University of Tennessee
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MATH 447 - Homework 6

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Section 3.3, Ex. 3

Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

Proof. We can demonstrate by induction that (x_n) is bounded below by 2. First note that $x_1 \geq 2$ by by hypothesis. Thus,

$$\begin{aligned} x_1 &\geq 2 \\ x_1 - 1 &\geq 1 \\ \sqrt{x_1 - 1} &\geq 1 \\ 1 + \sqrt{x_1 - 1} &\geq 1 + 1 \\ x_2 &\geq 2 \end{aligned}$$

so we have a basis for induction. Assume now that $x_n \geq 2$, and we will use this to demonstrate that $x_{n+1} \geq 2$:

$$\begin{aligned} x_n &\geq 2 \\ x_n - 1 &\geq 1 \\ \sqrt{x_n - 1} &\geq 1 \\ 1 + \sqrt{x_n - 1} &\geq 1 + 1 \\ x_{n+1} &\geq 2 \end{aligned}$$

Thus we have confirmed that (x_n) is bounded below by 2.

We can also demonstrate by induction that (x_n) is decreasing. Begin by noting that $x_1 \geq 2$, which implies that $x_1 - 1 \geq 1$. This tells us that $\sqrt{x_1} \geq 1$. Taking the difference of the previous equations tells us that $(x_1 - 1) - (\sqrt{x_1 - 1}) \geq 0$, and thus that $x_1 \geq 1 + \sqrt{x_1 - 1} = x_2$. Since $x_1 \geq x_2$, we have established a basis for induction.

Now assume that $x_{n-1} \geq x_n$ and recall that since 2 is a lower bound, $x_n - 1 \geq 1$. We aim to show that this implies $x_n \geq x_{n+1}$.

$$\begin{aligned}
x_{n-1} &\geq x_n \\
x_{n-1} - 1 &\geq x_n - 1 \\
\sqrt{x_{n-1} - 1} &\geq \sqrt{x_n - 1} \\
1 + \sqrt{x_{n-1} - 1} &\geq 1 + \sqrt{x_n - 1} \\
x_n &\geq x_{n+1}
\end{aligned}$$

Thus we have shown that (x_n) decreases.

Since (x_n) is decreasing, bounded below, and recursively defined, we may calculate its limit by acknowledging that $\lim(x_n) = \lim(x_{n+1})$, and for convenience we shall denote this $x*$. Thus,

$$\begin{aligned}
x^* &= 1 + \sqrt{x^* - 1} \\
x^* + 1 &= \sqrt{x^* - 1} \\
(x^* + 1)^2 &= x^* - 1 \\
x^{*2} - 3x^* + 2 &= 0
\end{aligned}$$

The solutions of the final quadratic equation are 1 and 2, however, since 2 is a lower bound for (x_n), we know its limit may not be 1. Thus, $\lim(x_n) = 2$.■

SECTION 3.3, EX. 4

Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

PROOF. $x_2 = \sqrt{3} < 2$, so we know that x_1 and x_2 are both less than 2. Assume $x_n < 2$, then

$$\begin{aligned}
x_n &< 2 \\
x_n + 2 &< 2 + 2 \\
\sqrt{x_n + 2} &< \sqrt{4} \\
x_{n+1} = \sqrt{x_n + 2} &< 2
\end{aligned}$$

so by induction we have that (x_n) is increasing and bounded above by 2. Since it is recursively defined, we may acknowledge that $\lim(x_n) = \lim(x_{n+1})$ and call that value x^*.

$$\begin{aligned} x^* &= \sqrt{2+x^*} \\ (x^*)^2 &= 2 + x* \\ (x^*)^2 - x* - 2 &= 0 \end{aligned}$$

so that solving with the quadratic formula gives us $x^* = 2$. ■

SECTION 3.3., EX. 5

Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit.

PROOF. The book tells us that $y_n \leq 1 + 2\sqrt{p}\ \forall n \in \mathbb{N}$, so (y_n) is bounded above. Since (y_n) is increasing, we know that $\lim(y_n) = \sup\{y_n : n \in \mathbb{N}\}$, and we call this value $y*$. Since this sequence is recursively defined, $\lim(y_{n+1}) = y^*$. Thus,

$$\begin{aligned} y^* &= \sqrt{p + y^*} \\ (y^*)^2 &= p + y^* \\ (y^*) - y^* &= p \\ (y^*)^2 - y^* - p &= 0 \end{aligned}$$

so by the quadratic formula, $y* \in \{\frac{1+\sqrt{1+4p}}{2}, \frac{1-\sqrt{1+4p}}{2}\}$. We can rule out the second element by noting that:

$$\begin{aligned} p &> 0 \\ 4p &> 0 \\ 1 + 4p &> 1 \\ \sqrt{1+4p} &> 1 \\ -\sqrt{1+4p} &< -1 \\ 1 - \sqrt{1+4p} &< 0 \end{aligned}$$

and since $y_n > 0$, it is clear that $y^* = \frac{1+\sqrt{1+4p}}{2}$.■

SECTION 3.3, EX. 9

Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup A$. Show there exists an increasing sequence (x_n) with $x_n \in A\ \forall n \in \mathbb{N}$ such that $u = \lim(x_n)$.

PROOF. By the Supremum Principle, for all $\epsilon > 0$, $\exists x_\epsilon \in A$ such that $u \geq x_\epsilon > x - \epsilon$. Pick some decreasing sequence of (ϵ_n) that converges to zero, and then $(x_n) = (x_{\epsilon n})$. ■

SECTION 3.3, EX. 11

Let $x_n := 1/1^1 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathbb{N}$. Prove that (x_n) is increasing and bounded, and hence converges.

PROOF. We want to show that (x_n) is increasing. First note that (x_n) is a sum of positive terms, it is positive. Thus, since $x_{n+1} = x_n + 1/(n+1)^2$, it is clear that (x_n) is increasing.

The book tells us that $1/k^2 \leq 1/k(k-1) = 1/(k-1) - 1/k$ when $k \geq 2$. Noting that $x_k - x_{k-1} = 1/k^2$ and also that $\lim(1/k^2) = 0$, we have that $\lim(x_k - x_{k-1}) = 0$. This tells us that for large enough k, we have $|x_k - x_{k-1}| < \epsilon$, so by the Cauchy Criterion (x_n) converges. This infers that it is bounded above. ■

SECTION 3.3, EX 2

Use the method of Example 3.4.3(b) to show that if $0 < c < 1$, then $\lim(c^{1/n}) = 1$.

PROOF. We begin by showing that (z_n) is increasing. Consider that

$$
\begin{gathered}
c < 1 \\
c^{1/n} < 1 \\
c^{1/n} \cdot c < 1 \cdot c < 1 \\
c^{1/n+1} < c < 1 \\
c^{(n+1)/n} < c < 1 \\
c^{1/n} < c^{1/(n+1)} < 1
\end{gathered}
$$

which tells us that (z_n) is both increasing and bounded above by 1. This confirms that $\lim(z_n)$ exists, so let us call it z^*. Thus, the subsequence (z_{2n}) exists and converges to z^*.

Since $z_{2n} = c^{1/2n} = \sqrt{c^{1/n}}$, we know $\lim(z_{2n}) = \sqrt{z^*}$. But since $\lim(z_{2n}) = z^*$, we have that $z^* = \sqrt{z^*}$. Thus, $z^* \in 0, 1$. Since $z_1 = c > 0$, and (z_n) is increasing, we know that $z^* = 1$. ■