The prefix membership problem for one-relator groups

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(joint work with Igor Dolinka)

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One-relator groups

Definition

A one-relator group is a group defined by a presentation of the form

$$\mathrm{Gp}\langle A \mid w = 1\rangle = \mathrm{FG}(A)/\langle\!\langle w \rangle\!\rangle$$

where A is a finite alphabet and $w \in (A \cup A^{-1})^*$.

- Magnus 1932: One-relator groups have decidable word problem.

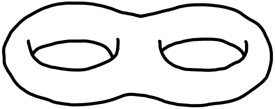
Example

- $\mathbb{Z} \times \mathbb{Z} = \mathrm{Gp}\langle x, y \mid [x, y] = 1\rangle$
  where $[x, y] = x^{-1}y^{-1}xy$.
- Baumslag–Solitar groups

  $$B(m, n) = \mathrm{Gp}\langle a, b | b^{-1}a^m ba^{-n} = 1\rangle$$

- Surface groups

  $$\mathrm{Gp}\langle a_1, \ldots, a_g, b_1, \ldots, b_g | [a_1, b_1] \ldots [a_g, b_g] = 1\rangle.$$



Submonoid and prefix membership problem

G - a finitely generated group with a finite group generating set A.

$\pi : (A \cup A^{-1})^* \to G$ – the canonical monoid homomorphism.

T – a finitely generated submonoid of G.

The membership problem for T in G is decidable if there is an algorithm which solves the following decision problem:

INPUT: A word $w \in (A \cup A^{-1})^*$.

QUESTION: $\pi(w) \in T$?

Prefix membership problem

Let $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ and set

$$P_w = \mathrm{Mon}\langle \mathrm{pref}(w) \rangle \leq G$$

the submonoid generated by the elements of G represented by prefixes of w. We call P_w the prefix monoid. Then G has decidable prefix membership problem if the membership problem for P_w in G is decidable.

Prefix membership problem

Example

If $G = \mathrm{Gp}\langle A \mid w = 1\rangle = \mathrm{Gp}\langle a, b \mid aba^{-1}b^{-1} = 1\rangle$ then $P_w = \mathrm{Mon}\langle a, ab, aba^{-1} = b\rangle = \mathrm{Mon}\langle a, b\rangle$ - the submonoid of all elements that can be written as positive words. The prefix membership problem is decidable in this case by rewriting to normal form $a^i b^j$ with $i, j \in \mathbb{Z}$ and checking $i, j \geq 0$. For example:

$$a^5 b^{-7} a^{-10} b^8 a^9 \in P_w, \quad \text{but} \quad a^3 b^5 a^{-5} b^2 \notin P_w.$$

Open problem: Does every one-relator group $\mathrm{Gp}\langle A \mid w = 1\rangle$ have decidable prefix membership problem?

Proved true in several cases e.g. when w satisfies...

- Idempotent word [Birget, Margolis, Meakin, 1993, 1994]
- w-strictly positive [Ivanov, Margolis, Meakin, 2001]
- Certain Adjan-type and Baumslag–Solitar type [Margolis, Meakin, Šunić, 2005]
- Certain small cancellation conditions [A. Juhász, 2012, 2014].

GROUPS

INVERSE MONOIDS

MONOIDS

$\mathrm{Gp}\langle A \mid w = 1\rangle$
DECIDABLE PREFIX MEMBERSHIP
W - REDUCED WORD

$\mathrm{Inv}\langle A \mid w = 1\rangle$
DECIDABLE WORD PROBLEM
W - REDUCED WORD

$\Longrightarrow$

$\mathrm{Mon}\langle A \mid u = v\rangle$
DECIDABLE WORD PROBLEM

$\Downarrow$

$\Downarrow$

$\mathrm{Gp}\langle A \mid w = 1\rangle$
DECIDABLE PREFIX MEMBERSHIP
W - CYCLICALLY REDUCED WORD

$\Longrightarrow$

$\mathrm{Inv}\langle A \mid w = 1\rangle$
DECIDABLE WORD PROBLEM
W - CYCLICALLY REDUCED WORD

IVANOV, MARGOLIS, MEAKIN (2001)

Reduced vs cyclically reduced words

$aba^{-1}ab$ - not reduced,

$abba^{-1}$ - reduced but not cyclically reduced

$aba^{-1}b^{-1}$ - cyclically reduced

Right-angled Artin groups

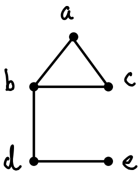
Definition

The right-angled Artin group $A(\Gamma)$ associated with the graph Γ is

$$\mathrm{Gp}\langle V\Gamma \mid uv = vu \text{ if and only if } \{u, v\} \in E\Gamma\rangle.$$

Example

$\Gamma =$



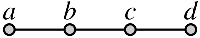
$$A(\Gamma) = \mathrm{Gp}\langle a, b, c, d, e \mid ac = ca,\ de = ed,\ ab = ba,\ bc = cb,\ bd = db \rangle$$

Right-angled Artin subgroups of one-relator groups

Theorem (RDG (2020))

$A(\Gamma)$ embeds into some one-relator group $\iff$ Γ is a finite forest.

Lohrey & Steinberg (2008) proved that $A(P_4)$ contains a finitely generated submonoid T in which membership is undecidable, where P_4 is the graph



Theorem (RDG (2020))

There is a one-relator group $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ with a fixed finitely generated submonoid $N \leq G$ such that the membership problem for N within G is undecidable.

Example

$\mathrm{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1\rangle$ is a one-relator group with this property.

Conservative factorisations

Definition

Let $w \in (A \cup A^{-1})^*$. Then for a factorisation

$$w \equiv w_1 w_2 \dots w_k$$

let $P(w_1, \dots, w_k)$ denote the submonoid of $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ generated by

$$\bigcup_{i=1}^{k} \mathrm{pref}(w_i).$$

- We always have $P_w \subseteq P(w_1, \dots, w_k)$ – since every prefix of w is a product of prefixes of the w_i

$$w_1 w_2 \dots w_{r-1} w_r' w_r'' w_{r+1} \dots w_k.$$

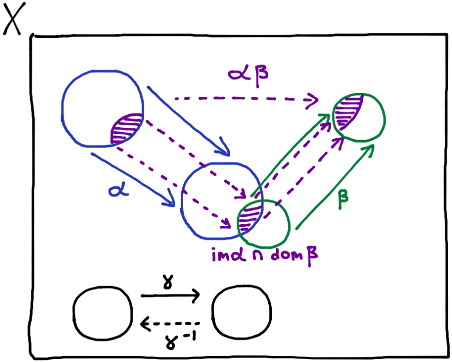
- If $P_w = P(w_1, \dots, w_k)$ then we say that the factorisation $w \equiv w_1 \dots w_k$ is conservative.

Inverse monoids

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

$m \in M$ is a right unit if there is an $n \in M$ such that $mn = 1$, left unit is defined analogously, and a unit is an element that is both a left and right unit.

Example: I_X = monoid of all partial bijections $X \to X$



Examples: In I_3

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & - & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & 2 \end{pmatrix}$$

Note:
$\gamma\gamma^{-1} = \mathrm{id}_{\mathrm{dom}\gamma}$

Inverse monoid presentations

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

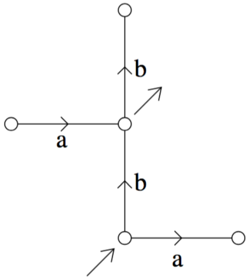
For all $x, y \in M$ we have

$$x = xx^{-1}x,\ (x^{-1})^{-1} = x,\ (xy)^{-1} = y^{-1}x^{-1},\ xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \qquad (\dagger)$$

$$\mathrm{Inv}\langle A \mid u_i = v_i\ (i \in I)\rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid u_i = v_i\ (i \in I) \cup (\dagger)\rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and x, y range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\mathrm{FIM}(A) = \mathrm{Inv}\langle A \mid \rangle$



Munn (1974)

Elements of $\mathrm{FIM}(A)$ can be represented using Munn trees. e.g. in $\mathrm{FIM}(a, b)$ we have $u = w$ where

$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$

$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$

Units and conservative factorisations

Definition

We say that a factorisation $w \equiv w_1 \dots w_m$ is unital if each factor w_i represents a unit of the inverse monoid $M = \mathrm{Inv}\langle A \mid w = 1 \rangle$. The $w_1, \dots, w_m$ are called invertible pieces.

Theorem (Dolinka & RDG (2021))

If $\mathrm{Inv}\langle A \mid w_1 \dots w_m = 1 \rangle$ is a unital factorisation then $\mathrm{Gp}\langle A \mid w_1 \dots w_m = 1 \rangle$ is a conservative factorisation.

- The converse is often true e.g. when $w \equiv w_1 \dots w_m$ is cyclically reduced.

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- The converse is often true e.g. when $w \equiv w_1 \dots w_m$ is cyclically reduced.

Adjan overlap method for computing invertible pieces based on the fact that for $\alpha, \beta, \gamma \in (A \cup A^{-1})^*$

$$\alpha\beta \text{ and } \beta\gamma \text{ both units} \Longrightarrow \alpha, \beta, \text{ and } \gamma \text{ are all units.}$$

Example

$abcdabcdcdab = 1$

$\Rightarrow ab$, $cdabcdcdab$, and $abcdabcdcd$ are all units.

$\Rightarrow ab$ and cd are units. So

$\mathrm{Gp}\langle A \mid (ab)(cd)(ab)(cd)(cd) = 1 \rangle$ is a conservative factorisation.

Computing unital factorisations

Benois method for computing invertible pieces method introduced in [RDG & Ruškuc, 2021] based on the facts that:

- The words $U = (\operatorname{pref} w \cup \operatorname{pref}(w^{-1}))^*$ are all right units in $\mathrm{Inv}\langle A \mid w = 1\rangle$.
- A word u is a unit if and only if u and u^{-1} are right units.
- If u is a right unit then $u = \mathrm{red}(u)$ (free reduction) in $\mathrm{Inv}\langle A \mid w = 1\rangle$.

We call it the Benois pieces computing algorithm since it makes use of the fact that the membership in $\mathrm{Mon}\langle U\rangle$ in the free group $\mathrm{FG}(A)$ is decidable, which which follows from

Theorem (Benois (1969))

Every free group of finite rank has a decidable rational subset[2] membership problem.

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[2]The class of The *rational subsets* of a group is the smallest set containing all finite subsets and is closed under union, product and submonoid generation.

O'Hare example (Margolis, Meakin and Stephen, 1987)

$$\mathrm{Inv}\langle a, b, c, d \mid abcdacdadabbcdacd = 1\rangle = \mathrm{Inv}\langle a, b, c, d \mid \alpha\beta\gamma\delta\beta = 1\rangle.$$

where

$$\alpha \equiv abcd,\ \beta \equiv acd,\ \gamma \equiv ad,\ \delta \equiv abbcd.$$

Claim:

$$r \equiv abcd \cdot acd \cdot ad \cdot abbcd \cdot acd = \alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \beta$$

is a unital factorization.

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where

$$\alpha \equiv abcd,\ \beta \equiv acd,\ \gamma \equiv ad,\ \delta \equiv abbcd.$$

Claim:

$$r \equiv abcd \cdot acd \cdot ad \cdot abbcd \cdot acd = \alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \beta$$

is a unital factorization. This follows from the calculations:

$$\alpha^{-1} \equiv \mathrm{red}(\ \beta^{-1} \cdot \alpha\beta \cdot (\alpha\beta\gamma\delta\beta)^{-1} \cdot (\alpha\beta\gamma)\ ) \Rightarrow \alpha \text{ is a unit.}$$

$$(\alpha\beta)^{-1} \equiv \mathrm{red}(\ \beta^{-1} \cdot \beta^{-1} \cdot \alpha\beta \cdot (\alpha\beta\gamma\delta\beta)^{-1} \cdot (\alpha\beta\gamma)\ ) \Rightarrow\ \alpha\beta \text{ is a unit.}$$

$$(\alpha\beta\gamma)^{-1} \equiv \mathrm{red}(\alpha \cdot \beta^{-1} \cdot \alpha\beta \cdot (\alpha\beta\gamma\delta\beta)^{-1}) \Rightarrow\ \alpha\beta\gamma \text{ is a unit.}$$

$$(\alpha\beta\gamma\delta)^{-1} \equiv \mathrm{red}((\alpha\beta) \cdot (\delta\beta)^{-1} \cdot \alpha \cdot (\alpha\beta\gamma\delta\beta)^{-1}) \Rightarrow\ \alpha\beta\gamma\delta \text{ is a unit.}$$

Conclusion:

$$\mathrm{Gp}\langle a,b,c,d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

is a conservative factorisation.

Prefix membership problem strategy

- Given $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$
- Use Adjan overlap and Benois pieces computing algorithms to compute unital factorisations

$$\mathrm{Inv}\langle A \mid w = 1 \rangle = \mathrm{Inv}\langle A \mid w_1 \dots w_m = 1 \rangle$$

which are in turn conserviate factorisations

$$\mathrm{Gp}\langle A \mid w = 1 \rangle = \mathrm{Gp}\langle A \mid w_1 \dots w_m = 1 \rangle.$$

- Seek conditions on the factors $w_1, \dots, w_k$ that allow us to solve the membership problem in the prefix monoid

$$P_w = P(w_1, \dots, w_k) = \mathrm{Mon}\langle \mathrm{pref}(w_1) \cup \mathrm{pref}(w_2) \cup \dots \cup \mathrm{pref}(w_k) \rangle.$$

- Apply this to deduce that $\mathrm{Inv}\langle A \mid w = 1 \rangle$ has decidable word problem (this implication holds e.g. when w is cyclically reduced).

Amalgamated free products

Definition

Let $H = \mathrm{Gp}\langle A \mid R \rangle$, $K = \mathrm{Gp}\langle B \mid Q \rangle$ with $A \cap B = \varnothing$. Suppose $f : L \to H$ and $g : L \to K$ are injective group homomorphisms Then

$$H *_L K = \mathrm{Gp}\langle A, B \mid R, Q, f(x) = g(x) \text{ for all } x \in L \rangle.$$

Theorem A (Dolinka & RDG (2021))

Let $G = H *_L K$, where L, H, K are finitely generated groups such that both H, K have decidable word problems, and the membership problem for L in both H and K is decidable. Let M be a submonoid of G such that:

(i) $L \subseteq M$;

(ii) both $M \cap H$ and $M \cap K$ are finitely generated and

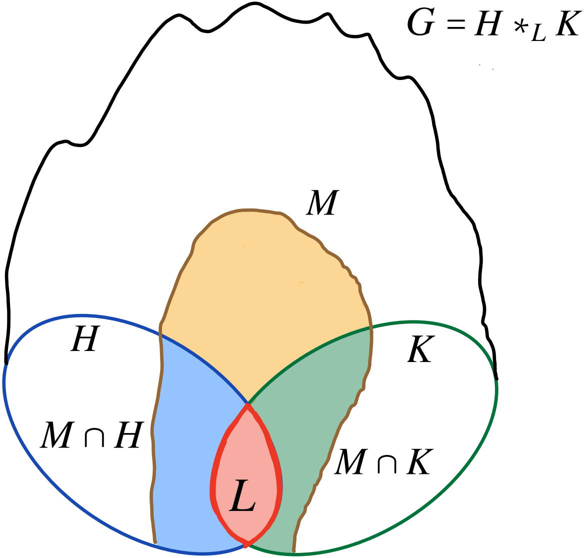
$$M = \mathrm{Mon}\langle (M \cap H) \cup (M \cap K) \rangle;$$

(iii) the membership problem for $M \cap H$ in H is decidable;

(iv) the membership problem for $M \cap K$ in K is decidable.

Then the membership problem for M in G is decidable.

Picture for Theorem A



Unique marker letter theorem

Theorem (Dolinka & RDG (2021))

Let $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ where $w \equiv w_1 \dots w_m$ is a conservative factorisation. Let $U = \{u_1, \dots, u_k\} \subseteq (A \cup A^{-1})^*$ be the words that appear as factors in this decomposition, that is, $w_i \in U \cup U^{-1}$ for $1 \leq i \leq m$. Suppose that

- for all $i \in \{1, \dots, k\}$ there is a letter $a_i \in A$ that appears exactly once in u_i and does not appear in any u_j for $j \neq i$.

Then $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem.

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- for all $i \in \{1, \dots, k\}$ there is a letter $a_i \in A$ that appears exactly once in u_i and does not appear in any u_j for $j \neq i$.

Then $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem.

Corollary

If the conditions above are satisfied and w is cyclically reduced then $\mathrm{Inv}\langle A \mid w = 1\rangle$ has decidable word problem.

Notes on proof:

- $G = FG(X_1) * H$ where H is a certain one-relator group.
- P_w = submonoid of $G = FG(X_1) * H$ generated by $Q \cup H$ where Q is a certain finite subset of $FG(X_1)$.
- Apply Magnus's Theorem, Benois' Theorem, and Theorem A.

Unique marker letter theorem example

Example

$$\mathrm{Gp}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle$$

has decidable prefix membership problem, since using the Adjan overlap method

$$(axb)(ayb)(ayb)(axb)(ayb)(axb)$$

is a conservative factorisation, and the factors axb and ayb have the unique marker letter property. It also follows that

$$\mathrm{Inv}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle.$$

has decidable word problem.

O’Hare example (Margolis, Meakin and Stephen, 1987)

$$\mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

is a conservative factorisation. The factors do not have the unique marker letter property.

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$$\mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

is a conservative factorisation. The factors do not have the unique marker letter property. However

$$\begin{aligned} G &= \mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle \\ &= \mathrm{Gp}\langle a, b, c, d \mid (aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad)(ad) \\ &\quad (aba^{-1})(aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad) = 1\rangle. \end{aligned}$$

where this is a conservative factorisation, and aba^{-1}, aca^{-1} and ad do satisfy the unique marker letter property.

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is a conservative factorisation. The factors do not have the unique marker letter property. However

$$G = \mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$
$$= \mathrm{Gp}\langle a, b, c, d \mid (aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad)(ad)$$
$$(aba^{-1})(aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad) = 1\rangle.$$

where this is a conservative factorisation, and aba^{-1}, aca^{-1} and ad do satisfy the unique marker letter property. The two prefix monoids are equal since, using the defining relation from the first presentation:

- $aba^{-1} = (abcd)(acd)^{-1} = (abcd)(abcd)(acd)(ad)(abbcd)$
- $aca^{-1} = (acd)(ad)^{-1} = (acd)(abbcd)(acd)(abcd)(acd)$

Conclusion:

$$\mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

has decidable prefix membership problem, and the corresponding inverse monoid has decidable word problem.

Disjoint alphabets theorem

Theorem (Dolinka & RDG (2021))

Let $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ where w is a cyclically reduced word and $w \equiv w_1 \dots w_m$ is a conservative factorisation. Let $U = \{u_1, \dots, u_k\} \subseteq (A \cup A^{-1})^*$ be the words that appear as factors in this decomposition, that is, $w_i \in U \cup U^{-1}$ for $1 \leq i \leq m$. Suppose $k \geq 2$ and that

- for any pair of distinct $i, j \in \{1, \dots, k\}$ the words u_i and u_j have no letters in common.

Then $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem.

Disjoint alphabets theorem

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Let $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ where w is a cyclically reduced word and $w \equiv w_1 \dots w_m$ is a conservative factorisation. Let $U = \{u_1, \dots, u_k\} \subseteq (A \cup A^{-1})^*$ be the words that appear as factors in this decomposition, that is, $w_i \in U \cup U^{-1}$ for $1 \leq i \leq m$. Suppose $k \geq 2$ and that

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Then $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ has decidable prefix membership problem.

Corollary

If the conditions above are satisfied and then $\mathrm{Inv}\langle A \mid w = 1 \rangle$ has decidable word problem.

Notes on proof:

- $G = FG(X_1) *_{A_1} (FG(X_2) *_{A_2} (\dots (FG(X_k) *_{A_k} G_k) \dots))$ a tower of amalgamated free products where G_k is a one-relator group related to the factorisation of w.
- Apply Magnus's Theorem, Benois' Theorem, and Theorem A.

Disjoint alphabets theorem example

Example

$$\mathrm{Gp}\langle a, b, c, d \mid ababcdcdababcdcdcdcdabab = 1\rangle$$

has decidable prefix membership problem, since using the Adjan overlap method

$$(abab)(cdcd)(abab)(cdcd)(cdcd)(abab)$$

is a conservative factorisation, and the factors *abab* and *cdcd* are over disjoint alphabets. It also follows that

$$\mathrm{Inv}\langle a, b, c, d \mid ababcdcdababcdcdcdcdabab = 1\rangle$$

has decidable word problem.

Amalgamated free products and HNN extensions

Amalgamated free products

The results above for the prefix membership problem are obtained by

- Decomposing $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ into smaller groups using amalgamated free products.
- Using the properties of the groups arising in this decomposition and information about the way the prefix monoid P_w sits inside this decomposition to show G has decidable prefix membership problem.

HNN extensions

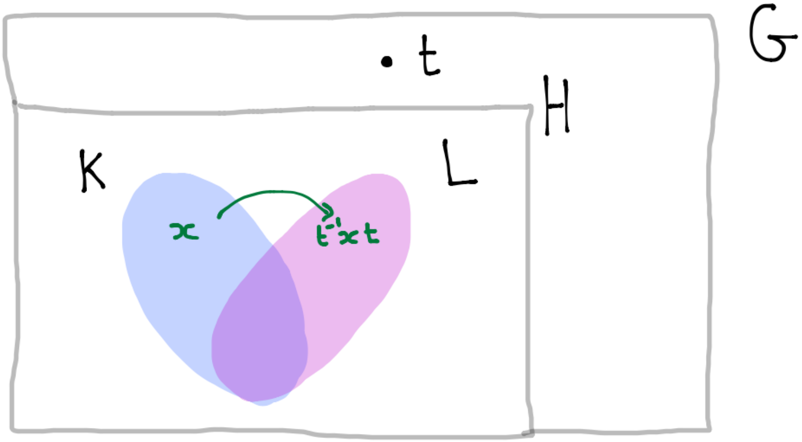
- It is natural to seek a similar approach but using HNN extensions to decompose the group $G = \mathrm{Gp}\langle A \mid w = 1\rangle$.
- In fact, the standard approach to one-relator groups is to decompose them using HNN extensions using a method of McCool and Schupp (1973) and Moldavanskiĭ (1967).

HNN-extensions of groups

$H \cong \mathrm{Gp}\langle A \mid R\rangle$, $\quad K, L \leq H$ with $K \cong L$. Let $\phi : K \to L$ be an isomorphism.
The HNN-extension of H with respect to ϕ is

$$G = H*_{t,\phi:K\to L} = \mathrm{Gp}\langle A,\ t \mid R,\ t^{-1}kt = \phi(k)\ (k \in K)\rangle$$

Fact: H embeds naturally into the HNN extension $G = H*_{t,\phi:K\to L}$.



McCool–Schupp approach to one-relator groups

Based on the following observation of Moldavanskiĭ (1967)

If $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ with $t \in A$ and where w has t-exponent sum zero (e.g. $w = atat^2a^2t^{-3}$). Then the following exist:

- a one-relator group $G' = \mathrm{Gp}\langle A' \mid w' = 1\rangle$ with w' shorter than w.
- sets $C, D \subseteq A'$ that form bases of free subgroups $FG(C), FG(D) \subseteq G'$
- an isomorphism $\phi : FG(C) \to FG(D)$, and
- an isomorphism $G \cong G' *_{t,\phi:FG(C)\to FG(D)}$.

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- sets $C, D \subseteq A'$ that form bases of free subgroups $FG(C), FG(D) \subseteq G'$
- an isomorphism $\phi : FG(C) \to FG(D)$, and
- an isomorphism $G \cong G' *_{t,\phi:FG(C)\to FG(D)}$.

Theorem (Dolinka & RDG (2021))

With the above notation, if G' is a free group and w is prefix t-positive then $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem.

Example: $G = \mathrm{Gp}\langle a, b, c, t \mid tbcbt^8bbct^{-6}ct^{-3}at^3bt^{-3}at^3ct^{-2}ct^{-1} = 1\rangle$ has decidable prefix membership problem since it is prefix t-positive and $G' = \mathrm{Gp}\langle a_0, b_{-9}, \ldots, b_{-1}, c_{-9}, \ldots, c_{-1} \mid b_{-1}c_{-1}b_{-1}b_{-9}^2c_{-9}c_{-3}a_0b_{-3}a_0c_{-3}c_{-1} = 1\rangle$ is a free group.

Theorem D

Theorem D (Dolinka & RDG (2021))

Let $G = H*_{t,\phi:K\to L}$ be an HNN extension of a finitely generated group H such that K, L are also finitely generated. Assume that the following conditions hold:

(i) the rational subset membership problem is decidable in H;

(ii) $K \leq H$ is effectively closed for rational intersections[3].

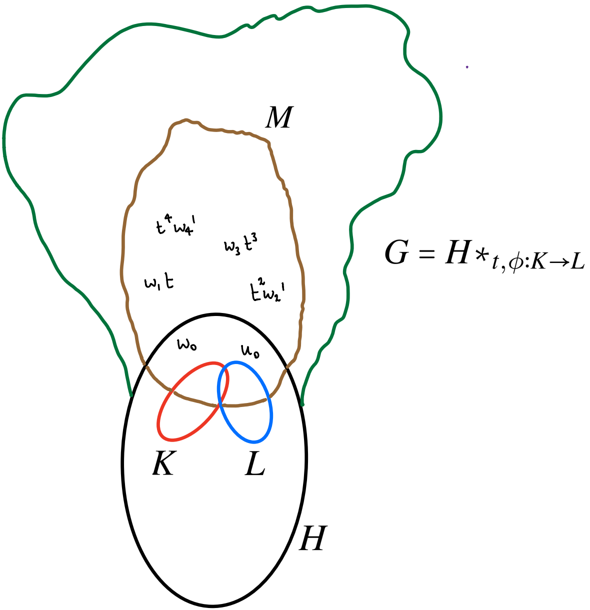
Then for any finite $W_0, W_1, \ldots, W_d, W_1', \ldots, W_d' \subseteq H$, $d \geq 0$, the membership problem for

$$M = \operatorname{Mon}\left\langle W_0 \cup W_1 t \cup W_2 t^2 \cup \cdots \cup W_d t^d \cup tW_1' \cup \cdots \cup t^d W_d' \right\rangle$$

in G is decidable.

[3] K in H is closed for rational intersections if $R \cap K \in \operatorname{Rat}(H)$ for all $R \in \operatorname{Rat}(H)$.

Picture for Theorem D



Further results

Our general results on the submonoid membership in amalgamated free products and HNN extensions can also be used to show the prefix membership problem is decidable for examples including:

- Cyclically pinched groups
  - $\mathrm{Gp}\langle X, Y \mid uv^{-1} = 1\rangle$ where $u \in (X \cup X^{-1})^*$ and $v \in (Y \cup Y^{-1})^*$.

  Including both the orientable surface group

  $$\mathrm{Gp}\langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid [a_1, b_1] \ldots [a_n, b_n] = 1\rangle$$

  and the non-orientable surface group

  $$\mathrm{Gp}\langle a_1, \ldots, a_n \mid a_1^2 \ldots a_n^2 = 1\rangle.$$

- Conjugacy pinched (including Baumslag–Solitar)
  - $\mathrm{Gp}\langle X \cup \{t\} \mid t^{-1}utv^{-1} = 1\rangle$ where $u, v \in (X \cup X^{-1})^*$ are nonempty reduced words.
- Adjan-type (several new cases)
  - $\mathrm{Gp}\langle X \mid uv^{-1} = 1\rangle$ where $u, v \in X^*$ are positive words such that the first letters of u, v are different, and also the last letters of u, v are different.

Limiting expectations

The following result shows that some conditions are needed on the defining relator word w for a positive answer to the prefix membership problem.

Theorem (Dolinka & RDG)

There is a finite alphabet A and a reduced word $w \in (A \cup A^{-1})^*$ such that $\mathrm{Gp}\langle A \mid w = 1 \rangle$ has undecidable prefix membership problem.

- The prefix membership problem for $\mathrm{Gp}\langle A \mid w = 1 \rangle$ remains open for cyclically reduced words w.

Future directions

Problem

Do one-relator groups $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ have decidable prefix membership problem if w is a cyclically reduced word?

- Find other combinatorial conditions on the pieces of conservative / unital factorisations that suffice to solve the prefix membership problem.
- Extend the HNN-extensions to all one-relator groups that are one step from being free via McCool–Schupp.
- Extend HNN-extension approach to examples higher in the hierarchy.
- Unify the two approaches above e.g. by proving analogous submonoid membership results for graph of groups constructions (i.e. Bass–Serre theory). This relates to:

  M. Kambites, P. V. Silva, B. Steinberg, On the rational subset problem for groups, *J. Algebra* 309 (2007), 622–639.