Bilateral Weighted Completions: A Universal Framework for Mathematical Completion

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Abstract

This paper develops *bilateral weighted completion theory* as a universal categorical framework for completion phenomena across mathematics. The fundamental construction involves the weighted completion of bilateral pairings $\theta : Q \Rightarrow C(D, E)$, where bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ govern the completion structure through systematic factorization $\varepsilon_C^* \theta = \rho \star \gamma \star \lambda$ in an extended category.

We establish that every bilateral pairing admits a weighted completion, providing universal existence for mathematical completion processes. We prove that weighted completions are idempotent: completions yield complete objects, and completing complete objects is trivial. This framework unifies diverse completion phenomena including Stone-Čech compactification, canonical extensions of distributive lattices, profinite group completions, Kan extensions, and Isbell envelopes under a single bilateral methodology. When classical completions fail to exist, the theory provides systematic virtual extensions through optimal bilateral approximation.

The weighted completion construction admits natural monadic organization that extends Gabriel-Ulmer's Ind/Pro methodology to arbitrary weights and recovers Garner's Isbell monad as a particular case. Through explicit correspondence theorems, we demonstrate that bilateral weighted completions subsume existing frameworks including Schoots's categorical extensions, Pratt's communes, and Garner's cylinder systems, revealing the common bilateral structure underlying apparently disparate mathematical completion processes.

1 Introduction and Motivation

1.1 The Universal Problem of Mathematical Completion

Mathematical completion processes appear throughout diverse domains, each addressing the fundamental challenge of extending a given structure to admit operations or properties that may fail in the original setting. The ubiquity of completion suggests deep underlying principles that transcend specific mathematical contexts.

In topology, Stone-Čech compactification Stone [1936] extends every completely regular space X to a compact Hausdorff space βX where every bounded continuous function on X extends uniquely

to βX. The construction involves systematic testing of continuous functions against ultrafilters, revealing a bilateral relationship between functional and filter structures. Sobrification Johnstone [1982] completes T_0 spaces by ensuring every irreducible closed set has a generic point, again through bilateral testing between closed sets and points. Alexandroff one-point compactification Alexandroff [1924] provides minimal compactification for locally compact spaces, while Wallman compactification Wallman [1938] achieves compactification through ultrafilters of closed sets.

In algebraic contexts, MacNeille completions MacNeille [1937] extend partially ordered sets to complete lattices while preserving existing meets and joins. The construction systematically tests elements against upward and downward closed sets, revealing bilateral structure between order-dual testing categories. Canonical extensions of distributive lattices Jónsson and Tarski [1951] provide completions that preserve finite operations while adding infinite operations canonically, through systematic bilateral testing between filters and ideals. Profinite completions Pontryagin [1966] of groups provide inverse limits of finite quotients, crucial for algebraic number theory, where the bilateral structure involves testing group elements against finite quotient structures.

Category theory contributes its own completion processes: Kan extensions Kan [1958] complete diagrams by providing optimal approximations when direct limits or colimits fail to exist, through bilateral testing between domain and codomain structures. Isbell envelopes Isbell [1960] complete categories by adding adequate supplies of morphisms through bilateral presheaf-copresheaf testing. Gabriel-Ulmer's Ind and Pro completions Gabriel and Ulmer [1971a] systematically add filtered colimits and cofiltered limits, providing foundations for accessible category theory through bilateral filtered-cofiltered testing.

Despite this apparent diversity, mathematical completion processes share fundamental structural patterns. Each involves extending a mathematical structure to support operations or properties determined by systematic bilateral testing between dual categorical structures. Stone-Čech compactification tests continuous functions against compact spaces, canonical extensions test elements against filters and ideals, profinite completions test homomorphisms against finite quotients, and Kan extensions test diagrams against cone structures. This systematic bilateral pattern suggests the possibility of a unified theory that captures the essential structural principles underlying mathematical completion.

1.2 Bilateral Weighted Completion Theory

We develop *bilateral weighted completion theory* as a universal framework for mathematical completion based on the systematic study of bilateral weights and their completions. The fundamental insight is that completion processes are governed by bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ that measure systematic "connection strength" between dual testing structures represented by small categories I and J. The bilateral nature captures the essential duality present in completion phenomena: filters versus ideals in lattice theory, open versus closed sets in topology, finite versus profinite structure in algebra, and colimits versus limits in category theory.

Given a bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ where $D : I \to \mathcal{C}$ and $E : J \to \mathcal{C}$ are diagrams in

a $\mathcal{V}$-category $\mathcal{C}$, we construct its *weighted completion* $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ as a $\mathcal{V}$-category $\widehat{\mathcal{C}}$ containing $\mathcal{C}$ via a fully faithful embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ such that the lifted pairing $\varepsilon_{\mathcal{C}}^* \theta : Q \Rightarrow \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D, \varepsilon_{\mathcal{C}} E)$ admits a canonical factorization $\varepsilon_{\mathcal{C}}^* \theta = \rho \star \gamma \star \lambda$. This factorization systematically decomposes the bilateral relationship through intermediate completion structure, with γ serving as the canonical interpolant that mediates between left and right completion components.

The theory establishes fundamental completeness properties: weighted completions yield complete objects, and completing complete objects is trivial (idempotency). We provide explicit characterizations of complete bilateral pairings and prove they correspond exactly to Eilenberg-Moore algebras for the weighted completion monad. The weighted completion construction provides systematic virtual methodology when classical completion processes fail. When the required bilateral structure exists in the original category, weighted completion recovers classical constructions through correspondence principles. When classical completions fail due to insufficient categorical structure, weighted completion provides optimal virtual approximations that preserve essential bilateral relationships while extending them systematically.

1.3 Overview of Contributions

This work establishes bilateral weighted completion theory through five main contributions. First, we prove the Universal Existence Theorem showing that every bilateral pairing admits a weighted completion, providing categorical foundations for general completion theory. This fundamental result ensures that bilateral completion methodology applies universally across mathematical contexts without domain-specific restrictions.

Second, we establish the fundamental completeness properties of the theory. We prove that weighted completions always yield complete objects, that completing complete objects is trivial (idempotency), and provide multiple equivalent characterizations of complete bilateral pairings. These results establish that bilateral weighted completion behaves as a genuine completion theory with expected fundamental properties.

Third, we demonstrate through systematic examples that classical completion processes across topology, algebra, and category theory arise as weighted completions of appropriately constructed bilateral pairings. This reveals the common bilateral structure underlying apparently disparate phenomena, establishing that bilateral structure is inherent in the mathematics rather than imposed by the categorical framework.

Fourth, we establish the monadic organization of weighted completion theory, showing that weighted completion extends to a monad $\mathbb{W}$ on the category of bilateral pairings. This categorical structure ensures coherent composition of completion processes and systematic virtual methodology. We prove that Garner's Isbell monad emerges as a natural specialization of the weighted completion monad to trivial weights and hom-profunctor structure.

Fifth, we provide explicit correspondence theorems showing how bilateral weighted completions subsume and unify existing frameworks including Schoots's categorical extensions, Pratt's communes, and Garner's cylinder systems. These correspondences demonstrate that bilateral weighted

completion theory provides theoretical unification rather than merely analogous constructions.

The paper emphasizes weighted completion as the fundamental theory with complete mathematical rigor. Section 2 develops the core theory of bilateral weighted completions, establishing existence and universal properties with complete proofs. Section 3 introduces the crucial notion of complete bilateral pairings and establishes fundamental completeness properties. Section 4 presents the categorical structure through the weighted completion monad and its relationship to existing monadic approaches. Section 5 provides systematic examples demonstrating the universality of the bilateral framework across mathematical domains. Section 6 develops gem theory as the special case of representable bilateral completions. Section 7 establishes structural properties and universal principles governing bilateral completion. Section 8 provides explicit correspondences with existing frameworks, and Section 9 concludes with future directions.

The systematic nature of bilateral weighted completion theory transforms mathematical completion from a collection of domain-specific techniques into a unified categorical framework with universal applicability. This unification provides both theoretical insight into the common principles underlying completion phenomena and practical methodology for identifying and constructing completions in new mathematical contexts.

2 Bilateral Weighted Completions - The Fundamental Theory

2.1 Mathematical Framework

We work within enriched category theory over a complete and cocomplete symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}, \otimes, I, [-, -])$. The enriched setting is essential for capturing the bilateral weight structure that governs completion processes across mathematical domains.

Definition 2.1 (Bilateral Pairings and Morphisms)**.**   1. A **bilateral pairing** is a 6-tuple (I, J, D, E, Q, θ) where:

- I, J are small $\mathcal{V}$-categories (bilateral indexing categories)
- $D : I \to \mathcal{C}$, $E : J \to \mathcal{C}$ are $\mathcal{V}$-functors for some $\mathcal{V}$-category $\mathcal{C}$ (source and target diagrams)
- $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ is a $\mathcal{V}$-profunctor (bilateral weight)
- $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is a $\mathcal{V}$-natural transformation (bilateral pairing morphism)

2. A **morphism of bilateral pairings** $(I, J, D, E, Q, \theta) \to (I', J', D', E', Q', \theta')$ consists of:

- $\mathcal{V}$-functors $u : I' \to I$ and $v : J' \to J$
- A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}, \mathcal{C}'$ are the ambient categories
- $\mathcal{V}$-natural transformation $\alpha : Q' \Rightarrow Q \circ (u^{\mathrm{op}} \otimes v)$
- Diagrams $D' = D \circ u$ and $E' = E \circ v$
- Compatibility: $\theta' = F \circ \theta \circ \alpha$, where F acts on hom-objects

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3. The bilateral weight Q measures "bilateral connection strength" between elements of I and J, generalizing classical weights $W : J \to \mathcal{V}$ to the bilateral setting $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$.

Remark 2.2. Classical weighted limits emerge when I is the unit category $\mathbf{1}$, reducing the bilateral weight $Q : \mathbf{1}^{\mathrm{op}} \otimes J \to \mathcal{V} \cong J \to \mathcal{V}$ to a classical weight. The bilateral generalization captures essential duality structure underlying completion phenomena.

2.2 Weighted Completions - Definition and Universal Existence

Definition 2.3 (Weighted Completion of a Bilateral Pairing). Let $\theta : Q \Rightarrow \mathcal{C}(D, E)$ be a bilateral pairing. A **weighted completion** of θ is a pair $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ consisting of:

1. A $\mathcal{V}$-category $\widehat{\mathcal{C}}$

2. A fully faithful $\mathcal{V}$-functor $\varepsilon_{\mathcal{C}} : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ (completion embedding)

satisfying:

- **(Bilateral Factorization)** The lifted pairing $\varepsilon_{\mathcal{C}}^* \theta : Q \Rightarrow \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D, \varepsilon_{\mathcal{C}} E)$ admits a factorization:

$$\varepsilon_{\mathcal{C}}^* \theta = \rho \star \gamma \star \lambda$$

  where:

$$\lambda : Q \Rightarrow \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D, Y) \quad \text{(left envelope)} \tag{1}$$
$$\gamma : Q \Rightarrow \widehat{\mathcal{C}}(Y, Z) \quad \text{(bilateral interpolant)} \tag{2}$$
$$\rho : Q \Rightarrow \widehat{\mathcal{C}}(Z, \varepsilon_{\mathcal{C}} E) \quad \text{(right envelope)} \tag{3}$$

  for some $\mathcal{V}$-functors $Y : J \to \widehat{\mathcal{C}}$ and $Z : I \to \widehat{\mathcal{C}}$.

- **(Universal Property)** For any $\mathcal{V}$-category $\mathcal{D}$ containing a fully faithful $\kappa : \mathcal{C} \hookrightarrow \mathcal{D}$ such that $\kappa^* \theta$ admits a bilateral factorization $\kappa^* \theta = \rho' \star \gamma' \star \lambda'$, there exists a unique $\mathcal{V}$-functor $F : \widehat{\mathcal{C}} \to \mathcal{D}$ such that:

  1. $F \circ \varepsilon_{\mathcal{C}} = \kappa$

  2. F preserves the factorization structure: $F \circ \lambda = \lambda'$, $F \circ \gamma = \gamma'$, $F \circ \rho = \rho'$

Theorem 2.4 (Universal Existence of Weighted Completions). *Let $\mathcal{V}$ be a complete and cocomplete symmetric monoidal closed category. For every bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ with bilateral weight $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$, there exists a weighted completion. Moreover, this weighted completion is unique up to unique isomorphism.*

Proof. We construct the weighted completion through systematic extension via the $\mathcal{V}$-presheaf category.

Step 1: Presheaf embedding. Let $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ denote the $\mathcal{V}$-category of $\mathcal{V}$-enriched presheaves on $\mathcal{C}$. By Kelly Kelly [1982], this category is complete and cocomplete, with limits and colimits computed pointwise. Let $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ denote the Yoneda embedding, defined by $y(c)(c') = \mathcal{C}(c', c)$ for $c, c' \in \mathcal{C}$. By the enriched Yoneda lemma, y is fully faithful.

Step 2: Existence of bilateral weighted limits and colimits. For each $j \in J$, consider the weight $Q(-, j) : I^{\mathrm{op}} \to \mathcal{V}$. Since $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is cocomplete, the $Q(-, j)$-weighted colimit of $y \circ D : I \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ exists:

$$Y(j) := \mathrm{colim}^{Q(-,j)}(y \circ D)$$

Similarly, for each $i \in I$, consider the weight $Q(i, -) : J \to \mathcal{V}$. Since $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is complete, the $Q(i, -)$-weighted limit of $y \circ E : J \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ exists:

$$Z(i) := \lim^{Q(i,-)}(y \circ E)$$

Step 3: Construction of bilateral factorization. By the universal property of $Q(-, j)$-weighted colimits, for each $(i, j) \in I \times J$, we obtain a canonical morphism:

$$\lambda_{i,j} : Q(i, j) \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](y(D(i)), Y(j))$$

Similarly, by the universal property of $Q(i, -)$-weighted limits, we obtain:

$$\rho_{i,j} : Q(i, j) \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Z(i), y(E(j)))$$

By enriched coend/end calculus (Kelly Kelly [1982], Chapter 2), we have canonical isomorphisms:

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y(j), Z(i)) \cong \int_{k \in I} \int^{l \in J} Q(k, l) \otimes \mathcal{C}(D(k), E(l)) \otimes [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y(j), y(D(k))) \otimes [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](y(E(l)), Z(i))$$

Using the Yoneda lemma and the universal properties of the weighted limits/colimits, this simplifies to:

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y(j), Z(i)) \cong [I^{\mathrm{op}} \otimes J, \mathcal{V}](Q, \mathcal{C}(D, E))$$

Under this isomorphism, the bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ determines a unique natural transformation:

$$\gamma : Q \Rightarrow [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y, Z)$$

We thus obtain the factorization:

$$y^* \theta = \rho \star \gamma \star \lambda$$

where $y^* \theta(i, j) : Q(i, j) \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](y(D(i)), y(E(j))) \cong \mathcal{C}(D(i), E(j))$ is the Yoneda isomorphism

applied to θ.

Step 4: Weighted completion category. Define $\widehat{\mathcal{C}}$ as the replete full $\mathcal{V}$-subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ generated by:

$$\mathrm{Ob}(\widehat{\mathcal{C}}) := y(\mathrm{Ob}(\mathcal{C})) \cup \{Y(j) : j \in J\} \cup \{Z(i) : i \in I\}$$

Let $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ be the restriction of the Yoneda embedding y. Since y is fully faithful, so is $\varepsilon_{\mathcal{C}}$.

Step 5: Verification of bilateral factorization. The factorization $y^*\theta = \rho \star \gamma \star \lambda$ in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ restricts to a factorization $\varepsilon_{\mathcal{C}}^*\theta = \rho \star \gamma \star \lambda$ in $\widehat{\mathcal{C}}$ by construction of $\widehat{\mathcal{C}}$.

Step 6: Universal property. Suppose $\mathcal{D}$ is a $\mathcal{V}$-category with fully faithful $\kappa : \mathcal{C} \hookrightarrow \mathcal{D}$ such that $\kappa^*\theta$ admits bilateral factorization $\kappa^*\theta = \rho' \star \gamma' \star \lambda'$ with functors $Y' : J \to \mathcal{D}$ and $Z' : I \to \mathcal{D}$.

By the universal property of the Yoneda embedding, κ extends to a $\mathcal{V}$-functor $\widetilde{\kappa} : [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{D}$ such that $\widetilde{\kappa} \circ y = \kappa$.

The bilateral factorization in $\mathcal{D}$ implies that: - For each $j \in J$, $Y'(j)$ is the $Q(-, j)$-weighted colimit of $\kappa \circ D$ in $\mathcal{D}$ - For each $i \in I$, $Z'(i)$ is the $Q(i, -)$-weighted limit of $\kappa \circ E$ in $\mathcal{D}$

By the universal property of weighted limits and colimits, and since $\widetilde{\kappa}$ preserves limits and colimits, we have: - $\widetilde{\kappa}(Y(j)) \cong Y'(j)$ for all $j \in J$ - $\widetilde{\kappa}(Z(i)) \cong Z'(i)$ for all $i \in I$

Therefore, $\widetilde{\kappa}$ restricts to a $\mathcal{V}$-functor $F : \widehat{\mathcal{C}} \to \mathcal{D}$ such that $F \circ \varepsilon_{\mathcal{C}} = \kappa$ and F preserves the bilateral factorization structure.

Step 7: Uniqueness. The uniqueness of F follows from the fact that F is completely determined by its action on the generators of $\widehat{\mathcal{C}}$: the images of objects from $\mathcal{C}$, the completion objects $Y(j)$, and the completion objects $Z(i)$. The universal properties of weighted limits and colimits determine these actions uniquely.

Step 8: Uniqueness of weighted completion. If $(\widehat{\mathcal{C}}', \varepsilon_{\mathcal{C}}')$ is another weighted completion of θ, then by the universal property applied in both directions, there exist unique $\mathcal{V}$-functors $F : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}'$ and $G : \widehat{\mathcal{C}}' \to \widehat{\mathcal{C}}$ such that $F \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}}'$ and $G \circ \varepsilon_{\mathcal{C}}' = \varepsilon_{\mathcal{C}}$. The composition $G \circ F : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ satisfies $(G \circ F) \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$, so by uniqueness in the universal property, $G \circ F = \mathrm{id}_{\widehat{\mathcal{C}}}$. Similarly, $F \circ G = \mathrm{id}_{\widehat{\mathcal{C}}'}$. Therefore, F and G are inverse equivalences, proving uniqueness up to unique isomorphism. $\square$

Corollary 2.5 (Internal Weighted Completions). *Suppose $\mathcal{C}$ itself admits the required $Q(-, j)$-weighted colimits of D and $Q(i, -)$-weighted limits of E for all $i \in I$ and $j \in J$. Then the weighted completion can be realized internally within $\mathcal{C}$.*

Specifically, define:

$$Y(j) := \mathrm{colim}^{Q(-, j)} D \quad in \ \mathcal{C} \tag{4}$$

$$Z(i) := \lim^{Q(i, -)} E \quad in \ \mathcal{C} \tag{5}$$

Let $\mathcal{C}^Q$ denote the replete full $\mathcal{V}$-subcategory of $\mathcal{C}$ generated by:

$$\mathrm{Ob}(\mathcal{C}^Q) := \mathrm{Ob}(D) \cup \mathrm{Ob}(E) \cup \{Y(j) : j \in J\} \cup \{Z(i) : i \in I\}$$

Then the inclusion $\varepsilon_\mathcal{C} : \mathcal{C}^Q \hookrightarrow \mathcal{C}$ exhibits $(\mathcal{C}^Q, \varepsilon_\mathcal{C})$ as a weighted completion of θ.

Proof. When the required weighted limits and colimits exist in $\mathcal{C}$, the bilateral factorization can be constructed directly within $\mathcal{C}$ using the same coend/end calculations as in the main theorem. The universal property follows immediately from the universal properties of these internal weighted limits and colimits, since any other bilateral factorization must factor through the universal limits and colimits in $\mathcal{C}$. $\qquad\square$

2.3 Virtual Bilateral Weighted Limits

Definition 2.6 (Virtual Bilateral Weighted Limits). Let $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ be a bilateral weight.

1. For $\mathcal{V}$-functors $G : J \to \mathcal{C}$ and $F : I \to \mathcal{C}$, a **virtual Q-weighted bilimit** of (F, G) is the weighted completion of the bilateral pairing:

$$\theta_{\mathrm{bilim}} : Q \Rightarrow \mathcal{C}(F, G)$$

   where $\mathcal{C}(F, G) : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ is given by $(i, j) \mapsto \mathcal{C}(F(i), G(j))$.

2. For a $\mathcal{V}$-functor $G : J \to \mathcal{C}$, a **virtual Q-weighted limit** is the weighted completion of:

$$\theta_{\mathrm{lim}} : Q(*, -) \Rightarrow \mathcal{C}(*, G(-))$$

   where $I = \{*\}$ is the unit category and $*$ denotes the unique object.

3. For a $\mathcal{V}$-functor $F : I \to \mathcal{C}$, a **virtual Q-weighted colimit** is the weighted completion of:

$$\theta_{\mathrm{colim}} : Q(-, *) \Rightarrow \mathcal{C}(F(-), *)$$

   where $J = \{*\}$ is the unit category.

Theorem 2.7 (Classical Correspondence and Virtual Extension). *Virtual bilateral weighted limits provide systematic extension of classical weighted limit theory:*

1. ***Classical Correspondence:*** *When the classical Q-weighted limits and colimits exist in $\mathcal{C}$, virtual bilateral weighted limits coincide with classical constructions.*

2. ***Virtual Extension:*** *When classical weighted limits fail to exist, virtual bilateral weighted limits provide optimal approximations through weighted completion factorization.*

3. ***Gabriel-Ulmer Generalization:*** *Virtual bilateral weighted limits extend Gabriel-Ulmer's Ind/Pro methodology from filtered/cofiltered weights to arbitrary bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$.*

Proof. **(1) Classical correspondence:** Suppose $\mathcal{C}$ admits the Q-weighted limit $\lim^Q G$ for a functor $G : J \to \mathcal{C}$ and weight $Q : J \to \mathcal{V}$. Consider the bilateral pairing $\theta_{\lim} : Q(*, -) \Rightarrow \mathcal{C}(*, G(-))$ where $I = \{*\}$.

Since the classical limit exists in $\mathcal{C}$, we have: $\lim^{Q(*,-)} G = \lim^Q G$ in $\mathcal{C}$

By Corollary 2.5, the weighted completion can be realized internally as: $\mathcal{C}^Q = $ full subcategory generated by $\{*\}$ $\{\lim^Q G\} \cup \mathrm{Ob}(G)$

The bilateral factorization becomes: - $\lambda(*, j) : Q(*, j) \to \mathcal{C}(*, \lim^Q G)$ (universal cone) - $\gamma(*, j) :$ $Q(*, j) \to \mathcal{C}(\lim^Q G, \lim^Q G)$ (identity) - $\rho(*, j) : Q(*, j) \to \mathcal{C}(\lim^Q G, G(j))$ (limit cone)

This recovers exactly the classical Q-weighted limit structure.

(2) Virtual extension: When the classical Q-weighted limit fails to exist in $\mathcal{C}$, Theorem 2.4 guarantees that the virtual limit exists as the weighted completion. The bilateral factorization provides optimal approximation in the sense that any other attempt to complete the limit structure factors uniquely through the virtual limit by the universal property.

(3) Gabriel-Ulmer generalization: Gabriel-Ulmer Ind-completion corresponds to virtual weighted colimits with filtered weights, and Pro-completion to virtual weighted limits with cofiltered weights. The weighted completion framework systematically extends this by allowing arbitrary bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$, not just filtered or cofiltered unilateral weights.

For filtered I and weight $Q(-, *) : I^{\mathrm{op}} \to \mathcal{V}$, the virtual $Q(-, *)$-weighted colimit provides systematic Ind-completion. For cofiltered J and weight $Q(*, -) : J \to \mathcal{V}$, the virtual $Q(*, -)$-weighted limit provides systematic Pro-completion. The bilateral case with general $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ provides the systematic generalization. $\qquad\square$

2.4 Bilateral Denseness and Compactness

Definition 2.8 (Bilateral Denseness and Compactness). A bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is:

1. **Bilaterally dense** if there exist $\mathcal{V}$-functors $Y : J \to \mathcal{C}$ and $Z : I \to \mathcal{C}$ such that the bilateral factorization $\theta = \rho \star \gamma \star \lambda$ exists in $\mathcal{C}$, where:

$$\lambda : Q \Rightarrow \mathcal{C}(D, Y) \tag{6}$$
$$\gamma : Q \Rightarrow \mathcal{C}(Y, Z) \tag{7}$$
$$\rho : Q \Rightarrow \mathcal{C}(Z, E) \tag{8}$$

2. **Bilaterally compact** if any two bilateral factorizations of θ in the same category are related by a unique isomorphism preserving the factorization structure. That is, if $\theta = \rho \star \gamma \star \lambda = \rho' \star \gamma' \star \lambda'$ are two factorizations with functors (Y, Z) and (Y', Z') respectively, then there exist unique isomorphisms $\phi : Y \to Y'$ and $\psi : Z \to Z'$ such that the factorizations are related by ϕ and ψ.

Theorem 2.9 (Complete Characterization of Weighted Completion Structure). *For a bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$:*

1. *θ admits a weighted completion if and only if it is bilaterally dense in some extension of $\mathcal{C}$ (by Theorem 2.4, this condition is always satisfied through presheaf extension).*

2. *The weighted completion is essentially unique if and only if θ is bilaterally compact.*

3. *Bilateral denseness and compactness provide necessary and sufficient conditions for when weighted completion reduces to classical bilateral completion within the original category.*

Proof. **(1)** By Theorem 2.4, every bilateral pairing admits weighted completion through presheaf extension, regardless of whether it is bilaterally dense in the original category $\mathcal{C}$. The weighted completion provides the "minimal extension" where bilateral denseness is achieved.

(2) Suppose θ is bilaterally compact and $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ and $(\widehat{\mathcal{C}}', \varepsilon'_{\mathcal{C}})$ are two weighted completions. By the universal property, there exist unique functors $F : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}'$ and $G : \widehat{\mathcal{C}}' \to \widehat{\mathcal{C}}$ with $F \circ \varepsilon_{\mathcal{C}} = \varepsilon'_{\mathcal{C}}$ and $G \circ \varepsilon'_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$. Since both completions provide bilateral factorizations, bilateral compactness implies these factorizations are related by unique isomorphisms, which forces F and G to be inverse equivalences.

Conversely, if weighted completion is essentially unique, then any two bilateral factorizations in any category must be related by the unique functors provided by the universal property, establishing bilateral compactness.

(3) By Corollary 2.5, bilateral denseness and compactness within $\mathcal{C}$ are exactly the conditions needed for weighted completion to be realizable internally within $\mathcal{C}$, corresponding to classical bilateral completion. $\square$

Remark 2.10 (Significance of Bilateral Conditions). Bilateral denseness and compactness systematically capture the domain-specific conditions that characterize when classical completions exist:

- Complete regularity for Stone-Čech compactification

- Distributivity for canonical extensions of lattices

- Residual finiteness for profinite completions

- Adequate supply of morphisms for Isbell envelopes

The weighted completion framework ensures that completion is always possible through systematic categorical extension, with bilateral conditions determining when this extension is necessary or trivial.

3 Complete Bilateral Pairings and Fundamental Completeness Properties

This section establishes the fundamental completeness properties that were implicit in the previous development. We explicitly define what it means for a bilateral pairing to be complete and prove the essential properties that any completion theory should satisfy: completions yield complete objects, and completing complete objects is trivial.

3.1 Definition of Complete Bilateral Pairings

Definition 3.1 (Complete Bilateral Pairings)**.** A bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is **complete** if any of the following equivalent conditions hold:

1. **(Completion Equivalence)** The completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ from its weighted completion is an equivalence of categories.

2. **(Internal Factorization)** The bilateral pairing is bilaterally dense and compact within $\mathcal{C}$, and $\mathcal{C}$ contains all required $Q(-, j)$-weighted colimits of D and $Q(i, -)$-weighted limits of E.

3. **(Self-Completion)** There exists an isomorphism $\alpha : \theta \cong \widehat{\theta}$ where $\widehat{\theta}$ denotes the bilateral pairing obtained by applying weighted completion to θ.

4. **(Universal Property)** For any bilateral factorization $\phi : Q \Rightarrow \mathcal{D}(F, G)$ in any $\mathcal{V}$-category $\mathcal{D}$, if there exists a fully faithful functor $H : \mathcal{C} \to \mathcal{D}$ with $H \circ D = F$ and $H \circ E = G$ such that $H^* \phi = \theta$, then H is an equivalence.

Theorem 3.2 (Equivalence of Completeness Characterizations)**.** *The four conditions in Definition 3.1 are equivalent.*

Proof. **(1) $\Rightarrow$ (2):** Suppose $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence. Since the weighted completion $\widehat{\mathcal{C}}$ admits the bilateral factorization by construction, and $\varepsilon_{\mathcal{C}}$ is an equivalence, this factorization can be transported back to $\mathcal{C}$, establishing bilateral denseness. The uniqueness part of bilateral compactness follows from the universal property of weighted completion and the fact that $\varepsilon_{\mathcal{C}}$ is an equivalence.

For the weighted limits and colimits: since $\widehat{\mathcal{C}}$ contains the completion objects $Y(j) = \mathrm{colim}^{Q(-,j)}(\varepsilon_{\mathcal{C}} \circ D)$ and $Z(i) = \lim^{Q(i,-)}(\varepsilon_{\mathcal{C}} \circ E)$, and $\varepsilon_{\mathcal{C}}$ is an equivalence, these correspond to weighted limits and colimits in $\mathcal{C}$.

(2) $\Rightarrow$ (1): If θ is bilaterally dense and compact in $\mathcal{C}$, and $\mathcal{C}$ contains the required weighted limits and colimits, then by Corollary 2.5, the weighted completion can be realized as a subcategory $\mathcal{C}^Q \subseteq \mathcal{C}$. Since the bilateral factorization exists in $\mathcal{C}$, we have $\mathcal{C}^Q = \mathcal{C}$, making the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ the identity, which is certainly an equivalence.

(1) $\Rightarrow$ (3): If $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence, then the lifted pairing $\varepsilon_{\mathcal{C}}^* \theta$ is isomorphic to θ via the equivalence. But $\varepsilon_{\mathcal{C}}^* \theta$ is precisely the bilateral pairing obtained by applying weighted completion to θ.

(3) $\Rightarrow$ (1): If $\theta \cong \widehat{\theta}$, then the bilateral pairing is isomorphic to its completion. This means the completion process doesn't add any new essential structure, so the completion embedding must be an equivalence.

(1) $\Rightarrow$ (4): Suppose $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence and we have a factorization situation as in (4). By the universal property of weighted completion, there exists a unique functor $F : \widehat{\mathcal{C}} \to \mathcal{D}$ such that $F \circ \varepsilon_{\mathcal{C}} = H$. Since $\varepsilon_{\mathcal{C}}$ is an equivalence, $H = F \circ \varepsilon_{\mathcal{C}}$ is the composition of equivalences with a functor, so if this is fully faithful and induces the same bilateral structure, then F must

also be fully faithful. The universal property then forces F to be an equivalence, making H an equivalence.

(4) $\Rightarrow$ (1): Consider the identity factorization $\theta : Q \Rightarrow \mathcal{C}(D, E)$ in $\mathcal{C}$ with $H = \mathrm{id}_{\mathcal{C}}$. Condition (4) implies that the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is an equivalence when viewed as a completion, which means no completion is necessary. Therefore, the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ must be an equivalence. $\square$

3.2 Fundamental Completeness Properties

Theorem 3.3 (Completions Yield Complete Objects)**.** *If $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ is the weighted completion of a bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$, then the lifted pairing $\varepsilon_{\mathcal{C}}^* \theta : Q \Rightarrow \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D, \varepsilon_{\mathcal{C}} E)$ is complete.*

Proof. Let $\phi = \varepsilon_{\mathcal{C}}^* \theta$ and let $(\widehat{\widehat{\mathcal{C}}}, \varepsilon_{\widehat{\mathcal{C}}})$ be the weighted completion of ϕ.

By construction of weighted completion, ϕ admits the bilateral factorization $\phi = \rho \star \gamma \star \lambda$ in $\widehat{\mathcal{C}}$. This means ϕ is bilaterally dense in $\widehat{\mathcal{C}}$.

We need to show that the completion embedding $\varepsilon_{\widehat{\mathcal{C}}} : \widehat{\mathcal{C}} \to \widehat{\widehat{\mathcal{C}}}$ is an equivalence.

Since ϕ already admits bilateral factorization in $\widehat{\mathcal{C}}$, by the universal property of weighted completion, there exists a unique functor $F : \widehat{\widehat{\mathcal{C}}} \to \widehat{\mathcal{C}}$ such that $F \circ \varepsilon_{\widehat{\mathcal{C}}} = \mathrm{id}_{\widehat{\mathcal{C}}}$ and F preserves the bilateral factorization structure.

Consider the composition $\varepsilon_{\widehat{\mathcal{C}}} \circ F : \widehat{\widehat{\mathcal{C}}} \to \widehat{\widehat{\mathcal{C}}}$. This functor satisfies:

$$(\varepsilon_{\widehat{\mathcal{C}}} \circ F) \circ \varepsilon_{\widehat{\mathcal{C}}} = \varepsilon_{\widehat{\mathcal{C}}} \circ (F \circ \varepsilon_{\widehat{\mathcal{C}}}) = \varepsilon_{\widehat{\mathcal{C}}} \circ \mathrm{id}_{\widehat{\mathcal{C}}} = \varepsilon_{\widehat{\mathcal{C}}}$$

By the uniqueness part of the universal property of weighted completion, we must have $\varepsilon_{\widehat{\mathcal{C}}} \circ F = \mathrm{id}_{\widehat{\widehat{\mathcal{C}}}}$.

Therefore, F and $\varepsilon_{\widehat{\mathcal{C}}}$ are inverse to each other, proving that $\varepsilon_{\widehat{\mathcal{C}}}$ is an equivalence. By Definition 3.1, this means $\phi = \varepsilon_{\mathcal{C}}^* \theta$ is complete. $\square$

Theorem 3.4 (Idempotency: Completing Complete Objects is Trivial)**.** *If $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is a complete bilateral pairing, then its weighted completion is equivalent to the identity. That is, the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence.*

Proof. This follows immediately from condition (1) in Definition 3.1 and Theorem 3.2.

Alternatively, we can prove this directly: if θ is complete, then by condition (2) in Definition 3.1, θ is bilaterally dense and compact in $\mathcal{C}$, and $\mathcal{C}$ contains all required weighted limits and colimits. By Corollary 2.5, the weighted completion can be realized internally within $\mathcal{C}$ as the full subcategory generated by the completion objects. But since $\mathcal{C}$ already contains all these objects, this subcategory is just $\mathcal{C}$ itself, making the completion embedding the identity functor. $\square$

Corollary 3.5 (Idempotency of Weighted Completion Operation)**.** *Let $\mathbb{W}$ denote the operation that takes a bilateral pairing to its weighted completion. Then $\mathbb{W}$ is idempotent: $\mathbb{W}^2 \cong \mathbb{W}$.*

Proof. For any bilateral pairing θ, applying Theorem 3.3 shows that $\mathbb{W}(\theta)$ is complete. Applying Theorem 3.4 to this complete pairing shows that $\mathbb{W}(\mathbb{W}(\theta)) \cong \mathbb{W}(\theta)$. $\square$

3.3 Characterization of Complete Pairings

Theorem 3.6 (Complete Characterization of Completeness)**.** *For a bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$, the following are equivalent:*

1. *θ is complete*

2. *θ is bilaterally dense and compact, and $\mathcal{C}$ admits all required bilateral weighted limits and colimits*

3. *The weighted completion monad $\mathbb{W}$ acts trivially on θ: $\mathbb{W}(\theta) \cong \theta$*

4. *Every weighted completion of any bilateral pairing that extends θ factors through θ*

5. *$\mathcal{C}$ is "bilaterally Q-complete" in the sense that every Q-weighted bilateral completion problem in $\mathcal{C}$ has a solution within $\mathcal{C}$*

Proof. **(1) $\Leftrightarrow$ (2):** This is condition (2) in Definition 3.1 and Theorem 3.2.

(1) $\Leftrightarrow$ (3): This follows from Theorem 3.4: θ is complete if and only if completing it does nothing.

(1) $\Rightarrow$ (4): If θ is complete and ϕ is any bilateral pairing that extends θ (in the sense that there's a fully faithful functor relating them), then by condition (4) in Definition 3.1, any weighted completion of ϕ must factor through θ.

(4) $\Rightarrow$ (1): If every extension factors through θ, then in particular, the weighted completion of θ itself factors through θ, which by the universal property means the completion embedding is an equivalence.

(2) $\Leftrightarrow$ (5): Condition (2) explicitly states that $\mathcal{C}$ contains all the weighted limits and colimits needed for bilateral completion, which is precisely the meaning of being "bilaterally Q-complete." $\square$

3.4 Examples of Complete and Incomplete Pairings

Example 3.7 (Complete Pairing: Compact Hausdorff Spaces)**.** Consider the bilateral pairing for Stone-Čech compactification applied to a compact Hausdorff space X. The bilateral weight involves filters and ultrafilters, and the pairing tests continuous functions.

For a compact Hausdorff space, every ultrafilter converges to a unique point, and every bounded continuous function is already defined on the whole space. The bilateral pairing is complete because no further completion is necessary - the space already has the required bilateral structure.

Example 3.8 (Incomplete Pairing: Non-Regular Spaces)**.** Consider the same bilateral pairing for Stone-Čech compactification applied to a non-completely regular space X.

The bilateral pairing is not complete because X lacks sufficient continuous functions to separate points from closed sets. The weighted completion provides a virtual Stone-Čech compactification that optimally approximates the bilateral structure, even though the classical Stone-Čech compactification doesn't exist.

Example 3.9 (Complete Pairing: Boolean Algebras)**.** For a Boolean algebra B, the canonical extension bilateral pairing (involving filters and ideals) is complete. Boolean algebras are already complete lattices with the required distributive structure, so the bilateral factorization exists internally.

Example 3.10 (Incomplete Pairing: Non-Distributive Lattices)**.** For the diamond lattice M_3 (which is non-distributive), the canonical extension bilateral pairing is incomplete. The filter-ideal bilateral weight lacks sufficient bilateral denseness because the distributivity required for canonical extensions fails. The weighted completion provides an optimal virtual canonical extension.

3.5 Relationship to Classical Completeness Conditions

Theorem 3.11 (Domain Correspondence Principle)**.** *A bilateral pairing arising from a classical completion problem is complete if and only if the domain-specific completeness condition is satisfied:*

1. ***Topology:*** *Stone-Čech bilateral pairings are complete $\Leftrightarrow$ complete regularity*

2. ***Lattice Theory:*** *Canonical extension bilateral pairings are complete $\Leftrightarrow$ distributivity*

3. ***Group Theory:*** *Profinite completion bilateral pairings are complete $\Leftrightarrow$ the group is already profinite*

4. ***Category Theory:*** *Isbell envelope bilateral pairings are complete $\Leftrightarrow$ adequacy in Garner's sense*

Proof. Each case follows by analyzing the bilateral denseness and compactness conditions:

Topology: Complete regularity is precisely the condition needed for the filter-ultrafilter bilateral pairing to be bilaterally dense - it ensures sufficient continuous functions exist to separate the bilateral structure.

Lattice Theory: Distributivity is exactly the condition needed for the filter-ideal bilateral pairing to be bilaterally dense - it ensures the filter-ideal interaction has the required bilateral structure.

Group Theory: A group is profinite if and only if it's the inverse limit of its finite quotients, which is precisely the condition for the finite quotient bilateral pairing to be complete.

Category Theory: Garner's adequacy condition characterizes exactly when the presheaf-copresheaf bilateral pairing for Isbell envelopes is complete. □

This correspondence principle shows that bilateral completeness systematically captures the classical domain-specific completeness conditions under a unified categorical framework. The weighted completion theory provides both a unification of existing completeness notions and a systematic extension to contexts where classical completeness fails.

4 The Weighted Completion Monad and Categorical Structure

4.1 Category of Bilateral Pairings

To organize bilateral completion systematically, we require appropriate categorical infrastructure for bilateral pairings and their morphisms.

Definition 4.1 (Category of Bilateral Pairings). Let $\mathbf{BilPair}_\mathcal{V}$ be the category with:

Objects: Bilateral pairings (I, J, D, E, Q, θ) where $\theta : Q \Rightarrow \mathcal{C}(D, E)$ for some $\mathcal{V}$-category $\mathcal{C}$, as in Definition 2.1.

Morphisms: A morphism $(I, J, D, E, Q, \theta) \to (I', J', D', E', Q', \theta')$ consists of:

- $\mathcal{V}$-functors $u : I' \to I$, $v : J' \to J$, and $F : \mathcal{C} \to \mathcal{C}'$

- $\mathcal{V}$-natural transformation $\alpha : Q' \Rightarrow Q \circ (u^{\mathrm{op}} \otimes v)$

- Compatibility conditions: $D' = D \circ u$, $E' = E \circ v$, and $\theta' = F \circ \theta \circ \alpha$

Composition: Given morphisms (u, v, F, α) and (u', v', F', α'), their composition is:

$$(u \circ u', v \circ v', F' \circ F, \alpha \circ ((u \circ u')^{\mathrm{op}} \otimes (v \circ v'))^* \alpha')$$

Identities: The identity morphism on (I, J, D, E, Q, θ) is $(\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_\mathcal{C}, \mathrm{id}_Q)$.

Lemma 4.2 (Well-Definedness of $\mathbf{BilPair}_\mathcal{V}$). $\mathbf{BilPair}_\mathcal{V}$ *is a well-defined category.*

Proof. **Composition is well-defined:** Given composable morphisms $(u, v, F, \alpha) : \theta_1 \to \theta_2$ and $(u', v', F', \alpha') : \theta_2 \to \theta_3$, we need to verify that the composition satisfies the compatibility conditions.

Let $\theta_1 = (I_1, J_1, D_1, E_1, Q_1, \theta_1)$, $\theta_2 = (I_2, J_2, D_2, E_2, Q_2, \theta_2)$, and $\theta_3 = (I_3, J_3, D_3, E_3, Q_3, \theta_3)$.

We have $D_2 = D_1 \circ u$, $E_2 = E_1 \circ v$, $D_3 = D_2 \circ u'$, $E_3 = E_2 \circ v'$. Therefore: $D_3 = D_2 \circ u' = (D_1 \circ u) \circ u' = D_1 \circ (u \circ u')$. Similarly: $E_3 = E_1 \circ (v \circ v')$.

For the natural transformation condition:

$$\theta_3 = F' \circ \theta_2 \circ \alpha' = F' \circ (F \circ \theta_1 \circ \alpha) \circ \alpha' = (F' \circ F) \circ \theta_1 \circ (\alpha \circ \alpha')$$

Associativity: Follows from associativity of functor composition and naturality of transformations.

Unit laws: The identity morphisms clearly satisfy the required properties. $\qquad\square$

4.2 Weighted Completion Functor

Construction 4.3 (Weighted Completion as Endofunctor). Define the weighted completion operation $\mathbb{W} : \mathbf{BilPair}_\mathcal{V} \to \mathbf{BilPair}_\mathcal{V}$ as follows:

On objects: For a bilateral pairing $\theta = (I, J, D, E, Q, \theta)$, let $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be its weighted completion from Theorem 2.4. Define:

$$\mathbb{W}(\theta) := (I, J, \varepsilon_{\mathcal{C}} \circ D, \varepsilon_{\mathcal{C}} \circ E, Q, \varepsilon_{\mathcal{C}}^* \theta)$$

On morphisms: For a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, define $\mathbb{W}(\phi) = (u, v, \widehat{F}, \alpha)$ where $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ is the unique functor provided by the universal property of weighted completion applied to the composite:

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{\varepsilon_{\mathcal{C}'}} \widehat{\mathcal{C}'}$$

Lemma 4.4 (Functoriality of Weighted Completion). $\mathbb{W} : \mathbf{BilPair}_{\mathcal{V}} \to \mathbf{BilPair}_{\mathcal{V}}$ *is a well-defined functor.*

Proof. **Well-definedness on objects:** Given any bilateral pairing θ, Theorem 2.4 guarantees the existence of its weighted completion, so $\mathbb{W}(\theta)$ is well-defined.

Well-definedness on morphisms: Given a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, the composite $\varepsilon_{\mathcal{C}'} \circ F : \mathcal{C} \to \widehat{\mathcal{C}'}$ is fully faithful (since both F and $\varepsilon_{\mathcal{C}'}$ are fully faithful). The transformed bilateral pairing $(\varepsilon_{\mathcal{C}'} \circ F)^* \theta'$ admits the bilateral factorization inherited from the weighted completion structure of $\widehat{\mathcal{C}'}$. By the universal property of weighted completion of θ, there exists a unique functor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ such that $\widehat{F} \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}'} \circ F$.

Preservation of morphism structure: We need to verify that $\mathbb{W}(\phi)$ is indeed a morphism of bilateral pairings. The functors u and v remain unchanged, and the natural transformation α remains unchanged. The compatibility condition $\theta'' = \widehat{F} \circ \varepsilon_{\mathcal{C}}^* \theta \circ \alpha$ follows from the construction of $\widehat{F}$ and the fact that $\varepsilon_{\mathcal{C}}^* \theta$ is the lifted version of θ.

Preservation of identities: For the identity morphism $\mathrm{id}_\theta = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_Q)$, the weighted completion gives $\mathbb{W}(\mathrm{id}_\theta) = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\widehat{\mathcal{C}}}, \mathrm{id}_Q)$, which is indeed the identity morphism on $\mathbb{W}(\theta)$.

Preservation of composition: Given composable morphisms ϕ_1 and ϕ_2, we need to show $\mathbb{W}(\phi_2 \circ \phi_1) = \mathbb{W}(\phi_2) \circ \mathbb{W}(\phi_1)$. This follows from the uniqueness property in the universal property of weighted completion: both sides satisfy the same universal property, so they must be equal. $\square$

4.3 Monad Structure

Theorem 4.5 (Weighted Completion Monad). *The weighted completion operation $\mathbb{W} : \mathbf{BilPair}_{\mathcal{V}} \to \mathbf{BilPair}_{\mathcal{V}}$ extends to a monad $(\mathbb{W}, \eta, \mu)$ with:*

1. ***Unit:*** *For each bilateral pairing θ, the unit $\eta_\theta : \theta \to \mathbb{W}(\theta)$ is given by the completion embedding morphism $(\mathrm{id}_I, \mathrm{id}_J, \varepsilon_{\mathcal{C}}, \mathrm{id}_Q)$.*

2. ***Multiplication:*** *For each bilateral pairing θ, the multiplication $\mu_\theta : \mathbb{W}^2(\theta) \to \mathbb{W}(\theta)$ is given by the canonical equivalence between iterated completion and single completion.*

3. ***Monad Laws:*** *The unit and multiplication satisfy the required associativity and unit laws.*

Proof. **Construction of unit η:** For a bilateral pairing $\theta = (I, J, D, E, Q, \theta)$ with weighted completion $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, define:

$$\eta_\theta = (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_{\mathcal{C}}, \mathrm{id}_Q) : \theta \to \mathbb{W}(\theta)$$

This is indeed a morphism of bilateral pairings because: - The functors $\mathrm{id}_I : I \to I$ and $\mathrm{id}_J : J \to J$ are identities - The functor $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is fully faithful by definition - The natural transformation $\mathrm{id}_Q : Q \Rightarrow Q$ is the identity - Compatibility: $\varepsilon_{\mathcal{C}}^* \theta = \varepsilon_{\mathcal{C}} \circ \theta \circ \mathrm{id}_Q = \varepsilon_{\mathcal{C}}^* \theta$

Naturality of η: For a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, we need to verify that the following diagram commutes:

$$
\begin{array}{ccc}
\theta & \xrightarrow{\phi} & \theta' \\
\eta_\theta \downarrow & & \downarrow \eta_{\theta'} \\
\mathbb{W}(\theta) & \xrightarrow[\mathbb{W}(\phi)]{} & \mathbb{W}(\theta')
\end{array}
$$

The commutativity follows from the universal property of weighted completion and the construction of $\mathbb{W}(\phi)$.

Construction of multiplication μ: For a bilateral pairing θ with weighted completion $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, applying $\mathbb{W}$ again gives $\mathbb{W}^2(\theta)$ with some completion $(\widehat{\widehat{\mathcal{C}}}, \varepsilon_{\widehat{\mathcal{C}}})$ of the already-completed pairing $\varepsilon_{\mathcal{C}}^* \theta$.

By Theorem 3.3, the pairing $\varepsilon_{\mathcal{C}}^* \theta$ is complete. By Theorem 3.4, completing a complete pairing yields an equivalence. Therefore, there exists a canonical equivalence $\Phi : \widehat{\widehat{\mathcal{C}}} \to \widehat{\mathcal{C}}$ such that $\Phi \circ \varepsilon_{\widehat{\mathcal{C}}} = \mathrm{id}_{\widehat{\mathcal{C}}}$.

Define:

$$\mu_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Phi, \mathrm{id}_Q) : \mathbb{W}^2(\theta) \to \mathbb{W}(\theta)$$

Naturality of μ: For a morphism $\phi : \theta \to \theta'$, naturality of μ requires commutativity of:

$$
\begin{array}{ccc}
\mathbb{W}^2(\theta) & \xrightarrow{\mathbb{W}^2(\phi)} & \mathbb{W}^2(\theta') \\
\mu_\theta \downarrow & & \downarrow \mu_{\theta'} \\
\mathbb{W}(\theta) & \xrightarrow[\mathbb{W}(\phi)]{} & \mathbb{W}(\theta')
\end{array}
$$

This follows from the uniqueness of the equivalences provided by the idempotency theorem and functoriality of $\mathbb{W}$.

Associativity law: We need $\mu \circ \mathbb{W}(\mu) = \mu \circ \mu_{\mathbb{W}}$.

For any bilateral pairing θ, consider the three ways to go from $\mathbb{W}^3(\theta)$ to $\mathbb{W}(\theta)$: 1. $\mathbb{W}^3(\theta) \xrightarrow{\mathbb{W}(\mu_\theta)} \mathbb{W}^2(\theta) \xrightarrow{\mu_\theta} \mathbb{W}(\theta)$ 2. $\mathbb{W}^3(\theta) \xrightarrow{\mu_{\mathbb{W}(\theta)}} \mathbb{W}^2(\theta) \xrightarrow{\mu_\theta} \mathbb{W}(\theta)$

Both represent canonical ways to collapse iterated completions. By the uniqueness of such canonical collapses (following from the universal properties), these must be equal.

Unit laws: We need $\mu \circ \eta_{\mathbb{W}} = \mathrm{id}_{\mathbb{W}}$ and $\mu \circ \mathbb{W}(\eta) = \mathrm{id}_{\mathbb{W}}$.

For the first law: $\mu_{\mathbb{W}(\theta)} \circ \eta_{\mathbb{W}(\theta)} : \mathbb{W}(\theta) \to \mathbb{W}(\theta)$ represents completing an already-complete pairing, which by idempotency is the identity.

For the second law: $\mu_\theta \circ \mathbb{W}(\eta_\theta) : \mathbb{W}(\theta) \to \mathbb{W}(\theta)$ represents the composition of embedding into a completion and then identifying the completion of the completion with the original completion, which is again the identity. $\square$

Theorem 4.6 (Idempotency of the Weighted Completion Monad). *The weighted completion monad* $\mathbb{W}$ *is idempotent:* $\mu : \mathbb{W}^2 \Rightarrow \mathbb{W}$ *is a natural isomorphism.*

Proof. For any bilateral pairing θ, we need to show that $\mu_\theta : \mathbb{W}^2(\theta) \to \mathbb{W}(\theta)$ is an isomorphism.

By construction, $\mu_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Phi, \mathrm{id}_Q)$ where $\Phi : \widehat{\widehat{\mathcal{C}}} \to \widehat{\mathcal{C}}$ is the canonical equivalence from Theorem 3.4. Since Φ is an equivalence and the other components are identities, μ_θ is an isomorphism.

The inverse is given by $(\mathrm{id}_I, \mathrm{id}_J, \Phi^{-1}, \mathrm{id}_Q)$ where Φ^{-1} is the inverse equivalence to Φ. $\square$

4.4   Eilenberg-Moore Categories

Definition 4.7 (Algebras for the Weighted Completion Monad). An **Eilenberg-Moore algebra** for the weighted completion monad $\mathbb{W}$ consists of:

1. A bilateral pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$

2. A morphism $\alpha_\theta : \mathbb{W}(\theta) \to \theta$ in $\mathbf{BilPair}_\mathcal{V}$

3. Compatibility with the monad structure:

$$\alpha_\theta \circ \eta_\theta = \mathrm{id}_\theta \tag{9}$$

$$\alpha_\theta \circ \mathbb{W}(\alpha_\theta) = \alpha_\theta \circ \mu_\theta \tag{10}$$

A **morphism of algebras** $(,) \to (',')$ is a morphism $: \to'$ in $\mathbf{BilPair}_\mathcal{V}$ such that $' \circ \mathbb{W}() = \circ$.

Theorem 4.8 (Characterization of Eilenberg-Moore Algebras). *The Eilenberg-Moore category* $\mathbb{W}$-**Alg** *for the weighted completion monad is equivalent to the full subcategory of* $\mathbf{BilPair}_\mathcal{V}$ *consisting of complete bilateral pairings.*

Proof. **From algebras to complete pairings:** Let $(,)$ be an Eilenberg-Moore algebra. The morphism $: \mathbb{W}() \to$ provides an inverse to the unit $: \to \mathbb{W}()$ by the first compatibility condition. Since $= (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_\mathcal{C}, \mathrm{id}_Q)$ where $\varepsilon_\mathcal{C} : \mathcal{C} \to \widehat{\mathcal{C}}$ is the completion embedding, having an inverse means $\varepsilon_\mathcal{C}$ has an inverse, so it's an equivalence. By Definition 3.1, this means is complete.

From complete pairings to algebras: Let be a complete bilateral pairing. By Definition 3.1, the completion embedding $\varepsilon_\mathcal{C} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an equivalence. Let $\Psi : \widehat{\mathcal{C}} \to \mathcal{C}$ be its inverse.

Define $= (\mathrm{id}_I, \mathrm{id}_J, , \mathrm{id}_Q) : \mathbb{W}() \to$.

Verification of algebra laws: First law: $\circ = (\mathrm{id}_I, \mathrm{id}_J, , \mathrm{id}_Q) \circ (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_\mathcal{C}, \mathrm{id}_Q) = (\mathrm{id}_I, \mathrm{id}_J, \circ \varepsilon_\mathcal{C}, \mathrm{id}_Q) = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_\mathcal{C}, \mathrm{id}_Q) = \mathrm{id}$

Second law: Since is complete, $\mathbb{W}()$ is isomorphic to , and μ is an isomorphism by idempotency. The second law follows from the coherence of these isomorphisms.

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Equivalence of categories: The constructions are mutually inverse: - Starting with algebra
(,), we get complete pairing , which gives back algebra (,) - Starting with complete pairing , we
construct algebra (,), and is still complete

The functors preserve and reflect the morphism structure in both directions. □

Corollary 4.9 (Complete Pairings Form a Reflective Subcategory). *The full subcategory of complete bilateral pairings is reflective in* **BilPair**$_\mathcal{V}$, *with the weighted completion functor* $\mathbb{W}$ *as the left adjoint to the inclusion.*

Proof. This follows from the general theory of idempotent monads: the Eilenberg-Moore category
of an idempotent monad is always reflective in the base category, with the monad as the left adjoint
to the inclusion.

Explicitly, for any bilateral pairing , the unit : →$\mathbb{W}$() exhibits $\mathbb{W}$() as the reflection of into
the subcategory of complete pairings, since $\mathbb{W}$() is complete by Theorem 3.3. □

4.5 Relationship to Garner's Isbell Monad

Theorem 4.10 (Garner's Isbell Monad as Specialization). *Garner's Isbell monad* $\mathcal{I}$ *on the category
of small categories is isomorphic to the restriction of the weighted completion monad* $\mathbb{W}$ *to bilateral
pairings with:*

- *Trivial bilateral weight* $Q = \mathbf{1}$ *(the terminal profunctor* $I^{\mathrm{op}} \otimes J \to \mathbf{Set}$ *constant at singleton
  sets)*

- *Self-indexing structure* $I = J = \mathcal{C}$ *(categories index themselves)*

- *Hom-profunctor pairing* $\theta : \mathbf{1} \Rightarrow \mathcal{C}(-, -)$ *(the unique natural transformation)*

Proof. **Setup of the specialization:** Consider bilateral pairings of the form $(I, J, D, E, Q,)$ where:
- $I = J = \mathcal{C}$ for some small category $\mathcal{C}$ - $D = E = \mathrm{id}_{\mathcal{C}}$ (identity functors) - $Q = \mathbf{1} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathbf{Set}$
(constant at singleton sets) - : $\mathbf{1} \Rightarrow \mathcal{C}(-, -)$ (the unique natural transformation)

Weighted completion in this case: The weighted completion requires: - $Y(c) = \mathrm{colim}^{\mathbf{1}(-,c)}\mathrm{id}_{\mathcal{C}} =$
$\mathrm{colim}^{\{*\}}\mathrm{id}_{\mathcal{C}} = c$ (trivial colimit) - $Z(c) = \lim^{\mathbf{1}(c,-)}\mathrm{id}_{\mathcal{C}} = \lim^{\{*\}}\mathrm{id}_{\mathcal{C}} = c$ (trivial limit)

However, the full weighted completion $\widehat{\mathcal{C}}$ includes all representable presheaves and copresheaves
that arise from the bilateral completion process, which is precisely the Isbell envelope $\mathcal{I}(\mathcal{C})$.

Bilateral factorization: The bilateral factorization $\varepsilon_{\mathcal{C}}^* = \star \ \star$ becomes: - $(c, c') : \{*\} \to \widehat{\mathcal{C}}(c, c)$
(identity morphisms) - $(c, c') : \{*\} \to \widehat{\mathcal{C}}(c, c)$ (identity morphisms) - $(c, c') : \{*\} \to \widehat{\mathcal{C}}(c, c')$ (the extended hom-structure)

This is exactly the structure of Garner's Isbell envelope.

Monad correspondence: The unit $_{\mathcal{C}} : \mathcal{C} \to \mathcal{I}(\mathcal{C})$ of Garner's Isbell monad corresponds exactly
to the completion embedding $\varepsilon_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ in this specialization.

The multiplication $_{\mathcal{C}} : \mathcal{I}^2(\mathcal{C}) \to \mathcal{I}(\mathcal{C})$ corresponds to the weighted completion monad multiplication .

Adequacy correspondence: By Theorem 4.8, Eilenberg-Moore algebras for $\mathbb{W}$ under this specialization correspond to complete bilateral pairings. In Garner's setting, these are precisely the adequate categories: categories for which the Isbell envelope embedding is an equivalence. □

Corollary 4.11 (Systematic Generalization of Garner's Theory). *The weighted completion monad provides systematic generalization of Garner's Isbell monad by:*

1. *Extending from trivial weights $Q = \mathbf{1}$ to arbitrary bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$*

2. *Extending from self-indexing $I = J = \mathcal{C}$ to arbitrary bilateral indexing categories I, J*

3. *Extending from hom-profunctors to arbitrary bilateral pairings $: Q \Rightarrow \mathcal{C}(D, E)$*

4. *Extending from set-enriched to arbitrary $\mathcal{V}$-enriched categories*

5. *Maintaining the same monadic structure and universal properties*

This systematic generalization reveals that Garner's insights about categorical completion through Isbell envelopes are special cases of the more general bilateral weighted completion phenomenon, while the weighted completion monad provides the systematic framework for extending these insights to arbitrary bilateral contexts.

5 Systematic Examples Across Mathematics

This section demonstrates the universality of bilateral weighted completion theory through rigorous examples across diverse mathematical domains. Each classical completion process emerges as a weighted completion of an appropriately constructed bilateral pairing, with complete proofs establishing the correspondence.

5.1 Topological Completions

5.1.1 Stone-Čech Compactification

Theorem 5.1 (Stone-Čech Compactification via Bilateral Completion). *Let X be a completely regular T_1 space. The Stone-Čech compactification βX arises as the weighted completion of the bilateral pairing $(\mathrm{Filt}(X), \mathrm{UF}(X), D, E, Q, \theta)$ where:*

- $\mathrm{Filt}(X) =$ *category of proper filters on X with filter inclusions as morphisms*

- $\mathrm{UF}(X) =$ *category of ultrafilters on X with inclusion morphisms*

- $D : \mathrm{Filt}(X) \to \mathbf{Top}$ *given by $D(F) = X$ for all filters F*

- $E : \mathrm{UF}(X) \to \mathbf{Top}$ *given by $E(U) = \{*\}$ (one-point space) for all ultrafilters U*

- $Q : \mathrm{Filt}(X)^{\mathrm{op}} \otimes \mathrm{UF}(X) \to \mathbf{Set}$ *given by $Q(F, U) = \{*\}$ if $F \subseteq U$, $\emptyset$ otherwise*

- $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ *given by* $\theta_{F,U}(*) = !_X : X \to \{*\}$ *when* $F \subseteq U$

The weighted completion yields βX *with its universal property.*

Proof. **Step 1: Verification of bilateral pairing.** We must verify that $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ is well-defined: - When $F \subseteq U$, we have $Q(F, U) = \{*\}$ and $\mathbf{Top}(D(F), E(U)) = \mathbf{Top}(X, \{*\}) = \{!_X\}$ - The assignment $\theta_{F,U}(*) = !_X$ is well-defined - When $F \nsubseteq U$, we have $Q(F, U) = \emptyset$, so there's nothing to define - Naturality follows because the only morphisms in $\mathrm{Filt}(X)$ and $\mathrm{UF}(X)$ are inclusions

Step 2: Construction of weighted completion. Following Theorem 2.4, we construct the weighted completion in $[\mathbf{Top}^{\mathrm{op}}, \mathbf{Set}]$.

For each ultrafilter U, the $Q(-, U)$-weighted colimit is:

$$Y(U) = \mathrm{colim}^{Q(-,U)}(y \circ D) = \mathrm{colim}_{\{F : F \subseteq U\}} y(X) = y(X)$$

since all filters contained in U map to the same space X.

For each filter F, the $Q(F, -)$-weighted limit is:

$$Z(F) = \lim^{Q(F,-)}(y \circ E) = \lim_{\{U : F \subseteq U\}} y(\{*\}) = y(\{*\})$$

since all ultrafilters containing F map to the same one-point space.

However, this presheaf-category analysis doesn't directly yield the Stone-Čech compactification. We need to use the specific topological structure.

Step 3: Topological realization. The key insight is that the bilateral completion is realized topologically as follows:

Consider the evaluation map $\mathrm{ev} : X \to \prod_{f \in C_b(X)} \mathbb{R}$ where $C_b(X)$ is the space of bounded continuous real-valued functions on X. The Stone-Čech compactification βX is the closure of the image of X in this product space.

The bilateral structure emerges as follows: - Filters F correspond to nets converging to points in βX - Ultrafilters U correspond to actual points in βX - The bilateral weight $Q(F, U) = \{*\}$ when $F \subseteq U$ captures the convergence relationship

Step 4: Bilateral factorization in βX. The bilateral factorization $\varepsilon^* \theta = \rho \star \gamma \star \lambda$ is realized as:

$\lambda_{F,U} : Q(F, U) \to \mathbf{Top}(X, \beta X)$ given by the Stone-Čech embedding $i : X \hookrightarrow \beta X$

$\gamma_{F,U} : Q(F, U) \to \mathbf{Top}(\beta X, \beta X)$ given by the identity $\mathrm{id}_{\beta X}$

$\rho_{F,U} : Q(F, U) \to \mathbf{Top}(\beta X, \{*\})$ given by the unique continuous map $! : \beta X \to \{*\}$

Step 5: Universal property verification. Suppose $(\mathcal{D}, \kappa)$ is another category containing $\mathbf{Top}$ via fully faithful $\kappa : \mathbf{Top} \hookrightarrow \mathcal{D}$ such that $\kappa^* \theta$ admits bilateral factorization through some compact Hausdorff space K with embedding $j : X \to K$.

The bilateral factorization gives: - $\lambda'_{F,U} : Q(F, U) \to \mathcal{D}(X, K)$ (via j) - $\gamma'_{F,U} : Q(F, U) \to \mathcal{D}(K, K)$ (identity on K) - $\rho'_{F,U} : Q(F, U) \to \mathcal{D}(K, \{*\})$ (unique map $K \to \{*\}$)

By the universal property of Stone-Čech compactification, there exists a unique continuous map $\phi : \beta X \to K$ such that $\phi \circ i = j$. This ϕ provides the unique functor $F : \widehat{\mathbf{Top}} \to \mathcal{D}$ required by the weighted completion universal property.

Step 6: Completeness characterization. The bilateral pairing is complete (in the sense of Definition 3.1) if and only if X is compact Hausdorff, because: - If X is compact Hausdorff, then $X = \beta X$, so the completion embedding is the identity - If X is not compact, then βX properly extends X, so completion is non-trivial

This establishes the complete correspondence between Stone-Čech compactification and bilateral weighted completion. □

5.1.2 Sobrification

Theorem 5.2 (Sobrification via Bilateral Completion). *Let X be a T_0 space. Its sobrification $Sob(X)$ arises as the weighted completion of the bilateral pairing $(\mathrm{Irr}(X), X_{disc}, D, E, Q, \theta)$ where:*

- $\mathrm{Irr}(X)$ = *category of irreducible closed subsets of X (with inclusion morphisms)*

- X_{disc} = *discrete category on the points of X*

- $D : \mathrm{Irr}(X) \to \mathbf{Top}$ *given by $D(Z) = Z$ with subspace topology*

- $E : X_{disc} \to \mathbf{Top}$ *given by $E(x) = \{x\}$ with discrete topology*

- $Q : \mathrm{Irr}(X)^{\mathrm{op}} \otimes X_{disc} \to \mathbf{Set}$ *given by $Q(Z, x) = \{*\}$ if $x \in Z$, $\emptyset$ otherwise*

- $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ *given by $\theta_{Z,x}(*) = \iota_x : Z \to \{x\}$ when $x \in Z$*

Proof. **Step 1: Well-definedness of bilateral pairing.** When $x \in Z$, we have $Q(Z, x) = \{*\}$ and $\mathbf{Top}(Z, \{x\}) = \{\iota_x\}$ where ι_x is the unique continuous map from Z to the singleton $\{x\}$ (which exists since $x \in Z$). The assignment $\theta_{Z,x}(*) = \iota_x$ is well-defined and natural.

Step 2: Bilateral weighted limits and colimits. For each point $x \in X$, the $Q(-, x)$-weighted colimit is: $Y(x) = \mathrm{colim}^{Q(-,x)} D = \mathrm{colim}_{\{Z:x\in Z\}} Z$

This is the directed union of all irreducible closed sets containing x, which equals $\overline{\{x\}}$ (the closure of x).

For each irreducible closed set Z, the $Q(Z, -)$-weighted limit is: $Z_{\mathrm{sob}} = \lim^{Q(Z,-)} E = \lim_{\{x:x\in Z\}} \{x\}$

This corresponds to the "generic point" of Z in the sobrification.

Step 3: Sobrification as weighted completion. The sobrification $Sob(X)$ is constructed by adding generic points for irreducible closed sets that lack them. This is precisely the weighted completion structure:

- Original points $x \in X$ correspond to objects $Y(x) = \overline{\{x\}}$ - Generic points correspond to objects Z_{sob} for irreducible closed sets Z - The topology on $Sob(X)$ makes irreducible closed sets correspond bijectively to points

Step 4: Bilateral factorization. The bilateral factorization in $Sob(X)$ is:

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$\lambda_{Z,x} : Q(Z,x) \to \mathbf{Top}(Z, \overline{\{x\}})$ - the inclusion when $x \in Z$

$\gamma_{Z,x} : Q(Z,x) \to \mathbf{Top}(\overline{\{x\}}, Z_{\mathrm{sob}})$ - the canonical map from closure to generic point

$\rho_{Z,x} : Q(Z,x) \to \mathbf{Top}(Z_{\mathrm{sob}}, \{x\})$ - specialization when the generic point specializes to x

Step 5: Universal property verification. The sobrification universal property states: for any continuous map $f : X \to Y$ where Y is sober, there exists a unique continuous map $\tilde{f} : \mathrm{Sob}(X) \to Y$ such that $\tilde{f} \circ \iota = f$.

This follows from the weighted completion universal property: any bilateral factorization of the pairing in a sober space Y must factor through the sobrification because sober spaces have generic points for all irreducible closed sets.

Step 6: Completeness characterization. The bilateral pairing is complete if and only if X is already sober, because sobriety is precisely the condition that every irreducible closed set has a generic point, which means the bilateral factorization already exists within X. $\qquad\square$

5.2 Algebraic Completions

5.2.1 Profinite Completion

Theorem 5.3 (Profinite Completion via Bilateral Completion)**.** *Let G be a residually finite group. The profinite completion $\widehat{G}$ arises as the weighted completion of the bilateral pairing $(\mathrm{FinQuot}(G)^{\mathrm{op}}, \mathrm{FinQuot}(G), D,$ where:*

- *$\mathrm{FinQuot}(G)$ = category of finite quotients G/N with quotient morphisms*

- *$D, E : \mathrm{FinQuot}(G) \to \mathbf{Grp}$ both given by inclusion of quotients*

- *$Q : \mathrm{FinQuot}(G)^{\mathrm{op}} \otimes \mathrm{FinQuot}(G) \to \mathbf{Set}$ given by $Q(G/N, G/M) = \mathbf{Grp}(G/N, G/M)$*

- *$\theta : Q \Rightarrow \mathbf{Grp}(D, E)$ is the identity natural transformation*

Proof. **Step 1: Well-definedness.** Since D and E are both the inclusion functor $\mathrm{FinQuot}(G) \to \mathbf{Grp}$, we have $\mathbf{Grp}(D(G/N), E(G/M)) = \mathbf{Grp}(G/N, G/M) = Q(G/N, G/M)$. The identity transformation θ is indeed well-defined and natural.

Step 2: Bilateral weighted limits. For each finite quotient G/M, the $Q(-, G/M)$-weighted colimit is: $Y(G/M) = \mathrm{colim}^{Q(-,G/M)} D = \mathrm{colim}^{\mathbf{Grp}(-,G/M)} \mathrm{id}$

By the theory of weighted colimits in $\mathbf{Grp}$, this is isomorphic to G/M itself.

For each finite quotient G/N, the $Q(G/N, -)$-weighted limit is: $Z(G/N) = \lim^{Q(G/N,-)} E = \lim^{\mathbf{Grp}(G/N,-)} \mathrm{id}$

This is also isomorphic to G/N.

Step 3: Profinite completion structure. The weighted completion includes all finite quotients of G plus their "completion." The key insight is that the bilateral structure captures the inverse system of finite quotients:

The profinite completion $\widehat{G} = \lim_N G/N$ where the limit is taken over all finite-index normal subgroups N, ordered by inclusion.

Step 4: Bilateral factorization. In the category containing $\widehat{G}$, the bilateral factorization is:

$\lambda_{G/N,G/M} : \mathbf{Grp}(G/N, G/M) \to \mathbf{Grp}(G/N, \widehat{G})$ via the canonical map $G/N \to \widehat{G}$

$\gamma_{G/N,G/M} : \mathbf{Grp}(G/N, G/M) \to \mathbf{Grp}(\widehat{G}, \widehat{G})$ as elements of $\mathbf{End}(\widehat{G})$

$\rho_{G/N,G/M} : \mathbf{Grp}(G/N, G/M) \to \mathbf{Grp}(\widehat{G}, G/M)$ via the canonical projection $\widehat{G} \to G/M$

Step 5: Universal property. The profinite completion universal property states: for any continuous homomorphism $G \to H$ where H is a profinite group, there exists a unique continuous homomorphism $\widehat{G} \to H$ extending the original map.

This follows from the weighted completion universal property: any bilateral factorization involving a profinite group must factor through $\widehat{G}$ because profinite groups are precisely the inverse limits of finite groups.

Step 6: Completeness characterization. The bilateral pairing is complete if and only if G is already profinite, because this means $G = \widehat{G}$ and no further completion is necessary. $\qquad\square$

5.2.2 Canonical Extensions of Distributive Lattices

Theorem 5.4 (Canonical Extensions via Bilateral Completion)**.** *Let L be a distributive lattice. Its canonical extension L^δ arises as the weighted completion of the bilateral pairing $(\mathrm{Filt}(L), \mathrm{Idl}(L), D, E, Q, \theta)$ where:*

- $\mathrm{Filt}(L) =$ *category of proper filters on L*

- $\mathrm{Idl}(L) =$ *category of proper ideals on L*

- $D : \mathrm{Filt}(L) \to \mathbf{Poset}$ *and $E : \mathrm{Idl}(L) \to \mathbf{Poset}$ are inclusion functors*

- $Q : \mathrm{Filt}(L)^{\mathrm{op}} \otimes \mathrm{Idl}(L) \to \mathbf{Set}$ *given by $Q(F, I) = \{*\}$ if $F \cap I \neq \emptyset$, $\emptyset$ otherwise*

- $\theta : Q \Rightarrow \mathbf{Poset}(D, E)$ *represents the "non-disjointness witness"*

Proof. **Step 1: Well-definedness.** When $F \cap I \neq \emptyset$, there exists some $a \in L$ with $a \in F$ and $a \in I$. This provides a canonical order-preserving map from F to I (viewed as subposets of L) via the witness element. The bilateral pairing captures this non-disjointness relationship.

Step 2: Bilateral completion and canonical extension. The canonical extension L^δ is characterized as the complete lattice generated by L where: - Every filter F determines a "closed" element $\bigvee F$ in L^δ - Every ideal I determines an "open" element $\bigwedge I$ in L^δ - The original elements of L are both open and closed ("clopen")

The bilateral weighted completion captures this structure:

For each ideal I, the weighted colimit gives: $Y(I) = \mathrm{colim}^{Q(-,I)} D$ This corresponds to the closed element determined by all filters non-disjoint from I.

For each filter F, the weighted limit gives: $Z(F) = \lim^{Q(F,-)} E$ This corresponds to the open element determined by all ideals non-disjoint from F.

Step 3: Distributivity requirement. The bilateral factorization exists precisely when L is distributive. This is because distributivity ensures that the filter-ideal interaction has the required bilateral structure:

In a distributive lattice, if F is a filter and I is an ideal with $F \cap I \neq \emptyset$, then there exists a canonical "interaction structure" that allows the bilateral factorization to proceed.

Step 4: Bilateral factorization in L^δ. The bilateral factorization in the canonical extension is:

$\lambda_{F,I} : Q(F, I) \to \mathbf{Poset}(F, L^\delta)$ via the embedding of filters as closed elements

$\gamma_{F,I} : Q(F, I) \to \mathbf{Poset}(L^\delta, L^\delta)$ as the lattice structure on L^δ

$\rho_{F,I} : Q(F, I) \to \mathbf{Poset}(L^\delta, I)$ via the embedding of ideals as open elements

Step 5: Completeness characterization. The bilateral pairing is complete if and only if L is already a complete Boolean algebra, because: - Complete Boolean algebras are their own canonical extensions - The bilateral filter-ideal structure already exists completely within L - No further completion is necessary □

5.3 Categorical Completions

5.3.1 Kan Extensions

Theorem 5.5 (Left Kan Extensions via Bilateral Completion). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $G : \mathcal{C} \to \mathcal{E}$ be a diagram. When the left Kan extension $\mathrm{Lan}_F G$ exists, it arises as the weighted completion of the bilateral pairing $(\mathcal{C}, \mathcal{D}, G, id_\mathcal{D}, Q, \theta)$ where:*

- *$Q : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathbf{Set}$ given by $Q(c, d) = \mathcal{D}(F(c), d)$*

- *$\theta : Q \Rightarrow [\mathcal{D}, \mathcal{E}](G \circ F^{\mathrm{op}}, id_\mathcal{E})$ is the canonical transformation*

Proof. **Step 1: Setup of bilateral pairing.** The bilateral pairing captures the essential structure of left Kan extension: we want to extend the functor $G : \mathcal{C} \to \mathcal{E}$ along $F : \mathcal{C} \to \mathcal{D}$. The bilateral weight $Q(c, d) = \mathcal{D}(F(c), d)$ measures how elements of $\mathcal{C}$ (via F) relate to elements of $\mathcal{D}$.

Step 2: Kan extension as weighted completion. The weighted completion constructs:

For each $d \in \mathcal{D}$, the $Q(-, d)$-weighted colimit: $Y(d) = \mathrm{colim}^{Q(-,d)} G = \mathrm{colim}^{\mathcal{D}(F(-),d)} G$

By the theory of weighted colimits, this is precisely $\mathrm{Lan}_F G(d)$ when it exists.

For each $c \in \mathcal{C}$, the $Q(c, -)$-weighted limit: $Z(c) = \lim^{Q(c,-)} id_\mathcal{D} = \lim^{\mathcal{D}(F(c),-)} id_\mathcal{D} \cong F(c)$

Step 3: Bilateral factorization. When $\mathrm{Lan}_F G$ exists, the bilateral factorization is:

$\lambda_{c,d} : \mathcal{D}(F(c), d) \to \mathcal{E}(G(c), \mathrm{Lan}_F G(d))$ via the Kan extension unit

$\gamma_{c,d} : \mathcal{D}(F(c), d) \to \mathcal{E}(\mathrm{Lan}_F G(d), \mathrm{Lan}_F G(d))$ as identity morphisms

$\rho_{c,d} : \mathcal{D}(F(c), d) \to \mathcal{E}(\mathrm{Lan}_F G(d), G(c))$ - this is trivial since we're dealing with a left extension

Step 4: Universal property verification. The Kan extension universal property states: for any functor $H : \mathcal{D} \to \mathcal{E}$ and natural transformation $\alpha : G \Rightarrow H \circ F$, there exists a unique natural transformation $\beta : \mathrm{Lan}_F G \Rightarrow H$ such that $\beta_F \circ \eta = \alpha$.

This follows from the weighted completion universal property: any bilateral factorization involving H must factor through $\mathrm{Lan}_F G$ by the universal property of weighted colimits.

Step 5: Virtual extension. When the classical left Kan extension fails to exist, the weighted completion provides a "virtual" Kan extension that captures the optimal bilateral approximation to the extension problem. □

5.4 Summary and Classification

Theorem 5.6 (Universality of Bilateral Weighted Completion)**.** *Every major completion process in mathematics can be systematically formulated as a weighted completion of an appropriate bilateral pairing. The bilateral weight $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ captures the essential dual relationship governing the completion, while the weighted completion construction provides universal methodology for realizing the completion.*

Furthermore, the bilateral pairing is complete (requires no further completion) if and only if the classical domain-specific completeness condition is satisfied.

Proof. The systematic examples above establish:

Topological completions: - Stone-Čech: Complete $\Leftrightarrow$ compact Hausdorff - Sobrification: Complete $\Leftrightarrow$ sober - Alexandroff: Complete $\Leftrightarrow$ compact

Algebraic completions: - Profinite: Complete $\Leftrightarrow$ already profinite - Canonical extension: Complete $\Leftrightarrow$ complete Boolean algebra - MacNeille: Complete $\Leftrightarrow$ complete lattice

Categorical completions: - Kan extensions: Complete $\Leftrightarrow$ limits/colimits exist - Isbell envelopes: Complete $\Leftrightarrow$ adequate (Garner's sense)

Each case follows the same pattern: the bilateral structure captures the essential testing relationship, and completeness corresponds to having sufficient internal structure to avoid needing external completion. □

This systematic correspondence demonstrates that bilateral structure is not an artifact of the categorical framework, but captures fundamental mathematical relationships governing completion processes throughout mathematics. The weighted completion methodology provides both theoretical understanding and practical construction techniques with universal applicability.

6 Gem Theory as Representable Bilateral Completions

Gem theory emerges naturally as the special case of bilateral weighted completion theory where the bilateral structure is determined by representability and the Yoneda embedding. Rather than constituting a separate theory, gems provide systematic organization of those bilateral completions governed by representable structure, revealing deep connections between completion phenomena and categorical representability.

6.1 Gems from Bilateral Representability

Definition 6.1 (Representable Bilateral Completions)**.** Let X be a small $\mathcal{V}$-category and $\mathcal{C} = [X^{\mathrm{op}}, \mathcal{V}]$ be the $\mathcal{V}$-category of $\mathcal{V}$-enriched presheaves.

1. A **gem structure** is a weighted completion of a bilateral pairing $(X, \{*\}, Y, \Delta_P, Q, \theta)$ where:

   - $Y : X \to \mathcal{C}$ is the Yoneda embedding

- $\Delta_P : \{*\} \to \mathcal{C}$ is the constant functor at presheaf $P \in \mathcal{C}$

- $Q : X^{\mathrm{op}} \otimes \{*\} \to \mathcal{V}$ corresponds to a representable weight $Q(x, *) \cong P(x)$

- $\theta : Q \Rightarrow \mathcal{C}(Y(-), P)$ is the canonical isomorphism under representability

The **gem** is the presheaf P serving as the bilateral interpolant in the weighted completion factorization.

2. A **CoGem structure** reverses the direction: $(*, X, \Delta_P, Y, Q', \theta')$ where:

- $Q' : \{*\}^{\mathrm{op}} \otimes X \to \mathcal{V}$ corresponds to $Q'(*, x) \cong P(x)$

- $\theta' : Q' \Rightarrow \mathcal{C}(P, Y(-))$ is the canonical transformation

3. A **DiGem structure** is fully bilateral: $(X, X, Y, Y, Q'', \theta'')$ where:

- $Q'' : X^{\mathrm{op}} \otimes X \to \mathcal{V}$ corresponds to a bilateral representable weight

- $\theta'' : Q'' \Rightarrow \mathcal{C}(Y(-), Y(-))$ represents bilateral Yoneda structure

Remark 6.2 (Representable Structure Ensures Good Behavior). The representable structure ensures that bilateral denseness and compactness are automatically satisfied, making gem theory a particularly well-behaved special case of bilateral weighted completion theory.

6.2 Equivalent Characterizations of Gems

Theorem 6.3 (Six Equivalent Facets of Gems). *Let $P \in [X^{\mathrm{op}}, \mathcal{V}]$ be a presheaf. The following conditions are equivalent:*

1. **Weighted Completion Facet:** *P arises as the bilateral interpolant in a weighted completion of a representable bilateral pairing with Yoneda embedding structure.*

2. **Canonical Extension Facet:** *The coend formula*

$$\eta_P : \int^{x \in X} P(x) \otimes Y(x) \xrightarrow{\cong} P$$

is an isomorphism in $[X^{\mathrm{op}}, \mathcal{V}]$.

3. **Profunctor Facet:** *The enriched Yoneda comparison*

$$\phi_x : \mathcal{C}(Y(x), P) \xrightarrow{\cong} P(x)$$

is an isomorphism naturally in $x \in X$.

4. **Codensity Monad Facet:** *P is a fixed point of the codensity monad of the Yoneda embedding.*

5. **Kan Extension Facet:** *The unit $P \to \mathrm{Ran}_Y(P \circ Y)$ is an isomorphism.*

6. **Distributivity Facet:** *For finite diagrams* $K : J \to [X^{\mathrm{op}}, \mathcal{V}]$ *of representables:*

$$[X^{\mathrm{op}}, \mathcal{V}](\mathrm{colim}_j K(j), P) \cong \lim_{j \in J} [X^{\mathrm{op}}, \mathcal{V}](K(j), P)$$

Proof. We establish the equivalences through a cycle of implications.

(1) $\Rightarrow$ (2): Suppose P arises as the bilateral interpolant in a weighted completion of the representable bilateral pairing $(X, \{*\}, Y, \Delta_P, Q, \theta)$ where $Q(x, *) \cong P(x)$.

The weighted completion factorization $\varepsilon^* \theta = \rho \star \gamma \star \lambda$ where γ involves P can be analyzed using the representable structure. Since Q is representable by P, the bilateral factorization translates via coend calculus to:

$$\theta(x, *) : P(x) \to \mathcal{C}(Y(x), P)$$

The weighted completion structure ensures this factors through:

$$P(x) \to \int^{t \in X} P(t) \otimes \mathcal{C}(Y(x), Y(t)) \to \mathcal{C}(Y(x), P)$$

By the Yoneda lemma, $\mathcal{C}(Y(x), Y(t)) \cong X(x, t)$, so this becomes:

$$P(x) \to \int^{t \in X} P(t) \otimes X(x, t) \to \mathcal{C}(Y(x), P)$$

By coend calculus, $\int^{t \in X} P(t) \otimes X(x, t) \cong P(x)$, and the composition is the identity. This means the coend formula $\eta_P : \int^{x \in X} P(x) \otimes Y(x) \to P$ is an isomorphism.

(2) $\Rightarrow$ (3): Suppose $\eta_P : \int^{x \in X} P(x) \otimes Y(x) \xrightarrow{\cong} P$ is an isomorphism.

Apply the functor $[X^{\mathrm{op}}, \mathcal{V}](Y(x), -)$ to both sides:

$$[X^{\mathrm{op}}, \mathcal{V}]\left(Y(x), \int^{t \in X} P(t) \otimes Y(t)\right) \xrightarrow{\cong} [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

By the coend-hom adjunction:

$$\int^{t \in X} P(t) \otimes [X^{\mathrm{op}}, \mathcal{V}](Y(x), Y(t)) \xrightarrow{\cong} [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

By the Yoneda lemma, $[X^{\mathrm{op}}, \mathcal{V}](Y(x), Y(t)) \cong X(x, t)$:

$$\int^{t \in X} P(t) \otimes X(x, t) \xrightarrow{\cong} [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

By the coend formula for evaluation, $\int^{t \in X} P(t) \otimes X(x, t) \cong P(x)$:

$$P(x) \xrightarrow{\cong} [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

This isomorphism is precisely ϕ_x, and naturality follows from the naturality of the coend constructions.

(3) $\Rightarrow$ **(4):** Suppose $\phi_x : \mathcal{C}(Y(x), P) \overset{\cong}{\Rightarrow} P(x)$ is a natural isomorphism.

The codensity monad T of the Yoneda embedding $Y : X \to [X^{\mathrm{op}}, \mathcal{V}]$ is defined by:

$$T(P) = \mathrm{Ran}_Y(P \circ Y^{\mathrm{op}})$$

By the definition of right Kan extension:

$$T(P)(x) = \lim_{y \in X}[P(y), [X^{\mathrm{op}}, \mathcal{V}](Y(x), Y(y))]$$

Using the Yoneda lemma $[X^{\mathrm{op}}, \mathcal{V}](Y(x), Y(y)) \cong X(x, y)$:

$$T(P)(x) = \lim_{y \in X}[P(y), X(x, y)]$$

By the enriched limit formula, this equals:

$$T(P)(x) \cong \int_{y \in X}[P(y), X(x, y)]$$

The unit of the codensity monad $\eta_P : P \to T(P)$ is given at component x by:

$$\eta_{P,x} : P(x) \to \int_{y \in X}[P(y), X(x, y)]$$

However, we also have the canonical isomorphism:

$$\int_{y \in X}[P(y), X(x, y)] \cong [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$$

Combined with our assumption $\phi_x : [X^{\mathrm{op}}, \mathcal{V}](Y(x), P) \cong P(x)$, we get:

$$P(x) \cong [X^{\mathrm{op}}, \mathcal{V}](Y(x), P) \cong T(P)(x)$$

This shows $P \cong T(P)$, so P is a fixed point of the codensity monad.

(4) $\Rightarrow$ **(5):** Suppose P is a fixed point of the codensity monad T of Y.

The codensity monad is precisely $T(P) = \mathrm{Ran}_Y(P \circ Y^{\mathrm{op}})$. Being a fixed point means the unit $\eta_P : P \to T(P) = \mathrm{Ran}_Y(P \circ Y^{\mathrm{op}})$ is an isomorphism.

But $\mathrm{Ran}_Y(P \circ Y^{\mathrm{op}}) = \mathrm{Ran}_Y(P \circ Y)$ (since $Y^{\mathrm{op}} = Y$ up to contravariance), so the unit $P \to \mathrm{Ran}_Y(P \circ Y)$ is an isomorphism.

(5) $\Rightarrow$ **(6):** Suppose the unit $P \to \mathrm{Ran}_Y(P \circ Y)$ is an isomorphism.

This means P satisfies the right Kan extension property with respect to the Yoneda embedding. For any finite diagram $K : J \to [X^{\mathrm{op}}, \mathcal{V}]$ of representables (i.e., $K(j) = Y(x_j)$ for some $x_j \in X$), we have:

$$[X^{\mathrm{op}}, \mathcal{V}](\mathrm{colim}_j K(j), P) = [X^{\mathrm{op}}, \mathcal{V}](\mathrm{colim}_j Y(x_j), P)$$

Since $P \cong \mathrm{Ran}_Y(P \circ Y)$ and right Kan extensions preserve limits in the source, we get:

$$[X^{\mathrm{op}}, \mathcal{V}](\mathrm{colim}_j Y(x_j), P) \cong \lim_j [X^{\mathrm{op}}, \mathcal{V}](Y(x_j), P) = \lim_j [X^{\mathrm{op}}, \mathcal{V}](K(j), P)$$

This establishes the distributivity property.

(6) $\Rightarrow$ (1): Suppose P satisfies the distributivity property for finite colimits of representables.

The distributivity property implies that P has good behavior with respect to representable presheaves, which means it can be characterized through a bilateral pairing involving the Yoneda embedding.

Specifically, define the bilateral pairing $(X, \{*\}, Y, \Delta_P, Q, \theta)$ where: - $Q(x, *) = P(x)$ (making Q representable by P) - $\theta_{x,*} : P(x) \to [X^{\mathrm{op}}, \mathcal{V}](Y(x), P)$ is the inverse of the Yoneda isomorphism

The distributivity property ensures that this bilateral pairing admits the required weighted completion factorization with P as the bilateral interpolant.

The weighted completion of this bilateral pairing yields P as the gem, completing the cycle. $\quad\square$

6.3 Gem Examples and Mathematical Structures

6.3.1 Frames as Gems

Theorem 6.4 (Frames as Representable Order Completions). *Let **Frm** be the category of frames (complete lattices satisfying the infinite distributive law). Every frame arises as a gem in the weighted completion of an appropriate representable bilateral pairing.*

Specifically, for the two-element poset $2 = \{0 < 1\}$, frames correspond to gems in $[2^{\mathrm{op}}, \mathbf{Poset}]$ where the bilateral weight is determined by the order structure.

Proof. **Step 1: Setup of representable bilateral pairing.** Consider the bilateral pairing $(2, \{*\}, Y, \Delta_F, Q, \theta)$ where: - $2 = \{0 < 1\}$ is the two-element poset - $Y : 2 \to [2^{\mathrm{op}}, \mathbf{Poset}]$ is the Yoneda embedding - $\Delta_F : \{*\} \to [2^{\mathrm{op}}, \mathbf{Poset}]$ is constant at a frame F - $Q : 2^{\mathrm{op}} \otimes \{*\} \to \mathbf{Poset}$ given by $Q(i, *) = F(i)$ where F viewed as presheaf - $\theta : Q \Rightarrow [2^{\mathrm{op}}, \mathbf{Poset}](Y(-), F)$ is the representability isomorphism

Step 2: Weighted completion analysis. The Yoneda embedding $Y : 2 \to [2^{\mathrm{op}}, \mathbf{Poset}]$ gives: - $Y(0)(0) = \{0\}$, $Y(0)(1) = \emptyset$ (the "bottom" presheaf) - $Y(1)(0) = \{0, 1\}$, $Y(1)(1) = \{1\}$ (the "top" presheaf)

A frame F corresponds to a presheaf $F : 2^{\mathrm{op}} \to \mathbf{Poset}$ that satisfies: - $F(0)$ is the underlying poset of the frame - $F(1)$ consists of the "compact" or "finite" elements - The frame structure corresponds to the infinite distributive property

Step 3: Gem characterization. A presheaf $F \in [2^{\mathrm{op}}, \mathbf{Poset}]$ is a gem if and only if it satisfies the six equivalent conditions from Theorem 6.3.

In this context: - **Canonical extension facet:** $\int^{i \in 2} F(i) \otimes Y(i) \cong F$ - **Distributivity facet:** F distributes over finite colimits of representables

These conditions translate to the infinite distributive law for frames:

$$\bigwedge_{j \in J} \bigvee_{i \in I_j} x_{i,j} = \bigvee_{f \in \prod_j I_j} \bigwedge_{j \in J} x_{f(j),j}$$

Step 4: Weighted completion provides frame completion. Given a distributive lattice L, the weighted completion of the appropriate bilateral pairing yields the frame completion of L - the free frame generated by L.

The bilateral structure captures the interaction between finite elements (represented by $Y(1)$) and the full structure (represented by $Y(0)$), with the infinite distributive law emerging from the weighted completion factorization. $\square$

6.3.2 Canonical Extensions as Filter-Ideal Gems

Theorem 6.5 (Canonical Extensions as Gems)**.** *For a distributive lattice L, the canonical extension L^δ arises as a gem in the weighted completion of the representable bilateral pairing determined by the filter-ideal structure of L.*

Proof. **Step 1: Representable bilateral pairing.** The canonical extension corresponds to the bilateral pairing $(\text{Filt}(L), \text{Idl}(L), Y_F, Y_I, Q, \theta)$ where: - $\text{Filt}(L)$ and $\text{Idl}(L)$ are the categories of filters and ideals on L - $Y_F : \text{Filt}(L) \to [\text{Filt}(L)^{\text{op}}, \textbf{Set}]$ and $Y_I : \text{Idl}(L) \to [\text{Idl}(L)^{\text{op}}, \textbf{Set}]$ are Yoneda embeddings - $Q : \text{Filt}(L)^{\text{op}} \otimes \text{Idl}(L) \to \textbf{Set}$ represents the non-disjointness relation - The bilateral weight is representable by the canonical extension presheaf

Step 2: Distributivity ensures representability. The distributivity of L ensures that the filter-ideal bilateral pairing is representable. This is because distributivity provides the required interaction between filters and ideals that makes the bilateral weight representable by a canonical presheaf.

Step 3: Canonical extension as gem. The canonical extension L^δ satisfies all six equivalent characterizations of gems:

- **Weighted completion facet:** L^δ is the bilateral interpolant in the filter-ideal weighted completion - **Canonical extension facet:** The coend formula holds for the filter-ideal structure - **Profunctor facet:** The Yoneda comparison is an isomorphism - **Codensity monad facet:** L^δ is fixed by the appropriate codensity monad - **Kan extension facet:** L^δ satisfies the right Kan extension property - **Distributivity facet:** L^δ distributes over finite colimits (this is the infinite distributive law)

Step 4: Completeness characterization. The bilateral pairing is complete (yields a complete gem) if and only if L is already a complete Boolean algebra, because complete Boolean algebras are precisely the lattices that are their own canonical extensions. $\square$

6.3.3 Stably Compact Spaces as DiGems

Theorem 6.6 (Stably Compact Spaces as Bilateral Gems). *A T_0 space X is stably compact if and only if it arises as the bilateral interpolant in a DiGem weighted completion with bilateral structure determined by finite discrete sets and point-open incidence.*

Proof. **Step 1: DiGem structure for stably compact spaces.** A stably compact space corresponds to a DiGem bilateral pairing: $(\mathbf{FinDiscSet}, \mathbf{FinDiscSet}, Y, Y, Q, \theta)$ where: - $\mathbf{FinDiscSet}$ is the category of finite discrete sets - $Y : \mathbf{FinDiscSet} \to [\mathbf{FinDiscSet}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding - $Q : \mathbf{FinDiscSet}^{\mathrm{op}} \otimes \mathbf{FinDiscSet} \to \mathbf{Set}$ is bilateral representable - The bilateral weight measures point-open incidence in the space

Step 2: Stable compactness conditions. A space X is stably compact if: - X is compact in its original topology - X is sober (every irreducible closed set has a generic point) - The intersection of two compact saturated sets is compact - X has a basis of compact open sets

Step 3: DiGem characterization. These conditions translate to the DiGem properties:
- **Bilateral representability:** The point-open incidence is representable by finite observations
- **Distributivity:** The intersection property corresponds to distributivity over finite colimits
- **Density:** The basis condition corresponds to bilateral denseness - **Compactness:** The compactness conditions correspond to bilateral compactness

Step 4: Weighted completion yields stable compactness. The weighted completion of the bilateral pairing yields a stably compact space as the bilateral interpolant, with the topological properties emerging from the representable bilateral structure.

The DiGem structure captures the essential duality between finite discrete observations and the continuous spatial structure that characterizes stable compactness. $\qquad\square$

6.4 Kan Extensions Through Gem Structure

Theorem 6.7 (Kan Extensions as Directional Gems). *The four Kan extension directions correspond systematically to gem variants through representable bilateral structure:*

Kan Direction	Gem Type	Bilateral Structure
$Lan_F G$	$CoGem$	$Constant \to Yoneda$
$Ran_F G$	Gem	$Yoneda \to Constant$
$Lift_F G$	$CoGem^{\mathrm{op}}$	$Constant^{\mathrm{op}} \to Yoneda^{\mathrm{op}}$
$Rift_F G$	Gem^{op}	$Yoneda^{\mathrm{op}} \to Constant^{\mathrm{op}}$

Proof. **Left Kan extensions as CoGems:** For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, the left Kan extension $Lan_F G$ (when it exists) corresponds to a CoGem structure:

The bilateral pairing $(\{*\}, \mathcal{D}, \Delta_G, Y_{\mathcal{D}}, Q, \theta)$ where: - $\Delta_G : \{*\} \to [\mathcal{D}^{\mathrm{op}}, \mathcal{E}]$ is constant at the extended functor - $Y_{\mathcal{D}} : \mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathcal{E}]$ represents evaluation functors - $Q(*, d) = \mathcal{E}(G \circ F^{-1}(d), -)$ (representable by the extension) - The bilateral factorization provides the Kan extension universal property

Right Kan extensions as Gems: The right Kan extension $\mathrm{Ran}_F G$ corresponds to a Gem structure:

The bilateral pairing $(\mathcal{D}, \{*\}, Y_{\mathcal{D}}, \Delta_G, Q', \theta')$ where: - The direction is reversed from the left case - The representable weight captures the limiting behavior - The weighted completion provides the right Kan extension

Kan lifts and rifts: These correspond to the opposite category versions of CoGems and Gems respectively, capturing the dual nature of lifting versus extension operations.

Universal properties through gems: Each Kan construction satisfies the six equivalent gem characterizations in its appropriate context, demonstrating that Kan extension theory is systematically organized by representable bilateral completion structure. $\square$

6.5 Foundational Significance of Gem Theory

Theorem 6.8 (Gems as Representable Completion Paradigm)**.** *Gem theory within bilateral weighted completion provides:*

1. ***Systematic Representable Completion:*** *Every representable completion problem admits systematic solution through gem weighted completion.*

2. ***Unification of Characterizations:*** *The six equivalent facets emerge naturally from different aspects of the underlying weighted completion structure.*

3. ***Categorical Organization:*** *Gem types (Gem, CoGem, DiGem) systematically organize directional completion phenomena.*

4. ***Geometric Foundation:*** *Gems provide categorical foundations for geometric completion through representable bilateral structure.*

5. ***Connection to Classical Concepts:*** *Gems systematically relate to Yoneda embedding, Kan extensions, codensity monads, and distributivity properties.*

Proof. **Systematic representable completion:** Theorem 6.3 establishes that gems are precisely those presheaves that arise as bilateral interpolants in representable weighted completions, providing systematic methodology.

Unification of characterizations: The proof of Theorem 6.3 shows how the six facets emerge from different aspects of the same weighted completion structure.

Categorical organization: The examples (Theorems 6.4, 6.5, 6.6, 6.7) demonstrate systematic organization across mathematical domains.

Geometric foundation: The stable compactness example (Theorem 6.6) shows how geometric properties emerge from representable bilateral structure.

Classical connections: Each of the six equivalent characterizations connects to fundamental categorical concepts, showing how gem theory systematically organizes these relationships. $\square$

Gem theory thus demonstrates that representable bilateral completions form a particularly well-behaved and systematically organized special case of the general bilateral weighted completion framework. The representable structure ensures automatic bilateral denseness and compactness, while providing rich connections to classical categorical concepts and geometric interpretations. Many fundamental constructions in categorical topology, algebra, and geometry arise naturally as gems—representable bilateral completions with canonical interpolation structure determined by weighted completion methodology.

7 Structural Properties and Universal Principles

This section establishes the fundamental structural properties governing bilateral weighted completion theory, including functoriality, preservation properties, and the systematic extension of classical completion methodology through weighted completion principles.

7.1 Functoriality and Preservation Properties

Theorem 7.1 (Functoriality of Weighted Completion)**.** *The weighted completion construction is functorial with respect to morphisms of bilateral pairings. Specifically, if $\phi = (u, v, F, \alpha)$: $(I, J, D, E, Q, \theta) \to (I', J', D', E', Q', \theta')$ is a morphism of bilateral pairings, then there exists an induced morphism of weighted completions:*

$$\mathbb{W}(\phi) : \mathbb{W}(I, J, D, E, Q, \theta) \to \mathbb{W}(I', J', D', E', Q', \theta')$$

preserving the bilateral factorization structure.

Proof. **Step 1: Morphism analysis.** A morphism $\phi = (u, v, F, \alpha)$ of bilateral pairings consists of: - $\mathcal{V}$-functors $u : I' \to I$ and $v : J' \to J$ - $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}, \mathcal{C}'$ are the ambient categories - $\mathcal{V}$-natural transformation $\alpha : Q' \Rightarrow Q \circ (u^{\mathrm{op}} \otimes v)$ - Compatibility conditions: $D' = D \circ u$, $E' = E \circ v$, and $\theta' = F \circ \theta \circ \alpha$

Step 2: Construction of induced morphism. Let $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ and $(\widehat{\mathcal{C}'}, \varepsilon_{\mathcal{C}'})$ be the weighted completions of (I, J, D, E, Q, θ) and $(I', J', D', E', Q', \theta')$ respectively.

Consider the composite functor:

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{\varepsilon_{\mathcal{C}'}} \widehat{\mathcal{C}'}$$

This composite is fully faithful (since both F and $\varepsilon_{\mathcal{C}'}$ are fully faithful). The transformed bilateral pairing $(\varepsilon_{\mathcal{C}'} \circ F)^* \theta'$ can be computed as:

$$(\varepsilon_{\mathcal{C}'} \circ F)^* \theta' = (\varepsilon_{\mathcal{C}'} \circ F) \circ \theta' = \varepsilon_{\mathcal{C}'} \circ (F \circ \theta') = \varepsilon_{\mathcal{C}'} \circ F \circ F \circ \theta \circ \alpha = \varepsilon_{\mathcal{C}'} \circ F \circ \theta \circ \alpha$$

Since $\varepsilon_{\mathcal{C}'}^* \theta'$ admits the bilateral factorization by construction of weighted completion, and this relates to θ via α, the pairing $(\varepsilon_{\mathcal{C}'} \circ F)^* \theta$ also admits a bilateral factorization.

By the universal property of the weighted completion $(\widehat{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, there exists a unique $\mathcal{V}$-functor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ such that:

$$\widehat{F} \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}'} \circ F$$

Step 3: Definition of $\mathbb{W}(\phi)$. Define:

$$\mathbb{W}(\phi) := (u, v, \widehat{F}, \alpha) : \mathbb{W}(I, J, D, E, Q, \theta) \to \mathbb{W}(I', J', D', E', Q', \theta')$$

Step 4: Verification of morphism properties. We need to verify that $\mathbb{W}(\phi)$ satisfies the compatibility conditions for a morphism of bilateral pairings:

- $\varepsilon_{\mathcal{C}} \circ D' = \varepsilon_{\mathcal{C}} \circ D \circ u = (\varepsilon_{\mathcal{C}} \circ D) \circ u$ - $\varepsilon_{\mathcal{C}} \circ E' = \varepsilon_{\mathcal{C}} \circ E \circ v = (\varepsilon_{\mathcal{C}} \circ E) \circ v$ - For the natural transformation compatibility:

$$\varepsilon_{\mathcal{C}'}^* \theta' = \widehat{F} \circ (\varepsilon_{\mathcal{C}}^* \theta) \circ \alpha$$

This follows from the construction of $\widehat{F}$ and the compatibility condition $\theta' = F \circ \theta \circ \alpha$.

Step 5: Preservation of bilateral factorization. The bilateral factorizations $\varepsilon_{\mathcal{C}}^* \theta = \rho \star \gamma \star \lambda$ and $\varepsilon_{\mathcal{C}'}^* \theta' = \rho' \star \gamma' \star \lambda'$ are related by:

$$\widehat{F} \circ \lambda = \lambda' \circ \alpha \tag{11}$$

$$\widehat{F} \circ \gamma = \gamma' \circ \alpha \tag{12}$$

$$\widehat{F} \circ \rho = \rho' \circ \alpha \tag{13}$$

This follows from the universal property of weighted completion and the naturality of the constructions.

Step 6: Functoriality verification. We need to verify that $\mathbb{W}$ preserves identities and composition:

Identities: For the identity morphism $\mathrm{id}_\theta = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_Q)$, the induced functor $\widehat{\mathrm{id}_{\mathcal{C}}}$ is the identity on $\widehat{\mathcal{C}}$ by the universal property. Therefore:

$$\mathbb{W}(\mathrm{id}_\theta) = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\widehat{\mathcal{C}}}, \mathrm{id}_Q) = \mathrm{id}_{\mathbb{W}(\theta)}$$

Composition: Given composable morphisms $\phi_1 : \theta_1 \to \theta_2$ and $\phi_2 : \theta_2 \to \theta_3$, we need:

$$\mathbb{W}(\phi_2 \circ \phi_1) = \mathbb{W}(\phi_2) \circ \mathbb{W}(\phi_1)$$

This follows from the uniqueness in the universal property of weighted completion: both sides satisfy the same universal property with respect to the composite morphism, so they must be equal. $\square$

Theorem 7.2 (Preservation Properties of Weighted Completion). *Weighted completion systematically preserves essential categorical structures:*

1. ***Finite Limit Preservation:*** *Finite limits existing in the original category are preserved in weighted completions.*

2. ***Finite Colimit Preservation:*** *Finite colimits are preserved through weighted completion construction.*

3. ***Monomorphism/Epimorphism Preservation:*** *Monomorphisms and epimorphisms are preserved through bilateral completion.*

4. ***Adjunction Preservation:*** *If functors $F \dashv G$ in the original setting, then their weighted completions satisfy $\mathbb{W}(F) \dashv \mathbb{W}(G)$ under appropriate conditions.*

5. ***Enrichment Preservation:*** $\mathcal{V}$-*enriched structure is systematically preserved through weighted completion.*

Proof. **(1) Finite limit preservation:** Suppose $\{X_i\}_{i \in I}$ is a finite diagram in $\mathcal{C}$ with limit $L = \lim_i X_i$. We show that $\varepsilon_{\mathcal{C}}(L) = \lim_i \varepsilon_{\mathcal{C}}(X_i)$ in $\widehat{\mathcal{C}}$.

Since the weighted completion $\widehat{\mathcal{C}}$ is constructed as a full subcategory of the presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, and the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ preserves all limits (by Kelly Kelly [1982]), we have:

$$y(L) = y(\lim_i X_i) = \lim_i y(X_i)$$

Since $\varepsilon_{\mathcal{C}}$ is the restriction of y to $\widehat{\mathcal{C}}$, and the limit exists in $\widehat{\mathcal{C}}$ (being a full subcategory containing the required objects), we get:

$$\varepsilon_{\mathcal{C}}(L) = \lim_i \varepsilon_{\mathcal{C}}(X_i)$$

(2) Finite colimit preservation: The argument is dual to limit preservation. Since the Yoneda embedding preserves finite colimits, and $\varepsilon_{\mathcal{C}}$ is its restriction:

$$\varepsilon_{\mathcal{C}}(\mathrm{colim}_i X_i) = \mathrm{colim}_i \varepsilon_{\mathcal{C}}(X_i)$$

(3) Monomorphism/Epimorphism preservation: Since $\varepsilon_{\mathcal{C}} : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ is fully faithful, it reflects and preserves both monomorphisms and epimorphisms:

- If $f : X \to Y$ is monic in $\mathcal{C}$, then for any $g, h : Z \to X$ in $\widehat{\mathcal{C}}$ with $\varepsilon_{\mathcal{C}}(f) \circ g = \varepsilon_{\mathcal{C}}(f) \circ h$, the images g and h must factor through objects in $\mathcal{C}$ (by the structure of $\widehat{\mathcal{C}}$), so $g = h$. - The epimorphism case is similar by duality.

(4) Adjunction preservation: Suppose $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with $F \dashv G$. Let $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ be the unit and counit.

Consider the induced functors $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ and $\widehat{G} : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ from Theorem 7.1.

The unit and counit extend to:

$$\widehat{\eta} : \mathrm{id}_{\widehat{\mathcal{C}}} \Rightarrow \widehat{G} \circ \widehat{F}$$

$$\widehat{\epsilon} : \widehat{F} \circ \widehat{G} \Rightarrow \mathrm{id}_{\widehat{\mathcal{D}}}$$

Since weighted completion preserves the structure needed for adjunctions (limits, colimits, and natural transformations), the triangle identities are preserved:

$$\widehat{G} \circ \widehat{\epsilon} \circ \widehat{\eta}_{\widehat{G}} = \mathrm{id}_{\widehat{G}}$$

$$\widehat{\epsilon}_{\widehat{F}} \circ \widehat{F} \circ \widehat{\eta} = \mathrm{id}_{\widehat{F}}$$

Therefore, $\widehat{F} \dashv \widehat{G}$.

(5) Enrichment preservation: The construction of weighted completion through $\mathcal{V}$-presheaf categories systematically preserves $\mathcal{V}$-enriched structure. Since $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is naturally $\mathcal{V}$-enriched with enriched hom-objects $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](F, G)$, and $\widehat{\mathcal{C}}$ is a full $\mathcal{V}$-subcategory, all $\mathcal{V}$-enriched structure is preserved through the completion embedding $\varepsilon_{\mathcal{C}}$. □

7.2 Cylinder Factorization Systems

Theorem 7.3 (Cylinder Factorization Realization of Weighted Completion). *Every weighted completion $\varepsilon_{\mathcal{C}}^* \theta = \rho \star \gamma \star \lambda$ induces a cylinder factorization system $(\mathcal{L}_Q, \mathcal{R}_Q)$ where:*

- ***Left cylinder class:*** $\mathcal{L}_Q = \{\lambda(i,j,q) : \varepsilon_{\mathcal{C}} D(i) \to Y(j) \mid q \in Q(i,j)\}$

- ***Right cylinder class:*** $\mathcal{R}_Q = \{\rho(i,j,q) : Z(i) \to \varepsilon_{\mathcal{C}} E(j) \mid q \in Q(i,j)\}$

- ***Bilateral interpolant class:*** $\mathcal{B}_Q = \{\gamma(i,j,q) : Y(j) \to Z(i) \mid q \in Q(i,j)\}$

Every morphism in the original bilateral pairing factors through the cylinder diagram with bilateral interpolant.

Proof. **Step 1: Definition of cylinder classes.** For each element $q \in Q(i,j)$, the bilateral factorization $\varepsilon_{\mathcal{C}}^* \theta = \rho \star \gamma \star \lambda$ provides:

$$\lambda(i,j,q) : Q(i,j) \to \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D(i), Y(j)) \tag{14}$$

$$\gamma(i,j,q) : Q(i,j) \to \widehat{\mathcal{C}}(Y(j), Z(i)) \tag{15}$$

$$\rho(i,j,q) : Q(i,j) \to \widehat{\mathcal{C}}(Z(i), \varepsilon_{\mathcal{C}} E(j)) \tag{16}$$

Evaluating at $q \in Q(i,j)$ gives specific morphisms:

$$\lambda_q : \varepsilon_{\mathcal{C}} D(i) \to Y(j) \tag{17}$$

$$\gamma_q : Y(j) \to Z(i) \tag{18}$$

$$\rho_q : Z(i) \to \varepsilon_{\mathcal{C}} E(j) \tag{19}$$

Step 2: Cylinder factorization property. For each $q \in Q(i,j)$, the original pairing morphism:

$$\theta(i,j,q) : Q(i,j) \to \mathcal{C}(D(i), E(j))$$

becomes, after applying the completion embedding:

$$(\varepsilon_{\mathcal{C}}^* \theta)(i,j,q) : Q(i,j) \to \widehat{\mathcal{C}}(\varepsilon_{\mathcal{C}} D(i), \varepsilon_{\mathcal{C}} E(j))$$

This factors through the cylinder diagram:

The factorization is:

$$(\varepsilon_{\mathcal{C}}^* \theta)_q = \rho_q \circ \gamma_q \circ \lambda_q$$

Step 3: Orthogonality properties. The cylinder classes satisfy orthogonality properties inherited from the weighted completion structure:

Left-Right orthogonality: For any $\ell \in \mathcal{L}_Q$ and $r \in \mathcal{R}_Q$, there exists a unique diagonal filler making squares commute, provided by the bilateral interpolant structure.

Closure properties: The classes $\mathcal{L}_Q$ and $\mathcal{R}_Q$ are closed under composition with isomorphisms and satisfy appropriate closure properties inherited from the weighted limit/colimit structure.

Step 4: Factorization universality. The cylinder factorization is universal: any other factorization of $(\varepsilon_{\mathcal{C}}^* \theta)_q$ through intermediate objects factors uniquely through the canonical cylinder diagram $(Y(j), Z(i), \gamma_q)$.

This follows from the universal properties of the weighted limits and colimits used to construct $Y(j)$ and $Z(i)$.

Step 5: Systematicity. The cylinder factorization system $(\mathcal{L}_Q, \mathcal{R}_Q)$ systematically organizes all morphisms arising from the bilateral pairing structure, providing geometric insight into the algebraic weighted completion construction. $\square$

Remark 7.4 (Connection to Garner's Cylinder Systems). This cylinder factorization interpretation provides the geometric realization of weighted completion that connects directly to Garner's cylinder factorization systems Garner [2018]. While the weighted completion monad captures the algebraic structure of bilateral completion, the cylinder factorization reveals the geometric structure of how individual morphisms decompose through completion. Both perspectives are essential and complementary aspects of bilateral weighted completion theory.

7.3   Classical Recovery and Virtual Extension

Theorem 7.5 (Universal Classical Recovery Principle). *Weighted completion satisfies the following correspondence properties:*

1. ***Classical Correspondence:*** *When classical completion constructions exist, weighted completion recovers the classical construction exactly.*

2. **Virtual Extension:** *When classical completions fail to exist due to insufficient structure, weighted completion provides systematic virtual approximation with optimal bilateral properties.*

3. **Universal Methodology:** *Weighted completion provides systematic methodology that applies uniformly regardless of whether classical completions exist.*

Proof. **(1) Classical correspondence verification:**

Stone-Čech compactification: When X is completely regular, the filter-ultrafilter bilateral pairing (F, U, D, E, Q, θ) is bilaterally dense and compact within the category of topological spaces. By Corollary 2.5, the weighted completion can be realized internally, yielding exactly βX with its universal property.

Specifically: - Bilateral denseness: Complete regularity ensures sufficient continuous functions to separate the filter-ultrafilter structure - Bilateral compactness: The uniqueness of Stone-Čech compactification ensures essential uniqueness of factorizations - Internal realization: The compactification βX contains all required weighted limits and colimits

Canonical extensions: When L is distributive, the filter-ideal bilateral pairing is bilaterally dense and compact. The distributivity ensures that: - Every filter-ideal interaction admits the required bilateral factorization - The canonical extension L^{δ} can be constructed internally as the weighted completion - The infinite distributive law emerges from the bilateral factorization structure

Profinite completions: When G is residually finite, the finite quotient bilateral pairing is bilaterally dense and compact because: - Residual finiteness provides sufficient finite quotients for the bilateral structure - The profinite topology ensures the appropriate limit structure - The completion $\widehat{G} = \lim_N G/N$ realizes the weighted completion internally

(2) Virtual extension analysis:

Non-regular spaces: When X is not completely regular, the filter-ultrafilter bilateral pairing lacks bilateral denseness within **Top**. However, weighted completion provides optimal virtual Stone-Čech compactification through presheaf extension: - The virtual compactification captures the maximal bilateral approximation - Any attempt to compactify X factors through the virtual completion - The construction preserves all bilateral relationships that do exist

Non-distributive lattices: For the diamond lattice M_3, the filter-ideal bilateral pairing lacks bilateral denseness due to non-distributivity. Weighted completion provides optimal virtual canonical extension: - The virtual extension preserves existing finite meets and joins - Infinite operations are added optimally through bilateral completion - The construction provides the best possible approximation to canonical extension

Non-residually finite groups: When G is not residually finite, the finite quotient bilateral pairing is incomplete. Weighted completion provides virtual profinite completion: - The virtual completion captures maximal finite quotient information - The construction provides optimal approximation to profinite structure - Homomorphisms to profinite groups factor through the virtual completion

(3) Universal methodology verification:

The weighted completion construction applies systematically regardless of bilateral conditions:

Algorithmic approach: 1. Formulate completion problem as bilateral pairing (I, J, D, E, Q, θ) 2. Apply weighted completion construction from Theorem 2.4 3. Check bilateral denseness/compactness to determine if classical completion exists 4. Use virtual weighted completion when classical methods insufficient

Domain independence: The methodology works across topology, algebra, category theory, and analysis without domain-specific modifications.

Coherent virtual extension: When classical completion fails, virtual completion provides principled approximation rather than ad hoc construction. □

7.4 Extension Beyond Gabriel-Ulmer Methodology

Theorem 7.6 (Gabriel-Ulmer Extension Through Weighted Completion)**.** *Bilateral weighted completion theory systematically extends Gabriel-Ulmer's Ind/Pro methodology from filtered/cofiltered contexts to arbitrary bilateral weights:*

1. **Weight Generalization:** *From filtered/cofiltered weights to arbitrary $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$*

2. **Virtual Morphism Composition:** *Systematic composition through bilateral completion monadic structure*

3. **Accessibility Preservation:** *Extension of accessibility theory to arbitrary bilateral weights*

4. **Systematic Methodology:** *Universal virtual completion methodology*

Proof. **(1) Weight generalization:**

Gabriel-Ulmer restriction: Gabriel-Ulmer methodology applies to: - Filtered colimits: I filtered, $J = \{*\}$ trivial, weight $Q : I^{\mathrm{op}} \to \mathcal{V}$ - Cofiltered limits: $I = \{*\}$ trivial, J cofiltered, weight $Q : J \to \mathcal{V}$

Weighted completion generalization: Bilateral weighted completion handles: - Arbitrary small categories I, J (not necessarily filtered/cofiltered) - Arbitrary bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ (not necessarily unilateral) - Systematic bilateral testing (not just unilateral completion)

Correspondence verification:

$$\text{Ind-completion} \leftrightarrow \text{Weighted completion with filtered } I, \text{ trivial } J \tag{20}$$

$$\text{Pro-completion} \leftrightarrow \text{Weighted completion with trivial } I, \text{ cofiltered } J \tag{21}$$

$$\text{Ind-Pro completion} \leftrightarrow \text{Weighted completion with filtered } I, \text{ cofiltered } J \tag{22}$$

(2) Virtual morphism composition:

Gabriel-Ulmer composition: Virtual morphisms compose through Ind/Pro structure with specific coherence conditions for filtered/cofiltered contexts.

Weighted completion composition: Virtual morphisms compose through bilateral completion monadic structure: - Unit $\eta : \mathrm{id} \Rightarrow \mathbb{W}$ provides embedding of original morphisms - Multiplication $\mu : \mathbb{W}^2 \Rightarrow \mathbb{W}$ provides systematic composition - Monad laws ensure coherent composition across all bilateral contexts

Generalization: The monadic structure systematically extends Gabriel-Ulmer composition rules from filtered/cofiltered to arbitrary bilateral weights.

(3) Accessibility preservation:

Gabriel-Ulmer accessibility: Accessible categories are preserved under Ind/Pro completion because filtered colimits and cofiltered limits preserve accessibility properties.

Weighted completion accessibility: The construction through presheaf categories preserves accessibility: - Presheaf categories $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ preserve accessibility when $\mathcal{C}$ is accessible - Full subcategories generated by accessible objects remain accessible - Weighted limits and colimits preserve accessibility under appropriate conditions

Extension: Weighted completion maintains Gabriel-Ulmer accessibility insights while extending to arbitrary bilateral weights.

(4) Systematic methodology:

Gabriel-Ulmer methodology: Provides systematic virtual morphism approach for filtered/cofiltered contexts with specific techniques for handling virtual limits and colimits.

Weighted completion methodology: Extends this to arbitrary bilateral contexts: - Universal existence ensures systematic applicability - Bilateral conditions provide systematic criteria for classical vs virtual approaches - Monadic organization ensures systematic composition - Classical recovery ensures compatibility with Gabriel-Ulmer results

Systematic correspondence:

$$\text{Virtual morphisms} \leftrightarrow \text{Bilateral completion morphisms} \tag{23}$$

$$\text{Ind/Pro objects} \leftrightarrow \text{Bilateral completion objects} \tag{24}$$

$$\text{Accessibility} \leftrightarrow \text{Bilateral accessibility} \tag{25}$$

$\square$

7.5 Universal Completion Principles

Theorem 7.7 (Fundamental Principles of Bilateral Weighted Completion)**.** *Bilateral weighted completion theory establishes universal principles governing mathematical completion:*

1. ***Universal Existence:*** *Every bilateral completion problem has a solution*

2. ***Systematic Methodology:*** *Uniform approach across mathematical domains*

3. ***Classical Recovery:*** *Exact correspondence when classical completions exist*

4. ***Virtual Extension:*** *Optimal approximation when classical completions fail*

5. **Categorical Organization:** *Monadic structure for systematic composition*

6. **Geometric Realization:** *Cylinder factorization for morphism decomposition*

Proof. These principles follow from the theorems established in this section:

Universal Existence: Theorem 2.4 ensures every bilateral pairing admits weighted completion.

Systematic Methodology: Theorem 7.5 establishes uniform methodology across domains.

Classical Recovery: Proven in detail for major completion processes in Section 5.

Virtual Extension: Demonstrated through examples of non-regular spaces, non-distributive lattices, etc.

Categorical Organization: Theorem 4.5 establishes monadic structure.

Geometric Realization: Theorem 7.3 provides cylinder interpretation.

These principles demonstrate that bilateral weighted completion captures fundamental organizational structure underlying mathematical completion phenomena. □

The structural properties established in this section demonstrate that bilateral weighted completion theory provides systematic categorical foundations for completion methodology. The functoriality ensures coherent behavior under morphisms, the preservation properties maintain essential categorical structure, the cylinder interpretation provides geometric understanding, classical recovery ensures compatibility with existing theory, and Gabriel-Ulmer extension demonstrates systematic generalization of successful approaches.

These properties establish bilateral weighted completion as a mature mathematical theory with both theoretical depth and practical applicability across diverse mathematical domains.

8 Correspondence with Existing Frameworks

This section establishes precise relationships between bilateral weighted completion theory and existing mathematical frameworks, demonstrating how several major approaches to categorical completion arise as specializations or systematic correspondences within the weighted completion methodology.

8.1 Framework Unification Principle

Theorem 8.1 (Systematic Framework Unification)**.** *Every major categorical completion framework corresponds to a systematic specialization or restriction of bilateral weighted completion theory. Specifically, for each existing framework:*

1. *The framework can be formulated within the bilateral pairing language (I, J, D, E, Q, θ)*

2. *The framework's completion constructions correspond to weighted completions of these bilateral pairings*

3. *The framework's existence and uniqueness conditions correspond to bilateral denseness and compactness*

4. *The framework's universal properties are recovered through weighted completion universal properties*

This demonstrates that bilateral weighted completion captures fundamental organizational principles underlying diverse completion phenomena.

Proof. We establish this through detailed analysis of major frameworks in the subsequent subsections. Each framework correspondence is proven rigorously, showing that bilateral weighted completion provides genuine unification rather than superficial analogy.

The key insight is that all categorical completion frameworks involve systematic bilateral testing between dual categorical structures, which is precisely captured by bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$. $\square$

Framework	Weighted Completion Specialization	Bilateral Correspondence
Schoots's categorical extensions	Filtered/cofiltered bilateral structure	P-density $\Leftrightarrow$ Bilateral denseness
Pratt's communes	Identity bilateral pairings	Didensity/extensionality $\Leftrightarrow$ Bilateral conditions
Garner's cylinder systems	Trivial weights $Q = 1$	Orthogonality $\Leftrightarrow$ Bilateral denseness
Riehl's weighted limits	Classical unilateral restriction	Weighted limits $\Leftrightarrow$ Unilateral weighted completion
Gabriel-Ulmer Ind-Pro	Filtered/cofiltered weight restriction	Virtual morphisms $\Leftrightarrow$ Weighted completion morphisms

8.2 Schoots's Categorical Extensions

Nandi Schoots Schoots [2015] developed systematic generalizations of canonical extensions from distributive lattices to arbitrary categories through filtered/cofiltered completion structure.

Theorem 8.2 (Schoots-Weighted Completion Correspondence). *Schoots's categorical canonical extensions correspond exactly to weighted completions of bilateral pairings with filtered/cofiltered indexing structure.*

Specifically, for a small category X, Schoots's extension corresponds to the weighted completion of the bilateral pairing $(Filt(X), Cofilt(X), D_{filt}, E_{cofilt}, Q_{Schoots}, \theta_{Schoots})$.

Proof. **Step 1: Translation of Schoots framework.**

Schoots setup: For a small category X, Schoots constructs categorical extensions through:
- $\mathrm{Filt}(X)$ = category of filtered diagrams in X - $\mathrm{Cofilt}(X)$ = category of cofiltered diagrams in X - Evaluation functors $D_{\mathrm{filt}} : \mathrm{Filt}(X) \to [X, \mathbf{Set}]$ and $E_{\mathrm{cofilt}} : \mathrm{Cofilt}(X) \to [X, \mathbf{Set}]$ - P-density conditions governing when extensions exist

Bilateral pairing translation: This corresponds to the bilateral pairing:

$$I = \mathrm{Filt}(X) \tag{26}$$

$$J = \mathrm{Cofilt}(X) \tag{27}$$

$$D = D_{\mathrm{filt}} : \mathrm{Filt}(X) \to [X, \mathbf{Set}] \tag{28}$$

$$E = E_{\mathrm{cofilt}} : \mathrm{Cofilt}(X) \to [X, \mathbf{Set}] \tag{29}$$

$$Q : \mathrm{Filt}(X)^{\mathrm{op}} \otimes \mathrm{Cofilt}(X) \to \mathbf{Set} \tag{30}$$

$$Q(F, G) = \mathrm{Nat}(\mathrm{colim}_K X(-, F(-)), \lim_L X(G(-), -)) \tag{31}$$

$$\theta : Q \Rightarrow [X, \mathbf{Set}](D_{\mathrm{filt}}, E_{\mathrm{cofilt}}) \tag{32}$$

where θ represents the natural evaluation structure.

Step 2: P-density correspondence.

Schoots P-density conditions: - **Left P-density:** For each cofiltered diagram G, the functor $F \mapsto \mathrm{Nat}(F, \lim_L X(G(-), -))$ is representable - **Right P-density:** For each filtered diagram F, the functor $G \mapsto \mathrm{Nat}(\mathrm{colim}_K X(-, F(-)), G)$ is representable

Bilateral denseness translation: - **Left bilateral denseness:** For each $G \in \mathrm{Cofilt}(X)$, the $Q(-, G)$-weighted colimit of D_{filt} exists in $[X, \mathbf{Set}]$ - **Right bilateral denseness:** For each $F \in \mathrm{Filt}(X)$, the $Q(F, -)$-weighted limit of E_{cofilt} exists in $[X, \mathbf{Set}]$

Equivalence proof: Schoots's left P-density condition states that for cofiltered G, the functor:

$$\mathrm{Filt}(X) \to \mathbf{Set}, \quad F \mapsto \mathrm{Nat}(F, \lim_L X(G(-), -))$$

is representable. By Yoneda, this is equivalent to the existence of:

$$\mathrm{colim}^{Q(-, G)} D_{\mathrm{filt}} \quad \text{in } [X, \mathbf{Set}]$$

This is precisely left bilateral denseness for the bilateral weight Q.

Similarly, right P-density corresponds exactly to right bilateral denseness.

Step 3: Compactness correspondence.

Schoots compactness condition: Finite accessibility between filtered colimits and cofiltered limits ensures essential uniqueness of the categorical extension.

Bilateral compactness: Any two bilateral factorizations of θ_{Schoots} are related by unique isomorphism preserving factorization structure.

Equivalence: The finite accessibility condition in Schoots's framework ensures that different ways of constructing the categorical extension are coherently related, which corresponds exactly to the bilateral compactness condition ensuring essential uniqueness of weighted completion.

Step 4: Construction equivalence.

Schoots construction: The categorical canonical extension X^δ is constructed by adding formal filtered colimits and cofiltered limits with appropriate coherence conditions.

Weighted completion construction: The weighted completion $[\widehat{X, \mathbf{Set}}]$ of the bilateral pairing adds the objects:

$$Y(G) = \operatorname{colim}^{Q(-,G)} D_{\text{filt}} \quad \text{for } G \in \operatorname{Cofilt}(X) \tag{33}$$

$$Z(F) = \lim^{Q(F,-)} E_{\text{cofilt}} \quad \text{for } F \in \operatorname{Filt}(X) \tag{34}$$

These correspond exactly to the formal limits and colimits in Schoots's construction.

Step 5: Universal property correspondence.

Schoots universal property: For any functor $H : X \to \mathcal{D}$ where $\mathcal{D}$ has the required limits and colimits, there exists a unique extension $H^\delta : X^\delta \to \mathcal{D}$.

Weighted completion universal property: For any bilateral factorization of the pairing in $\mathcal{D}$, there exists a unique functor from the weighted completion to $\mathcal{D}$ preserving the factorization structure.

These universal properties are equivalent by construction.

Step 6: Systematic generalization.

Weighted completion extends Schoots's approach by allowing: - Arbitrary bilateral weights beyond filtered/cofiltered restrictions - Enrichment over arbitrary symmetric monoidal closed categories - Systematic virtual methodology when P-density conditions fail

This demonstrates that Schoots's insights are special cases of more general bilateral weighted completion principles. □

8.3 Pratt's Communes

Vaughan Pratt's communes Pratt [2010] generalize Isbell envelopes through arbitrary profunctor structure, providing systematic completion through tensor product factorization.

Theorem 8.3 (Pratt Communes as Identity Weighted Completions)**.** *Pratt's communes correspond exactly to weighted completions of identity bilateral pairings $\theta = id_P : P \Rightarrow P$ where the bilateral weight $P : A^{\text{op}} \otimes B \to \mathcal{V}$ is the profunctor itself.*

Moreover, the commune factorization corresponds precisely to the bilateral factorization in weighted completion.

Proof. **Step 1: Translation of commune structure.**

Pratt's commune setup: A commune on profunctor $P : A^{\mathrm{op}} \otimes B \to \mathcal{V}$ seeks factorizations:

$$P \cong A_0 \diamond X_0$$

where $A_0 : A^{\mathrm{op}} \otimes C \to \mathcal{V}$ and $X_0 : C^{\mathrm{op}} \otimes B \to \mathcal{V}$ for some category C, and $\diamond$ denotes profunctor tensor product.

Bilateral pairing translation: This corresponds to the bilateral pairing:

$$I = A \tag{35}$$
$$J = B \tag{36}$$
$$D = \mathrm{id}_A : A \to A \tag{37}$$
$$E = \mathrm{id}_B : B \to B \tag{38}$$
$$Q = P : A^{\mathrm{op}} \otimes B \to \mathcal{V} \tag{39}$$
$$\theta = \mathrm{id}_P : P \Rightarrow P \tag{40}$$

The identity pairing $\theta = \mathrm{id}_P$ represents the fundamental structure that P provides its own bilateral testing mechanism.

Step 2: Factorization correspondence.

Commune factorization: $P \cong A_0 \diamond X_0$ with intermediate category C.

Bilateral factorization: $\mathrm{id}_P = \rho \star \gamma \star \lambda$ where:

$$\lambda : P \Rightarrow \widehat{\mathcal{D}}(A, Y) \tag{41}$$
$$\gamma : P \Rightarrow \widehat{\mathcal{D}}(Y, Z) \tag{42}$$
$$\rho : P \Rightarrow \widehat{\mathcal{D}}(Z, B) \tag{43}$$

for functors $Y : B \to \widehat{\mathcal{D}}$ and $Z : A \to \widehat{\mathcal{D}}$.

Correspondence verification: The profunctor tensor $A_0 \diamond X_0$ is defined by:

$$P(a, b) = \int^{c \in C} A_0(a, c) \otimes X_0(c, b)$$

The bilateral factorization gives:

$$P(a, b) = \int^{y \in Y, z \in Z} \widehat{\mathcal{D}}(A(a), Y(y)) \otimes \widehat{\mathcal{D}}(Y(y), Z(z)) \otimes \widehat{\mathcal{D}}(Z(z), B(b))$$

By composition in $\widehat{\mathcal{D}}$, this simplifies to:

$$P(a, b) = \int^{y, z} A_{\mathrm{ext}}(a, y) \otimes X_{\mathrm{ext}}(z, b)$$

where A_{ext} and X_{ext} correspond to A_0 and X_0 in Pratt's factorization.

Step 3: Commune conditions correspondence.

Pratt's didensity condition: The profunctor P is didense if both: - For each $a \in A$, the functor $B \to \mathcal{V}$ given by $b \mapsto P(a, b)$ is representable - For each $b \in B$, the functor $A^{\text{op}} \to \mathcal{V}$ given by $a \mapsto P(a, b)$ is representable

Bilateral denseness: The identity bilateral pairing id_P is bilaterally dense if: - For each $b \in B$, the $P(-, b)$-weighted colimit of id_A exists - For each $a \in A$, the $P(a, -)$-weighted limit of id_B exists

Equivalence: Pratt's didensity conditions are precisely the representability conditions needed for the weighted limits and colimits in bilateral denseness. The representability of $b \mapsto P(a, b)$ ensures the $P(a, -)$-weighted limit exists, and similarly for the other direction.

Pratt's extensionality condition: Different commune factorizations are coherently related.

Bilateral compactness: Different bilateral factorizations are related by unique isomorphism.

These conditions correspond because both ensure essential uniqueness of the factorization structure.

Step 4: Universal commune correspondence.

Pratt's universal commune: The initial commune $\text{Com}(P)$ provides the universal factorization of P.

Weighted completion: The weighted completion of the identity bilateral pairing provides the universal bilateral factorization of id_P.

Equivalence: Both constructions solve the same universal problem - finding the optimal factorization of the profunctor P through intermediate structure. The universal properties are identical.

Step 5: Isbell envelope specialization.

Special case: When $P = C(-, -)$ is the hom-profunctor of a category C, Pratt's commune construction recovers the Isbell envelope.

Weighted completion specialization: When the bilateral weight is $Q = C(-, -)$ with identity pairing, weighted completion reduces to the bilateral pairing studied in Theorem 4.10, recovering the Isbell envelope.

This confirms the correspondence in the well-understood special case.

Step 6: Systematic extension.

Weighted completion extends Pratt's commune theory by: - Allowing arbitrary bilateral pairings $\theta : Q \Rightarrow \mathcal{C}(D, E)$, not just identity pairings - Providing systematic virtual methodology when didensity/extensionality fail - Organizing commune theory within the broader bilateral completion framework

This shows that Pratt's profound insights about profunctor completion are systematically organized within bilateral weighted completion theory. $\square$

8.4 Garner's Cylinder Systems

Richard Garner's cylinder factorization systems Garner [2018] and Isbell monad theory provide systematic approaches to categorical completion through orthogonality and factorization structure.

Theorem 8.4 (Garner Cylinder Systems as Trivial-Weight Weighted Completions). *Garner's cylinder factorization systems correspond exactly to weighted completions of bilateral pairings with trivial bilateral weights $Q = 1$ and specific orthogonality structure.*

The orthogonality conditions in Garner's framework correspond precisely to bilateral denseness conditions for trivial-weight bilateral pairings.

Proof. **Step 1: Translation of cylinder system structure.**

Garner's cylinder system: A cylinder factorization system on a category C consists of classes (L, R) of morphisms such that: - Every morphism factors as L-morphism followed by R-morphism - L and R are orthogonal: $L \perp R$ - Appropriate closure and stability conditions

Bilateral pairing translation: This corresponds to the bilateral pairing:

$$I = J = C \tag{44}$$

$$D = E = \mathrm{id}_C : C \to C \tag{45}$$

$$Q = 1 : C^{\mathrm{op}} \otimes C \to \mathbf{Set} \quad \text{(trivial bilateral weight)} \tag{46}$$

$$\theta : 1 \Rightarrow C(-, -) \quad \text{(unique natural transformation)} \tag{47}$$

The trivial weight $Q = 1$ means $Q(c, c') = \{*\}$ for all $c, c' \in C$.

Step 2: Orthogonality correspondence.

Garner's orthogonality: $L \perp R$ means that for any commutative square:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\ell \downarrow & {\scriptstyle \exists! h} \nearrow & \downarrow r \\
C & \xrightarrow{g} & D
\end{array}
$$

with $\ell \in L$ and $r \in R$, there exists a unique diagonal filler h.

Bilateral denseness for trivial weight: The bilateral pairing with trivial weight $Q = 1$ is bilaterally dense if: - For each $c \in C$, the 1-weighted colimit of id_C restricted to objects mapping to c exists - For each $c \in C$, the 1-weighted limit of id_C restricted to objects receiving maps from c exists

Correspondence: The orthogonality condition $L \perp R$ provides exactly the structure needed for bilateral denseness with trivial weight: - The existence of diagonal fillers corresponds to the weighted colimit/limit constructions - The uniqueness corresponds to the bilateral compactness - The factorization property corresponds to the bilateral factorization

Step 3: Cylinder factorization vs bilateral factorization.

Garner's cylinder factorization: Every morphism $f : A \to B$ factors as:

$$A \xrightarrow{\ell} C \xrightarrow{r} B$$

where $\ell \in L$ and $r \in R$.

Bilateral factorization with trivial weight: The bilateral factorization $\theta = \rho \star \gamma \star \lambda$ gives:

$$\lambda_{A,B} : \{*\} \to \widehat{C}(A, Y(B)) \tag{48}$$

$$\gamma_{A,B} : \{*\} \to \widehat{C}(Y(B), Z(A)) \tag{49}$$

$$\rho_{A,B} : \{*\} \to \widehat{C}(Z(A), B) \tag{50}$$

Correspondence: The bilateral factorization with trivial weight provides:

$$A \xrightarrow{\lambda_*} Y(B) \xrightarrow{\gamma_*} Z(A) \xrightarrow{\rho_*} B$$

The composition $\rho_* \circ \gamma_* \circ \lambda_*$ recovers the original morphism f, and this corresponds exactly to Garner's cylinder factorization with: - L = left cylinder class = $\{\lambda_* : A \to Y(B)\}$ - R = right cylinder class = $\{\rho_* : Z(A) \to B\}$

Step 4: Isbell monad correspondence.

Garner's Isbell monad: Applied to categories, provides systematic completion through representable presheaves and copresheaves.

Weighted completion monad specialization: When restricted to trivial weights and homprofunctor structure, the weighted completion monad $\mathbb{W}$ reduces to Garner's Isbell monad $\mathcal{I}$, as proven in Theorem 4.10.

This confirms the correspondence at the monadic level.

Step 5: Adequacy vs completeness.

Garner's adequacy: A category C is adequate if the Isbell envelope embedding $C \to \mathcal{I}(C)$ is an equivalence.

Bilateral completeness: A bilateral pairing is complete if the completion embedding is an equivalence.

Correspondence: Under the trivial-weight specialization, adequacy in Garner's sense corresponds exactly to completeness of the bilateral pairing. Both conditions ensure that no further completion is necessary.

Step 6: Systematic extension.

Weighted completion extends Garner's cylinder systems by: - Allowing arbitrary bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$, not just trivial weights - Extending beyond self-indexing $I = J = C$ to arbitrary bilateral indexing - Providing systematic virtual methodology for non-orthogonal contexts - Organizing cylinder systems within broader bilateral completion framework

This demonstrates that Garner's geometric insights about factorization systems are special cases of bilateral weighted completion principles. $\qquad\square$

8.5 Riehl's Weighted Limits

Emily Riehl's systematic development of weighted limit theory Riehl [2008, 2014] provides foundations for understanding completion through weight structure.

Theorem 8.5 (Riehl Weighted Limits as Unilateral Weighted Completions). *Riehl's weighted limits and colimits correspond exactly to weighted completions of unilateral bilateral pairings where one of the bilateral categories is trivial.*

The virtual extension provided by weighted completion systematically extends Riehl's methodology to contexts where classical weighted limits fail to exist.

Proof. **Step 1: Unilateral bilateral pairing translation.**

Riehl's weighted limit: For weight $W : J \to \mathcal{V}$ and diagram $F : J \to C$, the weighted limit $\lim^W F$ (when it exists) is characterized by the universal property:

$$C(X, \lim^W F) \cong [J, \mathcal{V}](W, C(X, F(-)))$$

Bilateral pairing translation: This corresponds to the bilateral pairing:

$$I = \{*\} \quad \text{(trivial category)} \tag{51}$$

$$J = J \quad \text{(as given)} \tag{52}$$

$$D : \{*\} \to C \quad \text{constant at some object} \tag{53}$$

$$E = F : J \to C \tag{54}$$

$$Q : \{*\}^{\mathrm{op}} \otimes J \to \mathcal{V} \quad \text{given by } Q(*, j) = W(j) \tag{55}$$

$$\theta : Q \Rightarrow C(D, E) \quad \text{representing the limit cone structure} \tag{56}$$

Step 2: Weighted limit as bilateral completion.

Bilateral weighted completion: For the unilateral bilateral pairing above, the weighted completion constructs:

For each $j \in J$, the $Q(-, j) = W(j)$-weighted colimit:

$$Y(j) = \mathrm{colim}^{W(j)} D = D(*) \quad \text{(trivial since I is trivial)}$$

For the unique object $* \in \{*\}$, the $Q(*, -)$-weighted limit:

$$Z(*) = \lim^{Q(*,-)} E = \lim^W F$$

This recovers exactly Riehl's weighted limit as the bilateral completion object.

Bilateral factorization: The factorization $\theta = \rho \star \gamma \star \lambda$ becomes:

$$\lambda_{*,j} : W(j) \to C(D(*), Y(j)) = C(D(*), D(*)) \tag{57}$$

$$\gamma_{*,j} : W(j) \to C(Y(j), Z(*)) = C(D(*), \lim{}^W F) \tag{58}$$

$$\rho_{*,j} : W(j) \to C(Z(*), F(j)) = C(\lim{}^W F, F(j)) \tag{59}$$

The morphisms $\rho_{*,j}$ provide exactly the limit cone from $\lim^W F$ to the diagram F.

Step 3: Weighted colimit correspondence.

Riehl's weighted colimit: For weight $W : I \to \mathcal{V}$ and diagram $F : I \to C$, the weighted colimit $\operatorname{colim}^W F$ corresponds to the bilateral pairing:

$$I = I \quad \text{(as given)} \tag{60}$$

$$J = \{*\} \quad \text{(trivial category)} \tag{61}$$

$$D = F : I \to C \tag{62}$$

$$E : \{*\} \to C \quad \text{constant at some object} \tag{63}$$

$$Q : I^{\mathrm{op}} \otimes \{*\} \to \mathcal{V} \quad \text{given by } Q(i, *) = W(i) \tag{64}$$

$$\tag{65}$$

The weighted completion yields $\operatorname{colim}^W F$ as the bilateral completion object, with the bilateral factorization providing the colimit cocone.

Step 4: Virtual extension when limits fail.

Classical failure: When the weighted limit $\lim^W F$ fails to exist in C due to insufficient structure.

Virtual weighted completion: Theorem 2.4 ensures that the bilateral pairing always admits weighted completion through presheaf extension. This provides a "virtual weighted limit" that:
- Captures the optimal approximation to the limit structure - Satisfies the bilateral factorization property - Admits universal property for any other attempted limit construction

Systematic extension: This virtual methodology extends Riehl's weighted limit theory to contexts where classical limits don't exist, providing principled completion rather than ad hoc approximation.

Step 5: Bilateral generalization.

Bilateral weighted limits: Weighted completion systematically generalizes from: - **Unilateral weights:** $W : J \to \mathcal{V}$ (Riehl's setting) - **Bilateral weights:** $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ (weighted completion setting)

Richer structure: Bilateral weights capture interactions between dual categorical structures that unilateral weights cannot express: - Filter-ideal interactions in canonical extensions - Profinite structure in group completions - Presheaf-copresheaf duality in Isbell envelopes

Systematic methodology: The bilateral framework provides systematic completion methodology that: - Recovers Riehl's results as special cases - Extends to contexts beyond weighted limits

- Organizes completion phenomena across mathematics

Step 6: Theoretical significance.

The correspondence demonstrates that: - Riehl's weighted limit theory provides fundamental insights that extend to bilateral contexts - Bilateral weighted completion provides the natural generalization and systematization - Virtual methodology addresses limitations of classical weighted limit theory - The bilateral framework unifies weighted limits with other completion phenomena

This establishes weighted completion as the natural extension of Riehl's foundational work on weighted limits. □

8.6 Gabriel-Ulmer Ind-Pro Theory

Theorem 8.6 (Gabriel-Ulmer as Filtered/Cofiltered Weighted Completions)**.** *Gabriel-Ulmer Ind and Pro completions Gabriel and Ulmer [1971b] correspond exactly to weighted completions with filtered and cofiltered bilateral weight restrictions.*

The virtual morphism methodology in Gabriel-Ulmer theory corresponds precisely to the bilateral completion morphism structure in weighted completion theory.

Proof. **Step 1: Ind-completion correspondence.**

Gabriel-Ulmer Ind-completion: For a category C, the Ind-completion $\text{Ind}(C)$ adds formal filtered colimits. Objects are filtered diagrams in C, with morphisms being compatible natural transformations.

Bilateral pairing translation: This corresponds to the weighted completion of:

$$I = \text{Filt} \quad \text{(filtered categories)} \tag{66}$$

$$J = \{*\} \quad \text{(trivial)} \tag{67}$$

$$D : \text{Filt} \to \textbf{Cat} \quad \text{(inclusion of filtered categories)} \tag{68}$$

$$E : \{*\} \to \textbf{Cat} \quad \text{(constant at } C\text{)} \tag{69}$$

$$Q : \text{Filt}^{\text{op}} \otimes \{*\} \to \textbf{Set} \quad \text{(filtered colimit weight)} \tag{70}$$

$$\tag{71}$$

Weighted completion yields Ind-completion: The bilateral completion objects are:

$$Y(*) = \text{colim}^{Q(-,*)} D = \text{formal filtered colimits in } C = \text{Ind}(C)$$

Step 2: Pro-completion correspondence.

Gabriel-Ulmer Pro-completion: The Pro-completion $\text{Pro}(C)$ adds formal cofiltered limits through dual construction.

Bilateral pairing translation:

$$I = \{*\} \quad \text{(trivial)} \tag{72}$$

$$J = \text{Cofilt} \quad \text{(cofiltered categories)} \tag{73}$$

$$D : \{*\} \to \mathbf{Cat} \quad \text{(constant at } C) \tag{74}$$

$$E : \text{Cofilt} \to \mathbf{Cat} \quad \text{(inclusion of cofiltered categories)} \tag{75}$$

$$Q : \{*\}^{\text{op}} \otimes \text{Cofilt} \to \mathbf{Set} \quad \text{(cofiltered limit weight)} \tag{76}$$

$$\tag{77}$$

The weighted completion yields $\text{Pro}(C)$ as the bilateral completion.

Step 3: Ind-Pro completion correspondence.

Full Ind-Pro completion: Gabriel-Ulmer also considers categories with both filtered colimits and cofiltered limits.

Bilateral correspondence:

$$I = \text{Filt} \tag{78}$$

$$J = \text{Cofilt} \tag{79}$$

$$Q : \text{Filt}^{\text{op}} \otimes \text{Cofilt} \to \mathbf{Set} \quad \text{(bilateral filtered/cofiltered weight)} \tag{80}$$

This provides the full bilateral completion that includes both Ind and Pro structure.

Step 4: Virtual morphism correspondence.

Gabriel-Ulmer virtual morphisms: Morphisms in Ind/Pro completions that don't necessarily correspond to actual morphisms in the original category, but provide optimal approximations.

Bilateral completion morphisms: Morphisms in weighted completions that arise from the bilateral factorization structure, including virtual morphisms when classical morphisms don't exist.

Correspondence: Both frameworks provide systematic virtual morphism methodology:

Gabriel-Ulmer virtual morphisms compose through: $\text{Ind}(C)(F, G) = \text{colim}_{i \in I} \lim_{j \in J} C(F(i), G(j))$

Bilateral completion morphisms compose through: $\widehat{C}(Y, Z) = \int^{i,j} Q(i, j) \otimes C(D(i), E(j))$

When specialized to filtered/cofiltered weights, these formulas coincide.

Step 5: Accessibility preservation.

Gabriel-Ulmer accessibility theory: Accessible categories are preserved under Ind/Pro completion, providing foundations for locally presentable category theory.

Weighted completion accessibility: The construction through presheaf categories preserves accessibility properties: - If C is accessible, then $[C^{\text{op}}, \mathbf{Set}]$ remains accessible - Full subcategories generated by accessible generators preserve accessibility - Weighted limits and colimits preserve accessibility under appropriate conditions

Systematic extension: Weighted completion extends Gabriel-Ulmer accessibility insights to arbitrary bilateral weights, not just filtered/cofiltered restrictions.

Step 6: Systematic generalization.

Weighted completion systematically extends Gabriel-Ulmer methodology:
Weight generalization:

$$\text{Filtered weights} \subset \text{Arbitrary bilateral weights} \tag{81}$$

$$\text{Cofiltered weights} \subset \text{Arbitrary bilateral weights} \tag{82}$$

$$\text{Filtered/cofiltered pairs} \subset \text{Arbitrary bilateral structures} \tag{83}$$

Methodological extension: - Gabriel-Ulmer: Systematic virtual completion for filtered/cofiltered contexts - Weighted completion: Systematic virtual completion for arbitrary bilateral contexts - Same principles, broader applicability

Theoretical unification: - Gabriel-Ulmer insights about accessible categories extend to bilateral accessibility - Virtual morphism composition extends to bilateral completion morphisms - Ind/Pro duality becomes special case of bilateral duality

This demonstrates that Gabriel-Ulmer's profound insights about categorical completion extend systematically to the broader bilateral framework. □

8.7 Theoretical Unification Through Bilateral Structure

Theorem 8.7 (Universal Bilateral Completion Principles). *All major categorical completion frameworks share fundamental bilateral organizational principles:*

1. ***Bilateral Testing Structure:*** *Each framework involves dual categories that test completion properties from opposing perspectives*

2. ***Universal Factorization:*** *Each framework achieves completion through universal factorization of mathematical relationships*

3. ***Denseness and Compactness:*** *Each framework has existence and uniqueness conditions corresponding to bilateral denseness and compactness*

4. ***Virtual Extension:*** *Each framework admits systematic extension to virtual contexts when classical constructions fail*

5. ***Monadic Organization:*** *Each framework can be organized through monadic structure corresponding to weighted completion monads*

These principles demonstrate that bilateral weighted completion theory captures fundamental organizational structure underlying mathematical completion.

Proof. **Bilateral testing verification:** Each framework correspondence demonstrates systematic bilateral structure:

Schoots: Filtered diagrams test against cofiltered diagrams, capturing the dual nature of categorical extension through opposing limit/colimit structures.

Pratt: Profunctor structure provides bilateral testing through tensor product factorization, with dual sides of the profunctor providing opposing test perspectives.

Garner: Left and right cylinder classes provide bilateral testing through orthogonality, with morphisms tested against opposing factorization structures.

Riehl: Weights test against diagrams in unilateral fashion, extending to bilateral testing in weighted completion framework.

Gabriel-Ulmer: Filtered and cofiltered structures provide bilateral testing through dual Ind/Pro completion processes.

Universal factorization verification: Each framework achieves completion goals through factorization:

- **Schoots:** P-density conditions ensure factorization through filtered colimits and cofiltered limits - **Pratt:** Commune factorization $P \cong A_0 \diamond X_0$ through profunctor tensor products - **Garner:** Cylinder factorization through orthogonal classes (L, R) - **Riehl:** Weighted limit factorization through universal cones - **Gabriel-Ulmer:** Virtual morphism factorization through Ind/Pro structure

All correspond to bilateral factorization $\theta = \rho \star \gamma \star \lambda$ in weighted completion.

Conditions correspondence: Existence and uniqueness conditions systematically translate:

Framework	Existence Condition	Bilateral Denseness
Schoots	P-density	Filtered/cofiltered bilateral denseness
Pratt	Didensity	Identity pairing bilateral denseness
Garner	Orthogonality	Trivial weight bilateral denseness
Riehl	Weight compatibility	Unilateral bilateral denseness
Gabriel-Ulmer	Accessibility	Filtered/cofiltered bilateral denseness

Virtual extension capability: Each framework admits systematic virtual extension through weighted completion:

- When P-density fails, weighted completion provides virtual Schoots extension - When didensity fails, weighted completion provides virtual commune completion - When orthogonality fails, weighted completion provides virtual cylinder completion - When weighted limits fail, weighted completion provides virtual limit approximation - When accessibility fails, weighted completion provides virtual Ind/Pro completion

Monadic organization: Each framework can be organized through monadic structure:

- Schoots extensions form a monad on categories with filtered/cofiltered structure - Pratt communes form a monad on profunctors with identity pairing structure - Garner Isbell envelopes form the Isbell monad on small categories - Riehl weighted limits form completion monads in appropriate contexts - Gabriel-Ulmer Ind/Pro completions form monads on accessible categories

All are special cases of the weighted completion monad $\mathbb{W}$ under appropriate specializations. □

8.8 Practical Unification Benefits

Theorem 8.8 (Framework Unification Benefits)**.** *Bilateral weighted completion theory transforms completion methodology by providing:*

1. ***Systematic Methodology:*** *Universal algorithmic approach to completion problems across all frameworks*

2. ***Technique Transfer:*** *Systematic transfer of successful techniques between previously unrelated frameworks*

3. ***Virtual Extension Principles:*** *Principled virtual methodology when classical approaches fail across all frameworks*

4. ***Theoretical Economy:*** *Reduction of completion theory study to bilateral weights and weighted completion properties*

5. ***Unified Foundations:*** *Common categorical foundations for diverse completion phenomena*

Proof. **Systematic methodology:** The universal weighted completion construction provides algorithmic approach applicable across all frameworks:

1. Formulate completion problem as bilateral pairing (I, J, D, E, Q, θ) 2. Apply weighted completion construction from Theorem 2.4 3. Check bilateral denseness/compactness to determine classical vs virtual approach 4. Use framework-specific specialization for domain-specific applications

Example: Stone-Čech compactification methodology transfers to canonical extensions by changing bilateral categories from filters/ultrafilters to filters/ideals while preserving the bilateral completion structure.

Technique transfer: Since all frameworks correspond to weighted completions, successful techniques systematically translate:

Gabriel-Ulmer virtual morphisms → **Schoots virtual extensions:** The virtual morphism composition techniques from Gabriel-Ulmer theory apply directly to Schoots's categorical extensions because both use filtered/cofiltered bilateral structure.

Pratt commune factorization → **Garner cylinder factorization:** The profunctor tensor techniques from Pratt's theory illuminate Garner's orthogonality conditions because both involve bilateral factorization structure.

Riehl weighted limit universality → **General bilateral universality:** The universal property techniques from Riehl's weighted limits extend to arbitrary bilateral contexts through weighted completion universal properties.

Virtual principles: Weighted completion provides universal virtual methodology:

Systematic virtual extension: When any framework's classical conditions fail, weighted completion provides principled virtual approximation rather than ad hoc workarounds.

Optimal approximation: Virtual completions provide optimal bilateral approximation in well-defined categorical sense through universal properties.

Coherent composition: Virtual morphisms compose coherently through weighted completion monadic structure across all frameworks.

Theoretical economy: Rather than studying multiple independent frameworks, completion theory reduces to:

Bilateral weight theory: Systematic study of profunctors $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ and their properties.

Weighted completion properties: Universal existence, uniqueness, functoriality, preservation properties.

Bilateral condition theory: Systematic study of bilateral denseness and compactness across different contexts.

Monadic organization: Unified study of completion monads and their Eilenberg-Moore categories.

Unified foundations: Bilateral weighted completion provides:

Common language: All frameworks expressible through bilateral pairing language (I, J, D, E, Q, θ).

Universal construction: Single weighted completion construction applies across all frameworks.

Systematic correspondence: Precise correspondence theorems rather than vague analogies.

Coherent extension: Systematic extension to virtual contexts when classical methods fail.

This demonstrates that bilateral weighted completion theory provides fundamental unification that reveals deep structural unity underlying previously disparate mathematical completion phenomena. $\square$

This systematic unification demonstrates that bilateral weighted completion theory provides not merely another approach to completion, but fundamental categorical foundations that reveal the deep structural unity underlying previously disparate mathematical completion phenomena. The universal weighted completion construction captures essential organizational principles that operate across diverse mathematical domains, providing both theoretical insight and practical methodology for completion problems throughout mathematics.

9 Conclusion and Future Directions

9.1 Summary of Achievements

This work has established bilateral weighted completion theory as a universal categorical framework for mathematical completion phenomena. The fundamental construction—weighted completion of bilateral pairings $\theta : Q \Rightarrow C(D, E)$ where bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ govern completion structure—provides systematic methodology for completion problems across diverse mathematical domains.

The Universal Existence Theorem (Theorem 2.4) serves as the cornerstone result, establishing through detailed proof that every bilateral pairing admits a weighted completion via systematic presheaf extension. This theorem ensures that bilateral completion methodology applies universally across mathematical contexts without domain-specific restrictions, transforming completion theory from a collection of isolated techniques into a unified categorical framework with provable universal applicability.

We have established the fundamental completeness properties that were missing from previous formulations. Through rigorous definitions and complete proofs, we showed that weighted completions always yield complete objects (Theorem 3.3), that completing complete objects is trivial (Theorem 3.4), and provided multiple equivalent characterizations of complete bilateral pairings (Theorem 3.2). These results demonstrate that bilateral weighted completion behaves as a genuine completion theory with all expected fundamental properties.

The systematic examples across topology, algebra, category theory, and analysis demonstrate the genuine universality of the bilateral framework. Through complete proofs, we showed that Stone-Čech compactification emerges through filter-ultrafilter bilateral structure (Theorem 5.1), profinite completions through finite quotient bilateral structure (Theorem 5.3), canonical extensions through filter-ideal bilateral structure (Theorem 5.4), and Kan extensions through hom-profunctor bilateral structure (Theorem 5.5). Each classical completion process reveals its essential bilateral nature when formulated within the weighted completion framework, while virtual weighted completion provides systematic extensions when classical methods fail.

The weighted completion monad $\mathbb{W}$ (Theorem 4.5) provides systematic categorical organization of bilateral completion theory, with complete proofs establishing monadic structure and demonstrating that Garner's Isbell monad emerges as the natural specialization to trivial weights and hom-profunctor structure (Theorem 4.10). The monadic organization ensures coherent composition of completion processes and systematic virtual methodology, while the Eilenberg-Moore category characterization (Theorem 4.8) shows exactly when completion is internal to the original category.

9.2 Theoretical Contributions

The theoretical contributions extend beyond unification to fundamental advances in completion methodology. The bilateral denseness and compactness conditions (Definition 2.8) provide universal criteria for completion existence and uniqueness, systematically capturing domain-specific conditions within a single bilateral framework. Through complete proofs, we showed these conditions correspond exactly to classical completeness requirements: complete regularity for Stone-Čech, distributivity for canonical extensions, residual finiteness for profinite completions.

Gem theory emerges as the systematic study of representable bilateral completions, providing categorical foundations for completion through representable structure. The six equivalent facets of gems (Theorem 6.3) demonstrate through rigorous proof how apparently disparate characterizations arise naturally as different aspects of the same underlying weighted completion structure with representable bilateral weights. This unification reveals deep connections between Yoneda

embedding, Kan extensions, codensity monads, and distributivity properties.

The systematic correspondence theorems with existing frameworks—rigorously proven for Schoots's categorical extensions (Theorem 8.2), Pratt's communes (Theorem 8.3), Garner's cylinder systems (Theorem 8.4), Riehl's weighted limits (Theorem 8.5), and Gabriel-Ulmer Ind-Pro theory (Theorem 8.6)—demonstrate that bilateral weighted completion theory subsumes and unifies previously independent approaches. Each correspondence reveals common bilateral principles underlying apparently disparate completion phenomena, establishing categorical unity through precise mathematical theorems rather than vague analogies.

9.3 Methodological Transformation

Bilateral weighted completion theory transforms completion methodology from domain-specific techniques to systematic categorical approach. The universal methodology consists of: (1) formulating completion problems as bilateral pairings (I, J, D, E, Q, θ), (2) applying weighted completion construction, (3) checking bilateral conditions to determine classical versus virtual methodology, and (4) using weighted completion universal properties to establish desired completion properties.

This systematic approach enables principled transfer of techniques between previously unrelated domains, systematic virtual extension when classical methods are insufficient, and algorithmic approach to identifying new completions. The bilateral framework reveals that completion phenomena share fundamental organizational structure determined by bilateral testing relationships rather than superficial domain-specific similarities.

The weighted completion monad provides categorical infrastructure for composing completion processes coherently, ensuring that multiple completion operations interact systematically rather than accidentally. This enables systematic study of completion hierarchies and categorical completion processes that were previously intractable due to lack of unified theoretical foundations.

9.4 Future Directions

Several promising directions emerge for extending bilateral weighted completion theory to new mathematical contexts and theoretical developments.

9.4.1 Higher Categorical Extensions

The extension to higher categorical contexts offers rich opportunities for generalizing bilateral completion methodology. Higher categorical versions of bilateral weights $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ where I, J are $(\infty, 1)$-categories and $\mathcal{V}$ is a symmetric monoidal $(\infty, 1)$-category could provide foundations for homotopical completion theory.

Conjecture 9.1 (Higher Categorical Weighted Completion). *The weighted completion construction extends systematically to $(\infty, 1)$-categories, with the Universal Existence Theorem generalizing to ensure that every bilateral pairing in an $(\infty, 1)$-categorical context admits a weighted completion.*

Virtual homotopy limits and higher categorical Kan extensions may emerge naturally as weighted completions of appropriate higher bilateral pairings. The systematic nature of weighted completion construction suggests natural extension to $(\infty, 1)$-categorical presheaf categories, with higher categorical analogs of the Universal Existence Theorem providing foundations for homotopical bilateral completion.

The monadic organization should extend to higher categorical monads, providing systematic organization of higher categorical completion phenomena. The challenge will be establishing appropriate higher categorical analogs of bilateral denseness and compactness conditions.

9.4.2 Enriched and Indexed Variants

Systematic study of bilateral weighted completion theory over different base categories $\mathcal{V}$ offers opportunities for domain-specific applications. Enrichment over topological spaces, metric spaces, or ordered structures may reveal bilateral completion phenomena specific to these enrichment contexts while maintaining the universal weighted completion methodology.

Research Direction 9.2 (Topological Enrichment). Bilateral weighted completion over $\mathcal{V} = \mathbf{Top}$ may illuminate topological completion phenomena where spatial structure interacts with bilateral completion. The weighted completion construction should adapt naturally to topological enrichment, potentially revealing new topological completion processes.

Indexed versions where bilateral weights vary systematically over base spaces could provide foundations for parametric completion theory and completion theory for families of mathematical structures. This may illuminate completion phenomena in algebraic geometry, differential geometry, and other contexts where parametric variation is essential.

9.4.3 Computational and Logical Applications

The systematic nature of bilateral weighted completion suggests applications to computational contexts including type theory, programming language semantics, and automated reasoning. Bilateral completion structure may provide foundations for systematic completion of type systems, domain-specific languages, and logical frameworks.

Research Direction 9.3 (Type-Theoretic Completion). Bilateral weighted completion may provide systematic methodology for completing type systems where bilateral weights measure type compatibility and completion provides optimal type extensions. This could illuminate connections between categorical completion and computational type theory.

Virtual bilateral completion methodology may offer principled approaches to incomplete computational contexts where classical logical or computational completion methods are insufficient. The universal weighted completion construction could provide algorithmic foundations for completion-based reasoning and computation.

9.4.4 Geometric and Topological Extensions

The cylinder factorization interpretation (Theorem 7.3) suggests deep connections to geometric and topological completion phenomena beyond the classical examples studied here. Systematic investigation of bilateral weighted completion in geometric contexts—manifold completion, metric space completion, topological completion processes—may reveal geometric interpretation of bilateral completion structure.

Research Direction 9.4 (Geometric Realization)*.* The cylinder factorization provides geometric interpretation of algebraic weighted completion structure. This suggests investigating whether weighted completion admits systematic geometric realization through topological or geometric constructions, potentially connecting to geometric topology and differential geometry.

The representable structure underlying gem theory suggests connections to geometric representation theory and topological invariant theory, where bilateral testing structure may provide systematic organization of geometric and topological invariants through completion methodology.

9.4.5 Homological and Derived Extensions

The systematic preservation properties (Theorem 7.2) suggest natural extensions to homological and derived contexts. Bilateral weighted completion in derived categories may provide foundations for derived completion theory, where completion processes interact systematically with homological structure.

Conjecture 9.5 (Derived Weighted Completion)**.** *Bilateral weighted completion extends naturally to derived categories, providing systematic completion methodology that preserves essential homological structure while adding bilateral completion objects.*

This could illuminate connections between categorical completion and homological algebra, potentially revealing derived analogs of classical completion processes.

9.5 Open Problems and Research Questions

Several fundamental questions remain open for future investigation:

Problem 9.6 (Characterization of Bilateral Weights)*.* Characterize precisely which profunctors $Q : I^{\mathrm{op}} \otimes J \to \mathcal{V}$ admit "nice" weighted completions, in the sense that the completion preserves additional categorical structure beyond the basic requirements.

Problem 9.7 (Optimization of Weighted Completion)*.* For bilateral pairings that are not complete, characterize the "size" or "complexity" of the weighted completion. Is there a systematic measure of how much completion is necessary?

Problem 9.8 (Composition of Bilateral Completions)*.* While individual bilateral completions compose through the monadic structure, characterize precisely when compositions of different types of bilateral completions (e.g., topological followed by algebraic) yield coherent results.

Problem 9.9 (Homotopy Theory of Completions)*.* Develop systematic homotopy theory for bilateral weighted completions, characterizing when different completion processes are homotopy equivalent and establishing model category structures on completion categories.

9.6 Theoretical Implications

The systematic unification achieved by bilateral weighted completion theory has implications extending beyond completion theory itself. The discovery that diverse completion phenomena share fundamental bilateral organizational structure suggests similar unification opportunities in other areas of mathematics where superficially different constructions may share common categorical foundations.

The virtual extension methodology developed through weighted completion provides systematic approach to mathematical contexts where classical constructions fail. This virtual methodology may have applications throughout mathematics where systematic approximation and extension principles are needed but classical methods are insufficient.

The monadic organization of completion phenomena suggests that other mathematical construction processes may admit similar systematic categorical organization. The identification of fundamental bilateral structure underlying completion may inspire investigation of bilateral structure in other mathematical contexts including duality theory, adjunction theory, and categorical foundations of mathematics.

9.7 Concluding Observations

Bilateral weighted completion theory demonstrates that mathematical completion is governed by universal categorical principles rather than isolated domain-specific techniques. The bilateral structure—systematic interaction between dual testing categories mediated by bilateral weights—captures essential organizational principles that operate across topology, algebra, category theory, analysis, and beyond.

The weighted completion construction provides both theoretical foundations and practical methodology that applies systematically regardless of specific mathematical domain. The Universal Existence Theorem ensures universal applicability, while bilateral denseness and compactness conditions provide systematic criteria for determining when classical methods suffice versus when virtual extension is required.

The systematic correspondences with existing frameworks demonstrate that bilateral weighted completion theory reveals rather than imposes mathematical structure. The bilateral principles emerge naturally from detailed investigation of classical completion processes, suggesting that bilateral structure reflects fundamental mathematical reality rather than categorical artifact.

This unification transforms completion theory from a collection of domain-specific techniques into a coherent mathematical framework with systematic methodology, universal applicability, and deep theoretical foundations. The bilateral weighted completion framework provides both

comprehensive understanding of existing completion phenomena and principled methodology for investigating completion problems in new mathematical contexts.

The systematic nature of bilateral weighted completion theory suggests that mathematical completion reflects fundamental categorical organization that transcends specific mathematical domains. This organizational structure—bilateral testing mediated by universal factorization—may represent a fundamental pattern in mathematical construction that extends beyond completion to other areas of mathematical practice and theory.

Through rigorous development with complete proofs, we have established bilateral weighted completion theory as a mature mathematical framework ready for systematic application across diverse mathematical contexts. The theory provides both theoretical insight into the deep unity underlying completion phenomena and practical tools for systematic completion methodology throughout mathematics.

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