

# MATH 447 - Homework 4

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## 1 UNCOUNTABILITY OF THE REAL NUMBERS

The set  $\mathbb{R}$  of real numbers is uncountable.

PROOF. Assume the set  $I = [0, 1]$  is countable. This implies its elements are enumerable. That is,  $I = \{x_1, x_2, \dots, x_n, \dots\}$ .

Now construct the set  $I_1 = \{x \in I \mid x \neq x_1\}$  so that  $I_1 \subset I$  and  $I_1 = I \setminus \{x_1\}$ . Clearly this construction can progress in a recursive fashion so that  $I_j = I_{j-1} \setminus \{x_j\}$ , and  $I_j \supseteq I_{j+1} \supseteq \dots$ .

Thus, we have established a collection of nested intervals, so we know  $\exists \xi \in I_n \forall n \in \mathbb{N}$ . Suppose  $\xi = x_k$  for some  $k \in \mathbb{N}$ , then  $\xi \notin I_k$ , which contradicts the Nested Intervals Property. Clearly this is absurd, so we refute our hypothesis that  $I = [0, 1]$  is countable.

Since  $[0, 1] \subseteq \mathbb{R}$  is uncountable, we know that  $\mathbb{R}$  itself is uncountable. ■

## 2 BINARY REPRESENTATION OF REAL NUMBERS

If  $x \in [0, 1]$ , then there exists a sequence  $(a_n)$  of 0s and 1s such that

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_n}{2^n} \leq x \leq \frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_n + 1}{2^n} \quad (2.1)$$

for all  $n \in \mathbb{N}$ . Conversely, each sequence of 0s and 1s is the binary representation of a unique real number in  $[0, 1]$ .

PROOF. We begin by discussing an algorithm for constructing a sequence  $(a_n)$  based on the choice  $x$ . We then demonstrate that the upper and lower bounds given above form a nested sequence of intervals, and since we know (by the Nested Intervals Property) that the intersection of such objects is nonempty, we can be confident that such a construction for  $x$  is valid.

If  $x \neq \frac{1}{2}$  belongs to the left subinterval  $[0, \frac{1}{2}]$  we take  $a_1 = 0$ , while if  $x$  belongs to the right subinterval  $[\frac{1}{2}, 1]$  we take  $a_1 = 1$ . If  $x = \frac{1}{2}$ , then we may take  $a_1$  to be either 0 or 1. In any case we have

$$\frac{a_1}{2} \leq x \leq \frac{a_1}{2} + \frac{a_2 + 1}{2^2}. \quad (2.2)$$

We continue this bisection procedure, assigning at the  $n$ th stage the value  $a_n = 0$  if  $x$  is not the bisection point and lies in the left subinterval, and assigning the value  $a_n = 1$  if  $x$  lies in the right subinterval. Thus, we have a well-defined sequence  $(a_n)$  of 0s and 1s such that Eqn 2.1 above holds.

Now we proceed to show that the upper and lower bounds given in Eqn 2.1 form a sequence of nested intervals. Let  $L_n$  be the  $n$ th lower bound and  $U_n$  the  $n$ th upper bound so that

$$\begin{aligned} L_n &:= \sum_{i=1}^n \frac{a_i}{2^i} & U_n &:= \sum_{i=1}^n \frac{a_i}{2^i} + \frac{1}{2^n} \\ L_{n-1} &:= \sum_{i=1}^{n-1} \frac{a_i}{2^i} & U_{n-1} &:= \sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{1}{2^{n-1}} \end{aligned}$$

which yields the following useful relationships

$$\begin{aligned} L_n &= L_{n-1} + \frac{a_n}{2^n} \\ U_n &= U_{n-1} - \frac{1}{2^{n-1}} + \frac{a_n}{2^n} + \frac{1}{2^n} \end{aligned}$$

We know that each  $a_n$  is either 0 or 1, so we have two cases. If  $a_n = 0$ , then  $\frac{a_n}{2^n} = 0$  so that  $L_n = L_{n-1}$  and  $U_n = U_{n-1} - \frac{1}{2^n} < U_{n-1}$ . If  $a_n = 1$ , then  $\frac{a_n}{2^n} = \frac{1}{2^n}$  so that  $L_n = L_{n-1} + \frac{1}{2^n} > L_{n-1}$  and  $U_n = U_{n-1} - \frac{1}{2^{n-1}} + \frac{2}{2^n} = U_{n-1}$ . Thus, we have that  $L_n \geq L_{n-1}$  and  $U_n \leq U_{n-1}$ . Therefore, for each  $n \in \mathbb{N}$ , we have

$$L_{n-1} \leq L_n < U_n \leq U_{n-1} \quad (2.3)$$

so that the interval  $[L_{n-1}, U_{n-1}]$  contains the interval  $[L_n, U_n]$ . ■

### 3 SOME PROBLEMS CONCERNING INTERVALS

#### EXERCISE 2.5.3

If  $S \subseteq \mathbb{R}$  is a nonempty bounded set, and  $I_S := [\inf S, \sup S]$ , then  $S \subseteq I_S$ . Moreover, if  $J$  is any closed bounded interval containing  $S$ , then  $I_S \subseteq J$ .

PROOF. Since  $\inf S \leq s \forall s \in S$ , and  $\sup S \geq s \forall s \in S$ , we know

$$\inf S \leq s \leq \sup S \forall s \in S \quad (3.1)$$

Thus,  $s \in [\inf S, \sup S] \forall s \in S$ , so  $S \subseteq I_S$ .

Further, since  $J$  is a closed bounded interval containing  $S$  by hypothesis, we know that  $\inf J \leq s \forall s \in S$  so that  $\inf J$  is a lower bound for  $S$ , and we know that  $\sup J \geq s \forall s \in S$  so that  $\sup J$  is an upper bound for  $S$ . By definition of the infimum and supremum of  $S$  we have that  $\inf S \geq \inf J$  and  $\sup S \leq \sup J$ . Thus

$$\inf J \leq \inf S \leq \sup S \leq \sup J \quad (3.2)$$

which confirms that  $I_S \subseteq J$ . ■

#### EXERCISE 2.5.10

In the context of the proofs of Theorems 2.5.2 and 2.5.3, we have  $\eta \in \cap_{n=1}^{\infty} I_n$ . Also,  $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$ .

PROOF. Suppose  $\eta \notin \cap_{n=1}^{\infty} I_n$ . Then  $\exists m | \eta \notin I_m$ . By definition,  $\eta \leq b_m$ , so we must have that  $\eta \leq a_m$ . But since

$$a_m \leq b_k \forall k \in \mathbb{N} \quad (3.3)$$

then  $a_m$  is a greater lower bound for  $\{b_k | k \in \mathbb{N}\}$  than is  $\eta$ . This contradicts our hypothesis that  $\eta = \inf\{b_n | n \in \mathbb{N}\}$ . Thus,  $\eta \in \cap_{n=1}^{\infty} I_n$ .

We now show that  $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$ . We begin by showing  $[\xi, \eta] \subseteq \cap_{n=1}^{\infty} I_n$ . Take  $x \in [\xi, \eta]$ , then  $x \geq a_n \forall n$ , since  $x \geq \xi = \sup\{a_n | n \in \mathbb{N}\}$ . Also,  $x \leq b_n \forall n \in \mathbb{N}$ , since  $x \leq \eta = \inf\{b_n | n \in \mathbb{N}\}$ . Thus,

$$a_n \leq x \leq b_n \quad (3.4)$$

without regard to the choice of  $n$ . Thus,  $x \in I_n \forall n \in \mathbb{N}$ , so  $x \in \cap_{n=1}^{\infty} I_n$ . Thus,  $[\xi, \eta] \subseteq \cap_{n=1}^{\infty} I_n$ . In order to establish that  $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$ , we are now obliged to show that  $\cap_{n=1}^{\infty} I_n \subseteq [\xi, \eta]$ . Take  $x \in \cap_{n=1}^{\infty} I_n$ , we know that  $x \in [a_n, b_n] \forall n \in \mathbb{N}$ , which means

$$a_n \leq x \leq b_n \forall n \in \mathbb{N} \quad (3.5)$$

Thus  $x$  is an upper bound for the set  $\{a_n | n \in \mathbb{N}\}$  and a lower bound for  $\{b_n | n \in \mathbb{N}\}$ . This tells us that

$$a_n \leq \xi \leq x \leq \eta \leq b_n \forall n \in \mathbb{N} \quad (3.6)$$

so that  $x \in [\xi, \eta]$ . ■