A^{-1} + A^{-1}U(I - V^T A^{-1} U)^{-1} V^T A^{-1}
\end{aligned}
$$

Changes for Inverse Matrix

- Sherman-Morrison-Woodbury formula :

k x k Matrix

$$M^{-1} = (A - UV^T)^{-1}$$
$$= A^{-1} + A^{-1}U\boxed{(I - V^T A^{-1} U)^{-1}}V^T A^{-1}$$

- Complexity : $O(n^3) \rightarrow O(n^2 k)$

where m is the rank of UV^T and k << n

Reference : https://math.stackexchange.com/questions/1831840/time-complexity-of-inverting-an-n-times-n-matrix-which-is-the-sum-of-a-rank

Faster Algorithm

- Again, to solve $(A - uv^T)^{-1} = My = b$
- Implement by following steps :
  - Solve Az = u for z, so z = A^{-1}u
  - Solve Ax = b for x, so x = A^{-1}b
  - Compute $y = x + (\frac{v^T x}{1 - v^T z})z$

- (refer to textbook problem III.1 (4))

Verify the formula

- $y = x + (\frac{v^T x}{1 - v^T z})z$

$$= A^{-1}b + \frac{v^T A^{-1} b}{1 - v^T A^{-1} u}(A^{-1}u)$$

$$= (A^{-1} + \frac{A^{-1}uv^T A^{-1}}{1 - v^T A^{-1} u})b$$

$$= (A - uv^T)^{-1}b$$

Algorithm Complexity

- Asume A is already factorized or A^{-1} is known
- Implement by following steps :
  - Solve Az = u for z, so $z = A^{-1}u$ $\rightarrow \boxed{O(n^2)}$
  - Solve Ax = b for x, so $x = A^{-1}b$ $\rightarrow \boxed{O(n^2)}$
  - Compute $y = x + (\frac{v^T x}{1 - v^T z})z$ $\rightarrow O(n)$
- (refer to problem III.1 (4))

Total : $O(n^2)$

Quick Example

- Origin known equation : $Ax = b$

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

- Which we suppose we already know

$$A^{-1} = \frac{1}{8} \begin{bmatrix} 54 & -22 & 6 \\ -22 & 10 & -2 \\ 6 & -2 & 2 \end{bmatrix}$$

Quick Example

- Goal is now to solve : $(A - uv^T)y = b$

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -1 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

- Here choose $u = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Solution

- With the known we now follow the algorithm steps

$$Ax = b \Rightarrow x = A^{-1}b = \frac{1}{8}\begin{bmatrix} 54 & -22 & 6 \\ -22 & 10 & -2 \\ 6 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$Az = u \Rightarrow z = A^{-1}u = \frac{1}{8}\begin{bmatrix} 54 & -22 & 6 \\ -22 & 10 & -2 \\ 6 & -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Solution

- With the known we now follow the algorithm steps

$$y = x + \left(\frac{v^T x}{1 - v^T z}\right) z = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + \frac{2}{1 - 1/2} \begin{bmatrix} -3/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \boxed{\begin{bmatrix} -7 \\ 4 \\ 0 \end{bmatrix}}$$

Faster Algorithm

- Extend to rank k change $(A - UV^T)^{-1} = My = b$
- Implement by following steps :
  - Solve AZ = U for Z, so Z = $A^{-1}U$
  - Solve Ax = b for x, so x = $A^{-1}b$
  - Compute $y = x + \dfrac{V^T x}{I - V^T Z} Z$
- (refer to textbook problem III.1 (7))

Algorithm Complexity

- Asume A is already factorized or A^{-1} is known
- Implement by following steps :
  - Solve AZ = U for Z, so $Z = A^{-1}U$ $\rightarrow \boxed{O(n^2 k)}$
  - Solve Ax = b for x, so $x = A^{-1}b$ $\rightarrow O(n^2)$
  - Compute $y = x + \frac{V^T x}{I - V^T Z} Z$ $\rightarrow O(nk)$
- (refer to problem III.1 (4))

Total : $O(n^2 k)$

Usage

- Updating Least Squares :

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} A^T & v \end{bmatrix} \begin{bmatrix} A \\ v^T \end{bmatrix} \hat{x}_{new} = (A^T A + \boxed{v v^T}) \hat{x}_{new}$$

Summary

- When to use Woodbury formula :
  - Solving update of inverse matrix
- Benefit of using Woodbury formula :
  - Reduce complexity $O(n^3) \rightarrow O(n^2 k)$

Question