University of Tennessee
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MATH 447 - Homework 6

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Section 3.3, Ex. 3

Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

Proof. We can demonstrate by induction that (x_n) is bounded below by 2. First note that $x_1 \geq 2$ by by hypothesis. Thus,

$$x_1 \geq 2$$
$$x_1 - 1 \geq 1$$
$$\sqrt{x_1 - 1} \geq 1$$
$$1 + \sqrt{x_1 - 1} \geq 1 + 1$$
$$x_2 \geq 2$$

so we have a basis for induction. Assume now that $x_n \geq 2$, and we will use this to demonstrate that $x_{n+1} \geq 2$:

$$x_n \geq 2$$
$$x_n - 1 \geq 1$$
$$\sqrt{x_n - 1} \geq 1$$
$$1 + \sqrt{x_n - 1} \geq 1 + 1$$
$$x_{n+1} \geq 2$$

Thus we have confirmed that (x_n) is bounded below by 2.

We can also demonstrate by induction that (x_n) is decreasing. Begin by noting that $x_1 \geq 2$, which implies that $x_1 - 1 \geq 1$. This tells us that $\sqrt{x_1} \geq 1$. Taking the difference of the previous equations tells us that $(x_1 - 1) - (\sqrt{x_1 - 1}) \geq 0$, and thus that $x_1 \geq 1 + \sqrt{x_1 - 1} = x_2$. Since $x_1 \geq x_2$, we have established a basis for induction.

Now assume that $x_{n-1} \geq x_n$ and recall that since 2 is a lower bound, $x_n - 1 \geq 1$. We aim to show that this implies $x_n \geq x_{n+1}$.

$$\begin{aligned} x_{n-1} &\geq x_n \\ x_{n-1} - 1 &\geq x_n - 1 \\ \sqrt{x_{n-1} - 1} &\geq \sqrt{x_n - 1} \\ 1 + \sqrt{x_{n-1} - 1} &\geq 1 + \sqrt{x_n - 1} \\ x_n &\geq x_{n+1} \end{aligned}$$

Thus we have shown that (x_n) decreases.

Since (x_n) is decreasing, bounded below, and recursively defined, we may calculate its limit by acknowledging that $\lim(x_n) = \lim(x_{n+1})$, and for convenience we shall denote this $x*$. Thus,

$$\begin{aligned} x^* &= 1 + \sqrt{x^* - 1} \\ x^* + 1 &= \sqrt{x^* - 1} \\ (x^* + 1)^2 &= x^* - 1 \\ x^{*2} - 3x^* + 2 &= 0 \end{aligned}$$

The solutions of the final quadratic equation are 1 and 2, however, since 2 is a lower bound for (x_n), we know its limit may not be 1. Thus, $\lim(x_n) = 2.\blacksquare$

SECTION 3.3, EX. 4

Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

PROOF. $x_2 = \sqrt{3} < 2$, so we know that x_1 and x_2 are both less than 2. Assume $x_n < 2$, then

$$\begin{aligned} x_n &< 2 \\ x_n + 2 &< 2 + 2 \\ \sqrt{x_n + 2} &< \sqrt{4} \\ x_{n+1} = \sqrt{x_n + 2} &< 2 \end{aligned}$$

so by induction we have that (x_n) is increasing and bounded above by 2. Since it is recursively defined, we may acknowledge that $\lim(x_n) = \lim(x_{n+1})$ and call that value x^*.

$$\begin{aligned} x^* &= \sqrt{2 + x^*} \\ (x^*)^2 &= 2 + x* \\ (x^*)^2 - x * -2 &= 0 \end{aligned}$$

so that solving with the quadratic formula gives us $x^* = 2$. ■

SECTION 3.3., EX. 5

Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit.

PROOF. The book tells us that $y_n \le 1 + 2\sqrt{p}\ \forall n \in \mathbb{N}$, so (y_n) is bounded above. Since (y_n) is increasing, we know that $\lim(y_n) = \sup\{y_n : n \in \mathbb{N}\}$, and we call this value $y*$. Since this sequence is recursively defined, $\lim(y_{n+1}) = y^*$. Thus,

$$\begin{aligned} y^* &= \sqrt{p + y^*} \\ (y^*)^2 &= p + y^* \\ (y^*) - y^* &= p \\ (y^*)^2 - y^* - p &= 0 \end{aligned}$$

so by the quadratic formula, $y* \in \{\frac{1+\sqrt{1+4p}}{2}, \frac{1-\sqrt{1+4p}}{2}\}$. We can rule out the second element by noting that:

$$\begin{aligned} p &> 0 \\ 4p &> 0 \\ 1 + 4p &> 1 \\ \sqrt{1+4p} &> 1 \\ -\sqrt{1+4p} &< -1 \\ 1 - \sqrt{1+4p} &< 0 \end{aligned}$$

and since $y_n > 0$, it is clear that $y^* = \frac{1+\sqrt{1+4p}}{2}$.■

SECTION 3.3, EX. 9

Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup A$. Show there exists an increasing sequence (x_n) with $x_n \in A \; \forall n \in \mathbb{N}$ such that $u = \lim(x_n)$.

PROOF. By the Supremum Principle, for all $\epsilon > 0$, $\exists x_\epsilon \in A$ such that $u \geq x_\epsilon > x - \epsilon$. Pick some decreasing sequence of (ϵ_n) that converges to zero, and then $(x_n) = (x_{\epsilon n})$. ■

SECTION 3.3, EX. 11

Let $x_n := 1/1^1 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathbb{N}$. Prove that (x_n) is increasing and bounded, and hence converges.

PROOF. We want to show that (x_n) is increasing. First note that (x_n) is a sum of positive terms, it is positive. Thus, since $x_{n+1} = x_n + 1/(n+1)^2$, it is clear that (x_n) is increasing.

The book tells us that $1/k^2 \leq 1/k(k-1) = 1/(k-1) - 1/k$ when $k \geq 2$. Noting that $x_k - x_{k-1} = 1/k^2$ and also that $\lim(1/k^2) = 0$, we have that $\lim(x_k - x_{k-1}) = 0$. This tells us that for large enough k, we have $|x_k - x_{k-1}| < \epsilon$, so by the Cauchy Criterion (x_n) converges. This infers that it is bounded above. ■

SECTION 3.4, EX 2

Use the method of Example 3.4.3(b) to show that if $0 < c < 1$, then $\lim(c^{1/n}) = 1$.

PROOF. We begin by showing that (z_n) is increasing. Consider that

$$
\begin{gathered}
c < 1 \\
c^{1/n} < 1 \\
c^{1/n} \cdot c < 1 \cdot c < 1 \\
c^{1/n+1} < c < 1 \\
c^{(n+1)/n} < c < 1 \\
c^{1/n} < c^{1/(n+1)} < 1
\end{gathered}
$$

which tells us that (z_n) is both increasing and bounded above by 1. This confirms that $\lim(z_n)$ exists, so let us call it z^*. Thus, the subsequence (z_{2n}) exists and converges to z^*.

Since $z_{2n} = c^{1/2n} = \sqrt{c^{1/n}}$, we know $\lim(z_{2n}) = \sqrt{z^*}$. But since $\lim(z_{2n}) = z^*$, we have that $z^* = \sqrt{z^*}$. Thus, $z^* \in 0, 1$. Since $z_1 = c > 0$, and (z_n) is increasing, we know that $z^* = 1$. ■

Section 3.4, Ex 4

Part A

Show that $(x_n) = (1-(-1)^n + 1/n)$ diverges.

Proof. $\lim(x_{2n}) = \lim(1-1+1/2n) = 0$, but $\lim(x_{2n-1}) = \lim(1+1+\frac{1}{2n-1}) = 2$, so (x_n) contains two convergent subsequences whose limits are unequal. Thus, (x_n) diverges. ■

Part B

Show that $(y_n) = (\sin(\frac{n\pi}{4}))$ diverges.

Proof. $y_{8n} = \sin(\frac{8n\pi}{4}) = \sin(2n\pi) = 0$, which is a constant sequence. $y_{8n-1} = \sin(\frac{(8n-1)\pi}{4}) = \sin(2n\pi - \pi/4) = -1$, which is also a constant sequence. Thus, (y_n) contains two convergent subsequences whose limits are unequal. Thus, (y_n) diverges. ■

Section 3.4, Ex 5

Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the "shuffled" sequence $Z = (z_n)$ be defined by $z_{2n-1} := x_n$ and $z_{2n} := y_n$. Show that Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$.

Proof. We prove the logical equivalence of these statements in each direction separately.

($\rightarrow$) Assume that Z converges to z^*, then any subsequence of Z also converges to z^*. Thus, $\lim X = \lim Y = z^*$.

($\leftarrow$) Assume X and Y converge and that $\lim X = \lim Y$, but that $\lim Z = z^* \neq \lim X$. Then $|z^* - \lim X| = c$, so choose $\epsilon < c$, then $\exists M(\epsilon) n \in \mathbb{N}$ such that

$$|z_{2n} - \lim X| < \epsilon \text{ and } |z_{2n-1} - \lim X| < \epsilon$$

That is, all z_j for j even or odd lie in an ϵ-neighborhood of c, so that for any choice of $\delta > 0$, only finitely many z_j lie in $V_\delta(z^*)$. ■

Section 3.4, Ex 7

Part A

Find $\lim((1+1/n^2)^{n^2})$ if it exists.

Proof. The sequence $(1+1/n^2)^{n^2}$ is a subsequence of $(1+1/n)^n$. Since $(1+1/n)^n \to e$, we know that $(1+1/n^2)^{n^2} \to e$. ■

PART D

Find $\lim((1+2/n)^n)$ if it exists.

PROOF. Let $n = 2j$, then $(1+2/2j)^{2j} = (1+1/j)^{2j} = (1+1/j)^j \cdot (1+1/j)^j$. Since we know $\lim(1+1/j)^j = e$, then by the limit laws we know that $\lim(1+2/n)^n = e^2$. ■

PART C

Find $\lim((1+1/n^2)^{2n^2})$ if it exists.

PROOF. Take $m = n^2$, so that is becomes clear that we are working with a subsequence of $(1+1/m)^{2m}$. In Part D, we showed that $(1+1/m)^{2m} = (1+1/m) \cdot (1+1/m)$ and that the limit of this sequence is e^2. Thus, the $\lim((1+1/n^2)^{2n^2}) = e^2$. ■

PART B

Find $\lim(1+1/2n)^n$ if it exists.

PROOF. Take $m = 2n$, so that it becomes clear that $(1+1/2n)^{2n}$ is a subsequence of $(1+1/n)^n$ and thus $\lim(1+1/2n)^{2n} = e$. We know from the limit laws that the root of a convergent sequence converges to the root of the sequence's limit. Thus, $\lim(\sqrt{(1+1/2n)^{2n}}) = \lim(1+1/2n)^n = \sqrt{e}$. ■