Memorial University of Newfoundland
Department of Physics and Physical Oceanography
Physics 3900 Laboratory

The Period Doubling Route to Chaos

Objective

To investigate the period doubling route to chaos by numerical investigation of the logistic map and by observation of period doubling in the response, to driving voltage amplitude, of a diode-inductance circuit.

Introduction

Systems with dynamics described by nonlinear equations of motion can display chaotic behaviour despite that behavior being completely determined by the parameters of the motion. Chaotic behaviour is characterized by:

(1) Long term unpredictability. The equations of motion describing a chaotic system are completely deterministic and the observed behaviour is not random. The inability to predict the future state of a chaotic system by extrapolating from its past behaviour reflects the complicated nature of the motion.

(2) Extreme sensitivity to initial conditions. In the chaotic regime, two trajectories that start with initial conditions that differ by arbitrarily small amounts will diverge exponentially with time (or iteration number for the logistic map example to be described below). Consider a situation in which the variables describing two possible states of a dynamic system at time t are $x_1(t)$ and $x_2(t)$ and where the initial states of these trajectories, $x_1(0)$ and $x_2(0)$ respectively, are arbitrarily close. The difference between these trajectories will evolve in time according to

$$|x_1(t) - x_2(t)| \propto \exp \alpha t$$

where α is the Lyapunov exponent. The transition from "normal" dynamical behaviour to chaotic behaviour corresponds to a change in the Lyapunov exponent from negative to positive.

Chaotic behaviour has been studied experimentally and theoretically. It has also been the subject of many popular treatments in part because of the images that can be obtained by graphing variables generated by chaotic systems. You should take advantage of some of the many sources of information about chaotic behaviour including some of the references listed at the end of this lab description.

This lab consists of numerical and experimental components. In the numerical part, you will use a spreadsheet (or your own code in a suitable programming environment) to investigate the chaotic behaviour obtained by iterating a nonlinear system known as the logistic map. In particular, you will investigate how the results of iterating the logistic map evolve from periodic to chaotic behaviour through a series of period doubling transitions as a control parameter is varied. In the experimental part of the lab, you will study the response of a circuit, containing a diode and an inductor, as the amplitude of a driving oscillator signal is varied. You will find that the voltage across the diode also evolves from periodic to chaotic behaviour through a series of period doubling transitions as the amplitude of the driving voltage is changed.

The logistic map and the period-doubling route to chaos

The iterative function

$$x_{n+1} = 4\lambda x_n \left(1 - x_n\right),$$

where the control parameter λ is constrained by $0 \le \lambda \le 1$ and variable x is restricted to the range $0 \le x \le 1$ is known as the logistic map. Starting from an arbitrary initial value of x, and allowing a sufficiently large number of iterations to occur first, may result in the system settling into either periodic or chaotic behaviour. Depending on the value of λ, successive iterations, beyond the initial transient behaviour, can return a fixed value of x, known as a *fixed point* or an *attractor*, a recurring sequence of x values forming a *limit cycle*, or a sequence of x values that never recurs. The latter situation is an example of chaotic behaviour. The number of initial iterations that must be completed before the system settles into an attractor can range from 10s to 100s depending on the proximity of the control parameter λ to its value, λ_C, at which chaotic behaviour sets in.

The logistic map is interesting because it provides an example of the *period-doubling route to chaos*. You can see that $x = 0$ is always a fixed point of the logistic map but we will be interested in its behaviour for non-zero initial values of x. For $\lambda < 0.25$, iteration of the logistic map from a non-zero initial values of x still settles at $x = 0$. For $0.25 < \lambda < 0.75$, iteration gives a single fixed value. In the range $0.75 < \lambda < 0.86$, where the upper limit is approximate here, iterations alternate between 2 values of x once the initial transient is over. In effect, the period of the output has *doubled* from one iteration to two iterations. As λ is increased beyond $\lambda \approx 0.86$, the period doubles again and again, for successively smaller steps in λ, until the period is effectively infinite and the behaviour is chaotic. If λ_i is the value of the control parameter at the onset of period-2^i behaviour, then a constant known as *Feigenbaum's constant* is defined as

$$q = \frac{\lambda_i - \lambda_{i-1}}{\lambda_{i+1} - \lambda_i}.$$

Theoretically, Feigenbaum's constant is found to approach $\delta = 4.6692...$ and is predicted to be universal in the sense that any nonlinear phenomenon approaching chaos by period doubling should display the same value of δ.
Figure 1 shows the result of iterating the logistic map for $\lambda = 0.87$ and an initial $x_0 = 0.2$. Values of x_n for iterations $897 \le n < 1007$ are plotted with connecting lines to make the periodicity more apparent. For this value of λ, the period is 4 iterations.

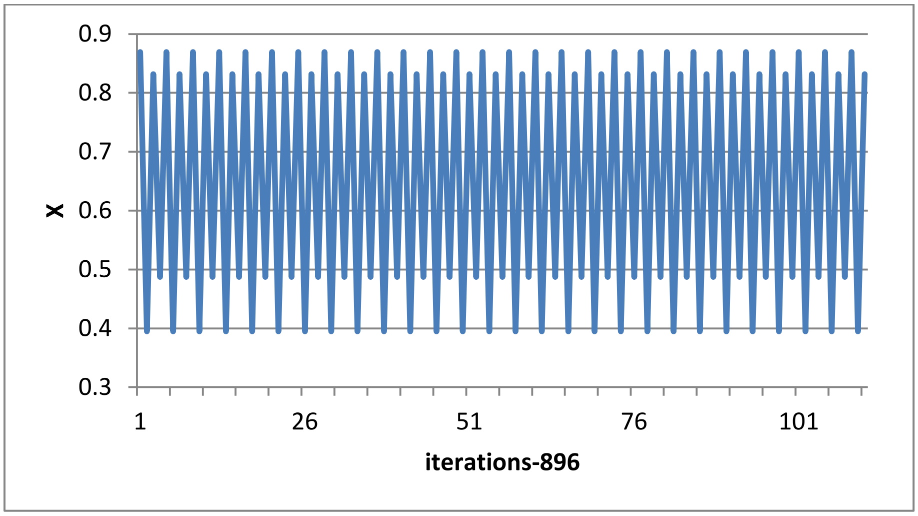

Figure 1

Values of x_n for iterations $897 \le n < 1007$ for the logistic map with $\lambda = 0.87$ and an initial $x_0 = 0.2$. Values are plotted with connecting lines to make the periodicity more apparent

The logistic map is often represented by plotting of all of the values of x_n obtained, after the initial transient, for a given value of λ. Care must be taken to avoid using data from iterations that fall within the initial transient. It is also necessary to include sufficient iterations to ensure that at least one full period is obtained, for values of λ having an attractor, of that enough iterations are obtained to characterize chaotic behaviour for higher values of λ. Figure 2 shows a representation of the logistic map for $0.7 < \lambda < 0.925$. For $0.88 < \lambda$, the results of 64 iterations, starting at iteration 899, are plotted.

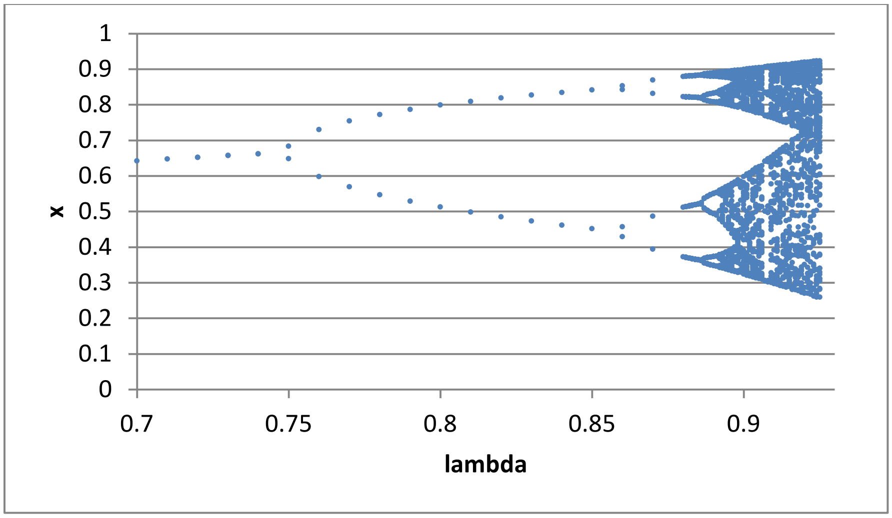

Figure 2
A representation of the logistic map for $0.7 < \lambda < 0.925$.

Procedure

1. Numerical study of the logistic map and the period-doubling route to chaos

For this part of the lab, you can use a commercial spreadsheet program (Excel or the Open Office analog) or you can write your own code in a programming environment of your choice. The suggested approach here assumes the use of a spreadsheet. The objective is to explore the period-doubling route to chaos, as exemplified by the logicstic map, and obtain estimates of the Feigenbaum constant and the dependence of the Lyapunov exponent on λ for this system.

The first step is to set up a spreadsheet to do about 1024 iterations of the logistic map by defining a constant (λ), an initial value for x_0, and then the formula for x_1 in a cell. Suitably copying that cell (being careful to ensure the preservation of the constant value) will give you a simple way to do about 1024 iterations of the logistic map equation. You will probably want to use the resulting data in different ways so you should save the spreadsheet, and 2 copies with alternate names, in a suitable folder.

With the first version of your spreadsheet, some range of iterations and plot them, as shown in Figure 1, for a few values of λ. Note how the behaviour changes from small iteration numbers to larger iteration numbers where the transient behaviour has died out. For the rest of the steps, it is suggested that you plot the last 128 iterations to avoid, as much as possible, interference form the transient behaviour. You may wish to record some of the more interesting patterns.

Vary λ over the range $0 < \lambda < 1$, with a few different initial values for x_0, and note how the periodicity of the iteration changes. Try to get a rough idea of the values of λ at which period doubling occurs. You can also try to find the critical value, λ_C, at which the period becomes infinite (very large). This corresponds to the onset of chaotic behaviour.

You should next try to obtain estimates of the Feigenbaum constant. Trying to identify the onset of period doubling from plots of x versus iteration number is difficult so it is recommended that you use one of your duplicate spreadsheets for the following. On your spreadsheet, create columns containing $x_{i+1} - x_i$, $x_{i+2} - x_i$, $x_{i+4} - x_i$, $x_{i+8} - x_i$, $x_{i+16} - x_i$, and $x_{i+32} - x_i$. Generate a plot of the last 128 points in each of these columns. Observe what happens for values of λ that give different periodicities and use your findings to determine the first 5-6 values of λ_i (along with estimates of uncertainty). This will allow you to make a few estimates of the Feigenbaum constant.

You can use your third copy of the initial spreadsheet to investigate the behaviour of the Lyapunov constant. It is suggested that you set up two columns of iterations in your spreadsheet using initial values that differ by a small amount (i.e. $\Delta x_0 \approx 10^{-6}$). Set up a third column to calculate iteration-by-iteration differences and plot those differences. Because x is bounded, those differences will not diverge but you should be able to draw some inferences about the Lyapunov exponent from trends in the dependence of Δx on iteration number.

2. Period doubling in the diode-inductance circuit

A pn junction diode is a semiconducting device formed by a semiconductor (like silicon) in which one end (the n-type end) contains electron-donating impurities (like phosphorus) and the other end (the p-type end) contains electron accepting impurities (like boron). Such a device conducts current when the p-type end of the junction is biased positively with respect to the n-type end (forward-bias) but not when biased in the other direction (reverse-bias). If you are not already familiar with diodes, you should review the properties of pn junction diodes in a standard electronics text. Figure 3 shows the symbol for a pn junction diode and the behaviour of the current-voltage characteristic for forward-bias and reverse-bias conditions.

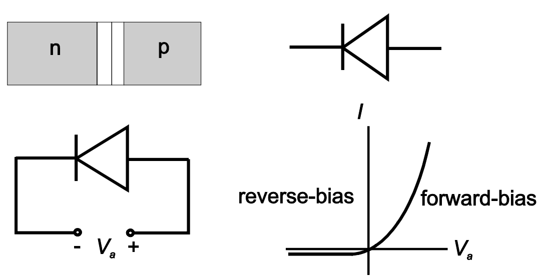

Figure 3

Representations of a pn junction diode and a schematic IV characteristic.

When forward-biased, voltage across the diode is limited to a small voltage drop (about 0.6 V for the 1N400X diodes we will use). When reverse-biased, the diode looks like a small capacitance (about 15 pF for the 1N4001 and 1N4002 diodes and about 8 pF for the 1N4004 and 1N4007 diodes). The behaviour we will study here is likely a consequence of the non-zero time required for switching of the diode form capacitive to conducting (see reference 2).

Figure 4 shows the series resistor-inductance-diode circuit that will be studied in this experiment. The input voltage is supplied by a Stanford Research Systems DS345 Frequency Synthesizer and the Digital Oscilloscope is a TDS 410 (with Fourier transform capabilities). The resistor is 100 Ω and the variable inductor can be started at 30 mH.

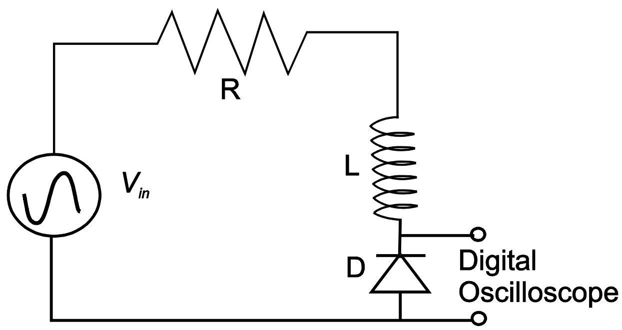

Figure 4
The diode-inductance circuit.

A reverse-biased diode has a small capacitance *C*. As a result, the diode-inductance does have a resonant frequency but because the diode is conducting for part of each cycle, the resonant frequency is higher than the frequency, $\omega_0 = (LC)^{-1/2}$, that would be obtained using just the reverse-biased junction capacitance. Using a small input voltage (say 0.2 V peak-to-peak), determine the resonant frequency for the circuit with a 1N4002 diode. By observing the voltage across the diode using the digital oscilloscope, you should find a frequency between 60 kHz and 80 kHz. For the following part of the experiment, set the oscillator frequency to something in the range 90-95% of the resonant frequency.

You can now observe the behaviour of the voltage across the diode as a function of the input voltage. You should set up the oscilloscope so that one trace (channel 1) is displaying the voltage across the diode. You should set up the other channel to display the Fourier transform of the input signal. You can do this by selecting “More” for the second channel. With the “More” channel activated, you can select “Math 2, FFT (Chan 1)”. You can also select a windowing function (start with “Hanning”) and then select “OK

Create Math WFM". You should think about the purpose of the windowing function. Once you are comfortable with the oscilloscope, you can try other windowing functions. It is important to note that the horizontal and vertical range adjustments on the oscilloscope affect whichever channel is currently activated as indicated by the green light beside either "Channel 1" or "More". A good starting point for the channel 1 settings is a vertical range setting of 2-5 V per division and a horizontal range setting of 50 μs. For the Math WFM settings, active when "More" is illuminated, a good starting point is 20 dB for the vertical and 10 kHz for the horizontal scales.

Now you can observe the voltage across the diode as a function of the input voltage. For low input voltages, the peak voltage across the diode will be the same for each oscillation and the Fourier transform will show peaks at multiples of the driving frequency. At the first period doubling transition, the amplitude of the voltage across the diode will begin to vary between alternate cycles and peaks will appear in the Fourier transform at odd multiples of half the driving frequency (corresponding to twice the original period). You should be able to observe at least three period-doubling transitions before the voltage across the diode begins to behave chaotically. You may also see other periodicities at higher driving amplitudes. Describe your observations.

Starting again from a small driving voltage, you should obtain a quantitative record the voltages across the diode for increasing drive voltage amplitudes and plot these as a function of the input voltage. You may find it useful to use the "Run/Stop" button and the horizontal cursor function for these measurements. You should also determine, as carefully as possible, the values of the drive voltage corresponding to each period-doubling transition and use these to obtain an estimate of the Feigenbaum constant.

You can then try varying some of the parameters of the experiment (diode, offset from resonant frequency, series resistance, etc.) to see how they affect the observation of period-doubling. You may also find it interesting to experiment with different oscilloscope settings to see how they affect your ability to detect period doubling. Using an analog oscilloscope in *x-y* mode, you might also find it interesting to observe period doubling while using the input voltage to drive the *x* axis and the diode voltage to drive the *y* axis.

Suggested Reading:

(1) For information about pn-junction diodes, refer to an electronics text or a semiconductor device text such as D. Neamen, *An introduction to semiconductor devices*, (McGraw Hill, New York, 2006).

(2) K. Briggs, "Simple experiments in chaotic dynamics", *Am. J. Phys.* **55**, 1083 (1987).

(3) C. Cooper, "Experimental observation and characterization of chaotic properties in a driven infrared diode circuit", *Chaos Solitons & Fractals*. **24**, 157 (2005).

(4) J. Gleik, *Chaos-Making a New Science*, (Viking, New York, 1987).