MIMO System Analysis

Multivariable Control ECSE 6460

Mini Project 2

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CONTENTS

Abstract

This report will cover the modeling and analysis of the MIMO system seen in Figure 1. A model will be derived from the plant circuit. This model will include an exogenous, parametric, and control input, and a performance, parametric, and sensor output. The plant will also be analyzed for the best sensor and actuator placement. MIMO robustness will also be analyzed.

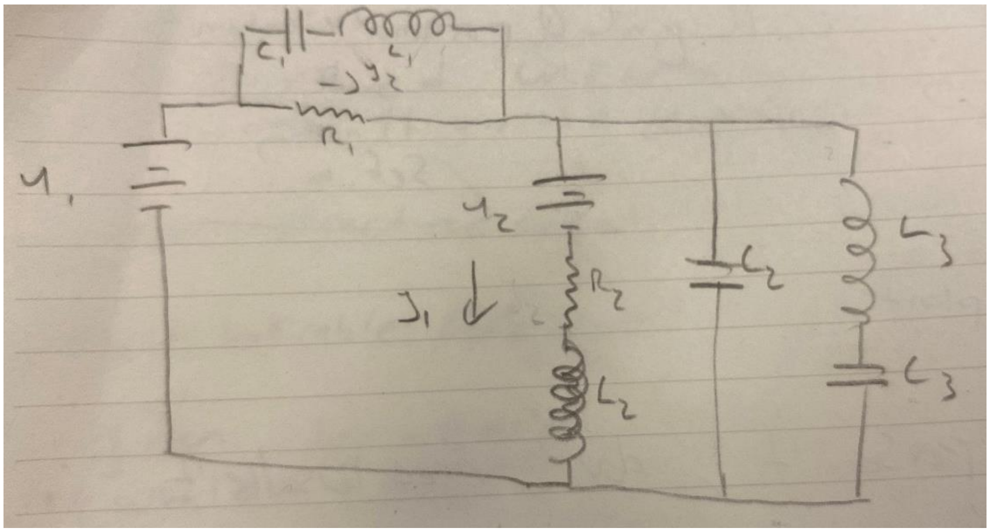

Figure 1: Project 2 Plant

Modeling the System

P22 models strictly the plant dynamics, P models all aspects of the model including noise, references, performance outputs etc. P fits the general plant model as seen in Figure 2. P22 is the map from u to y.

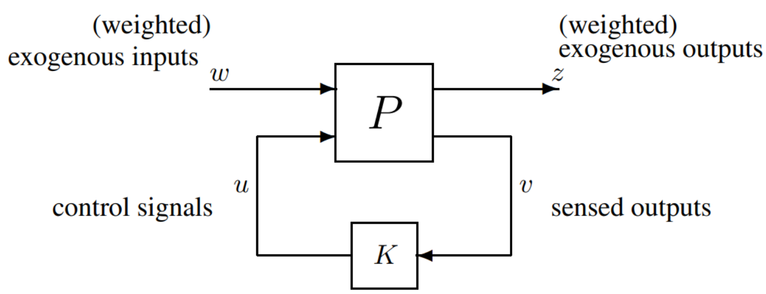

Figure 2: General P Model

Modeling 22

To obtain the dynamics of the MIMO system, a similar approach to the SISO system was used. The model is assumed to be LTI, and the circuit has no dependent sources, thus superposition can be used to obtain the transfer functions between every source and output. Using the impedance of the components as seen in equations (1),(2), and (3). Using these, and simple circuit reduction techniques, each circuit can be reduced to a simple impedance circuit, and the transfer function can be obtained using Ohm’s Law, as seen in Figure 3.

$$Z_L = sL \tag{1}$$

$$Z_C = \frac{1}{sC} \tag{2}$$

$$Z_R = R \tag{3}$$

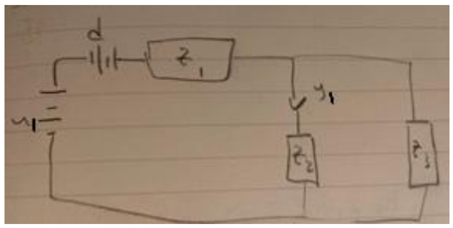

Figure 3 a): Circuit From u_1 to y_1

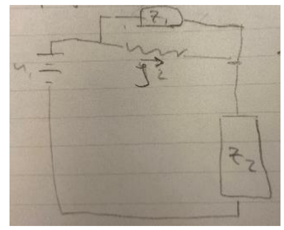

Figure 3 b): Circuit From u_1 to y_2

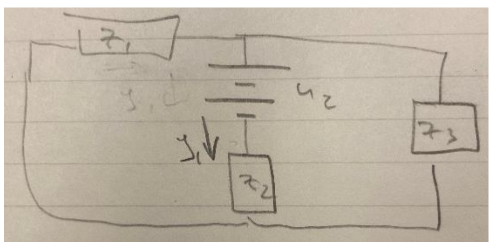

Figure 3 c): Circuit From u_2 to y_1

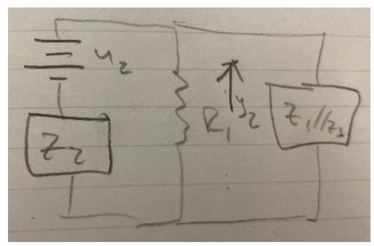

Figure 3 d): Circuit From u_2 to y_2

Figure 3: Reduced Circuits

The transfer functions were then put into MATLAB as a single MIMO system, whose state space form can be seen in Figure 4, and transfer functions can be seen in Figure 5. There are four uncontrollable and unobservable states that can be removed with the MATLAB minreal command, this minimum state space representation can be seen in Figure 6.

A =

	x1	x2	x3	x4	x5	x6	x7	x8
x1	-2002	-320.3	-610.5	-23.84	0	0	0	0
x2	1024	0	0	0	0	0	0	0
x3	0	1024	0	0	0	0	0	0
x4	0	0	32	0	0	0	0	0
x5	0	0	0	0	-2002	-320.3	-610.5	-23.84
x6	0	0	0	0	1024	0	0	0
x7	0	0	0	0	0	1024	0	0
x8	0	0	0	0	0	0	32	0

B =

	u1	u2
x1	2	0
x2	0	0
x3	0	0
x4	0	0
x5	0	4
x6	0	0
x7	0	0
x8	0	0

C =

	x1	x2	x3	x4	x5	x6	x7	x8
y1	0	0.01953	1.907	0.0596	-5	-0.009766	-1.526	-0.0298
y2	-0.01	-0.9961	-1.908	-0.0596	0	-0.009766	0	-0.0298

D =

	u1	u2
y1	0	0
y2	0.01	0

Figure 4: P22 Full

```
From input 1 to output...
                   40 s^2 + 4e06 s + 4e06
 1:  -----------------------------------------------
     s^4 + 2002 s^3 + 328000 s^2 + 6.402e08 s + 8e08

         s^4 + 2000 s^3 + 124000 s^2 + 2.4e08 s + 4e08
 2:  -------------------------------------------------------
     100 s^4 + 200200 s^3 + 3.28e07 s^2 + 6.402e10 s + 8e10

From input 2 to output...
             -20 s^3 - 40 s^2 - 6.4e06 s - 4e06
 1:  -----------------------------------------------
     s^4 + 2002 s^3 + 328000 s^2 + 6.402e08 s + 8e08

                       -40 s^2 - 4e06
 2:  -----------------------------------------------
     s^4 + 2002 s^3 + 328000 s^2 + 6.402e08 s + 8e08
```

Figure 5: P22 Transfer Functions

A =

	x1	x2	x3	x4
x1	-1838	-62.24	-574.9	27.14
x2	345.8	126.4	-1108	110.4
x3	-473.2	271.9	-278.3	-22.6
x4	75.24	-63.16	201.8	-11.9

B =

	u1	u2
x1	0.1104	3.669
x2	-0.3963	-0.02131
x3	-0.9056	1.035
x4	0.1608	-0.06106

C =

	x1	x2	x3	x4
y1	-5.03	1.865	-1.467	-0.2123
y2	-0.2545	-1.947	0.8712	0.1598

D =

	u1	u2
y1	0	0
y2	0.01	0

Figure 6: P22 Minimum Realization

To ensure that this is the correct plant, the super-positioned circuits were recreated using LTSpice and the bode plots were compared between the MATLAB model and the circuit, as seen in Figure 7 and Figure 8. These bode plots matched for every input-output pair.

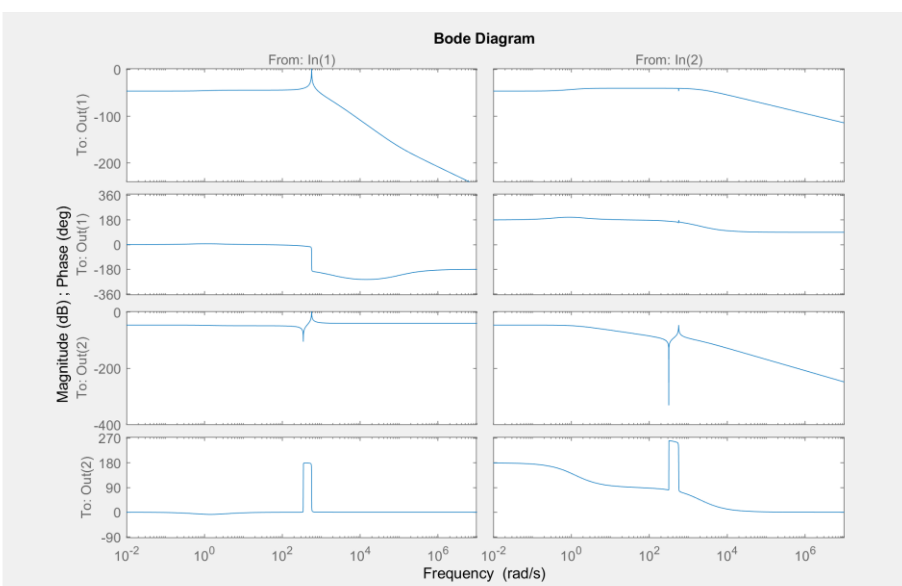

Figure 7: MATLAB Model Bode Plot

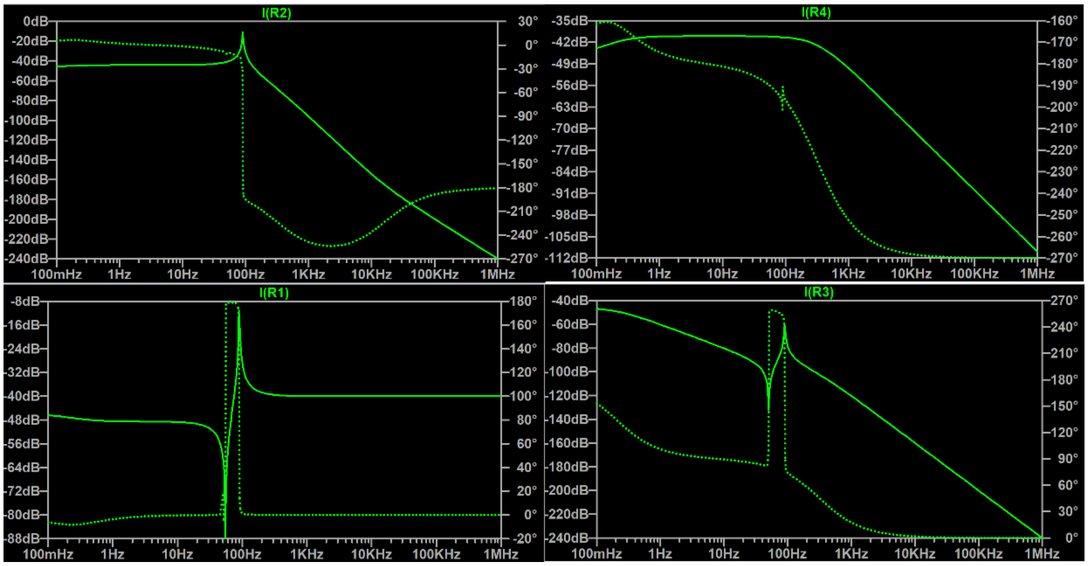

Figure 8: LTSpice Bode Plot

The P22 system can be represented in various ways, one way is the Smith-McMillan form. The Smith-McMillan form diagonalizes the system with the transform UGV = S/d, where d is the denominator of G. The S here for our system is as seen in Figure 9.

```
/                              1                                                     \
| -------------------------------------------------------------,              0       |
|  4               3               2                                                  |
| s   + 2.0020e+03 s   + 3.2800e+05 s   + 6.4020e+08 s + 8.0000e+08                   |
|                                                                                     |
|                                                              2                      |
\                               0,                            s   + 1.0000e+05 s + 1.0000e+05 /
```

Figure 9: Smith-McMillan Form

Coprime factorization is a representation of the system such that G = $N_R D_R^{-1} = D_L^{-1} N_L$, and D and N are coprime to each other. Using the MATLAB rncf, we can obtain N_R to be as seen in Figure 10, and D_R to be as seen in Figure 11.

```
From input 1 to output...
              39.965 (s+1.001e05) (s+1998) (s+1.25) (s+1) (s^2 + 2.651s + 3.202e05)
 1:  -------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

     0.0099995 (s+1998)^2 (s+1.668) (s+1.25) (s^2 + 0.3242s + 1.2e05) (s^2 + 2.651s + 3.202e05)
 2:  ------------------------------------------------------------------------------------------
      (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

From input 2 to output...
        -20 (s+1998) (s+1.25) (s+0.625) (s^2 + 4.303s + 3.2e05) (s^2 + 2.651s + 3.202e05)
 1:  -------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

     1.6268e-05 (s-2.459e06) (s+1998) (s+1.25) (s^2 - 0.301s + 1e05) (s^2 + 2.651s + 3.202e05)
 2:  ---------------------------------------------------------------------------------------
      (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)
```

Figure 10: Nr

```
From input 1 to output...
               (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 2.651s + 3.202e05)^2
 1:  ---------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

     0.0016269 (s+1998) (s+1.25) (s+1.062) (s^2 + 2.651s + 3.202e05) (s^2 + 4149s + 1.635e07)
 2:  ----------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

From input 2 to output...
     0.0016268 (s+1998) (s+1.25) (s+0.8769) (s^2 + 2.651s + 3.202e05) (s^2 + 101.4s + 8.24e06)
 1:  ------------------------------------------------------------------------------------------
      (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

           (s+1998)^2 (s+1.25)^2 (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)
 2:  ---------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)
```

Figure 11: Dr

Finally, we can also look at a stable coprime factorization. rncf also returns the combination of N_r and D_r and is guaranteed to be stable, as is seen in Figure 12. Note that this is using the full P22, not the minimized, hence the 8 states. Note that a coprime factorization is stable if N_r and D_r have no common unstable zeros, by observing the transmission zeros, as seen in Figure 13, we can see the only common zeros are in the left half plane, therefore this is a stable coprime factorization.

```
From input 1 to output...
               (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 2.651s + 3.202e05)^2
 1:  ---------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

     0.0016269 (s+1998) (s+1.25) (s+1.062) (s^2 + 2.651s + 3.202e05) (s^2 + 4149s + 1.635e07)
 2:  ----------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

          39.965 (s+1.001e05) (s+1998) (s+1.25) (s+1) (s^2 + 2.651s + 3.202e05)
 3:  ---------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

     0.0099995 (s+1998)^2 (s+1.668) (s+1.25) (s^2 + 0.3242s + 1.2e05) (s^2 + 2.651s + 3.202e05)
 4:  ------------------------------------------------------------------------------------------
      (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

From input 2 to output...
     0.0016268 (s+1998) (s+1.25) (s+0.8769) (s^2 + 2.651s + 3.202e05) (s^2 + 101.4s + 8.24e06)
 1:  ------------------------------------------------------------------------------------------
      (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

           (s+1998)^2 (s+1.25)^2 (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)
 2:  ---------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

        -20 (s+1998) (s+1.25) (s+0.625) (s^2 + 4.303s + 3.2e05) (s^2 + 2.651s + 3.202e05)
 3:  ---------------------------------------------------------------------------------------
     (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)

     1.6268e-05 (s-2.459e06) (s+1998) (s+1.25) (s^2 - 0.301s + 1e05) (s^2 + 2.651s + 3.202e05)
 4:  ------------------------------------------------------------------------------------------
      (s+1998) (s+1998) (s+1.25) (s+1.25) (s^2 + 5.579s + 3.202e05) (s^2 + 2.651s + 3.202e05)
```

Figure 12: Stable Coprime Realization

```
>> tzero(Dr)

ans =

    -1.2504 +         0i
    -1.3256 +    565.86i
    -1.3256 -    565.86i
    -1998.1 +         0i
    -1998.1 +         0i
    -1.3256 +    565.86i
    -1.3256 -    565.86i
    -1.2504 +         0i
```

```
>> tzero(Nr)

ans =

       -1998.1 +         0i
       -1.3256 +    565.86i
       -1.3256 -    565.86i
    1.7565e-13 +    346.41i
    1.7565e-13 -    346.41i
    -1.361e-14 +         0i
       -1.2504 +         0i
```

Figure 13: Dr and Nr Transmission Zeros

Modeling P

P contains two inputs (w and u) and two outputs (z and v). W is the exogenous input, and contains r, n_1, n_2 and d. U is the control input, it contains control signals u_1 and u_2. Z is the performance output, it contains error in y_1 (y_1-r), and scaled control signals ($\epsilon_1 u_1$ and $\epsilon_2 u_2$). Finally V is the controller input, it contains error in y_1 with noise (y_1-r+n_1), and y_2 with noise (y_2+n_2). P is of the form seen in Figure 14, where A, B2, C2, and D22, are knows from P22. B1, C1, D11, D12, and D21 are filled in by matching the above equations to the correct matrices.

A	B_1	B_2
C_1	D_{11}	D_{12}
C_2	D_{21}	D_{22}

Figure 14: P State Space Structure

To solve B1, we can note that d is the in the same location as u_1, and thus will share the same dynamics of u_1. Note that r, n_1 and n_2 have no effect on the B1 matrix, B1 can be seen below in Figure 15. Note that the d column here matches the u_1 column seen in B2 in Figure 6.

```
B1 =

         0         0         0    0.1104
         0         0         0   -0.3963
         0         0         0   -0.9056
         0         0         0    0.1608
```

Figure 15: B1

To solve C1, note that the structure of z contains y_1 in the first element. Similar to B1, we can reuse the y_1 element in C2 in constructing C1. Note that z does not contain any other elements pertaining to C, thus the C1 matrix is as seen in Figure 16. Note that the first row is the same as the first row from C1 as seen in Figure 6.

```
C1 =

   -5.0296    1.8649   -1.4667   -0.2123
         0         0         0         0
         0         0         0         0
```

Figure 16: C1

To solve D11, we observe the effects of w in z. The only w term in z is a '-r.' Thus the D11 matrix is as seen below in Figure 17.

$$D11 = \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Figure 17: D11

To solve D12, we observe the effects of u in z. There are two u terms in z ($\epsilon_1 u_1$ and $\epsilon_2 u_2$), thus the D12 matrix is as seen below in Figure 18, this is when ϵ_1 and ϵ_2 are evaluated to be 1.

$$D12 = \begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}$$

Figure 18: D12

To solve D21, we observe the effects of w in v. Note that v has a '$-r+n_1$' in the first term and a 'n_1' in the second term. Thus the D21 matrix is as seen in Figure 19.

$$D21 = \begin{matrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

Figure 19: D21

With all the parts derived, the overall P matrix can be constructed as laid out in Figure 14. The overall P matrix is as seen below in Figure 20.

A =

	x1	x2	x3	x4
x1	-1838	-62.24	-574.9	27.14
x2	345.8	126.4	-1108	110.4
x3	-473.2	271.9	-278.3	-22.6
x4	75.24	-63.16	201.8	-11.9

B =

	u1	u2	u3	u4	u5	u6
x1	0	0	0	0.1104	0.1104	3.669
x2	0	0	0	-0.3963	-0.3963	-0.02131
x3	0	0	0	-0.9056	-0.9056	1.035
x4	0	0	0	0.1608	0.1608	-0.06106

C =

	x1	x2	x3	x4
y1	-5.03	1.865	-1.467	-0.2123
y2	0	0	0	0
y3	0	0	0	0
y4	-5.03	1.865	-1.467	-0.2123
y5	-0.2545	-1.947	0.8712	0.1598

D =

	u1	u2	u3	u4	u5	u6
y1	-1	0	0	0	0	0
y2	0	0	0	0	1	0
y3	0	0	0	0	0	1
y4	-1	1	0	0	0	0
y5	0	0	1	0	0.01	0

Figure 20: P State Space Model

Sensor & Actuator Placement

To determine the best sensor placement (y_1 or y_2) and the best actuator placement (u_1 or u_2), the controllability and observability of the system will be examined. The singular values of the gramians of the SISIO systems are seen below in Table 1. States three and four are both weakly observable and controllable and will be ignored. States one and two are easily observable across the board, however, u_1 has provides considerably more controllability.

	u1->y1	u1->y2	u2->y1	u2->y2
Observability	2.8109	3.0356	2.8109	3.0356
	0.6291	0.6778	0.6291	0.6778
	0.0069	0.0004	0.0069	0.0004
	0.0004	0	0.0004	0
Controllability	0.6552	0.6552	0.014	0.014
	0.1476	0.1476	0.0036	0.0036
	0.0009	0.0009	0	0
	0	0	0	0

Table 1: Controllability and Observability Gramiams

Parametric Uncertainties

To isolate the parametric uncertainties of the plant, it is easiest to re-derive the plant in state space form. This is done by setting the states to be the currents through the inductors and the voltage across the capacitors, and using the physics definitions of each to relate them to each other (4) and (5). Using this method, the state space representation seen in Figure 21 is obtained. Using this method, the uncertainty can be pulled out and represented in the form seen in Figure 22. This results a system block diagram as seen in Figure 23. The B_Δ, C_Δ, and D_Δ matrices can be pulled out of the state space representation by separating the uncertainty into its own terms and setting Y_Δ and u_Δ equal to the parametric matrices where they show up in the dynamics. Using these the overall P matrix including uncertainty is as seen in Figure 24, Figure 25, Figure 26, and Figure 27. In this format the Δ matrix is diagonal in each uncertainty.

$$v = L\frac{di}{dt} \tag{4}$$

$$\mathrm{i} = \mathrm{C}\,\frac{\mathrm{dv}}{\mathrm{dt}} \tag{5}$$

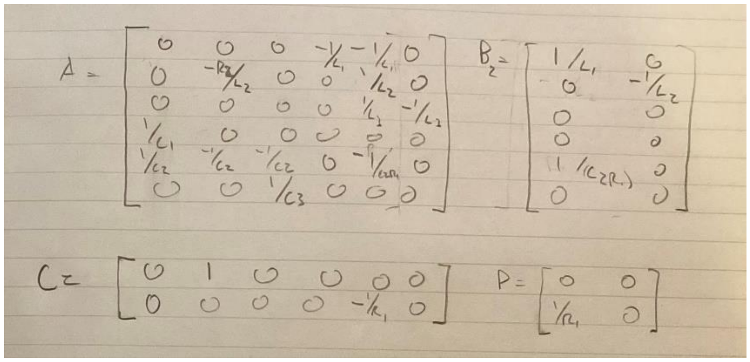

Figure 21: State Space Derived Representation

A	B_Δ	B_1	B_2
C_Δ	$D_{\Delta\Delta}$	$D_{\Delta 1}$	$D_{\Delta 2}$
C_1	$D_{1\Delta}$	D_{11}	D_{12}
C_2	$D_{2\Delta}$	D_{21}	D_{22}

Figure 22: State Space Format With Uncertainty

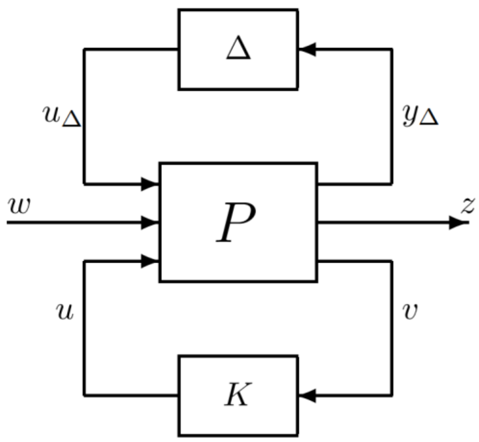

Figure 23: Uncertainty P Block Diagram

A =

	x1	x2	x3	x4	x5	x6
x1	0	0	0	-1000	-1000	0
x2	0	-2000	0	0	20	0
x3	0	0	0	0	100	-100
x4	100	0	0	0	0	0
x5	200	-200	-200	0	-2	0
x6	0	0	1000	0	0	0

Figure 24: State Space Derived A

B =

	u1	u2	u3	u4	u5	u6	u7	u8
x1	0	-1414	0	0	0	1000	1000	0
x2	0	0	0	0	0	0	0	-20
x3	0	0	0	0	0	0	0	0
x4	0	0	0	0	0	0	0	0
x5	-263.2	0	0	0	0	2	2	0
x6	0	0	0	0	0	0	0	0

BΔ

Figure 25: B Delta

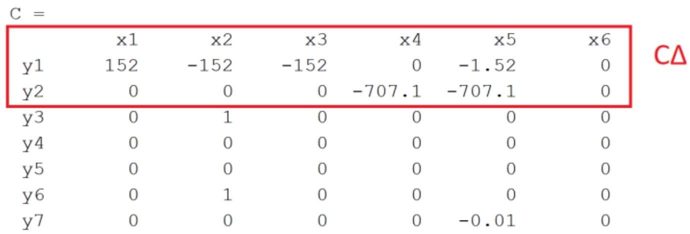

Figure 26: C Delta

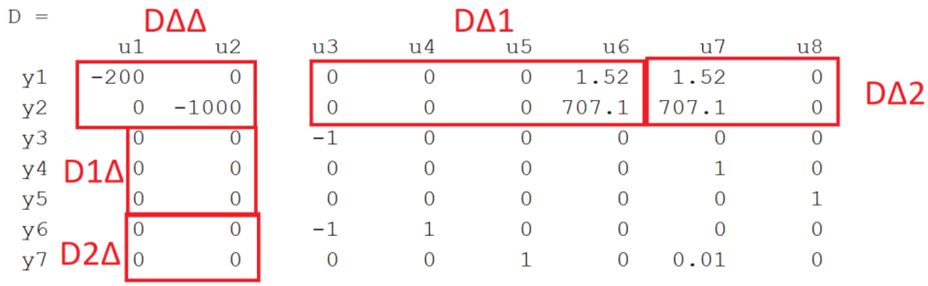

Figure 27: D Delta

P22 Poles and Transmission Zeros

For a MIMO system, the poles of the entire system are the same. Thus, the poles of P22 are the same as the poles for each individual SISO system. Transmission zeros are zeros that result in a zero output across all outputs of the system. Going from the Smith-McMillan form, as seen in Figure 9, we should have four poles and two transmission zeros in our minimized system. Calculating them we achieve four poles, but three transmission zeros, as seen in Table 2. The poles match the SISO Poles, however the transmission zeros do not match the SISO zeros. The transmission zeros match the zeros of G21, and nothing else.

P22 Poles			P22 Transmission Zeros		
-1998.1	+	0i	2.42E-13	+	346.41i
-1.3256	+	565.86i	2.42E-13	-	346.41i
-1.3256	-	565.86i	7.23E-14	+	0i
-1.2504	+	0i			

Table 2: P22 Poles and Transmission Zeros

All Stabilizing Controllers

Given a plant, it is possible to obtain a generalized formula to obtain all possible stabilizing controllers. Take any Q matrix, the stabilizing controller is as in equation (6). F_l is a lower linear fractional transformation on $\overline{K}_0$ and Q. $\overline{K}_0$ can be obtained as seen in (7). L and F are obtained as is seen in (8) and (9). M_r, N_r, M_l, and N_l are the right and left coprime factorizations of G. Y_r, X_r, Y_l, and X_l are obtained as seen in (10). Using the provided Q factorization function, if we supply a Q of identity, the stabilizing controller K for the minimum realization of P22 is as seen in Figure 28.

$$K = F_l(\overline{K}_0, Q)$$

(6)

$$\bar{K}_0 = \left[\begin{array}{c|c} A+BF+LC+LDF & \begin{bmatrix} -L & B+LD \end{bmatrix} \\ \hline \begin{bmatrix} F \\ -(C+DF) \end{bmatrix} & \begin{bmatrix} 0 & I \\ I & -D \end{bmatrix} \end{array}\right]$$

(7)

$$\begin{bmatrix} M_r & Y_r \\ N_r & X_r \end{bmatrix} = \left[\begin{array}{c|c} A+BF & \begin{bmatrix} B & -L \end{bmatrix} \\ \hline \begin{bmatrix} F \\ C+DF \end{bmatrix} & \begin{bmatrix} I & 0 \\ D & I \end{bmatrix} \end{array}\right]$$

(8)

$$\begin{bmatrix} X_\ell & Y_\ell \\ -N_\ell & M_\ell \end{bmatrix} = \left[\begin{array}{c|c} A+LC & \begin{bmatrix} -(B+LD) & L \end{bmatrix} \\ \hline \begin{bmatrix} F \\ C \end{bmatrix} & \begin{bmatrix} I & 0 \\ -D & I \end{bmatrix} \end{array}\right]$$

(9)

$$\begin{bmatrix} X_\ell & Y_\ell \\ -N_\ell & M_\ell \end{bmatrix} \begin{bmatrix} M_r & Y_r \\ N_r & X_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

(10)

```
K =

  A =
           x1       x2       x3       x4
   x1   -1838   -55.05     -578    26.52
   x2   353.2    119.9    -1105    111.2
   x3  -475.7    275.3   -283.1    -22.6
   x4   74.54    -62.6      202   -12.01

  B =
            u1        u2
   x1  -0.0209     3.651
   x2    1.503    -1.906
   x3   -0.706     1.053
   x4  -0.1208    0.1761

  C =
            x1        x2        x3        x4
   y1    4.156    -1.892     4.543   -0.1421
   y2   0.1887     1.978   -0.9271   -0.1596

  D =
          u1     u2
   y1      1      0
   y2  -0.01      1
```

Figure 28: Stabilizing K for Identity Q

Project 1 Optimization

H_2 and H_{inf} norms are very powerful in evaluating and optimizing control systems. The H_2 norm generally tells how good the tracking is. The H_{inf} norm generally tells how robust the system is. To optimize the project 1 controller on the H_2 and H_{inf} norm, gradient descent on H_2 and H_{inf} could be used. The parameters to be tuned could be location and depth of the notch filter, and P, I, and D terms. A potential loss function could be (11), where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are weights on the norms. I would keep $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, relatively close together, perhaps 0.6 and 0.4 respectively to slightly prioritize tracking over robustness. The gradient of each parameter would be obtained with respect to the loss function, then parameter updates can occur using the gradients, an example of updating P can be seen in (12). This method is essentially a boiled down, single layer neural network to train the controller parameters. If the controller becomes larger, this can be expanded to multiple layers or to contain more parameters as needed. Extra terms can also be added to the loss function to incorporate various norms: the norm of d to y, or n to y. This would add disturbance, and noise terms to the loss function such that it could optimize around those too if desired.

$$Loss = \alpha H_2 + \beta H_{inf} \tag{11}$$

$$P_t = P_{t-1} - \eta \frac{\partial P}{\partial Loss} \tag{12}$$

Project 1 Evaluation

Using a simple H_2 and H_{inf} norm on the closed loop transfer function we obtain the H_2 norm to be 18.513, and the H_{inf} norm to be 2.0111 for the disturbance rejection controller. This tells us that the system is relatively robust overall. If we were using the loss function as described in (11), our loss would be 11.912. If we look at the H_2 and H_{inf} norm of the disturbance to output transfer function, we get closed loop values of 5.2082 and 38.8791 respectively. This tells us that the controller is not very robust to noise. This makes sense as when introducing disturbance into the system, it did not behave nearly as well as without it.

Robustness Evaluation

Using the K controller seen in Figure 28, the SISO Nyquist plots can be seen below in Figure 29. We can see that in general all four SISO systems are generally robust, not getting close to the -1 point. The u1 controllers are slightly less robust in phase, and the y_2 controllers are slightly less robust in magnitude. However in general all of these SISO controllers would need a great deal of phase and gain uncertainty to become unstable.

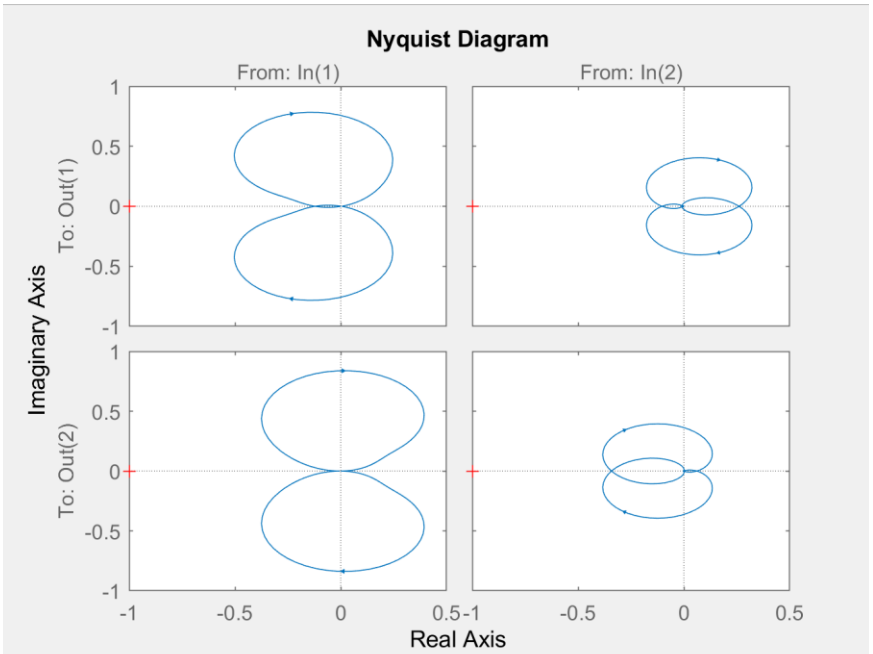

Figure 29: SISO Nyquist Plots

Robustness can also be observed using a MIMO Nyquist plot, as seen in Figure 30. The multivariable Nyquist also does not approach the -1 point. Also implying that this controller is robust.

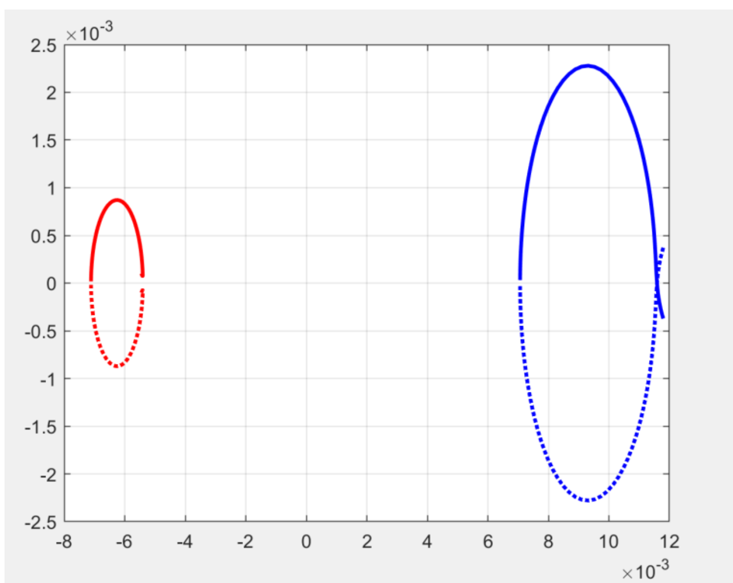

Figure 30: Multivariable Nyquist

Parametric Robustness Evaluation

Small Gain Theorem

The small gain theorem states that the system has robust stability if the H infinity norm of the system times the norm of the uncertainty is less than 1. The norm of the closed loop system with the stabilizing controller seen in Figure 28 is 2.9647. The norm of the uncertainties would be the maximum uncertainties. Thus this system is robustly stable for all uncertainties of less than 33%. This would likely mean that the system in general is robustly stable as the common uncertainty for electrical components is ±20%.

Passivity Theorem

The Passivity Theorem states that the overall system is stable if the system and uncertainty are strictly positive real. This system fails the passivity test. This is derived using the isPassive MATLAB command on the closed loop system.

Kharitonov's Theorem

The closed loop characteristic equation can be written with uncertainty terms included. This the characteristic equation of said closed loop would have coefficients with uncertainty. Kharitonov states that if four key equations, as seen in Figure 31, are Hurwitz, then the system is stable. Thus, to evaluate stability under uncertainty, one would update the coefficients respectively and calculate the results.

$$
\begin{aligned}
\pi_1(s) &= a_n^- s^n + a_{n-1}^- s^{n-1} + a_{n-2}^+ s^{n-2} + a_{n-3}^+ s^{n-3} + \ldots \quad (--++) \\
\pi_2(s) &= a_n^+ s^n + a_{n-1}^+ s^{n-1} + a_{n-2}^- s^{n-2} + a_{n-3}^- s^{n-3} + \ldots \quad (++--) \\
\pi_3(s) &= a_n^+ s^n + a_{n-1}^- s^{n-1} + a_{n-2}^- s^{n-2} + a_{n-3}^+ s^{n-3} + \ldots \quad (+--+) \\
\pi_4(s) &= a_n^- s^n + a_{n-1}^+ s^{n-1} + a_{n-2}^+ s^{n-2} + a_{n-3}^- s^{n-3} + \ldots \quad (-++-)
\end{aligned}
$$

Figure 31: Kharitonov's Equations

The closed loop system's characteristic equation evaluates to Figure 32. Substituting maximum and minimum values of 1.2 and 0.8 for k1 and k2 yields us with all positive roots for the Kharitonov equations, which would imply that the system is not stable with 20% uncertainty. This proves contrary to the conclusion from the small gain theorem. I believe there is an error in my Kharitonov implementation.

```
 14                          13
s   + (0.2353 k2 + 6.0165e+03) s   + (960.2610 k2 - 30.8718 k1 - 1.5770 k1 k2 + 1.3170e+07)

   12                                                                      11
  s   + (3.8474e+06 k1 + 1.1866e+06 k2 - 4.5895e+04 k1 k2 + 1.4621e+10) s   + (1.5775e+10 k1 + 7.4885e+08 k2 - 1.6613e+08 k1 k2 + 1.3405e+13)

   10                                                                      9
  s   + (1.8920e+13 k1 + 7.0658e+11 k2 - 1.9353e+11 k1 k2 + 1.1116e+16) s  + (1.1918e+16 k1 + 1.5528e+14 k2 - 1.2378e+14 k1 k2 + 5.0777e+18)

  8                                                                      7
  s  + (1.2515e+19 k1 + 1.5078e+17 k2 - 1.2606e+17 k1 k2 + 3.6699e+21) s  + (2.7441e+21 k1 - 2.5825e+18 k2 - 2.8253e+19 k1 k2 + 8.0661e+23)

  6                                                                      5
  s  + (2.7059e+24 k1 + 1.3744e+22 k2 - 2.7164e+22 k1 k2 + 5.3500e+26) s  + (1.7673e+26 k1 - 2.7904e+24 k2 - 1.8109e+24 k1 k2 + 4.2971e+28)

  4                                                                      3
  s  + (1.6398e+29 k1 + 4.5277e+26 k2 - 1.6449e+27 k1 k2 + 2.6516e+31) s  + (5.7252e+29 k1 - 1.6322e+29 k2 - 5.7485e+27 k1 k2 + 9.8865e+31)

  2
  s  + (6.6396e+29 k1 - 4.1030e+29 k2 - 6.6746e+27 k1 k2 + 1.2320e+32) s + 2.5522e+29 k1 - 2.5696e+29 k2 - 2.5696e+27 k1 k2 + 5.1207e+31
```

Figure 32: Evaluated Characteristic Equation

Conclusion and Future Directions

This project serves as a foundation for MIMO analysis. These tools will be useful in the development of MIMO control systems. Modeling uncertainty, performance outputs, exogenous inputs were all used to gain an accurate model of the plant. This plant was represented in various ways, including coprime factorization and Smith-McMillan form. Norms were explored as methods to gauge tracking and robustness, as well as optimization of controllers. Finally MIMO robustness was explored using various methods.

References

[1] "Control System Toolbox," Control System Toolbox Documentation, https://www.mathworks.com/help/control/ (accessed Feb. 4, 2024).

[2] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*. Hoboken, NJ: John Wiley, 2005.

Appendix

GitHub containing MATLAB code:

https://github.com/robertdabney/mvar_control/tree/main/mini_project_2