

# Sentential Natural Deduction: Additional Exercises B

Robert Craven

## Exercise B.1

With premises  $\neg p$  and  $\neg\neg q \rightarrow \neg\neg p$ , give as many kinds of derivation possible of 6 lines or fewer. (6 lines, including the premises. So apply rules in as many different ways as you can, in the space of 4 extra lines.)

*Solution:*

Here, for a start, are a selection of those for just 4 lines or fewer, and which just use (R), (DN1) and (CON). (This isn't all of the ones possible for *those* limitations.) If we used 6 lines, and all of the rules, there would be infinitely many more.

|  |       |
|--|-------|
| 1. $\neg p$                            | (P1)  |
| 2. $\neg\neg q \rightarrow \neg\neg p$ | (P2)  |
| 3. $\neg p$                            | 1,(R) |

|  |       |
|--|-------|
| 1. $\neg p$                            | (P1)  |
| 2. $\neg\neg q \rightarrow \neg\neg p$ | (P2)  |
| 3. $\neg\neg q \rightarrow \neg\neg p$ | 2,(R) |

|  |         |
|--|---------|
| 1. $\neg p$                            | (P1)    |
| 2. $\neg\neg q \rightarrow \neg\neg p$ | (P2)    |
| 3. $\neg\neg\neg p$                    | 1,(DN1) |

|  |         |
|--|---------|
| 1. $\neg p$                            | (P1)    |
| 2. $\neg\neg q \rightarrow \neg\neg p$ | (P2)    |
| 3. $\neg\neg\neg p$                    | 1,(DN1) |
| 4. $\neg\neg\neg\neg p$                | 3,(DN1) |

|  |         |
|--|---------|
| 1. $\neg p$                                      | (P1)    |
| 2. $\neg\neg q \rightarrow \neg\neg p$           | (P2)    |
| 3. $\neg\neg(\neg\neg q \rightarrow \neg\neg p)$ | 2,(DN1) |

|  |         |
|--|---------|
| 1. $\neg p$  | (P1)    |
| 2. $\neg\neg q \rightarrow \neg\neg p$                   | (P2)    |
| 3. $\neg\neg(\neg\neg q \rightarrow \neg\neg p)$         | 2,(DN1) |
| 4. $\neg\neg\neg\neg(\neg\neg q \rightarrow \neg\neg p)$ | 3,(DN1) |

|  |           |
|--|-----------|
| 1. $\neg p$                            | (P1)      |
| 2. $\neg\neg q \rightarrow \neg\neg p$ | (P2)      |
| 3. $\neg p \wedge \neg p$              | 1,1,(CON) |

|   |           |
|---|-----------|
| 1. $\neg p$                               | (P1)      |
| 2. $\neg\neg q \rightarrow \neg\neg p$    | (P2)      |
| 3. $\neg p \wedge \neg p$                 | 1,1,(CON) |
| 4. $\neg p \wedge (\neg p \wedge \neg p)$ | 1,3,(CON) |

|    |  |           |
|----|--|-----------|
| 1. | $\neg p$                               | (P1)      |
| 2. | $\neg \neg q \rightarrow \neg \neg p$  | (P2)      |
| 3. | $\neg p \wedge \neg p$                 | 1,1,(CON) |
| 4. | $(\neg p \wedge \neg p) \wedge \neg p$ | 3,1,(CON) |

  

|    |  |           |
|----|--|-----------|
| 1. | $\neg p$   | (P1)      |
| 2. | $\neg \neg q \rightarrow \neg \neg p$                  | (P2)      |
| 3. | $\neg p \wedge \neg p$                                 | 1,1,(CON) |
| 4. | $(\neg p \wedge \neg p) \wedge (\neg p \wedge \neg p)$ | 3,3,(CON) |

  

|    |  |           |
|----|--|-----------|
| 1. | $\neg p$   | (P1)      |
| 2. | $\neg \neg q \rightarrow \neg \neg p$  | (P2)      |
| 3. | $(\neg \neg q \rightarrow \neg \neg p) \wedge (\neg \neg q \rightarrow \neg \neg p)$ | 2,2,(CON) |

  

|    |                                       |           |
|----|---------------------------------------|-----------|
| 1. | $\neg p$                              | (P1)      |
| 2. | $\neg \neg q \rightarrow \neg \neg p$ | (P2)      |
| 3. | $\neg \neg p$                         | 1,(DN1)   |
| 4. | $\neg p \wedge \neg \neg p$           | 1,3,(CON) |

  

|    |                                       |           |
|----|---------------------------------------|-----------|
| 1. | $\neg p$                              | (P1)      |
| 2. | $\neg \neg q \rightarrow \neg \neg p$ | (P2)      |
| 3. | $\neg \neg p$                         | 1,(DN1)   |
| 4. | $\neg \neg p \wedge \neg p$           | 3,1,(CON) |

  

|    |                                       |           |
|----|---------------------------------------|-----------|
| 1. | $\neg p$                              | (P1)      |
| 2. | $\neg \neg q \rightarrow \neg \neg p$ | (P2)      |
| 3. | $\neg \neg p$                         | 1,(DN1)   |
| 4. | $\neg \neg p \wedge \neg \neg p$      | 3,3,(CON) |

  

|    |  |           |
|----|--|-----------|
| 1. | $\neg p$   | (P1)      |
| 2. | $\neg \neg q \rightarrow \neg \neg p$                      | (P2)      |
| 3. | $\neg \neg p$  | 1,(DN1)   |
| 4. | $\neg \neg p \wedge (\neg \neg q \rightarrow \neg \neg p)$ | 3,2,(CON) |

  

|    |  |           |
|----|--|-----------|
| 1. | $\neg p$   | (P1)      |
| 2. | $\neg \neg q \rightarrow \neg \neg p$                      | (P2)      |
| 3. | $\neg \neg p$  | 1,(DN1)   |
| 4. | $\neg \neg p \wedge (\neg \neg q \rightarrow \neg \neg p)$ | 3,2,(CON) |

## Exercise B.2

Show the following.

1.  $p \rightarrow q \vdash \neg p \vee q$ . (Use any of the derived rules on pp. 18–19 of the notes, as well as any non-derived rules.)

*Solution:*

Here's the short solution which a number of people found in the tutorial, using (CD) but no derived rules:

|    |                           |            |
|----|---------------------------|------------|
| 1. | $p \rightarrow q$         | (P1)       |
| 2. | $\parallel p$             | (Ass,CD 1) |
| 3. | $\parallel q$             | 2,1,(MP)   |
| 4. | $\parallel \neg p \vee q$ | 3,(ADD2)   |
| 5. | $\parallel \neg p$        | (Ass,CD 2) |
| 6. | $\parallel \neg p \vee q$ | 5,(ADD1)   |
| 7. | $\neg p \vee q$           | 2–6,(CD)   |

Here is the longer solution I had, which uses the derived rule (DM2):

|    |                                      |           |
|----|--------------------------------------|-----------|
| 1. | $p \rightarrow q$                    | (P1)      |
| 2. | $\parallel \neg(\neg p \vee q)$      | (Ass,IP)  |
| 3. | $\parallel \neg\neg p \wedge \neg q$ | 2,(DM2)   |
| 4. | $\parallel \neg\neg p$               | 3,(SIMP1) |
| 5. | $\parallel p$                        | 4,(DN2)   |
| 6. | $\parallel q$                        | 5,1,(MP)  |
| 7. | $\parallel \neg q$                   | 3,(SIMP2) |
| 8. | $\parallel q \wedge \neg q$          | 6,7,(CON) |
| 9. | $\neg p \vee q$                      | 2-8,(IP)  |

2.  $\neg(p \rightarrow q) \vdash p$ . (Just use the regular, non-derived rules.)

*Solution:*

|    |  |           |
|----|--|-----------|
| 1. | $\neg(p \rightarrow q)$                                    | (P1)      |
| 2. | $\parallel \neg p$   | (Ass,IP)  |
| 3. | $\parallel \parallel p$                                    | (Ass,CP)  |
| 4. | $\parallel \parallel q$                                    | 3,2,(ECQ) |
| 5. | $\parallel p \rightarrow q$                                | 3-4,(CP)  |
| 6. | $\parallel (p \rightarrow q) \wedge \neg(p \rightarrow q)$ | 5,1,(CON) |
| 7. | $p$  | 2-6,(IP)  |

3. Show  $\neg\neg p \vdash p$  **without** using (DN2), (ECQ), or (IP).

*Solution:*

|    |                           |            |
|----|---------------------------|------------|
| 1. | $\neg\neg p$              | (P1)       |
| 2. | $\parallel p$             | (Ass,CD 1) |
| 3. | $\parallel p \vee \neg p$ | 2,(ADD1)   |
| 4. | $\parallel \neg p$        | (Ass,CD 2) |
| 5. | $\parallel p \vee \neg p$ | 4,(ADD2)   |
| 6. | $p \vee \neg p$           | 2-5,(CD)   |
| 7. | $p$                       | 6,1,(DS2)  |

### Exercise B.3

Show that the following derivable rules, from pp. 18–19 of the notes, obtain. (Once you’ve derived a rule, you can use it for the proofs in later parts of this exercise.)

1. (IDEMP2- $\vee$ )  $A \vee A \vdash A$ .

*Solution:*

|    |                          |           |
|----|--------------------------|-----------|
| 1. | $A \vee A$               | (P1)      |
| 2. | $\parallel A$            | (Ass,CP)  |
| 3. | $\parallel A$            | 2,(R)     |
| 4. | $A \rightarrow A$        | 2-3,(CP)  |
| 5. | $A \vee A \rightarrow A$ | 4,4,(DIS) |
| 6. | $A$                      | 1,5,(MP)  |

2. (DM1)  $\neg(A \wedge B) \vdash \neg A \vee \neg B$ .

*Solution:*

|     |  |            |
|-----|--|------------|
| 1.  | $\neg(A \wedge B)$   | (P1)       |
| 2.  | $\parallel \neg(\neg A \vee \neg B)$                                       | (Ass,IP)   |
| 3.  | $\parallel \parallel \neg A$   | (Ass,IP)   |
| 4.  | $\parallel \parallel \neg A \vee \neg B$                                   | 3,(ADD1)   |
| 5.  | $\parallel \parallel (\neg A \vee \neg B) \wedge \neg(\neg A \vee \neg B)$ | 4,2,(CON)  |
| 6.  | $\parallel A$  | 3-5,(IP)   |
| 7.  | $\parallel \parallel \neg B$   | (Ass,IP)   |
| 8.  | $\parallel \parallel \neg A \vee \neg B$                                   | 7,(ADD2)   |
| 9.  | $\parallel \parallel (\neg A \vee \neg B) \wedge \neg(\neg A \vee \neg B)$ | 8,2,(CON)  |
| 10. | $\parallel B$  | 7-9,(IP)   |
| 11. | $\parallel A \wedge B$   | 6,10,(CON) |
| 12. | $\parallel (A \wedge B) \wedge \neg(A \wedge B)$                           | 11,1,(CON) |
| 13. | $\neg A \vee \neg B$   | 2-12,(IP)  |

3. (DM2)  $\neg(A \vee B) \vdash \neg A \wedge \neg B$ .

*Solution:*

|     |  |            |
|-----|--|------------|
| 1.  | $\neg(A \vee B)$                             | (P1)       |
| 2.  | $\parallel \neg \neg A$                      | (Ass,IP)   |
| 3.  | $\parallel A$                                | 2,(DN2)    |
| 4.  | $\parallel A \vee B$                         | 3,(ADD1)   |
| 5.  | $\parallel (A \vee B) \wedge \neg(A \vee B)$ | 4,1,(CON)  |
| 6.  | $\neg A$                                     | 2-5,(IP)   |
| 7.  | $\parallel \neg \neg B$                      | (Ass,IP)   |
| 8.  | $\parallel B$                                | 7,(DN2)    |
| 9.  | $\parallel A \vee B$                         | 8,(ADD2)   |
| 10. | $\parallel (A \vee B) \wedge \neg(A \vee B)$ | 9,1,(CON)  |
| 11. | $\neg B$                                     | 7-10,(IP)  |
| 12. | $\neg A \wedge \neg B$                       | 6,11,(CON) |

4. (DIST1)  $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$ .

*Solution:*

|     |  |             |
|-----|--|-------------|
| 1.  | $A \wedge (B \vee C)$  | (P1)        |
| 2.  | $A$  | 1,(SIMP1)   |
| 3.  | $B \vee C$   | 1,(SIMP2)   |
| 4.  | $\parallel \neg((A \wedge B) \vee (A \wedge C))$               | (Ass,IP)    |
| 5.  | $\parallel \parallel \neg(A \wedge B) \wedge \neg(A \wedge C)$ | 4,(DM2)     |
| 6.  | $\parallel \parallel \neg(A \wedge B)$                         | 5,(SIMP1)   |
| 7.  | $\parallel \parallel \neg A \vee \neg B$                       | 6,(DM1)     |
| 8.  | $\parallel \parallel \neg \neg A$                              | 2,(DN1)     |
| 9.  | $\parallel \parallel \neg B$                                   | 7,8,(DS1)   |
| 10. | $\parallel C$  | 3,9,(DS1)   |
| 11. | $\parallel \parallel \neg(A \wedge C)$                         | 5,(SIMP2)   |
| 12. | $\parallel \parallel \neg A \vee \neg C$                       | 11,(DM1)    |
| 13. | $\parallel \parallel \neg \neg A$                              | 2,(DN1)     |
| 14. | $\parallel \parallel \neg C$                                   | 12,13,(DS1) |
| 15. | $\parallel C \wedge \neg C$                                    | 10,14,(CON) |
| 16. | $(A \wedge B) \vee (A \wedge C)$                               | 4-15,(IP)   |

5. (DIST2)  $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$ .

*Solution:*

|     |  |             |
|-----|--|-------------|
| 1.  | $A \vee (B \wedge C)$  | (P1)        |
| 2.  | $\parallel A$  | (Ass,CP)    |
| 3.  | $\parallel A \vee B$   | 2,(ADD1)    |
| 4.  | $\parallel A \vee C$   | 2,(ADD1)    |
| 5.  | $\parallel (A \vee B) \wedge (A \vee C)$                       | 3,4,(CON)   |
| 6.  | $A \rightarrow (A \vee B) \wedge (A \vee C)$                   | 2-5,(CP)    |
| 7.  | $\parallel B \wedge C$   | (Ass,CP)    |
| 8.  | $\parallel B$  | 7,(SIMP1)   |
| 9.  | $\parallel C$  | 7,(SIMP2)   |
| 10. | $\parallel A \vee B$   | 8,(ADD2)    |
| 11. | $\parallel A \vee C$   | 9,(ADD2)    |
| 12. | $\parallel (A \vee B) \wedge (A \vee C)$                       | 10,11,(CON) |
| 13. | $B \wedge C \rightarrow (A \vee B) \wedge (A \vee C)$          | 7-12,(CP)   |
| 14. | $A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)$ | 6,13,(DIS)  |
| 15. | $(A \vee B) \wedge (A \vee C)$                                 | 1,14,(MP)   |

#### Exercise B.4

In the notes (p. 19), it's shown that (DIS), (R), and (MT) are unnecessary: if we remove them all together, we have an extensionally equivalent system. Show the following other rules are also unnecessary (i.e., derive the following without using any of (DIS), (R), (MT), (ECQ), (DS2)). **NOTE:** don't use any derived rules for this part.

1. (ECQ)  $A, \neg A \vdash B$ .

*Solution:*

|    |                             |           |
|----|-----------------------------|-----------|
| 1. | $A$                         | (P1)      |
| 2. | $\neg A$                    | (P2)      |
| 3. | $\parallel \neg B$          | (Ass,IP)  |
| 4. | $\parallel A \wedge \neg A$ | 1,2,(CON) |
| 5. | $B$                         | 3-4,(IP)  |

2. (DS2)  $A \vee B, \neg B \vdash A$ .

*Solution:*

|    |                             |           |
|----|-----------------------------|-----------|
| 1. | $A \vee B$                  | (P1)      |
| 2. | $\neg B$                    | (P2)      |
| 3. | $\parallel \neg A$          | (Ass,IP)  |
| 4. | $\parallel B$               | 1,(DS1)   |
| 5. | $\parallel B \wedge \neg B$ | 3,4,(CON) |
| 6. | $A$                         | 3-5,(IP)  |