Modular Forms

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Defining Modular Forms

In this presentation, I will be working with the modular group $SL_2(\mathbb{Z}) = \Gamma_1$

Definition: Modular form

f is a modular form of weight k if

1. f is holomorphic on $\mathbb{H}$
2. f continues to be holomorphic as $z \longrightarrow i\infty$
3. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1$

Simplifying the full Modular Group

Theorem

Γ_1 is a group generated by $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

This simplifies proving a function is a modular form when working with the full modular group because now we only need to check that the desired behavior for composition with Mobius transformations from the matrix group works for the transformations corresponding with S and T. That is $S(z) = f(\frac{-1}{z}) = z^k f(z)$ and $T(z) = f(z+1) = f(z)$. Note this means modular forms are periodic and have Fourier series!

Dimension of $M_k(\Gamma_1)$

Modular forms of a fixed weight form a vector space over the
Complex numbers. The dimension of $M_k(\Gamma_1)$ is 0 for all
negative and odd values of k. Otherwise the dimension is given
by the following formula:

$$dim(M_k(\Gamma_1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 12 \ mod(12) \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 12 \ mod(12) \end{cases}$$

Defining Eisenstein Series

There are different ways of defining the Eisenstein Series, but they all differ by a constant. Here is a common definition for series of even weight $k > 2$:

Eisenstein Series

$$G_k(\tau) = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}$$

This series converges absolutely for $k > 2$.

Transformation under T

$$G_k(\tau + 1) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m(\tau+1)+n)^k}$$

$$= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+m+n)^k}$$

Define $\mu = m + n$ (Note $(m,n) \neq (0,0) \Rightarrow (m,\mu) \neq (0,0)$)

$$= \sum_{\substack{(m,\mu)\in\mathbb{Z}^2 \\ (m,\mu)\neq(0,0)}} \frac{1}{(m\tau+\mu)^k} = G_k(\tau)$$

Transformation under S

$$G_k\left(\frac{-1}{\tau}\right) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m(\frac{-1}{\tau})+n)^k}$$

$$= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{\tau^k}{(-m+n\tau)^k} = \tau^k \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(-m+n\tau)^k}$$

$\text{Define}(\mu,\eta)=(n,-m)(\text{Note}(m,n)\neq(0,0)\Rightarrow(\mu,\eta)\neq(0,0))$

$$= \tau^k \sum_{\substack{(\mu,\eta)\in\mathbb{Z}^2 \\ (\mu,\eta)\neq(0,0)}} \frac{1}{(\mu\tau+\eta)^k} = \tau^k G_k(\tau)$$

Proving the Growth Condition

$$\lim_{\Im(\tau)\to\infty} \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{1}{(m\tau + n)^k}$$

$$= \lim_{\Im(\tau)\to\infty} \sum_{n\in\mathbb{Z}/\{0\}} \frac{1}{n^k} + \sum_{\substack{m\in\mathbb{Z}/\{0\}\\n\in\mathbb{Z}}} \frac{1}{(m\tau + n)^k}$$

$$= \sum_{n\in\mathbb{Z}/\{0\}} \frac{1}{n^k}$$

1. Since the series converges uniformly on any compact subset of $\mathbb{H}$, it is holomorphic

2. The series transforms properly for S and T $\Rightarrow$ satisfies Modularity Condition

3. The series satisfies the growth condition

$\therefore$ G_k is a modular form for $k > 2$. Moreover, it is a modular form of weight k.