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# MATH 447 - Homework 4

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## 1 Uncountability of the Real Numbers

The set $\mathbb{R}$ of real numbers is uncountable.

Proof. Assume the set $I = [0,1]$ is countable. This implies its elements are enumerable. That is, $I = \{x_1, x_2, \ldots, x_n, \ldots\}$.

Now construct the set $I_1 = \{x \in I | x \neq x_1\}$ so that $I_1 \subset I$ and $I_1 = I \setminus \{x_1\}$. Clearly this construction can progress in a recursive fashion so that $I_j = I_{j-1} \setminus \{x_j\}$, and $I_j \supseteq I_{j+1} \supseteq \cdots$.

Thus, we have establshed a collection of nested intervals, so we know $\exists \xi \in I_n \forall n \in \mathbb{N}$. Suppose $\xi = x_k$ for some $k \in \mathbb{N}$, then $\xi \notin I_k$, which contradicts the Nested Intervals Property. Clearly this is absurd, so we refute our hypothesis that $I = [0,1]$ is countable.

Since $[0,1] \subseteq \mathbb{R}$ is uncountable, we know that $\mathbb{R}$ itself is uncountable.■

## 2 Binary representation of Real Numbers

If $x \in [0,1]$, then there exists a sequence $(a_n)$ of 0s and 1s such that

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} \leq x \leq \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n + 1}{2^n} \tag{2.1}$$

for all $n \in \mathbb{N}$. Conversely, each sequence of 0s and 1s is the binary representation of a unique real number in $[0,1]$.

PROOF. We begin by discussing an algorithm for constructing a sequence $(a_n)$ based on the choice $x$. We then demonstrate that the upper and lower bounds given above form a nested sequence of intervals, and since we know (by the Nested Intervals Property) that the intersection of such objects is nonempty, we can be confident that such a construction for $x$ is valid.

If $x \neq \frac{1}{2}$ belongs to the left subinterval $[0, \frac{1}{2}]$ we take $a_1 = 0$, while if $x$ belongs to the right subinterval $[\frac{1}{2}, 1]$ we take $a_1 = 1$. If $x = \frac{1}{2}$, then we may take $a_1$ to be either 0 or 1. In any case we have

$$\frac{a_1}{2} \leq x \leq \frac{a_1}{2} + \frac{a_2+1}{2^2}. \tag{2.2}$$

We continue this bisection procedure, assigning at the $n$th stage the value $a_n = 0$ if $x$ is not the bisection point and lies in the left subinterval, and assigning the value $a_n = 1$ if $x$ lies in the right subinterval. Thus, we have a well-defined sequence $(a_n)$ of 0s and 1s such that Eqn 2.1 above holds.

Now we proceed to show that the upper and lower bounds given in Eqn 2.1 form a sequence of nested intervals. Let $L_n$ be the $n$th lower bound and $U_n$ the $n$th upper bound so that

$$L_n := \sum_1^n \frac{a_i}{2^i} \qquad U_n := \sum_1^n \frac{a_i}{2^i} + \frac{1}{2^n}$$
$$L_{n-1} := \sum_1^{n-1} \frac{a_i}{2^i} \qquad U_{n-1} := \sum_1^{n-1} \frac{a_i}{2^i} + \frac{1}{2^{n-1}}$$

which yields the following useful relationships

$$\begin{aligned} L_n &= L_{n-1} + \frac{a_n}{2^n} \\ U_n &= U_{n-1} - \frac{1}{2^{n-1}} + \frac{a_n}{2^n} + \frac{1}{2^n} \end{aligned}$$

We know that each $a_n$ is either 0 or 1, so we have two cases. If $a_n = 0$, then $\frac{a_n}{2^n} = 0$ so that $L_n = L_{n-1}$ and $U_n = U_{n-1} - \frac{1}{2^n} < U_{n-1}$. If $a_n = 1$, then $\frac{a_n}{2^n} = \frac{1}{2^n}$ so that $L_n = L_{n-1} + \frac{1}{2^n} > L_{n-1}$ and $U_n = U_{n-1} - \frac{1}{2^{n-1}} + \frac{2}{2^n} = U_{n-1}$. Thus, we have that $L_n \geq L_{n-1}$ and $U_n \leq U_{n-1}$. Therefore, for each $n \in \mathbb{N}$, we have

$$L_{n-1} \leq L_n < U_n \leq U_{n-1} \tag{2.3}$$

so that the interval $[L_{n-1}, U_{n-1}]$ contains the interval $[L_n, U_n]$.■

## 3 Some problems concerning Intervals

### Exercise 2.5.3

If $S \subseteq \mathbb{R}$ is a nonempty bounded set, and $I_S := [\inf S, \sup S]$, then $S \subseteq I_S$. Moreover, if $J$ is any closed bounded interval containing $S$, then $I_S \subseteq J$.

Proof. Since $\inf S \leq s \forall s \in S$, and $\sup S \geq s \forall s \in S$, we know

$$\inf S \leq s \leq \sup S \forall s \in S \tag{3.1}$$

Thus, $s \in [\inf S, \sup S] \forall s \in S$, so $S \subseteq I_S$.

Further, since $J$ is a closed bounded interval containing $S$ by hypothesis, we know that $\inf J \leq s \forall s \in S$ so that $\inf J$ is a lower bound for $S$, and we know that $\sup J \geq s \forall s \in S$ so that $\sup J$ is an upper bound for $S$. By definition of the infimum and supremum of $S$ we have that $\inf S \geq \inf J$ and $\sup S \leq \sup J$. Thus

$$\inf J \leq \inf S \leq \sup S \leq \sup J \tag{3.2}$$

which confirms that $I_S \subseteq J$. ■

### Exercise 2.5.10

In the context of the proofs of Theorems 2.5.2 and 2.5.3, we have $\eta \in \cap_{n=1}^{\infty} I_n$. Also, $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$.

Proof. Suppose $\eta \notin \cap_{n=1}^{\infty} I_n$. Then $\exists m | \eta \notin I_m$. By definition, $\eta \leq b_m$, so we must have that $\eta \leq a_m$. But since

$$a_m \leq b_k \forall k \in \mathbb{N} \tag{3.3}$$

then $a_m$ is a greater lower bound for $\{b_k | k \in \mathbb{N}\}$ than is $\eta$. This contradicts our hypothesis that $\eta = \inf\{b_n | n \in \mathbb{N}\}$. Thus, $\eta \in \cap_{n=1}^{\infty} I_n$.

We now show that $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$. We begin by showing $[\xi, \eta] \subseteq \cap_{n=1}^{\infty} I_n$. Take $x \in [\xi, \eta]$, then $x \geq a_n \forall n$, since $x \geq \xi = \sup\{a_n | n \in \mathbb{N}\}$. Also, $x \leq b_n \forall n \in \mathbb{N}$, since $x \leq \eta = \inf\{b_n | n \in \mathbb{N}\}$. Thus,

$$a_n \leq x \leq b_n \tag{3.4}$$

without regard to the choice of $n$. Thus, $x \in I_n \forall n \in \mathbb{N}$, so $x \in \cap_{n=1}^{\infty} I_n$. Thus, $[\xi, \eta] \subseteq \cap_{n=1}^{\infty} I_n$. In order to establish that $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$, we are now obliged to show that $\cap_{n=1}^{\infty} I_n \subseteq [\xi, \eta]$. Take $x \in \cap_{n=1}^{\infty} I_n$, we know that $x \in [a_n, b_n] \forall n \in \mathbb{N}$, which means

$$a_n \le x \le b_n \forall n \in \mathbb{N} \tag{3.5}$$

Thus $x$ is an upper bound for the set $\{a_n | n \in \mathbb{N}\}$ and a lower bound for $\{b_n | n \in \mathbb{N}\}$. This tells us that

$$a_n \le \xi \le x \le \eta \le b_n \forall n \in \mathbb{N} \tag{3.6}$$

so that $x \in [\xi, \eta]$. ∎

# 4 Some problems concerning Sequences and Their Limits

### Exercise 3.1.4

For any $b \in \mathbb{R}$, $\lim(b/n) = 0$.

Proof. Consider first that, $\forall n \in \mathbb{N}$, $|b/n| \le |b/n|$, and since $n > 0$, we can even say that $|b/n - 0| \le |b| \cdot 1/n$. We can leverage the fact that $\lim(1/n) = 0$ together with Theorem 3.1.10 (by taking $C = |b|$ and $m = 1$) to conclude that $\lim(b/n) = 0$ for any $b \in \mathbb{R}$. ∎

### Exercise 3.1.5

Use the definition of the limit of a sequence to establish the following limits:

#### Part a

$\lim(\frac{n}{n^2+1}) = 0$

Proof. Choose $\epsilon > 0$, the by the Archimedean Property, $\exists K(\epsilon)$ such that $\frac{1}{K(\epsilon)} < \epsilon$. Clearly,

$$\frac{1}{K(\epsilon) + \frac{1}{K(\epsilon)}} < \frac{1}{K(\epsilon)} < \epsilon \tag{4.1}$$

Since $K(\epsilon) \in \mathbb{N}$ by definition, we know it is nonzero, so we may employ the identity $1 = K(\epsilon)/K(\epsilon)$ as follows

$$
\begin{aligned}
1 \cdot \frac{1}{K(\epsilon) + \frac{1}{K(\epsilon)}} &< \frac{1}{K(\epsilon)} < \epsilon \\
\left(\frac{K(\epsilon)}{K(\epsilon)}\right) \cdot \frac{1}{K(\epsilon) + \frac{1}{K(\epsilon)}} &< \frac{1}{K(\epsilon)} < \epsilon \\
\frac{K(\epsilon)}{(K(\epsilon))^2 + 1} &< \frac{1}{K(\epsilon)} < \epsilon \\
\frac{K(\epsilon)}{(K(\epsilon))^2 + 1} &< \epsilon
\end{aligned}
$$

We are guaranteed that $\frac{K(\epsilon)}{K(\epsilon)^2+1}$ is positive, so

$$
\begin{aligned}
\frac{K(\epsilon)}{(K(\epsilon))^2 + 1} &< \epsilon \\
\left|\frac{K(\epsilon)}{(K(\epsilon))^2 + 1} - 0\right| &< \epsilon
\end{aligned}
$$

Lastly, since $n/(n^2+1) < K(\epsilon)/(K(\epsilon)^2+1)$ when $n \geq K(\epsilon)$, we have that $\lim(\frac{n}{n^2+1}) = 0$. ■

## PART D

$\lim\left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$

PROOF. Given $\epsilon > 0$, we want to obtain the inequality

$$
\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| < \epsilon \tag{4.2}
$$

when $n$ is sufficiently large. We first simplify the expression on the left:

$$
\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{n^2 - n^2 - 3/2}{2n^2+3}\right| = \frac{3/2}{2n^2+3} = \frac{1}{\frac{4}{3}n^2 + 2} < \frac{1}{n}.
$$

Now if the inequality $1/n < \epsilon$ is satisfied, then the inequality 4.2 holds. Thus if $1/K < \epsilon$, then for any $n \geq K$, we also have $1/n < \epsilon$ and hence 4.2 holds. Therefore the limit of the sequence is $\frac{1}{2}$. ■

# EXERCISE 3.1.6

## PART C

$\lim\left(\frac{\sqrt{n}}{n+1}\right) = 0$

PROOF. Given $\epsilon > 0$, we want to obtain the inequality

$$\left|\frac{\sqrt{n}}{n+1} - 0\right| < \epsilon \tag{4.3}$$

when $n$ is sufficiently large. We note that by the Archimedean Property we have that $\exists t_\epsilon$ such that $1/t_\epsilon < \epsilon$. Thus, take $K(\epsilon) := t_\epsilon^2$ so that we have $1/\sqrt{K(\epsilon)} < \epsilon$. Thus,

$$\epsilon > \frac{1}{\sqrt{K(\epsilon)}} = \frac{\sqrt{K(\epsilon)}}{K(\epsilon)} > \frac{\sqrt{K(\epsilon)}}{K(\epsilon)+1}.$$

Lastly, since $\frac{\sqrt{K(\epsilon)}}{K(\epsilon)+1} > \frac{\sqrt{n}}{n+1}$ when $n > K(\epsilon)$, we have

$$\epsilon > \frac{\sqrt{n}}{n+1} = \left|\frac{\sqrt{n}}{n+1} - 0\right|$$

when $n > K(\epsilon)$. Thus, the limit is 0. ■

## PART D

$\lim\left(\frac{(-1)^n n}{n^2+1}\right) = 0$

PROOF. Given $\epsilon > 0$, we want to obtain the inequality

$$\left|\frac{(-1)^n n}{n^2+1} - 0\right| < \epsilon.$$

We know from Part A of Exercise 3.1.5 that $\lim\left(\frac{n}{n^2+1}\right) = 0$, and from that fact we can deduce the following:

$$\epsilon > \left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} = \left|(-1)^n \frac{n}{n^2+1}\right| = \left|\frac{(-1)^n n}{n^2+1} - 0\right|.$$

and thus the limit is also 0. ■

## EXERCISE 3.1.8

Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Provide an example to show that the convergence of $(|x_n|)$ need not imply the convergence of $(x_n)$.

PROOF. We shall demonstrate this by showing that each statement implies the other. Begin by assuming $\lim(x_n) = 0$, so that we have

$$\epsilon > |x_n - 0| = |x_n| = |x_n| - 0 = ||x_n| - 0|$$

so that we have $||x_n| - 0| < \epsilon$ which implies $\lim(|x_n|) = 0$.

Now we work the other direction. Assume that $\lim(|x_n|) = 0$, so then

$$\epsilon > ||x_n| - 0| = |x_n| - 0 = |x_n| = |x_n - 0|$$

which tells us $\lim(x_n) = 0$. Since each statement implies the other, we have demonstrated their logical equivalence. ■

EXAMPLE. Let $x_n := (-1)^n$. Then $|x_n| = 1$, so that the constant sequence $(|x_n|) = (1)$ converges. However, for any $\epsilon < 2$, there are infinitely many terms not satisfying $|x_n - L| < \epsilon$ for any proposed limit $L$.