UNIVERSITY OF TENNESSEE
DEPARTMENT OF MATHEMATICS

MATH 447 - Homework 5

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SECTION 3.1

PROBLEM 10

Prove that if $\lim(x_n) = 0$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

PROOF. By the definition of limit, for each positive ϵ such an $M(\epsilon)$ can be found. Thus, take ϵ to be smaller than x, so that $x_n \in V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ whenever $n \geq M$. Since $\epsilon < x$, $x - \epsilon \in \mathbb{P}$ so $x_n > 0$.■

PROBLEM 14

Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$.

PROOF. We know that $n^{1/n} > 1\ \forall n \in \mathbb{N}$. This implies that $n^{1/n} = 1 + k_n$ for some $k_n > 0$, so that we have $bn^{1/n} = b + bk_n$. Thus, $nb^n = (b + bk_n)^n$.

By the Binomial Theorem we know that

$$nb^n = (b + bk_n)^n = b^n + nb^{n-1}bk_n + \frac{1}{2}n(n-1)b^{n-2}b^2{k_n}^2 + \cdots$$

which tells us that $nb^n \geq b^n + \frac{1}{2}n(n-1)b^n{k_n}^2$. Factoring out b^n and subtracting 1 yields

$$(n-1) \geq \frac{1}{2}n(n-1){k_n}^2$$

so that we have $\frac{2}{n} \geq {k_n}^2$.

Problem 17

Hint

If $n \geq 3$, then $0 < 2^n / n! \leq 2(\frac{2}{3})^{n-2}$.

Proof. Let $n = 3$, then $\frac{2^3}{3!} = \frac{8}{6} \leq 2(\frac{2}{3}) = \frac{4}{3} = \frac{8}{6}$. Thus, we have established a basis for induction.

Now assume that $0 < \frac{2^{n-1}}{(n-1)!} \leq 2\left(\frac{2}{3}\right)^{n-3}$. Multiplying by 2 yields $0 < \frac{2^n}{(n-1)!} \leq 2\left(\frac{2^{n-2}}{3^{n-3}}\right)$. Since $n \geq 3$, we have $0 < \frac{1}{n} \leq \frac{1}{3}$ so that

$$0 < \frac{2^n}{(n-1)!} \cdot \frac{1}{n} = \frac{2^n}{n!} \leq 2 \cdot \left(\frac{2^{n-2}}{3^{n-3}}\right) \cdot \frac{1}{3} = 2\left(\frac{2}{3}\right)^{n-2}$$

whence $0 < \frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$. Therefore the hypothesis holds by the Principal of Mathematical Inducation. ■

Main Problem

Show that $\lim(2^n / n!) = 0$.

Proof. We know that $0 < \frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$ when $n \geq 3$, so if we can show that $2\left(\frac{2}{3}\right)^{n-2} \to 0$, then $\frac{2^n}{n!} \to 0$ by the Squeeze Theorem.

Take $b = 2/3$ and $n = m-2$, then by Exercise 3.1.14 we know $nb^n \to 0$. Since $2 \leq m-2 \; \forall m > 3$, we have that $0 < 2(\frac{2}{3})^{m-2} \leq (m-2)b^{m-2}$ so that $2\left(\frac{2}{3}\right)^{m-2} \to 0$. Thus, by the Squeeze Theorem, $2^n / n! \to 0$. ■

Problem 18

If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

Proof. Pick $\epsilon < \frac{1}{2}x$, then by the definition of limit $\exists K$ such that $x_n \in V_\epsilon(x) \forall n \geq K$. Since $V_\epsilon(x) = (x - x/2, x + x/2) = (x/2, 3x/2)$, we know $V_\epsilon(x) \subseteq (x/2, 2x)$, demonstrating that such a K exists. ■

Section 3.2

Problem 7

If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Thm 3.2.3 cannot be used.

REMARK. We may not apply Theorem 3.2.3 directly because it requires that (b_n) is convergent, but we know only that (b_n) is bounded. However, we may employ Theorem 3.2.3 for certain features of our argument.

PROOF. Since (b_n) is bounded, we have

$$I_b = \inf(b_n) \le b_j \le \sup(b_n) = S_b \ \forall j \in \mathbb{N}$$

which implies

$$I_b|a_j| = \inf(b_n)|a_j| \le b_j \cdot |a_j| \le \sup(b_n) \cdot |a_j| = S_b \cdot |a_j| \ \forall j \in \mathbb{N}$$

Thus if $\inf(b_n)|a_j| \to 0$ and $\sup(b_n)|a_j| \to 0$, we will be confident that $b_j|a_j| \to 0$.

Since $\inf(b_n)$ and $\sup(b_n)$ are constants, and since we know $|a_j| \to 0$ due to the fact that $a_j \to 0$, we have that $\inf(b_n)|a_j| \to 0$ and $\sup(b_n)|a_j| \to 0$ by Theorem 3.2.3. Thus, $b_j \cdot |a_j| \to 0$.

Therefore, given $\epsilon > 0$, we know $\exists K(\epsilon)$ such that $\left|b_j \cdot |a_j| - 0\right| < \epsilon \ \forall j \ge K(\epsilon)$.

What we want to show is that $|b_j \cdot a_j - 0| < \epsilon \ \forall j \ge M(\epsilon)$. By the above work, we know $|b_j \cdot |a_j| - 0| < \epsilon$ which tells us that $|b_j \cdot |a_j - 0| < \epsilon$ and thereforce $\lim(a_n b_n) = 0$. ■

PROBLEM 9

Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n} y_n)$ converges. Find the limit.

PROOF. We want to show that there exists (and we can find) some L such that given $\epsilon > 0$, $|\sqrt{n+1} - \sqrt{n} - L| < \epsilon$. Using the Ratio Test,

$$\frac{y_{n+1}}{y_n} = \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}}$$

we consider the conjugate of y_{n+1}

$$\frac{y_{n+1}}{y_n} = \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}} \left(\frac{\sqrt{n+2} + \sqrt{n+1}}{\sqrt{n+2} + \sqrt{n+1}} \right)$$

In this case, the numerator simplifies to 1, while the denominator resolves to the following expression which we call ϕ_n:

$$\phi_n = \sqrt{n+1}\left(\sqrt{n+2} + \sqrt{n+1} - 1\right) - \sqrt{n}(\sqrt{n+2})$$

Our immediate goal is to show that $\frac{y_{n+1}}{y_n} = 1/\phi_n$ converges. Well, we know $\sqrt{n+1} > \sqrt{n}$, and since $\sqrt{n+1} - 1$ is positive, we know that $\sqrt{n+2} + \sqrt{n+1} - 1 > \sqrt{n+2}$. Multiplying these two facts gives us

$$\sqrt{n+1}(\sqrt{n+2}+\sqrt{n+1}-1) > \sqrt{n}\sqrt{n+2}$$

By definition, subtracting the right hand side of the above inequality from the left yields a positive number, and since this difference is exactly ϕ_n, we know that $\phi_n > 0$.

By the Archimedean Property, we know that for each $n \in \mathbb{N}$ there is a least $M > \phi_n$ so that $K = M - 1 \leq \phi_n$. Thus, $1/K \geq 1/\phi_n$. Since $\phi_n > 0$, we know $1/n \geq 1/\phi_n > 0$ when $n \geq K$ so that $1/\phi_n \to 0$. Since $0 < 1$, we know by the ratio test that y_n converges and $\lim(y_n) = 0$.■

PROBLEM 11

PART A

Find $\lim\left((3\sqrt{n})^{1/2n}\right)$

PROOF. We know that $\sqrt[n]{3\sqrt{n}} > 0$ since the domain function $3\sqrt{n} > 0$. So if $\sqrt[n]{3\sqrt{n}}$ can be shown to converge to L_1, then $\sqrt{\sqrt[n]{3\sqrt{n}}} \to L_0 = \sqrt{L_1}$.

$$\sqrt[n]{3\sqrt{n}} = (3\sqrt{n})^{1/n} = 3^{1/n}\sqrt{n}^{1/n} = 3^{1/n}n^{1/2n} = 3^{1/n}\sqrt{n^{1/n}}$$

If $3^{1/n}$ and $\sqrt{n^{1/n}}$ can be shown to converge to L_3 and L_2 respectively, then $L_1 = L_3 \cdot L_2$, so this is our goal.

Since $n^{1/n} > 0$ and we have shown by earlier work that $n^{1/n} \to 1$, then by Theorem 3.2.10 we have that $\sqrt{n^{1/n}} \to \sqrt{1}$ or rather $L_2 = 1$.

Since $1 < 3 < n$ whenever $n \geq 3$, we know that $1^{1/n} < 3^{1/n} < n^{1/n}$ whenever $n \geq 3$. Thus, since $n^{1/n} \to 1$, we have that $3^{1/n} \to 1$, so $L_3 = 1$.

Thus, $L_1 = L_3L_2 = 1 \cdot 1 = 1$, so that $L_0 = \sqrt{L_1} = \sqrt{1} = 1$. Thus, $\lim\left((3\sqrt{n})^{1/2n}\right) = 1$. ■

PART B

Find $(\sqrt{n^2+5n} - n)$

PROOF.

PROBLEM 12

If $0 < a < b$, determine $\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.

PROOF. First note that

$$\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)=\left(\frac{a^{n+1}}{a^n+b^n}+\frac{b^{n+1}}{a^n+b^n}\right)=a\left(\frac{a^n}{a^n+b^n}\right)+b\left(\frac{b^n}{a^n+b^n}\right)$$

and since $a^n+b^n=a^n(1+(b/a)^n)=b^n((a/b)^n+1)$ we can write

$$\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)=a\left(\frac{1}{1+(b/a)^n}\right)+b\left(\frac{1}{(a/b)^n+1}\right)$$

which encourages us to focus our discussion on $(b/a)^n$ and $(a/b)^n$.

Since $b>a>0$, we know that $b/a>1$ so that $(b/a)^n+1$ is unbounded. The Archimedean Property guarantees us an M bigger than $(b/a)^n+1$, so that $1/M>\frac{1}{(b/a)^n+1}$ for all sufficiently large n. Thus $\frac{1}{(b/a)^n+1}\to 0$ and so does $\frac{a}{(b/a)^n+1}$ since it is a constant multiple.

Knowing that $b>a>0$ also tells us that $a/b<1$ so that $a/b\in(0,1)$. We know from earlier work that $\lim(nc^n)=0$ when $c\in(0,1)$, and since $(a/b)^n<n(a/b)^n$ for all n, we have that $\lim((a/b)^n)=0$ as well. Thus, $\lim(\frac{b}{(a/b)^n+1})=b$.

Since we have expressed our original sequence in terms of the sum of two convergent sequences, we have that

$$\lim\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)=b\ \blacksquare$$

PROBLEM 13

If $a>0$, $b>0$, show that $\lim\left(\sqrt{(n+a)(n+b)}-n\right)=(a+b)/2$.

PROOF. Consider multiplying by a conjugate:

$$\begin{gathered}\sqrt{(n+a)(n+b)}-n\\ (\sqrt{(n+a)(n+b)}-n)\left(\frac{\sqrt{(n+a)(n+b)}+n}{\sqrt{(n+a)(n+b)}+n}\right)\\ \frac{(n+a)(n+b)-n^2}{\sqrt{(n+a)(n+b)}+n}\\ \frac{n(a+b)+ab}{\sqrt{(n+a)(n+b)}+n}\\ \frac{n(a+b)}{\sqrt{(n+a)(n+b)}+n}+\frac{ab}{\sqrt{(n+a)(n+b)}+n}\end{gathered}$$

We note that the second term $(ab)/(\sqrt{(n+a)(n+b)}+n)$ has a fixed numerator and an unbounded denominator, so the Archimedean Property affirms that the quotient will tend towards zero. Thus we break apart the first term into

$$\frac{n(a+b)}{\sqrt{(n+a)(n+b)}+n} = \frac{na}{\sqrt{(n+a)(n+b)}+n} + \frac{nb}{\sqrt{(n+a)(n+b)}+n}$$

We note that since na, nb, and ab are all positive, we have

$$n^2 + na + nb + ab > n^2$$

so we know that $\sqrt{(n+a)(n+b)} > n$. Thus,

$$\frac{na}{\sqrt{(n+a)(n+b)}+n} + \frac{nb}{\sqrt{(n+a)(n+b)}+n} \le \frac{na}{2n} + \frac{nb}{2n} = \frac{a+b}{2}$$

Problem 14

Part A

Use the Squeeze Theorem to find $\lim(n^{1/n^2})$.

Proof. We know from earlier work that $\lim(n^{1/n}) = 1$. We can infer from this that $n > n^{1/n}$ for sufficiently large values of n. This implies that $n^{1/n} > n^{1/n^2}$.

Also, consider that $n > 1^{n^2}$. This lets us know that $n^{1/n^2} > 1$. Therefore $1 \le n^{1/n^2} \le n^{1/n}$ for all sufficiently large n, and since $n^{1/n} \to 1$, we may employ the Squeeze Theorem to assert that $n^{1/n^2} \to 1$. ■

Part B

Use the Squeeze Theorem to find $\lim((n!)^{1/n^2})$.

Proof. Well, $n^{1/n^2} \le (n!)^{1/n^2}$ since $n \le n!\ \forall n$. We alos know that $(n!)^{1/n^2} \le (n!)^{1/n}$ since $n! \le (n!)^n\ \forall n$. Moreover, since $n! \le n^n$, we know $\sqrt[n]{n!} \le n$. This implies $(n!)^{1/n} \le n$ which in turn asserts that $(n!)^{1/n^2} \le n^{1/n}$. Therefore we have

$$n^{1/n^2} \le (n!)^{1/n^2} \le n^{1/n}$$

and since $n^{1/n^2} \to 1$ and $n^{1/n} \to 1$, we know by the Squeeze Theorem that $(n!)^{1/n^2} \to 1$. ■

Problem 15

Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.

Proof.

Problem 23

Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent.

PROOF.

PROBLEM 24

Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \mathrm{mid}\{x_n, y_n, z_n\}$ is also convergent.

PROOF.