

Subgroups of inverse monoids via the geometry of their Cayley graphs

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(joint work with Mark Kambites)

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Presentations and the word problem

$$\langle A \mid R \rangle = \langle \underbrace{a_1, \dots, a_n}_{\text{letters / generators}} \mid \underbrace{u_1 = v_1, \dots, u_m = v_m}_{\text{words / defining relations}} \rangle$$

- Defines the monoid $M = A^* / \rho$ where ρ is the equivalence relation on the free monoid A^* of all words over A where two words are in the same equivalence class (i.e. they represent the same element of M) if one can be transformed into the other by applying the relations R .

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- Is $abaa = aaba$?

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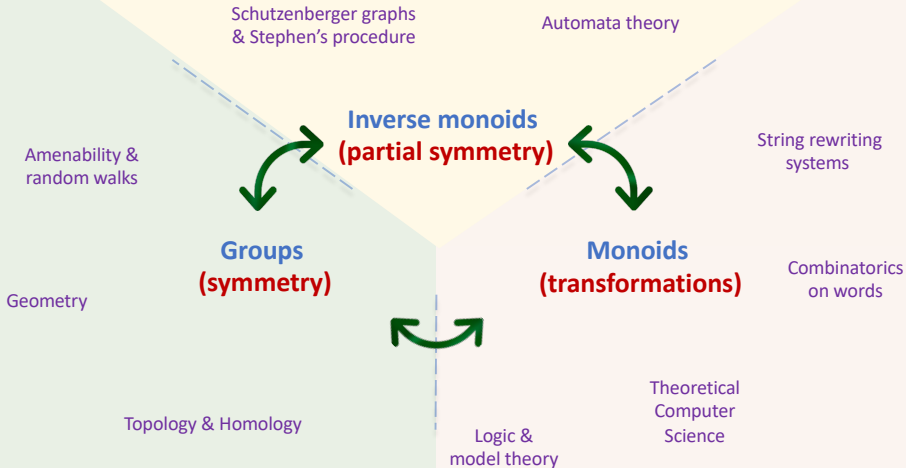
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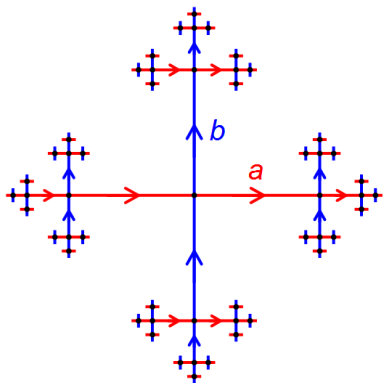
Longstanding open problem

Is the word problem decidable for one-relation monoids $\langle A \mid u = v \rangle$?



The free group F_2

$$F_2 = \text{Mon}\langle a, a^{-1}, b, b^{-1} \mid aa^{-1} = 1, a^{-1}a = 1, bb^{-1} = 1, b^{-1}b = 1 \rangle$$



Fact: Every word w over the generators is equal in F_2 to a unique reduced word $\text{red}(w)$ with no occurrences of xx^{-1} or $x^{-1}x$.

e.g. $abb^{-1}aab^{-1}ba^{-1}b = aab$.

This solves **the word problem** in F_2 :

$w_1 = w_2$ in $F_2 \Leftrightarrow \text{red}(w_1) = \text{red}(w_2)$ as words

The reduced words give **normal forms** for elements of the free group.

Cayley graph of F_2

Vertices: Elements of F_2 .

Edges: $g \xrightarrow{x} gx$ for $g \in F_2$ and $x \in A$.

Group presentations

$$G \cong \text{Gp}\langle A \mid u_i = v_i \ (i \in I) \rangle \quad (u_i, v_i \in (A \cup A^{-1})^*)$$

Elements of G - equivalence classes of words over $A \cup A^{-1}$ where $u = v$ in $G \Leftrightarrow$ we can transform u into v by applying defining relations or relations from the free group.

Example

The free group F_2 is defined by $\text{Gp}\langle a, b \mid \rangle$.

Example

The free abelian group $\mathbb{Z} \times \mathbb{Z}$ is defined by $\text{Gp}\langle a, b \mid ab = ba \rangle$.

Normal forms: $a^i b^j$ ($i, j \in \mathbb{Z}$)

Definition

G is **finitely presented** if $G \cong \text{Gp}\langle A \mid R \rangle$ with $|A| < \infty$ and $|R| < \infty$.

Subgroups of finitely presented groups

Question: What are the finitely generated subgroups of finitely presented groups?

- ▶ There are finitely generated subgroups of finitely presented groups that are not finitely presented
 - ▶ e.g. [Grunewald \(1978\)](#): shows $F_2 \times F_2$ has such a subgroup.
- ▶ [Neumann \(1937\)](#): There are uncountably many 2-generator groups.
⇒ Not every finitely generated group embeds in a finitely presented group.

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Computing subsets of $(A \cup A^{-1})^*$

- ▶ A set W of words is **computably enumerable** if there is an algorithm which takes any word u as input and, if u is a member of W , then the algorithm eventually halts and says YES; otherwise it runs forever.
- ▶ A set W of words is **computable** if there is an algorithm which takes any word u as input, terminates after a finite amount of time and decides whether or not the word u belongs to W , returning either YES or NO.

Important fact: There exist sets of words that are c.e. but not computable.

The Higman Embedding Theorem

Definition

A **recursive presentation** for a finitely generated group is a presentation on a finite number of generators such that the set of defining relators is computably enumerable.

Theorem (Higman (1961))

A finitely generated group G can be embedded in some finitely presented group if and only if G can be recursively presented.

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Example

For any computably enumerable subset S of \mathbb{N}

$$\text{Gp}\langle a, b, c, d \mid a^{-i}ba^i = c^{-i}dc^i \ (i \in S) \rangle$$

is recursively presented and hence embeds in a finitely presented group.

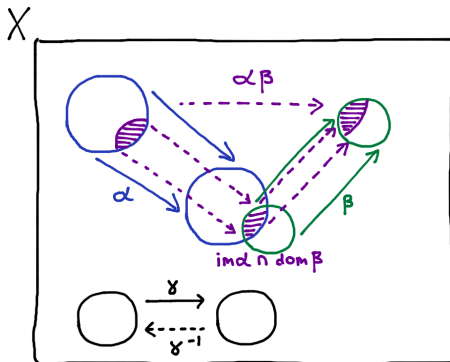
- ▶ Choosing S to be computably enumerable but non-computable
 - \Rightarrow This group has undecidable word problem.
 - \Rightarrow There are finitely presented groups with undecidable word problem.

Inverse monoids

An **inverse monoid** is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

$m \in M$ is a **right unit** if there is an $n \in M$ such that $mn = 1$, **left unit** is defined analogously, and a **unit** is an element that is both a left and right unit.

Example: I_X = monoid of all partial bijections $X \rightarrow X$



Examples: In I_3

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \\ 3 & - & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & - & 1 \\ - & 1 & - \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & 2 \end{pmatrix}$$

Note:

$$\gamma\gamma^{-1} = \text{id}_{\text{dom}\gamma}$$

Group of units of I_X : is the symmetric group S_X .

Inverse monoid presentations

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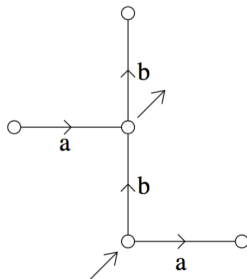
For all $x, y \in M$ we have

$$x = xx^{-1}x, (x^{-1})^{-1} = x, (xy)^{-1} = y^{-1}x^{-1}, xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \quad (\dagger)$$

$$\text{Inv}\langle A \mid u_i = v_i \ (i \in I) \rangle = \text{Mon}\langle A \cup A^{-1} \mid u_i = v_i \ (i \in I) \cup (\dagger) \rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and x, y range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\text{FIM}(A) = \text{Inv}\langle A \mid \rangle$



Munn (1974)

Elements of $\text{FIM}(A)$ can be represented using Munn trees. e.g. in $\text{FIM}(a, b)$ we have $u = w$ where

$$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$$

$$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$$

Special inverse monoids

Definition

A finitely presented **special inverse monoid** is one defined by a presentation of the form

$$\text{Inv}\langle A \mid w_1 = 1, \dots, w_k = 1 \rangle.$$

Motivation from the theory of one-relator monoids and groups

Theorem (Ivanov, Margolis, Meakin (2001))

If the word problem is decidable for all inverse monoids of the form $\text{Inv}\langle A \mid r = 1 \rangle$, with r a reduced word, then the word problem is also decidable for every one-relation monoid $\text{Mon}\langle A \mid u = v \rangle$.

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Theorem (Adjan (1966))

The group of units G of a one-relator monoid $M = \text{Mon}\langle A \mid r = 1 \rangle$ is a one-relator group. Furthermore, M has decidable word problem.

Aim: Study the subgroup structure of finitely presented special inverse monoids.

Units of special inverse monoids

Monoids

Theorem (Makanin (1966))

The group of units G of $M = \text{Mon}\langle A \mid r_1 = 1, \dots, r_k = 1 \rangle$ is finitely presented by a presentation with k -relators. Furthermore, M has decidable word problem if and only if G has decidable word problem.

Example

The group of units of $M = \text{Mon}\langle a, b, c, d \mid abab = 1, abcdabcdabcd = 1 \rangle$ is $G = \text{Gp}\langle X, Y \mid X^2 = 1, (XY)^3 = 1 \rangle$.

Inverse monoids

Theorem (Ivanov, Margolis, Meakin (2001))

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There is a finitely presented special inverse monoid $\text{Inv}\langle A \mid r_1 = 1, \dots, r_k = 1 \rangle$ whose group of units is not finitely presented.

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Notes:

- ▶ Equivalently, by Higman, these are exactly the finitely generated subgroups of finitely presented groups.
- ▶ One direction of the proof is straightforward: Since the group of units is a finitely generated subgroup of a finitely presented inverse monoid it follows quickly it must itself be recursively presented.
- ▶ The other direction requires a construction to realise each such group as the group of units.
- ▶ For this we use the theory of Schützenberger graphs.

Schützenberger graphs

Definition

The **Schützenberger graph** $ST(1)$ of $M = \text{Inv}\langle A \mid r_1 = 1, \dots, r_k = 1 \rangle$ is the subgraph of the Cayley graph of M induced on the set of right units of M .

Theorem (Stephen (1990))

The group of units of $M = \text{Inv}\langle A \mid r_1 = 1, \dots, r_k = 1 \rangle$ is isomorphic to the group $\text{Aut}(ST(1))$ of label-preserving automorphisms of the Schützenberger graph $ST(1)$.

Stephen's procedure

The Schützenberger graph $ST(1)$ can be obtained as the limit of a sequence of labelled digraphs obtained by an iterative construction given by successively applying operations called **expansions** and **Stallings foldings**.

Example - Stephen's Procedure

$$\text{Inv}\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$$

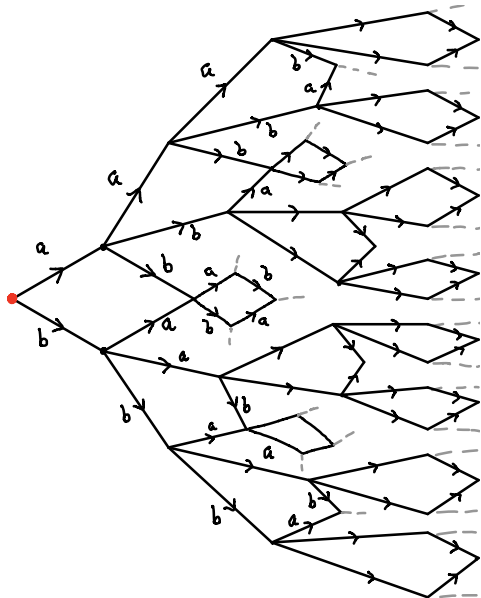
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Expansions: Attach a simple closed path labelled by r at a vertex (if one does not already exist).

Stallings foldings: Identify two edges with the same label and the same initial or terminal vertex.

This process may not stop.
Stephen shows that the

- ▶ process is confluent &
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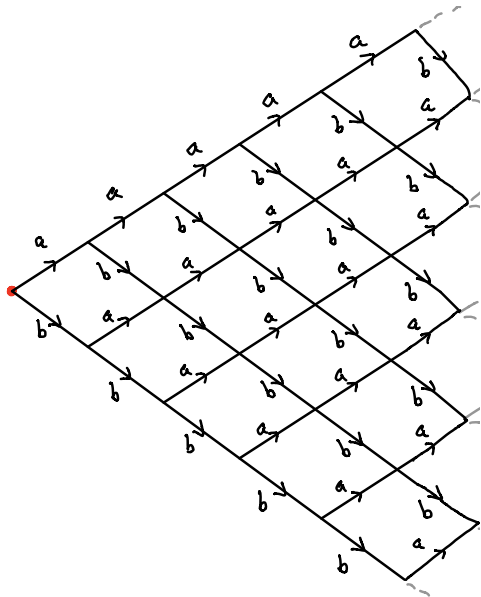
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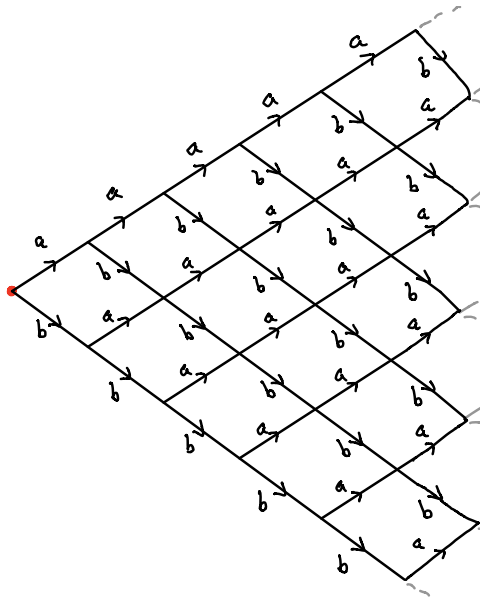
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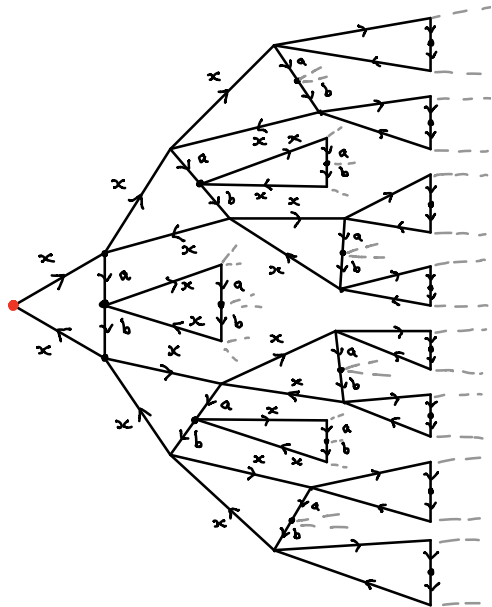
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 $\text{Aut}(ST(1)) = \{1\}$.

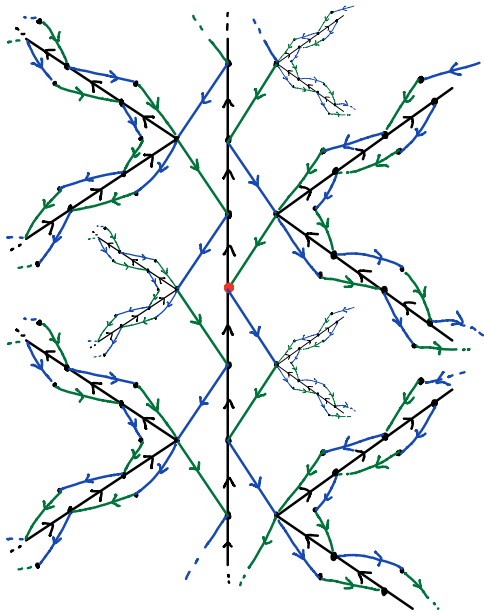


Example - a non-trivial group of units



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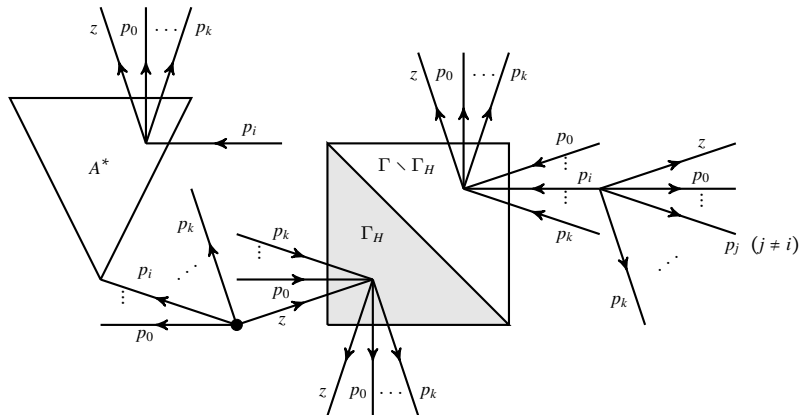
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The group of units is

$$\text{Aut}(\text{ST}(1)) \cong \mathbb{Z}$$

the infinite cyclic group.

Our construction - the general idea



- ▶ Given $G = \text{Gp}\langle A \mid R \rangle$ finitely presented, and $H \leq G$ a finitely generated subgroup, we construct $\text{Inv}\langle A \mid w_1 = 1, \dots, w_k = 1 \rangle$ whose $ST(1)$ has the above structure where Γ is the Cayley graph of G with respect to A , and Γ_H is the subgraph induced on the subset H of vertices.
- ▶ We prove every automorphism of $ST(1)$ fixes Γ_H setwise and deduce $\text{Aut}(ST(1)) \cong \text{Aut}(\Gamma_H) \cong H$.

One-relator case

Theorem (Adjan (1966))

The group of units G of a one-relator monoid $M = \text{Mon}\langle A \mid r = 1 \rangle$ is a one-relator group.

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There exists a one-relator special inverse monoid $M = \text{Inv}\langle A \mid r = 1 \rangle$ whose group of units G is not a one-relator group.

Question: Is the group of units of $\text{Inv}\langle A \mid r = 1 \rangle$ always finitely presented?

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Question (Baumslag (1973))

Is every one-relator group coherent?

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If all one-relator special inverse monoids $\text{Inv}\langle A \mid r = 1 \rangle$ have finitely presented groups of units then all one-relator groups are coherent.

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- ▶ [Linton \(2023\)](#) Proved **all** one-relator groups are coherent.

Maximal subgroups in general

Definition

For any idempotent $e = e^2$ in an inverse monoid M define

$$H_e = \{m \in M : mm^{-1} = e = m^{-1}m\}.$$

Then H_e is a group called a **group \mathcal{H} -class** of M .

e.g. H_1 is the group of units of M .

Definition

A **recursive presentation** for a (countable but not necessarily finitely generated) group is a presentation of the form $\text{Gp}\langle A \mid R \rangle$ where A is either finite or $A = \{a_i : i \in \mathbb{N}\}$ and R is a computably enumerable subset of $(A \cup A^{-1})^*$.

Theorem (RDG & Kambites (2023))

The possible group \mathcal{H} -classes of finitely presented special inverse monoids are exactly the (not necessarily finitely generated) recursively presented groups.

One-relator case (maximal subgroups)

Theorem (RDG & Kambites (2023))

Every finitely generated subgroup of a one-relator group arises as a group \mathcal{H} -class in a one-relator special inverse monoid.

Question: Is every group \mathcal{H} -class of $\text{Inv}\langle A \mid r = 1 \rangle$ necessarily finitely generated?

The above question would be answered negatively if the answer to the following is yes:

Question: Does there exist a one-relator group $G = \text{Gp}\langle A \mid w = 1 \rangle$ with a finitely generated subgroup $H \leq G$ and an element $g \in G$ such that $H \cap gHg^{-1}$ is not finitely generated.

- ▶ This relates to the **Howson property** - which asks that $H \cap K$ is finitely generated whenever H and K both are.
 - ▶ There are one-relator groups (even hyperbolic ones) that do not have the Howson property – [Karrass & Solitar \(1969\)](#), [Kapovich \(1999\)](#).