UNIVERSITY OF TENNESSEE
DEPARTMENT OF MATHEMATICS

MATH 447 - Homework 5

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SECTION 3.1

PROBLEM 10. Prove that if $\lim(x_n) = 0$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

PROOF. By the definition of limit, for each positive ϵ such an $M(\epsilon)$ can be found. Thus, take ϵ to be smaller than x, so that $x_n \in V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ whenever $n \geq M$. Since $\epsilon < x$, $x - \epsilon \in \mathbb{P}$ so $x_n > 0$. ■

PROBLEM 14. Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$.

PROOF. We first begin by proving the following Lemma: If $(k_n)^2 \to 0$ and $k_n < 0$, then $k_n \to 0$.

By hypothesis, given an $\epsilon > 0$, there exists some $M(\epsilon) \in \mathbb{N}$ such that $|(k_n)^2 - 0| < \epsilon$ whenever $n \geq M(\epsilon)$. Further, $|(k_n)^2 - 0| = ||k_n| \cdot |k_n|| = |k_n|^2$ so that we have $|k_n| < \sqrt{\epsilon}$. Since the absolute value of k_n is smaller than any arbitrary positive number, and we know that $k_n < 0$ for all choices of n, we have that $k_n \to 0$.

Now we may proceed with the main portion of our argument. Since $b \in (0,1)$, we may define a sequence (k_n) such that $b^{1/n} = 1 + k_n$ for some $k_n < 0$. This implies that $b = (1+k_n)^n$.

The Binomial Theorem tells us that $b = 1 + nk_n + \frac{1}{2}n(n-1)(k_n)^2 + \cdots \geq 1 + \frac{1}{2}n(n-1)(k_n)^2$, so we have that $b - 1 \geq \frac{1}{2}n(n-1)(k_n)^2$. Thus, $\frac{2b-2}{n(n-1)} \geq (k_n)^2$.

We know from previous work that $\frac{1}{n(n-1)} \to 0$, and since $\frac{2b-2}{n(n-1)}$ is a constant times a convergent expression, we know that its limit is zero as well. Since $(k_n)^2$ is bounded above by a sequence which converges to zero, and below by the constant sequence (0), we have that $(k_n)^2 \to 0$. Thus, by our earlier Lemma, k_n