UNIVERSITY OF TENNESSEE
DEPARTMENT OF MATHEMATICS

MATH 447 - Homework 5

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SECTION 3.1

PROBLEM 10

Prove that if $\lim(x_n) = 0$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

PROOF. By the definition of limit, for each positive ϵ such an $M(\epsilon)$ can be found. Thus, take ϵ to be smaller than x, so that $x_n \in V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ whenever $n \geq M$. Since $\epsilon < x$, $x - \epsilon \in \mathbb{P}$ so $x_n > 0$. ■

PROBLEM 14

Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$.

PROOF. We first begin by proving the following Lemma: If $(k_n)^2 \to 0$ and $k_n < 0$, then $k_n \to 0$.

By hypothesis, given an $\epsilon > 0$, there exists some $M(\epsilon) \in \mathbb{N}$ such that $|(k_n)^2 - 0| < \epsilon$ whenever $n \geq M(\epsilon)$. Further, $|(k_n)^2 - 0| = ||k_n| \cdot |k_n|| = |k_n|^2$ so that we have $|k_n| < \sqrt{\epsilon}$. Since the absolute value of k_n is smaller than any arbitrary positive number, and we know that $k_n < 0$ for all choices of n, we have that $k_n \to 0$.

Now we may proceed with the main portion of our argument. Since $b \in (0,1)$, we may define a sequence (k_n) such that $b^{1/n} = 1 + k_n$ for some $k_n < 0$. This implies that $b = (1+k_n)^n$.

The Binomial Theorem tells us that $b = 1 + nk_n + \frac{1}{2}n(n-1)(k_n)^2 + \cdots \geq 1 + \frac{1}{2}n(n-1)(k_n)^2$, so we have that $b - 1 \geq \frac{1}{2}n(n-1)(k_n)^2$. Thus, $\frac{2b-2}{n(n-1)} \geq (k_n)^2$.

We know from previous work that $\frac{1}{n(n-1)} \to 0$, and since $\frac{2b-2}{n(n-1)}$ is a constant times a convergent expression, we know that its limit is zero as well. Since $(k_n)^2$ is bounded above by a sequence which converges to zero, and below by the constant sequence (0), we have that $(k_n)^2 \to 0$. Thus, by our earlier Lemma, k_n

Problem 17

Hint

If $n \geq 3$, then $0 < 2^n/n! \leq 2(\frac{2}{3})^{n-2}$.

Proof.

Main Problem

Show that $\lim(2^n/n!) = 0$.

Proof.

Problem 18

If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

Proof.

Section 3.2

Problem 7

If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Thm 3.2.3 cannot be used.

Proof.

Problem 9

Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}y_n)$ converges. Find the limit.

Proof.

Problem 11

Part A

Find $\lim\left((3\sqrt{n})^{1/2n}\right)$

Proof.

Part B

Find $(\sqrt{n^2+5n}-n)$

Proof.

Problem 12

If $0<a<b$, determine $\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.

Proof.

Problem 13

If $a>0$, $b>0$, show that $\lim\left(\sqrt{(n+a)(n+b)}-n\right)=(a+b)/2$.

Proof.

Problem 14

Part A

Use the Squeeze Theorem to find $\lim(n^{1/n^2})$.

Proof.

Part B

Use the Squeeze Theorem to find $\lim((n!)^{1/n^2})$.

Proof.

Problem 15

Show that if $z_n := (a^n+b^n)^{1/n}$ where $0<a<b$, then $\lim(z_n)=b$.

Proof.

Problem 23

Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent.

Proof.

Problem 24

Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \mathrm{mid}\{x_n, y_n, z_n\}$ is also convergent.

Proof.