Infinite monoids as geometric objects

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Groups, monoids, and geometry

Gromov - “Infinite groups as geometric objects” International Congress of Mathematicians address in Warsaw, 1984

There are two main inter-related strands in geometric group theory

1. one seeks to understand groups by studying their actions on appropriate spaces, and
2. one seeks understanding from the intrinsic geometry of finitely generated groups endowed with word metrics.

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2. one seeks understanding from the intrinsic geometry of finitely generated groups endowed with word metrics.

How about monoids and semigroups?

1. To what extent can we gain information about finitely generated monoids by studying their actions on geometric objects?
2. How much algebraic information about finitely generated monoids is encoded in the geometry of their Cayley graphs?

General philosophy

Algebra	Combinatorics
Groups	Graphs
Monoids / Semigroups	Digraphs

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?? = directed metric spaces = semimetric spaces

Semimetric space = a set equipped with an asymmetric, partially-defined distance function.

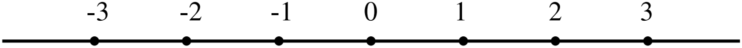
Cayley graphs and the notion of quasi-isometry

G - group, generated by a finite set $A \subseteq G$,
Assume $1 \notin A$, and $A = A^{-1}$.

This gives rise to a metric space (G, d_A) with word metric d_A.
Points: G
Distance: $d_A(g, h)$ the minimum length of a word $a_1 a_2 \cdots a_r \in A^*$ with the property that $g a_1 a_2 \cdots a_r = h$.

The Cayley graph $\Gamma(G, A)$
Vertices: G
Edges: $g \sim h \Leftrightarrow h = ga$ for some $a \in A$

The (?) Cayley graph of a group



$$\mathbb{Z} = \langle \pm\{1\} \rangle$$

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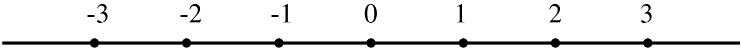

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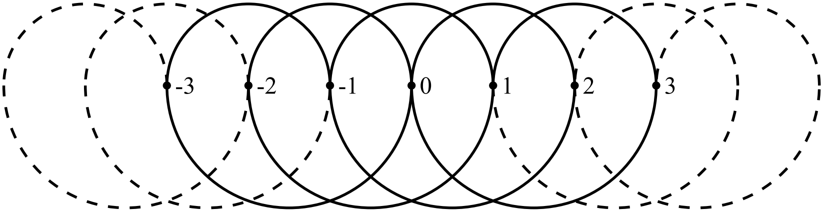

$$\mathbb{Z} = \langle \pm\{2, 3\} \rangle$$

Conclusion

Changing the finite generating set can result in spaces that are not isometric.

Idea: These two spaces look the same when viewed from far enough away. This idea is formalised via the notion of quasi-isometry.

Quasi-isometry for metric spaces

Definition

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is a quasi-isometric embedding if there exist constants $\lambda \geqslant 1$ and $C \geqslant 0$ such that

$$\frac{1}{\lambda} d_X(a, b) - C \;\leqslant\; d_Y(f(a), f(b)) \;\leqslant\; \lambda d_X(a, b) + C,$$

for all $a, b \in X$.

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(X, d_X) and (Y, d_Y) are quasi-isometric if in addition there is a constant $D \geqslant 0$ such that every point in Y has a distance at most D from some point in the image $f(X)$.

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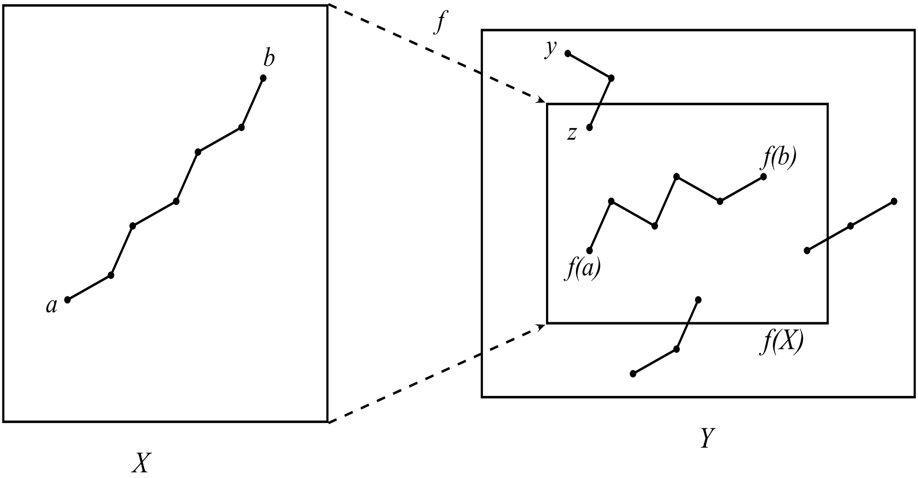
- Quasi-isometry is an equivalence relation between metric spaces, which ignores finite details.



f
y
b
z
f(b)
f(a)
a
f(X)
Y
X

Quasi-isometry for finitely generated groups

Proposition

Let A and B be two finite generating sets for the group G.
Then the metric spaces (G, d_A) and (G, d_B) are quasi-isometric.

The quasi-isometry class of a group

Given a finitely generated group G, the metric space (G, d_A) is well defined up to quasi-isometry by the group G alone.

In particular, given two finitely generated groups G and H, one may ask whether they are quasi-isometric or not, without reference to any specific choice of finite generating sets.

Examples of quasi-isometric groups

The quasi-isometry class of the trivial group $G = \{1\}$ is precisely the class of all finite groups.

Let G be a finitely generated group and H be a subgroup of finite index in G. Then H is finitely generated and quasi-isometric to G.

Let G be a finitely generated group and let N be a finite normal subgroup of G. Then G/N is a finitely generated group quasi-isometric to G.

For $k, l \geqslant 2$, the free groups F_k and F_l are quasi-isometric to each other.

Free abelian groups $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are quasi-isometric $\Leftrightarrow m = n$.

Tigers, Lions and Frogs

Tigers and lions look similar and, genetically, they have a lot on common.

Tigers and frogs on the other hand...

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Tigers and frogs on the other hand...

Quasi-isometric groups look similar and, algebraically, they have a lot in common.

Quasi-isometry invariants

Definition

A property (P) of finitely generated groups is said to be a quasi-isometry invariant if, whenever G_1 and G_2 are quasi-isometric finitely generated groups,

$$G_1 \text{ has property } (P) \iff G_2 \text{ has property } (P).$$

Examples of quasi-isometry invariants

(i) Finite (ii) Infinite virtually cyclic (iii) Finitely presented (iv) Virtually abelian (v) Virtually nilpotent (vi) Virtually free (vii) Amenable (viii) Hyperbolic (ix) Accessible (x) Type of growth (xi) Finitely presented with solvable word problem (xii) Satisfying the homological finiteness condition F_n or the condition FP_n (xiii) Number of ends.

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?? = directed metric spaces = semimetric spaces

Semimetric space = a set equipped with an asymmetric, partially-defined distance function.

Cayley graphs of semigroups and monoids

$$
\begin{tikzcd}
1 \arrow[r, dotted, bend left] \arrow[d] & b \arrow[l, bend left] \arrow[r, dotted, bend left] & b^2 \arrow[l, bend left] \arrow[r, dotted, bend left] & b^3 \arrow[l, bend left] \arrow[r, dotted, bend left] & b^4 \arrow[l, bend left] \arrow[r, dotted, bend left] & b^5 \arrow[l, bend left] \\
c \arrow[r, dotted, bend left] \arrow[d] & cb \arrow[l, bend left] \arrow[r, dotted, bend left] & cb^2 \arrow[l, bend left] \arrow[r, dotted, bend left] & cb^3 \arrow[l, bend left] \arrow[r, dotted, bend left] & cb^4 \arrow[l, bend left] \arrow[r, dotted, bend left] & cb^5 \arrow[l, bend left] \\
c^2 \arrow[r, dotted, bend left] \arrow[d] & c^2b \arrow[l, bend left] \arrow[r, dotted, bend left] & c^2b^2 \arrow[l, bend left] \arrow[r, dotted, bend left] & c^2b^3 \arrow[l, bend left] \arrow[r, dotted, bend left] & c^2b^4 \arrow[l, bend left] \arrow[r, dotted, bend left] & c^2b^5 \arrow[l, bend left] \\
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c^5 \arrow[r, dotted, bend left] & c^5b \arrow[l, bend left] \arrow[r, dotted, bend left] & c^5b^2 \arrow[l, bend left] \arrow[r, dotted, bend left] & c^5b^3 \arrow[l, bend left] \arrow[r, dotted, bend left] & c^5b^4 \arrow[l, bend left] \arrow[r, dotted, bend left] & c^5b^5 \arrow[l, bend left]
\end{tikzcd}
$$

The bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$

Semimetric spaces

Definition (Semimetric space)

A semimetric space is a pair (X, d) where X is a set, and

$$d : X \times X \to \mathbb{R}^{\infty} = \mathbb{R}^{\geqslant 0} \cup \{\infty\}$$

is a function satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$

for all $x, y, z \in X$.

Here $\mathbb{R}^{\infty} = \mathbb{R}^{\geqslant 0} \cup \{\infty\}$ with the obvious order, and we set

$$\infty + x = x + \infty = y\infty = \infty y = \infty$$

for all $x \in \mathbb{R}^{\infty}$ and $y \in \mathbb{R}^{\infty} \backslash \{0\}$.

Monoids as semimetric spaces

M - monoid generated by a finite set A.

This gives rise to a semimetric space (M, d_A) with word semimetric d_A.
Points: M
Directed distance: $d_A(x, y)$ the minimum length of a word $a_1 a_2 \cdots a_r \in A^*$ with the property that $x a_1 a_2 \cdots a_r = y$, or ∞ if there is no such word.

The (right) Cayley graph $\Gamma(M, A)$
Vertices: M
Directed edges: $x \to y$ iff $y = xa$ for some $a \in A$

Quasi-isometry for semimetric spaces

Definition

Let (X, d_X) and (Y, d_Y) be two semimetric spaces. A map $f : X \to Y$ is a quasi-isometric embedding if there exist constants $\lambda \geqslant 1$ and $C \geqslant 0$ such that

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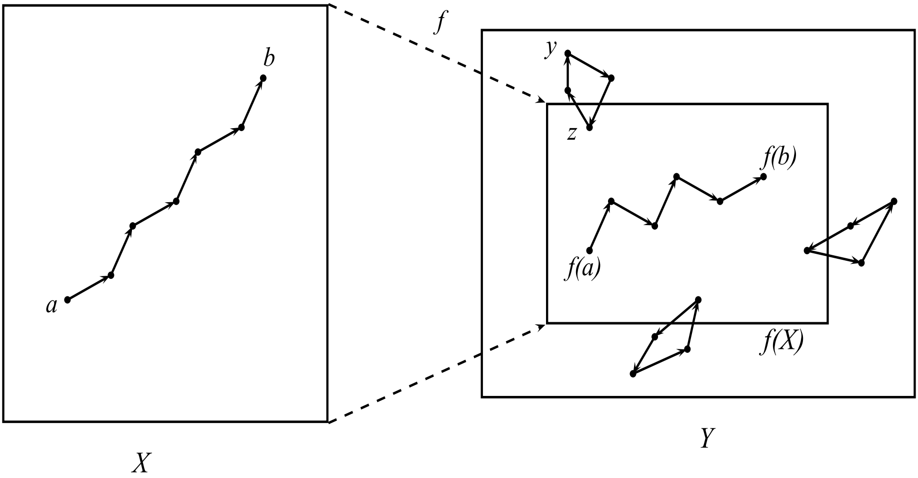
The semimetric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if in addition there is a constant $D \geqslant 0$ such that for every $y \in Y$ there exists a $z \in f(X)$ such that

$$d_Y(y, z) \leqslant D \quad \text{and} \quad d_Y(z, y) \leqslant D.$$

- Quasi-isometry is an equivalence relation between semimetric spaces.


f
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Quasi-isometry for finitely generated monoids

Proposition

Let A and B be two finite generating sets for a monoid M.
Then the semimetric space (M, d_A) is quasi-isometric to the semimetric space (M, d_B).

The quasi-isometry class of a monoid

The semimetric space (M, d_A) is well defined up to quasi-isometry by the finitely generated monoid M alone.

In particular, given two finitely generated monoids M and N, one may ask whether they are quasi-isometric or not, without reference to any specific choice of finite generating sets.

Quasi-isometric monoids

Quasi-isomtries between monoids preserve right ideal structure.

Two finite monoids M and N are quasi-isometric if and only if the partially ordered sets or $\mathcal{R}$-classes $M/\mathcal{R}$ and $N/\mathcal{R}$ are isomorphic.

If M is a finitely generated monoid, and η is a congruence on M, then M and M/η are quasi-isometric if there is a bound on the diameter of the η-classes of M.

Finitely generated free monoids are quasi-isometric if and only if they have the same rank.

Finitely generated free commutative monoids are quasi-isometric if and only if they have the same rank.

Quasi-isometry invariants of monoids

Theorem

The following properties are all quasi-isometry invariants of finitely generated monoids:

- Finiteness;
- Number of right ideals;
- Being a group (for monoids);
- Being right simple (for semigroups);
- Type of growth;
- Number of ends (in the sense of Jackson and Kilibarda (2009)).

Finite presentability and the word problem

Q1 Is finite presentability a quasi-isometry invariant of finitely generated monoids?

Q2 Is being finitely presented with solvable word problem a quasi-isometry invariant of finitely generated monoids?

One should not expect yes to be the answer for monoids in general. Indeed:

- Jenni Awang (St Andrews) recently found a counterexample to (Q1);
- (Q2) is still open, but we have constructed an example to show that having solvable word problem is not a quasi-isometry invariant of finitely generated monoids.

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Idea: Investigate (Q1) and (Q2) for classes of monoids that lie between monoids and groups, such as:

1. Left cancellative monoids
2. Monoids with finitely many left and right ideals

Obviously any group satisfies both (1) and (2).

Left cancellative semigroups and monoids

Left cancellativity: $ab = ac \Rightarrow b = c$.
Right cancellativity, and cancellativity are defined analogously.

Interesting classes of cancellative monoids

- Divisibility monoids (Droste & Kuske (2001));
- Garside monoids; includes, spherical Artin monoids, Braid monoids of complex reflection groups etc. (Dehornoy & Paris (1999)).

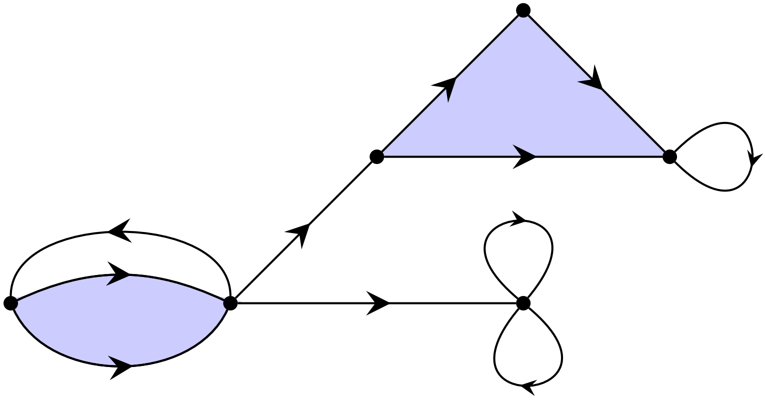
One-relator monoids

- Adyan and Oganesyan (1987): Decidability of the word problem for one relator monoids is reducible to the left cancellative case.
- Motivates the development of new methods for approaching the word problem for finitely presented left cancellative monoids.

Directed 2-complexes (Guba & Sapir (2006))

Definition

A directed 2-complex is a digraph Γ, together with a set F of faces.

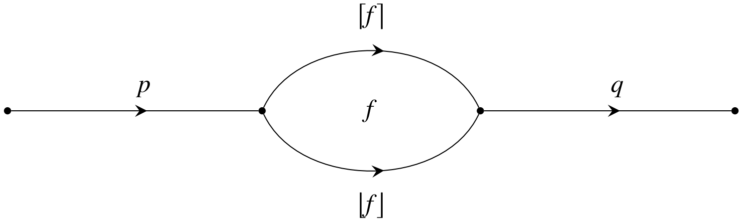


2-paths = paths between paths

K - a directed 2-complex, with underlying digraph Γ, and set of faces F
The 1-paths in K are the paths in Γ.

Definition (2-path)

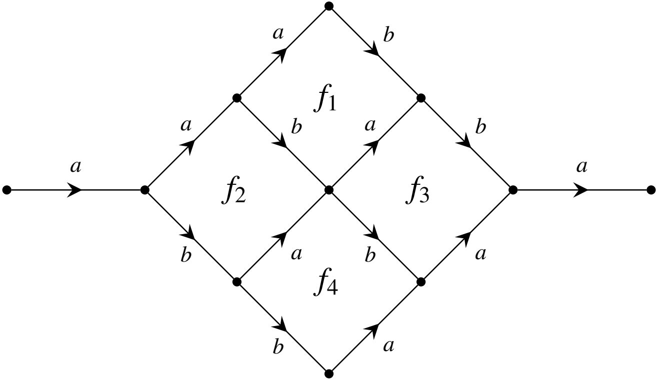
An atomic 2-path δ is a triple (p, f, q) where p, q are 1-paths, $f \in F$ and:



A 2-path in K is then a sequence $\delta = \delta_1 \delta_2 \dots \delta_n$ of composable atomic 2-paths.

Two paths p, q in K are homotopic if there is a 2-path between them.

2-path example

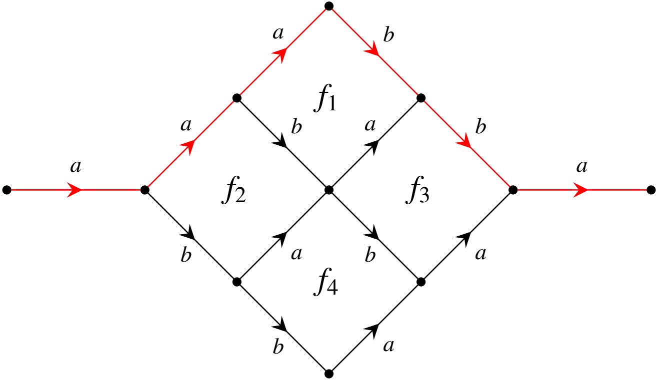


Γ - right Cayley graph of the monoid $\langle a, b \mid ab = ba \rangle$.

$K_4(\Gamma)$ - directed 2-complex with underlying digraph Γ and face set F given by adding a face for every pair $p \parallel q$ of parallel paths with $|p| + |q| \leqslant 4$.

Diagram illustrates a 2-path δ of length 4 in $K_4(\Gamma)$.

2-path example



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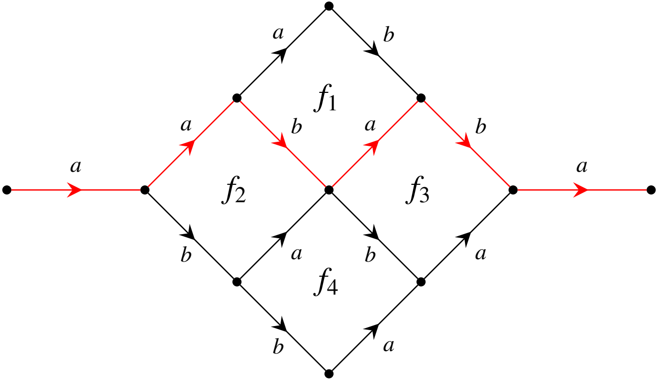

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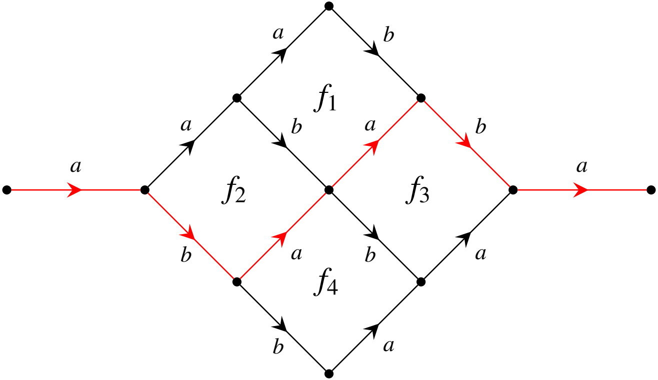

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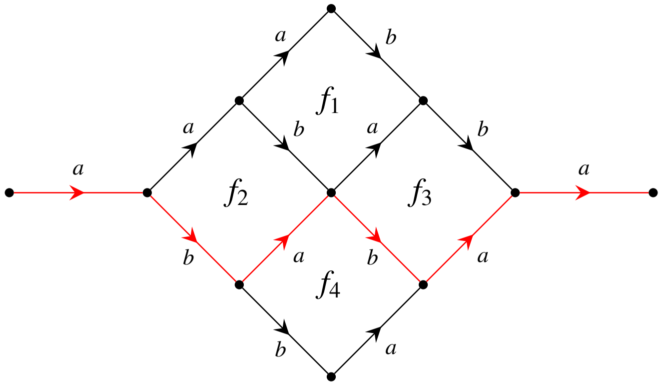

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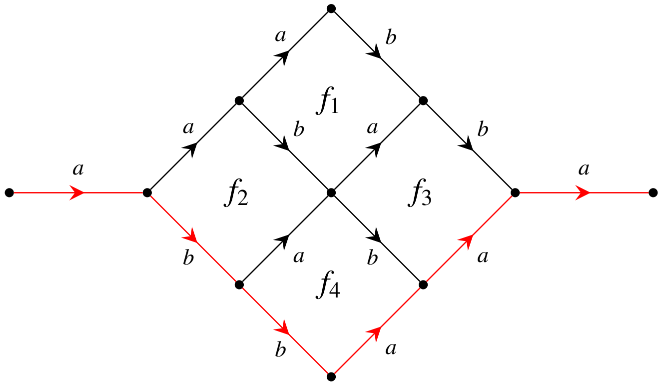

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Directed homotopy and simple connectedness

Directed homotopy

A directed 2-complex K is directed simply connected if for every pair $p \parallel q$ of parallel paths, p and q are homotopic in K.

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A directed 2-complex K is directed simply connected if for every pair $p \parallel q$ of parallel paths, p and q are homotopic in K.

Quasi-simple connectedness

Γ - digraph, $n \in \mathbb{N}$
$K_n(\Gamma) =$ directed 2-complex with underlying digraph Γ and face set

$$F = \{(p, q) \mid p \text{ and } q \text{ are parallel paths in } \Gamma \text{ with } |p| + |q| \leqslant n\}.$$

We say the digraph Γ is quasi-simply-connected if $K_n(\Gamma)$ is directed simply connected for some n.

Note: $K_n(\Gamma)$ is the natural directed analogue of the Rips complex from geometric group theory.

Finite presentability and the word problem

Theorem

Let S be a left cancellative monoid generated by a finite set A. Then:

- S is finitely presented $\Leftrightarrow \Gamma(S, A)$ is quasi-simply-connected.

Proposition

The property of being quasi-simply-connected is a quasi-isometry invariant of directed graphs.

Theorem

Let M and N be left cancellative, finitely generated monoids which are quasi-isometric. Then M is finitely presentable $\Leftrightarrow$ N is finitely presentable.

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By defining and studying Dehn functions of directed 2-complexes and their behaviour under quasi-isometry, one can show:

Theorem

Let M and N be left cancellative, finitely presentable monoids which are quasi-isometric. Then M has solvable word problem if and only if N has solvable word problem.

Monoids with finitely many left and right ideals

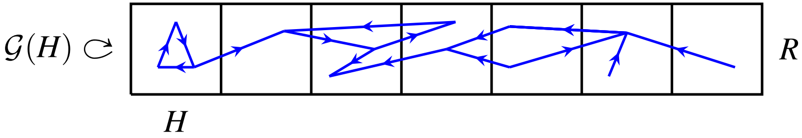
We established a Švarc–Milnor Lemma for groups acting on geodesic semimetric spaces. Applying this result to Schützenberger groups acting on Schützenberger graphs leads to the following.

Theorem

Let M be a finitely generated monoid with finitely many left and right ideals. Then M is finitely presented if and only if all right Schützenberger graphs of M are quasi-simply-connected.

Theorem

For finitely generated monoids with finitely many left and right ideals, finite presentability is a quasi-isometry invariant.



Future directions

- Are there other natural classes of monoids for which the properties of being
  (a) finitely presented;
  (b) finitely presented with solvable word problem;

  are quasi-isometry invariants?
- Investigate other properties from the point of view of quasi-isometry (e.g. Amenable semigroups (Day (1957)) / Følner conditions in digraphs and semimetric spaces).
- We have restricted our attention to the geometry of right Cayley graphs only. What if one considers the geometry of right *and* left Cayley graphs simultaneously?