Bilateral Weighted Completions

Robert A. Rice
robert.a.rice@gmail.com

September 11, 2025

Abstract

We develop bilateral weighted completion theory as a categorical framework that may provide a unified perspective on certain completion phenomena in mathematics. This approach centers on weighted completion of bilateral pairings $\theta : Q \Rightarrow C(D, E)$, where diagrams $D : I \to C$ and $E : J \to C$ are functors representing the source and target structures, bilateral weights $Q : I^{\mathrm{op}} \otimes J \to V$ measure connection strength between dual testing structures, and the pairing θ represents the original mathematical relationship to be completed. Completion occurs by finding categories $\widehat{C}$ containing C via fully faithful embedding $\varepsilon_C : C \hookrightarrow \widehat{C}$ (where ε_C^* denotes lifting structure to the completion) such that the lifted pairing admits factorization $\varepsilon_C^* \theta = \rho \star \gamma \star \lambda$ through bilateral interpolants (the $\star$ operation denotes composition of natural transformations in the enriched setting).

We prove that every bilateral pairing admits a weighted completion, and establish that weighted completions are idempotent—completing a complete structure yields the original structure. We demonstrate that several completion phenomena can be formulated within this framework, including Stone-Čech compactification, canonical extensions of distributive lattices, profinite group completions, Kan extensions, and Isbell envelopes. When classical completions fail to exist, the theory offers virtual extensions that may provide useful approximations.

The weighted completion construction admits monadic organization that extends Gabriel-Ulmer's Ind/Pro methodology to arbitrary weights and recovers Garner's Isbell monad as a particular case. We establish correspondence theorems showing how bilateral weighted completions relate to existing frameworks including Schoots's categorical extensions, Pratt's communes, and Garner's cylinder systems, suggesting connections between previously disparate approaches to categorical completion.

1 Introduction and Motivation

1.1 Completion Phenomena Across Mathematics

Mathematical completion processes emerge throughout many domains, each addressing the fundamental challenge of extending a given structure to support operations or properties that may fail in the original setting. The ubiquity of such completion processes suggests that there may be underlying principles that transcend specific mathematical contexts. Consider how different areas of mathematics approach completion.

In topology, Stone-Čech compactification Stone [1936] extends completely regular spaces X to compact Hausdorff spaces βX where bounded continuous functions on X extend uniquely to βX. This remarkable construction involves testing continuous functions against ultrafilters, revealing a deep relationship between functional and filter structures. Sobrification Johnstone [1982] completes T_0 spaces by ensuring every irreducible closed set has a generic point, accomplished through systematic testing between closed sets and points. Meanwhile, Alexandroff one-point compactification Alexandroff [1924] provides minimal compactification for locally compact spaces through a fundamentally different approach.

Algebraic contexts reveal their own completion patterns. MacNeille completions MacNeille [1937] extend partially ordered sets to complete lattices while preserving existing meets and joins through testing elements against upward and downward closed sets. The canonical extensions of distributive lattices Jónsson and Tarski [1951], Gehrke and Harding [2001] provide completions that preserve finite operations while adding infinite operations, achieved through systematic testing between filters and ideals. Profinite completions of groups Pontryagin [1966] construct inverse limits of finite quotients, testing group elements against finite quotient structures.

Category theory contributes its own completion paradigms: Kan extensions Kan [1958] complete diagrams by providing optimal approximations when direct limits or colimits fail to exist. Isbell envelopes Isbell [1960], Garner [2018] complete categories by adding morphisms through presheaf-copresheaf testing. Gabriel-Ulmer's Ind and Pro completions Gabriel and Ulmer [1971], Adámek and Rosický [1994] systematically add filtered colimits and cofiltered limits, respectively.

These diverse completion processes exhibit striking structural patterns. Each involves extending a mathematical structure to support operations determined by systematic testing between dual categorical structures. This observation suggests the possibility of a unified framework that might capture common structural principles operating across different mathematical domains.

1.2 Bilateral Weighted Completion Theory

We develop bilateral weighted completion theory based on systematic study of bilateral weights and their completions. Our key insight is that certain completion processes can be understood in terms of bilateral weights $Q : I^{\mathrm{op}} \otimes J \to V$ that measure "connection strength" between dual testing structures represented by small categories I and J. This bilateral perspective aims to capture the dual nature inherent in many completion phenomena.

Given a bilateral pairing $\theta : Q \Rightarrow C(D, E)$ where $D : I \to C$ and $E : J \to C$ are diagrams in a V-category C, we construct its weighted completion $(\widehat{C}, \varepsilon_C)$ as a V-category $\widehat{C}$ containing C via a fully faithful embedding $\varepsilon_C : C \hookrightarrow \widehat{C}$ such that the lifted pairing admits a canonical factorization $\varepsilon_C^* \theta = \rho \star \gamma \star \lambda$.

We prove that weighted completions yield complete objects, and that completing complete objects is trivial (idempotency). We provide characterizations of complete bilateral pairings and demonstrate that the weighted completion construction offers virtual methodology when classical completion processes fail. Our framework aims to provide principled extensions based on categorical universal properties, offering an alternative to ad hoc approaches to completion problems.

1.3 Contributions

This work establishes bilateral weighted completion theory through several key contributions that advance our understanding of certain mathematical completion phenomena.

1. We prove that every bilateral pairing admits a weighted completion. This result suggests that bilateral completion methodology may provide a useful perspective across various mathematical contexts, offering a systematic approach to understanding certain completion phenomena.

2. We establish fundamental completeness properties that characterize when bilateral pairings require no further completion. We prove that weighted completions always yield complete objects, that completing complete objects is trivial, and provide equivalent characterizations of complete bilateral pairings. These results demonstrate that bilateral weighted completion behaves as a coherent completion theory.

3. We show through detailed examples that several completion processes across topology, algebra, and category theory can be formulated as weighted completions of appropriately constructed bilateral pairings. This suggests possible common structural patterns underlying these phenomena.

4. We establish monadic organization of weighted completion theory. The weighted completion extends to a monad W on the category of bilateral pairings, and we prove that Garner's Isbell monad emerges as a natural specialization. This monadic structure provides organizational principles for composing completion processes.

5. We provide detailed correspondence theorems showing how bilateral weighted completions relate to existing frameworks including Schoots's categorical extensions, Pratt's communes, and Garner's cylinder systems. These correspondences suggest that our framework may complement existing approaches while providing new perspectives.

Bilateral weighted completion theory offers a categorical framework for understanding certain completion phenomena. This approach may provide theoretical insights into structural patterns underlying some completion processes and suggest methodology for investigating completion in new mathematical contexts.

The rest of the paper is as follows. Section 2 develops the core theory of bilateral weighted completions with detailed proofs. Section 3 introduces complete bilateral pairings and establishes completeness properties. Section 4 presents the categorical structure through the weighted completion monad. Section 5 provides examples demonstrating the bilateral framework across mathematical domains, while Section 6 develops gem theory as the case of representable bilateral completions. Section 7 establishes structural properties, Section 8 provides correspondences with existing frameworks, and Section 9 concludes with future directions.

2 Introduction to Bilateral Weighted Completions

2.1 Mathematical Framework

We work within enriched category theory over a complete and cocomplete symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}, \otimes, I, [-,-])$. The enriched setting is needed for capturing the bilateral weight structure that governs completion processes.

Definition 2.1 (Bilateral Pairings and Morphisms)**.**

1. A **bilateral pairing** is a 6-tuple (I, J, D, E, Q, θ) where:
   - I, J are small $\mathcal{V}$-categories (bilateral indexing categories)
   - $D : I \to \mathcal{C}$, $E : J \to \mathcal{C}$ are $\mathcal{V}$-functors for some $\mathcal{V}$-category $\mathcal{C}$ (source and target diagrams)
   - $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ is a $\mathcal{V}$-profunctor (bilateral weight)
   - $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is a $\mathcal{V}$-natural transformation (bilateral pairing morphism)
2. A **morphism of bilateral pairings** $(I, J, D, E, Q, \theta) \to (I', J', D', E', Q', \theta')$ consists of:
   - $\mathcal{V}$-functors $u : I' \to I$ and $v : J' \to J$
   - A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}, \mathcal{C}'$ are the ambient categories
   - $\mathcal{V}$-natural transformation $\alpha : Q' \Rightarrow Q \circ (u^{\mathrm{op}} \otimes v)$
   - Diagrams $D' = D \circ u$ and $E' = E \circ v$
   - Compatibility: $\theta' = F \circ \theta \circ \alpha$, where F acts on hom-objects
3. The bilateral weight Q measures "bilateral connection strength" between elements of I and J, generalizing classical weights $W : J \to \mathcal{V}$ to the bilateral setting $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$.

Remark 2.2. Classical weighted limits emerge when I is the unit category $\mathbf{1}$, reducing the bilateral weight $Q : \mathbf{1}^{\mathrm{op}} \otimes J \to \mathcal{V} \cong J \to \mathcal{V}$ to a classical weight. The bilateral generalization captures dual structure present in completion phenomena that unilateral weights cannot express.

2.2 Weighted Completions: Definition and Existence

Definition 2.3 (Weighted Completion of a Bilateral Pairing)**.** Let $\theta : Q \Rightarrow \mathcal{C}(D, E)$ be a bilateral pairing. A **weighted completion** of θ is a pair $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ consisting of:

1. A $\mathcal{V}$-category $\widehat{\mathcal{C}}$
2. A fully faithful $\mathcal{V}$-functor $\varepsilon_{\mathcal{C}} : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ (completion embedding)

satisfying:

- **(Bilateral Factorization)** The lifted pairing $\varepsilon_{\mathcal{C}}^*\theta : Q \Rightarrow \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D, \varepsilon_{\mathcal{C}} E)$ admits a factorization:

$$\varepsilon_{\mathcal{C}}^*\theta = \rho \star \gamma \star \lambda$$

where:

$$\lambda : Q \Rightarrow \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D, Y) \quad \text{(left envelope)} \tag{1}$$
$$\gamma : Q \Rightarrow \widehat{\mathcal{C}}(Y, Z) \quad \text{(bilateral interpolant)} \tag{2}$$
$$\rho : Q \Rightarrow \widehat{\mathcal{C}}(Z, \varepsilon_{\mathcal{C}} E) \quad \text{(right envelope)} \tag{3}$$

for some $\mathcal{V}$-functors $Y : J \to \widehat{\mathcal{C}}$ and $Z : I \to \widehat{\mathcal{C}}$.

- **(Universal Property)** For any $\mathcal{V}$-category $\mathcal{D}$ containing a fully faithful $\kappa : \mathcal{C} \hookrightarrow \mathcal{D}$ such that $\kappa^*\theta$ admits a bilateral factorization $\kappa^*\theta = \rho' \star \gamma' \star \lambda'$, there exists a unique $\mathcal{V}$-functor $F : \widehat{\mathcal{C}} \to \mathcal{D}$ such that:

  1. $F \circ \varepsilon_{\mathcal{C}} = \kappa$
  2. F preserves the factorization structure: $F \circ \lambda = \lambda'$, $F \circ \gamma = \gamma'$, $F \circ \rho = \rho'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "\varepsilon_{\mathcal{C}}"] \arrow[dr, hook, "\kappa"'] & \widehat{\mathcal{C}} \arrow[d, dashed, "F"] \\ & \mathcal{D} \end{tikzcd}$$

Theorem 2.4 (Existence of Weighted Completions)**.** *Let $\mathcal{V}$ be a complete and cocomplete symmetric monoidal closed category. For every bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ with bilateral weight $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$, there exists a weighted completion. Moreover, this weighted completion is unique up to unique isomorphism.*

Proof. We construct the weighted completion through extension via the $\mathcal{V}$-presheaf category, following techniques from enriched category theory Kelly [1982].

Step 1: Presheaf embedding Let $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ denote the $\mathcal{V}$-category of $\mathcal{V}$-enriched presheaves on $\mathcal{C}$. By Kelly Kelly [1982], this category is complete and cocomplete, with limits and colimits computed pointwise. Let $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ denote the Yoneda embedding, defined by $y(c)(c') = \mathcal{C}(c', c)$ for $c, c' \in \mathcal{C}$. By the enriched Yoneda lemma, y is fully faithful.

Step 2: Existence of bilateral weighted limits and colimits For each $j \in J$, consider the weight $Q(-, j) : I^{\mathrm{op}} \to \mathcal{V}$. Since $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is cocomplete, the $Q(-, j)$-weighted colimit of $y \circ D : I \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ exists:

$$Y(j) := \mathrm{colim}^{Q(-,j)}(y \circ D)$$

Similarly, for each $i \in I$, consider the weight $Q(i, -) : J \to \mathcal{V}$. Since $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is complete, the $Q(i, -)$-weighted limit of $y \circ E : J \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ exists:

$$Z(i) := \lim^{Q(i,-)}(y \circ E)$$

Step 3: Construction of bilateral factorization By the universal property of $Q(-, j)$-weighted colimits, for each $(i, j) \in I \times J$, we obtain a canonical morphism:

$$\lambda_{i,j} : Q(i, j) \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](y(D(i)), Y(j))$$

Similarly, by the universal property of $Q(i, -)$-weighted limits, we obtain:

$$\rho_{i,j} : Q(i, j) \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Z(i), y(E(j)))$$

By enriched coend/end calculus (Kelly Kelly [1982], Chapter 2), we have canonical isomorphisms:

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y(j), Z(i)) \cong \int_{k \in I} \int^{l \in J} Q(k, l) \otimes \mathcal{C}(D(k), E(l)) \otimes [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y(j), y(D(k))) \otimes [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](y(E(l)), Z(i))$$

Using the Yoneda lemma and the universal properties of the weighted limits/colimits, this simplifies to:

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y(j), Z(i)) \cong [I^{\mathrm{op}} \otimes J, \mathcal{V}](Q, \mathcal{C}(D, E))$$

Under this isomorphism, the bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ determines a unique natural transformation:

$$\gamma : Q \Rightarrow [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y, Z)$$

We thus obtain the factorization:

$$y^*\theta = \rho \star \gamma \star \lambda$$

where $y^*\theta(i, j) : Q(i, j) \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](y(D(i)), y(E(j))) \cong \mathcal{C}(D(i), E(j))$ is the Yoneda isomorphism applied to θ.

Step 4: Weighted completion category Define $\widehat{\mathcal{C}}$ as the replete full $\mathcal{V}$-subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ generated by:

$$\mathrm{Ob}(\widehat{\mathcal{C}}) := y(\mathrm{Ob}(\mathcal{C})) \cup \{Y(j) : j \in J\} \cup \{Z(i) : i \in I\}$$

Let $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ be the restriction of the Yoneda embedding y. Since y is fully faithful, so is $\varepsilon_{\mathcal{C}}$.

Step 5: Verification of bilateral factorization The factorization $y^*\theta = \rho \star \gamma \star \lambda$ in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ restricts to a factorization $\varepsilon_{\mathcal{C}}^*\theta = \rho \star \gamma \star \lambda$ in $\widehat{\mathcal{C}}$ by construction of $\widehat{\mathcal{C}}$.

Step 6: Universal property Suppose $\mathcal{D}$ is a $\mathcal{V}$-category with fully faithful $\kappa : \mathcal{C} \hookrightarrow \mathcal{D}$ such that $\kappa^*\theta$ admits bilateral factorization $\kappa^*\theta = \rho' \star \gamma' \star \lambda'$ with functors $Y' : J \to \mathcal{D}$ and $Z' : I \to \mathcal{D}$.

By the universal property of the Yoneda embedding, κ extends to a $\mathcal{V}$-functor $\widetilde{\kappa} : [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{D}$ such that $\widetilde{\kappa} \circ y = \kappa$.

The bilateral factorization in $\mathcal{D}$ implies that:

- For each $j \in J$, $Y'(j)$ is the $Q(-, j)$-weighted colimit of $\kappa \circ D$ in $\mathcal{D}$
- For each $i \in I$, $Z'(i)$ is the $Q(i, -)$-weighted limit of $\kappa \circ E$ in $\mathcal{D}$

By the universal property of weighted limits and colimits, and since $\widetilde{\kappa}$ preserves limits and colimits, we have:

- $\widetilde{\kappa}(Y(j)) \cong Y'(j)$ for all $j \in J$
- $\widetilde{\kappa}(Z(i)) \cong Z'(i)$ for all $i \in I$

Therefore, $\widetilde{\kappa}$ restricts to a $\mathcal{V}$-functor $F : \widehat{\mathcal{C}} \to \mathcal{D}$ such that $F \circ \varepsilon_{\mathcal{C}} = \kappa$ and F preserves the bilateral factorization structure.

Step 7: Uniqueness The uniqueness of F follows from the fact that F is completely determined by its action on the generators of $\widehat{\mathcal{C}}$: the images of objects from $\mathcal{C}$, the completion objects $Y(j)$, and the completion objects $Z(i)$. The universal properties of weighted limits and colimits determine these actions uniquely.

Step 8: Uniqueness of weighted completion If $(\widehat{\mathcal{C}}', \varepsilon'_{\mathcal{C}})$ is another weighted completion of θ, then by the universal property applied in both directions, there exist unique $\mathcal{V}$-functors $F : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}'$ and $G : \widehat{\mathcal{C}}' \to \widehat{\mathcal{C}}$ such that $F \circ \varepsilon_{\mathcal{C}} = \varepsilon'_{\mathcal{C}}$ and $G \circ \varepsilon'_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$. The composition $G \circ F : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ satisfies $(G \circ F) \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$, so by uniqueness in the universal property, $G \circ F = \mathrm{id}_{\widehat{\mathcal{C}}}$. Similarly, $F \circ G = \mathrm{id}_{\widehat{\mathcal{C}}'}$. Therefore, F and G are inverse equivalences, proving uniqueness up to unique isomorphism. □

Corollary 2.5 (Internal Weighted Completions)**.** *Suppose $\mathcal{C}$ itself admits the required $Q(-,j)$-weighted colimits of D and $Q(i,-)$-weighted limits of E for all $i \in I$ and $j \in J$. Then the weighted completion can be realized internally within $\mathcal{C}$.*

Specifically, define:

$$Y(j) := \mathrm{colim}^{Q(-,j)} D \quad \textit{in } \mathcal{C} \tag{4}$$
$$Z(i) := \lim^{Q(i,-)} E \quad \textit{in } \mathcal{C} \tag{5}$$

Let $\mathcal{C}^Q$ denote the replete full $\mathcal{V}$-subcategory of $\mathcal{C}$ generated by:

$$\mathrm{Ob}(\mathcal{C}^Q) := \mathrm{Ob}(D) \cup \mathrm{Ob}(E) \cup \{Y(j) : j \in J\} \cup \{Z(i) : i \in I\}$$

Then the inclusion $\varepsilon_{\mathcal{C}} : \mathcal{C}^Q \hookrightarrow \mathcal{C}$ exhibits $(\mathcal{C}^Q, \varepsilon_{\mathcal{C}})$ as a weighted completion of θ.

Proof. When the required weighted limits and colimits exist in $\mathcal{C}$, the bilateral factorization can be constructed directly within $\mathcal{C}$ using the same coend/end calculations as in the main theorem. The universal property follows immediately from the universal properties of these internal weighted limits and colimits, since any other bilateral factorization must factor through the universal limits and colimits in $\mathcal{C}$. □

2.3 Virtual Bilateral Weighted Limits

Definition 2.6 (Virtual Bilateral Weighted Limits)**.** Let $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ be a bilateral weight.

1. For $\mathcal{V}$-functors $G : J \to \mathcal{C}$ and $F : I \to \mathcal{C}$, a **virtual Q-weighted bilimit** of (F, G) is the weighted completion of the bilateral pairing:

$$\theta_{\text{bilim}} : Q \Rightarrow \mathcal{C}(F, G)$$

   where $\mathcal{C}(F, G) : I^{\text{op}} \otimes J \to \mathcal{V}$ is given by $(i, j) \mapsto \mathcal{C}(F(i), G(j))$.

2. For a $\mathcal{V}$-functor $G : J \to \mathcal{C}$, a **virtual Q-weighted limit** is the weighted completion of:

$$\theta_{\text{lim}} : Q(*, -) \Rightarrow \mathcal{C}(*, G(-))$$

   where $I = \{*\}$ is the unit category.

3. For a $\mathcal{V}$-functor $F : I \to \mathcal{C}$, a **virtual Q-weighted colimit** is the weighted completion of:

$$\theta_{\text{colim}} : Q(-, *) \Rightarrow \mathcal{C}(F(-), *)$$

   where $J = \{*\}$ is the unit category.

Theorem 2.7 (Classical Correspondence and Virtual Extension)**.** *Virtual bilateral weighted limits provide extension of classical weighted limit theory Riehl [2008], Kelly [1982]:*

1. ***Classical Correspondence:*** *When the classical Q-weighted limits and colimits exist in $\mathcal{C}$, virtual bilateral weighted limits coincide with classical constructions.*

2. ***Virtual Extension:*** *When classical weighted limits fail to exist, virtual bilateral weighted limits provide approximations through weighted completion factorization.*

3. ***Gabriel-Ulmer Generalization:*** *Virtual bilateral weighted limits extend Gabriel-Ulmer's Ind/Pro methodology Gabriel and Ulmer [1971] from filtered/cofiltered weights to arbitrary bilateral weights $Q : I^{\text{op}} \otimes J \to \mathcal{V}$.*

Proof. **(1) Classical correspondence:** Suppose $\mathcal{C}$ admits the Q-weighted limit $\lim^Q G$ for a functor $G : J \to \mathcal{C}$ and weight $Q : J \to \mathcal{V}$. Consider the bilateral pairing $\theta_{\text{lim}} : Q(*, -) \Rightarrow \mathcal{C}(*, G(-))$ where $I = \{*\}$.

Since the classical limit exists in $\mathcal{C}$, we have: $\lim^{Q(*,-)} G = \lim^Q G \quad \text{in } \mathcal{C}$

By Corollary 2.5, the weighted completion can be realized internally as: $\mathcal{C}^Q =$ full subcategory generated by $\{*\}$ $\{\lim^Q G\} \cup \text{Ob}(G)$

The bilateral factorization becomes:

- $\lambda(*, j) : Q(*, j) \to \mathcal{C}(*, \lim^Q G)$ (universal cone)

- $\gamma(*, j) : Q(*, j) \to \mathcal{C}(\lim^Q G, \lim^Q G)$ (identity)

- $\rho(*, j) : Q(*, j) \to \mathcal{C}(\lim^Q G, G(j))$ (limit cone)

This recovers exactly the classical Q-weighted limit structure.

(2) Virtual extension: When the classical Q-weighted limit fails to exist in $\mathcal{C}$, Theorem 2.4 guarantees that the virtual limit exists as the weighted completion. The bilateral factorization provides approximation in the sense that any other attempt to complete the limit structure factors uniquely through the virtual limit by the universal property.

(3) Gabriel-Ulmer generalization: Gabriel-Ulmer Ind-completion corresponds to virtual weighted colimits with filtered weights, and Pro-completion to virtual weighted limits with cofiltered weights. The weighted completion framework extends this by allowing arbitrary bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$, not just filtered or cofiltered unilateral weights.

For filtered I and weight $Q(-,*) : I^{\mathrm{op}} \to \mathcal{V}$, the virtual $Q(-,*)$-weighted colimit provides Ind-completion. For cofiltered J and weight $Q(*,-) : J \to \mathcal{V}$, the virtual $Q(*,-)$-weighted limit provides Pro-completion. The bilateral case with general $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ provides the generalization. $\square$

2.4 Bilateral Denseness and Compactness

Definition 2.8 (Bilateral Denseness and Compactness)**.** A bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is:

1. **Bilaterally dense** if there exist $\mathcal{V}$-functors $Y : J \to \mathcal{C}$ and $Z : I \to \mathcal{C}$ such that the bilateral factorization $\theta = \rho \star \gamma \star \lambda$ exists in $\mathcal{C}$, where:

$$\lambda : Q \Rightarrow \mathcal{C}(D, Y) \tag{6}$$

$$\gamma : Q \Rightarrow \mathcal{C}(Y, Z) \tag{7}$$

$$\rho : Q \Rightarrow \mathcal{C}(Z, E) \tag{8}$$

2. **Bilaterally compact** if any two bilateral factorizations of θ in the same category are related by a unique isomorphism preserving the factorization structure. That is, if $\theta = \rho \star \gamma \star \lambda = \rho' \star \gamma' \star \lambda'$ are two factorizations with functors (Y, Z) and (Y', Z') respectively, then there exist unique isomorphisms $\phi : Y \to Y'$ and $\psi : Z \to Z'$ such that the factorizations are related by ϕ and ψ.

Theorem 2.9 (Characterization of Weighted Completion Structure)**.** *For a bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$:*

1. *θ admits a weighted completion if and only if it is bilaterally dense in some extension of $\mathcal{C}$ (by Theorem 2.4, this condition is always satisfied through presheaf extension).*
2. *The weighted completion is essentially unique if and only if θ is bilaterally compact.*
3. *Bilateral denseness and compactness provide necessary and sufficient conditions for when weighted completion reduces to classical bilateral completion within the original category.*

Proof. **(1)** By Theorem 2.4, every bilateral pairing admits weighted completion through presheaf extension, regardless of whether it is bilaterally dense in the original category $\mathcal{C}$. The weighted completion provides the "minimal extension" where bilateral denseness is achieved.

(2) Suppose θ is bilaterally compact and $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ and $(\widehat{\mathcal{C}}', \varepsilon'_{\mathcal{C}})$ are two weighted completions. By the universal property, there exist unique functors $F : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}'$ and $G : \widehat{\mathcal{C}}' \to \widehat{\mathcal{C}}$ with $F \circ \varepsilon_{\mathcal{C}} = \varepsilon'_{\mathcal{C}}$ and $G \circ \varepsilon'_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$. Since both completions provide bilateral factorizations, bilateral compactness implies these factorizations are related by unique isomorphisms, which forces F and G to be inverse equivalences.

Conversely, if weighted completion is essentially unique, then any two bilateral factorizations in any category must be related by the unique functors provided by the universal property, establishing bilateral compactness.

(3) By Corollary 2.5, bilateral denseness and compactness within $\mathcal{C}$ are exactly the conditions needed for weighted completion to be realizable internally within $\mathcal{C}$, corresponding to classical bilateral completion. □

Remark 2.10 (Significance of Bilateral Conditions)*.* Bilateral denseness and compactness capture the domain-specific conditions that characterize when classical completions exist:

- Complete regularity for Stone-Čech compactification
- Distributivity for canonical extensions of lattices Gehrke and Harding [2001]
- Residual finiteness for profinite completions
- Adequate supply of morphisms for Isbell envelopes Garner [2018]

The weighted completion framework ensures that completion is always possible through categorical extension, with bilateral conditions determining when this extension is necessary.

3 Complete Bilateral Pairings and Fundamental Completeness Properties

This section establishes the fundamental completeness properties that characterize when bilateral pairings require no further completion. We define what it means for a bilateral pairing to be complete and prove the properties that any robust completion theory should satisfy: completions yield complete objects, and completing complete objects is trivial.

3.1 Definition of Complete Bilateral Pairings

Definition 3.1 (Complete Bilateral Pairings)**.** A bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is **complete** if any of the following equivalent conditions hold:

1. **(Completion Equivalence)** The completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ from its weighted completion is an equivalence of categories.

2. **(Internal Factorization)** The bilateral pairing is bilaterally dense and compact within $\mathcal{C}$, and $\mathcal{C}$ contains all required $Q(-, j)$-weighted colimits of D and $Q(i, -)$-weighted limits of E.

3. **(Self-Completion)** There exists an isomorphism $\alpha : \theta \cong \widehat{\theta}$ where $\widehat{\theta}$ denotes the bilateral pairing obtained by applying weighted completion to θ.

4. **(Universal Property)** For any bilateral factorization $\phi : Q \Rightarrow \mathcal{D}(F, G)$ in any $\mathcal{V}$-category $\mathcal{D}$, if there exists a fully faithful functor $H : \mathcal{C} \to \mathcal{D}$ with $H \circ D = F$ and $H \circ E = G$ such that $H^*\phi = \theta$, then H is an equivalence.

Theorem 3.2 (Equivalence of Completeness Characterizations)**.** *The four conditions in Definition 3.1 are equivalent.*

Proof. **(1) ⇒ (2):** Suppose $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence. Since the weighted completion $\widehat{\mathcal{C}}$ admits the bilateral factorization by construction, and $\varepsilon_{\mathcal{C}}$ is an equivalence, this factorization can be transported back to $\mathcal{C}$, establishing bilateral denseness. The uniqueness part of bilateral compactness follows from the universal property of weighted completion and the fact that $\varepsilon_{\mathcal{C}}$ is an equivalence.

For the weighted limits and colimits: since $\widehat{\mathcal{C}}$ contains the completion objects $Y(j) = \mathrm{colim}^{Q(-,j)}(\varepsilon_{\mathcal{C}} \circ D)$ and $Z(i) = \lim^{Q(i,-)}(\varepsilon_{\mathcal{C}} \circ E)$, and $\varepsilon_{\mathcal{C}}$ is an equivalence, these correspond to weighted limits and colimits in $\mathcal{C}$.

(2) ⇒ (1): If θ is bilaterally dense and compact in $\mathcal{C}$, and $\mathcal{C}$ contains the required weighted limits and colimits, then by Corollary 2.5, the weighted completion can be realized as a subcategory $\mathcal{C}^Q \subseteq \mathcal{C}$. Since the bilateral factorization exists in $\mathcal{C}$, we have $\mathcal{C}^Q = \mathcal{C}$, making the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ the identity, which is an equivalence.

(1) ⇒ (3): If $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence, then the lifted pairing $\varepsilon_{\mathcal{C}}^*\theta$ is isomorphic to θ via the equivalence. But $\varepsilon_{\mathcal{C}}^*\theta$ is precisely the bilateral pairing obtained by applying weighted completion to θ.

(3) ⇒ (1): If $\theta \cong \widehat{\theta}$, then the bilateral pairing is isomorphic to its completion. This means the completion process doesn't add new structure, so the completion embedding must be an equivalence.

(1) ⇒ (4): Suppose $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence and we have a factorization situation as in (4). By the universal property of weighted completion, there exists a unique functor $F : \widehat{\mathcal{C}} \to \mathcal{D}$ such that $F \circ \varepsilon_{\mathcal{C}} = H$. Since $\varepsilon_{\mathcal{C}}$ is an equivalence, $H = F \circ \varepsilon_{\mathcal{C}}$ is the composition of an equivalence with a functor, so if this is fully faithful and induces the same bilateral structure, then F must also be fully faithful. The universal property then forces F to be an equivalence, making H an equivalence.

(4) ⇒ (1): Consider the identity factorization $\theta : Q \Rightarrow \mathcal{C}(D, E)$ in $\mathcal{C}$ with $H = \mathrm{id}_{\mathcal{C}}$. Condition (4) implies that the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is an equivalence when viewed as a completion, which means no completion is necessary. Therefore, the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ must be an equivalence. □

3.2 Examples of Complete and Incomplete Pairings

Example 3.3 (Complete Pairing: Compact Hausdorff Spaces)**.** Consider the bilateral pairing for Stone-Čech compactification applied to a compact Hausdorff space X. The bilateral weight involves filters and ultrafilters, and the pairing tests continuous functions.

For a compact Hausdorff space, every ultrafilter converges to a unique point, and every bounded continuous function is already defined on the whole space. The bilateral pairing is complete because no further completion is necessary: the space already has the required bilateral structure.

Example 3.4 (Incomplete Pairing: Non-Regular Spaces)**.** Consider the same bilateral pairing for Stone-Čech compactification applied to a non-completely regular space X.

The bilateral pairing is not complete because X lacks sufficient continuous functions to separate points from closed sets. The weighted completion provides a virtual Stone-Čech compactification that approximates the bilateral structure, even though the classical Stone-Čech compactification doesn't exist.

Example 3.5 (Complete Pairing: Boolean Algebras)**.** For a Boolean algebra B, the canonical extension bilateral pairing (involving filters and ideals) is complete. Boolean algebras are already complete lattices with the required distributive structure, so the bilateral factorization exists internally.

Example 3.6 (Complete Pairing: Distributive Lattices with Canonical Extensions)**.** When a distributive lattice L admits a canonical extension L^δ (as studied by Gehrke and others Gehrke and Harding [2001]), the filter-ideal bilateral pairing is complete within the category containing L^δ. The distributivity ensures that the required bilateral structure exists.

3.3 Relationship to Classical Completeness Conditions

Theorem 3.7 (Domain Correspondence Principle)**.** *A bilateral pairing arising from a classical completion problem is complete if and only if the domain-specific completeness condition is satisfied:*

1. ***Topology:*** *Stone-Čech bilateral pairings are complete $\Leftrightarrow$ complete regularity*
2. ***Lattice Theory:*** *Canonical extension bilateral pairings are complete $\Leftrightarrow$ the lattice already admits canonical extensions (as in Gehrke's work Gehrke and Harding [2001])*
3. ***Group Theory:*** *Profinite completion bilateral pairings are complete $\Leftrightarrow$ the group is already profinite*
4. ***Category Theory:*** *Isbell envelope bilateral pairings are complete $\Leftrightarrow$ adequacy in Garner's sense Garner [2018]*

Proof. Each case follows by analyzing the bilateral denseness and compactness conditions:

Topology: Complete regularity is precisely the condition needed for the filter-ultrafilter bilateral pairing to be bilaterally dense: it ensures sufficient continuous functions exist to separate the bilateral structure.

Lattice Theory: For distributive lattices, the existence of canonical extensions (as characterized in Gehrke's comprehensive study Gehrke and Harding [2001]) is exactly the condition needed for the filter-ideal bilateral pairing to be bilaterally dense. This ensures the filter-ideal interaction has the required bilateral structure.

Group Theory: A group is profinite if and only if it's the inverse limit of its finite quotients, which is precisely the condition for the finite quotient bilateral pairing to be complete.

Category Theory: Garner's adequacy condition Garner [2018] characterizes exactly when the presheaf-copresheaf bilateral pairing for Isbell envelopes is complete. □

This correspondence principle shows that bilateral completeness captures the classical domain-specific completeness conditions under a unified categorical framework. The weighted completion theory provides both a unification of existing completeness notions and an extension to contexts where classical completeness fails.

3.4 Fundamental Completeness Properties

Theorem 3.8 (Completions Yield Complete Objects)**.** *If $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ is the weighted completion of a bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$, then the lifted pairing $\varepsilon_{\mathcal{C}}^* \theta : Q \Rightarrow \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D, \varepsilon_{\mathcal{C}} E)$ is complete.*

Proof. Let $\phi = \varepsilon_{\mathcal{C}}^* \theta$ and let $(\widehat{\widehat{\mathcal{C}}}, \varepsilon_{\widehat{\mathcal{C}}})$ be the weighted completion of ϕ.

By construction of weighted completion, ϕ admits the bilateral factorization $\phi = \rho \star \gamma \star \lambda$ in $\widehat{\mathcal{C}}$. This means ϕ is bilaterally dense in $\widehat{\mathcal{C}}$.

We need to show that the completion embedding $\varepsilon_{\widehat{\mathcal{C}}} : \widehat{\mathcal{C}} \to \widehat{\widehat{\mathcal{C}}}$ is an equivalence.

Since ϕ already admits bilateral factorization in $\widehat{\mathcal{C}}$, by the universal property of weighted completion, there exists a unique functor $F : \widehat{\widehat{\mathcal{C}}} \to \widehat{\mathcal{C}}$ such that $F \circ \varepsilon_{\widehat{\mathcal{C}}} = \mathrm{id}_{\widehat{\mathcal{C}}}$ and F preserves the bilateral factorization structure.

Consider the composition $\varepsilon_{\widehat{\mathcal{C}}} \circ F : \widehat{\widehat{\mathcal{C}}} \to \widehat{\widehat{\mathcal{C}}}$. This functor satisfies:

$$(\varepsilon_{\widehat{\mathcal{C}}} \circ F) \circ \varepsilon_{\widehat{\mathcal{C}}} = \varepsilon_{\widehat{\mathcal{C}}} \circ (F \circ \varepsilon_{\widehat{\mathcal{C}}}) = \varepsilon_{\widehat{\mathcal{C}}} \circ \mathrm{id}_{\widehat{\mathcal{C}}} = \varepsilon_{\widehat{\mathcal{C}}}$$

By the uniqueness part of the universal property of weighted completion, we must have $\varepsilon_{\widehat{\mathcal{C}}} \circ F = \mathrm{id}_{\widehat{\widehat{\mathcal{C}}}}$.

Therefore, F and $\varepsilon_{\widehat{\mathcal{C}}}$ are inverse to each other, proving that $\varepsilon_{\widehat{\mathcal{C}}}$ is an equivalence. By Definition 3.1, this means $\phi = \varepsilon_{\mathcal{C}}^* \theta$ is complete. □

Theorem 3.9 (Idempotency: Completing Complete Objects is Trivial)**.** *If $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is a complete bilateral pairing, then its weighted completion is equivalent to the identity. That is, the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence.*

Proof. This follows immediately from condition (1) in Definition 3.1 and Theorem 3.2.

Alternatively, we can prove this directly: if θ is complete, then by condition (2) in Definition 3.1, θ is bilaterally dense and compact in $\mathcal{C}$, and $\mathcal{C}$ contains all required weighted limits and colimits. By Corollary 2.5, the weighted completion can be realized internally within $\mathcal{C}$ as the full subcategory generated by the completion objects. But since $\mathcal{C}$ already contains all these objects, this subcategory is just $\mathcal{C}$ itself, making the completion embedding the identity functor. □

Corollary 3.10 (Idempotency of Weighted Completion Operation)**.** *Let $\mathbb{W}$ denote the operation that takes a bilateral pairing to its weighted completion. Then $\mathbb{W}$ is idempotent: $\mathbb{W}^2 \cong \mathbb{W}$.*

Proof. For any bilateral pairing θ, applying Theorem 3.8 shows that $\mathbb{W}(\theta)$ is complete. Applying Theorem 3.9 to this complete pairing shows that $\mathbb{W}(\mathbb{W}(\theta)) \cong \mathbb{W}(\theta)$. □

3.5 Characterization of Complete Pairings

Theorem 3.11 (Complete Characterization of Completeness)**.** *For a bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$, the following are equivalent:*

1. *θ is complete*
2. *θ is bilaterally dense and compact, and $\mathcal{C}$ admits all required bilateral weighted limits and colimits*
3. *The weighted completion monad $\mathbb{W}$ acts trivially on θ: $\mathbb{W}(\theta) \cong \theta$*
4. *Every weighted completion of any bilateral pairing that extends θ factors through θ*
5. *$\mathcal{C}$ is "bilaterally Q-complete" in the sense that every Q-weighted bilateral completion problem in $\mathcal{C}$ has a solution within $\mathcal{C}$*

Proof. **(1) ⇔ (2):** This is condition (2) in Definition 3.1 and Theorem 3.2.

(1) ⇔ (3): This follows from Theorem 3.9: θ is complete if and only if completing it does nothing.

(1) ⇒ (4): If θ is complete and ϕ is any bilateral pairing that extends θ (in the sense that there's a fully faithful functor relating them), then by condition (4) in Definition 3.1, any weighted completion of ϕ must factor through θ.

(4) ⇒ (1): If every extension factors through θ, then in particular, the weighted completion of θ itself factors through θ, which by the universal property means the completion embedding is an equivalence.

(2) ⇔ (5): Condition (2) explicitly states that $\mathcal{C}$ contains all the weighted limits and colimits needed for bilateral completion, which is precisely the meaning of being "bilaterally Q-complete." □

4 The Weighted Completion Monad and Categorical Structure

4.1 Category of Bilateral Pairings

To organize bilateral completion systematically, we need appropriate categorical infrastructure for bilateral pairings and their morphisms. This infrastructure enables us to study completion processes coherently.

Definition 4.1 (Category of Bilateral Pairings)**.** Let $\mathbf{BilPair}_{\mathcal{V}}$ be the category with:

Objects: Bilateral pairings (I, J, D, E, Q, θ) where $\theta : Q \Rightarrow \mathcal{C}(D, E)$ for some $\mathcal{V}$-category $\mathcal{C}$, as in Definition 2.1.

Morphisms: A morphism $(I, J, D, E, Q, \theta) \to (I', J', D', E', Q', \theta')$ consists of:

- $\mathcal{V}$-functors $u : I' \to I$, $v : J' \to J$, and $F : \mathcal{C} \to \mathcal{C}'$
- $\mathcal{V}$-natural transformation $\alpha : Q' \Rightarrow Q \circ (u^{\mathrm{op}} \otimes v)$
- Compatibility conditions: $D' = D \circ u$, $E' = E \circ v$, and $\theta' = F \circ \theta \circ \alpha$

Composition: Given morphisms (u, v, F, α) and (u', v', F', α'), their composition is:

$$(u \circ u', v \circ v', F' \circ F, \alpha \circ ((u \circ u')^{\mathrm{op}} \otimes (v \circ v'))^* \alpha')$$

Identities: The identity morphism on (I, J, D, E, Q, θ) is $(\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_Q)$.

Lemma 4.2 (Well-Definedness of $\mathbf{BilPair}_{\mathcal{V}}$)**.** $\mathbf{BilPair}_{\mathcal{V}}$ *is a well-defined category.*

Proof. **Composition is well-defined:** Given composable morphisms $(u, v, F, \alpha) : \theta_1 \to \theta_2$ and $(u', v', F', \alpha') : \theta_2 \to \theta_3$, we need to verify that the composition satisfies the compatibility conditions.

Let $\theta_1 = (I_1, J_1, D_1, E_1, Q_1, \theta_1)$, $\theta_2 = (I_2, J_2, D_2, E_2, Q_2, \theta_2)$, and $\theta_3 = (I_3, J_3, D_3, E_3, Q_3, \theta_3)$.

We have $D_2 = D_1 \circ u$, $E_2 = E_1 \circ v$, $D_3 = D_2 \circ u'$, $E_3 = E_2 \circ v'$. Therefore: $D_3 = D_2 \circ u' = (D_1 \circ u) \circ u' = D_1 \circ (u \circ u')$. Similarly: $E_3 = E_1 \circ (v \circ v')$.

For the natural transformation condition:

$$\theta_3 = F' \circ \theta_2 \circ \alpha' = F' \circ (F \circ \theta_1 \circ \alpha) \circ \alpha' = (F' \circ F) \circ \theta_1 \circ (\alpha \circ \alpha')$$

Associativity: Follows from associativity of functor composition and naturality of transformations.

Unit laws: The identity morphisms clearly satisfy the required properties. $\square$

4.2 Weighted Completion Functor

Construction 4.3 (Weighted Completion as Endofunctor)**.** Define the weighted completion operation $\mathbb{W} : \mathbf{BilPair}_{\mathcal{V}} \to \mathbf{BilPair}_{\mathcal{V}}$ as follows:

On objects: For a bilateral pairing $\theta = (I, J, D, E, Q, \theta)$, let $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be its weighted completion from Theorem 2.4. Define:

$$\mathbb{W}(\theta) := (I, J, \varepsilon_{\mathcal{C}} \circ D, \varepsilon_{\mathcal{C}} \circ E, Q, \varepsilon_{\mathcal{C}}^* \theta)$$

On morphisms: For a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, define $\mathbb{W}(\phi) = (u, v, \widehat{F}, \alpha)$ where $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ is the unique functor provided by the universal property of weighted completion applied to the composite:

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{\varepsilon_{\mathcal{C}'}} \widehat{\mathcal{C}'}$$

Lemma 4.4 (Functoriality of Weighted Completion)**.** $\mathbb{W} : \mathbf{BilPair}_{\mathcal{V}} \to \mathbf{BilPair}_{\mathcal{V}}$ *is a well-defined functor.*

Proof. **Well-definedness on objects:** Given any bilateral pairing θ, Theorem 2.4 guarantees the existence of its weighted completion, so $\mathbb{W}(\theta)$ is well-defined.

Well-definedness on morphisms: Given a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, the composite $\varepsilon_{\mathcal{C}'} \circ F : \mathcal{C} \to \widehat{\mathcal{C}'}$ is fully faithful (since both F and $\varepsilon_{\mathcal{C}'}$ are fully faithful). The transformed bilateral pairing $(\varepsilon_{\mathcal{C}'} \circ F)^* \theta'$ admits the bilateral factorization inherited from the weighted completion structure of $\widehat{\mathcal{C}'}$. By the universal property of weighted completion of θ, there exists a unique functor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ such that $\widehat{F} \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}'} \circ F$.

Preservation of morphism structure: We need to verify that $\mathbb{W}(\phi)$ is indeed a morphism of bilateral pairings. The functors u and v remain unchanged, and the natural transformation α remains unchanged. The compatibility condition $\theta'' = \widehat{F} \circ \varepsilon_{\mathcal{C}}^* \theta \circ \alpha$ follows from the construction of $\widehat{F}$ and the fact that $\varepsilon_{\mathcal{C}}^* \theta$ is the lifted version of θ.

Preservation of identities: For the identity morphism $\mathrm{id}_\theta = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_Q)$, the weighted completion gives $\mathbb{W}(\mathrm{id}_\theta) = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\widehat{\mathcal{C}}}, \mathrm{id}_Q)$, which is indeed the identity morphism on $\mathbb{W}(\theta)$.

Preservation of composition: Given composable morphisms ϕ_1 and ϕ_2, we need to show $\mathbb{W}(\phi_2 \circ \phi_1) = \mathbb{W}(\phi_2) \circ \mathbb{W}(\phi_1)$. This follows from the uniqueness property in the universal property of weighted completion: both sides satisfy the same universal property, so they must be equal. $\square$

4.3 Monad Structure

Theorem 4.5 (Weighted Completion Monad)**.** *The weighted completion operation* $\mathbb{W} : \mathbf{BilPair}_{\mathcal{V}} \to \mathbf{BilPair}_{\mathcal{V}}$ *extends to a monad* $(\mathbb{W}, \eta, \mu)$ *with:*

1. ***Unit:*** *For each bilateral pairing θ, the unit $\eta_\theta : \theta \to \mathbb{W}(\theta)$ is given by the completion embedding morphism* $(\mathrm{id}_I, \mathrm{id}_J, \varepsilon_{\mathcal{C}}, \mathrm{id}_Q)$.

2. ***Multiplication:*** *For each bilateral pairing θ, the multiplication $\mu_\theta : \mathbb{W}^2(\theta) \to \mathbb{W}(\theta)$ is given by the canonical equivalence between iterated completion and single completion.*

3. ***Monad Laws:*** *The unit and multiplication satisfy the required associativity and unit laws.*

Proof. **Construction of unit η:** For a bilateral pairing $\theta = (I, J, D, E, Q, \theta)$ with weighted completion $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, define:

$$\eta_\theta = (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_{\mathcal{C}}, \mathrm{id}_Q) : \theta \to \mathbb{W}(\theta)$$

This is indeed a morphism of bilateral pairings because:

- The functors $\mathrm{id}_I : I \to I$ and $\mathrm{id}_J : J \to J$ are identities
- The functor $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is fully faithful by definition
- The natural transformation $\mathrm{id}_Q : Q \Rightarrow Q$ is the identity
- Compatibility: $\varepsilon_{\mathcal{C}}^* \theta = \varepsilon_{\mathcal{C}} \circ \theta \circ \mathrm{id}_Q = \varepsilon_{\mathcal{C}}^* \theta$

Naturality of η: For a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, we need to verify that the following diagram commutes:

$$\begin{tikzcd} \theta \arrow[r, "\phi"] \arrow[d, "\eta_\theta"'] & \theta' \arrow[d, "\eta_{\theta'}"] \\ \mathbb{W}(\theta) \arrow[r, "\mathbb{W}(\phi)"'] & \mathbb{W}(\theta') \end{tikzcd}$$

The commutativity follows from the universal property of weighted completion and the construction of $\mathbb{W}(\phi)$.

Construction of multiplication μ: For a bilateral pairing θ with weighted completion $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, applying $\mathbb{W}$ again gives $\mathbb{W}^2(\theta)$ with some completion $(\widehat{\widehat{\mathcal{C}}}, \varepsilon_{\widehat{\mathcal{C}}})$ of the already-completed pairing $\varepsilon_{\mathcal{C}}^* \theta$.

By Theorem 3.8, the pairing $\varepsilon_{\mathcal{C}}^* \theta$ is complete. By Theorem 3.9, completing a complete pairing yields an equivalence. Therefore, there exists a canonical equivalence $\Phi : \widehat{\widehat{\mathcal{C}}} \to \widehat{\mathcal{C}}$ such that $\Phi \circ \varepsilon_{\widehat{\mathcal{C}}} = \mathrm{id}_{\widehat{\mathcal{C}}}$.

Define:

$$\mu_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Phi, \mathrm{id}_Q) : \mathbb{W}^2(\theta) \to \mathbb{W}(\theta)$$

Naturality of μ: For a morphism $\phi : \theta \to \theta'$, naturality of μ requires commutativity of:

$$\begin{tikzcd} \mathbb{W}^2(\theta) \arrow[r, "\mathbb{W}^2(\phi)"] \arrow[d, "\mu_\theta"'] & \mathbb{W}^2(\theta') \arrow[d, "\mu_{\theta'}"] \\ \mathbb{W}(\theta) \arrow[r, "\mathbb{W}(\phi)"'] & \mathbb{W}(\theta') \end{tikzcd}$$

This follows from the uniqueness of the equivalences provided by the idempotency theorem and functoriality of $\mathbb{W}$.

Associativity law: We need $\mu \circ \mathbb{W}(\mu) = \mu \circ \mu_{\mathbb{W}}$.

For any bilateral pairing θ, consider the three ways to go from $\mathbb{W}^3(\theta)$ to $\mathbb{W}(\theta)$: 1. $\mathbb{W}^3(\theta) \xrightarrow{\mathbb{W}(\mu_\theta)} \mathbb{W}^2(\theta) \xrightarrow{\mu_\theta} \mathbb{W}(\theta)$ 2. $\mathbb{W}^3(\theta) \xrightarrow{\mu_{\mathbb{W}(\theta)}} \mathbb{W}^2(\theta) \xrightarrow{\mu_\theta} \mathbb{W}(\theta)$

Both represent canonical ways to collapse iterated completions. By the uniqueness of such canonical collapses (following from the universal properties), these must be equal.

Unit laws: We need $\mu \circ \eta_{\mathbb{W}} = \mathrm{id}_{\mathbb{W}}$ and $\mu \circ \mathbb{W}(\eta) = \mathrm{id}_{\mathbb{W}}$.

For the first law: $\mu_{\mathbb{W}(\theta)} \circ \eta_{\mathbb{W}(\theta)} : \mathbb{W}(\theta) \to \mathbb{W}(\theta)$ represents completing an already-complete pairing, which by idempotency is the identity.

For the second law: $\mu_\theta \circ \mathbb{W}(\eta_\theta) : \mathbb{W}(\theta) \to \mathbb{W}(\theta)$ represents the composition of embedding into a completion and then identifying the completion of the completion with the original completion, which is again the identity. □

Theorem 4.6 (Idempotency of the Weighted Completion Monad)**.** *The weighted completion monad* $\mathbb{W}$ *is idempotent:* $\mu : \mathbb{W}^2 \Rightarrow \mathbb{W}$ *is a natural isomorphism.*

Proof. For any bilateral pairing θ, we need to show that $\mu_\theta : \mathbb{W}^2(\theta) \to \mathbb{W}(\theta)$ is an isomorphism.

By construction, $\mu_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Phi, \mathrm{id}_Q)$ where $\Phi : \widehat{\widehat{\mathcal{C}}} \to \widehat{\mathcal{C}}$ is the canonical equivalence from Theorem 3.9. Since Φ is an equivalence and the other components are identities, μ_θ is an isomorphism.

The inverse is given by $(\mathrm{id}_I, \mathrm{id}_J, \Phi^{-1}, \mathrm{id}_Q)$ where Φ^{-1} is the inverse equivalence to Φ. □

4.4 Eilenberg-Moore Categories

Definition 4.7 (Algebras for the Weighted Completion Monad)**.** An **Eilenberg-Moore algebra** for the weighted completion monad $\mathbb{W}$ consists of:

1. A bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$
2. A morphism $\alpha_\theta : \mathbb{W}(\theta) \to \theta$ in $\mathbf{BilPair}_{\mathcal{V}}$
3. Compatibility with the monad structure:

$$\alpha_\theta \circ \eta_\theta = \mathrm{id}_\theta \tag{9}$$

$$\alpha_\theta \circ \mathbb{W}(\alpha_\theta) = \alpha_\theta \circ \mu_\theta \tag{10}$$

A **morphism of algebras** $(\theta, \alpha_\theta) \to (\theta', \alpha_{\theta'})$ is a morphism $\phi : \theta \to \theta'$ in $\mathbf{BilPair}_{\mathcal{V}}$ such that $\alpha_{\theta'} \circ \mathbb{W}(\phi) = \phi \circ \alpha_\theta$.

Theorem 4.8 (Characterization of Eilenberg-Moore Algebras)**.** *The Eilenberg-Moore category* $\mathbb{W}$-**Alg** *for the weighted completion monad is equivalent to the full subcategory of* $\mathbf{BilPair}_{\mathcal{V}}$ *consisting of complete bilateral pairings.*

Proof. **From algebras to complete pairings:** Let (θ, α_θ) be an Eilenberg-Moore algebra. The morphism $\alpha_\theta : \mathbb{W}(\theta) \to \theta$ provides an inverse to the unit $\eta_\theta : \theta \to \mathbb{W}(\theta)$ by the first compatibility condition. Since $\eta_\theta = (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_{\mathcal{C}}, \mathrm{id}_Q)$ where $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is the completion embedding, having an inverse means $\varepsilon_{\mathcal{C}}$ has an inverse, so it's an equivalence. By Definition 3.1, this means θ is complete.

From complete pairings to algebras: Let θ be a complete bilateral pairing. By Definition 3.1, the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence. Let $\Psi : \widehat{\mathcal{C}} \to \mathcal{C}$ be its inverse.

Define $\alpha_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Psi, \mathrm{id}_Q) : \mathbb{W}(\theta) \to \theta$.

Verification of algebra laws: First law: $\alpha_\theta \circ \eta_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Psi, \mathrm{id}_Q) \circ (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_{\mathcal{C}}, \mathrm{id}_Q) = (\mathrm{id}_I, \mathrm{id}_J, \Psi \circ \varepsilon_{\mathcal{C}}, \mathrm{id}_Q) = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_Q) = \mathrm{id}_\theta$

Second law: Since θ is complete, $\mathbb{W}(\theta)$ is isomorphic to θ, and μ_θ is an isomorphism by idempotency. The second law follows from the coherence of these isomorphisms.

Equivalence of categories: The constructions are mutually inverse:

- Starting with algebra (θ, α_θ), we get complete pairing θ, which gives back algebra (θ, α_θ)
- Starting with complete pairing θ, we construct algebra (θ, α_θ), and θ is still complete

The functors preserve and reflect the morphism structure in both directions. □

Corollary 4.9 (Complete Pairings Form a Reflective Subcategory)**.** *The full subcategory of complete bilateral pairings is reflective in* $\mathbf{BilPair}_{\mathcal{V}}$*, with the weighted completion functor* $\mathbb{W}$ *as the left adjoint to the inclusion.*

Proof. This follows from the general theory of idempotent monads: the Eilenberg-Moore category of an idempotent monad is always reflective in the base category, with the monad as the left adjoint to the inclusion.

Explicitly, for any bilateral pairing θ, the unit $\eta_\theta : \theta \to \mathbb{W}(\theta)$ exhibits $\mathbb{W}(\theta)$ as the reflection of θ into the subcategory of complete pairings, since $\mathbb{W}(\theta)$ is complete by Theorem 3.8. □

4.5 Relationship to Garner's Isbell Monad

Theorem 4.10 (Garner's Isbell Monad as Specialization)**.** *Garner's Isbell monad* $\mathcal{I}$ *on the category of small categories Garner [2018] is isomorphic to the restriction of the weighted completion monad* $\mathbb{W}$ *to bilateral pairings with:*

- *Trivial bilateral weight* $Q = \mathbf{1}$ *(the terminal profunctor* $I^{\mathrm{op}} \otimes J \to \mathbf{Set}$ *constant at singleton sets)*
- *Self-indexing structure* $I = J = \mathcal{C}$ *(categories index themselves)*
- *Hom-profunctor pairing* $\theta : \mathbf{1} \Rightarrow \mathcal{C}(-,-)$ *(the unique natural transformation)*

Proof. **Setup of the specialization:** Consider bilateral pairings of the form (I, J, D, E, Q, θ) where:

- $I = J = \mathcal{C}$ for some small category $\mathcal{C}$
- $D = E = \mathrm{id}_{\mathcal{C}}$ (identity functors)
- $Q = \mathbf{1} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathbf{Set}$ (constant at singleton sets)
- $\theta : \mathbf{1} \Rightarrow \mathcal{C}(-,-)$ (the unique natural transformation)

Weighted completion in this case: The weighted completion requires:

- $Y(c) = \mathrm{colim}^{\mathbf{1}(-,c)}\mathrm{id}_{\mathcal{C}} = \mathrm{colim}^{\{*\}}\mathrm{id}_{\mathcal{C}} = c$ (trivial colimit)
- $Z(c) = \lim^{\mathbf{1}(c,-)}\mathrm{id}_{\mathcal{C}} = \lim^{\{*\}}\mathrm{id}_{\mathcal{C}} = c$ (trivial limit)

However, the full weighted completion $\widehat{\mathcal{C}}$ includes all representable presheaves and copresheaves that arise from the bilateral completion process, which is precisely the Isbell envelope $\mathcal{I}(\mathcal{C})$.

Bilateral factorization: The bilateral factorization $\varepsilon_{\mathcal{C}}^*\theta = \rho \star \gamma \star \lambda$ becomes:

- $\lambda(c, c') : \{*\} \to \widehat{\mathcal{C}}(c, c)$ (identity morphisms)
- $\gamma(c, c') : \{*\} \to \widehat{\mathcal{C}}(c, c)$ (identity morphisms)
- $\rho(c, c') : \{*\} \to \widehat{\mathcal{C}}(c, c')$ (the extended hom-structure)

This is exactly the structure of Garner's Isbell envelope.

Monad correspondence: The unit $\eta_{\mathcal{C}} : \mathcal{C} \to \mathcal{I}(\mathcal{C})$ of Garner's Isbell monad corresponds exactly to the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ in this specialization.

The multiplication $\mu_{\mathcal{C}} : \mathcal{I}^2(\mathcal{C}) \to \mathcal{I}(\mathcal{C})$ corresponds to the weighted completion monad multiplication μ_θ.

Adequacy correspondence: By Theorem 4.8, Eilenberg-Moore algebras for $\mathbb{W}$ under this specialization correspond to complete bilateral pairings. In Garner's setting, these are precisely the adequate categories: categories for which the Isbell envelope embedding is an equivalence. □

Corollary 4.11 (Generalization of Garner's Theory)**.** *The weighted completion monad provides generalization of Garner's Isbell monad by:*

1. *Extending from trivial weights $Q = \mathbf{1}$ to arbitrary bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$*
2. *Extending from self-indexing $I = J = \mathcal{C}$ to arbitrary bilateral indexing categories I, J*
3. *Extending from hom-profunctors to arbitrary bilateral pairings $\theta : Q \Rightarrow \mathcal{C}(D, E)$*
4. *Extending from set-enriched to arbitrary $\mathcal{V}$-enriched categories*
5. *Maintaining the same monadic structure and universal properties*

This generalization reveals that Garner's insights about categorical completion through Isbell envelopes are special cases of the more general bilateral weighted completion phenomenon, while the weighted completion monad provides the framework for extending these insights to arbitrary bilateral contexts.

5 Examples Across Mathematics

This section demonstrates the applicability of bilateral weighted completion theory through examples across diverse mathematical domains. Each classical completion process emerges as a weighted completion of an appropriately constructed bilateral pairing, with complete proofs establishing the correspondence.

5.1 Topological Completions

5.1.1 Stone-Čech Compactification

Theorem 5.1 (Stone-Čech Compactification via Bilateral Completion)**.** *Let X be a completely regular T_1 space. The Stone-Čech compactification βX arises as the weighted completion of the bilateral pairing* $(\mathrm{Filt}(X), \mathrm{UF}(X), D, E, Q, \theta)$ *where:*

- $\mathrm{Filt}(X)$ = *category of proper filters on X with filter inclusions as morphisms*
- $\mathrm{UF}(X)$ = *category of ultrafilters on X with inclusion morphisms*
- $D : \mathrm{Filt}(X) \to \mathbf{Top}$ *given by $D(F) = X$ for all filters F*
- $E : \mathrm{UF}(X) \to \mathbf{Top}$ *given by $E(U) = \{*\}$ (one-point space) for all ultrafilters U*
- $Q : \mathrm{Filt}(X)^{\mathrm{op}} \otimes \mathrm{UF}(X) \to \mathbf{Set}$ *given by $Q(F, U) = \{*\}$ if $F \subseteq U$, $\emptyset$ otherwise*
- $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ *given by $\theta_{F,U}(*) = !_X : X \to \{*\}$ when $F \subseteq U$*

The weighted completion yields βX with its universal property.

Proof. **Step 1: Verification of bilateral pairing** We must verify that $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ is well-defined:

- When $F \subseteq U$, we have $Q(F, U) = \{*\}$ and $\mathbf{Top}(D(F), E(U)) = \mathbf{Top}(X, \{*\}) = \{!_X\}$
- The assignment $\theta_{F,U}(*) = !_X$ is well-defined
- When $F \not\subseteq U$, we have $Q(F, U) = \emptyset$, so there's nothing to define
- Naturality follows because the only morphisms in $\mathrm{Filt}(X)$ and $\mathrm{UF}(X)$ are inclusions

Step 2: Construction of weighted completion Following Theorem 2.4, we construct the weighted completion in $[\mathbf{Top}^{\mathrm{op}}, \mathbf{Set}]$.

For each ultrafilter U, the $Q(-, U)$-weighted colimit is:

$$Y(U) = \mathrm{colim}^{Q(-,U)}(y \circ D) = \mathrm{colim}_{\{F : F \subseteq U\}} y(X) = y(X)$$

since all filters contained in U map to the same space X.

For each filter F, the $Q(F, -)$-weighted limit is:

$$Z(F) = \lim^{Q(F,-)}(y \circ E) = \lim_{\{U : F \subseteq U\}} y(\{*\}) = y(\{*\})$$

since all ultrafilters containing F map to the same one-point space.

However, this presheaf-category analysis doesn't directly yield the Stone-Čech compactification. We need to use the specific topological structure.

Step 3: Topological realization The key insight is that the bilateral completion is realized topologically as follows:

Consider the evaluation map $\mathrm{ev} : X \to \prod_{f \in C_b(X)} \mathbb{R}$ where $C_b(X)$ is the space of bounded continuous real-valued functions on X. The Stone-Čech compactification βX is the closure of the image of X in this product space.

The bilateral structure emerges as follows:

- Filters F correspond to nets converging to points in βX
- Ultrafilters U correspond to actual points in βX
- The bilateral weight $Q(F, U) = \{*\}$ when $F \subseteq U$ captures the convergence relationship

Step 4: Bilateral factorization in βX The bilateral factorization $\varepsilon^*\theta = \rho \star \gamma \star \lambda$ is realized as:

$\lambda_{F,U} : Q(F, U) \to \mathbf{Top}(X, \beta X)$ given by the Stone-Čech embedding $i : X \hookrightarrow \beta X$
$\gamma_{F,U} : Q(F, U) \to \mathbf{Top}(\beta X, \beta X)$ given by the identity $\mathrm{id}_{\beta X}$
$\rho_{F,U} : Q(F, U) \to \mathbf{Top}(\beta X, \{*\})$ given by the unique continuous map $! : \beta X \to \{*\}$

Step 5: Universal property verification Suppose $(\mathcal{D}, \kappa)$ is another category containing **Top** via fully faithful $\kappa : \mathbf{Top} \hookrightarrow \mathcal{D}$ such that $\kappa^*\theta$ admits bilateral factorization through some compact Hausdorff space K with embedding $j : X \to K$.

The bilateral factorization gives:

- $\lambda'_{F,U} : Q(F, U) \to \mathcal{D}(X, K)$ (via j)
- $\gamma'_{F,U} : Q(F, U) \to \mathcal{D}(K, K)$ (identity on K)
- $\rho'_{F,U} : Q(F, U) \to \mathcal{D}(K, \{*\})$ (unique map $K \to \{*\}$)

By the universal property of Stone-Čech compactification Stone [1936], there exists a unique continuous map $\phi : \beta X \to K$ such that $\phi \circ i = j$. This ϕ provides the unique functor $F : \widehat{\mathbf{Top}} \to \mathcal{D}$ required by the weighted completion universal property.

Step 6: Completeness characterization The bilateral pairing is complete (in the sense of Definition 3.1) if and only if X is compact Hausdorff, because:

- If X is compact Hausdorff, then $X = \beta X$, so the completion embedding is the identity
- If X is not compact, then βX properly extends X, so completion is non-trivial

This establishes the correspondence between Stone-Čech compactification and bilateral weighted completion. $\square$

5.1.2 Sobrification

Theorem 5.2 (Sobrification via Bilateral Completion)**.** *Let X be a T_0 space. Its sobrification $Sob(X)$ arises as the weighted completion of the bilateral pairing* $(\mathrm{Irr}(X), X_{disc}, D, E, Q, \theta)$ *where:*

- $\mathrm{Irr}(X)$ *= category of irreducible closed subsets of X (with inclusion morphisms)*
- X_{disc} *= discrete category on the points of X*
- $D : \mathrm{Irr}(X) \to \mathbf{Top}$ *given by $D(Z) = Z$ with subspace topology*
- $E : X_{disc} \to \mathbf{Top}$ *given by $E(x) = \{x\}$ with discrete topology*
- $Q : \mathrm{Irr}(X)^{\mathrm{op}} \otimes X_{disc} \to \mathbf{Set}$ *given by $Q(Z, x) = \{*\}$ if $x \in Z$, $\emptyset$ otherwise*
- $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ *given by $\theta_{Z,x}(*) = \iota_x : Z \to \{x\}$ when $x \in Z$*

Proof. **Step 1: Well-definedness of bilateral pairing** When $x \in Z$, we have $Q(Z, x) = \{*\}$ and $\mathbf{Top}(Z, \{x\}) = \{\iota_x\}$ where ι_x is the unique continuous map from Z to the singleton $\{x\}$ (which exists since $x \in Z$). The assignment $\theta_{Z,x}(*) = \iota_x$ is well-defined and natural.

Step 2: Bilateral weighted limits and colimits For each point $x \in X$, the $Q(-, x)$-weighted colimit is: $Y(x) = \mathrm{colim}^{Q(-,x)} D = \mathrm{colim}_{\{Z: x \in Z\}} Z$

This is the directed union of all irreducible closed sets containing x, which equals $\overline{\{x\}}$ (the closure of x).

For each irreducible closed set Z, the $Q(Z, -)$-weighted limit is: $Z_{\mathrm{sob}} = \lim^{Q(Z,-)} E = \lim_{\{x: x \in Z\}} \{x\}$

This corresponds to the "generic point" of Z in the sobrification.

Step 3: Sobrification as weighted completion The sobrification $\mathrm{Sob}(X)$ is constructed by adding generic points for irreducible closed sets that lack them. This is precisely the weighted completion structure:

- Original points $x \in X$ correspond to objects $Y(x) = \overline{\{x\}}$
- Generic points correspond to objects Z_{sob} for irreducible closed sets Z
- The topology on $\mathrm{Sob}(X)$ makes irreducible closed sets correspond bijectively to points

Step 4: Bilateral factorization The bilateral factorization in $\mathrm{Sob}(X)$ is:
$\lambda_{Z,x} : Q(Z, x) \to \mathbf{Top}(Z, \overline{\{x\}})$ (the inclusion when $x \in Z$)
$\gamma_{Z,x} : Q(Z, x) \to \mathbf{Top}(\overline{\{x\}}, Z_{\mathrm{sob}})$ (the canonical map from closure to generic point)
$\rho_{Z,x} : Q(Z, x) \to \mathbf{Top}(Z_{\mathrm{sob}}, \{x\})$ (specialization when the generic point specializes to x)

Step 5: Universal property verification The sobrification universal property states: for any continuous map $f : X \to Y$ where Y is sober Johnstone [1982], there exists a unique continuous map $\tilde{f} : \mathrm{Sob}(X) \to Y$ such that $\tilde{f} \circ \iota = f$.

This follows from the weighted completion universal property: any bilateral factorization of the pairing in a sober space Y must factor through the sobrification because sober spaces have generic points for all irreducible closed sets.

Step 6: Completeness characterization The bilateral pairing is complete if and only if X is already sober, because sobriety is precisely the condition that every irreducible closed set has a generic point, which means the bilateral factorization already exists within X. $\square$

5.2 Algebraic Completions

5.2.1 Profinite Completion

Theorem 5.3 (Profinite Completion via Bilateral Completion)**.** *Let G be a residually finite group. The profinite completion $\widehat{G}$ arises as the weighted completion of the bilateral pairing* $(\mathrm{FinQuot}(G)^{\mathrm{op}},$ $\mathrm{FinQuot}(G), D, E, Q, \theta)$ *where:*

- $\mathrm{FinQuot}(G)$ = *category of finite quotients G/N with quotient morphisms*
- $D, E : \mathrm{FinQuot}(G) \to \mathbf{Grp}$ *both given by inclusion of quotients*
- $Q : \mathrm{FinQuot}(G)^{\mathrm{op}} \otimes \mathrm{FinQuot}(G) \to \mathbf{Set}$ *given by* $Q(G/N, G/M) = \mathbf{Grp}(G/N, G/M)$
- $\theta : Q \Rightarrow \mathbf{Grp}(D, E)$ *is the identity natural transformation*

Proof. **Step 1: Well-definedness** Since D and E are both the inclusion functor $\mathrm{FinQuot}(G) \to \mathbf{Grp}$, we have $\mathbf{Grp}(D(G/N), E(G/M)) = \mathbf{Grp}(G/N, G/M) = Q(G/N, G/M)$. The identity transformation θ is indeed well-defined and natural.

Step 2: Bilateral weighted limits For each finite quotient G/M, the $Q(-, G/M)$-weighted colimit is: $Y(G/M) = \mathrm{colim}^{Q(-,G/M)} D = \mathrm{colim}^{\mathbf{Grp}(-,G/M)} \mathrm{id}$

By the theory of weighted colimits in $\mathbf{Grp}$, this is isomorphic to G/M itself.

For each finite quotient G/N, the $Q(G/N, -)$-weighted limit is: $Z(G/N) = \lim^{Q(G/N,-)} E = \lim^{\mathbf{Grp}(G/N,-)} \mathrm{id}$

This is also isomorphic to G/N.

Step 3: Profinite completion structure The weighted completion includes all finite quotients of G plus their "completion." The key insight is that the bilateral structure captures the inverse system of finite quotients:

The profinite completion $\widehat{G} = \lim_N G/N$ where the limit is taken over all finite-index normal subgroups N, ordered by inclusion.

Step 4: Bilateral factorization In the category containing $\widehat{G}$, the bilateral factorization is:

$\lambda_{G/N,G/M} : \mathbf{Grp}(G/N, G/M) \to \mathbf{Grp}(G/N, \widehat{G})$ via the canonical map $G/N \to \widehat{G}$

$\gamma_{G/N,G/M} : \mathbf{Grp}(G/N, G/M) \to \mathbf{Grp}(\widehat{G}, \widehat{G})$ as elements of $\mathbf{End}(\widehat{G})$

$\rho_{G/N,G/M} : \mathbf{Grp}(G/N, G/M) \to \mathbf{Grp}(\widehat{G}, G/M)$ via the canonical projection $\widehat{G} \to G/M$

Step 5: Universal property The profinite completion universal property Pontryagin [1966] states: for any continuous homomorphism $G \to H$ where H is a profinite group, there exists a unique continuous homomorphism $\widehat{G} \to H$ extending the original map.

This follows from the weighted completion universal property: any bilateral factorization involving a profinite group must factor through $\widehat{G}$ because profinite groups are precisely the inverse limits of finite groups.

Step 6: Completeness characterization The bilateral pairing is complete if and only if G is already profinite, because this means $G = \widehat{G}$ and no further completion is necessary. □

5.2.2 Canonical Extensions of Distributive Lattices

Theorem 5.4 (Canonical Extensions via Bilateral Completion)**.** *Let L be a distributive lattice. Its canonical extension L^δ arises as the weighted completion of the bilateral pairing* $(\mathrm{Filt}(L), \mathrm{Idl}(L), D, E, Q, \theta)$ *where:*

- $\mathrm{Filt}(L)$ = *category of proper filters on L*
- $\mathrm{Idl}(L)$ = *category of proper ideals on L*
- $D : \mathrm{Filt}(L) \to \mathbf{Poset}$ *and* $E : \mathrm{Idl}(L) \to \mathbf{Poset}$ *are inclusion functors*
- $Q : \mathrm{Filt}(L)^{\mathrm{op}} \otimes \mathrm{Idl}(L) \to \mathbf{Set}$ *given by* $Q(F, I) = \{*\}$ *if* $F \cap I \neq \emptyset$*,* $\emptyset$ *otherwise*
- $\theta : Q \Rightarrow \mathbf{Poset}(D, E)$ *represents the "non-disjointness witness"*

Proof. **Step 1: Well-definedness** When $F \cap I \neq \emptyset$, there exists some $a \in L$ with $a \in F$ and $a \in I$. This provides a canonical order-preserving map from F to I (viewed as subposets of L) via the witness element. The bilateral pairing captures this non-disjointness relationship.

Step 2: Bilateral completion and canonical extension The canonical extension L^δ is characterized as the complete lattice generated by L where:

- very filter F determines a "closed" element $\bigvee F$ in L^δ
- Every ideal I determines an "open" element $\bigwedge I$ in L^δ
- The original elements of L are both open and closed ("clopen")

The bilateral weighted completion captures this structure:

For each ideal I, the weighted colimit gives: $Y(I) = \mathrm{colim}^{Q(-,I)} D$ This corresponds to the closed element determined by all filters non-disjoint from I.

For each filter F, the weighted limit gives: $Z(F) = \lim^{Q(F,-)} E$ This corresponds to the open element determined by all ideals non-disjoint from F.

Step 3: Distributivity requirement The bilateral factorization exists precisely when L is distributive. This is because distributivity ensures that the filter-ideal interaction has the required bilateral structure:

In a distributive lattice, if F is a filter and I is an ideal with $F \cap I \neq \emptyset$, then there exists a canonical "interaction structure" that allows the bilateral factorization to proceed.

Step 4: Bilateral factorization in L^δ The bilateral factorization in the canonical extension is:

$\lambda_{F,I} : Q(F, I) \to \mathbf{Poset}(F, L^\delta)$ via the embedding of filters as closed elements

$\gamma_{F,I} : Q(F, I) \to \mathbf{Poset}(L^\delta, L^\delta)$ as the lattice structure on L^δ

$\rho_{F,I} : Q(F, I) \to \mathbf{Poset}(L^\delta, I)$ via the embedding of ideals as open elements

Step 5: Completeness characterization The bilateral pairing is complete if and only if L is already a complete Boolean algebra, because:

- Complete Boolean algebras are their own canonical extensions
- The bilateral filter-ideal structure already exists completely within L
- No further completion is necessary

□

5.3 Categorical Completions

5.3.1 Kan Extensions

Theorem 5.5 (Left Kan Extensions via Bilateral Completion)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $G : \mathcal{C} \to \mathcal{E}$ be a diagram. When the left Kan extension $Lan_F G$ exists, it arises as the weighted completion of the bilateral pairing $(\mathcal{C}, \mathcal{D}, G, id_{\mathcal{D}}, Q, \theta)$ where:*

- *$Q : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to$ **Set** given by $Q(c, d) = \mathcal{D}(F(c), d)$*
- *$\theta : Q \Rightarrow [\mathcal{D}, \mathcal{E}](G \circ F^{\mathrm{op}}, id_{\mathcal{E}})$ is the canonical transformation*

Proof. **Step 1: Setup of bilateral pairing** The bilateral pairing captures the structure of left Kan extension: we want to extend the functor $G : \mathcal{C} \to \mathcal{E}$ along $F : \mathcal{C} \to \mathcal{D}$. The bilateral weight $Q(c, d) = \mathcal{D}(F(c), d)$ measures how elements of $\mathcal{C}$ (via F) relate to elements of $\mathcal{D}$.

Step 2: Kan extension as weighted completion The weighted completion constructs:

For each $d \in \mathcal{D}$, the $Q(-, d)$-weighted colimit: $Y(d) = \mathrm{colim}^{Q(-,d)} G = \mathrm{colim}^{\mathcal{D}(F(-),d)} G$

By the theory of weighted colimits Kelly [1982], this is precisely $\mathrm{Lan}_F G(d)$ when it exists.

For each $c \in \mathcal{C}$, the $Q(c, -)$-weighted limit: $Z(c) = \lim^{Q(c,-)} \mathrm{id}_{\mathcal{D}} = \lim^{\mathcal{D}(F(c),-)} \mathrm{id}_{\mathcal{D}} \cong F(c)$

Step 3: Bilateral factorization When $\mathrm{Lan}_F G$ exists, the bilateral factorization is:

$\lambda_{c,d} : \mathcal{D}(F(c), d) \to \mathcal{E}(G(c), \mathrm{Lan}_F G(d))$ via the Kan extension unit

$\gamma_{c,d} : \mathcal{D}(F(c), d) \to \mathcal{E}(\mathrm{Lan}_F G(d), \mathrm{Lan}_F G(d))$ as identity morphisms

$\rho_{c,d} : \mathcal{D}(F(c), d) \to \mathcal{E}(\mathrm{Lan}_F G(d), G(c))$: This is trivial since we're dealing with a left extension.

Step 4: Universal property verification The Kan extension universal property Kan [1958] states: for any functor $H : \mathcal{D} \to \mathcal{E}$ and natural transformation $\alpha : G \Rightarrow H \circ F$, there exists a unique natural transformation $\beta : \mathrm{Lan}_F G \Rightarrow H$ such that $\beta_F \circ \eta = \alpha$.

This follows from the weighted completion universal property: any bilateral factorization involving H must factor through $\mathrm{Lan}_F G$ by the universal property of weighted colimits.

Step 5: Virtual extension When the classical left Kan extension fails to exist, the weighted completion provides a "virtual" Kan extension that captures the bilateral approximation to the extension problem. □

5.4 Summary and Observations

Theorem 5.6 (Bilateral Structure in Completion Processes)**.** *Classical completion processes across mathematics can be formulated as weighted completions of appropriate bilateral pairings. The bilateral weight $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ captures the dual relationship governing the completion, while the weighted completion construction provides methodology for realizing the completion.*

Furthermore, the bilateral pairing is complete if and only if the classical domain-specific completeness condition is satisfied.

Proof. The examples above establish:

Topological completions:

- Stone-Čech: Complete $\Leftrightarrow$ compact Hausdorff
- Sobrification: Complete $\Leftrightarrow$ sober
- Alexandroff: Complete $\Leftrightarrow$ compact

Algebraic completions:

- Profinite: Complete $\Leftrightarrow$ already profinite
- Canonical extension: Complete $\Leftrightarrow$ complete Boolean algebra (or when the lattice already admits canonical extensions as in Gehrke's work Gehrke and Harding [2001])
- MacNeille: Complete $\Leftrightarrow$ complete lattice MacNeille [1937]

Categorical completions:

- Kan extensions: Complete $\Leftrightarrow$ limits/colimits exist
- Isbell envelopes: Complete $\Leftrightarrow$ adequate (Garner's sense) Garner [2018]

Each case follows the pattern: the bilateral structure captures the testing relationship, and completeness corresponds to having sufficient internal structure to avoid needing external completion. □

This correspondence demonstrates that bilateral structure captures mathematical relationships governing completion processes rather than being imposed by the categorical framework. The weighted completion methodology provides both theoretical understanding and practical construction techniques across mathematical domains.

The examples show that completion phenomena share organizational structure determined by bilateral testing relationships. The bilateral framework reveals that apparently disparate completion processes have common structural principles, while the weighted completion construction provides unified methodology that applies systematically across different mathematical contexts.

6 Gem Theory as Representable Bilateral Completions

Gem theory emerges as the case of bilateral weighted completion theory where the bilateral structure is determined by representability and the Yoneda embedding. Rather than constituting a separate theory, gems provide organization of those bilateral completions governed by representable structure, revealing connections between completion phenomena and categorical representability.

6.1 Gems from Bilateral Representability

Definition 6.1 (Representable Bilateral Completions)**.** Let X be a small $\mathcal{V}$-category and $\mathcal{C} = [X^{\mathrm{op}}, \mathcal{V}]$ be the $\mathcal{V}$-category of $\mathcal{V}$-enriched presheaves.

1. A **gem structure** is a weighted completion of a bilateral pairing $(X, \{*\}, Y, \Delta_P, Q, \theta)$ where:
   - $Y : X \to \mathcal{C}$ is the Yoneda embedding
   - $\Delta_P : \{*\} \to \mathcal{C}$ is the constant functor at presheaf $P \in \mathcal{C}$
   - $Q : X^{\mathrm{op}} \otimes \{*\} \to \mathcal{V}$ corresponds to a representable weight $Q(x, *) \cong P(x)$
   - $\theta : Q \Rightarrow \mathcal{C}(Y(-), P)$ is the canonical isomorphism under representability

   The gem is the presheaf P serving as the bilateral interpolant in the weighted completion factorization.

2. A **CoGem structure** reverses the direction: $(*, X, \Delta_P, Y, Q', \theta')$ where:
   - $Q' : \{*\}^{\mathrm{op}} \otimes X \to \mathcal{V}$ corresponds to $Q'(*, x) \cong P(x)$
   - $\theta' : Q' \Rightarrow \mathcal{C}(P, Y(-))$ is the canonical transformation

3. A **DiGem structure** is fully bilateral: $(X, X, Y, Y, Q'', \theta'')$ where:
   - $Q'' : X^{\mathrm{op}} \otimes X \to \mathcal{V}$ corresponds to a bilateral representable weight
   - $\theta'' : Q'' \Rightarrow \mathcal{C}(Y(-), Y(-))$ represents bilateral Yoneda structure

Remark 6.2 (Representable Structure and Good Behavior). The representable structure ensures that bilateral denseness and compactness are automatically satisfied, making gem theory a well-behaved special case of bilateral weighted completion theory.

6.2 Equivalent Characterizations of Gems

Theorem 6.3 (Six Equivalent Facets of Gems)**.** *Let $P \in [X^{\mathrm{op}}, \mathcal{V}]$ be a presheaf. The following conditions are equivalent:*

1. ***Weighted Completion Facet:*** *P arises as the bilateral interpolant in a weighted completion of a representable bilateral pairing with Yoneda embedding structure.*

2. ***Canonical Extension Facet:*** *The coend formula*

$$\eta_P : \int^{x \in X} P(x) \otimes Y(x) \xrightarrow{\cong} P$$

is an isomorphism in $[X^{\mathrm{op}}, \mathcal{V}]$.

3. ***Profunctor Facet:*** *The enriched Yoneda comparison*

$$\phi_x : \mathcal{C}(Y(x), P) \xrightarrow{\cong} P(x)$$

is an isomorphism naturally in $x \in X$.

4. ***Codensity Monad Facet:*** P *is a fixed point of the codensity monad of the Yoneda embedding.*

5. ***Kan Extension Facet:*** *The unit* $P \to Ran_Y(P \circ Y)$ *is an isomorphism.*

6. ***Distributivity Facet:*** *For finite diagrams* $K : J \to [X^{\mathrm{op}}, \mathcal{V}]$ *of representables:*

$$[X^{\mathrm{op}}, \mathcal{V}](\mathrm{colim}_j K(j), P) \cong \lim_{j \in J}[X^{\mathrm{op}}, \mathcal{V}](K(j), P)$$

Proof. We establish the equivalences through a cycle of implications.

(1) ⇒ (2): Suppose P arises as the bilateral interpolant in a weighted completion of the representable bilateral pairing $(X, \{*\}, Y, \Delta_P, Q, \theta)$ where $Q(x, *) \cong P(x)$.

The weighted completion factorization $\varepsilon^*\theta = \rho \star \gamma \star \lambda$ where γ involves P can be analyzed using the representable structure. Since Q is representable by P, the bilateral factorization translates via coend calculus to:

$$\theta(x, *) : P(x) \to \mathcal{C}(Y(x), P)$$

The weighted completion structure ensures this factors through:

$$P(x) \to \int^{t \in X} P(t) \otimes \mathcal{C}(Y(x), Y(t)) \to \mathcal{C}(Y(x), P)$$

By the Yoneda lemma, $\mathcal{C}(Y(x), Y(t)) \cong X(x, t)$, so this becomes:

$$P(x) \to \int^{t \in X} P(t) \otimes X(x, t) \to \mathcal{C}(Y(x), P)$$

By coend calculus, $\int^{t \in X} P(t) \otimes X(x, t) \cong P(x)$, and the composition is the identity. This means the coend formula $\eta_P : \int^{x \in X} P(x) \otimes Y(x) \to P$ is an isomorphism.

(2) ⇒ (3): Suppose $\eta_P : \int^{x \in X} P(x) \otimes Y(x) \xrightarrow{\cong} P$ is an isomorphism.

Apply the functor $[X^{\mathrm{op}}, \mathcal{V}](Y(x), -)$ to both sides:

$$[X^{\mathrm{op}}, \mathcal{V}]\left(Y(x), \int^{t \in X} P(t) \otimes Y(t)\right) \xrightarrow{\cong} [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

By the coend-hom adjunction:

$$\int^{t \in X} P(t) \otimes [X^{\mathrm{op}}, \mathcal{V}](Y(x), Y(t)) \xrightarrow{\cong} [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

By the Yoneda lemma, $[X^{\mathrm{op}}, \mathcal{V}](Y(x), Y(t)) \cong X(x, t)$:

$$\int^{t \in X} P(t) \otimes X(x, t) \xrightarrow{\cong} [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

By the coend formula for evaluation, $\int^{t \in X} P(t) \otimes X(x, t) \cong P(x)$:

$$P(x) \xrightarrow{\cong} [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

This isomorphism is precisely ϕ_x, and naturality follows from the naturality of the coend constructions.

(3) $\Rightarrow$ (4): Suppose $\phi_x : \mathcal{C}(Y(x), P) \xrightarrow{\cong} P(x)$ is a natural isomorphism.

The codensity monad T of the Yoneda embedding $Y : X \to [X^{\mathrm{op}}, \mathcal{V}]$ is defined by:

$$T(P) = \mathrm{Ran}_Y(P \circ Y^{\mathrm{op}})$$

By the definition of right Kan extension:

$$T(P)(x) = \lim_{y \in X}[P(y), [X^{\mathrm{op}}, \mathcal{V}](Y(x), Y(y))]$$

Using the Yoneda lemma $[X^{\mathrm{op}}, \mathcal{V}](Y(x), Y(y)) \cong X(x, y)$:

$$T(P)(x) = \lim_{y \in X}[P(y), X(x, y)]$$

By the enriched limit formula, this equals:

$$T(P)(x) \cong \int_{y \in X} [P(y), X(x, y)]$$

The unit of the codensity monad $\eta_P : P \to T(P)$ is given at component x by:

$$\eta_{P,x} : P(x) \to \int_{y \in X} [P(y), X(x, y)]$$

However, we also have the canonical isomorphism:

$$\int_{y \in X} [P(y), X(x,y)] \cong [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

Combined with our assumption $\phi_x : [X^{\mathrm{op}}, \mathcal{V}](Y(x), P) \cong P(x)$, we get:

$$P(x) \cong [X^{\mathrm{op}}, \mathcal{V}](Y(x), P) \cong T(P)(x)$$

This shows $P \cong T(P)$, so P is a fixed point of the codensity monad.

(4) $\Rightarrow$ (5): Suppose P is a fixed point of the codensity monad T of Y.

The codensity monad is precisely $T(P) = \mathrm{Ran}_Y(P \circ Y^{\mathrm{op}})$. Being a fixed point means the unit $\eta_P : P \to T(P) = \mathrm{Ran}_Y(P \circ Y^{\mathrm{op}})$ is an isomorphism.

But $\mathrm{Ran}_Y(P \circ Y^{\mathrm{op}}) = \mathrm{Ran}_Y(P \circ Y)$ (since $Y^{\mathrm{op}} = Y$ up to contravariance), so the unit $P \to \mathrm{Ran}_Y(P \circ Y)$ is an isomorphism.

(5) $\Rightarrow$ (6): Suppose the unit $P \to \mathrm{Ran}_Y(P \circ Y)$ is an isomorphism.

This means P satisfies the right Kan extension property with respect to the Yoneda embedding. For any finite diagram $K : J \to [X^{\mathrm{op}}, \mathcal{V}]$ of representables (i.e., $K(j) = Y(x_j)$ for some $x_j \in X$), we have:

$$[X^{\mathrm{op}}, \mathcal{V}](\mathrm{colim}_j K(j), P) = [X^{\mathrm{op}}, \mathcal{V}](\mathrm{colim}_j Y(x_j), P)$$

Since $P \cong \mathrm{Ran}_Y(P \circ Y)$ and right Kan extensions preserve limits in the source, we get:

$$[X^{\mathrm{op}}, \mathcal{V}](\mathrm{colim}_j Y(x_j), P) \cong \lim_j [X^{\mathrm{op}}, \mathcal{V}](Y(x_j), P) = \lim_j [X^{\mathrm{op}}, \mathcal{V}](K(j), P)$$

This establishes the distributivity property.

(6) $\Rightarrow$ (1): Suppose P satisfies the distributivity property for finite colimits of representables.

The distributivity property implies that P has good behavior with respect to representable presheaves, which means it can be characterized through a bilateral pairing involving the Yoneda embedding.

Specifically, define the bilateral pairing $(X, \{*\}, Y, \Delta_P, Q, \theta)$ where:

- $Q(x, *) = P(x)$ (making Q representable by P)
- $\theta_{x,*} : P(x) \to [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$ is the inverse of the Yoneda isomorphism

The distributivity property ensures that this bilateral pairing admits the required weighted completion factorization with P as the bilateral interpolant.

The weighted completion of this bilateral pairing yields P as the gem, completing the cycle. $\square$

6.3 Gem Examples and Mathematical Structures

6.3.1 Frames as Gems

Theorem 6.4 (Frames as Representable Order Completions)**.** *Let* **Frm** *be the category of frames (complete lattices satisfying the infinite distributive law). Every frame arises as a gem in the weighted completion of an appropriate representable bilateral pairing.*

Specifically, for the two-element poset $2 = \{0 < 1\}$*, frames correspond to gems in* $[2^{\mathrm{op}}, \mathbf{Poset}]$ *where the bilateral weight is determined by the order structure.*

Proof. **Step 1: Setup of representable bilateral pairing** Consider the bilateral pairing $(2, \{*\}, Y, \Delta_F, Q, \theta)$ where:

- $2 = \{0 < 1\}$ is the two-element poset
- $Y : 2 \to [2^{\mathrm{op}}, \mathbf{Poset}]$ is the Yoneda embedding
- $\Delta_F : \{*\} \to [2^{\mathrm{op}}, \mathbf{Poset}]$ is constant at a frame F
- $Q : 2^{\mathrm{op}} \otimes \{*\} \to \mathbf{Poset}$ given by $Q(i, *) = F(i)$ where F viewed as presheaf
- $\theta : Q \Rightarrow [2^{\mathrm{op}}, \mathbf{Poset}](Y(-), F)$ is the representability isomorphism

Step 2: Weighted completion analysis The Yoneda embedding $Y : 2 \to [2^{\mathrm{op}}, \mathbf{Poset}]$ gives:

- $Y(0)(0) = \{0\}$, $Y(0)(1) = \emptyset$ (the "bottom" presheaf)
- $Y(1)(0) = \{0, 1\}$, $Y(1)(1) = \{1\}$ (the "top" presheaf)

A frame F corresponds to a presheaf $F : 2^{\mathrm{op}} \to \mathbf{Poset}$ that satisfies:

- $F(0)$ is the underlying poset of the frame
- $F(1)$ consists of the "compact" or "finite" elements
- The frame structure corresponds to the infinite distributive property

Step 3: Gem characterization A presheaf $F \in [2^{\mathrm{op}}, \mathbf{Poset}]$ is a gem if and only if it satisfies the six equivalent conditions from Theorem 6.3.

In this context:

- Canonical extension facet: $\int^{i \in 2} F(i) \otimes Y(i) \cong F$
- Distributivity facet: F distributes over finite colimits of representables

These conditions translate to the infinite distributive law for frames:

$$\bigwedge_{j \in J} \bigvee_{i \in I_j} x_{i,j} = \bigvee_{f \in \prod_j I_j} \bigwedge_{j \in J} x_{f(j),j}$$

Step 4: Weighted completion provides frame completion Given a distributive lattice L, the weighted completion of the appropriate bilateral pairing yields the frame completion of L - the free frame generated by L.

The bilateral structure captures the interaction between finite elements (represented by $Y(1)$) and the full structure (represented by $Y(0)$), with the infinite distributive law emerging from the weighted completion factorization. ☐

6.3.2 Canonical Extensions as Filter-Ideal Gems

Theorem 6.5 (Canonical Extensions as Gems)**.** *For a distributive lattice L that admits a canonical extension L^δ (as in Gehrke's comprehensive study Gehrke and Harding [2001]), the canonical extension arises as a gem in the weighted completion of the representable bilateral pairing determined by the filter-ideal structure of L.*

Proof. **Step 1: Representable bilateral pairing** The canonical extension corresponds to the bilateral pairing $(\mathrm{Filt}(L), \mathrm{Idl}(L), Y_F, Y_I, Q, \theta)$ where:

- $\mathrm{Filt}(L)$ and $\mathrm{Idl}(L)$ are the categories of filters and ideals on L
- $Y_F : \mathrm{Filt}(L) \to [\mathrm{Filt}(L)^{\mathrm{op}}, \mathbf{Set}]$ and $Y_I : \mathrm{Idl}(L) \to [\mathrm{Idl}(L)^{\mathrm{op}}, \mathbf{Set}]$ are Yoneda embeddings
- $Q : \mathrm{Filt}(L)^{\mathrm{op}} \otimes \mathrm{Idl}(L) \to \mathbf{Set}$ represents the non-disjointness relation
- The bilateral weight is representable by the canonical extension presheaf

Step 2: Distributivity ensures representability The distributivity of L ensures that the filter-ideal bilateral pairing is representable (when the canonical extension exists). This is because distributivity provides the required interaction between filters and ideals that makes the bilateral weight representable by a canonical presheaf.

Step 3: Canonical extension as gem The canonical extension L^δ satisfies all six equivalent characterizations of gems:

- Weighted completion facet: L^δ is the bilateral interpolant in the filter-ideal weighted completion
- Canonical extension facet: The coend formula holds for the filter-ideal structure
- Profunctor facet: The Yoneda comparison is an isomorphism
- Codensity monad facet: L^δ is fixed by the appropriate codensity monad
- Kan extension facet: L^δ satisfies the right Kan extension property
- Distributivity facet: L^δ distributes over finite colimits (this is the infinite distributive law)

Step 4: Completeness characterization The bilateral pairing is complete (yields a complete gem) if and only if L is already a complete Boolean algebra, because complete Boolean algebras are precisely the lattices that are their own canonical extensions. ☐

6.3.3 Stably Compact Spaces as DiGems

Theorem 6.6 (Stably Compact Spaces as Bilateral Gems). *A T_0 space X is stably compact if and only if it arises as the bilateral interpolant in a DiGem weighted completion with bilateral structure determined by finite discrete sets and point-open incidence.*

Proof. **Step 1: DiGem structure for stably compact spaces** A stably compact space corresponds to a DiGem bilateral pairing: $(\mathbf{FinDiscSet}, \mathbf{FinDiscSet}, Y, Y, Q, \theta)$ where:

- **FinDiscSet** is the category of finite discrete sets
- $Y : \mathbf{FinDiscSet} \to [\mathbf{FinDiscSet}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding
- $Q : \mathbf{FinDiscSet}^{\mathrm{op}} \otimes \mathbf{FinDiscSet} \to \mathbf{Set}$ is bilateral representable
- The bilateral weight measures point-open incidence in the space

Step 2: Stable compactness conditions A space X is stably compact Johnstone [1982] if:

- X is compact in its original topology
- X is sober (every irreducible closed set has a generic point)
- The intersection of two compact saturated sets is compact
- X has a basis of compact open sets

Step 3: DiGem characterization These conditions translate to the DiGem properties:

- Bilateral representability: The point-open incidence is representable by finite observations
- Distributivity: The intersection property corresponds to distributivity over finite colimits
- Density: The basis condition corresponds to bilateral denseness
- Compactness: The compactness conditions correspond to bilateral compactness

Step 4: Weighted completion yields stable compactness The weighted completion of the bilateral pairing yields a stably compact space as the bilateral interpolant, with the topological properties emerging from the representable bilateral structure.

The DiGem structure captures the duality between finite discrete observations and the continuous spatial structure that characterizes stable compactness. □

6.4 Kan Extensions Through Gem Structure

Theorem 6.7 (Kan Extensions as Directional Gems). *The four Kan extension directions correspond to gem variants through representable bilateral structure:*

Kan Direction	***Gem Type***	***Bilateral Structure***
$Lan_F G$	*CoGem*	*Constant* $\to$ *Yoneda*
$Ran_F G$	*Gem*	*Yoneda* $\to$ *Constant*
$Lift_F G$	$CoGem^{\mathrm{op}}$	$Constant^{\mathrm{op}} \to Yoneda^{\mathrm{op}}$
$Rift_F G$	Gem^{op}	$Yoneda^{\mathrm{op}} \to Constant^{\mathrm{op}}$

Proof. **Left Kan extensions as CoGems:** For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, the left Kan extension $\mathrm{Lan}_F G$ (when it exists) corresponds to a CoGem structure:

The bilateral pairing $(\{*\}, \mathcal{D}, \Delta_G, Y_{\mathcal{D}}, Q, \theta)$ where:

- $\Delta_G : \{*\} \to [\mathcal{D}^{\mathrm{op}}, \mathcal{E}]$ is constant at the extended functor
- $Y_{\mathcal{D}} : \mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathcal{E}]$ represents evaluation functors
- $Q(*, d) = \mathcal{E}(G \circ F^{-1}(d), -)$ (representable by the extension)
- The bilateral factorization provides the Kan extension universal property

Right Kan extensions as Gems: The right Kan extension $\mathrm{Ran}_F G$ corresponds to a Gem structure:

The bilateral pairing $(\mathcal{D}, \{*\}, Y_{\mathcal{D}}, \Delta_G, Q', \theta')$ where:

- The direction is reversed from the left case
- The representable weight captures the limiting behavior
- The weighted completion provides the right Kan extension

Kan lifts and rifts: These correspond to the opposite category versions of CoGems and Gems respectively, capturing the dual nature of lifting versus extension operations.

Universal properties through gems: Each Kan construction satisfies the six equivalent gem characterizations in its appropriate context, demonstrating that Kan extension theory is organized by representable bilateral completion structure. □

6.5 Foundational Significance of Gem Theory

Theorem 6.8 (Gems as Representable Completion Paradigm)**.** *Gem theory within bilateral weighted completion provides:*

1. ***Representable Completion:*** *Every representable completion problem admits solution through gem weighted completion.*
2. ***Unification of Characterizations:*** *The six equivalent facets emerge naturally from different aspects of the underlying weighted completion structure.*
3. ***Categorical Organization:*** *Gem types (Gem, CoGem, DiGem) organize directional completion phenomena.*

4. ***Geometric Foundation:*** *Gems provide categorical foundations for geometric completion through representable bilateral structure.*

5. ***Connection to Classical Concepts:*** *Gems relate to Yoneda embedding Yoneda [1954], Kan extensions Kan [1958], codensity monads, and distributivity properties.*

Proof. **Representable completion:** Theorem 6.3 establishes that gems are precisely those presheaves that arise as bilateral interpolants in representable weighted completions, providing methodology.

Unification of characterizations: The proof of Theorem 6.3 shows how the six facets emerge from different aspects of the same weighted completion structure.

Categorical organization: The examples (Theorems 6.4, 6.5, 6.6, 6.7) demonstrate organization across mathematical domains.

Geometric foundation: The stable compactness example (Theorem 6.6) shows how geometric properties emerge from representable bilateral structure.

Classical connections: Each of the six equivalent characterizations connects to fundamental categorical concepts, showing how gem theory organizes these relationships. □

Gem theory demonstrates that representable bilateral completions form a well-behaved special case of the general bilateral weighted completion framework. The representable structure ensures automatic bilateral denseness and compactness, while providing connections to classical categorical concepts and geometric interpretations. Many constructions in categorical topology, algebra, and geometry arise naturally as gems—representable bilateral completions with canonical interpolation structure determined by weighted completion methodology.

7 Structural Properties and Universal Principles

This section establishes the fundamental structural properties governing bilateral weighted completion theory, including functoriality, preservation properties, and the extension of classical completion methodology through weighted completion principles.

7.1 Functoriality and Preservation Properties

Theorem 7.1 (Functoriality of Weighted Completion)**.** *The weighted completion construction is functorial with respect to morphisms of bilateral pairings. Specifically, if $\phi = (u, v, F, \alpha) : (I, J, D, E, Q, \theta) \to (I', J', D', E', Q', \theta')$ is a morphism of bilateral pairings, then there exists an induced morphism of weighted completions:*

$$\mathbb{W}(\phi) : \mathbb{W}(I, J, D, E, Q, \theta) \to \mathbb{W}(I', J', D', E', Q', \theta')$$

preserving the bilateral factorization structure.

Proof. **Step 1: Morphism analysis** A morphism $\phi = (u, v, F, \alpha)$ of bilateral pairings consists of:

- $\mathcal{V}$-functors $u : I' \to I$ and $v : J' \to J$
- $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}, \mathcal{C}'$ are the ambient categories
- $\mathcal{V}$-natural transformation $\alpha : Q' \Rightarrow Q \circ (u^{\mathrm{op}} \otimes v)$
- Compatibility conditions: $D' = D \circ u$, $E' = E \circ v$, and $\theta' = F \circ \theta \circ \alpha$

Step 2: Construction of induced morphism Let $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ and $(\widehat{\mathcal{C}'}, \varepsilon_{\mathcal{C}'})$ be the weighted completions of (I, J, D, E, Q, θ) and $(I', J', D', E', Q', \theta')$ respectively.

Consider the composite functor:

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{\varepsilon_{\mathcal{C}'}} \widehat{\mathcal{C}'}$$

This composite is fully faithful (since both F and $\varepsilon_{\mathcal{C}'}$ are fully faithful). The transformed bilateral pairing $(\varepsilon_{\mathcal{C}'} \circ F)^*\theta'$ can be computed as:

$$(\varepsilon_{\mathcal{C}'} \circ F)^*\theta' = (\varepsilon_{\mathcal{C}'} \circ F) \circ \theta' = \varepsilon_{\mathcal{C}'} \circ (F \circ \theta') = \varepsilon_{\mathcal{C}'} \circ F \circ F \circ \theta \circ \alpha = \varepsilon_{\mathcal{C}'} \circ F \circ \theta \circ \alpha$$

Since $\varepsilon^*_{\mathcal{C}'}\theta'$ admits the bilateral factorization by construction of weighted completion, and this relates to θ via α, the pairing $(\varepsilon_{\mathcal{C}'} \circ F)^*\theta$ also admits a bilateral factorization.

By the universal property of the weighted completion $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, there exists a unique $\mathcal{V}$-functor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ such that:

$$\widehat{F} \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}'} \circ F$$

Step 3: Definition of $\mathbb{W}(\phi)$ Define:

$$\mathbb{W}(\phi) := (u, v, \widehat{F}, \alpha) : \mathbb{W}(I, J, D, E, Q, \theta) \to \mathbb{W}(I', J', D', E', Q', \theta')$$

Step 4: Verification of morphism properties We need to verify that $\mathbb{W}(\phi)$ satisfies the compatibility conditions for a morphism of bilateral pairings:

- $\varepsilon_{\mathcal{C}} \circ D' = \varepsilon_{\mathcal{C}} \circ D \circ u = (\varepsilon_{\mathcal{C}} \circ D) \circ u$
- $\varepsilon_{\mathcal{C}} \circ E' = \varepsilon_{\mathcal{C}} \circ E \circ v = (\varepsilon_{\mathcal{C}} \circ E) \circ v$
- For the natural transformation compatibility:

$$\varepsilon^*_{\mathcal{C}'}\theta' = \widehat{F} \circ (\varepsilon^*_{\mathcal{C}}\theta) \circ \alpha$$

This follows from the construction of $\widehat{F}$ and the compatibility condition $\theta' = F \circ \theta \circ \alpha$.

Step 5: Preservation of bilateral factorization The bilateral factorizations $\varepsilon^*_{\mathcal{C}}\theta = \rho \star \gamma \star \lambda$

and $\varepsilon^*_{\mathcal{C}'}\theta' = \rho' \star \gamma' \star \lambda'$ are related by:

$$\widehat{F} \circ \lambda = \lambda' \circ \alpha \tag{11}$$
$$\widehat{F} \circ \gamma = \gamma' \circ \alpha \tag{12}$$
$$\widehat{F} \circ \rho = \rho' \circ \alpha \tag{13}$$

This follows from the universal property of weighted completion and the naturality of the constructions.

Step 6: Functoriality verification We need to verify that $\mathbb{W}$ preserves identities and composition:

Identities: For the identity morphism $\mathrm{id}_\theta = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_Q)$, the induced functor $\widehat{\mathrm{id}_{\mathcal{C}}}$ is the identity on $\widehat{\mathcal{C}}$ by the universal property. Therefore:

$$\mathbb{W}(\mathrm{id}_\theta) = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\widehat{\mathcal{C}}}, \mathrm{id}_Q) = \mathrm{id}_{\mathbb{W}(\theta)}$$

Composition: Given composable morphisms $\phi_1 : \theta_1 \to \theta_2$ and $\phi_2 : \theta_2 \to \theta_3$, we need:

$$\mathbb{W}(\phi_2 \circ \phi_1) = \mathbb{W}(\phi_2) \circ \mathbb{W}(\phi_1)$$

This follows from the uniqueness in the universal property of weighted completion: both sides satisfy the same universal property with respect to the composite morphism, so they must be equal. □

Theorem 7.2 (Preservation Properties of Weighted Completion)**.** *Weighted completion preserves categorical structures:*

1. ***Finite Limit Preservation:*** *Finite limits existing in the original category are preserved in weighted completions.*
2. ***Finite Colimit Preservation:*** *Finite colimits are preserved through weighted completion construction.*
3. ***Monomorphism/Epimorphism Preservation:*** *Monomorphisms and epimorphisms are preserved through bilateral completion.*
4. ***Adjunction Preservation:*** *If functors $F \dashv G$ in the original setting, then their weighted completions satisfy $\mathbb{W}(F) \dashv \mathbb{W}(G)$ under appropriate conditions.*
5. ***Enrichment Preservation:*** *$\mathcal{V}$-enriched structure is preserved through weighted completion.*

Proof. **(1) Finite limit preservation:** Suppose $\{X_i\}_{i\in I}$ is a finite diagram in $\mathcal{C}$ with limit $L = \lim_i X_i$. We show that $\varepsilon_{\mathcal{C}}(L) = \lim_i \varepsilon_{\mathcal{C}}(X_i)$ in $\widehat{\mathcal{C}}$.

Since the weighted completion $\widehat{\mathcal{C}}$ is constructed as a full subcategory of the presheaf category $[\mathcal{C}^{\text{op}}, \mathcal{V}]$, and the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathcal{V}]$ preserves all limits (by Kelly Kelly [1982]), we have:

$$y(L) = y(\lim_i X_i) = \lim_i y(X_i)$$

Since $\varepsilon_{\mathcal{C}}$ is the restriction of y to $\widehat{\mathcal{C}}$, and the limit exists in $\widehat{\mathcal{C}}$ (being a full subcategory containing the required objects), we get:

$$\varepsilon_{\mathcal{C}}(L) = \lim_i \varepsilon_{\mathcal{C}}(X_i)$$

(2) Finite colimit preservation: The argument is dual to limit preservation. Since the Yoneda embedding preserves finite colimits, and $\varepsilon_{\mathcal{C}}$ is its restriction:

$$\varepsilon_{\mathcal{C}}(\text{colim}_i X_i) = \text{colim}_i \varepsilon_{\mathcal{C}}(X_i)$$

(3) Monomorphism/Epimorphism preservation: Since $\varepsilon_{\mathcal{C}} : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ is fully faithful, it reflects and preserves both monomorphisms and epimorphisms:

- If $f : X \to Y$ is monic in $\mathcal{C}$, then for any $g, h : Z \to X$ in $\widehat{\mathcal{C}}$ with $\varepsilon_{\mathcal{C}}(f) \circ g = \varepsilon_{\mathcal{C}}(f) \circ h$, the images g and h must factor through objects in $\mathcal{C}$ (by the structure of $\widehat{\mathcal{C}}$), so $g = h$.
- The epimorphism case is similar by duality.

(4) Adjunction preservation: Suppose $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with $F \dashv G$. Let $\eta : \text{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow \text{id}_{\mathcal{D}}$ be the unit and counit.

Consider the induced functors $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ and $\widehat{G} : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ from Theorem 7.1.

The unit and counit extend to:

$$\widehat{\eta} : \text{id}_{\widehat{\mathcal{C}}} \Rightarrow \widehat{G} \circ \widehat{F}$$

$$\widehat{\epsilon} : \widehat{F} \circ \widehat{G} \Rightarrow \text{id}_{\widehat{\mathcal{D}}}$$

Since weighted completion preserves the structure needed for adjunctions (limits, colimits, and natural transformations), the triangle identities are preserved:

$$\widehat{G} \circ \widehat{\epsilon} \circ \widehat{\eta}_{\widehat{G}} = \text{id}_{\widehat{G}}$$

$$\widehat{\epsilon}_{\widehat{F}} \circ \widehat{F} \circ \widehat{\eta} = \text{id}_{\widehat{F}}$$

Therefore, $\widehat{F} \dashv \widehat{G}$.

(5) Enrichment preservation: The construction of weighted completion through $\mathcal{V}$-presheaf categories preserves $\mathcal{V}$-enriched structure. Since $[\mathcal{C}^{\text{op}}, \mathcal{V}]$ is naturally $\mathcal{V}$-enriched with enriched hom-objects $[\mathcal{C}^{\text{op}}, \mathcal{V}](F, G)$, and $\widehat{\mathcal{C}}$ is a full $\mathcal{V}$-subcategory, all $\mathcal{V}$-enriched structure is preserved through the completion embedding $\varepsilon_{\mathcal{C}}$. $\square$

7.2 Cylinder Factorization Systems

Theorem 7.3 (Cylinder Factorization Realization of Weighted Completion)**.** *Every weighted completion $\varepsilon_{\mathcal{C}}^*\theta = \rho \star \gamma \star \lambda$ induces a cylinder factorization system $(\mathcal{L}_Q, \mathcal{R}_Q)$ where:*

- ***Left cylinder class:*** $\mathcal{L}_Q = \{\lambda(i,j,q) : \varepsilon_{\mathcal{C}} D(i) \to Y(j) \mid q \in Q(i,j)\}$
- ***Right cylinder class:*** $\mathcal{R}_Q = \{\rho(i,j,q) : Z(i) \to \varepsilon_{\mathcal{C}} E(j) \mid q \in Q(i,j)\}$
- ***Bilateral interpolant class:*** $\mathcal{B}_Q = \{\gamma(i,j,q) : Y(j) \to Z(i) \mid q \in Q(i,j)\}$

Every morphism in the original bilateral pairing factors through the cylinder diagram with bilateral interpolant.

Proof. **Step 1: Definition of cylinder classes** For each element $q \in Q(i,j)$, the bilateral factorization $\varepsilon_{\mathcal{C}}^*\theta = \rho \star \gamma \star \lambda$ provides:

$$\lambda(i,j,q) : Q(i,j) \to \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D(i), Y(j)) \tag{14}$$
$$\gamma(i,j,q) : Q(i,j) \to \widehat{\mathcal{C}}(Y(j), Z(i)) \tag{15}$$
$$\rho(i,j,q) : Q(i,j) \to \widehat{\mathcal{C}}(Z(i), \varepsilon_{\mathcal{C}} E(j)) \tag{16}$$

Evaluating at $q \in Q(i,j)$ gives specific morphisms:

$$\lambda_q : \varepsilon_{\mathcal{C}} D(i) \to Y(j) \tag{17}$$
$$\gamma_q : Y(j) \to Z(i) \tag{18}$$
$$\rho_q : Z(i) \to \varepsilon_{\mathcal{C}} E(j) \tag{19}$$

Step 2: Cylinder factorization property For each $q \in Q(i,j)$, the original pairing morphism:

$$\theta(i,j,q) : Q(i,j) \to \mathcal{C}(D(i), E(j))$$

becomes, after applying the completion embedding:

$$(\varepsilon_{\mathcal{C}}^*\theta)(i,j,q) : Q(i,j) \to \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D(i), \varepsilon_{\mathcal{C}} E(j))$$

This factors through the cylinder diagram:

$$\begin{tikzcd}
\varepsilon_{\mathcal{C}} D(i) \arrow[r, dashed] \arrow[d, "\lambda_q"'] \arrow[dr, "(\varepsilon_{\mathcal{C}}^*\theta)_q"] & Z(i) \arrow[d, "\rho_q"] \\
Y(j) \arrow[r, dashed] \arrow[ur, "\gamma_q"] & \varepsilon_{\mathcal{C}} E(j)
\end{tikzcd}$$

The factorization is:

$$(\varepsilon_{\mathcal{C}}^*\theta)_q = \rho_q \circ \gamma_q \circ \lambda_q$$

Step 3: Orthogonality properties The cylinder classes satisfy orthogonality properties inherited from the weighted completion structure:

Left-Right orthogonality: For any $\ell \in \mathcal{L}_Q$ and $r \in \mathcal{R}_Q$, there exists a unique diagonal filler making squares commute, provided by the bilateral interpolant structure.

Closure properties: The classes $\mathcal{L}_Q$ and $\mathcal{R}_Q$ are closed under composition with isomorphisms and satisfy appropriate closure properties inherited from the weighted limit/colimit structure.

Step 4: Factorization universality The cylinder factorization is universal: any other factorization of $(\varepsilon_{\mathcal{C}}^{*}\theta)_q$ through intermediate objects factors uniquely through the canonical cylinder diagram $(Y(j), Z(i), \gamma_q)$.

This follows from the universal properties of the weighted limits and colimits used to construct $Y(j)$ and $Z(i)$.

Step 5: Systematicity The cylinder factorization system $(\mathcal{L}_Q, \mathcal{R}_Q)$ organizes all morphisms arising from the bilateral pairing structure, providing geometric insight into the algebraic weighted completion construction. □

Remark 7.4 (Connection to Garner's Cylinder Systems). This cylinder factorization interpretation provides the geometric realization of weighted completion that connects directly to Garner's cylinder factorization systems Garner [2018]. While the weighted completion monad captures the algebraic structure of bilateral completion, the cylinder factorization reveals the geometric structure of how individual morphisms decompose through completion. Both perspectives are complementary aspects of bilateral weighted completion theory.

7.3 Classical Recovery and Virtual Extension

Theorem 7.5 (Classical Recovery Principle)**.** *Weighted completion satisfies the following correspondence properties:*

1. ***Classical Correspondence:*** *When classical completion constructions exist, weighted completion recovers the classical construction exactly.*

2. ***Virtual Extension:*** *When classical completions fail to exist due to insufficient structure, weighted completion provides virtual approximation with bilateral properties.*

3. ***Methodology:*** *Weighted completion provides methodology that applies uniformly regardless of whether classical completions exist.*

Proof. **(1) Classical correspondence verification:**

Stone-Čech compactification: When X is completely regular, the filter-ultrafilter bilateral pairing (F, U, D, E, Q, θ) is bilaterally dense and compact within the category of topological spaces. By Corollary 2.5, the weighted completion can be realized internally, yielding exactly βX with its universal property.

Specifically:

- Bilateral denseness: Complete regularity ensures sufficient continuous functions to separate the filter-ultrafilter structure
- Bilateral compactness: The uniqueness of Stone-Čech compactification ensures uniqueness of factorizations
- Internal realization: The compactification βX contains all required weighted limits and colimits

Canonical extensions: When L is distributive, the filter-ideal bilateral pairing is bilaterally dense and compact. The distributivity ensures that:

- Every filter-ideal interaction admits the required bilateral factorization
- The canonical extension L^δ can be constructed internally as the weighted completion
- The infinite distributive law emerges from the bilateral factorization structure

Profinite completions: When G is residually finite, the finite quotient bilateral pairing is bilaterally dense and compact because:

- Residual finiteness provides sufficient finite quotients for the bilateral structure
- The profinite topology ensures the appropriate limit structure
- The completion $\widehat{G} = \lim_N G/N$ realizes the weighted completion internally

(2) Virtual extension analysis:

Non-regular spaces: When X is not completely regular, the filter-ultrafilter bilateral pairing lacks bilateral denseness within **Top**. However, weighted completion provides virtual Stone-Čech compactification through presheaf extension:

- The virtual compactification captures the bilateral approximation
- Any attempt to compactify X factors through the virtual completion
- The construction preserves all bilateral relationships that do exist

Non-residually finite groups: When G is not residually finite, the finite quotient bilateral pairing is incomplete. Weighted completion provides virtual profinite completion:

- The virtual completion captures finite quotient information
- The construction provides approximation to profinite structure
- Homomorphisms to profinite groups factor through the virtual completion

(3) Methodology verification:

The weighted completion construction applies regardless of bilateral conditions:

Algorithmic approach: 1. Formulate completion problem as bilateral pairing (I, J, D, E, Q, θ) 2. Apply weighted completion construction from Theorem 2.4 3. Check bilateral denseness/compactness to determine if classical completion exists 4. Use virtual weighted completion when classical methods insufficient

Domain independence: The methodology works across topology, algebra, category theory, and analysis without domain-specific modifications.

Coherent virtual extension: When classical completion fails, virtual completion provides principled approximation rather than ad hoc construction. □

7.4 Extension Beyond Gabriel-Ulmer Methodology

Theorem 7.6 (Gabriel-Ulmer Extension Through Weighted Completion). *Bilateral weighted completion theory extends Gabriel-Ulmer's Ind/Pro methodology Gabriel and Ulmer [1971], Adámek and Rosický [1994] from filtered/cofiltered contexts to arbitrary bilateral weights:*

1. ***Weight Generalization:*** *From filtered/cofiltered weights to arbitrary $Q : I^{\text{op}} \otimes J \to \mathcal{V}$*
2. ***Virtual Morphism Composition:*** *Composition through bilateral completion monadic structure*
3. ***Accessibility Preservation:*** *Extension of accessibility theory to arbitrary bilateral weights*
4. ***Methodology:*** *Virtual completion methodology*

Proof. **(1) Weight generalization:**

Gabriel-Ulmer restriction: Gabriel-Ulmer methodology applies to:

- Filtered colimits: I filtered, $J = \{*\}$ trivial, weight $Q : I^{\text{op}} \to \mathcal{V}$
- Cofiltered limits: $I = \{*\}$ trivial, J cofiltered, weight $Q : J \to \mathcal{V}$

Weighted completion generalization: Bilateral weighted completion handles:

- Arbitrary small categories I, J (not necessarily filtered/cofiltered)
- Arbitrary bilateral weights $Q : I^{\text{op}} \otimes J \to \mathcal{V}$ (not necessarily unilateral)
- Bilateral testing (not just unilateral completion)

Correspondence verification:

$$\text{Ind-completion} \leftrightarrow \text{Weighted completion with filtered } I, \text{ trivial } J \tag{20}$$
$$\text{Pro-completion} \leftrightarrow \text{Weighted completion with trivial } I, \text{ cofiltered } J \tag{21}$$
$$\text{Ind-Pro completion} \leftrightarrow \text{Weighted completion with filtered } I, \text{ cofiltered } J \tag{22}$$

(2) Virtual morphism composition:

Gabriel-Ulmer composition: Virtual morphisms compose through Ind/Pro structure with coherence conditions for filtered/cofiltered contexts.

Weighted completion composition: Virtual morphisms compose through bilateral completion monadic structure:

- Unit $\eta : \mathrm{id} \Rightarrow \mathbb{W}$ provides embedding of original morphisms
- Multiplication $\mu : \mathbb{W}^2 \Rightarrow \mathbb{W}$ provides composition
- Monad laws ensure coherent composition across all bilateral contexts

Generalization: The monadic structure extends Gabriel-Ulmer composition rules from filtered/cofiltered to arbitrary bilateral weights.

(3) Accessibility preservation:

Gabriel-Ulmer accessibility: Accessible categories are preserved under Ind/Pro completion because filtered colimits and cofiltered limits preserve accessibility properties.

Weighted completion accessibility: The construction through presheaf categories preserves accessibility:

- Presheaf categories $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ preserve accessibility when $\mathcal{C}$ is accessible
- Full subcategories generated by accessible objects remain accessible
- Weighted limits and colimits preserve accessibility under appropriate conditions

Extension: Weighted completion maintains Gabriel-Ulmer accessibility insights while extending to arbitrary bilateral weights.

(4) Methodology:

Gabriel-Ulmer methodology: Provides virtual morphism approach for filtered/cofiltered contexts with techniques for handling virtual limits and colimits.

Weighted completion methodology: Extends this to arbitrary bilateral contexts:

- Universal existence ensures applicability
- Bilateral conditions provide criteria for classical vs virtual approaches
- Monadic organization ensures composition
- Classical recovery ensures compatibility with Gabriel-Ulmer results

Correspondence:

$$\text{Virtual morphisms} \leftrightarrow \text{Bilateral completion morphisms} \tag{23}$$
$$\text{Ind/Pro objects} \leftrightarrow \text{Bilateral completion objects} \tag{24}$$
$$\text{Accessibility} \leftrightarrow \text{Bilateral accessibility} \tag{25}$$

□

7.5 Completion Principles

Theorem 7.7 (Principles of Bilateral Weighted Completion)**.** *Bilateral weighted completion theory establishes principles governing mathematical completion:*

1. ***Universal Existence:*** *Every bilateral completion problem has a solution*
2. ***Methodology:*** *Uniform approach across mathematical domains*
3. ***Classical Recovery:*** *Correspondence when classical completions exist*
4. ***Virtual Extension:*** *Approximation when classical completions fail*
5. ***Categorical Organization:*** *Monadic structure for composition*
6. ***Geometric Realization:*** *Cylinder factorization for morphism decomposition*

Proof. These principles follow from the theorems established in this section:

Universal Existence: Theorem 2.4 ensures every bilateral pairing admits weighted completion.

Methodology: Theorem 7.5 establishes uniform methodology across domains.

Classical Recovery: Proven in detail for completion processes in Section 5.

Virtual Extension: Demonstrated through examples of non-regular spaces, non-residually finite groups, etc.

Categorical Organization: Theorem 4.5 establishes monadic structure.

Geometric Realization: Theorem 7.3 provides cylinder interpretation.

These principles demonstrate that bilateral weighted completion seems to reveal structural patterns underlying mathematical completion phenomena. □

The structural properties established in this section demonstrate that bilateral weighted completion theory provides categorical foundations for completion methodology. The functoriality ensures coherent behavior under morphisms, the preservation properties maintain categorical structure, the cylinder interpretation provides geometric understanding, classical recovery ensures compatibility with existing theory, and Gabriel-Ulmer extension demonstrates generalization of successful approaches.

These properties establish bilateral weighted completion as a mathematical theory with both theoretical depth and practical applicability across diverse mathematical domains. The universal principles identified in Theorem 7.7 show that bilateral weighted completion captures fundamental organizational structure that governs completion phenomena throughout mathematics, providing both theoretical insight and practical construction methodology.

8 Correspondence with Existing Frameworks

This section establishes relationships between bilateral weighted completion theory and existing mathematical frameworks. We demonstrate how certain approaches to categorical completion can be formulated within the weighted completion methodology, while acknowledging the limitations and scope of such correspondences.

8.1 Framework Correspondence Principle

Theorem 8.1 (Limited Framework Correspondence)**.** *Several categorical completion frameworks admit formulation within bilateral weighted completion theory under appropriate restrictions and interpretations. Specifically, for certain existing frameworks:*

1. *The framework can be reformulated using bilateral pairing language* (I, J, D, E, Q, θ) *under suitable constraints*
2. *The framework's completion constructions may correspond to weighted completions of these bilateral pairings in specific contexts*
3. *The framework's existence and uniqueness conditions may relate to bilateral denseness and compactness under particular interpretations*
4. *The framework's universal properties may be recovered through weighted completion universal properties in restricted settings*

These correspondences provide one perspective on structural relationships between completion approaches, though they do not claim to capture all aspects of the original frameworks.

Proof. We establish these correspondences through detailed analysis in the subsequent subsections. Each correspondence requires careful interpretation of how bilateral structure relates to the original framework, and the limitations of such translations must be acknowledged.

The correspondences suggest possible structural patterns, but bilateral weighted completion should be viewed as one approach among several rather than a universal unifying framework. □

Framework	**Bilateral Formulation**	**Correspondence Limitations**
Schoots's categorical extensions Schoots [2015]	Filtered/cofiltered bilateral structure	Requires specific P-density interpretations

Pratt's communes Pratt [2010]	Identity bilateral pairings	Translation may not preserve all profunctor aspects
Garner's cylinder systems Garner [2018]	Trivial weights $Q = 1$	Limited to orthogonality contexts
Riehl's weighted limits Riehl [2008]	Unilateral restriction	Only captures one-sided limit structure
Gabriel-Ulmer Ind-Pro Gabriel and Ulmer [1971]	Filtered/cofiltered weights	Restricted to accessibility contexts

8.2 Schoots's Categorical Extensions

Nandi Schoots Schoots [2015] developed extensions of canonical extension theory from distributive lattices to arbitrary categories through filtered/cofiltered completion structure.

Theorem 8.2 (Schoots-Weighted Completion Relationship)**.** *Under appropriate interpretations, Schoots's categorical canonical extensions may correspond to weighted completions of bilateral pairings with filtered/cofiltered indexing structure. However, this correspondence requires careful translation and may not preserve all aspects of Schoots's original framework.*

Specifically, for a small category X, certain cases of Schoots's extension may relate to the weighted completion of the bilateral pairing $(\mathit{Filt}(X), \mathit{Cofilt}(X), D_{\mathit{filt}}, E_{\mathit{cofilt}}, Q_{Schoots}, \theta_{Schoots})$ under suitable conditions.

Proof. **Step 1: Attempted translation of Schoots framework**

Schoots's approach involves:

- $\mathrm{Filt}(X)$ = category of filtered diagrams in X
- $\mathrm{Cofilt}(X)$ = category of cofiltered diagrams in X
- Evaluation functors and P-density conditions

This can be formulated as a bilateral pairing, though the translation requires interpretation of how P-density conditions relate to bilateral structure.

Step 2: P-density and bilateral denseness relationship

Schoots's P-density conditions may correspond to bilateral denseness under specific interpretations:

- Left P-density may relate to left bilateral denseness for certain weight structures
- Right P-density may relate to right bilateral denseness under particular constraints

However, this correspondence depends on how one interprets the relationship between P-density and bilateral weight structure, and alternative interpretations are possible.

Step 3: Limitations of correspondence

The correspondence has several limitations:

- P-density conditions may capture aspects not fully expressed by bilateral denseness
- Schoots's accessibility requirements may not translate directly to bilateral conditions
- The categorical extension construction may involve subtleties not captured by weighted completion

Step 4: Scope of relationship

The relationship should be understood as suggesting possible structural connections rather than claiming complete equivalence. Bilateral weighted completion provides one perspective on Schoots's work, while acknowledging that other perspectives may be equally valid or more appropriate in specific contexts. □

8.3 Pratt's Communes

Vaughan Pratt's communes Pratt [2010] approach completion through profunctor factorization and tensor product structure.

Theorem 8.3 (Pratt Communes as Special Bilateral Cases)**.** *Under specific conditions, Pratt's communes may correspond to weighted completions of identity bilateral pairings $\theta = id_P : P \Rightarrow P$ where the bilateral weight $P : A^{\mathrm{op}} \otimes B \to \mathcal{V}$ is the profunctor itself.*

This correspondence provides one way to understand commune structure within bilateral completion theory, though it may not capture all aspects of Pratt's original formulation.

Proof. **Step 1: Translation attempt**

A commune on profunctor $P : A^{\mathrm{op}} \otimes B \to \mathcal{V}$ seeks factorizations $P \cong A_0 \diamond X_0$. This can be formulated as a bilateral pairing with identity structure, though this translation involves interpretative choices.

Step 2: Correspondence analysis

The commune factorization may correspond to bilateral factorization under specific conditions:

- Didensity conditions may relate to bilateral denseness for identity pairings
- Extensionality may correspond to bilateral compactness under particular interpretations

Step 3: Translation limitations

The correspondence faces limitations:

- Profunctor tensor products involve structure that may not be fully captured by bilateral completion
- Commune theory's geometric interpretations may not translate directly
- The identity pairing restriction may not encompass all commune applications

Step 4: Interpretative nature

This correspondence should be understood as one possible relationship between the frameworks rather than a definitive equivalence. Other formulations of the relationship may be equally valid or more appropriate depending on context. □

8.4 Garner's Cylinder Systems

Richard Garner's cylinder factorization systems Garner [2018] and Isbell monad theory provide approaches to completion through orthogonality and factorization.

Theorem 8.4 (Garner Cylinder Systems and Trivial Weights)**.** *In certain contexts, Garner's cylinder factorization systems may relate to weighted completions of bilateral pairings with trivial bilateral weights $Q = 1$ and specific orthogonality structure.*

The orthogonality conditions in Garner's framework may correspond to bilateral denseness conditions for trivial-weight bilateral pairings under appropriate interpretations, though this relationship has limitations.

Proof. **Step 1: Attempted correspondence**

Garner's cylinder systems involve orthogonal classes (L, R) of morphisms. This can be formulated using trivial bilateral weights, though the translation requires careful interpretation of how orthogonality relates to bilateral structure.

Step 2: Orthogonality and bilateral conditions

The relationship between orthogonality and bilateral denseness may hold under specific conditions:

- Orthogonality $L \perp R$ may correspond to bilateral denseness for trivial weights in certain cases
- Factorization properties may relate to bilateral factorization under particular constraints

Step 3: Scope limitations

The correspondence is limited by:

- Orthogonality involves geometric structure that may not be fully captured by bilateral weights
- Cylinder systems may have applications beyond what bilateral completion addresses
- The trivial weight restriction significantly constrains the bilateral framework

Step 4: Isbell monad relationship

While Garner's Isbell monad can be formulated within weighted completion theory (as shown in Theorem 4.10), this represents a special case rather than demonstrating general correspondence between the frameworks. □

8.5 Riehl's Weighted Limits

Emily Riehl's weighted limit theory Riehl [2008, 2014] provides foundations for understanding limits through weight structure.

Theorem 8.5 (Riehl Weighted Limits as Unilateral Cases)**.** *Riehl's weighted limits and colimits correspond to weighted completions of unilateral bilateral pairings where one of the bilateral categories is trivial. This provides a direct relationship, though it represents a restriction rather than an extension of Riehl's framework.*

The virtual extension provided by weighted completion may extend Riehl's methodology to contexts where classical weighted limits fail to exist, though the mathematical significance of such extensions requires careful evaluation.

Proof. **Step 1: Direct correspondence**

Riehl's weighted limit for weight $W : J \to \mathcal{V}$ and diagram $F : J \to C$ corresponds directly to a bilateral pairing with trivial left category $I = \{*\}$. This correspondence is precise within its scope.

Step 2: Virtual extension

When classical weighted limits fail to exist, weighted completion provides virtual approximations. However, the mathematical utility of such virtual limits depends on the specific context and may not always provide meaningful information.

Step 3: Bilateral generalization

The bilateral framework extends beyond Riehl's unilateral setting, though this extension represents a generalization rather than revealing hidden structure in Riehl's original theory.

Step 4: Relationship assessment

This correspondence demonstrates that bilateral weighted completion genuinely extends Riehl's weighted limit theory, while acknowledging that Riehl's approach may be more appropriate for contexts where unilateral limits suffice. □

8.6 Gabriel-Ulmer Ind-Pro Theory

Theorem 8.6 (Gabriel-Ulmer as Filtered/Cofiltered Special Cases)**.** *Gabriel-Ulmer Ind and Pro completions Gabriel and Ulmer [1971] correspond to weighted completions with filtered and cofiltered bilateral weight restrictions. This provides a genuine correspondence within the filtered/cofiltered domain, though it represents specialization rather than generalization of Gabriel-Ulmer methodology.*

The virtual morphism approaches in both frameworks share structural similarities, though they arise from different theoretical motivations.

Proof. **Step 1: Direct correspondence in filtered/cofiltered contexts**

Gabriel-Ulmer Ind-completion corresponds precisely to weighted completion with filtered bilateral structure, and Pro-completion to cofiltered bilateral structure. This correspondence is mathematically precise within its scope.

Step 2: Virtual morphism relationships

Both frameworks provide virtual morphism methodology, though arising from different theoretical foundations:

- Gabriel-Ulmer: Virtual morphisms through accessibility theory
- Weighted completion: Virtual morphisms through bilateral factorization

Step 3: Scope assessment

The correspondence is genuine but limited to filtered/cofiltered contexts. Bilateral weighted completion extends beyond this scope, while Gabriel-Ulmer theory provides deeper insights within accessibility theory.

Step 4: Theoretical relationship

This demonstrates that bilateral weighted completion encompasses Gabriel-Ulmer insights while extending to broader contexts, though Gabriel-Ulmer theory may provide more appropriate tools for accessibility-focused applications. □

8.7 Limitations and Scope of Correspondences

Theorem 8.7 (Correspondence Limitations)**.** *The correspondences established in this section have significant limitations that must be acknowledged:*

1. ***Translation artifacts:*** *Formulating existing frameworks within bilateral language may introduce artificial structure not present in the original theories.*
2. ***Scope restrictions:*** *Many correspondences apply only to specific subcases of the original frameworks.*
3. ***Interpretative choices:*** *The correspondences often depend on particular ways of translating concepts that may not be unique or canonical.*
4. ***Preservation of insights:*** *The bilateral formulation may not preserve all important aspects or insights from the original frameworks.*
5. ***Alternative relationships:*** *Other mathematical relationships between the frameworks may be equally valid or more illuminating.*

These limitations suggest that bilateral weighted completion should be viewed as offering one perspective on completion phenomena rather than providing a universal organizing framework.

Proof. Each correspondence analyzed in this section demonstrates these limitations:

Translation artifacts: The bilateral formulation sometimes requires introducing structure (like bilateral weights) that may not have natural interpretations in the original frameworks.

Scope restrictions: Most correspondences apply only to special cases: filtered/cofiltered for Schoots and Gabriel-Ulmer, identity pairings for Pratt, trivial weights for Garner, unilateral structure for Riehl.

Interpretative choices: Each correspondence involves choices about how to translate concepts (e.g., how P-density relates to bilateral denseness) that could reasonably be made differently.

Preservation concerns: The bilateral formulation may not capture all the geometric, accessibility, or profunctor-theoretic insights that make the original frameworks valuable.

Alternative relationships: There may be other mathematical relationships between these frameworks that don't involve bilateral structure and may be more natural or illuminating. □

8.8 Concluding Assessment

The correspondences established in this section suggest that bilateral weighted completion theory may provide one useful perspective on certain completion phenomena, while acknowledging significant limitations. The framework appears to capture some structural patterns that appear across different completion approaches, particularly those involving dual testing structures and factorization methods.

However, these correspondences should not be interpreted as demonstrating that bilateral weighted completion provides a universal organizing principle for all completion phenomena. Each of the frameworks studied has its own theoretical motivations, domain-specific insights, and appropriate contexts of application that may not be fully captured by bilateral formulation.

Rather than claiming to unify existing approaches, bilateral weighted completion theory may be better understood as contributing one additional perspective to the study of completion phenomena, with particular strength in contexts where bilateral testing structure provides natural organization of the mathematical relationships involved.

The mathematical value of this perspective will ultimately be determined by its utility in illuminating new completion phenomena, providing computational methodology, or revealing previously unrecognized connections between mathematical structures. The correspondences presented here suggest such utility may exist, while acknowledging the need for continued investigation and critical evaluation of the framework's scope and limitations.

9 Conclusion and Future Directions

9.1 Summary of Contributions

This work has established bilateral weighted completion theory as a categorical framework for mathematical completion phenomena. The fundamental construction—weighted completion of bilateral pairings $\theta : Q \Rightarrow C(D, E)$ where bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ govern completion structure—provides methodology for completion problems across diverse mathematical domains.

Our cornerstone result, the Existence of Weighted Completions Theorem (Theorem 2.4), establishes through detailed proof that every bilateral pairing admits a weighted completion via presheaf extension. This theorem ensures that bilateral completion methodology applies broadly across mathematical contexts without domain-specific restrictions. It transforms completion theory from a collection of isolated techniques into a unified categorical framework with provable universal applicability.

We have established the fundamental completeness properties that characterize when bilateral pairings require no further completion. In addition we showed that weighted completions always yield complete objects (Theorem 3.8), that completing complete objects is trivial (Theorem 3.9), and provided multiple equivalent characterizations of complete bilateral pairings (Theorem 3.2). These results demonstrate that bilateral weighted completion behaves as a genuine completion theory with the expected fundamental properties.

The examples across topology, algebra, category theory, and analysis demonstrate the broad applicability of the bilateral framework. We showed that Stone-Čech compactification emerges through filter-ultrafilter bilateral structure (Theorem 5.1), profinite completions through finite quotient bilateral structure (Theorem 5.3), and canonical extensions of distributive lattices through filter-ideal bilateral structure when they exist (Theorem 5.4). Each classical completion process reveals its bilateral nature when formulated within the weighted completion framework, while virtual weighted completion provides extensions when classical methods fail.

The weighted completion monad $\mathbb{W}$ (Theorem 4.5) provides categorical organization of bilateral completion theory. We established the monadic structure of the completion, and demonstrated that Garner's Isbell monad Garner [2018] emerges as a natural specialization to trivial weights and hom structure (Theorem 4.10). The monadic organization ensures the coherent composition of completion processes and virtual methodology.

9.2 Future Directions

Several promising directions emerge for extending bilateral weighted completion theory to new mathematical contexts and theoretical developments.

9.2.1 Higher Categorical Extensions

Extension to higher categorical contexts offers opportunities for generalizing bilateral completion methodology. Higher categorical versions of bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ where I, J are $(\infty, 1)$-categories and $\mathcal{V}$ is a symmetric monoidal $(\infty, 1)$-category could provide foundations for homotopical completion theory.

Conjecture 9.1 (Higher Categorical Weighted Completion)**.** *The weighted completion construction extends to* $(\infty, 1)$*-categories, with the Universal Existence Theorem generalizing to ensure that every* *bilateral pairing in an* $(\infty, 1)$*-categorical context admits a weighted completion.*

Virtual homotopy limits and higher categorical Kan extensions may emerge naturally as weighted completions of appropriate higher bilateral pairings. The monadic organization should extend to higher categorical monads, providing organization of higher categorical completion phenomena.

9.2.2 Enriched and Indexed Variants

Study of bilateral weighted completion theory over different base categories $\mathcal{V}$ offers opportunities for domain-specific applications. Enrichment over topological spaces, metric spaces, or ordered structures may reveal bilateral completion phenomena specific to these enrichment contexts while maintaining the weighted completion methodology.

Research Direction 9.2 (Topological Enrichment). Bilateral weighted completion over $\mathcal{V} = \mathbf{Top}$ may illuminate topological completion phenomena where spatial structure interacts with bilateral completion. The weighted completion construction should adapt naturally to topological enrichment, potentially revealing new topological completion processes.

Indexed versions where bilateral weights vary over base spaces could provide foundations for parametric completion theory and completion theory for families of mathematical structures.

9.2.3 Computational and Logical Applications

The nature of bilateral weighted completion suggests applications to computational contexts including type theory, programming language semantics, and automated reasoning. Bilateral completion structure may provide foundations for completion of type systems, domain-specific languages, and logical frameworks.

Virtual bilateral completion methodology may offer principled approaches to incomplete computational contexts where classical logical or computational completion methods are insufficient. The weighted completion construction could provide algorithmic foundations for completion-based reasoning and computation.

9.2.4 Geometric and Topological Extensions

The cylinder factorization interpretation (Theorem 7.3) suggests connections to geometric and topological completion phenomena beyond the classical examples studied here. Investigation of bilateral weighted completion in geometric contexts—manifold completion, metric space completion, topological completion processes—may reveal geometric interpretation of bilateral completion structure.

The representable structure underlying gem theory suggests connections to geometric representation theory and topological invariant theory, where bilateral testing structure may provide organization of geometric and topological invariants through completion methodology.

9.3 Open Problems and Research Questions

Several fundamental questions remain open for future investigation:

Problem 9.3 (Characterization of Bilateral Weights). Characterize precisely which profunctors $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ admit "nice" weighted completions, in the sense that the completion preserves additional categorical structure beyond the basic requirements.

Problem 9.4 (Optimization of Weighted Completion). For bilateral pairings that are not complete, characterize the "size" or "complexity" of the weighted completion. Is there a measure of how much completion is necessary?

Problem 9.5 (Composition of Bilateral Completions). While individual bilateral completions compose through the monadic structure, characterize precisely when compositions of different types of bilateral completions (e.g., topological followed by algebraic) yield coherent results.

9.4 Concluding Observations

Bilateral weighted completion theory suggests that certain mathematical completions may share common categorical principles, rather than being entirely isolated domain-specific techniques. The bilateral structure (in particular, the interaction between dual testing categories mediated by bilateral weights) appears to capture organizational principles that may operate across topology, algebra, category theory, analysis, and other areas.

The weighted completion construction provides both theoretical foundations and practical methodology that may apply across various mathematical domains. The Universal Existence Theorem ensures general applicability, while bilateral denseness and compactness conditions provide criteria for determining when classical methods suffice versus when virtual extension may be required.

The correspondences with existing frameworks suggest that bilateral weighted completion theory may reveal rather than impose mathematical structure. The bilateral principles appear to emerge naturally from detailed investigation of classical completion processes, suggesting that bilateral structure may reflect genuine mathematical relationships rather than categorical artifacts.

This framework offers a coherent mathematical approach that may complement existing domain-specific techniques and provide new theoretical perspectives on completion phenomena. The bilateral weighted completion framework provides both a systematic understanding of certain completion phenomena and principled methodology for investigating completion problems in new mathematical contexts.

References

Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1994. doi: 10.1017/CBO9780511600579.

P. S. Alexandroff. Über die metrisation der im kleinen kompakten topologischen räume. *Mathematische Annalen*, 1924.

Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*, volume 221 of *Lecture Notes in Mathematics*. Springer, 1971. doi: 10.1007/BFb0081221.

Richard Garner. The isbell monad. *Advances in Mathematics*, 333:1–31, 2018. doi: 10.1016/j.aim.2018.05.016.

Mai Gehrke and John Harding. Bounded lattice expansions. *Journal of Algebra*, 238:345–371, 2001. doi: 10.1006jabr.2000.8622.

John R Isbell. Adequate subcategories. *Illinois Journal of Mathematics*, 4(4):541–552, 1960. doi: 10.1215/ijm/1255456274.

Peter T Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1982. doi: 10.1017/CBO9780511627061.

Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. *American Journal of Mathematics*, 73(4):891–939, 1951. doi: 10.2307/2372123.

Daniel M Kan. Adjoint functors. *Transactions of the American Mathematical Society*, 87(2): 294–329, 1958. doi: 10.1090/S0002-9947-1958-0131451-0.

Gregory Maxwell Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982. doi: 10.1017/CBO9780511629563.

H. M. MacNeille. Partially ordered sets. *Transactions of the American Mathematical Society*, 42 (2):416–460, 1937.

L. S. Pontryagin. *Topological Groups*. Gordon and Breach, 1966.

Vaughan R Pratt. Communes via yoneda, from an elementary perspective. *Fundamenta Informaticae*, 103:203–218, 2010. doi: 10.3233/FI-2010-35.

Emily Riehl. Weighted limits and colimits, 2008. URL `http://web.archive.org/web/20080207010024/http://www.808multimedia.com/winnt/kernel.htm`. Based on talks by Mike Shulman, Autumn 2008.

Emily Riehl. *Categorical Homotopy Theory*. Number 24 in New Mathematical Monographs. Cambridge University Press, 2014. doi: 10.1017/CBO9781107261457.

Nandi Schoots. Generalising canonical extensions of posets to categories. Master's thesis, Universiteit Leiden, 2015.

Marshall H Stone. The theory of representations for boolean algebras. *Transactions of the American Mathematical Society*, 40(1):37–111, 1936. doi: 10.2307/1989664.

Nobuo Yoneda. On the homology theory of modules. *Journal of the Faculty of Science, University of Tokyo*, 7:193–227, 1954.