UNIVERSITY OF TENNESSEE
DEPARTMENT OF MATHEMATICS

# MATH 447 - Homework 5

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## SECTION 3.1

### PROBLEM 10

Prove that if $\lim(x_n) = 0$ and if $x > 0$, then there exists a natural number $M$ such that $x_n > 0$ for all $n \geq M$.

PROOF. By the definition of limit, for each positive $\epsilon$ such an $M(\epsilon)$ can be found. Thus, take $\epsilon$ to be smaller than $x$, so that $x_n \in V_\epsilon(x) = (x - \epsilon, x + \epsilon)$ whenever $n \geq M$. Since $\epsilon < x$, $x - \epsilon \in \mathbb{P}$ so $x_n > 0$. ∎

### PROBLEM 14

Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$.

PROOF. We know that $n^{1/n} > 1 \; \forall n \in \mathbb{N}$. This implies that $n^{1/n} = 1 + k_n$ for some $k_n > 0$, so that we have $bn^{1/n} = b + bk_n$. Thus, $nb^n = (b + bk_n)^n$.

By the Binomial Theorem we know that

$$nb^n = (b + bk_n)^n = b^n + nb^{n-1}bk_n + \frac{1}{2}n(n-1)b^{n-2}b^2k_n{}^2 + \cdots$$

which tells us that $nb^n \geq b^n + \frac{1}{2}n(n-1)b^nk_n{}^2$. Factoring out $b^n$ and subtracting 1 yields

$$(n-1) \geq \frac{1}{2}n(n-1)k_n{}^2$$

so that we have $\frac{2}{n} \geq k_n{}^2$.

## PROBLEM 17

### HINT

If $n \geq 3$, then $0 < 2^n/n! \leq 2(\frac{2}{3})^{n-2}$.

PROOF.    Let $n = 3$, then $\frac{2^3}{3!} = \frac{8}{6} \leq 2(\frac{2}{3}) = \frac{4}{3} = \frac{8}{6}$. Thus, we have established a basis for induction.

Now assume that $0 < \frac{2^{n-1}}{(n-1)!} \leq 2\left(\frac{2}{3}\right)^{n-3}$. Multiplying by 2 yields $0 < \frac{2^n}{(n-1)!} \leq 2\left(\frac{2^{n-2}}{3^{n-3}}\right)$. Since $n \geq 3$, we have $0 < \frac{1}{n} \leq \frac{1}{3}$ so that

$$0 < \frac{2^n}{(n-1)!} \cdot \frac{1}{n} = \frac{2^n}{n!} \leq 2 \cdot \left(\frac{2^{n-2}}{3^{n-3}}\right) \cdot \frac{1}{3} = 2\left(\frac{2}{3}\right)^{n-2}$$

whence $0 < \frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$.  Therefore the hypothesis holds by the Principal of Mathematical Inducation. ∎

### MAIN PROBLEM

Show that $\lim(2^n/n!) = 0$.

PROOF.

## PROBLEM 18

If $\lim(x_n) = x > 0$, show that there exists a natural number $K$ such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

PROOF.

# SECTION 3.2

## PROBLEM 7

If $(b_n)$ is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$.  Explain why Thm 3.2.3 cannot be used.

PROOF.

## PROBLEM 9

Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}\,y_n)$ converges. Find the limit.

PROOF.

## PROBLEM 11

### PART A

Find $\lim\left((3\sqrt{n})^{1/2n}\right)$

PROOF.

### PART B

Find $(\sqrt{n^2+5n}-n)$

PROOF.

## PROBLEM 12

If $0 < a < b$, determine $\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.

PROOF.

## PROBLEM 13

If $a > 0$, $b > 0$, show that $\lim\left(\sqrt{(n+a)(n+b)}-n\right) = (a+b)/2$.

PROOF.

## PROBLEM 14

### PART A

Use the Squeeze Theorem to find $\lim(n^{1/n^2})$.

PROOF.

### PART B

Use the Squeeze Theorem to find $\lim((n!)^{1/n^2})$.

PROOF.

## PROBLEM 15

Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.

PROOF.

## PROBLEM 23

Show that if $(x_n)$ and $(y_n)$ are convergent sequences, then the sequences $(u_n)$ and $(v_n)$ defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent.

PROOF.

## PROBLEM 24

Show that if $(x_n)$, $(y_n)$, $(z_n)$ are convergent sequences, then the sequence $(w_n)$ defined by $w_n := \operatorname{mid}\{x_n, y_n, z_n\}$ is also convergent.

PROOF.