DS6040

M3 HW

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May 26, 2025

Problem 2: Posterior Derivation with Nonzero Mean μ

We are given:

- $y_i \mid \theta \sim \text{Normal}(\mu, \theta)$, with known $\mu \neq 0$
- Prior: $\theta \sim \text{Inverse-Gamma}(a, b)$

Likelihood

The likelihood of $y_1, \ldots, y_n$ given θ is:

$$L(y \mid \theta) \propto \theta^{-n/2} \exp\left(-\frac{1}{2\theta}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

Define:

$$s = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu)^2 \quad \text{(sample variance relative to known mean } \mu\text{)}$$

Multiply both sides by n, then:

$$\sum_{i=1}^{n}(y_i - \mu)^2 = ns$$

Prior

Given in Problem 1 (2), the prior is:

$$\pi(\theta) \propto \theta^{-(a+1)} \exp\left(-\frac{b}{\theta}\right)$$

Posterior

By Bayes' Rule, the posterior is proportional to the product of the likelihood and the prior and we replace our prior and likelihood:

$$\begin{aligned}
\pi(\theta \mid y) &\propto L(y \mid \theta) \cdot \pi(\theta) \\
&\propto \theta^{-n/2} \exp\left(-\frac{ns}{2\theta}\right) \cdot \theta^{-(a+1)} \exp\left(-\frac{b}{\theta}\right) \\
&= \theta^{-(a+n/2+1)} \exp\left(-\frac{b + ns/2}{\theta}\right)
\end{aligned}$$

Conclusion

The posterior distribution is still Inverse-Gamma:

$$\theta \mid y \sim \text{Inverse-Gamma}\left(a + \frac{n}{2},\ b + \frac{ns}{2}\right)$$

The only change from the zero-mean case is that the sufficient statistic in the likelihood becomes $\sum(y_i - \mu)^2$ instead of $\sum y_i^2$, which adjusts the scale parameter in the posterior.