University of Tennessee
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MATH 447 - Homework 6

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Section 3.3, Ex. 3

Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

Proof. We can demonstrate by induction that (x_n) is bounded below by 2. First note that $x_1 \geq 2$ by by hypothesis. Thus,

$$\begin{aligned} x_1 &\geq 2 \\ x_1 - 1 &\geq 1 \\ \sqrt{x_1 - 1} &\geq 1 \\ 1 + \sqrt{x_1 - 1} &\geq 1 + 1 \\ x_2 &\geq 2 \end{aligned}$$

so we have a basis for induction. Assume now that $x_n \geq 2$, and we will use this to demonstrate that $x_{n+1} \geq 2$:

$$\begin{aligned} x_n &\geq 2 \\ x_n - 1 &\geq 1 \\ \sqrt{x_n - 1} &\geq 1 \\ 1 + \sqrt{x_n - 1} &\geq 1 + 1 \\ x_{n+1} &\geq 2 \end{aligned}$$

Thus we have confirmed that (x_n) is bounded below by 2.

We can also demonstrate by induction that (x_n) is decreasing. Begin by noting that $x_1 \geq 2$, which implies that $x_1 - 1 \geq 1$. This tells us that $\sqrt{x_1} \geq 1$. Taking the difference of the previous equations tells us that $(x_1 - 1) - (\sqrt{x_1 - 1}) \geq 0$, and thus that $x_1 \geq 1 + \sqrt{x_1 - 1} = x_2$. Since $x_1 \geq x_2$, we have established a basis for induction.

Now assume that $x_{n-1} \geq x_n$ and recall that since 2 is a lower bound, $x_n - 1 \geq 1$. We aim to show that this implies $x_n \geq x_{n+1}$.

$$
\begin{aligned}
x_{n-1} &\geq x_n \\
x_{n-1} - 1 &\geq x_n - 1 \\
\sqrt{x_{n-1} - 1} &\geq \sqrt{x_n - 1} \\
1 + \sqrt{x_{n-1} - 1} &\geq 1 + \sqrt{x_n - 1} \\
x_n &\geq x_{n+1}
\end{aligned}
$$

Thus we have shown that (x_n) decreases.

Since (x_n) is decreasing, bounded below, and recursively defined, we may calculate its limit by acknowledging that $\lim(x_n) = \lim(x_{n+1})$, and for convenience we shall denote this $x*$. Thus,

$$
\begin{aligned}
x^* &= 1 + \sqrt{x^* - 1} \\
x^* + 1 &= \sqrt{x^* - 1} \\
(x^* + 1)^2 &= x^* - 1 \\
x^{*2} - 3x^* + 2 &= 0
\end{aligned}
$$

The solutions of the final quadratic equation are 1 and 2, however, since 2 is a lower bound for (x_n), we know its limit may not be 1. Thus, $\lim(x_n) = 2$.■

Section 3.3, Ex. 4

Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

Proof. $x_2 = \sqrt{3} < 2$, so we know that x_1 and x_2 are both less than 2. Assume $x_n < 2$, then

$$
\begin{aligned}
x_n &< 2 \\
x_n + 2 &< 2 + 2 \\
\sqrt{x_n + 2} &< \sqrt{4} \\
x_{n+1} = \sqrt{x_n + 2} &< 2
\end{aligned}
$$

so by induction we have that (x_n) is increasing and bounded above by 2. Since it is recursively defined, we may acknowledge that $\lim(x_n) = \lim(x_{n+1})$ and call that value x^*.

$$x^* = \sqrt{2+x^*}$$
$$(x^*)^2 = 2 + x*$$
$$(x^*)^2 - x * -2 = 0$$

so that solving with the quadratic formula gives us $x^* = 2$. ■