Solutions to Exercises 12 for *Introduction to Logic*

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Exercise 12.1

What is the semantic value of an individual constant? What is the extension of a one-place predicate? What is the extension of an n-ary predicate?

Solution:

The semantic value of an individual constant is a member of the domain D, i.e., something d such that $d \in D$.

The extension of a one-place predicate is a set of members of D, i.e., a set of things in D (those members of which the predicate is true). Thus where $\mathfrak{I}$ is an interpretation $\langle D, \phi \rangle$, and P a one-place predicate, then $\phi(P) \subseteq D$.

The extension of an n-ary predicate is a set of n-tuples of members of D. Let P be an n-ary predicate. Then $\phi(P)$ is a set, each member of which has the form $\langle d_1, \ldots, d_n \rangle$, where all of $d_1, \ldots, d_n$ are in D. So $\phi(P) \subseteq D^n$.

Exercise 12.2

What is n-th Cartesian product

$$D^n = \underbrace{D \times \cdots \times D}_{n \text{ times}}$$

of the set D?

Solution:

D^n is the set containing all n-tuples of members of D. In other words, it's the set of *all* n-tuples $\langle d_1, \ldots, d_n \rangle$, where $d_1, \ldots, d_n \in D$.

So, for example, if $D = \{\mathit{Frege}, \mathit{Russell}\}$, then

$$\begin{aligned} D^2 = \{\ & \langle \mathit{Frege}, \mathit{Frege} \rangle, \\ & \langle \mathit{Frege}, \mathit{Russell} \rangle, \\ & \langle \mathit{Russell}, \mathit{Frege} \rangle, \\ & \langle \mathit{Russell}, \mathit{Russell} \rangle\ \} \end{aligned}$$

$$\begin{aligned} D^3 = \{\ & \langle \mathit{Frege}, \mathit{Frege}, \mathit{Frege} \rangle, \\ & \langle \mathit{Frege}, \mathit{Frege}, \mathit{Russell} \rangle, \\ & \langle \mathit{Frege}, \mathit{Russell}, \mathit{Frege} \rangle, \\ & \langle \mathit{Frege}, \mathit{Russell}, \mathit{Russell} \rangle, \\ & \langle \mathit{Russell}, \mathit{Frege}, \mathit{Frege}, \rangle, \\ & \langle \mathit{Russell}, \mathit{Frege}, \mathit{Russell} \rangle, \\ & \langle \mathit{Russell}, \mathit{Russell}, \mathit{Frege} \rangle, \\ & \langle \mathit{Russell}, \mathit{Russell}, \mathit{Russell} \rangle\ \} \end{aligned}$$

and so on.

The role D^n plays is that each n-ary predicate, P, in the language, is assigned a set of members of D^n. For example, suppose we wanted to represent 'was born before' by the two-place predicate P. Then we would let $\phi(P) = \{\langle \mathit{Frege}, \mathit{Russell} \rangle\}$, because Frege was born before Russell (so $\langle \textit{Frege}, \textit{Russell}\rangle$ has to be a member of $\phi(P)$), but of no *other* pair in D^2 is 'was born before' true. Similarly, if we wanted to represent 'is the same person as' by the two-place predicate Q, then we would have $\phi(Q) = \{\langle \textit{Frege}, \textit{Frege}\rangle, \langle \textit{Russell}, \textit{Russell}\rangle\}$, because Frege is the same person as Frege, and Russell is the same person as Russell.

Exercise 12.3

Let S be the set $S = \{a, b\}$. What is S^2 ? What is S^3?

Solution:

S^2 is $\{\langle x, y\rangle : x, y \in S\}$, which is the set of pairs of members in S. This is just the set $\{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle\}$.

S^3 is $\{\langle x, y, z\rangle : x, y, z \in S\}$: the set of triples of members of S. Given the particular S, this is $\{\langle a, a, a\rangle, \langle a, a, b\rangle, \langle a, b, a\rangle, \langle a, b, b\rangle, \langle b, a, a\rangle, \langle b, a, b\rangle, \langle b, b, a\rangle, \langle b, b, b\rangle\}$.

NOTE: you can probably see the pattern in general. S^n is just $\{\langle x_1, \ldots, x_n\rangle : x_1, \ldots, x_n \in S\}$, and writing out which that set is can always be done according to a simple rule.

Exercise 12.4

Let $\mathfrak{I}$ be the interpretation $\mathfrak{I} = \langle D, \phi\rangle$:

- $D = \{\textit{Merkel}, \textit{Obama}, \textit{Rouhani}, \textit{Silverman}\}$ (i.e. Angela Merkel, Barack Obama, Hassan Rouhani, Sarah Silverman)
- $\phi(a) = \textit{Merkel}$
- $\phi(b) = \textit{Obama}$
- $\phi(c) = \textit{Rouhani}$
- $\phi(d) = \textit{Silverman}$
- $\phi(F) = \{\textit{Merkel}, \textit{Silverman}\} = \{d \in D : d \text{ is female}\}$
- $\phi(S) = \{\textit{Merkel}, \textit{Obama}, \textit{Rouhani}\} = \{d \in D : d \text{ is a head of state}\}$
- $\phi(H) = \{\textit{Merkel}, \textit{Obama}, \textit{Rouhani}, \textit{Silverman}\} = \{d \in D : d \text{ is a human}\}$
- $\phi(N) = \emptyset = \{d \in D : d \text{ is a number}\}$
- $\begin{aligned}\phi(Y) = \{&\langle \textit{Silverman}, \textit{Obama}\rangle, \langle \textit{Silverman}, \textit{Merkel}\rangle, \langle \textit{Silverman}, \textit{Rouhani}\rangle,\\ &\langle \textit{Obama}, \textit{Merkel}\rangle, \langle \textit{Obama}, \textit{Rouhani}\rangle, \langle \textit{Merkel}, \textit{Rouhani}\rangle\}\\ = \{&\langle d_1, d_2\rangle \in D^2 : d_1 \text{ is younger than } d_2\}\end{aligned}$

On the basis of $\mathfrak{I}$, which of the following sentences are true? Which ones are false?

Solution:

(a) $S(a)$

True. (Because $\phi(a) \in \phi(S)$. Given the values of 'a' and 'S', the formula represents that *Merkel is a head of state.*)

(b) $S(b)$

True. (Because $\phi(b) \in \phi(S)$: *Obama is a head of state.*)

(c) $S(d)$

False. (Because it's not the case that $\phi(d) \in \phi(S)$: Sarah Silverman isn't, alas, a head of state.)

(d) $N(a)$

False. (We don't have that $\phi(a) \in \phi(N)$. Merkel isn't a number!)

(e) $F(c)$

False. ($\phi(c) \notin \phi(F)$. It's not the case that Rouhani is female.)

(f) $\neg F(b)$

True. (Because $\neg F(b)$ is true iff $F(b)$ is false. And $F(b)$ is *indeed* false, because $\phi(b)$ is not in $\phi(F)$. It's not the case that Obama is female.)

(g) $Y(a, b)$

False. (Because $Y(a, b)$ is true just if $\langle \phi(a), \phi(b) \rangle \in \phi(Y)$. But since $\phi(a) = \textit{Merkel}$, $\phi(b) = \textit{Obama}$, then $Y(a, b)$ is true just in case $\langle \textit{Merkel}, \textit{Obama} \rangle$ is in $\phi(Y)$. But it's not. It's not the case that Merkel is younger than Obama.)

(h) $Y(b, a) \wedge (Y(c, d) \vee \neg N(d))$

True. (The formula has the form of a conjunction, so both conjuncts must be true, i.e., both $Y(b, a)$ and $Y(c, d) \vee \neg N(d)$ must be true. Consider first $Y(b, a)$: this is true iff $\langle \phi(b), \phi(a) \rangle \in \phi(Y)$, i.e., iff $\langle \textit{Obama}, \textit{Merkel} \rangle \in \phi(Y)$. This is so. (Obama is younger than Merkel.) Now consider $Y(c, d) \vee \neg N(d)$. Since this is a disjunction, for it to be true requires at least one of the disjuncts to be true. Is $Y(c, d)$ true? No, because we don't have that $\langle \phi(c), \phi(d) \rangle \in \phi(Y)$: Rouhani isn't younger than Silverman. How about the second disjunct? Is $\neg N(d)$ true? It is, iff $N(d)$ is false, i.e., iff it's false that $\phi(d) \in \phi(N)$. This *is* false, because Silverman isn't a number. This means that the second disjunct is true, so the whole formula is true.—The formula represents that *Obama is younger than Merkel, and either Rouhani is younger than Silberman or Silverman isn't a number.*)

(i) $\neg S(c) \to \neg F(a)$

True. (Is the antecedent, $\neg S(c)$, true? It is, iff $S(c)$ is false, i.e., iff $\phi(c)$ is not a member of $\phi(S)$. But $\phi(c)$, i.e., Rouhani, *is* a member of $\phi(S)$. So $S(c)$ is true, and so $\neg S(c)$ is false. But a conditional with a false antecedent, is true—from the truth-table for '$\to$'. So the formula is true. It represents that *if Rouhani isn't a head of state, then Merkel isn't female.*)

(j) $\forall x H(x)$

True. ($\forall x H(x)$ is true iff everything in D is in $\phi(H)$. But D and $\phi(H)$ are, in $\mathfrak{I}$, the same set: they're both $\{\textit{Merkel}, \textit{Obama}, \textit{Rouhani}, \textit{Silverman}\}$. So the statement is true. It represents that *everything is human*—and that is true in a situation where all that exists are just the four humans of our D.)

(k) $\forall x S(x)$

False. ($\forall x S(x)$ is true iff everything in D is also in $\phi(S)$, i.e., iff for all $x \in D$, $x \in \phi(S)$. But that's false, because $\textit{Silverman} \in D$, but it's not the case that $\textit{Silverman} \in \phi(S)$—Silverman isn't a head of state. The formula represents that everything is a head of state.)

(l) $\exists x N(x)$

False. (The formula represents that *something is a number*. But in the situation of our D, it's only *Merkel*, *Obama*, *Rouhani* and *Silverman* that exist. In this situation, there's nothing that's a number. Now $\exists x N(x)$ is true iff there's a member of D which is also in $\phi(N)$; since $\phi(N)$ is $\emptyset$, the empty set, this must be false.)

(m) $\exists x (H(x) \wedge \neg N(x))$

True. (The formula is true iff there is some member of D, x, which is both in $\phi(H)$, and not in $\phi(N)$. In fact, in our $\mathfrak{I}$, *everything* is in $\phi(H)$ and *nothing* is in $\phi(N)$. So any of *Merkel*, *Obama*, *Rouhani* and *Silverman* suffice for making the formula true. It represents that some human isn't a number.)

(n) $\neg \exists x \neg N(x)$

False. (The formula is true iff $\exists x \neg N(x)$ is false. But $\exists x \neg N(x)$ is *true* iff there is a member of D which is not in $\phi(N)$. There is such a member of D—indeed, all of them, as we saw in (m)—then $\exists x \neg N(x)$ is true. So $\neg \exists x \neg N(x)$ is false. It represents that there's nothing which isn't a number.)

(o) $\exists y(S(y) \wedge H(y))$

True. (This is true iff there is some $y \in D$ such that $S(y) \wedge H(y)$. So it's true iff there is some $y \in D$ such that $S(y)$ is true and $H(y)$ is true. We can show this by picking such a y. Consider *Merkel*. Clearly, *Merkel* $\in \phi(S)$ and *Merkel* $\in \phi(H)$. So there is a $y \in D$ that's a member of both $\phi(S)$ and $\phi(H)$. So the formula is true. It represents that *something is a human head of state*.

(p) $\forall x(F(x) \to S(x))$

False. (The formula is true iff, for every $x \in D$, if $F(x)$ then $S(x)$. In other words: it must be true of everything in D, that *if* that thing is in $\phi(F)$, *then* it is in $\phi(S)$. Let's check this by considering all the members of D in turn. *Merkel is* in $\phi(F)$, but she's also in $\phi(S)$, so of her it's true that if she's in $\phi(F)$, then she's in $\phi(S)$. *Obama* and *Rouhani* are both not in $\phi(F)$: so of each of them it's true that, if they're in $\phi(F)$, they're in $\phi(S)$—just from the truth-table for '$\to$', because a false antecedent makes the whole conditional true. Finally, what of *Silverman*? We know that *Silverman* $\in \phi(F)$, but—as has been lamented often enough—*Silverman* is *not* in $\phi(S)$. So she's a counterexample: she falsifies our formula $\forall x(F(x) \to S(x))$. The formula represents that all women are heads of state.

(q) $\exists x Y(x, a)$

True. (This is true iff there is a member of D, x, such that $\langle x, \phi(a)\rangle$ is in $\phi(Y)$. Since $\phi(a)$ is just *Merkel*, there clearly is such a member of D: take *Obama*, for instance. We know that $\langle$*Obama*, *Merkel*$\rangle$ is in $\phi(Y)$. So the formula is true. It represents that something is younger than Merkel.

(r) $\forall x(Y(x, b) \to \neg S(x))$

True. (The formula is true iff everything $x \in D$ is such that, *if* $Y(x, b)$, then $\neg S(x)$. Let's check this. If we take x to be any of *Merkel*, *Obama* and *Rouhani*, then it would be false that $Y(x, b)$—because none of $\langle$*Merkel*, *Obama*$\rangle$, $\langle$*Obama*, *Obama*$\rangle$ and $\langle$*Rouhani*, *Obama*$\rangle$ are members of $\phi(Y)$. In these cases, the antecedent of $Y(x, b) \to \neg S(x)$ is false, so the whole conditional is true. The only way of assigning a value to x so as to make $Y(x, b)$ true is if we take x to be *Silverman*: we have $\langle$*Silverman*, $\phi(b)\rangle \in \phi(Y)$. Further, we *don't* have *Silverman* $\in \phi(S)$: Silverman isn't a head of state. This means that, still taking x to be Silverman, $\neg S(x)$ is true. So taking x to be Silverman, both the antecedent and consequent come out true. So in this case, the whole conditional is true. The formula represents that *Everything younger than Obama isn't a head of state.*)

(s) $Y(a, b) \to \exists x(S(x) \wedge N(x))$

True. (Remember the truth-table for '$\to$': if the antecedent is false, then the whole conditional is true. Our formula itself is a conditional: the main connective is '$\to$'. Now, $Y(a, b)$ is true iff $\langle\phi(a), \phi(b)\rangle \in \phi(Y)$. Since $\langle\phi(a), \phi(b)\rangle$ is just $\langle$*Merkel*, *Obama*$\rangle$, then by looking at $\phi(Y)$, we can immediately see that $\langle\phi(a), \phi(b)\rangle$ isn't a member. So $Y(a, b)$ is false. Since it's false, the whole conditional is false. We don't need to check the truth the value of the consequent. The whole formula represents that *if Merkel is younger than Obama, then a number is a head of state.*)

(t) $\forall x Y(x, c)$

False. (The formula is true iff all things x in D are such that $Y(x, c)$, i.e., iff all members x of D are such that $\langle x, \phi(c)\rangle \in \phi(Y)$. But $\phi(c) =$ *Rouhani*. So we need to check whether $\langle$*Merkel*, *Rouhani*$\rangle$, $\langle$*Obama*, *Rouhani*$\rangle$, $\langle$*Rouhani*, *Rouhani*$\rangle$ and $\langle$*Silverman*, *Rouhani*$\rangle$ are all in $\phi(Y)$. Only the first, second and fourth of are in $\phi(Y)$; $\langle$*Rouhani*, *Rouhani*$\rangle \notin \phi(Y)$. (Rouhani isn't younger than himself!) So the formula is false. It represents that *everything is younger than Rouhani.*)

(u) $\exists y \forall x Y(x, y)$

False. (What does it say? That there is something, y, such that everything is younger than y. This can't be true: because nothing is younger than itself, so that for anything

y we choose, there will always be something that's not younger than y, namely, y itself. Let's go through the steps a bit more formally. The formula is true iff there is some $y \in D$, such that $\forall x Y(x, y)$. So it's true iff there is some $y \in D$, such that for every $x \in D$, $\langle x, y \rangle \in \phi(Y)$. But there is no such y, because for any y, $\langle y, y \rangle \notin \phi(Y)$. The formula represents that *something is younger than everything*, and that is clearly false.)

(v) $\exists y \forall x (x \neq y \to Y(x, y))$

True. (First notice the similarity to (u). The important difference here is that the formula is claiming that *there's something younger than everything* ***apart from itself***— the 'apart from itself' is what the '$x \neq y$' ensures. So, if there is a youngest among *Merkel*, *Obama*, *Rouhani* and *Silverman*, then it will show that the formula is true. But *Silverman* is youngest. Let's observe that $\forall x(x \neq d \to Y(x, d))$, which represents that *everything apart from Silverman is younger than Silverman.* This is true, because for all $x \in D$, if $x \neq d$ is true, then $Y(x, d)$ is true. So $\exists y \forall x(x \neq y \to Y(x, y))$.)

(w) $\forall x(N(x) \to F(x))$

True. (The formula is true iff everything x in D is such that, if $N(x)$ then $F(x)$. So it's true iff everything x in D is such that, if $x \in \phi(N)$, then $x \in \phi(F)$. Now, $\phi(N) = \emptyset$: $\phi(N)$ is the empty set; and $\phi(F)$ is the set $\{\textit{Merkel}, \textit{Silverman}\}$. So the statement is true iff for everything $x \in \emptyset$, then $x \in \{\textit{Merkel}, \textit{Silverman}\}$. This is true: since there is *nothing* in the empty set, then it's true 'vacuously' that everything in the empty set is in the other set. Another way to think about this is: the only way to *falsify* the claim that everything in the empty set is in $\phi(F)$, would be to find an example of something in $\emptyset$ but not in $\phi(F)$. But it's obvious that this can't be done: there's no such example, because there's nothing in the empty set *at all.* So the formula is true. It represents that *all the numbers are female.*)