DS6040
M3 HW

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Problem 5: Posterior for θ when Var$(y \mid \theta) = \sigma^2 \neq 1$

We assume the same model as in Problem 4, but now the variance of the data is $\sigma^2 \neq 1$, where σ^2 is known.

Model

- $y_i \mid \theta \sim \text{Normal}(\theta, \sigma^2)$
- $\theta \sim \text{Normal}(a, b)$

Posterior Derivation

Likelihood:

$$L(y \mid \theta) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_i (y_i - \theta)^2\right]$$

From Problem 4 (2) we can define:

$$L(y \mid \theta) \propto \exp\left[-\frac{n}{2\sigma^2}(\bar{x} - \theta)^2\right]$$

Prior:

$$\pi(\theta) \propto \exp\left[-\frac{1}{2b}(\theta - a)^2\right]$$

Posterior:

$$\pi(\theta \mid y) \propto L(y \mid \theta) \cdot \pi(\theta)$$

$$\propto \exp\left[-\frac{n}{2\sigma^2}(\theta - \bar{x})^2 - \frac{1}{2b}(\theta - a)^2\right]$$

This is again a Normal distribution, so the posterior is:

$$\theta \mid y_1, \ldots, y_n \sim \text{Normal}\left(\bar{x}\left(\frac{n/\sigma^2}{n/\sigma^2 + 1/b}\right) + a\left(\frac{1/b}{n/\sigma^2 + 1/b}\right),\ \frac{1}{n/\sigma^2 + 1/b}\right)$$