Membership problems in one-relator groups and the word problem for one-relator inverse monoids

Robert D. Gray

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University of East Anglia

One-relator groups

Definition

A one-relator group is a group defined by a presentation of the form

$$\mathrm{Gp}\langle A \mid r = 1\rangle = \mathrm{FG}(A)/\langle\!\langle r\rangle\!\rangle$$

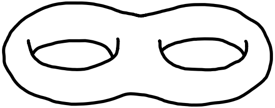
where A is a finite alphabet and $r \in (A \cup A^{-1})^*$.

- Magnus 1932: One-relator groups have decidable word problem.

Example

- $\mathbb{Z} \times \mathbb{Z} = \mathrm{Gp}\langle x, y \mid [x, y] = 1\rangle$
  where $[x, y] = x^{-1}y^{-1}xy$.
- Baumslag–Solitar groups

  $$B(m, n) = \mathrm{Gp}\langle a, b | b^{-1}a^m ba^{-n} = 1\rangle$$



- Surface groups

  $$\mathrm{Gp}\langle a_1, \dots, a_g, b_1, \dots, b_g | [a_1, b_1] \dots [a_g, b_g] = 1\rangle.$$

Special inverse monoids

An inverse monoid is a monoid M such that for every $m \in M$ there is a unique $m^{-1} \in M$ such that $mm^{-1}m = m$ and $m^{-1}mm^{-1} = m^{-1}$.

Definition (One-relator special inverse monoids)

$$\mathrm{Inv}\langle A \mid r = 1 \rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid r = 1, \quad x = xx^{-1}x, \quad xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \rangle$$

where x, y range over all words from $(A \cup A^{-1})^*$.

Example

The bicyclic monoid is defined by $\mathrm{Inv}\langle a \mid aa^{-1} = 1 \rangle$.

Problem

Investigate the word problem for $\mathrm{Inv}\langle A \mid r = 1 \rangle$.

Theorem (Ivanov, Margolis, Meakin (2001))

If the word problem is decidable for all inverse monoids of the form $\mathrm{Inv}\langle A \mid r = 1 \rangle$, with r a reduced word in $(A \cup A^{-1})^*$, then the word problem is also decidable for every one-relator monoid $\mathrm{Mon}\langle A \mid u = v \rangle$.

Right units and units

Let $M = \mathrm{Inv}\langle A \mid r = 1 \rangle$.

An element $m \in M$ is a right unit (or right invertible) if there is an $n \in M$ such that $mn = 1$, left unit is defined analogously, and a unit is an element that is both a left and right unit.

The right units form a submonoid of M and the units form a subgroup of M.

Key theme

We can gain insight into the word problem for $M = \mathrm{Inv}\langle A \mid r = 1 \rangle$ by studying the right units, and units, of M.

Facts:

- m is a right unit if and only if $mm^{-1} = 1$;
- m is a unit if and only if m and m^{-1} are both right units.

Right units and units

The right unit problem for $M = \mathrm{Inv}\langle A \mid r = 1\rangle$ is decidable if there is an algorithm which solves the following decision problem:

INPUT: A word $w \in (A \cup A^{-1})^*$.

QUESTION: Does w represent a right unit of M?

Fact

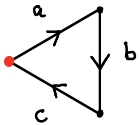
If $M = \mathrm{Inv}\langle A \mid w = 1\rangle$ has decidable word problem then M has decidable right unit problem.

Proof: $w \in (A \cup A^{-1})^*$ is a right unit if and only if $ww^{-1} = 1$ in M which can be checked since the word problem is decidable.

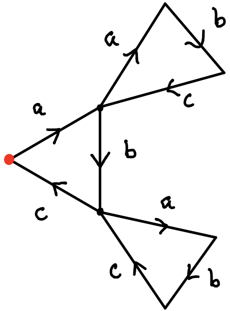
- The right unit problem is a kind of membership problem - it asks whether an element belongs to a particular subset of the monoid.
- The right unit problem can be viewed geometrically:
  - Construct the Schützenberger graph $S\Gamma(1)$ using Stephen's procedure.
  - $w \in (A \cup A^{-1})^*$ is a right unit $\Leftrightarrow$ w can be read from the origin in $S\Gamma(1)$.

Right unit problem geometrically - Example 1

$\mathrm{Inv}\langle a, b, c \mid abc = 1 \rangle$

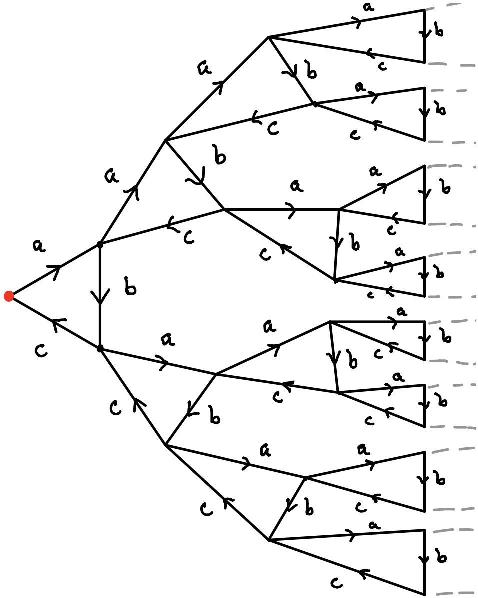


Right unit problem geometrically - Example 1



$\mathrm{Inv}\langle a, b, c \mid abc = 1\rangle$

Right unit problem geometrically - Example 1



$\mathrm{Inv}\langle a, b, c \mid abc = 1\rangle$

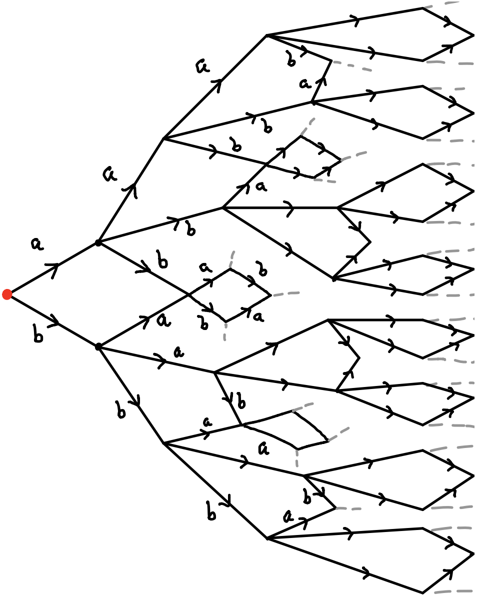
$w \in (A \cup A^{-1})^*$ is a right unit $\Leftrightarrow$ w can be read from the origin in $S\Gamma(1)$.

Examples

$ac^{-1}aba$ is a right unit.

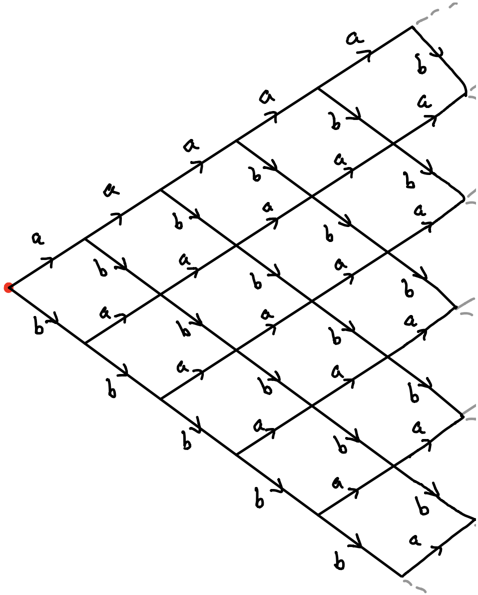
$abac$ is not a right unit.

Right unit problem geometrically - Example 2



$\mathrm{Inv}\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$

Right unit problem geometrically - Example 2



$\mathrm{Inv}\langle a, b \mid aba^{-1}b^{-1} = 1\rangle$

$w \in (A \cup A^{-1})^*$ is a right unit $\Leftrightarrow$ w can be read from the origin in $S\Gamma(1)$.

Examples

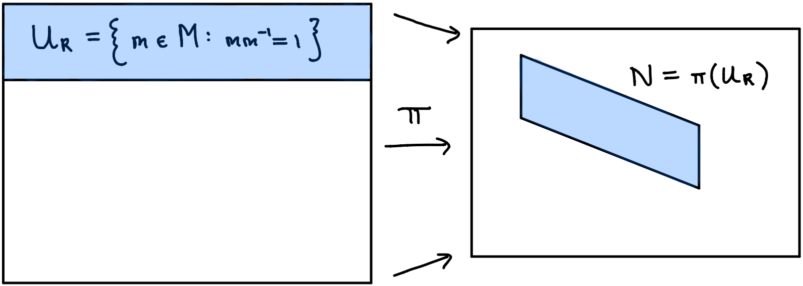
$aaba^{-1}a^{-1}$ is a right unit.

Note: This word cannot be read in the previous unfolded graph.

$bab^{-1}b^{-1}a$ is not a right unit.

Membership problems in one-relator groups

$$M = \mathrm{Inv}\langle A \mid r = 1 \rangle \longrightarrow G = \mathrm{Gp}\langle A \mid r = 1 \rangle$$



If M has decidable word problem

$\Longrightarrow$ membership problem for $U_R \leq M$ is decidable

since for $w \in (A \cup A^{-1})^*$ $\boxed{w \in U_R \iff ww^{-1} = 1}$

(sometimes)

$\rightsquigarrow$ membership problem for $N \leq G$ is decidable

Right-angled Artin groups

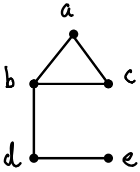
Definition

The right-angled Artin group $A(\Gamma)$ associated with the graph Γ is

$$\mathrm{Gp}\langle V\Gamma \mid uv = vu \text{ if and only if } \{u, v\} \in E\Gamma\rangle.$$

Example

$\Gamma =$



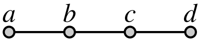
$$A(\Gamma) = \mathrm{Gp}\langle a, b, c, d, e \mid ac = ca,\ de = ed,\ ab = ba,\ bc = cb,\ bd = db \rangle$$

Right-angled Artin subgroups of one-relator groups

Theorem (RDG (2020))

$A(\Gamma)$ embeds into some one-relator group $\Longleftrightarrow$ Γ is a finite forest.

Lohrey & Steinberg (2008) proved that $A(P_4)$ contains a finitely generated submonoid T in which membership is undecidable, where P_4 is the graph



- Since P_4 is a tree there is a one-relator group $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ and an embedding $\theta : A(P_4) \hookrightarrow G$.
- Then $N = \theta(T)$ is a finitely generated submonoid of G in which membership is undecidable.

Theorem (RDG (2020))

There is a one-relator group $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ with a fixed finitely generated submonoid $N \leq G$ such that the membership problem for N within G is undecidable.

$S\Gamma(1)$ gets complicated

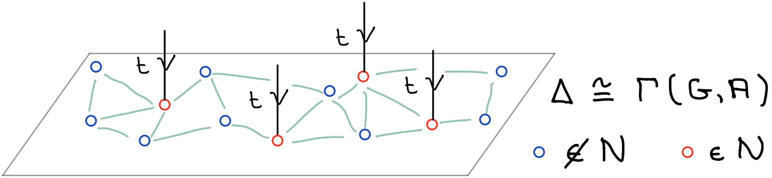
Theorem (RDG (2020))

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid r = 1\rangle$ with undecidable right unit problem.

Idea of the proof:

- Take $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ with $N \leq G$ in which membership is undecidable.
- Construct $M = \mathrm{Inv}\langle A, t \mid s = 1\rangle$ such that $S\Gamma(1)$ contains a copy Δ of the Cayley graph $\Gamma(G, A)$ of G where for the vertices v in Δ we have

$$\exists \text{ a } t\text{-edge into } v \iff v \in N \leq G.$$



- $\therefore$ cannot decide membership in $S\Gamma(1)$.

The word problem

Corollary (RDG (2020))

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid r = 1\rangle$ with undecidable word problem.

- The known examples all have the property that r is not a reduced word.

Reduced vs cyclically reduced words

$aba^{-1}ab$ - not reduced,
$abba^{-1}$ - reduced but not cyclically reduced
$aba^{-1}b^{-1}$ - cyclically reduced

Key question

For which words $w \in (A \cup A^{-1})^*$ does $\mathrm{Inv}\langle A \mid w = 1\rangle$ have decidable word problem? In particular is the word problem always decidable when w is (a) reduced or (b) cyclically reduced?

Recall: A positive answer to (a) would imply the word problem is also decidable for every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$.

Prefix membership problem

Proposition (Ivanov, Margolis, Meakin (2001)) The right units of $\mathrm{Inv}\langle A \mid w = 1\rangle$ are generated by the prefixes of w.

Prefix membership problem

Let $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ and set

$$P_w = \mathrm{Mon}\langle \mathrm{pref}(w)\rangle \leq G$$

the submonoid generated by the elements of G represented by prefixes of w. We call P_w the prefix monoid. Then G has decidable prefix membership problem if the membership problem for P_w in G is decidable.

Theorem (Ivanov, Margolis, Meakin (2001))

If w is a cyclically reduced word then the word problem for $\mathrm{Inv}\langle A \mid w = 1\rangle$ is decidable if $\mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem.

- In fact the theorem holds for any $\mathrm{Inv}\langle A \mid w = 1\rangle$ that is E-unitary.

Problem

Investigate the prefix membership problem for $\mathrm{Gp}\langle A \mid w = 1\rangle$.

Conservative factorisations

Definition

Let $w \in (A \cup A^{-1})^*$. Then for a factorisation

$$w \equiv w_1 w_2 \dots w_k$$

let $P(w_1, \dots, w_k)$ denote the submonoid of $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ generated by

$$\bigcup_{i=1}^{k} \mathrm{pref}(w_i).$$

- We always have $P_w \subseteq P(w_1, \dots, w_k)$ – since every prefix of w is a product of prefixes of the w_i

$$w_1 w_2 \dots w_{r-1} w_r' w_r'' w_{r+1} \dots w_k.$$

- If $P_w = P(w_1, \dots, w_k)$ then we say that the factorisation $w \equiv w_1 \dots w_k$ is conservative.

Units and conservative factorisations

Definition

We say that a factorisation $w \equiv w_1 \dots w_m$ is unital if each factor w_i represents a unit of the inverse monoid $M = \mathrm{Inv}\langle A \mid w = 1 \rangle$. The $w_1, \dots, w_m$ are called invertible pieces.

Theorem (Dolinka & RDG (2021))

If $\mathrm{Inv}\langle A \mid w_1 \dots w_m = 1 \rangle$ is a unital factorisation then $\mathrm{Gp}\langle A \mid w_1 \dots w_m = 1 \rangle$ is a conservative factorisation.

- The converse is often true e.g. when $w \equiv w_1 \dots w_m$ is cyclically reduced.

Computing unital factorisations of $\mathrm{Inv}\langle A \mid w = 1\rangle$

Adjan overlap method based on the fact that for $\alpha, \beta, \gamma \in (A \cup A^{-1})^*$

$$\alpha\beta \text{ and } \beta\gamma \text{ both units} \Longrightarrow \alpha,\ \beta, \text{ and } \gamma \text{ are all units.}$$

Example

$\mathrm{Inv}\langle A \mid (ab)(cd)(ab)(cd)(cd)(ab) = 1\rangle$ is a unital factorisation.
$\therefore$ the prefix monoid of $\mathrm{Gp}\langle A \mid (ab)(cd)(ab)(cd)(cd)(ab) = 1\rangle$ is
$P = \mathrm{Mon}\langle a, ab, c, cd\rangle$.

Benois method introduced in [RDG & Ruškuc, 2021] based on a theorem of Benois (1969) about membership problem in rational subsets[1] of free groups.

Example

The Benois method can be applied to show

$$\mathrm{Inv}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

is a unital factorisation. This presentation was first introduced in (Margolis, Meakin and Stephen, 1987).

[1] The class of The *rational subsets* of a group is the smallest set containing all finite subsets and is closed under union, product and submonoid generation.

Prefix membership problem strategy

- Given $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$
- Use Adjan overlap and Benois pieces computing algorithms to compute unital factorisations

$$\mathrm{Inv}\langle A \mid w = 1 \rangle = \mathrm{Inv}\langle A \mid w_1 \ldots w_m = 1 \rangle$$

  which are in turn conserviate factorisations

$$\mathrm{Gp}\langle A \mid w = 1 \rangle = \mathrm{Gp}\langle A \mid w_1 \ldots w_m = 1 \rangle.$$

- Seek conditions on the factors $w_1, \ldots, w_k$ that allow us to solve the membership problem in the prefix monoid

$$P_w = P(w_1, \ldots, w_k) = \mathrm{Mon}\langle \mathrm{pref}(w_1) \cup \mathrm{pref}(w_2) \cup \ldots \cup \mathrm{pref}(w_k) \rangle.$$

- Apply this to deduce that $\mathrm{Inv}\langle A \mid w = 1 \rangle$ has decidable word problem (this implication holds e.g. when w is cyclically reduced).

Change of basis theorem

Theorem (RDG & Ruškuc (2021))

If $M = \mathrm{Inv}\langle A \mid w = 1 \rangle = \mathrm{Inv}\langle A \mid w_1 \dots w_m = 1 \rangle$ is a unital factorisation with all w_i being reduced words, then for any finite set $V \subseteq (A \cup A^{-1})^*$ of reduced words that generates the same subgroup of the free group F_A as $\{w_1, \dots, w_k\}$ there is another presentation for M with unital factorisation

$$\mathrm{Inv}\langle A \mid v_1 \dots v_k = 1 \rangle$$

where the v_i come from V.

- e.g. We could take V to be a Nielsen reduced set of generators.

Example

$$M = \mathrm{Inv}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1 \rangle$$

is a unital factorisation. We have

$$\mathrm{Gp}\langle abcd, acd, ad, abbcd \rangle = \mathrm{Gp}\langle aba^{-1}, aca^{-1}, ad \rangle \leq F_{a,b,c,d},$$

and

$$M = \mathrm{Inv}\langle a, b, c, d \mid (aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad)$$
$$(aba^{-1})(aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad) = 1 \rangle.$$

Unique marker letter theorem

Theorem (Dolinka & RDG (2021))

Let $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ where $w \equiv w_1 \dots w_m$ is a conservative factorisation. Let $U = \{u_1, \dots, u_k\} \subseteq (A \cup A^{-1})^*$ be the words that appear as factors in this decomposition, that is, $w_i \in U \cup U^{-1}$ for $1 \leq i \leq m$. Suppose that

- for all $i \in \{1, \dots, k\}$ there is a letter $a_i \in A$ that appears exactly once in u_i and does not appear in any u_j for $j \neq i$.

Then $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem.

Corollary

If the conditions above are satisfied and w is cyclically reduced then $\mathrm{Inv}\langle A \mid w = 1\rangle$ has decidable word problem.

- In fact the corollary holds provided $\mathrm{Inv}\langle A \mid w = 1\rangle$ is E-unitary.

Unique marker letter theorem example

Example

$$\mathrm{Gp}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle$$

has decidable prefix membership problem, since using the Adjan overlap method

$$(axb)(ayb)(ayb)(axb)(ayb)(axb)$$

is a conservative factorisation, and the factors axb and ayb have the unique marker letter property. It also follows that

$$\mathrm{Inv}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle.$$

has decidable word problem.

Disjoint alphabets theorem

Theorem (Dolinka & RDG (2021))

Let $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ where w is a cyclically reduced word and $w \equiv w_1 \ldots w_m$ is a conservative factorisation. Let $U = \{u_1, \ldots, u_k\} \subseteq (A \cup A^{-1})^*$ be the words that appear as factors in this decomposition, that is, $w_i \in U \cup U^{-1}$ for $1 \leq i \leq m$. Suppose $k \geq 2$ and that

- for any pair of distinct $i, j \in \{1, \ldots, k\}$ the words u_i and u_j have no letters in common.

Then $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem.

Corollary

If the conditions above are satisfied and then $\mathrm{Inv}\langle A \mid w = 1\rangle$ has decidable word problem.

Disjoint alphabets theorem example

Example

$$\mathrm{Gp}\langle a, b, c, d \mid ababcdcdababcdcdcdcdabab = 1\rangle$$

has decidable prefix membership problem, since using the Adjan overlap method

$$(abab)(cdcd)(abab)(cdcd)(cdcd)(abab)$$

is a conservative factorisation, and the factors $abab$ and $cdcd$ are over disjoint alphabets. It also follows that

$$\mathrm{Inv}\langle a, b, c, d \mid ababcdcdababcdcdcdcdabab = 1\rangle$$

has decidable word problem.

Limiting expectations

The following result shows that some conditions are needed on the defining relator word w for a positive answer to the prefix membership problem.

Theorem (Dolinka & RDG)

There is a finite alphabet A and a reduced word $w \in (A \cup A^{-1})^*$ such that $\mathrm{Gp}\langle A \mid w = 1\rangle$ has undecidable prefix membership problem.

- The prefix membership problem for $\mathrm{Gp}\langle A \mid w = 1\rangle$ remains open for cyclically reduced words w.

GROUPS | INVERSE MONOIDS | MONOIDS

~~$\mathrm{Gp}\langle A \mid w = 1\rangle$~~
~~DECIDABLE PREFIX MEMBERSHIP~~
~~W - REDUCED WORD~~

$\Downarrow$

$\mathrm{Gp}\langle A \mid w = 1\rangle$
DECIDABLE PREFIX MEMBERSHIP
W - CYCLICALLY REDUCED WORD

$\Rightarrow$

$\mathrm{Inv}\langle A \mid w = 1\rangle$
DECIDABLE WORD PROBLEM
W - CYCLICALLY REDUCED WORD

IVANOV, MARGOLIS, MEAKIN (2001)

$\mathrm{Inv}\langle A \mid w = 1\rangle$
DECIDABLE WORD PROBLEM
W - REDUCED WORD

$\Rightarrow$

$\mathrm{Mon}\langle A \mid u = v\rangle$
DECIDABLE WORD PROBLEM

$\Downarrow$ (from $\mathrm{Inv}\langle A \mid w = 1\rangle$, W - reduced word, to $\mathrm{Inv}\langle A \mid w = 1\rangle$, W - cyclically reduced word)

Reduced vs cyclically reduced words

$aba^{-1}ab$	- not reduced,
$abba^{-1}$	- reduced but not cyclically reduced
$aba^{-1}b^{-1}$	- cyclically reduced

Amalgamated free products and HNN extensions

Amalgamated free products

The results above for the prefix membership problem are obtained by

- Decomposing $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ into smaller groups using amalgamated free products.
- Using the properties of the groups arising in this decomposition and information about the way the prefix monoid P_w sits inside this decomposition to show G has decidable prefix membership problem.

HNN extensions

- We also have a similar approach giving results that use HNN extensions to decompose the group $G = \mathrm{Gp}\langle A \mid w = 1\rangle$.
- In fact, the standard approach to one-relator groups is to decompose them using HNN extensions using a method of McCool and Schupp (1973) and Moldavanskiĭ (1967).

Further results

Our general results on the submonoid membership in amalgamated free products and HNN extensions can also be used to show the prefix membership problem is decidable for examples including:

- Cyclically pinched groups
  - $\mathrm{Gp}\langle X, Y \mid uv^{-1} = 1\rangle$ where $u \in (X \cup X^{-1})^*$ and $v \in (Y \cup Y^{-1})^*$.

  Including both the orientable surface group

  $$\mathrm{Gp}\langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid [a_1, b_1] \ldots [a_n, b_n] = 1\rangle$$

  and the non-orientable surface group

  $$\mathrm{Gp}\langle a_1, \ldots, a_n \mid a_1^2 \ldots a_n^2 = 1\rangle.$$

- Conjugacy pinched (including Baumslag–Solitar)
  - $\mathrm{Gp}\langle X \cup \{t\} \mid t^{-1}utv^{-1} = 1\rangle$ where $u, v \in (X \cup X^{-1})^*$ are nonempty reduced words.
- Adjan-type (several new cases)
  - $\mathrm{Gp}\langle X \mid uv^{-1} = 1\rangle$ where $u, v \in X^*$ are positive words such that the first letters of u, v are different, and also the last letters of u, v are different.

Open problems

Problem

For which words $w \in (A \cup A^{-1})^*$ does $\mathrm{Inv}\langle A \mid w = 1\rangle$ have decidable word problem? In particular is the word problem always decidable when w is (a) reduced or (b) cyclically reduced?

Note: A positive answer to (a) would imply the word problem is also decidable for every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$.

Problem

Do one-relator groups $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ have decidable prefix membership problem if w is a cyclically reduced word?

Problem

Characterise the one-relator groups with decidable submonoid membership problem.

Problem

Is the subgroup membership problem decidable for one-relator groups?

Under the hood

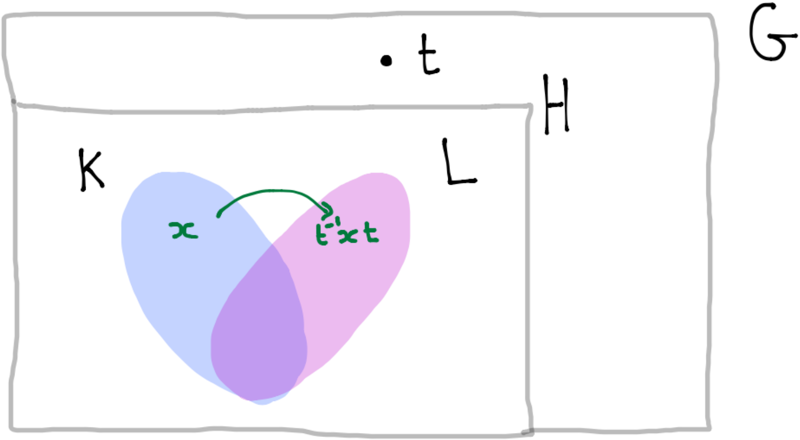
Below are the general results used to prove some of the results above about the prefix membership problem.

HNN-extensions of groups

$H \cong \mathrm{Gp}\langle A \mid R \rangle$, $\quad K, L \leq H$ with $K \cong L$. Let $\phi : K \to L$ be an isomorphism.
The HNN-extension of H with respect to ϕ is

$$G = H *_{t,\phi:K\to L} = \mathrm{Gp}\langle A,\ t \mid R,\ t^{-1}kt = \phi(k)\ (k \in K)\rangle$$

Fact: H embeds naturally into the HNN extension $G = H *_{t,\phi:K\to L}$.



McCool–Schupp approach to one-relator groups

Based on the following observation of Moldavanskiĭ (1967)

If $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ with $t \in A$ and where w has t-exponent sum zero (e.g. $w = atat^2a^2t^{-3}$). Then the following exist:

- a one-relator group $G' = \mathrm{Gp}\langle A' \mid w' = 1\rangle$ with w' shorter than w.
- sets $C, D \subseteq A'$ that form bases of free subgroups $FG(C), FG(D) \subseteq G'$
- an isomorphism $\phi : FG(C) \to FG(D)$, and
- an isomorphism $G \cong G' *_{t,\phi:FG(C)\to FG(D)}$.

Theorem (Dolinka & RDG (2021))

With the above notation, if G' is a free group and w is prefix t-positive then $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem.

Example: $G = \mathrm{Gp}\langle a, b, c, t \mid tbcbt^8bbct^{-6}ct^{-3}at^3bt^{-3}at^3ct^{-2}ct^{-1} = 1\rangle$ has decidable prefix membership problem since it is prefix t-positive and $G' = \mathrm{Gp}\langle a_0, b_{-9}, \ldots, b_{-1}, c_{-9}, \ldots, c_{-1} \mid b_{-1}c_{-1}b_{-1}b_{-9}^2c_{-9}c_{-3}a_0b_{-3}a_0c_{-3}c_{-1} = 1\rangle$ is a free group.

Amalgamated free products

Definition

Let $H = \mathrm{Gp}\langle A \mid R\rangle$, $K = \mathrm{Gp}\langle B \mid Q\rangle$ with $A \cap B = \varnothing$. Suppose $f : L \to H$ and $g : L \to K$ are injective group homomorphisms Then

$$H *_L K = \mathrm{Gp}\langle A, B \mid R, Q, f(x) = g(x) \text{ for all } x \in L\rangle.$$

Theorem A (Dolinka & RDG (2021))

Let $G = H *_L K$, where L, H, K are finitely generated groups such that both H, K have decidable word problems, and the membership problem for L in both H and K is decidable. Let M be a submonoid of G such that:

(i) $L \subseteq M$;

(ii) both $M \cap H$ and $M \cap K$ are finitely generated and

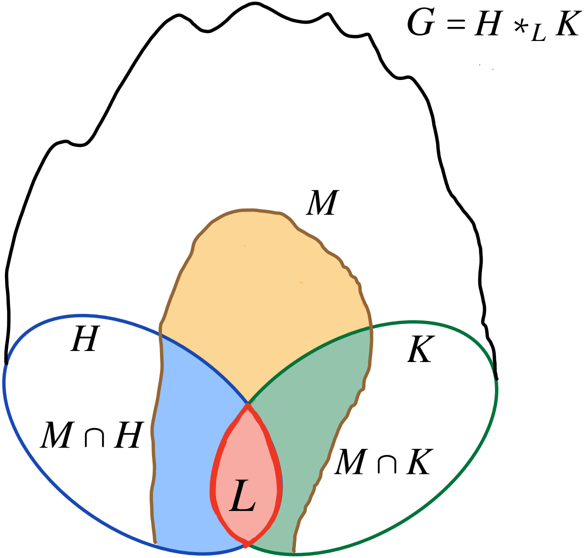
$$M = \mathrm{Mon}\langle (M \cap H) \cup (M \cap K)\rangle;$$

(iii) the membership problem for $M \cap H$ in H is decidable;

(iv) the membership problem for $M \cap K$ in K is decidable.

Then the membership problem for M in G is decidable.

Picture for Theorem A



Theorem D (Dolinka & RDG (2021))

Let $G = H*_{t,\phi:K\to L}$ be an HNN extension of a finitely generated group H such that K, L are also finitely generated. Assume that the following conditions hold:

(i) the rational subset membership problem is decidable in H;

(ii) $K \leq H$ is effectively closed for rational intersections[2].

Then for any finite $W_0, W_1, \ldots, W_d, W_1', \ldots, W_d' \subseteq H$, $d \geq 0$, the membership problem for

$$M = \mathrm{Mon}\left\langle W_0 \cup W_1 t \cup W_2 t^2 \cup \cdots \cup W_d t^d \cup t W_1' \cup \cdots \cup t^d W_d' \right\rangle$$

in G is decidable.

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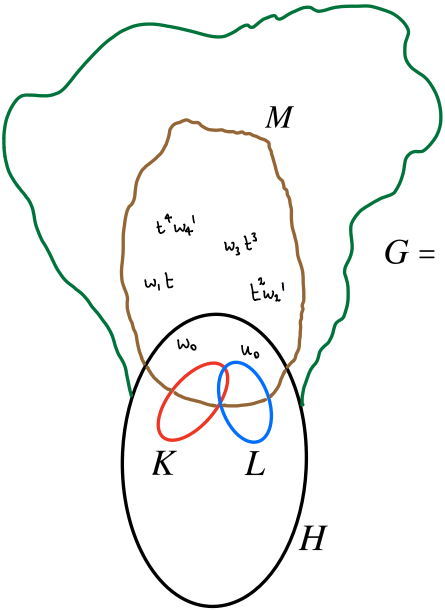
[2] K in H is closed for rational intersections if $R \cap K \in \mathrm{Rat}(H)$ for all $R \in \mathrm{Rat}(H)$.

Picture for Theorem D



$G = H *_{t,\phi:K \to L}$