Modular Forms

Robert Sweeney Blanco

Advisor: Brandon Williams

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Defining Modular Forms

In this presentation, I will be working with the modular group $SL_2(\mathbb{Z}) = \Gamma_1$

Definition: Modular form

$f(z)$ is a modular form of weight k if

1. f is holomorphic on $\mathbb{H}$
2. f continues to be holomorphic as $\Im(z) \longrightarrow \infty$
3. $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1$

Simplifying the full Modular Group

Theorem

Γ_1 is a group generated by $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

This simplifies proving a function is a modular form when working with the full modular group because now we only need to check that the desired behavior for composition with Mobius transformations from the matrix group works for the transformations corresponding with S and T. That is $S(z) = f(\frac{-1}{z}) = z^k f(z)$ and $T(z) = f(z+1) = f(z)$. Note this means modular forms are periodic and have Fourier series!

Dimension of $M_k(\Gamma_1)$

Modular forms of a fixed weight form a vector space over the Complex numbers. The dimension of $M_k(\Gamma_1)$ is 0 for all negative and odd values of k. Otherwise the dimension is given by the following formula:

$$dim(M_k(\Gamma_1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \; mod(12) \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \; mod(12) \end{cases}$$

Defining Eisenstein Series

There are different ways of defining the Eisenstein Series, but they all differ by a constant. Here is a common definition for series of even weight $k > 2$:

Eisenstein Series

$$G_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

This series converges absolutely for $k > 2$.

Transformation under T

$$G_k(z+1) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m(z+1)+n)^k}$$

$$= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+m+n)^k}$$

Define $\mu = m+n$ (Note $(m,n)\neq(0,0) \Rightarrow (m,\mu)\neq(0,0)$)

$$= \sum_{\substack{(m,\mu)\in\mathbb{Z}^2 \\ (m,\mu)\neq(0,0)}} \frac{1}{(mz+\mu)^k} = G_k(z)$$

Transformation under S

$$G_k\left(\frac{-1}{z}\right) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m(\frac{-1}{z}) + n)^k}$$

$$= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{z^k}{(-m + nz)^k} = z^k \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(-m + nz)^k}$$

Define$(\mu, \eta) = (n, -m)$(Note$(m, n) \neq (0, 0) \Rightarrow (\mu, \eta) \neq (0, 0)$)

$$= z^k \sum_{\substack{(\mu,\eta)\in\mathbb{Z}^2 \\ (\mu,\eta)\neq(0,0)}} \frac{1}{(\mu z + \eta)^k} = z^k G_k(z)$$

Proving the Growth Condition

$$\lim_{\Im(z)\to\infty} \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

$$= \lim_{\Im(z)\to\infty} \left[\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}} \frac{1}{n^k} + \sum_{\substack{(m,n)\in\mathbb{Z}^2\\m\neq 0}} \frac{1}{(mz+n)^k}\right]$$

$$= \sum_{\substack{n\in\mathbb{Z}\\n\neq 0}} \frac{1}{n^k} = 2\sum_{\substack{n\in\mathbb{N}\\n\neq 0}} \frac{1}{n^k} = 2\zeta(k)$$

Proof Conclusion

1. Since the series converges uniformly on any compact subset of $\mathbb{H}$, it is holomorphic
2. The series transforms properly for S and $T \Rightarrow$ satisfies Modularity Condition
3. The series satisfies the growth condition

$\therefore$ G_k is a modular form for $k > 2$. Moreover, it is a modular form of weight k.

Fourier Expansion for Eisenstein Series

Since Eisenstein Series for even $k > 2$ are modular forms, they have a unique Fourier expansion.

Fourier Definition of Eisenstein Series

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{d,n \geq 1} n^{k-1} e^{nd2\pi iz}$$

Where B_k is the kth Bernoulli number.

Eisenstein Identities

The set of modular forms $M_*(\Gamma_1)$ form a ring. Moreover, the weight of the product of modular forms is the sum of their weights. This paves the way for several easy identities:

1. E_4^2 is a modular form of weight 8, both E_4^2 and E_8 have Fourier series that start with 1, and $dim(M_8(\Gamma_1)) = 1 \Rightarrow E_4^2 \equiv E_8$
2. $E_4 \cdot E_6$ is a modular form of weight 10, the product of their Fourier series has a leading term of 1, and $dim(M_{10}(\Gamma_1)) = 1 \Rightarrow E_4 \cdot E_6 \equiv E_{10}$

Discriminant Function

A famous modular form of weight 12 is the Discriminant Function.

Discriminant Function

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}$$

Fourier Expansion of Discriminant Function

Since the Discriminant function is a modular form, it has a Fourier expansion. The coefficients of the series can be calculated with the Ramanujan tau function $\tau(n)$.

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2n\pi iz} = e^{2\pi iz} - 24e^{4\pi iz} + 252e^{6\pi iz} - \ldots$$

Discriminant Function identity

Note that $E_4^3 = (1 + 240e^{2\pi iz} + ...)^3 = 1 + 720e^{2\pi iz} + ...$ and $E_6^2 = (1 - 540e^{2\pi iz} + ...)^2 = 1 - 1008e^{2n\pi iz} + ...$ are not scalar multiples of each other. Since $dim(M_12(\Gamma_1)) = 2$, both E_4^3 and E_6^2 span $M_{12}(\Gamma_1)$, and thus there is a linear combination of them that equal $\Delta(z)$. Since the first term in the Fourier series of $\Delta(z)$ is $e^{2\pi iz}$, it must be some scalar multiple of the difference of E_4^3 and E_6^2 (to get rid of the leading one). The first term of $E_4^3 - E_6^2$ is $1728e^{2\pi iz}$, so we need to scale down by 1728, which will match the first term of $\Delta(z)$ and thus we are done. $\therefore \Delta(z) = \frac{E_4^3 - E_6^2}{1728}$.