

MATH 447 - Homework 4

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1 UNCOUNTABILITY OF THE REAL NUMBERS

The set \mathbb{R} of real numbers is uncountable.

PROOF. Assume the set $I = [0, 1]$ is countable. This implies its elements are enumerable. That is, $I = \{x_1, x_2, \dots, x_n, \dots\}$.

Now construct the set $I_1 = \{x \in I \mid x \neq x_1\}$ so that $I_1 \subset I$ and $I_1 = I \setminus \{x_1\}$. Clearly this construction can progress in a recursive fashion so that $I_j = I_{j-1} \setminus \{x_j\}$, and $I_j \supseteq I_{j+1} \supseteq \dots$.

Thus, we have established a collection of nested intervals, so we know $\exists \xi \in I_n \forall n \in \mathbb{N}$. Suppose $\xi = x_k$ for some $k \in \mathbb{N}$, then $\xi \notin I_k$, which contradicts the Nested Intervals Property. Clearly this is absurd, so we refute our hypothesis that $I = [0, 1]$ is countable.

Since $[0, 1] \subseteq \mathbb{R}$ is uncountable, we know that \mathbb{R} itself is uncountable. ■

2 BINARY REPRESENTATION OF REAL NUMBERS

If $x \in [0, 1]$, then there exists a sequence (a_n) of 0s and 1s such that

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_n}{2^n} \leq x \leq \frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_n + 1}{2^n} \quad (2.1)$$

for all $n \in \mathbb{N}$. Conversely, each sequence of 0s and 1s is the binary representation of a unique real number in $[0, 1]$.

PROOF. We begin by discussing an algorithm for constructing a sequence (a_n) based on the choice x . We then demonstrate that the upper and lower bounds given above form a nested sequence of intervals, and since we know (by the Nested Intervals Property) that the intersection of such objects is nonempty, we can be confident that such a construction for x is valid.

If $x \neq \frac{1}{2}$ belongs to the left subinterval $[0, \frac{1}{2}]$ we take $a_1 = 0$, while if x belongs to the right subinterval $[\frac{1}{2}, 1]$ we take $a_1 = 1$. If $x = \frac{1}{2}$, then we may take a_1 to be either 0 or 1. In any case we have

$$\frac{a_1}{2} \leq x \leq \frac{a_1}{2} + \frac{a_2 + 1}{2^2}. \quad (2.2)$$

We continue this bisection procedure, assigning at the n th stage the value $a_n = 0$ if x is not the bisection point and lies in the left subinterval, and assigning the value $a_n = 1$ if x lies in the right subinterval. Thus, we have a well-defined sequence (a_n) of 0s and 1s such that Eqn 2.1 above holds.

Now we proceed to show that the upper and lower bounds given in Eqn 2.1 form a sequence of nested intervals. Let L_n be the n th lower bound and U_n the n th upper bound so that

$$\begin{aligned} L_n &:= \sum_{i=1}^n \frac{a_i}{2^i} & U_n &:= \sum_{i=1}^n \frac{a_i}{2^i} + \frac{1}{2^n} \\ L_{n-1} &:= \sum_{i=1}^{n-1} \frac{a_i}{2^i} & U_{n-1} &:= \sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{1}{2^{n-1}} \end{aligned}$$

which yields the following useful relationships

$$\begin{aligned} L_n &= L_{n-1} + \frac{a_n}{2^n} \\ U_n &= U_{n-1} - \frac{1}{2^{n-1}} + \frac{a_n}{2^n} + \frac{1}{2^n} \end{aligned}$$

We know that each a_n is either 0 or 1, so we have two cases. If $a_n = 0$, then $\frac{a_n}{2^n} = 0$ so that $L_n = L_{n-1}$ and $U_n = U_{n-1} - \frac{1}{2^n} < U_{n-1}$. If $a_n = 1$, then $\frac{a_n}{2^n} = \frac{1}{2^n}$ so that $L_n = L_{n-1} + \frac{1}{2^n} > L_{n-1}$ and $U_n = U_{n-1} - \frac{1}{2^{n-1}} + \frac{2}{2^n} = U_{n-1}$. Thus, we have that $L_n \geq L_{n-1}$ and $U_n \leq U_{n-1}$. Therefore, for each $n \in \mathbb{N}$, we have

$$L_{n-1} \leq L_n < U_n \leq U_{n-1} \quad (2.3)$$

so that the interval $[L_{n-1}, U_{n-1}]$ contains the interval $[L_n, U_n]$. ■

3 SOME PROBLEMS CONCERNING INTERVALS

EXERCISE 2.5.3

If $S \subseteq \mathbb{R}$ is a nonempty bounded set, and $I_S := [\inf S, \sup S]$, then $S \subseteq I_S$. Moreover, if J is any closed bounded interval containing S , then $I_S \subseteq J$.

PROOF. Since $\inf S \leq s \forall s \in S$, and $\sup S \geq s \forall s \in S$, we know

$$\inf S \leq s \leq \sup S \forall s \in S \quad (3.1)$$

Thus, $s \in [\inf S, \sup S] \forall s \in S$, so $S \subseteq I_S$.

Further, since J is a closed bounded interval containing S by hypothesis, we know that $\inf J \leq s \forall s \in S$ so that $\inf J$ is a lower bound for S , and we know that $\sup J \geq s \forall s \in S$ so that $\sup J$ is an upper bound for S . By definition of the infimum and supremum of S we have that $\inf S \geq \inf J$ and $\sup S \leq \sup J$. Thus

$$\inf J \leq \inf S \leq \sup S \leq \sup J \quad (3.2)$$

which confirms that $I_S \subseteq J$. ■

EXERCISE 2.5.10

In the context of the proofs of Theorems 2.5.2 and 2.5.3, we have $\eta \in \cap_{n=1}^{\infty} I_n$. Also, $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$.

PROOF. Suppose $\eta \notin \cap_{n=1}^{\infty} I_n$. Then $\exists m | \eta \notin I_m$. By definition, $\eta \leq b_m$, so we must have that $\eta \leq a_m$. But since

$$a_m \leq b_k \forall k \in \mathbb{N} \quad (3.3)$$

then a_m is a greater lower bound for $\{b_k | k \in \mathbb{N}\}$ than is η . This contradicts our hypothesis that $\eta = \inf\{b_n | n \in \mathbb{N}\}$. Thus, $\eta \in \cap_{n=1}^{\infty} I_n$.

We now show that $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$. We begin by showing $[\xi, \eta] \subseteq \cap_{n=1}^{\infty} I_n$. Take $x \in [\xi, \eta]$, then $x \geq a_n \forall n$, since $x \geq \xi = \sup\{a_n | n \in \mathbb{N}\}$. Also, $x \leq b_n \forall n \in \mathbb{N}$, since $x \leq \eta = \inf\{b_n | n \in \mathbb{N}\}$. Thus,

$$a_n \leq x \leq b_n \quad (3.4)$$

without regard to the choice of n . Thus, $x \in I_n \forall n \in \mathbb{N}$, so $x \in \cap_{n=1}^{\infty} I_n$. Thus, $[\xi, \eta] \subseteq \cap_{n=1}^{\infty} I_n$. In order to establish that $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$, we are now obliged to show that $\cap_{n=1}^{\infty} I_n \subseteq [\xi, \eta]$. Take $x \in \cap_{n=1}^{\infty} I_n$, we know that $x \in [a_n, b_n] \forall n \in \mathbb{N}$, which means

$$a_n \leq x \leq b_n \forall n \in \mathbb{N} \quad (3.5)$$

Thus x is an upper bound for the set $\{a_n | n \in \mathbb{N}\}$ and a lower bound for $\{b_n | n \in \mathbb{N}\}$. This tells us that

$$a_n \leq \xi \leq x \leq \eta \leq b_n \forall n \in \mathbb{N} \quad (3.6)$$

so that $x \in [\xi, \eta]$. ■