UNIVERSITY OF TENNESSEE
DEPARTMENT OF MATHEMATICS

MATH 447 - Homework 5

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SECTION 3.1

PROBLEM 10

Prove that if $\lim(x_n) = 0$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

PROOF. By the definition of limit, for each positive ϵ such an $M(\epsilon)$ can be found. Thus, take ϵ to be smaller than x, so that $x_n \in V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ whenever $n \geq M$. Since $\epsilon < x$, $x - \epsilon \in \mathbb{P}$ so $x_n > 0$.■

PROBLEM 14

Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$.

PROOF. We know that $n^{1/n} > 1 \; \forall n \in \mathbb{N}$. This implies that $n^{1/n} = 1 + k_n$ for some $k_n > 0$, so that we have $bn^{1/n} = b + bk_n$. Thus, $nb^n = (b + bk_n)^n$.

By the Binomial Theorem we know that

$$nb^n = (b+bk_n)^n = b^n + nb^{n-1}bk_n + \frac{1}{2}n(n-1)b^{n-2}b^2{k_n}^2 + \cdots$$

which tells us that $nb^n \geq b^n + \frac{1}{2}n(n-1)b^n{k_n}^2$. Factoring out b^n and subtracting 1 yields

$$(n-1) \geq \frac{1}{2}n(n-1){k_n}^2$$

so that we have $\frac{2}{n} \geq {k_n}^2$.

Problem 17

Hint

If $n \geq 3$, then $0 < 2^n / n! \leq 2(\frac{2}{3})^{n-2}$.

Proof. Let $n = 3$, then $\frac{2^3}{3!} = \frac{8}{6} \leq 2(\frac{2}{3}) = \frac{4}{3} = \frac{8}{6}$. Thus, we have established a basis for induction.

Now assume that $0 < \frac{2^{n-1}}{(n-1)!} \leq 2\left(\frac{2}{3}\right)^{n-3}$. Multiplying by 2 yields $0 < \frac{2^n}{(n-1)!} \leq 2\left(\frac{2^{n-2}}{3^{n-3}}\right)$. Since $n \geq 3$, we have $0 < \frac{1}{n} \leq \frac{1}{3}$ so that

$$0 < \frac{2^n}{(n-1)!} \cdot \frac{1}{n} = \frac{2^n}{n!} \leq 2 \cdot \left(\frac{2^{n-2}}{3^{n-3}}\right) \cdot \frac{1}{3} = 2\left(\frac{2}{3}\right)^{n-2}$$

whence $0 < \frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$. Therefore the hypothesis holds by the Principal of Mathematical Inducation. ■

Main Problem

Show that $\lim(2^n / n!) = 0$.

Proof. We know that $0 < \frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$ when $n \geq 3$, so if we can show that $2\left(\frac{2}{3}\right)^{n-2} \to 0$, then $\frac{2^n}{n!} \to 0$ by the Squeeze Theorem.

Take $b = 2/3$ and $n = m-2$, then by Exercise 3.1.14 we know $nb^n \to 0$. Since $2 \leq m-2\ \forall m > 3$, we have that $0 < 2(\frac{2}{3})^{m-2} \leq (m-2)b^{m-2}$ so that $2\left(\frac{2}{3}\right)^{m-2} \to 0$. Thus, by the Squeeze Theorem, $2^n / n! \to 0$. ■

Problem 18

If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

Proof. Pick $\epsilon < \frac{1}{2}x$, then by the definition of limit $\exists K$ such that $x_n \in V_\epsilon(x) \forall n \geq K$. Since $V_\epsilon(x) = (x - x/2, x + x/2) = (x/2, 3x/2)$, we know $V_\epsilon(x) \subseteq (x/2, 2x)$, demonstrating that such a K exists. ■

Section 3.2

Problem 7

If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Thm 3.2.3 cannot be used.

PROOF.

PROBLEM 9

Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}\, y_n)$ converges. Find the limit.

PROOF.

PROBLEM 11

PART A

Find $\lim\left((3\sqrt{n})^{1/2n}\right)$

PROOF.

PART B

Find $(\sqrt{n^2+5n} - n)$

PROOF.

PROBLEM 12

If $0 < a < b$, determine $\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.

PROOF.

PROBLEM 13

If $a > 0$, $b > 0$, show that $\lim\left(\sqrt{(n+a)(n+b)} - n\right) = (a+b)/2$.

PROOF.

PROBLEM 14

PART A

Use the Squeeze Theorem to find $\lim(n^{1/n^2})$.

PROOF.

PART B

Use the Squeeze Theorem to find $\lim((n!)^{1/n^2})$.

PROOF.

PROBLEM 15

Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.

PROOF.

PROBLEM 23

Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent.

PROOF.

PROBLEM 24

Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \operatorname{mid}\{x_n, y_n, z_n\}$ is also convergent.

PROOF.