Solutions to Exercises 10 for *Introduction to Logic*

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Exercise 10.1

Represent the following sentences in the language of predicate logic.

(In the following, '$H(x)$' will represent 'x is human', etc.)

(a) Nouar wins the election only if an engineer is president.

$W(n, e) \to \exists x(E(x) \land P(x))$.

(b) Everyone has a beloved (i.e. everyone has someone whom they love).

$\forall x(H(x) \to \exists y(H(y) \land L(x, y)))$.

(c) Everyone loves everyone.

$\forall x(H(x) \to \forall y(H(y) \to L(x, y)))$.

If you have $\forall x \forall y(H(x) \land H(y) \to L(x, y))$, then that's a close second.

(d) No one loves everyone.

$\neg\exists x(H(x) \land \forall y(H(y) \to L(x, y)))$.

(e) Someone is loved by everyone.

$\exists x(H(x) \land \forall y(H(y) \to L(y, x)))$.

(f) Someone loves everyone.

$\exists x(H(x) \land \forall y(H(y) \to L(x, y)))$.

(g) Everyone loves a lover.

$\forall x(H(x) \land \exists y(H(y) \land L(x, y)) \to \forall z(H(z) \to L(z, x)))$

That's my preferred answer.

But—if you think about it a bit more—you might suppose the sentence is ambiguous: it can be read in various different ways. For (i) it can be varied what the understanding of 'lover' is, and (ii) the understanding of the overall logical structure can vary.

First consider (i). It could be said that the most natural reading **(a)** is that a lover is someone who loves someone else, i.e., someone who loves another human being. On this reading, we could represent 'x is a lover' by '$\exists y(H(y) \land L(x, y))$'. But there's at least one more possibility. For example: I love books—I'm a book-lover. So **(b)** a lover could be someone who loves *something* else. On this reading, we could represent 'x is a lover' by '$\exists y L(x, y)$'.

Now for (ii), and bearing in mind that 'lover' can be understood in the two different ways we've just seen. There are three different readings of the overall structure. The **first**, most likely reading of 'Everyone loves a lover' makes it mean the same as 'It's true of all lovers that everyone loves each of them', i.e., 'Everyone loves anyone who's a lover'. On the (a) reading of 'lover', this makes the sentence be represented as

$$\forall x(H(x) \land \exists y(H(y) \land L(x, y)) \to \forall z(H(z) \to L(z, x)))$$

and that's perhaps the most likely reading. On the (b) reading of 'lover', we could represent the sentence as

$$\forall x(H(x) \land \exists y L(x, y) \to \forall z(H(z) \to L(z, x))).$$

On the **second**, less likely reading of the overall structure, 'Everyone loves a lover' means that everyone has at least one lover they love. It could be a different lover in each case: Frieda might love Diego (who's a lover), Ludwig might love Ben (who's a lover), and so on. But even though both Diego and Ben are lovers, it's not required—as it was in the first reading—that Frieda and everyone else love both of them. On the (a) reading of 'lover', this second overall reading would be represented as

$$\forall x(H(x) \to \exists y(H(y) \land \exists z(H(z) \land L(y,z)) \land L(x,z))).$$

And on the (b) reading of 'lover', we represent the sentence as

$$\forall x(H(x) \to \exists y((H(y) \land \exists z L(y,z)) \land L(x,z))).$$

But wait, there's also a **third** reading of the overall structure of the sentence. On this third reading, the sentence 'Everyone loves a lover' says that there's some one lover whom everybody loves. That is: there has to be at least one lover who's in the lucky, or unlucky position of being loved by *everyone*. This third reading is perhaps even less likely than the two others, but is still possible. On the (a) version of 'lover', it gets represented as

$$\exists x((H(x) \land \exists y(H(y) \land L(x,y))) \land \forall z(H(x) \to L(z,x)))$$

and on the (b) version of 'lover' it gets represented as

$$\exists x((H(x) \land \exists y L(x,y)) \land \forall z(H(x) \to L(z,x))).$$

Now, we've seen six different ways of understanding the sentence—based on different understandings of its overall logical structure, and of how best to represent that someone is a lover. The lesson to draw from all this complication is: logic is, amongst other things, an incredible way of showing you the structure of what you actually think. We all know the difference between each of these readings intuitively; logic makes all those differences explicit.

(h) Not all love is requited.

$\neg\forall x\forall y(L(x,y) \to L(y,x))$.

Notice that this is just the (tragic) denial of symmetry as a property of love. If we'd wanted to represent 'No love is requited', that would be '$\forall x\forall y(L(x,y) \to \neg L(y,x))$'.

(i) Herb loves everyone but himself.

$\forall x(H(x) \to (L(h,x) \leftrightarrow \neg I(x,h)))$

Here, '$I(x,y)$' represents 'x is identical with y'. So the answer says: of everything, it's true that if that thing is a human, then Herb loves that thing if and only if the thing is not identical with Herb. 'x is identical with y' is usually represented as '$x = y$': so if you'd written $\forall x(H(x) \to (L(h,x) \leftrightarrow \neg x = y))$ then that is just as good.

(j) There is exactly one lover.

(Here I use '$x = y$' again for 'x is identical with y'.) Recall, from the answer for (g), that a lover can be someone who loves *someone* else, or someone who loves *something* else. On the first understanding, the answer is

$$\exists x((H(x) \land \exists y(H(y) \land L(x,y))) \land \forall z(H(z) \land \exists w(H(w) \land L(z,w)) \to z = x)).$$

On the second understanding of what a lover is, the answer is

$$\exists x((H(x) \land \exists y L(x,y)) \land \forall z(H(z) \land \exists w L(z,w) \to z = x)).$$

But represent 'x is a lover' by something more brief, '$F(x)$', and this will make what's going on clearer. The representation is then

$$\exists x(F(x) \land \forall z(F(z) \to z = x)).$$

What does this say? That there is at least one lover to which anything else that's a lover is identical. But if there's a lover to which any other lover is identical, then there can only be one lover! So the representation is correct. Let's argue another way that it's correct. The formula says that there's at least one lover to whom any other lover is identical. Pick one of these lovers that the formula says exists, and let's call this lover 'Aphrodite'. Then Aphrodite is a lover, and any lover must be identical with Aphrodite, i.e., any lover must just *be* Aphrodite. So there's exactly one lover: Aphrodite. (Note: the process of picking a new name for something said to exist by an existential quantifier has a name—it's called 'Skolemization', after the Norwegian logican, Thoralf Skolem. That's just what I did with 'Aphrodite'.)