Stochastic Calculus

Robert Sweeney Blanco

Advisor: Milind Hegde

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Ordinary Differential Equations

In a Calculus or ODE course, you'll see problems of the following form

$$\begin{cases} \frac{dx}{dt} = f(x(t)) \quad (t > 0) \\ x(0) = x_0 \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$
$x_0 \in \mathbb{R}$, and
$x : [0, \infty) \to \mathbb{R}$ is the solution function.

Stochastic Differential Equations

In a Stochastic Calculus, we'll be studying differential equations with random noise

$$\begin{cases} dX(t) = f(X(t))dt + g(X(t))dB_t \quad (t > 0) \\ X(0) = x_0 \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$
$g : \mathbb{R} \to \mathbb{R}$
$x_0 \in \mathbb{R}$
B_t is Brownian motion and
$X : [0, \infty) \to \mathbb{R}$ is the solution stochastic process.

Brownian Motion

Stochastic Calculus

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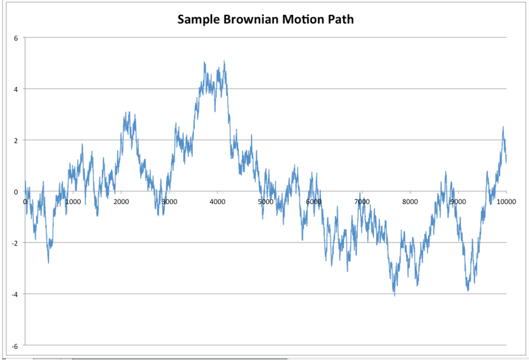
Brownian Motion is one of the most famous continuous time stochastic processes.



Problem: B_t does not have bounded variation!

Example SDE

Let $P(t)$ denote the random price of a stock at time $t \geq 0$

$$\begin{cases} dP = \mu P \; dt + \sigma P \; dB_t & (t > 0) \\ P(0) = p_0 \end{cases}$$

For some some constants $\mu > 0$ (drift) and σ (volatility)

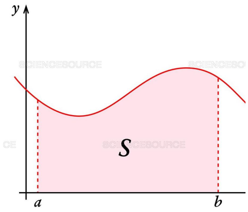
Defining $M^2([a, b])$

In this presentation we will work with real valued stochastic processes $X = (\Omega, \mathscr{F}, \mathscr{F}_t, (X_t), \mathbb{P})$ mostly in the space of M^2.

Definition of $M^2([a, b])$

We say X_t is in $M^2([a, b])$ if

$$\mathbb{E}\Big[\int_a^b |X_s|^2 ds\Big] < +\infty$$

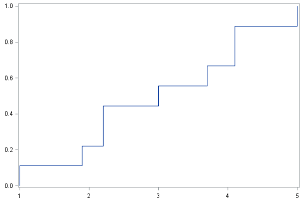


Defining Elementary Processes

Definition of Elementary Processes

We say that $X_t \in M^2([t_0, t_n])$ is an elementary process if

$$X_t = \sum_{i=1}^{n-1} X_{t_i} \mathbb{1}_{[t_i, t_{i+1})}$$



Defining Stochastic Integral for Elementary Processes

Definition of Stochastic Integral for Elementary Processes

If $X_t \in M^2([t_0, t_n])$ is an elementary process

$$\int_{t_0}^{t_n} X_t \; dB_t = \sum_{i=0}^{n-1} X_{t_i}(B_{t_{i+1}} - B_{t_i})$$

Note the similarity with the Riemann–Stieltjes integral.

Convergence of Elementary Processes

Elementary processes are dense in $M^2([a, b])$

If $X_t \in M^2([a, b])$, then there exists a sequence of elementary processes $(X_n)_n$ such that

$$\lim_{n \to \infty} \mathbb{E}[\int_a^b |X_t - X_n(t)|^2 dt] = 0$$

Defining the Stochastic Integral

Definition of the Stochastic Integral

If $X_t \in M^2([a,b])$

$$\int_a^b X_t \, dB_t = \lim_{n\to\infty} \int_a^b (X_t)_n \, dB_t$$

Defining the Stochastic Differential

Definition of the Stochastic Differential

We say the stochastic differential of a stochastic process X_t is

$$dX_t = F_t dt + G_t dB_t$$

if for every $0 \leq t_1 < t_2 \leq T$

$$X_{t_2} - X_{t_1} = \int_{t_1}^{t_2} F_t dt + \int_{t_1}^{t_2} G_t \; dB_t$$

for some $F \in M^2([0, T])$ and $G \in M^2([0, T])$.

Stochastic Product Rule

Product Rule

Suppose

$$dX_t = F_t dt + G_t dB_t$$

$$dY_t = H_t dt + J_t dB_t$$

then,

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + G_t J_t dt$$

Ito's Formula

Ito's formula (also known as Ito's Lemma) is the analogue of the chain rule.

Ito's Formula

Suppose $f : \mathbb{R}_x \times \mathbb{R}_t^+ \to \mathbb{R}$ is a continuous function, once continuous differentiable in t and twice in x and

$$dX_t = F_t dt + G_t dB_t$$

then,

$$df(X_t, t) = \frac{\partial f}{\partial t}(X_t, t)dt + \frac{\partial f}{\partial x}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t, t)G_t^2 dt$$

Example Revisited

Recall the SDE example given at the beginning

$$\begin{cases} dP = \mu P \; dt + \sigma P \; dB_t & (t > 0) \\ P(0) = p_0 \end{cases}$$

We will show that

$$P(t) = p_0 e^{\sigma B_t + (\mu - \frac{\sigma^2}{2})t}$$

Example Revisited

Define $f(x,t) = p_0 e^{\sigma x + (\mu - \frac{\sigma^2}{2})t}$

$\frac{\partial f}{\partial t} = p_0(\mu - \frac{\sigma^2}{2}) e^{\sigma x + (\mu - \frac{\sigma^2}{2})t}$

$\frac{\partial f}{\partial x} = p_0 \sigma e^{\sigma x + (\mu - \frac{\sigma^2}{2})t}$

$\frac{\partial^2 f}{\partial x^2} = p_0 \sigma^2 e^{\sigma x + (\mu - \frac{\sigma^2}{2})t}$

Clearly $P(t) = f(B_t, t)$ so Ito's formula,

$$\begin{aligned} df(B_t,t) &= p_0(\mu - \frac{\sigma^2}{2}) e^{\sigma B_t + (\mu - \frac{\sigma^2}{2})t} dt + p_0 \sigma e^{\sigma B_t + (\mu - \frac{\sigma^2}{2})t} dB_t + \frac{1}{2} p_0 \sigma^2 e^{\sigma B_t + (\mu - \frac{\sigma^2}{2})t} dt \\ &= p_0 \mu e^{\sigma B_t + (\mu - \frac{\sigma^2}{2})t} dt + p_0 \sigma e^{\sigma B_t + (\mu - \frac{\sigma^2}{2})t} dB_t \\ &= \mu P \, dt + \sigma P \, dB_t \end{aligned} \tag{1}$$

We can verify that this satisfies the initial condition
$P(0) = p_0 e^{\sigma B_0 + (\mu - \frac{\sigma^2}{2})0} = p_0$