

Undecidability of the word problem for one-relator inverse monoids via right-angled Artin subgroups of one-relator groups

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Word problem for one-relator groups and monoids

	$\text{Gp}\langle A \mid w = 1 \rangle$ $\text{FG}(A)/\langle\langle w \rangle\rangle$	$\text{Mon}\langle A \mid w = 1 \rangle$ $A^*/\langle\langle (w, 1) \rangle\rangle$	$\text{Inv}\langle A \mid w = 1 \rangle$ $\text{FIM}(A)/\langle\langle (w, 1) \rangle\rangle$
Word problem decidable	Magnus (1932) ✓	Adjan (1966) ✓	?

Theorem (Scheiblich (1973) & Munn (1974))

Free inverse monoids have decidable word problem.

Conjecture (Margolis, Meakin, Stephen (1987))

If $M = \text{Inv}\langle A \mid w = 1 \rangle$, then the word problem for M is decidable.

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Theorem (Ivanov, Margolis, Meakin (2001))

If the word problem is decidable for all inverse monoids of the form $\text{Inv}\langle A \mid w = 1 \rangle$ then the word problem is also decidable for every one-relator monoid $\text{Mon}\langle A \mid u = v \rangle$.

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Proved true in many cases e.g. when w satisfies...

- ▶ Idempotent word [Birget, Margolis, Meakin, 1993, 1994]
- ▶ w -strictly positive [Ivanov, Margolis, Meakin, 2001]
- ▶ Adjan or Baumslag-Solitar type [Margolis, Meakin, Šuník, 2005]
- ▶ Sparse word [Hermiller, Lindblad, Meakin, 2010]
- ▶ Certain small cancellation conditions [A. Juhász, 2012, 2014]

Word problem for one-relator groups and monoids

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Theorem (RDG (2019))

There is a one-relator inverse monoid $\text{Inv}\langle A \mid w = 1 \rangle$ with undecidable word problem.

Word problem for one-relator groups and monoids

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Theorem (RDG (2019))

There is a one-relator inverse monoid $\text{Inv}\langle A \mid w = 1 \rangle$ with undecidable word problem.

Ingredients for the proof:

- ▶ Submonoid membership problem for one relator groups.
- ▶ HNN-extensions and free products of groups.
- ▶ Right-angled Artin groups (RAAGs).
- ▶ Right units of special inverse monoids

$$\text{Inv}\langle A \mid w_1 = 1, w_2 = 1, \dots, w_k = 1 \rangle$$

and Stephen's procedure for constructing Schützenberger graphs.

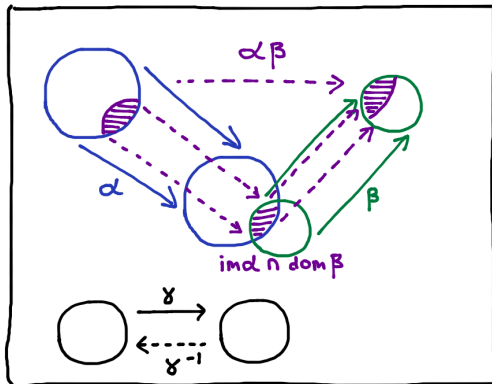
- ▶ Properties of E -unitary inverse monoids.

Inverse monoids

An **inverse monoid** is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

Example: I_X = monoid of all partial bijections $X \rightarrow X$

X



Examples: In I_3

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & - & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & 2 \end{pmatrix}$$

Note:

$$\gamma\gamma^{-1} = \text{id}_{\text{dom}\gamma}$$

Inverse monoid presentations

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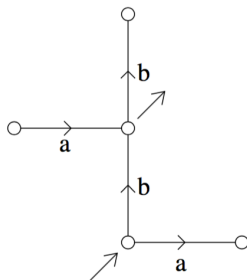
For all $x, y \in M$ we have

$$x = xx^{-1}x, (x^{-1})^{-1} = x, (xy)^{-1} = y^{-1}x^{-1}, xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \quad (\dagger)$$

$$\text{Inv}\langle A \mid u_i = v_i \ (i \in I) \rangle = \text{Mon}\langle A \cup A^{-1} \mid u_i = v_i \ (i \in I) \cup (\dagger) \rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and x, y range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\text{FIM}(A) = \text{Inv}\langle A \mid \rangle$



Munn (1974)

Elements of $\text{FIM}(A)$ can be represented using Munn trees. e.g. in $\text{FIM}(a, b)$ we have $u = w$ where

$$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$$

$$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$$

The word problem

M - a finitely generated monoid with a finite generating set A .

$\pi : A^* \rightarrow M$ - the canonical monoid homomorphism.

The **monoid M has decidable word problem** if there is an algorithm which solves the following decision problem:

INPUT: Two words $u, v \in A^*$.

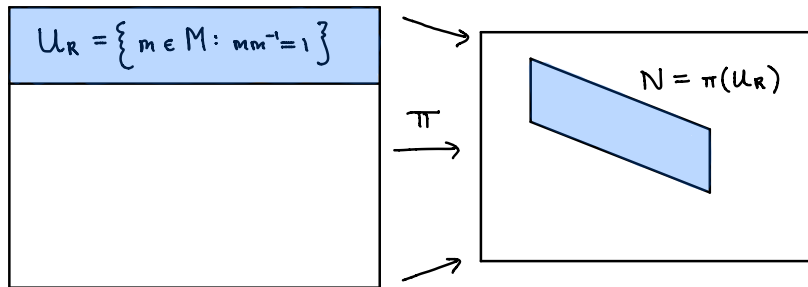
QUESTION: $\pi(u) = \pi(v)$? i.e. do u and v represent the same element of the monoid M ?

For a **group** or an **inverse monoid** with generating set A the word problem is defined in the same way except the input is two words $u, v \in (A \cup A^{-1})^*$.

Example. The **bicyclic monoid** $\text{Inv}\langle a \mid aa^{-1} = 1 \rangle$ has decidable word problem.

Proof strategy

$$M = \text{Inv}\langle A \mid r=1 \rangle \longrightarrow G = \text{Gp}\langle A \mid r=1 \rangle$$



If M has decidable word problem

\Rightarrow membership problem for $U_R \leq M$ is decidable

since for $w \in (A \cup A^{-1})^*$

$$w \in U_R \iff ww^{-1} = 1$$

(sometimes)

\rightsquigarrow membership problem for $N \leq G$ is decidable

RAAGs induced subgraphs and subgroups

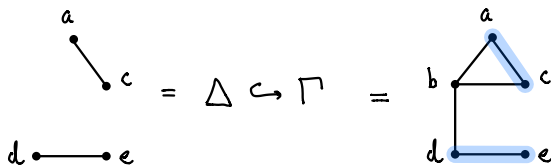
Definition

The **right-angled Artin group** $A(\Gamma)$ associated with the graph Γ is

$$\text{Gp}\langle VT \mid uv = vu \text{ if and only if } \{u, v\} \in E\Gamma \rangle.$$

Fact: If Δ is an induced subgraph of Γ then the embedding $\Delta \rightarrow \Gamma$ induces an embedding $A(\Delta) \rightarrow A(\Gamma)$.

Example



$$\begin{array}{l|l} A(\Delta) = \text{Gp} \langle a, c, d, e & ac = ca, de = ed \rangle \\ \hookrightarrow A(\Gamma) = \text{Gp} \langle a, b, c, d, e & \begin{array}{l} ac = ca, de = ed, \\ ab = ba, bc = cb, \\ bd = db \end{array} \end{array}$$

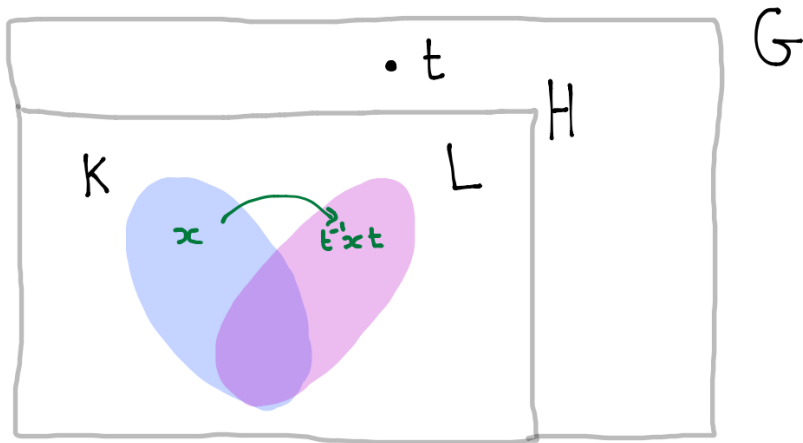
HNN-extensions of groups

$H \cong \text{Gp}\langle A \mid R \rangle$, $K, L \leq H$ with $K \cong L$. Let $\phi : K \rightarrow L$ be an isomorphism.

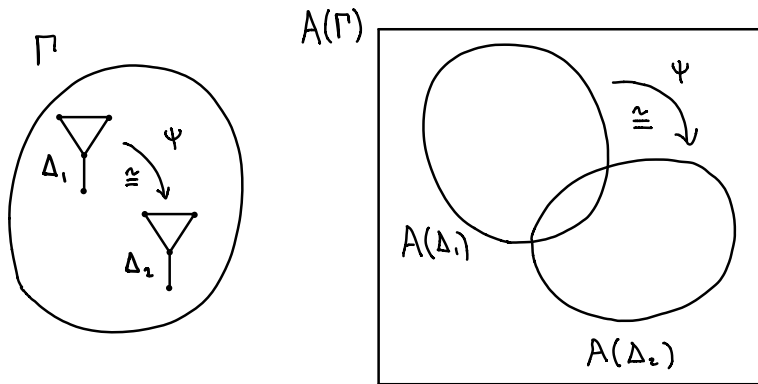
The **HNN-extension** of H with respect to ϕ is

$$G = \text{HNN}(H, \phi) = \text{Gp}\langle A, t \mid R, t^{-1}kt = \phi(k) \ (k \in K) \rangle$$

Fact: H embeds naturally into the HNN extension $G = \text{HNN}(H, \phi)$.



HNN-extensions of RAAGs



Definition

Γ - finite graph, $\psi : \Delta_1 \rightarrow \Delta_2$ an isomorphism between finite induced subgraphs.

$A(\Gamma, \psi)$ is defined to be the HNN-extension of $A(\Gamma)$ with respect to the isomorphism $A(\Delta_1) \rightarrow A(\Delta_2)$ induced by ψ .

Fact: $A(\Gamma)$ embeds naturally into $A(\Gamma, \psi)$.

HNN-extension of $A(P_4)$ over $A(P_3)$

Let P_4 be the graph



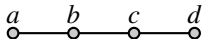
$$A(P_4) = \text{Gp}\langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle.$$

Δ_1 - subgraph induced by $\{a, b, c\}$, Δ_2 subgraph induced by $\{b, c, d\}$,

$\psi : \Delta_1 \rightarrow \Delta_2$ - the isomorphism $a \mapsto b$, $b \mapsto c$, and $c \mapsto d$.

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Then the HNN-extension $A(P_4, \psi)$ of $A(P_4)$ with respect to ψ is

$$\begin{aligned} & A(P_4, \psi) \\ = & \text{Gp}\langle a, b, c, d, t \mid ab = ba, bc = cb, cd = dc, tat^{-1} = b, tbt^{-1} = c, tct^{-1} = d \rangle \end{aligned}$$

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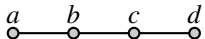
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Conclusion

$A(P_4)$ embeds into the one-relator group

$$A(P_4, \psi) = \text{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1 \rangle.$$

Submonoid membership problem

G - a finitely generated group with a finite group generating set A .

$\pi : (A \cup A^{-1})^* \rightarrow G$ - the canonical monoid homomorphism.

T - a finitely generated submonoid of G .

The **membership problem for T within G is decidable** if there is an algorithm which solves the following decision problem:

INPUT: A word $w \in (A \cup A^{-1})^*$.

QUESTION: $\pi(w) \in T$?

Theorem B

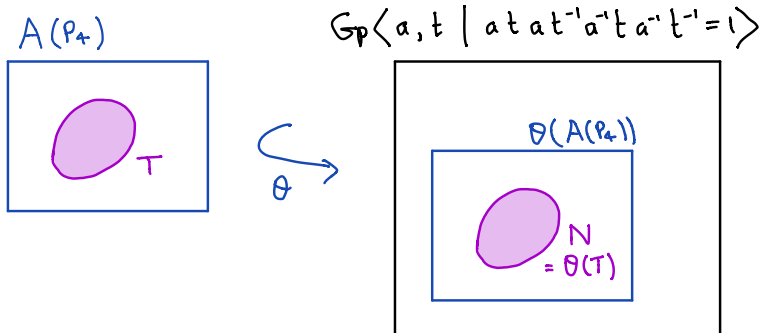
Let G be the one-relator group $\text{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1 \rangle$. Then there is a fixed finitely generated submonoid N of G such that the membership problem for N within G is undecidable.

Proof of Theorem B

Theorem B

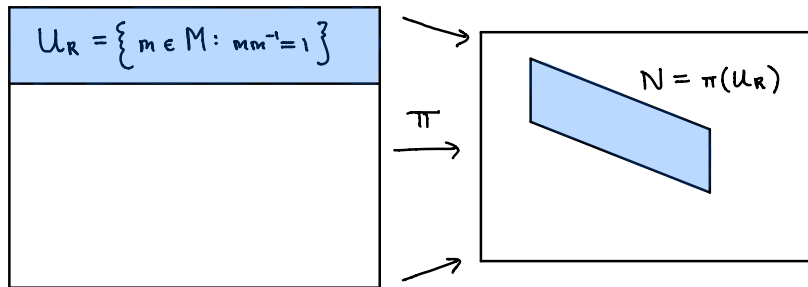
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Proof. By [Lohrey & Steinberg, 2008] there is a finitely generated submonoid T of $A(P_4)$ such that the membership problem for T within $A(P_4)$ is undecidable. Let $\theta : A(P_4) \rightarrow G$ be an embedding. Then $N = \theta(T)$ is a finitely generated submonoid of G such that the membership problem for N within G is undecidable.



Proof strategy

$$M = \text{Inv}\langle A \mid r=1 \rangle \longrightarrow G = \text{Gp}\langle A \mid r=1 \rangle$$



If M has decidable word problem

\Rightarrow membership problem for $U_R \leq M$ is decidable

since for $w \in (A \cup A^{-1})^*$

$$w \in U_R \iff ww^{-1} = 1$$

(sometimes)

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General observations about inverse monoids

S – an inverse monoid generated by A , $E(S)$ – set of idempotents,

$U_R \leq S$ – right units = submonoid of right invertible elements.

- ▶ If $e \in E(S)$ and $e \in U_R$ then $e = 1$.
- ▶ **Two relations for the price of one:** If e is an idempotent in $\text{FIM}(A)$ and $r \in (A \cup A^{-1})^*$ then

$$\text{Inv}\langle A \mid er = 1 \rangle = \text{Inv}\langle A \mid e = 1, r = 1 \rangle.$$

- ▶ $e \in (A \cup A^{-1})^*$ is an idempotent in $\text{FIM}(A)$ if and only if e freely reduces to 1 in the free group $\text{FG}(A)$. e.g.

$$x^{-1}y^{-1}xx^{-1}yzz^{-1}x \in E(\text{FIM}(x, y, z)).$$

A general construction

For any $r, w_1, \dots, w_k \in (A \cup A^{-1})^*$, with $A = \{a_1, \dots, a_n\}$, set e equal to $a_1 a_1^{-1} \dots a_n a_n^{-1} (tw_1 t^{-1})(tw_1^{-1} t^{-1})(tw_2 t^{-1})(tw_2^{-1} t^{-1}) \dots (tw_k t^{-1})(tw_k^{-1} t^{-1}) a_n^{-1} a_n \dots a_1^{-1} a_1$

where t is a new symbol. Then

$$\begin{aligned} M &= \text{Inv}\langle A, t \mid er = 1 \rangle \\ &= \text{Inv}\langle A, t \mid r = 1, aa^{-1} = 1, a^{-1}a = 1 \ (a \in A), (tw_i t^{-1})(tw_i t^{-1})^{-1} = 1 \ (1 \leq i \leq k) \rangle \\ &\cong \text{Gp}\langle A \mid r = 1 \rangle * \text{FIM}(t) \ / \ \{(tw_i t^{-1})(tw_i t^{-1})^{-1} = 1 \ (1 \leq i \leq k)\}. \end{aligned}$$

Key claim

Let T be the submonoid of $G = \text{Gp}\langle A \mid r = 1 \rangle$ generated by $\{w_1, w_2, \dots, w_k\}$. Then for all $u \in (A \cup A^{-1})^*$ we have

$$u \in T \text{ in } G \iff tut^{-1} \in U_R \text{ in } M.$$

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$$u \in T \text{ in } G \iff tut^{-1} \in U_R \text{ in } M.$$

Theorem

If $M = \text{Inv}\langle A, t \mid er = 1 \rangle$ has decidable word problem then the membership problem for T within $G = \text{Gp}\langle A \mid r = 1 \rangle$ is decidable.

Proof strategy refined

$$M = \text{Inv} \langle A, t \mid e_r = 1 \rangle \longrightarrow G_p \langle A, t \mid r = 1 \rangle$$

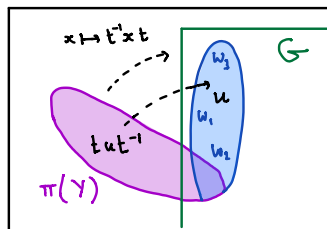
$$= G_p \langle A \mid r = 1 \rangle * FG(t)$$

$$= G$$

$tut^{-1} = \gamma$

$U_R = \{ m \in M : mm^{-1} = 1 \}$

tvt^{-1}



If M has decidable word problem

\Rightarrow membership problem for $U_R \leq M$ is decidable

$\Rightarrow \forall u \in (A \cup A^{-1})^*$ can decide $tut^{-1} \in U_R ?$

\Rightarrow
(by key claim) can decide $u \in T = \text{Mon} \langle w_1, \dots, w_k \rangle \leq G$

Tying things together

Thoerem A

There is a one-relator inverse monoid $\text{Inv}\langle A \mid w = 1 \rangle$ with undecidable word problem.

Proof.

Let $A = \{a, z\}$ and let G be the one-relator group

$$\text{Gp}\langle a, z \mid aza z^{-1} a^{-1} z a^{-1} z^{-1} = 1 \rangle.$$

Let $W = \{w_1, \dots, w_k\}$ be a finite subset of $(A \cup A^{-1})^*$ such that the membership problem for $T = \text{Mon}\langle W \rangle$ within G is undecidable. Such a set W exists by Theorem B. Set e to be the idempotent word

$$aa^{-1}zz^{-1}(tw_1t^{-1})(tw_1^{-1}t^{-1})(tw_2t^{-1})(tw_2^{-1}t^{-1}) \dots (tw_kt^{-1})(tw_k^{-1}t^{-1})z^{-1}za^{-1}a.$$

Then by the above theorem the one-relator inverse monoid

$$\text{Inv}\langle a, z, t \mid eaza z^{-1} a^{-1} z a^{-1} z^{-1} = 1 \rangle$$

has undecidable word problem. This completes the proof. □

What next?

Reduced vs cyclically reduced words

$aba^{-1}ab$ - not reduced,

$abba^{-1}$ - reduced but not cyclically reduced

$aba^{-1}b^{-1}$ - cyclically reduced

Key question

For which words $w \in (A \cup A^{-1})^*$ does $\text{Inv}\langle A \mid w = 1 \rangle$ have decidable word problem? In particular is the word problem always decidable when w is

(a) **reduced** or (b) **cyclically reduced**?

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Note: A positive answer to (a) would imply the word problem is also decidable for every one-relator monoid $\text{Mon}\langle A \mid u = v \rangle$.

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Definition

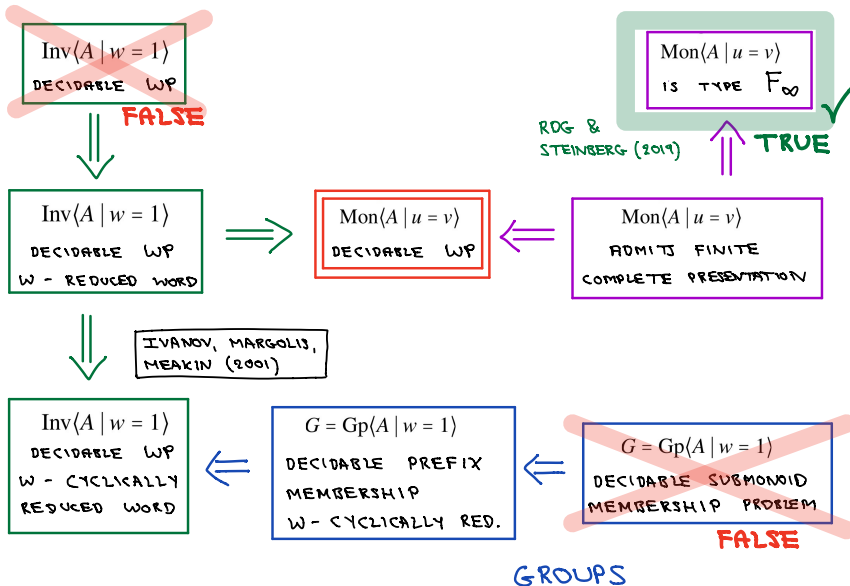
The **prefix submonoid** P_w of $\text{Gp}\langle A \mid w = 1 \rangle$ is the submonoid generated by all prefixes of the word w .

Theorem (Ivanov, Margolis and Meakin (2001))

Let $w \in (A \cup A^{-1})^*$ be cyclically reduced. If $\text{Gp}\langle A \mid w = 1 \rangle$ has decidable prefix membership problem (e.g. can decide membership in P_w) then $\text{Inv}\langle A \mid w = 1 \rangle$ has decidable word problem.

INVERSE MONOIDS

MONOIDS



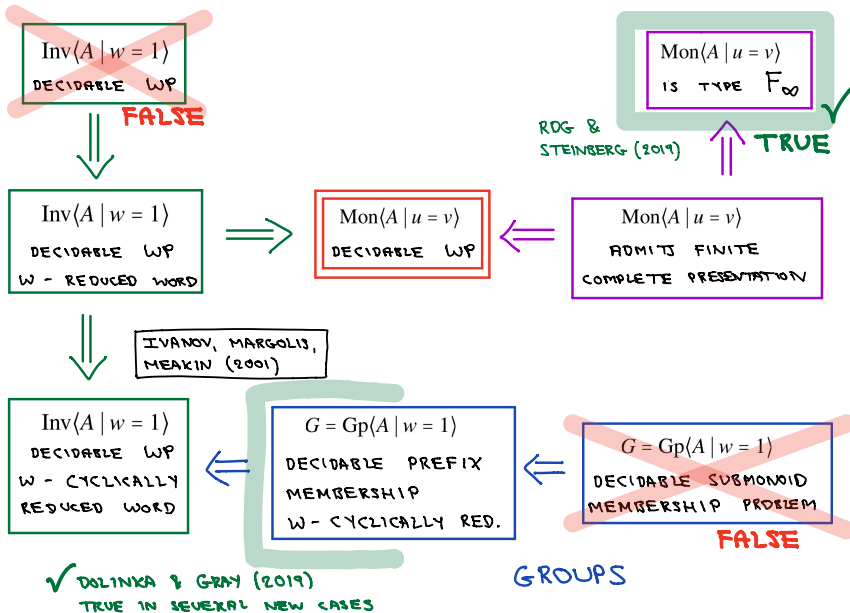
Approaching the prefix membership problem for one-relator groups

The general idea:

- ▶ Let $G = \text{Gp}\langle A \mid w = 1 \rangle$ and suppose that w contains a letter with exponent sum zero (e.g. $w = bt^{-1}at^2bt^{-1}a$ has t -exponent sum zero).
- ▶ Then by [McCool and Schupp \(1973\)](#) (based on an observation due to [Moldavanskii \(1967\)](#)) the group G can be expressed as an HNN extension of a one-relator group H with a defining relator shorter than w . (e.g. in the above example we write $bt^{-1}at^2bt^{-1}a = b(t^{-1}at)(tbt^{-1})a$ and H has defining relator $b_0a_1b_{-1}a_0$).
- ▶ The aim is then to relate the prefix membership problem for $G = \text{Gp}\langle A \mid w = 1 \rangle$ to the prefix membership problem for H .
- ▶ This leads to questions about the membership problem for certain submonoids of certain HNN extensions of groups.
- ▶ [Dolinka & RDG \(2019\)](#) - we have some new positive results in several cases.

INVERSE MONOIDS

MONOIDS



More open problems

Problem

Characterise the one-relator groups with decidable submonoid membership problem.

Problem

Characterise the one-relator groups with decidable rational subset membership problem.

Problem

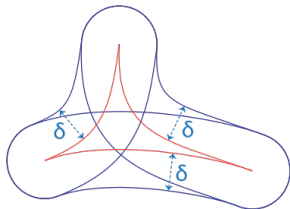
Is the subgroup membership problem decidable for one-relator groups?

Further applications

- ▶ In my paper there is actually a more general construction which shows how to encode the submonoid membership problem for any n -relator group into the word problem of an n -relator inverse monoid.
- ▶ This general construction has since been applied in other ways e.g. In recent joint work with [Pedro Silva](#) and [Nora Szakacs](#) we have combined that construction with the **Rips construction** to prove:

Theorem (RDG, Silva & Szakacs (2019))

There is a finitely presented inverse monoid $M = \text{Inv}\langle A \mid w_1 = 1, \dots, w_k = 1 \rangle$ such that all strongly connected components of the directed Cayley graph of M are δ -hyperbolic, but M has undecidable word problem.



- ▶ In contrast we prove that if every strongly connected component is quasi-isometric to a tree then the monoid has decidable word problem.