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Countably infinite set-homogeneous structures

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Homogeneity

A relational structure M is *homogeneous* if any isomorphism between finite substructures of M extends to an automorphism of M. That is, every partial finite symmetry extends to a global symmetry.

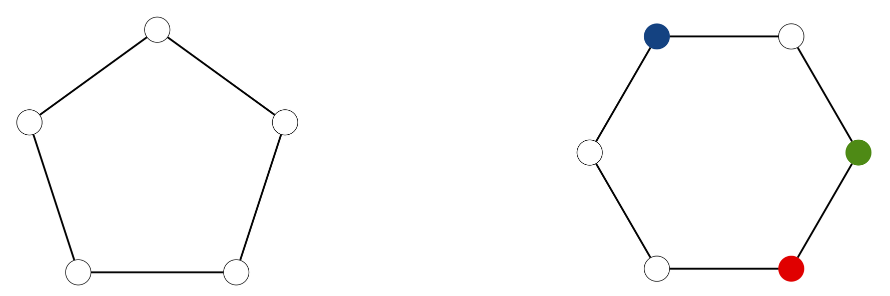
- Dates back to pioneering work of Fraïssé (1953).

Homogeneous structures:

- have lots of symmetry
- often have rich and interesting automorphism groups
- provide a meeting-point of ideas from combinatorics, model theory, and permutation group theory

Examples

- **Finite graphs**



- Pentagon is homogeneous
- Hexagon is not homogeneous: (•, •) ↦ (•, •) does not extend to an automorphism

- **Finite digraphs**
- Lachlan (1982) - the following digraph is homogeneous

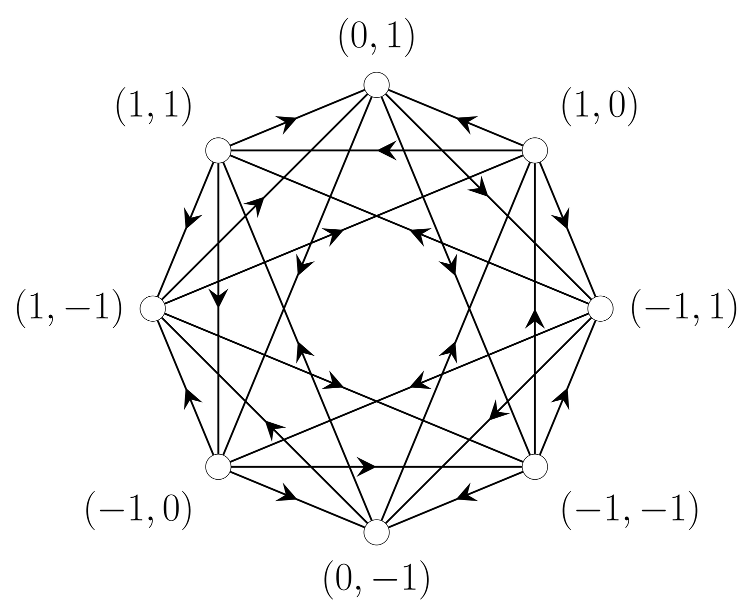
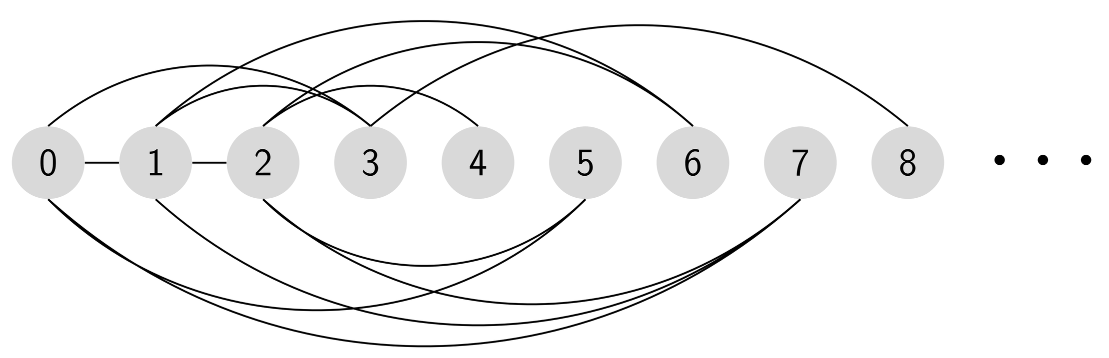


Figure: The vertices are the 8 non-zero vectors in $GF(3)^2$ and there is an arc from u to v if $|u^T\ v^T| = 1$ (where $|u^T\ v^T|$ is the determinant of the 2×2 matrix with columns u^T and v^T).

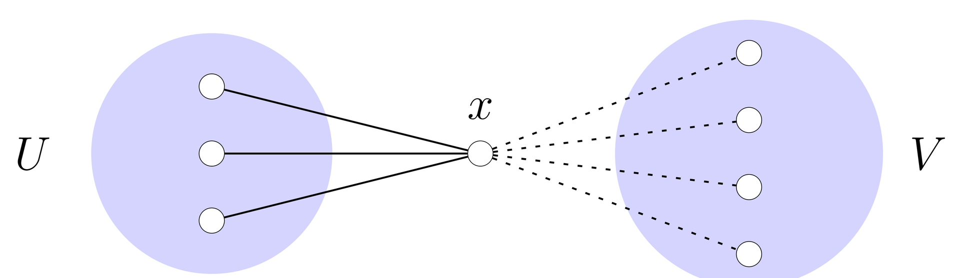
- **The random graph**

In 1964 Rado constructed a countably infinite graph R as follows: The vertex set is the set of natural numbers (including zero). For $i, j \in \mathbb{N}$, $i < j$, then i and j are joined if and only if the ith digit in j (in base 2) is 1.



Rado's graph R satisfies the following condition

(*) Given any two finite disjoint sets U and V of vertices, there is a vertex x joined to every vertex in U and to no vertex in V.



A back-and-forth argument shows Rado's graph is the unique countable graph (up to isomorphism) satisfying condition (*).

- This can be used to show that R is homogeneous.

Theorem (Erdős and Rényi (1963)). If a countable random graph is chosen by selecting edges independently with probability $\frac{1}{2}$ from all pairs of vertices, the resulting graph is isomorphic to R with probability 1.

- Thus, the infinite homogeneous graph R is *the* countable random graph.

Set-homogeneity

A relational structure M is *set-homogeneous* if, whenever U and V are isomorphic finite substructures, there is an automorphism g of M with $U^g = V$.

- Concept originally due to Fraïssé and Pouzet
- Is a natural weakening of homogeneity

Main question

How much stronger is homogeneity than set-homogeneity?

- **Finite graphs**

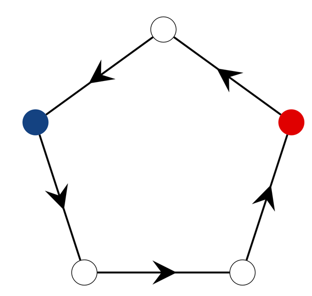
Ronse (1978) and Enomoto (1981): for finite graphs set-homogeneity is equivalent to homogeneity.

Finite set-homogeneous digraphs

- **Finite digraphs**

The directed 5-cycle:

- is a set-homogeneous digraph
- is not homogeneous: (•, •) ↦ (•, •) does not extend to an automorphism



Theorem 1

Let D be a finite set-homogeneous digraph. Then either D is homogeneous or D is isomorphic to the directed 5-cycle.

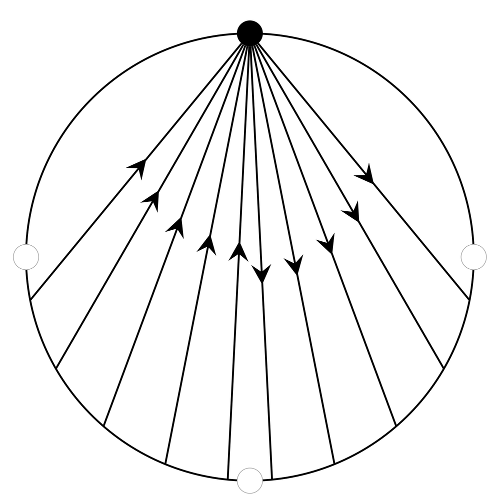
Infinite set-homogeneous digraphs

Let us consider two constructions which give examples of countable set-homogeneous digraphs which are not 2-homogeneous, and therefore are not homogeneous.

Circular structures

Let $T(4)$ be the digraph obtained by distributing countably many points densely around the unit circle with

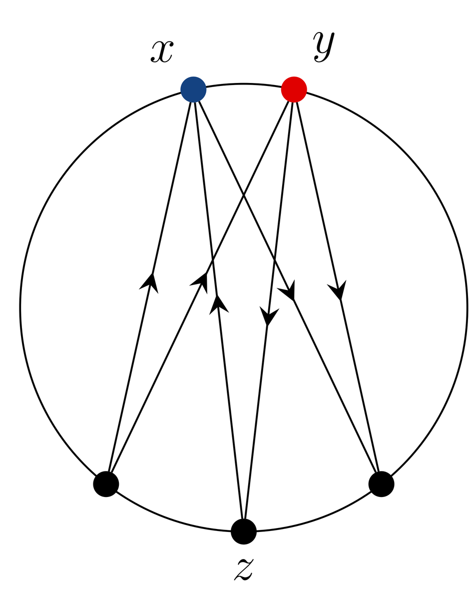
- no two making an angle of π or $\pi/2$ at the centre, and
- $x \to y$ if and only if $\pi/2 < \arg(x/y) < \pi$.



By a back-and-forth argument, this construction for $T(4)$ determines a unique digraph.

Lemma. The digraph $T(4)$ is set-homogeneous but not 2-homogeneous.

- set-homogeneity: shown by "expanding" $T(4)$ to a homogeneous structure
- not 2-homogeneous: if $x, y \in T(4)$ with $0 < \arg(x/y) < \pi/2$, then
- $\exists z\ (y \to z \to x)$ but $\nexists z\ (x \to z \to y)$



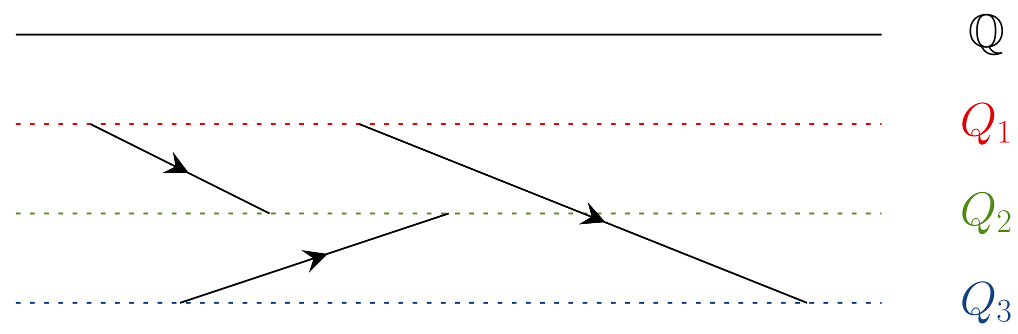
- Thus, $(x.y) \mapsto (y, x)$ does not extend to an automorphism.
- $T(4)$ is set-homogeneous but not 2-homogeneous, and has primitive automorphism group.

Colouring the rationals

- Let $2 \leq n \leq \aleph_0$ and $\{Q_i : 1 \leq i \leq n\}$ be a partition of $\mathbb{Q}$ into n dense codense sets.

Define a digraph R_n with domain $\mathbb{Q}$, putting $a \to b$ if and only if $a < b$ and there is no i such that $a, b \in Q_i$.

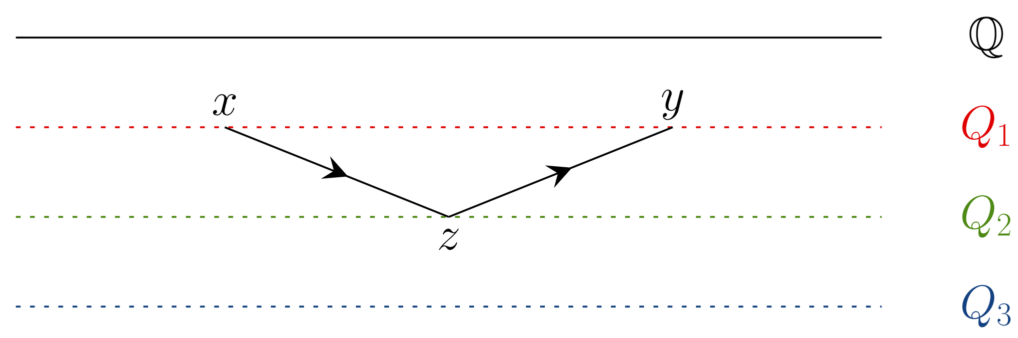
- Viewing the Q_i as colours, R_n may be thought of as $\mathbb{Q}$ interdensely coloured by n colours, with arcs strictly increasing and between vertices of different colours.



By a back-and-forth argument, this construction for R_n determines a unique digraph.

Lemma. The digraphs R_n (for $n \geq 2$) are set-homogeneous but not 2-homogeneous.

- set-homogeneity: shown by "expanding" R_n to a homogeneous structure
- not 2-homogeneous: if $x, y \in Q_1$ with $x < y$ then there is $z \in Q_2$ with $x \to z \to y$ but no z with $y \to z \to x$



- Thus $(x, y) \mapsto (y, x)$ does not extend to an automorphism.
- R_n (for $n \geq 2$) is set-homogeneous but not 2-homogeneous, and has imprimitive automorphism group.

Theorem 2

Let D be a countably infinite set-homogeneous digraph which is not 2-homogeneous. Then D is isomorphic to $T(4)$ or to R_n for some $n \geq 2$.

Open problems

- Is there a countably infinite tournament that is set-homogeneous but not homogeneous?
- Classify the countably infinite set-homogeneous digraphs.

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