Crystal monoids and crystal bases: rewriting systems and biautomatic structures for Plactic monoids

Robert D. Gray[1]
University of East Anglia

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[1](joint work with A. J. Cain and A. Malheiro)

Presentations

$$\langle A \mid R \rangle = \langle \underbrace{a_1, \ldots, a_n}_{\text{letters / generators}} \mid \underbrace{u_1 = v_1, \ldots, u_m = v_m}_{\text{words / defining relations}} \rangle$$

- Defines the semigroup $S = A^+/\rho$ where ρ is the smallest congruence on the free semigroup A^+ containing R.
- S is the free-est / largest (in terms of homomorphic images) semigroup generated by A in which the generators satisfy all the relations R.

How to think about S

- Elements of S are equivalence classes of words over A.
- Two words are in the same equivalence class (i.e. they represent the same element of S) if one can be transformed into the other by applying the relations R.

Definition: S is finitely presented if both A and R finite.

Presentations

Example: $S \cong \langle A \mid R \rangle = \langle a, b \mid ab = ba \rangle$

Words $u, v \in A^+$ represent the same element of S if u can be transformed into v by a finite number of applications of the relations.

$$\text{e.g.} \quad abaa = aaba = aaab, \qquad abb \neq aab.$$

Fact: Every word $u \in A^+$ is equal in S to a unique word of the form $a^i b^j$, and these normal forms multiply as

$$(a^{i_1} b^{j_1})(a^{i_2} b^{j_2}) = a^{i_1+i_2} b^{j_1+j_2}.$$

Conclusion:

$$S \cong [(\mathbb{N}, +) \times (\mathbb{N}, +)] \setminus \{(0,0)\}.$$

The word problem: For any $u, v \in \{a, b\}^+$ we have

$$\begin{array}{rl} u = v \iff & u \text{ and } v \text{ have the same number occurrences of the letter } a \\ \& & u \text{ and } v \text{ have the same number occurrences of the letter } b. \end{array}$$

The word problem for semigroups and groups

Definition

A semigroup S with a finite generating set A has decidable word problem if there is an algorithm which for any two words $w_1, w_2 \in A^+$ decides whether or not they represent the same element of S.

Example. $S \cong \langle a, b \mid ab = ba \rangle$ has decidable word problem.

Some history

- Markov (1947) and Post (1947): first examples of finitely presented semigroups with undecidable word problem;
- Turing (1950): finitely presented cancellative semigroup with undecidable word problem;
- Novikov (1955) and Boone (1958): finitely presented group with undecidable word problem.

Complete rewriting systems

A - alphabet, $R \subseteq A^* \times A^*$ - rewrite rules, $\langle A \mid R \rangle$ - rewriting system
Write $r = (r_{+1}, r_{-1}) \in R$ as $r_{+1} \to r_{-1}$.

Define a binary relation $\to_R$ on A^* by

$$u \to_R v \iff u \equiv w_1 r_{+1} w_2 \text{ and } v \equiv w_1 r_{-1} w_2$$

for some $(r_{+1}, r_{-1}) \in R$ and $w_1, w_2 \in A^*$.

$\xrightarrow{*}_R$ is the transitive and reflexive closure of $\to_R$

Noetherian: No infinite descending chain

$$w_1 \to_R w_2 \to_R \cdots \to_R w_n \to_R \cdots$$

Confluent: Whenever

$$u \xrightarrow{*}_R v \text{ and } u \xrightarrow{*}_R v'$$

there is a word $w \in A^*$:

$$v \xrightarrow{*}_R w \text{ and } v' \xrightarrow{*}_R w$$

Definition: $\langle A \mid R \rangle$ is a finite complete rewriting system if it is complete (noetherian and confluent) and $|A| < \infty$ and $|R| < \infty$.

Complete rewriting systems

Example (Free commutative semigroup)

$\langle a, b \mid ba \to ab \rangle$

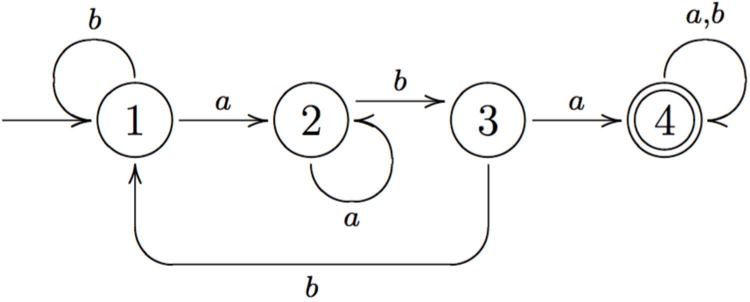
Normal forms (irreducibles) = $\{a^i b^j : i, j \geq 0\}$

Example (Free group)

$\langle a, a^{-1}, b, b^{-1} \mid aa^{-1} \to 1,\ a^{-1}a \to 1,\ bb^{-1} \to 1,\ b^{-1}b \to 1 \rangle.$

Normal forms (irreducibles) = { freely reduced words }.

Finite state automata



- Alphabet: $A = \{a, b\}$.
- $L(\mathcal{A}) \subseteq A^*$ - language of words recognised by the automaton $\mathcal{A}$
  - e.g. here $aba \in L(\mathcal{A})$ while $abba \notin L(\mathcal{A})$
- Languages recognised by finite state automata are the regular languages.

Automatic structures

Automatic groups and monoids

Defining property: $\exists$ a regular language $L \subseteq A^*$ such that every element has at least one representative in L, and $\forall a \in A \cup \{\epsilon\}$, there is a finite automaton recognising pairs from L that differ by multiplication by a.

- Automatic groups
  - Capture a large class of groups with easily decidable word problem
  - Examples: finite groups, free groups, free abelian groups, various small cancellation groups, Artin groups of finite and large type, Braid groups, hyperbolic groups.
- Automatic semigroups and monoids
  - Classes of monoids that have been shown to be automatic include divisibility monoids and singular Artin monoids of finite type.

Rewriting systems & automatic structures

Finite complete rewriting systems

- Decidable word problem: Relations $u \to_R v$ ordered, converge to unique normal forms.
- Applications to homological / homotopical finiteness properties
  - FP_∞: Anick (1986), collapsing schemes Brown (1992), Discrete Morse theory Forman (2002).
  - FDT: Squier (1994), diagram groups Guba and Sapir (1997).

Automaticity

- Decidable word problem: Automatic structure $\Rightarrow$ quadratic time word problem solution (Campbell et al. (2001)).

Applications to semigroup algebras: KS where $S \cong \langle A | R \rangle$

- Finite Gröbner–Shirshov bases
- Automaton algebras (see Ufnarovskii (1995) and some recent work of Okniński (2014)).

The Plactic monoid

The Plactic monoid $\mathrm{Pl}(A_n)$ is a finitely presented monoid with elements in correspondence with the set of semistandard Young tableaux. It has word problem solvable in quadratic time (Schensted algorithm).

E. Zelmanov (Lisbon, 2011) asked "Are Plactic monoids automatic?"

A. J. Cain, R. D. Gray, A. Malheiro
Finite Gröbner-Shirshov bases for Plactic algebras and biautomatic structures for Plactic monoids.
Journal of Algebra Vol. 423, 2015, pp. 37–53.

This work has since led us in several directions:

1. Other related monoids: Chinese, hypoplactic, Sylvester monoids, ...
2. Homogeneous monoids (motivated by recent work on homogeneous semigroup algebras by Cedó, Jaszuńska, Jespers, Kubat, Okniński)
3. Crystals basis theory.

Plactic monoid

Let $\mathcal{A}_n$ be the finite ordered alphabet $\{1 < 2 < \ldots < n\}$.

I want to give three different ways of defining a certain equivalence relation $\sim$ on the free monoid $\mathcal{A}_n^*$ of all words:

1. Presentation (Knuth relations)
2. Tableaux (Schensted insertion algorithm)
3. Crystal bases (in the sense of Kashiwara)

We call $\sim$ the Plactic congruence and the resulting quotient monoid $\mathrm{Pl}(A_n) = \mathcal{A}_n^*/\sim$ is called the Plactic monoid (of rank n).

Plactic monoid via Knuth relations

Definition

Let $\mathcal{A}_n$ be the finite ordered alphabet $\{1 < 2 < \ldots < n\}$.
Let $\mathcal{R}$ be the set of defining relations:

$$zxy = xzy \quad \text{and} \quad yzx = yxz \qquad x < y < z,$$
$$xyx = xxy \quad \text{and} \quad xyy = yxy \qquad x < y.$$

The Plactic monoid $\mathrm{Pl}(A_n)$ is defined by the presentation $\langle \mathcal{A}_n | \mathcal{R} \rangle$.

$$\text{e.g. } 212313 =_{\mathrm{Pl}(A_n)} 212133$$

- The relations in this presentation are called the Knuth relations.
- The Plactic monoid obviously has decidable word problem.

A (semi-standard) tableau

1	1	1	2	2	4	4
2	2	3	3			
4	5	5	6			
6	8					

Properties

- Is a filling of the Young diagram with symbols from the alphabet $\mathcal{A}_n$.
- Rows read left-to-right are non-decreasing.
- Columns read down are strictly increasing.
- Never have a longer row below a strictly shorter one.

Schensted column insertion algorithm

- Associates to each word $w \in \mathcal{A}_n^*$ a tableau $P(w)$.
- The algorithm which produces $P(w)$ is recursive.

Input: Any letter $x \in \mathcal{A}_n$ and a tableau T.

Output: A new tableau denoted $x \to T$.

The idea: Suppose $T = C_1 C_2 \dots C_r$ where the C_i are the columns of T.

- We try to insert the box $\boxed{x}$ under the column C_1 if we can.
- If this fails, the box $\boxed{x}$ will be put into column C_1 higher up and will "bump out" to the right a box $\boxed{y}$ where y is the minimal letter in C_1 such that $x \leq y$.
- We then take the bumped out box $\boxed{y}$ and try and insert it under the column C_2, and so on...

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\begin{array}{|c|}\hline 2\\\hline\end{array}\,,\; \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,,\; \begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\,,\; \begin{array}{|c|c|c|}\hline 1&2&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\,,\; \begin{array}{|c|c|c|}\hline 1&2&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\,,\; \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&\multicolumn{1}{c|}{3}\\\cline{1-2} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array} = P(w).$$

Observation: $231 = 213$ is one of the Knuth relations in the presentation of the Plactic monoid and $P(231) = P(213)$:

$$\begin{array}{|c|}\hline 2\\\hline\end{array}\,,\; \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,,\; \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} = P(231), \qquad \begin{array}{|c|}\hline 2\\\hline\end{array}\,,\; \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,,\; \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} = P(213).$$

Theorem (Lascoux and Shützenberger (1981))

Define a relation $\sim$ on $\mathcal{A}_n^*$ by

$$u \sim w \Leftrightarrow P(u) = P(w).$$

Then $\sim$ is the Plactic congruence and $\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ is the Plactic monoid.

The Plactic monoid via tableaux

$\mathrm{w}(T)$ = the word obtained by reading the columns of a tableau T from right to left and top to bottom (Japanese reading).

Example: If $T = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ then $\mathrm{w}(T) = 415123$.

Theorem (Lascoux and Shützenberger (1981))

For any word $u \in \mathcal{A}_n^*$ we have

- $u = \mathrm{w}(P(u))$ in the Plactic monoid $\mathrm{Pl}(A_n)$ and
- $P(u)$ is the unique tableau such that this is true.

Conclusion: the set of word readings of tableaux is a set of normal forms for the elements of the Plactic monoid. So, the Plactic monoid is the monoid of tableaux:

Elements	The set of all tableaux over $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$.
Products	Computed using Schensted insertion.

The Plactic monoid

- Has origins in work of Schensted (1961) and Knuth (1970) concerned with combinatorial problems on Young tableaux.
- Later studied in depth by Lascoux and Shützenberger (1981).

Due to close relations to Young tableaux, has become a tool in several aspects of representation theory and algebraic combinatorics.

Applications of the Plactic monoid

- proof of the Littlewood–Richardson rule for Schur functions (an important result in the theory of symmetric functions);
  - see appendix of J. A. Green's "Polynomial representations of GL_n".
- a combinatorial description of the Kostka–Foulkes polynomials, which arise as entries of the character table of the finite linear groups.

M. P. Schützenberger 'Pour le monoïde plaxique' (1997)

Argues that the Plactic monoid ought to be considered as "one of the most fundamental monoids in algebra".

Crystals

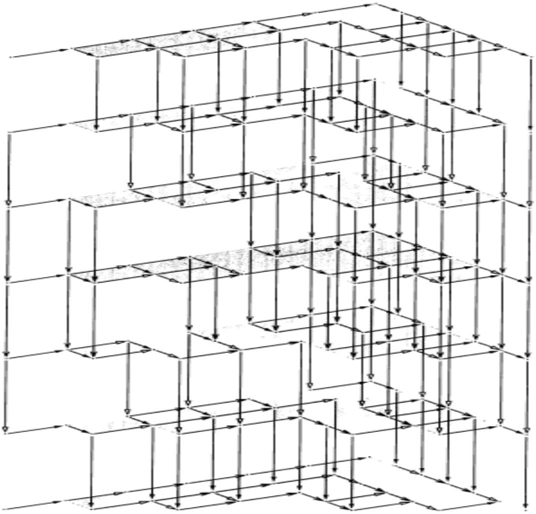


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Fig 8.4 from Hong and Kang's book *An introduction to quantum groups and crystal bases.*

Crystal graphs

(following Kashiwara and Nakashima (1994))

Idea: Define a directed labelled digraph Γ_{A_n} with the properties:

- Vertex set = $\mathcal{A}_n^*$
- Each directed edge is labelled by a symbol from the label set $I = \{1, 2, \ldots, n-1\}$.
- For each vertex $u \in \mathcal{A}_n^*$ every $i \in I$ there is at most one directed edge labelled by i leaving u, and there is at most one directed edge labelled by i entering u,

$$u \xrightarrow{\quad i \quad} v \,, \qquad w \xrightarrow{\quad i \quad} u$$

- If $u \xrightarrow{\quad i \quad} v$ then $|u| = |v|$, so words in the same component have the same length as each other. In particular, connected components are all finite.

Building the crystal graph Γ_{A_n}

$$\mathcal{A}_n = \{1 < 2 < \ldots < n\}$$

We begin by specifying structure on the words of length one

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

This is known as a Crystal basis.

Kashiwara operators

For each $i \in \{1, \ldots, n-1\}$ we define partial maps $\tilde{e}_i$ and $\tilde{f}_i$ on the letters $\mathcal{A}_n$ called the Kashiwara crystal graph operators. For each edge

$$a \xrightarrow{i} b \; ,$$

we define $\tilde{f}_i(a) = b$ and $\tilde{e}_i(b) = a$.

The crystal graph Γ_{A_n}

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

$$a \xrightarrow{i} \tilde{f}_i(a) \quad , \qquad \tilde{e}_i(b) \xrightarrow{i} b$$

Kashiwara operators on words

For $u, v \in \mathcal{A}_n^+$ define inductively

$$\tilde{e}_i(uv) = \begin{cases} u\,\tilde{e}_i(v) & \text{if } \varphi_i(u) < \epsilon_i(v) \\ \tilde{e}_i(u)\,v & \text{if } \varphi_i(u) \geq \epsilon_i(v) \end{cases}, \quad \tilde{f}_i(uv) = \begin{cases} \tilde{f}_i(u)\,v & \text{if } \varphi_i(u) > \epsilon_i(v) \\ u\tilde{f}_i(v) & \text{if } \varphi_i(u) \leq \epsilon_i(v) \end{cases}.$$

where ϵ_i and φ_i are auxiliary maps defined by

$$\epsilon_i(w) = \max\{k \in \mathbb{N} \cup \{0\} : \underbrace{\tilde{e}_i \cdots \tilde{e}_i}_{k \text{ times}}(w) \text{ is defined}\}$$

$$\varphi_i(w) = \max\{k \in \mathbb{N} \cup \{0\} : \underbrace{\tilde{f}_i \cdots \tilde{f}_i}_{k \text{ times}}(w) \text{ is defined}\}$$

The crystal graph Γ_{A_n}

Definition

The crystal graph Γ_{A_n} is the directed labelled graph with:

- Vertex set: $\mathcal{A}_n^*$
- Directed labelled edges: for $u \in \mathcal{A}_n^*$

$$u \xrightarrow{i} \tilde{f}_i(u), \qquad \tilde{e}_i(u) \xrightarrow{i} u$$

Note: When defined $\tilde{e}_i(\tilde{f}_i(u)) = u$ and $\tilde{f}_i(\tilde{e}_i(u)) = u$.

Note: It follows from the recursive nature of the definition of the Kashiwara operators that (when defined) we have $\tilde{e}_i(u) = u'\tilde{e}_i(a)u''$ for some decomposition $u \equiv u'au''$ where a is a single letter.

Example: Practical computation of $\tilde{e}_i(u)$ and $\tilde{f}_i(u)$

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$a \xrightarrow{i} \tilde{f}_i(a), \qquad \tilde{e}_i(b) \xrightarrow{i} b$$

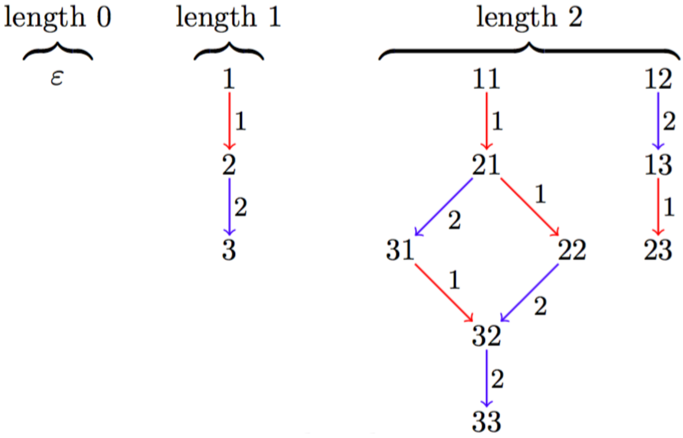
Example

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

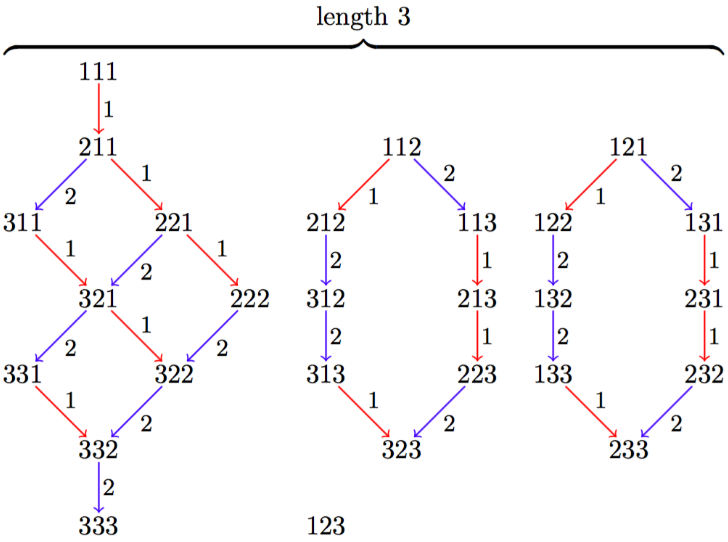
$$\begin{array}{ccccccccccc}
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\
- & - & + & & + & - & & - & + & - & + \\
- & - & \cancel{+} & & \cancel{+} & \cancel{-} & & \cancel{-} & \cancel{+} & \cancel{-} & + \\
- & - & & & & & & & & & +
\end{array}$$

$$\begin{array}{ccccccccccc}
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 3 = \tilde{f}_2(u) \\
3 & 2 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 = \tilde{e}_2(u)
\end{array}$$

Part of the crystal graph for $\mathcal{A}_3 = \{1 < 2 < 3\}$



Part of the crystal graph for $\mathcal{A}_3 = \{1 < 2 < 3\}$



Plactic monoid via crystals

Definition: Two connected components $B(w)$ and $B(w')$ of Γ_{A_n} are isomorphic if there is a label-preserving digraph isomorphism $f : B(w) \to B(w')$.

Fact: In Γ_{A_n} if $B(w) \cong B(w')$ then there is a unique isomorphism $f : B(w) \to B(w')$.

Theorem (Kashiwara and Nakashima (1994))

Let Γ_{A_n} be the crystal graph with crystal basis

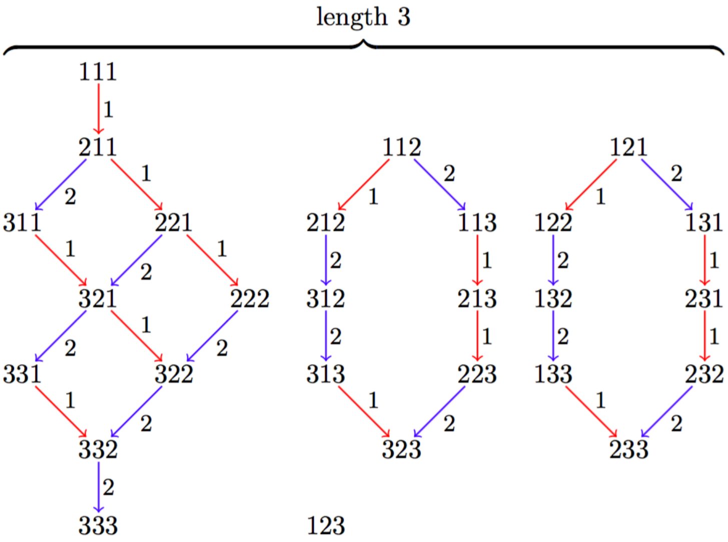
$$1 \xrightarrow{\ 1\ } 2 \xrightarrow{\ 2\ } \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

Define a relation $\sim$ on $\mathcal{A}_n^*$ by

$$u \sim w \Leftrightarrow \exists \text{ an isomorphism } f : B(u) \to B(w) \text{ with } f(u) = w.$$

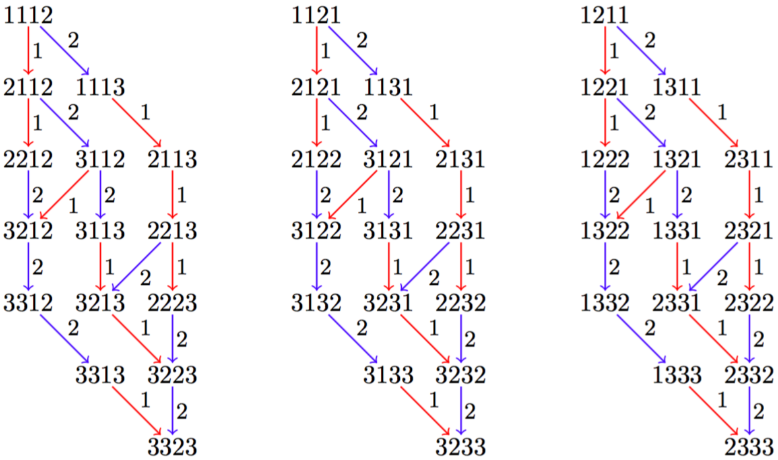
Then $\sim$ is the Plactic congruence and $\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ is the Plactic monoid.

Knuth relations via crystal isomorphisms[3]



[3](**Confession:** I lied a bit. Actually, crystal isomorphisms must also preserve “weight”. For $\mathrm{Pl}(A_n)$ weight preserving means “content preserving”.)

Three isomorphic components for $\mathcal{A}_3 = \{1 < 2 < 3\}$.



2113, 2131, and 2311 all represent the same element.

Where do crystals come from?

J. Hong, S.-J. Kang,
Introduction to Quantum Groups and Crystal Bases.
Stud. Math., vol. 42, Amer. Math. Soc., Providence, RI, 2002.

- Take a “nice” Lie algebra $\mathfrak{g}$. Nice means symmetrizable Kac-Moody Lie algebra e.g. a finite-dimensional semisimple Lie algebra.
- Crystal bases are bases of $U_q(\mathfrak{g})$-modules at $q = 0$ that satisfy certain axioms.
  - Kashiwara (1991): proves existence and uniqueness of crystal bases of finite dimensional representations of $U_q(\mathfrak{g})$.
- Every crystal basis has the structure of a coloured digraph (called a crystal graph). The structure of these coloured digraphs has been explicitly determined for certain semisimple Lie algebras.
- The crystal constructed from the crystal basis using Kashiwara operators is then a useful combinatorial tool for studying representations of $U_q(\mathfrak{g})$.

Crystal bases and crystal monoids

Lie algebra type	Crystal basis	Monoid
A_n: $\mathfrak{sl}_{n+1}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$	$\mathrm{Pl}(A_n)$
B_n: $\mathfrak{so}_{2n+1}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} 0 \xrightarrow{n} \overline{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(B_n)$
C_n: $\mathfrak{sp}_{2n}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} \overline{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(C_n)$
D_n: $\mathfrak{so}_{2n}$	$\begin{tikzcd}[column sep=small] & & & & \overline{n} \arrow[dr, "n"] & & & & \\ 1 \arrow[r, "1"] & 2 \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & n-1 \arrow[ur, "n-1"] \arrow[dr, "n"'] & & \overline{n-1} \arrow[r, "n-2"] & \cdots \arrow[r, "2"] & \overline{2} \arrow[r, "1"] & \overline{1} \\ & & & & n \arrow[ur, "n-1"'] & & & & \end{tikzcd}$	$\mathrm{Pl}(D_n)$
G_2	$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{1} 0 \xrightarrow{1} \overline{3} \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(G_2)$

Crystal monoids in general

- Crystal basis = a directed labelled graph, vertex set X, label set I, such that for all $x \in X$ and $i \in I$:
  - there is at most one edge starting at x labelled by i and at most one edge ending at x labelled by i, and
  - there is no infinite path made up of edges labelled by i.
- A weight function $\mathrm{wt} : X^* \to P$, where P is called the weight monoid, such that there is a partial order $\leq$ on P such that:

$$\mathrm{wt}(u) < \mathrm{wt}(\tilde{e}_i(u)) \quad \& \quad \mathrm{wt}(\tilde{f}_i(u)) < \mathrm{wt}(u).$$

- Construct a (weighted) crystal graph Γ_X from this data
  - Vertex set: X^*
  - Directed labelled edges: determined by $\tilde{e}_i, \tilde{f}_i$

Definition (Crystal monoid)

Let Γ_X be a crystal graph. Define a relation $\approx$ on X^* where two words are related if they lie in the same position of isomorphic components of the crystal graph Γ_X. Then $\approx$ is a congruence on X^* and $X^* / \approx$ is called the crystal monoid of Γ_X.

Crystal monoids in general

Γ_X - a crystal graph, $X/\approx$ - the associated crystal monoid

Crystal monoids provide a link between:

$$\text{Kashiwara crystals} \quad \longleftrightarrow \quad \begin{array}{l}\text{Combinatorial semigroup theory,} \\ \text{combinatorics on words, and} \\ \text{formal language theory}\end{array}$$

Proposition

If a crystal monoid arises from a finite crystal basis, and has a weight monoid in which multiplication is computable, then it has decidable word problem.

- In particular, if the weight monoid is $\mathbb{Z}^m$ for some m then the crystal monoid has decidable word problem (this is the case for all of the types A_n, B_n, C_n D_n and G_2 above).

General Problem: To what extent can rewriting systems and automata be used to compute efficiently with crystals and crystal monoids?

Some known results

Known results on crystals A_n, B_n, C_n, D_n, or G_2 and their crystal monoids:

1. Crystal bases - combinatorial description Kashiwara and Nakashima (1994).
2. Tableaux theory and Schensted-type insertion algorithms - Kashiwara and Nakashima (1994), Lecouvey (2002, 2003, 2007).
3. Finite presentations for $\mathrm{Pl}(X) = X^* / \approx$ via Knuth-type relations - Lecouvey (2002, 2003, 2007).

Building on the work above, we have developed the theory of complete rewriting systems and automatic structures for all of these types of crystal monoids.

Rewriting systems and biautomatic structures

Theorem (Cain, RG, Malheiro (2014))

For any $X \in \{A_n, B_n, C_n, D_n, G_2\}$, there is a finite complete rewriting system (Σ, T) that presents $\mathrm{Pl}(X)$.

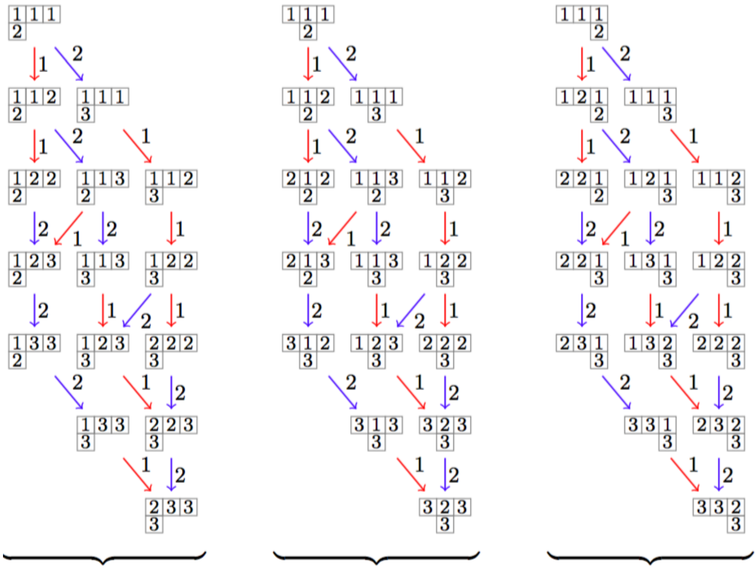
- Builds on our earlier results on the Plactic monoid $\mathrm{Pl}(A_n)$.
- In each case there is a tableau theory. Admissible columns are columns of tableaux.
- *Key idea:* work with the larger generating set Σ of admissible columns.

Theorem (Cain, RG, Malheiro (2014))

The monoids $\mathrm{Pl}(A_n)$, $\mathrm{Pl}(B_n)$, $\mathrm{Pl}(C_n)$, $\mathrm{Pl}(D_n)$, and $\mathrm{Pl}(G_2)$ are all biautomatic. In particular each of these monoids has word problem that is solvable in quadratic time.

- Biautomatic = the strongest form of automaticity for monoids.
- Language of representatives $=$ language of irreducible words of the rewriting systems (Σ, T) above.

Kashiwara operators preserve shape

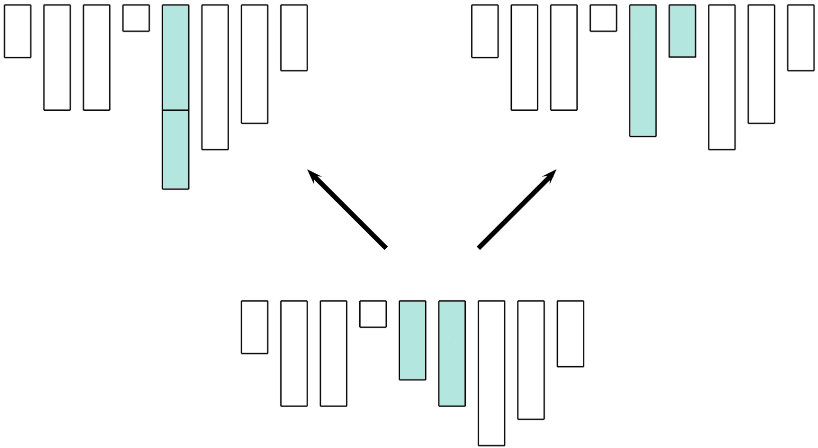


Tableaux of the same shape

Tabloids of the same shape

Tabloids of the same shape

Rewriting tabloids



- Multiplying two adjacent admissible columns of a tabloid brings us one step closer to being a tableau.

Crystal-theoretic consequences

Corollary

For the crystal graphs of types A_n, B_n, C_n, D_n, or G_2, there is a quadratic-time algorithm that takes as input two vertices and decides whether they lie in the same position in isomorphic components.

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Ongoing and future work

- Are there any further consequences that can be drawn from our results
  - For crystals? For Lie theory?
- What do these results say about the Plactic algebras of Littelmann (1996)? (Related to this is a recent paper of Hage (2015)).

We are developing further the general theory of crystal monoids.

- Examples of crystal monoids (with weight monoid $\mathbb{Z}^m$)
  - free monoids, the bicyclic monoid, the Thompson monoid, ...
- Conditions on the crystal basis that guarantee unique highest weight words in the crystal components
  - It does not suffice to assume the basis has unique highest weight.
- When are crystal monoids finite presented?
- When do they admit finite complete presentations / have automatic structures?
- What can we say about complexity of the word problem?
- When do we have a tableaux theory?

Appendix

Biautomaticity - formal definiton

Let A be an alphabet and let $\$$ be a new symbol not in A. Define the mapping $\delta_{\mathrm{R}} : A^* \times A^* \to ((A \cup \{\$\}) \times (A \cup \{\$\}))^*$ by

$$(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto \begin{cases} (u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\ (u_1, v_1) \cdots (u_n, v_n)(u_{n+1}, \$) \cdots (u_m, \$) & \text{if } m > n, \\ (u_1, v_1) \cdots (u_m, v_m)(\$, v_{m+1}) \cdots (\$, v_n) & \text{if } m < n, \end{cases}$$

and the mapping $\delta_{\mathrm{L}} : A^* \times A^* \to ((A \cup \{\$\}) \times (A \cup \{\$\}))^*$ by

$$(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto \begin{cases} (u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\ (u_1, \$) \cdots (u_{m-n}, \$)(u_{m-n+1}, v_1) \cdots (u_m, v_n) & \text{if } m > n, \\ (\$, v_1) \cdots (\$, v_{n-m})(u_1, v_{n-m+1}) \cdots (u_m, v_n) & \text{if } m < n, \end{cases}$$

where $u_i, v_i \in A$.

Biautomaticity - formal definiton

Let M be a monoid. Let A be a finite alphabet representing a set of generators for M and let $L \subseteq A^*$ be a regular language such that every element of M has at least one representative in L. For each $a \in A \cup \{\varepsilon\}$, define the relations

$$L_a = \{(u, v) : u, v \in L, ua =_M v\}$$
$$_aL = \{(u, v) : u, v \in L, au =_M v\}.$$

The pair (A, L) is a biautomatic structure for M if $L_a\delta_{\mathrm{R}}$, ${}_aL\delta_{\mathrm{R}}$, $L_a\delta_{\mathrm{L}}$, and ${}_aL\delta_{\mathrm{L}}$ are regular languages over $(A \cup \{\$\}) \times (A \cup \{\$\})$ for all $a \in A \cup \{\varepsilon\}$.

A monoid M is biautomatic if it admits a biautomatic structure with respect to some generating set.