University of Tennessee
Department of Mathematics

MATH 447 - Homework 6

Robert D. French

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Section 3.3, Ex. 3

Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

Proof. We can demonstrate by induction that (x_n) is bounded below by 2. First note that $x_1 \geq 2$ by by hypothesis. Thus,

$$\begin{aligned} x_1 &\geq 2 \\ x_1 - 1 &\geq 1 \\ \sqrt{x_1 - 1} &\geq 1 \\ 1 + \sqrt{x_1 - 1} &\geq 1 + 1 \\ x_2 &\geq 2 \end{aligned}$$

so we have a basis for induction. Assume now that $x_n \geq 2$, and we will use this to demonstrate that $x_{n+1} \geq 2$:

$$\begin{aligned} x_n &\geq 2 \\ x_n - 1 &\geq 1 \\ \sqrt{x_n - 1} &\geq 1 \\ 1 + \sqrt{x_n - 1} &\geq 1 + 1 \\ x_{n+1} &\geq 2 \end{aligned}$$

Thus we have confirmed that (x_n) is bounded below by 2.

We can also demonstrate by induction that (x_n) is decreasing. Begin by noting that $x_1 \geq 2$, which implies that $x_1 - 1 \geq 1$. This tells us that $\sqrt{x_1} \geq 1$. Taking the difference of the previous equations tells us that $(x_1 - 1) - (\sqrt{x_1 - 1}) \geq 0$, and thus that $x_1 \geq 1 + \sqrt{x_1 - 1} = x_2$. Since $x_1 \geq x_2$, we have established a basis for induction.

Now assume that $x_{n-1} \geq x_n$ and recall that since 2 is a lower bound, $x_n - 1 \geq 1$. We aim to show that this implies $x_n \geq x_{n+1}$.

$$\begin{aligned} x_{n-1} &\geq x_n \\ x_{n-1} - 1 &\geq x_n - 1 \\ \sqrt{x_{n-1} - 1} &\geq \sqrt{x_n - 1} \\ 1 + \sqrt{x_{n-1} - 1} &\geq 1 + \sqrt{x_n - 1} \\ x_n &\geq x_{n+1} \end{aligned}$$

Thus we have shown that (x_n) decreases.

Since (x_n) is decreasing, bounded below, and recursively defined, we may calculate its limit by acknowledging that $\lim(x_n) = \lim(x_{n+1})$, and for convenience we shall denote this $x*$. Thus,

$$\begin{aligned} x^* &= 1 + \sqrt{x^* - 1} \\ x^* + 1 &= \sqrt{x^* - 1} \\ (x^* + 1)^2 &= x^* - 1 \\ x^{*2} - 3x^* + 2 &= 0 \end{aligned}$$

The solutions of the final quadratic equation are 1 and 2, however, since 2 is a lower bound for (x_n), we know its limit may not be 1. Thus, $\lim(x_n) = 2$.