Solutions to Exercises 11 for *Introduction to Logic*

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Exercise 11.1

What is it for an individual variable to have a free (unbound) occurrence in a formula? What is it for an individual variable to have a bound occurrence in a formula?

Solution:

A variable v has a *free* occurrence in a formula A iff there's an occurrence of v in A which (i) doesn't immediately follow a quantifier, and (ii) doesn't occur in the scope of a quantificational expression of the form $\exists v$ or $\forall v$.

A variable v has a *bound* occurrence in a formula A iff there's an occurrence where (i) v occurs immediately after a quantifier $\exists$ or $\forall$, or (ii) v occurs in the scope of a quantificational expression $\exists v$ or $\forall v$.

Extra comments: the (ii) clause of the definition of there being a bound occurrence of a variable is actually redundant. Why? Because suppose that v has a bound occurrence in A, and the reason is (ii): v occurs in the scope of a quantificational expression $\exists v$ or $\forall v$. But then there must *be* a quantificational expression $\exists v$ or $\forall v$, for the occurrence of v to be in the scope of. This means that (i) (in the definition for 'bound') must be true. Since (ii) implies (i), (ii) is redundant, and we *could* just have used (i).

Exercise 11.2

What is the scope of the occurrence of a quantificational expression (recall that we took a quantificational expression to be a quantifier—$\exists$ or $\forall$—along with the variable it binds (e.g. $\exists x$, $\forall z$))?

Solution:

If, within a formula A, there's a subformula '$\exists xB$' or '$\forall xB$', then 'B' is the *scope* of this occurrence of the quantificational expression.

For example, in '$\forall xP(x) \wedge \exists xQ(x)$', the scope of the '$\forall x$' is the occurrence of '$P(x)$'; and the scope of the '$\exists x$' is the occurrence of '$Q(x)$'. Or in '$\forall x\forall yR(x, y)$', the scope of the '$\forall x$' is the occurrence of '$\forall yR(x, y)$'. Or in '$\forall x(P(x) \wedge Q(a))$', the scope of the '$\forall x$' is '$(P(x) \wedge Q(a))$'.

Exercise 11.3

Which occurrences of individual variables in the following formulas are free? Which ones are bound?

Solution:

Free occurrences are in **bold red**; bound are in underlined blue.

(a) $P(a_2)$

(b) $\forall \underline{x_1}(R(\underline{x_1}, a_1) \rightarrow P(\underline{x_1}))$

(c) $\forall \underline{x_1}(R(\underline{x_1}, \boldsymbol{x_3}) \rightarrow P(\underline{x_1}))$

(d) $\exists \underline{x_3}(P(\boldsymbol{x_1}) \wedge \forall \underline{x_2}R(\underline{x_3}, \underline{x_2}))$

(e) $\forall \underline{x_5}(P(\boldsymbol{x_1}) \rightarrow \exists \underline{x_2}(R(\underline{x_2}, \underline{x_5}) \wedge \forall \underline{x_1}S(\underline{x_2}, \underline{x_1})))$

(f) $\exists \underline{x_1}(P(\underline{x_1}) \wedge R(a_7, \underline{x_1})) \wedge S(a_8, \boldsymbol{x_1})$

Exercise 11.4

What does it mean for a formula to be open? What is a sentence (or closed formula)?

Solution:

A formula A is *open* iff there is a variable which has at least one free occurrence in A.

A formula A is *closed* iff there is no variable with a free occurrence in A.

Extra comments: What's the importance of a formula's being open or closed? It's only a closed formula which is capable of being true or false. An open formula is incomplete in a certain way, and that's indicated by the presence of the unbound variable. Roughly, just as if we take away 'Everything' from 'Everything is illuminated' we get something that's not true or false, if we take away '$\forall x$' from '$\forall x Illuminated(x)$' we get '$Illuminated(x)$', which isn't in itself true or false.

Exercise 11.5

Which of the following strings of symbols are formulas of predicate logic? Which ones are open and which are sentences (or closed)?

Solution:

(a) $(R(a_1, a_3) \land \exists x_1 P(x_1))$
Closed formula.

(b) $\exists x_1 (R(x_1, a_3) \land P(x_1))$
Closed formula.

(c) $\exists x_1 (R(x_1, a_3) \land P(x_2))$
Open formula.

(d) $\exists x_1 (R(x_1, a_3) \land \forall x_2 P(x_2))$
Closed formula.

(e) $\exists x_1 \forall x_2 \exists x_4 (R(x_4, x_2) \lor P(x_1))$
Closed formula.

(f) $\forall x_5 R(a_1, a_5)$
Closed formula.

(g) $\forall x_5 R(a_1, x_5)$
Closed formula.

(h) $\forall x_5 (R(a_1, x_5) \to \exists x_3 P(x_3))$
Closed formula.

(i) $P(x_1) \land p$
Not a formula.

(j) $P(x_1) \land \forall x_1 P(x_1)$
Open formula.

(k) $\exists \forall x_2 R(x_2, a_{47355})$
Not a formula.

(l) $\forall x_{18} P^7(x_{18}, x_{18}, x_{18}, x_{18}, x_{18}, x_{18}, x_{18})$
Closed formula.

(m) $\forall x_{24} P^9(x_{24}, x_{24}, x_{24}, x_{24}, x_{24}, x_{24}, x_{24}, x_{24})$

Closed formula.

(Well: I suppose there could be an argument here. If the '9' superscript on the 'P^9' successfully signifies that the arity of the predicate is 9, then the string above is *not* a formula of predicate logic. Because where the arity is 9, then the predicate *must* be followed by a list of 9 singular terms.

However, it's also been said that we can choose our descriptive language as we see fit: we aren't required to *mark* the arity accurately on the predicate name itself. If that's right, then I stand by the string's being a closed formula, on the understanding that 'P^9' here has an arity of 8.)

(n) $\exists x_1 \neg \forall x_2 (P(x_1) \rightarrow Q(x_2))$

Closed formula.

(o) $\exists x_1 \neg\neg (\forall x_2 (P(x_1) \rightarrow Q(x_2)))$

Closed formula.

(p) $\forall x_3 \exists x_3 R(x_3, x_4)$

Open formula.

(q) $\neg \forall x_1 (P(x_1))$

Not a formula. (We don't need brackets around the '$P(x_1)$'.

(r) $\exists x_5 (P(x_5) \wedge \forall x_2 (Q(x_2 \rightarrow P(a_8)))$

Not a formula.