UNIVERSITY OF TENNESSEE
DEPARTMENT OF MATHEMATICS

MATH 447 - Homework 5

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SECTION 3.1

PROBLEM 10

Prove that if $\lim(x_n) = 0$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

PROOF. By the definition of limit, for each positive ϵ such an $M(\epsilon)$ can be found. Thus, take ϵ to be smaller than x, so that $x_n \in V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ whenever $n \geq M$. Since $\epsilon < x$, $x - \epsilon \in \mathbb{P}$ so $x_n > 0$.■

PROBLEM 14

Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$.

PROOF. We know that $n^{1/n} > 1\ \forall n \in \mathbb{N}$. This implies that $n^{1/n} = 1 + k_n$ for some $k_n > 0$, so that we have $bn^{1/n} = b + bk_n$. Thus, $nb^n = (b + bk_n)^n$.

By the Binomial Theorem we know that

$$nb^n = (b+bk_n)^n = b^n + nb^{n-1}bk_n + \frac{1}{2}n(n-1)b^{n-2}b^2{k_n}^2 + \cdots$$

which tells us that $nb^n \geq b^n + \frac{1}{2}n(n-1)b^n{k_n}^2$. Factoring out b^n and subtracting 1 yields

$$(n-1) \geq \frac{1}{2}n(n-1){k_n}^2$$

so that we have $\frac{2}{n} \geq {k_n}^2$.

Problem 17

Hint

If $n \geq 3$, then $0 < 2^n/n! \leq 2(\frac{2}{3})^{n-2}$.

Proof. Let $n = 3$, then $\frac{2^3}{3!} = \frac{8}{6} \leq 2(\frac{2}{3}) = \frac{4}{3} = \frac{8}{6}$. Thus, we have established a basis for induction.

Now assume that $0 < \frac{2^{n-1}}{(n-1)!} \leq 2\left(\frac{2}{3}\right)^{n-3}$. Multiplying by 2 yields $0 < \frac{2^n}{(n-1)!} \leq 2\left(\frac{2^{n-2}}{3^{n-3}}\right)$. Since $n \geq 3$, we have $0 < \frac{1}{n} \leq \frac{1}{3}$ so that

$$0 < \frac{2^n}{(n-1)!} \cdot \frac{1}{n} = \frac{2^n}{n!} \leq 2 \cdot \left(\frac{2^{n-2}}{3^{n-3}}\right) \cdot \frac{1}{3} = 2\left(\frac{2}{3}\right)^{n-2}$$

whence $0 < \frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$. Therefore the hypothesis holds by the Principal of Mathematical Induction. ■

Main Problem

Show that $\lim(2^n/n!) = 0$.

Proof. We know that $0 < \frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$ when $n \geq 3$, so if we can show that $2\left(\frac{2}{3}\right)^{n-2} \to 0$, then $\frac{2^n}{n!} \to 0$ by the Squeeze Theorem.

Take $b = 2/3$ and $n = m-2$, then by Exercise 3.1.14 we know $nb^n \to 0$. Since $2 \leq m-2\ \forall m > 3$, we have that $0 < 2(\frac{2}{3})^{m-2} \leq (m-2)b^{m-2}$ so that $2\left(\frac{2}{3}\right)^{m-2} \to 0$. Thus, by the Squeeze Theorem, $2^n/n! \to 0$. ■

Problem 18

If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

Proof. Pick $\epsilon < \frac{1}{2}x$, then by the definition of limit $\exists K$ such that $x_n \in V_\epsilon(x) \forall n \geq K$. Since $V_\epsilon(x) = (x - x/2, x + x/2) = (x/2, 3x/2)$, we know $V_\epsilon(x) \subseteq (x/2, 2x)$, demonstrating that such a K exists. ■

Section 3.2

Problem 7

If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Thm 3.2.3 cannot be used.

REMARK. We may not apply Theorem 3.2.3 directly because it requires that (b_n) is convergent, but we know only that (b_n) is bounded. However, we may employ Theorem 3.2.3 for certain features of our argument.

PROOF. Since (b_n) is bounded, we have

$$I_b = \inf(b_n) \le b_j \le \sup(b_n) = S_b \ \forall j \in \mathbb{N}$$

which implies

$$I_b|a_j| = \inf(b_n)|a_j| \le b_j \cdot |a_j| \le \sup(b_n) \cdot |a_j| = S_b \cdot |a_j| \ \forall j \in \mathbb{N}$$

Thus if $\inf(b_n)|a_j| \to 0$ and $\sup(b_n)|a_j| \to 0$, we will be confident that $b_j|a_j| \to 0$.

Since $\inf(b_n)$ and $\sup(b_n)$ are constants, and since we know $|a_j| \to 0$ due to the fact that $a_j \to 0$, we have that $\inf(b_n)|a_j| \to 0$ and $\sup(b_n)|a_j| \to 0$ by Theorem 3.2.3. Thus, $b_j \cdot |a_j| \to 0$.

Therefore, given $\epsilon > 0$, we know $\exists K(\epsilon)$ such that $\left|b_j \cdot |a_j| - 0\right| < \epsilon \ \forall j \ge K(\epsilon)$.

What we want to show is that $|b_j \cdot a_j - 0| < \epsilon \ \forall j \ge M(\epsilon)$. By the above work, we know $|b_j \cdot |a_j| - 0| < \epsilon$ which tells us that $|b_j \cdot |a_j - 0| < \epsilon$ and therefore $\lim(a_n b_n) = 0$. ■

PROBLEM 9

Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n} y_n)$ converges. Find the limit.

PROOF. We want to show that there exists (and we can find) some L such that given $\epsilon > 0$, $|\sqrt{n+1} - \sqrt{n} - L| < \epsilon$. Using the Ratio Test,

$$\frac{y_{n+1}}{y_n} = \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}}$$

we consider the conjugate of y_{n+1}

$$\frac{y_{n+1}}{y_n} = \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}} \left(\frac{\sqrt{n+2} + \sqrt{n+1}}{\sqrt{n+2} + \sqrt{n+1}} \right)$$

In this case, the numerator simplifies to 1, while the denominator resolves to the following expression which we call ϕ_n:

$$\phi_n = \sqrt{n+1}\left(\sqrt{n+2} + \sqrt{n+1} - 1\right) - \sqrt{n}(\sqrt{n+2})$$

Our immediate goal is to show that $\frac{y_{n+1}}{y_n} = 1/\phi_n$ converges. Well, we know $\sqrt{n+1} > \sqrt{n}$, and since $\sqrt{n+1} - 1$ is positive, we know that $\sqrt{n+2} + \sqrt{n+1} - 1 > \sqrt{n+2}$. Multiplying these two facts gives us

$$\sqrt{n+1}(\sqrt{n+2}+\sqrt{n+1}-1) > \sqrt{n}\sqrt{n+2}$$

By definition, subtracting the right hand side of the above inequality from the left yields a positive number, and since this difference is exactly ϕ_n, we know that $\phi_n > 0$.

By the Archimedean Property, we know that for each $n \in \mathbb{N}$ there is a least $M > \phi_n$ so that $K = M - 1 \leq \phi_n$. Thus, $1/K \geq 1/\phi_n$. Since $\phi_n > 0$, we know $1/n \geq 1/\phi_n > 0$ when $n \geq K$ so that $1/\phi_n \to 0$. Since $0 < 1$, we know by the ratio test that y_n converges and $\lim(y_n) = 0$.■

Problem 11

Part A

Find $\lim\left((3\sqrt{n})^{1/2n}\right)$

Proof. We know that $\sqrt[n]{3\sqrt{n}} > 0$ since the domain function $3\sqrt{n} > 0$. So if $\sqrt[n]{3\sqrt{n}}$ can be shown to converge to L_1, then $\sqrt{\sqrt[n]{3\sqrt{n}}} \to L_0 = \sqrt{L_1}$.

$$\sqrt[n]{3\sqrt{n}} = (3\sqrt{n})^{1/n} = 3^{1/n}\sqrt{n}^{1/n} = 3^{1/n}n^{1/2n} = 3^{1/n}\sqrt{n^{1/n}}$$

If $3^{1/n}$ and $\sqrt{n^{1/n}}$ can be shown to converge to L_3 and L_2 respectively, then $L_1 = L_3 \cdot L_2$, so this is our goal.

Since $n^{1/n} > 0$ and we have shown by earlier work that $n^{1/n} \to 1$, then by Theorem 3.2.10 we have that $\sqrt{n^{1/n}} \to \sqrt{1}$ or rather $L_2 = 1$.

Since $1 < 3 < n$ whenever $n \geq 3$, we know that $1^{1/n} < 3^{1/n} < n^{1/n}$ whenever $n \geq 3$. Thus, since $n^{1/n} \to 1$, we have that $3^{1/n} \to 1$, so $L_3 = 1$.

Thus, $L_1 = L_3L_2 = 1 \cdot 1 = 1$, so that $L_0 = \sqrt{L_1} = \sqrt{1} = 1$. Thus, $\lim\left((3\sqrt{n})^{1/2n}\right) = 1$. ■

Part B

Find $(\sqrt{n^2+5n} - n)$

Proof.

Problem 12

If $0 < a < b$, determine $\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.

PROOF. First note that

$$\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)=\left(\frac{a^{n+1}}{a^n+b^n}+\frac{b^{n+1}}{a^n+b^n}\right)=a\left(\frac{a^n}{a^n+b^n}\right)+b\left(\frac{b^n}{a^n+b^n}\right)$$

and since $a^n+b^n=a^n(1+(b/a)^n)=b^n((a/b)^n+1)$ we can write

$$\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)=a\left(\frac{1}{1+(b/a)^n}\right)+b\left(\frac{1}{(a/b)^n+1}\right)$$

which encourages us to focus our discussion on $(b/a)^n$ and $(a/b)^n$.

Since $b>a>0$, we know that $b/a>1$ so that $(b/a)^n+1$ is unbounded. The Archimedean Property guarantees us an M bigger than $(b/a)^n+1$, so that $1/M>\frac{1}{(b/a)^n+1}$ for all sufficiently large n. Thus $\frac{1}{(b/a)^n+1}\to 0$ and so does $\frac{a}{(b/a)^n+1}$ since it is a constant multiple.

Knowing that $b>a>0$ also tells us that $a/b<1$ so that $a/b\in(0,1)$. We know from earlier work that $\lim(nc^n)=0$ when $c\in(0,1)$, and since $(a/b)^n<n(a/b)^n$ for all n, we have that $\lim((a/b)^n)=0$ as well. Thus, $\lim(\frac{b}{(a/b)^n+1})=b$.

Since we have expressed our original sequence in terms of the sum of two convergent sequences, we have that

$$\lim\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)=b\ \blacksquare$$

PROBLEM 13

If $a>0$, $b>0$, show that $\lim\left(\sqrt{(n+a)(n+b)}-n\right)=(a+b)/2$.

PROOF. Consider multiplying by a conjugate:

$$\begin{gathered}\sqrt{(n+a)(n+b)}-n\\ (\sqrt{(n+a)(n+b)}-n)\left(\frac{\sqrt{(n+a)(n+b)}+n}{\sqrt{(n+a)(n+b)}+n}\right)\\ \frac{(n+a)(n+b)-n^2}{\sqrt{(n+a)(n+b)}+n}\\ \frac{n(a+b)+ab}{\sqrt{(n+a)(n+b)}+n}\\ \frac{n(a+b)}{\sqrt{(n+a)(n+b)}+n}+\frac{ab}{\sqrt{(n+a)(n+b)}+n}\end{gathered}$$

We note that the second term $(ab)/(\sqrt{(n+a)(n+b)}+n)$ has a fixed numerator and an unbounded denominator, so the Archimedean Property affirms that the quotient will tend towards zero. Thus we break apart the first term into

$$\frac{n(a+b)}{\sqrt{(n+a)(n+b)}+n} = \frac{na}{\sqrt{(n+a)(n+b)}+n} + \frac{nb}{\sqrt{(n+a)(n+b)}+n}$$

We note that since na, nb, and ab are all positive, we have

$$n^2 + na + nb + ab > n^2$$

so we know that $\sqrt{(n+a)(n+b)} > n$. Thus,

$$\frac{na}{\sqrt{(n+a)(n+b)}+n} + \frac{nb}{\sqrt{(n+a)(n+b)}+n} \leq \frac{na}{2n} + \frac{nb}{2n} = \frac{a+b}{2}$$

Problem 14

Part A

Use the Squeeze Theorem to find $\lim(n^{1/n^2})$.

Proof. We know from earlier work that $\lim(n^{1/n}) = 1$. We can infer from this that $n > n^{1/n}$ for sufficiently large values of n. This implies that $n^{1/n} > n^{1/n^2}$.

Also, consider that $n > 1^{n^2}$. This lets us know that $n^{1/n^2} > 1$. Therefore $1 \leq n^{1/n^2} \leq n^{1/n}$ for all sufficiently large n, and since $n^{1/n} \to 1$, we may employ the Squeeze Theorem to assert that $n^{1/n^2} \to 1$. ■

Part B

Use the Squeeze Theorem to find $\lim((n!)^{1/n^2})$.

Proof. Well, $n^{1/n^2} \leq (n!)^{1/n^2}$ since $n \leq n!\ \forall n$. We alos know that $(n!)^{1/n^2} \leq (n!)^{1/n}$ since $n! \leq (n!)^n\ \forall n$. Moreover, since $n! \leq n^n$, we know $\sqrt[n]{n!} \leq n$. This implies $(n!)^{1/n} \leq n$ which in turn asserts that $(n!)^{1/n^2} \leq n^{1/n}$. Therefore we have

$$n^{1/n^2} \leq (n!)^{1/n^2} \leq n^{1/n}$$

and since $n^{1/n^2} \to 1$ and $n^{1/n} \to 1$, we know by the Squeeze Theorem that $(n!)^{1/n^2} \to 1$. ■

Problem 15

Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.

Proof. We begin by noting that since $a < b$, we have

$$(b^n)^{1/n} \leq (a^n + b^n)^{1/n} \leq (b^n + b^n)^{1/n}$$

$(b^n)^{1/n} = b$ by definition. $(b^n + b^n)^{1/n} = (2b^n)^{1/n} = \sqrt[n]{2}b$. Since $sqrt[n]2 \to 1$, we know that $\sqrt[n]{2}b \to b$. Thus, z_n is bounded above and below by sequences that converge to b, so by the Sandwich Theorem, $z_n \to b$. ■

PROBLEM 23

Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent.

PROOF. By definition, $\max\{x_n, y_n\} = \frac{1}{2}(x_n + y_n + |x_n - y_n|)$. Thus, because sums, differences, absolute values, and scalar multiples of convergent sequences are themselves convergent, we have that $\max\{x_n, y_n\}$ converges.

Likewise, $\min\{x_n, y_n\} = \frac{1}{2}(x_n + y_n - |x_n - y_n|)$, which is again a composition of operations known to map convergent sequences to convergent sequences. Thus, $\min\{x_n, y_n\}$ is itself convergent. ■

PROBLEM 24

Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \text{mid}\{x_n, y_n, z_n\}$ is also convergent.

PROOF. By definition, the mid function is a composition of min and max functions. In the previous problem we showed that min and max of convergent sequences are convergent themselves, so we may apply this argument iteratively to demonstrate that w_n converges as well. ■