Solutions to Exercises 9 for *Introduction to Logic*

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Preliminary note on Qs 9.4 and 9.5.
You might wonder why 'Some empiricists are wrong' is represented as '$\exists x(E(x) \wedge W(x))$', whereas 'All empiricists are wrong' is represented as '$\forall x(E(x) \rightarrow W(x))$': why the '$\wedge$' in one case, but the '$\rightarrow$' in the other?

First, let me try to strengthen your intuitions that this *is* the right way to represent them. 'Some empiricists are wrong' is true iff there's at least one empiricist who is wrong, i.e., it's true iff there's at least one thing such that it is an empiricist *and* it is wrong. That last way of putting the claim reveals how we should represent it: the 'and' is there for all to see. It must be '$\exists x(E(x) \wedge W(x))$'.

How about 'All empiricists are wrong'? This says that everything is such that, if it's an empiricist, then it's wrong. Wrongness is only attributed here to every empiricist: there's a condition on those things which are being said to be wrong (the condition is, that they're empiricists). It's this condition which the '$\rightarrow$' ensures (it's called a 'material conditional').

Let's now check that the alternatives don't work. Consider first what would happen if we tried to represent 'All empiricists are wrong' as '$\forall x(E(x) \wedge W(x))$'. What does the formula actually say? Pretty clearly, it says: 'Everything is such that it's an empiricist and it's wrong', i.e., it says that 'Everything is a wrong empiricist'. But that's not what we wanted to say! 'Everything is a wrong empiricist' would mean that everything is an empiricist, and it would also mean that everything is wrong. So this alternative is no good, because we were just saying that all the *empiricists* are wrong.

How about the second alternative? The suggestion would be that we try to represent 'Some empiricists are wrong' as '$\exists x(E(x) \rightarrow W(x))$'. This clearly means: at least one thing is such that, *if* that thing is an empiricist, then that thing is wrong. But consider Kant: he's not an empiricist. Let's ignore the issue of whether he's wrong or not (phew!). Since Kant isn't an empiricist, it's true of him that, if he *is* an empiricist, then he's wrong. (Remember the truth-table for $p \rightarrow q$? If p is false, then the whole thing is true.) So since it's true of Kant that, if he's an empiricist, then he's wrong—it's true because he's not an empiricist at all—then there is at least one thing that is such that, if it's an empiricist, then it's wrong. Therefore the existence of something that's not an empiricist can make '$\exists x(E(x) \rightarrow W(x))$' true. So it can't be that this formula represents 'Some empiricists are wrong', because it's not the case that some empiricists are wrong just because some one non-empiricist exists.

Exercise 9.1

Explain what an atomic formula in the language of predicate logic amounts to.

Solution:

An atomic formula in predicate logic is an expression '$P(t_1, \ldots, t_n)$'. Here 'P' is an n-ary predicate, i.e., a general term which *must* be followed by n singular terms in formulas.

Suppose that 'P' is an 1-ary (also known as 'monadic', 'unary') predicate, and 'Q' is a 2-ary (also known as 'dyadic', 'binary') predicate. Then the following are atomic formulas: '$P(x)$', '$P(y)$', '$P(a)$', '$Q(x, y)$', '$Q(x, x)$'. The following are *not* atomic formulas: '$P(Q)$', '$\exists x P(x)$', '$\neg P(a)$', '$P(x, y)$'.

Exercise 9.2

Represent the following sentences in the language of predicate logic.

a. Anna is smart.

   $S(a)$.

b. Herb loves Anna.

   $L(h, a)$.

c. Anna drives Herb from Hull to Harare.

   $D(a, h, u, a)$.

d. Anna and Herb are driving from Hull to Harare.

   On reflection, I think I prefer '$D(a, u, a) \wedge D(h, u, a)$' for this.

   I originally had '$D(a, h, u, a)$', thinking that it was implied that they were driving together, and that if they're driving together then we can't 'break them apart' and say, e.g., that 'Anna is driving from Hull to Harare'—$D(a, u, a)$. But actually it seems to me that *even if* they're on a road trip together, we can still say that Anna is driving from Hull to Harare, and that this should be implied.

e. Anna works with Rachel in Manchester.

   $W(a, r, m)$.

Exercise 9.3

What quantifiers are there in the language of predicate logic? Explain what they mean.

Solution:

Predicate logic has the universal quantifier '$\forall$' and the existential quantifer '$\exists$'.

'$\forall$' means *everything.* Suppose $P(x)$ means *x is a planet*; then '$\forall x P(x)$' means *Everything is a planet.* Or suppose '$Q(a, x)$' means *Andrew likes x.* Then '$\forall x Q(a, x)$' means *Andrew likes everything.* (I'm supposing 'a' refers to Andrew.) Alternatively, '$\forall x Q(x, a)$' would mean *Everything likes Andrew.*

'$\exists$' means *something* or *at least one thing.* Then, with the same examples as before, '$\exists x P(x)$' means *Something is a planet*—true, I hope you'll agree. And '$\exists x Q(a, x)$' means *Andrew likes something* or *There's at least one thing that Andrew likes.* (Let's hope this is true too, for Andrew's life-satisfaction.)

Exercise 9.4

How are existential sentences containing more than one general term generally represented?

Often, as existential formulas containing a conjunction. Here, the idea is that the *main propositional connective* is '$\wedge$'. If, for example, there are just the monadic predicate P and the monadic predicate Q corresponding to the general terms, then an existential sentence containing these predicates will probably be '$\exists x(P(x) \wedge Q(x))$'. If '$P(x)$' means *$x$ is a dog* and '$Q(x)$' means *x is hungry*, then '$\exists x(P(x) \wedge Q(x))$' means *there's a hungry dog* or *there exists at least one hungry dog.*

NOTE: this rule isn't without exceptions. For instance, we would formalize 'There's something that's not a hungry dog' as '$\exists x \neg(P(x) \wedge Q(x))$'. This doesn't conform to the rule, because the main connective here is '$\neg$'.

Exercise 9.5

How are universal sentences containing more than one general term generally represented?

Often, as universal formulas containing a conditional. Again, here the idea is that the *main propositional connective* is '$\to$'. Consider, with 'P' and 'Q' as for quesion **9.4**, the sentence 'Everything that's a dog is hungry' (you could perhaps say this is as: 'All the dogs are hungry'). We would represent this as: '$\forall x(P(x) \to Q(x))$'.

NOTE: again, there are exceptions. For instance, consider ‘Everything is a hungry dog’. This would be ‘$\forall x(P(x) \wedge Q(x))$’. Notice the difference. ‘Everything that’s a dog is hungry’ doesn’t exclude the possibility that there are cats which aren’t hungry. It just says all the *dogs* are hungry. But ‘Everything is a hungry dog’ *does* exclude there being cats which aren’t hungry.

Exercise 9.6

Represent the following sentences in the language of predicate logic.

(I’ll suppose that we’re not representing them all at once in predicate logic, so I can reuse predicate letters and constants.)

a. There are no snakes in New Zealand.

   $\neg\exists x(S(x) \wedge I(x, n))$, where ‘$n$’ represents New Zealand, and ‘$I(x, y)$’ represents that x is in y.

   Second prize goes to: $\neg\exists x(S(x) \wedge N(x))$, where ‘$N(x)$’ represents that x is in New Zealand.

b. All inhabitants of Camden are happy.

   $\forall x(I(x, c) \rightarrow H(x))$, where ‘$c$’ represents Camden and ‘$I(x, y)$’ represents that x is an inhabitant of y.

   Second prize goes to: $\forall x(C(x) \rightarrow H(x))$, where ‘$C(x)$’ represents that x is an inhabitant of Camden.

c. All inhabitants of Camden are cheering, but some inhabitants of Islington are angry.

   $\forall x(I(x, c) \rightarrow C(x)) \wedge \exists x(I(x, i) \wedge A(x))$.

d. All Austrians are not philosophers.

   $\forall x(A(x) \rightarrow \neg P(x))$.

e. Not all Austrians are not philosophers.

   $\neg\forall x(A(x) \rightarrow \neg P(x))$.

f. It’s not the case that no Austrians are philosophers.

   $\neg\neg\exists x(A(x) \wedge P(x))$.

g. Some Austrians are philosophers.

   $\exists x(A(x) \wedge P(x))$.

h. There are Austrians who are not philosophers.

   $\exists x(A(x) \wedge \neg P(x))$.

i. It is not the case that there are Austrians who are philosophers.

   $\neg\exists x(A(x) \wedge P(x))$.

j. It is not the case that there are Austrians who are not not philosophers.

   $\neg\exists x(A(x) \wedge \neg\neg P(x))$.

k. All inhabitants of Salzburg are Austrians and Europeans.

   $\forall x(I(x, s) \rightarrow A(x) \wedge E(x))$.

l. Some Irish people live in New Jersey or in Massachusetts.

   $\exists x(I(x) \wedge P(x) \wedge (L(x, n) \vee L(x, m)))$.

m. Someone loves Herb.

   $\exists x L(x, h)$.

n. Some philosophers love Herb.

   $\exists x(P(x) \wedge L(x, h))$.

o. Everyone who is a philosopher or a logician loves Anna and Rachel.

   $\forall x(P(x) \vee L(x) \rightarrow O(x, a) \wedge O(x, r))$.