UNIVERSITY OF TENNESSEE, DEPARTMENT OF MATHEMATICS

MATH 447 - Homework 4

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1 UNCOUNTABILITY OF THE REAL NUMBERS

The set $\mathbb{R}$ of real numbers is uncountable.

PROOF. Assume the set $I = [0,1]$ is countable. This implies its elements are enumerable. That is, $I = \{x_1, x_2, \ldots, x_n, \ldots\}$.

Now construct the set $I_1 = \{x \in I | x \neq x_1\}$ so that $I_1 \subset I$ and $I_1 = I \setminus \{x_1\}$. Clearly this construction can progress in a recursive fashion so that $I_j = I_{j-1} \setminus \{x_j\}$, and $I_j \supseteq I_{j+1} \supseteq \cdots$.

Thus, we have establshed a collection of nested intervals, so we know $\exists \xi \in I_n \forall n \in \mathbb{N}$. Suppose $\xi = x_k$ for some $k \in \mathbb{N}$, then $\xi \notin I_k$, which contradicts the Nested Intervals Property. Clearly this is absurd, so we refute our hypothesis that $I = [0,1]$ is countable.

Since $[0,1] \subseteq \mathbb{R}$ is uncountable, we know that $\mathbb{R}$ itself is uncountable.■

2 BINARY REPRESENTATION OF REAL NUMBERS

If $x \in [0,1]$, then there exists a sequence (a_n) of 0s and 1s such that

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} \leq x \leq \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n + 1}{2^n} \tag{2.1}$$

for all $n \in \mathbb{N}$. Conversely, each sequence of 0s and 1s is the binary representation of a unique real number in $[0,1]$.

PROOF. We begin by discussing an algorithm for constructing a sequence (a_n) based on the choice x. We then demonstrate that the upper and lower bounds given above form a nested sequence of intervals, and since we know (by the Nested Intervals Property) that the intersection of such objects is nonempty, we can be confident that such a construction for x is valid.

If $x \neq \frac{1}{2}$ belongs to the left subinterval $[0, \frac{1}{2}]$ we take $a_1 = 0$, while if x belongs to the right subinterval $[\frac{1}{2}, 1]$ we take $a_1 = 1$. If $x = \frac{1}{2}$, then we may take a_1 to be either 0 or 1. In any case we have

$$\frac{a_1}{2} \leq x \leq \frac{a_1}{2} + \frac{a_2 + 1}{2^2}. \tag{2.2}$$

We continue this bisection procedure, assigning at the nth stage the value $a_n = 0$ if x is not the bisection point and lies in the left subinterval, and assigning the value $a_n = 1$ if x lies in the right subinterval. Thus, we have a well-defined sequence (a_n) of 0s and 1s such that Eqn 2.1 above holds.

Now we proceed to show that the upper and lower bounds given in Eqn 2.1 form a sequence of nested intervals. Let L_n be the nth lower bound and U_n the nth upper bound so that

$$L_n := \sum_1^n \frac{a_i}{2^i} \qquad U_n := \sum_1^n \frac{a_i}{2^i} + \frac{1}{2^n}$$

$$L_{n-1} := \sum_1^{n-1} \frac{a_i}{2^i} \qquad U_{n-1} := \sum_1^{n-1} \frac{a_i}{2^i} + \frac{1}{2^{n-1}}$$

which yields the following useful relationships

$$L_n = L_{n-1} + \frac{a_n}{2^n}$$

$$U_n = U_{n-1} - \frac{1}{2^{n-1}} + \frac{a_n}{2^n} + \frac{1}{2^n}$$

We know that each a_n is either 0 or 1, so we have two cases. If $a_n = 0$, then $\frac{a_n}{2^n} = 0$ so that $L_n = L_{n-1}$ and $U_n = U_{n-1} - \frac{1}{2^n} < U_{n-1}$. If $a_n = 1$, then $\frac{a_n}{2^n} = \frac{1}{2^n}$ so that $L_n = L_{n-1} + \frac{1}{2^n} > L_{n-1}$ and $U_n = U_{n-1} - \frac{1}{2^{n-1}} + \frac{2}{2^n} = U_{n-1}$. Thus, we have that $L_n \geq L_{n-1}$ and $U_n \leq U_{n-1}$. Therefore, for each $n \in \mathbb{N}$, we have

$$L_{n-1} \leq L_n < U_n \leq U_{n-1} \tag{2.3}$$

so that the interval $[L_{n-1}, U_{n-1}]$ contains the interval $[L_n, U_n]$.■