

## APPENDIX C - BASIC RELIABILITY EQUATIONS

$$F(t) = \int_0^t f(t)dt ; 1.0 - R(t) ; 1.0 - e^{-\int_0^t h(t)dt} ; 1.0 - e^{-H(t)} ; 1.0 - e^{-\int_0^t \frac{1+L'(\tau)}{L(\tau)}d\tau}$$

An equation in which the independent variable is "time" and in which the dependent variable is the fraction failed is known as the Cumulative distribution function or C.D.F. Denoted by:  $F(t)$  = Fraction failed within total time "t".

Probabilistically:  $P(T \leq t)$

Properties: (1)  $F(t)$  is continuous for all "t".

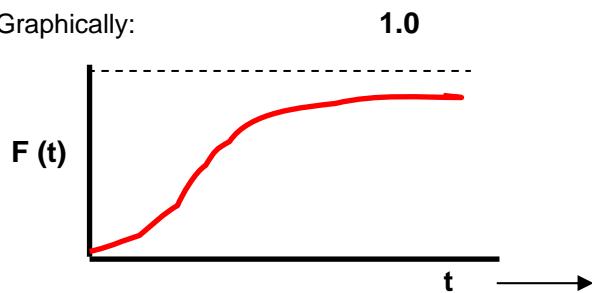
(2)  $F(t) < F(t')$  for all  $t < t'$ .

(3)  $F(t)$  is a monotonically increasing function on the interval  $[0, \infty]$  so that

$$F(\infty) = 1.0$$

$$F(0) = 0 \implies \text{Nothing has failed at time "0".} \implies \text{Everything has failed at time "\infty"}$$

Graphically:



$$f(t) = F'(t) ; -R'(t) ; -\frac{d}{dt}[e^{-H(t)}] ; -\frac{d}{dt}[e^{-\int_0^t h(t)dt}] ;$$

$$h(t) * R(t); \frac{1 + L'(t)}{L(t)} e^{-\int_0^t \frac{1+L'(\tau)}{L(\tau)}d\tau}$$

If we differentiate a C.D.F. with respect to "t", we obtain a **Probability Density function or p.d.f.**. Denoted by:  $f(t)$  = Rate of change of the C.D.F.; It measures the "slope" of the cumulative curve.

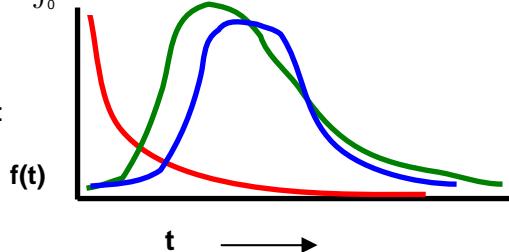
Probabilistically:  $P(T = t)$

Properties: (1)  $f(t)$  is continuous for all "t".  
 (2)  $f(t) > 0$  for all  $t$ .

$$(3) \int_0^t f(t)dt = 1.0 \text{ area under the curve such that the area under the curve from "0" to "a" } = F(a).$$

$$\text{slope} = f(t) = \frac{dF(t)}{dt}$$

Graphically:



$$R(t) = 1.0 - F(t); 1.0 - \int_0^t f(t)dt; e^{-H(t)}; e^{-\int_0^t h(t)dt}; f(t)/h(t);$$

$$F'(t)/h(t); e^{-\int_0^t \frac{1+L'(\tau)}{L(\tau)} d\tau}$$

If we take the complement of the C.D.F. "F(t)" at some point in time "t", we obtain the **"Reliability function" at "t"**, or the fraction unfailed at time "t". Denoted by:  $R(t)$  = Fraction unfailed at time "t."

Probabilistically:  $P(T > t)$

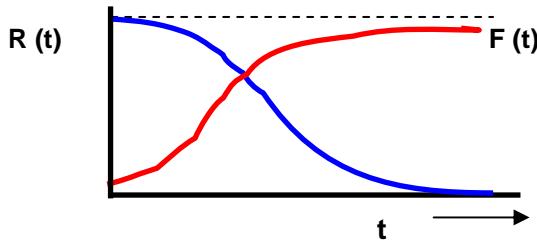
Properties: (1)  $R(t)$  is continuous for all "t".  
 (2)  $R(t) < R(t')$  for all  $t < t'$ .  
 (3)  $R(t)$  is a monotonically decreasing function on the interval  $[0, \infty]$

Such that  $R(0) = 1.0 \Rightarrow$  Nothing has failed at time "0".

$R(\infty) = 0.0 \Rightarrow$  Everything has failed at time " $\infty$ ".

Graphically:

$$R(t) = 1.0 - F(t)$$

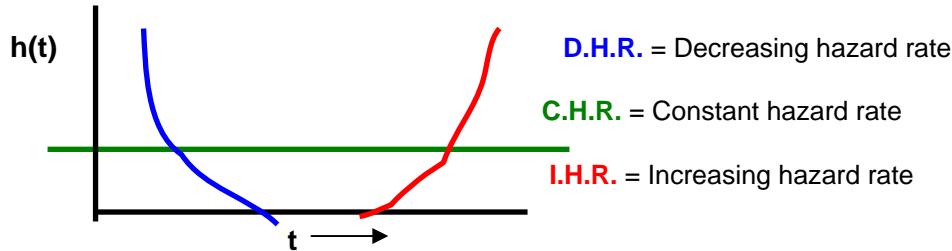


The **instantaneous hazard rate** at any point in time is  $f(x)/F(x)$ . Like a density function, a failure rate is a complete descriptor of a life distribution. Denoted by:  $h(t)$  = Rate of change of failure at time "t."

Probabilistically:  $P(T = t)/P(T > t)$

Properties: (1)  $h(t)$  is piece-wise continuous for all "t" greater than or equal to "0."  
 (2)  $h(t)$  is an increasing / decreasing /constant function on the interval  $[0, \infty]$

Graphically:

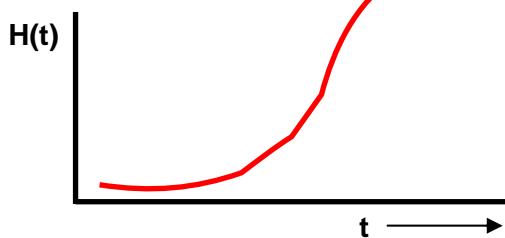


$$H(t) = \int_0^t h(\tau) d\tau ; \quad \int_0^t \frac{f(\tau)}{1.0 - F(\tau)} d\tau ; \quad \ln \frac{1.0}{R(t)} ; -\ln[R(t)]; \quad \ln F(t); \quad \int_0^t \frac{1+L'(\tau)}{L(\tau)} d\tau$$

The cumulative hazard rate at any point in time is  $\int_0^t h(\tau) d\tau$  denoted by:  $H(t)$  = Cumulative failure rate at time "t."

Properties: (1)  $H(t)$  is piece-wise continuous for all "t" greater than or equal to "0."  
 (2)  $H(t)$  is an increasing function on the interval  $[0, \infty]$

The average hazard rate on an interval  $(t_1, t_2]$  is  $AHR(t_1, t_2) = \frac{H(t_2) - H(t_1)}{t_2 - t_1}$



$$L(t) = \frac{\int_t^\infty f(\tau) d\tau}{\int_t^\infty f(\tau) d\tau} = 1.0 - \frac{1}{R(t)} \int_t^\infty R(\tau) d\tau; \quad \frac{1}{R(t)} \int_t^\infty R(\tau) d\tau; \quad \frac{\int_t^\infty e^{-\int_0^\tau h(y) dy} d\tau}{e^{-\int_0^t h(\tau) d\tau}}; \quad e^{H(t)} * \int_t^\infty e^{H(\tau)} d\tau$$

At time "t", the probability that the component or system has remaining (residual) life " $t^*$ " is denoted by:  
 $L(t)$  = Remaining / residual life at " $t^*$ ".

Probabilistically:  $P(T > t+t^*) / P(T > t) = R(t+x) / R(x)$

## Special Conditional Random Variables

Probability of a component or system of age "t" surviving some future time period "t\*":

$$P(T > t^* | T > t) = R(t^*) / R(t)$$

Probability that surviving life after burn in is greater than the warranty period

where  $t_b$  = tentative burn in time and

$t_w$  = warranty time/period

$$P(t - t_b > t_w | t > t_b) = \frac{R(t_w + t_b)}{R(t_b)}$$

## BASIC MEASURES OF CENTRAL TENDENCY

$$E[T] = \mu = \int_0^{\infty} t * f(t) dt ; \quad \int_0^{\infty} R(t) dt \quad \text{If it exists. Think "Center of Mass."}$$

If "T" is a continuous random variable signifying Time to Failure, then the expected or Mean time to failure

random variable is used frequently as an indicator for system reliability. In general, the first moment about the

origin is defined as the mean. Denoted by:  $E [T] = \text{Mean} = M.T.T.F.$  or

Discrete case:  $E(T) = \sum_{i=1}^n x_i * p(x_i)$

$$E(T) = \mu = \int_{-\infty}^{+\infty} t * f(t) dt$$

### Properties:

$$E[c] = c$$

$$E[c + X] = c + E[X] \quad \longrightarrow$$

for "c" a constant.

$$E[cX] = c * E[X]$$

$$E[X + Y] = E[X] + E[Y] \quad \} \quad \longrightarrow$$

Used in statistical tolerancing.

$$E[X - Y] = E[X] - E[Y]$$

If X and Y are statistically independent.

$$E(XY) = E(X) * E(Y)$$

$$E[c * g(T)] = c * E[g(T)]$$

$$E[g_1(T) + g_2(T)] = E[g_1(T)] + E[g_2(T)]$$

### Sample Mean or Arithmetic Mean

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

### Population Mean

$$\mu = \frac{\sum_{i=1}^N x_i}{N}$$

### Geometric Mean

$$\bar{x}_g = \sqrt[N]{\prod_{i=1}^N x_i}$$

$$= \text{antilog}\left(\frac{1}{N} \sum_{i=1}^N \ln x_i\right)$$

### Harmonic Mean

$$\bar{x}_h = \frac{N}{\sum_{i=1}^N \frac{1}{x_i}}$$

Mode (Measure of central tendency on a nominal scale).

On a p.d.f. plot, the location on the x-axis corresponding to the highest failure frequency is the mode. Also referenced as that value measurement of a sample that appears most frequently.

Median: On the C.D.F. plot, the location on the x-axis where  $F(x) = .50$  is known as the median.

For "ordered" sample data, it corresponds to the following position within the ordered data values -

For "n" even

$$Md = \frac{\tilde{x}_{(N/2)} + \tilde{x}_{(N/2+1)}}{2}$$

For "n" odd

$$Md = \tilde{x}_{((N+1)/2)}$$

## BASIC MEASURES OF DISPERSION

$$V[T] = \sigma^2 = \int_{-\infty}^{\infty} (t - \mu)^2 * f(t) dt$$

Think "Distance from Center of Mass."

The second moment of a random variable about the origin is defined as the **variance**.

Denoted by:  $V[T] = E[(T - E(T))^2] = E[T^2] - (E[T])^2 = \sigma^2$

The **standard deviation** is denoted by:  
 $\sqrt{E[(T - E(T))^2]} = \sigma [T]$

**Properties:**  
except

- (1)  $V[c] = 0$
- (2)  $V[c * T + b] = c^2 * V[T]$
- (3)  $V[c + T] = V[T]$
- (4)  $V[T_1 \pm T_2] = V[T_1] + V[T_2]$  whenever  
 $T_1$  and  $T_2$  are independent.  
 $= V[T_1] + V[T_2] + 2 * Cov(T_1, T_2)$  otherwise.

**Properties:** Same as Variance case

take the square root. Property (3) of the Variance case used in statistical tolerancing.

### Sample Variance Deviation

$$s^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1} = \frac{SS}{n-1}$$

### Sample Standard Deviation

$$s = \sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}} = \sqrt{\frac{SS}{n-1}}$$

### Sample Average

$$AD = \frac{\sum_{i=1}^n |X_i - Md|}{n}$$

### Population Variance

$$\sigma^2 = \frac{\sum_{i=1}^N (X_i - \mu)^2}{N} = \frac{SS}{N}$$

### Population Standard Deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^N (X_i - \mu)^2}{N}} = \sqrt{\frac{SS}{N}}$$

### Sample Mean Deviation

$$MD = \frac{\sum_{i=1}^n |X_i - \bar{X}|}{n}$$

where SS = Sum of Squares.

### Population Average Deviation

$$AD = .7979\sigma$$

### Population Mean Deviation

$$MD = E |x - \mu|$$

### Range (on an interval)

$$X_{\max} - X_{\min}$$

## BASIC MEASURES OF DISPERSION (cont.)

### Coefficient of Variation Standard

$$CV = \eta = \frac{\sigma}{\mu} \approx \frac{\sigma_{\bar{x}}}{\bar{x}} = \frac{s}{\bar{x}}$$

NOTE :  $\frac{1}{CV} = S/N\text{Ratio}$

### Sample Standard error of the Mean

$$s_{\bar{x}} = \frac{s}{\sqrt{n}}$$

### Population error of the mean

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

## Skewness

The third moment of a random variable about the origin divided by the cube of the standard deviation is defined as the skewness.

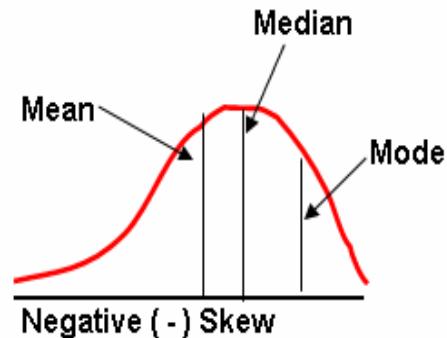
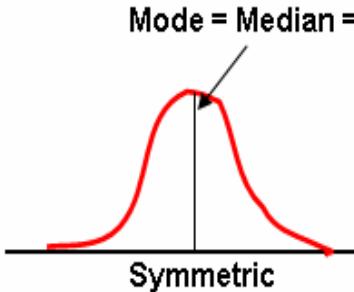
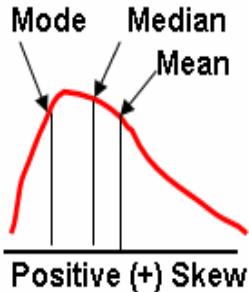
Population coefficient of skewness denoted by:

$$\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\int_{-\infty}^{\infty} (t - \mu)^3 * f(t) dt}{\sigma^3}$$

Sample coefficient of skewness denoted by:

$$\hat{\alpha}_3 = \frac{3 * (\bar{X} - Md)}{s} = \frac{\sum x_i^3 - [3 \sum x_i * \sum x_i^2 / N] + [2(\sum x_i)^3 / N^3]}{N}$$

It is an indicator of where a majority of the data lies within a distribution (p.d.f.). It conveys Location.



## **BASIC MEASURES OF DISPERSION (cont.)**

### **Kurtosis**

The fourth moment of a random variable about the origin divided by the variance squared is defined as the **kurtosis**.

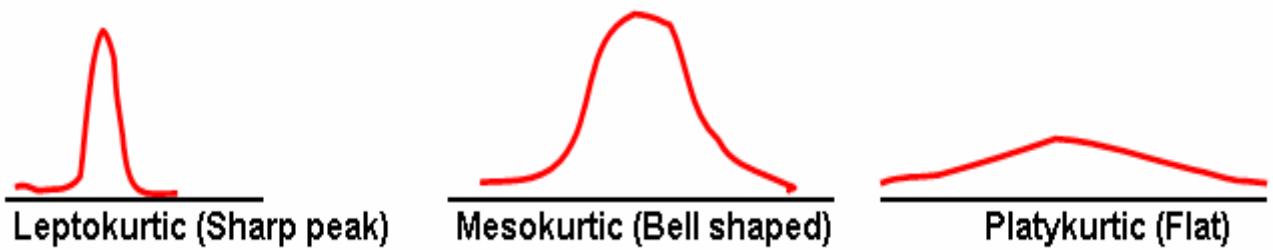
Population coefficient of kurtosis denoted by:

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{\int_{-\infty}^{\infty} (t - \mu)^4 * f(t) dt}{\sigma^4}$$

Sample coefficient of kurtosis denoted by:

$$\hat{\alpha}_4 = \frac{\sum x_i^4 - [4 \sum x_i * \sum x_i^3 / N] + 6((\sum x_i)^2 * \sum x_i^2 / N^2) - [3(\sum x_i)^4 / N^4]}{N}$$

It is an indicator of where a majority of the data lies within a distribution (p.d.f.). It conveys **Location**.



## **Estimators (Biased / Unbiased)**

An estimator is said to be **Biased** if and only if

$$b(\theta) = E(\hat{\theta}) - \theta \neq 0 \quad \text{or} \quad E(\hat{\theta}) \neq \theta$$

An estimator is said to be **Unbiased** if and only if

$$E(\hat{\theta}) - \theta \equiv 0 \quad \text{or} \quad E(\hat{\theta}) \equiv \theta \quad \text{or} \quad b(\theta) = 0.$$

Probability is the theory of modeling uncertainty. The problem in probability is to say something about an outcome when given a probabilistic model. The set of all possible outcomes is known to follow a certain probability law. On the other hand, the problem in Statistics is to say something about a population given a sample of outcomes; The goal of statistics is the following: "We take a sample from a population and estimate its statistics (parameters) so that we may attempt to make an inference about the population's behavior (parameters)." The "behavior" of the population may be defined with a "probabilistic model" (distribution). Examples of estimators for a population parameter vs. the "true" population parameters::

From  $N(\mu, \sigma)$  we have  $\hat{\theta} = (\bar{X}, s, s^2)$  vs.  $\theta = (\mu, \sigma, \sigma^2)$

## Maximum Likelihood Estimators

For a given "p.d.f.", the most probable estimator value (if it exists) for a given parameter. A "supremum".  
Properties (if it exists for a given distribution):

- It is invariant.
- It is the Uniformly Minimum Variance Unbiased Estimator (UMVUE).
- It is unique.
- It is asymptotically normal.
- It is asymptotically efficient.

