On connected-homogeneity in graphs and partial orders

Robert Gray

UNIVERSITY OF LEEDS

Combinatorics of Arc-Transitive Graphs and Partial Orders, August 2007

Outline

Outline

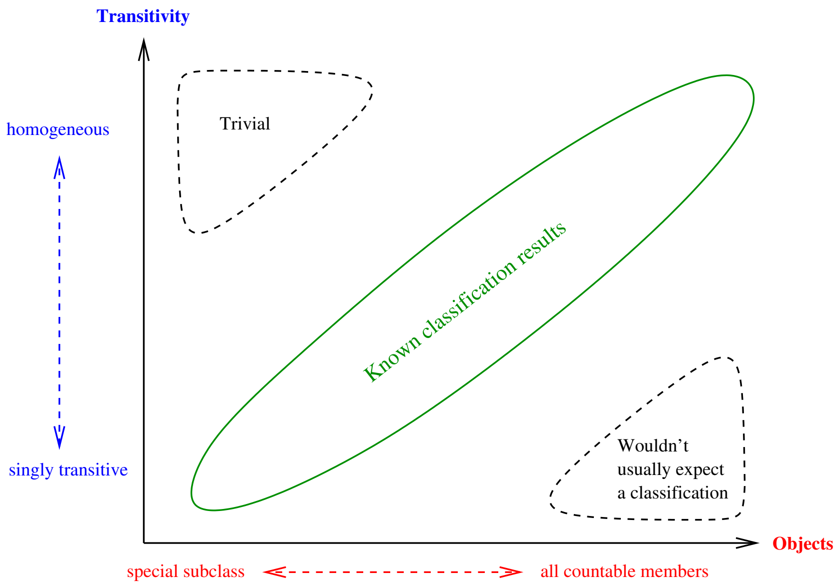
Structures with symmetry

- Roughly speaking, the ‘more’ symmetry a mathematical object has the ‘larger’ its automorphism group will be (and vice versa).

- **Aim.** To obtain classifications of families of structures with a high degree of symmetry.

- In each case we impose a transitivity assumption on the automorphism groups of the structures and then attempt to describe all (countable) structures satisfying the property.

Range of classification problems



Outline

Homogeneous graphs

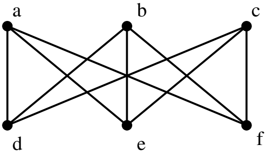
Definition

A graph Γ is called homogeneous if any isomorphism between finite induced subgraphs extends to an automorphism of the graph.

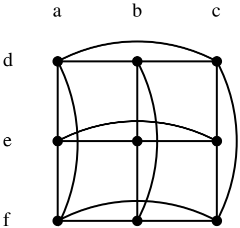
Homogeneity is the *strongest* possible symmetry condition we can impose.

Example

The line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$ is a finite homogeneous graph.



$K_{3,3}$



$L(K_{3,3})$

Classification of finite homogeneous graphs

Gardiner classified the finite homogeneous graphs.

Theorem (Gardiner (1976))

A finite graph is homogeneous if and only if it is isomorphic to one of the following:

1. *finitely many disjoint copies of a complete graph K_r (or its complement, complete multipartite graph)*
2. *the pentagon C_5*
3. *line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$.*

An infinite homogeneous graph

Definition (The random graph R)

Constructed by Rado in 1964. The vertex set is the natural numbers (including zero).

For $i, j \in \mathbb{N}$, $i < j$, then i and j are joined if and only if the ith digit in j (in base 2, reading right-to-left) is 1.

Example

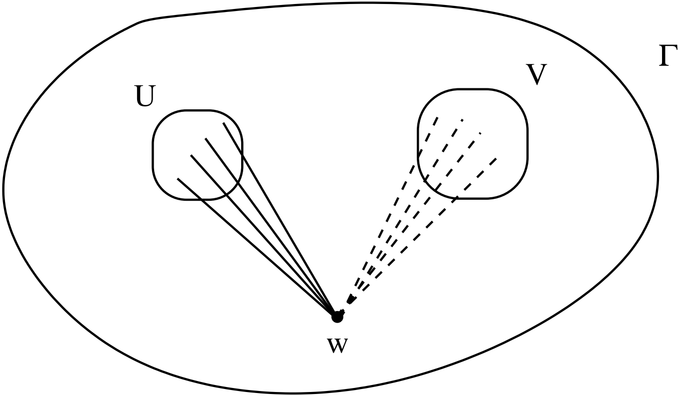
Since $88 = 8 + 16 + 64 = 2^3 + 2^4 + 2^6$ the numbers less that 88 that are adjacent to 88 are just $\{3, 4, 6\}$.

Of course, many numbers greater than 88 will also be adjacent to 88 (for example 2^{88}).

The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets U and V of vertices, there exists a vertex w adjacent to every vertex in U and to no vertex in V.



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Theorem

There exists a countably infinite graph R satisfying property (*), and it is unique up to isomorphism. The graph R is homogeneous.

Existence. The random graph R defined above satisfies property (*).

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Existence. The random graph R defined above satisfies property (*).

Uniqueness and homogeneity. Both follow from a back-and-forth argument. Property (*) is used to extend the domain (or range) of any isomorphism between finite substructures one vertex at a time.

Building homogeneous graphs: Fraïssé's theorem

- The age of a graph Γ is the class of isomorphism types of its finite induced subgraphs.
- e.g. the age of the random graph R is the class of *all* finite graphs.

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Fraïssé (1953) - gives necessary and sufficient conditions for a class $\mathcal{C}$ of finite graphs to be the age of a countably infinite homogeneous graph M. The key condition is the amalgamation property.

If Fraïssé's conditions hold, then M is unique, $\mathcal{C}$ is called a Fraïssé class, and M is called the Fraïssé limit of the class $\mathcal{C}$.

Homogeneous graphs

Examples

- The class of all finite graphs is a Fraïssé class. Its Fraïssé limit is the random graph R.
- The class of all finite graphs not embedding K_n (for some fixed n) is a Fraïssé class. We call the Fraïssé limit the countable generic K_n-free graph.

Theorem (Lachlan and Woodrow (1980))

Let Γ be a countably infinite homogeneous graph. Then Γ is isomorphic to one of: the random graph, a disjoint union of complete graphs (or its complement), the generic K_n-free graph (or its complement).

Outline

Connected-homogeneous graphs

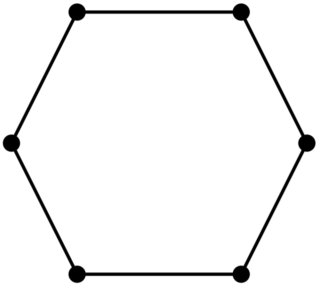
Definition

A graph Γ is connected-homogeneous if any isomorphism between *connected* finite induced subgraphs extends to an automorphism.

Example

The hexagon C_6 is connected-homogeneous

Use rotations and reflections



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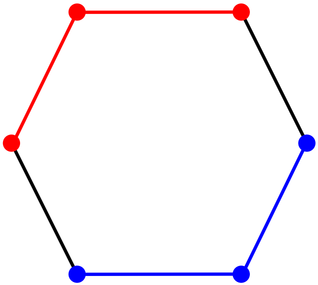

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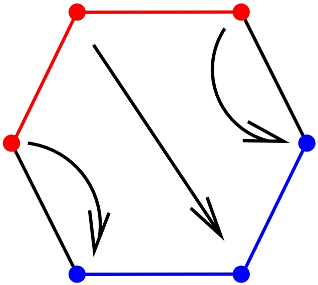

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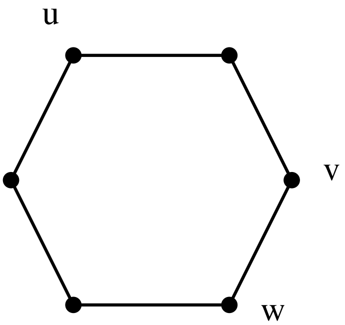
Example

On the other hand the hexagon is not homogeneous.

There is no automorphism α such that $(u, v)^{\alpha} = (u, w)$.



Connected-homogeneous graphs

Connected-homogeneity...

1. is a natural weakening of homogeneity;
2. gives a class of graphs that lie between the (already classified) homogeneous graphs and the (not yet classified) distance-transitive graphs.

$$\text{homogeneous} \Rightarrow \text{connected-homogeneous} \Rightarrow \text{distance-transitive}$$

(A graph is distance-transitive if for any two pairs (u, v) and (u', v') with $d(u, v) = d(u', v')$, where d denotes distance in the graph, there is an automorphism taking u to u' and v to v'.)

Finite connected-homogeneous graphs

Gardiner classified the finite connected-homogeneous graphs.

Theorem (Gardiner (1978))

A finite graph is connected-homogeneous if and only if it is isomorphic to a disjoint union of copies of one of the following:

1. *a finite homogeneous graph*
2. *bipartite "complement of a perfect matching" (obtained by removing a perfect matching from a complete bipartite graph $K_{s,s}$)*
3. *cycle C_n*
4. *the line graph $L(K_{s,s})$ of a complete bipartite graph $K_{s,s}$*
5. *Petersen's graph*
6. *the graph obtained by identifying antipodal vertices of the 5-dimensional cube Q_5*

Treelike examples

Definition (Tree)

A tree is a connected graph without cycles. A tree is regular if all vertices have the same degree. We use T_r to denote a regular tree of valency r.

A graph is locally finite if each of its vertices has finite valency.

Fact. A regular tree T_r ($r \in \mathbb{N}$) is an example of an infinite locally-finite connected-homogeneous graph.

Definition (Semiregular tree)

$T_{a,b}$: A tree $T = X \cup Y$ where $X \cup Y$ is a bipartition, all vertices in X have degree a, and all in Y have degree b.

Locally finite infinite connected-homogeneous graphs

Let $r, l \in \mathbb{N}$ ($l \geq 2$)

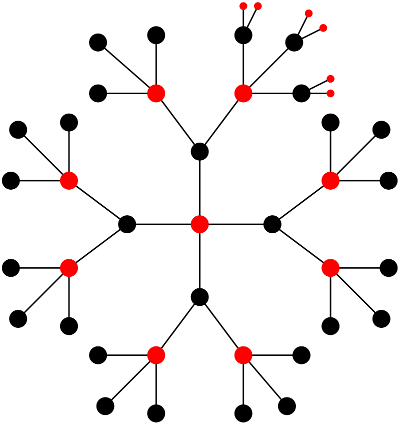
Take the bipartite semiregular tree $T_{r+1,l}$.

The graph $X_{r,l}$ is given by:

Vertices = bipartite block of $T_{r+1,l}$ of vertices of degree l.

Edges = adjacent in $X_{r,l}$ if their distance in the tree is 2.

(Macpherson (1982) proved that every connected infinite locally-finite distance transitive graph has this form)



$T_{3,4}$

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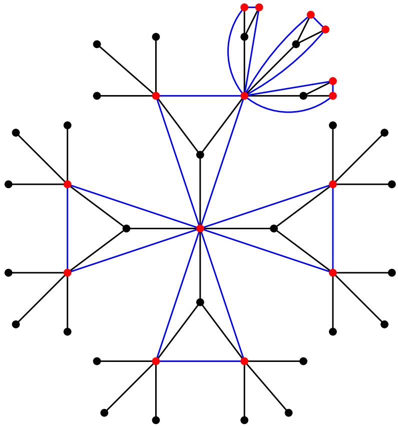

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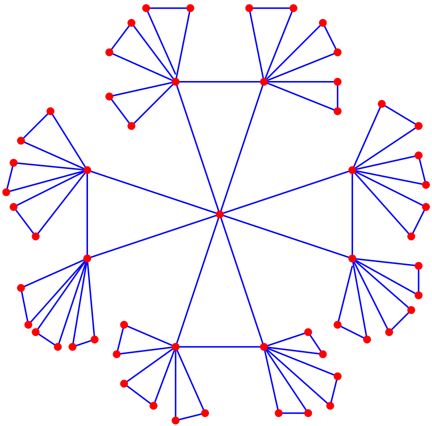

$X_{2,4}$

Infinite connected-homogeneous graphs

Theorem (RG, Macpherson (2007))

A countable graph is connected-homogeneous if and only if it is isomorphic to the disjoint union of a finite or countable number of copies of one of the following:

1. *a finite connected-homogeneous graph;*
2. *a homogeneous graph;*
3. *the random bipartite graph;*
4. *bipartite infinite complement of a perfect matching;*
5. *the line graph of the infinite complete bipartite graph $K_{\aleph_0,\aleph_0}$;*
6. *a treelike graph X_{κ_1,κ_2} with $\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$.*

Weaker forms of homogeneity

Let Γ be a graph and let $k \in \mathbb{N}$.

Definition

Γ is k-homogeneous if all isomorphisms between induced subgraphs of size k extend to automorphisms of the graph Γ.

Γ is k-transitive if for any two isomorphic induced subgraphs A and B of Γ, each of size k, at least one isomorphism between A and B extends to an automorphism of Γ.

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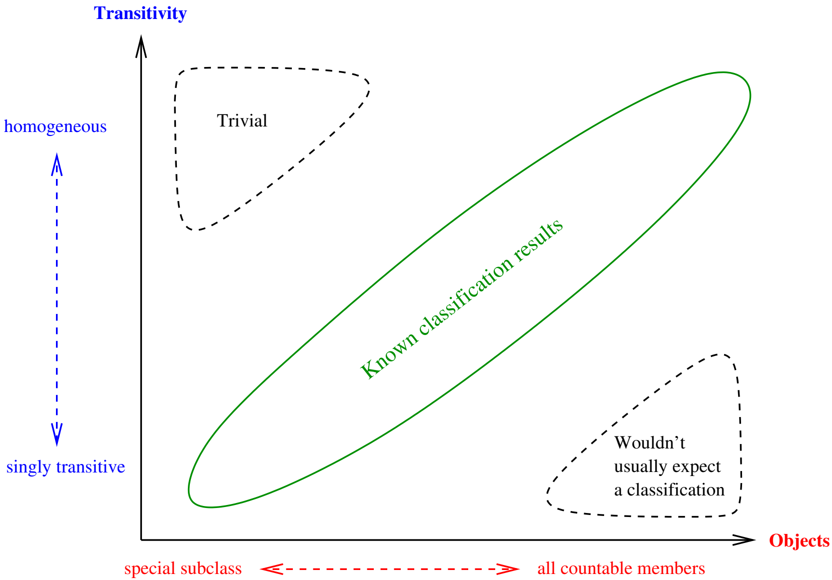
Strongest

$$\boxed{\begin{array}{ccc} \text{Homogeneous} & \Rightarrow & \text{Connected-homogeneous} \\ \Downarrow & & \Downarrow \\ k\text{-homogeneous} & \Rightarrow & k\text{-}CS\text{-homogeneous} \\ \Downarrow & & \Downarrow \\ k\text{-transitive} & \Rightarrow & k\text{-}CS\text{-transitive} \end{array}}$$

Weakest

Classification problems



Outline

Number of ends of a graph

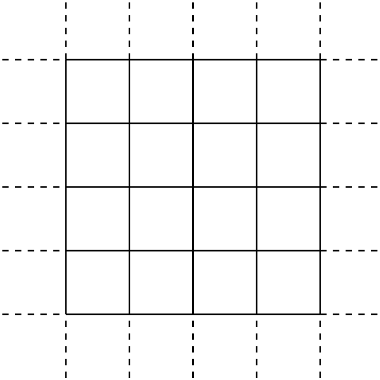
Definition

The number of ends of graph is the least upper bound (possibly ∞) of the number of infinite connected components that can be obtained by removing finitely many vertices.

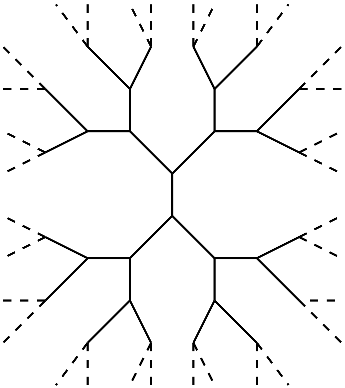
Theorem (Diestel, Jung, Möller (1993))

A connected vertex transitive graph has either 1, 2 *or* ∞ *many ends.*

Examples: A grid, a tree and a line



Grid has 1 end



Tree has ∞ many ends

Line has 2 ends

s-arc-transitivity

Definition

- An s-arc in a graph is a sequence $v_0, \ldots, v_s$ of vertices such that v_i is adjacent to v_{i+1} for all $0 \le i \le s-1$, and $v_j \ne v_{j+2}$ for $0 \le j \le s-2$.
- A graph is s-arc transitive if given any two s-arcs $v_0, \ldots, v_s$ and $u_0, \ldots, u_s$ there is an automorphism α such that

$$v_i^\alpha = u_i \quad (0 \le i \le s).$$

Fact. For locally finite graphs with more than one end s-arc-transitivity is a very restrictive condition.

Locally finite s-arc-transitive graphs

Let Γ be a locally finite connected graph with more than one end.

Theorem (Thomassen–Woess (93))

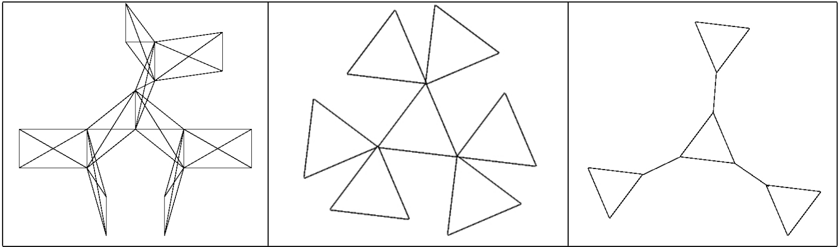
If Γ is 2-arc transitive then Γ is a regular tree.

Theorem (Thomassen–Woess (93))

If Γ is 1-arc transitive and all vertices have degree r, where r is a prime, then Γ is a regular tree.

Using ideas developed by Möller (1992) it is possible to obtain a classification in the case that Γ is 3-CS-transitive.

3-CS-transitive graphs



$K_4(r)$ with $r = 3$	$X_{r,l}$ with $r = 2$ and $l = 3$	Y_r with $r = 3$

Figure: Local structure of the graphs $K_4(3)$, $X_{2,3}$ and Y_3.

3-CS-transitive graphs

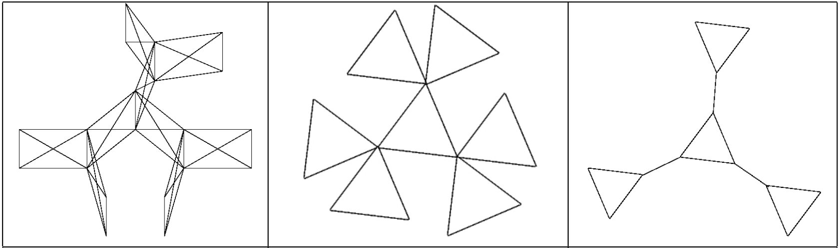


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Figure: Local structure of the graphs $K_4(3)$, $X_{2,3}$ and Y_3.

Theorem (RG (2007))

Let Γ be a connected locally finite graph with more than one end. Then Γ is 3-CS-transitive if and only if it is isomorphic to one of the following:

1. $X_{r,l}$ *($r \geq 1$, $l \geq 2$)*
2. Y_r *($r \geq 3$)*
3. $K_4(r)$ *($r \geq 1$).*

Outline

Cycle-free partial orders

The definition of cycle-free partial order is given in terms of an extension of a poset called its Dedekind–MacNeille completion.

Definition

A poset $P = (P, \leq)$ is called Dedekind–MacNeille complete if:

1. any maximal chain is Dedekind-complete (so non-empty bounded subsets have suprema and infima);
2. any two-element subset bounded above has a supremum;
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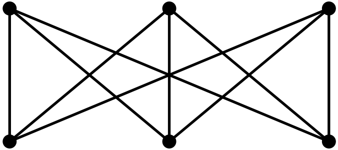
Fact. For any poset M there is a unique minimal extension M^D of M which is Dedekind–MacNeille complete. We call M^D the Dedekind–MacNeille completion of M.

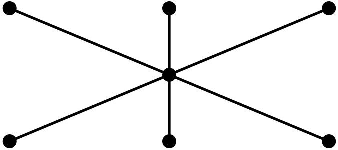
Examples of completions

The poset

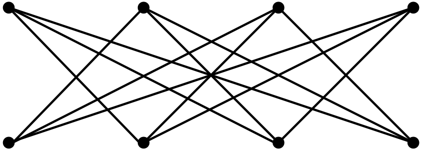


$(P, \leq)$

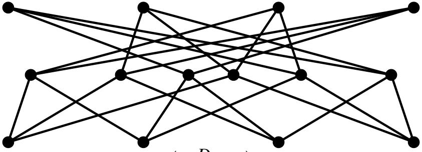
Its completion



$(P^D, \leq)$

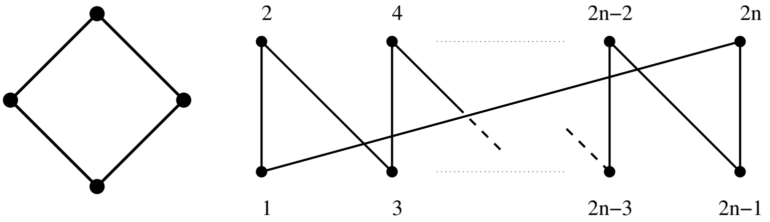


$(Q, \leq)$



$(Q^D, \leq)$

Cycle-free partial orders (*CFPO*s)



Diamond $\qquad$ $2n$-Crown

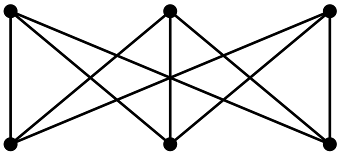
Definition

A poset P is called cycle-free if its completion P^D does not embed a diamond or $2n$-crown (for any $n \geq 3$).

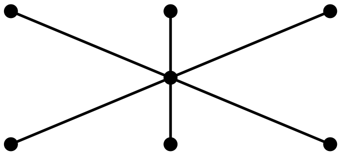
In other words, P is cycle-free provided that its completion does not contain any 'cycles'.

Examples of completions

P is a *CFPO* since its completion P^D embeds no diamonds and no crowns.

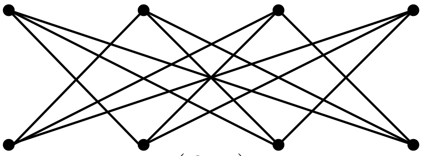


$(P, \leq)$

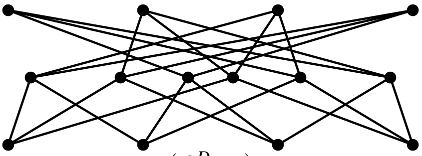


$(P^D, \leq)$

Q is not a *CFPO* since its completion Q^D embeds a diamond.



$(Q, \leq)$



$(Q^D, \leq)$

Connection with bipartite graphs

Theorem (Warren 1997)

Let M be an infinite CFPO all of whose chains are finite. If M is k-CS-transitive for some $k \geq 2$ and C is a maximal chain in M, then $|C| = 2$.

- So k-CS-transitive ($k \geq 2$) finite chain *CFPO*s can be thought of both as partial orders and as bipartite graphs.

- The classification of countably infinite k-CS-transitive *CFPO*s ($k \geq 3$) is complete (due to Creed, Truss, and Warren).

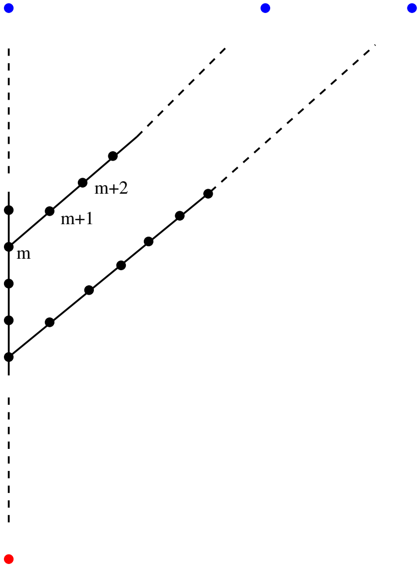
Building *CFPO*s from chains

Begin with $(\mathbb{Z}, \leq)$

Adjoin minimal and maximal elements α and β so that

$\alpha < \mathbb{Z} < \beta$.

Building *CFPO*s from chains



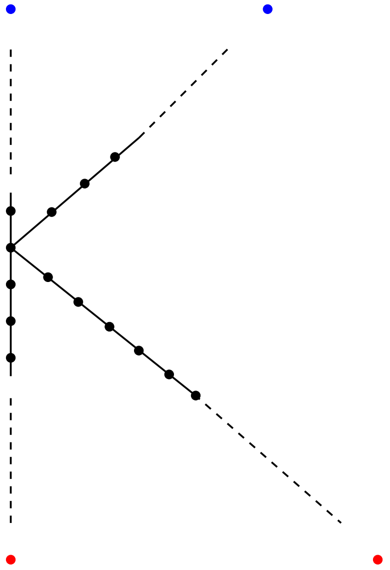
For each integer m in the original copy of $\mathbb{Z}$, adjoin a new copy of

$$[m, m+1, m+2, \ldots] \cup \beta$$

above m.

Now each point of the original copy of $\mathbb{Z}$ ramifies upwards with order 2.

Building *CFPO*s from chains



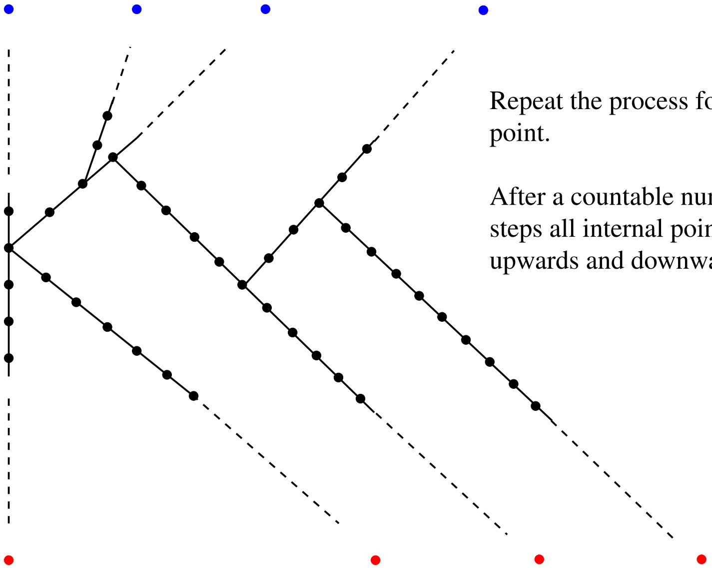
Dually for each integer m on the original copy of $\mathbb{Z}$, adjoin a new copy of

$\alpha \cup [\ldots, m-2, m-1, m]$

below m.

Now each point of the original copy of $\mathbb{Z}$ ramifies downwards with order 2.

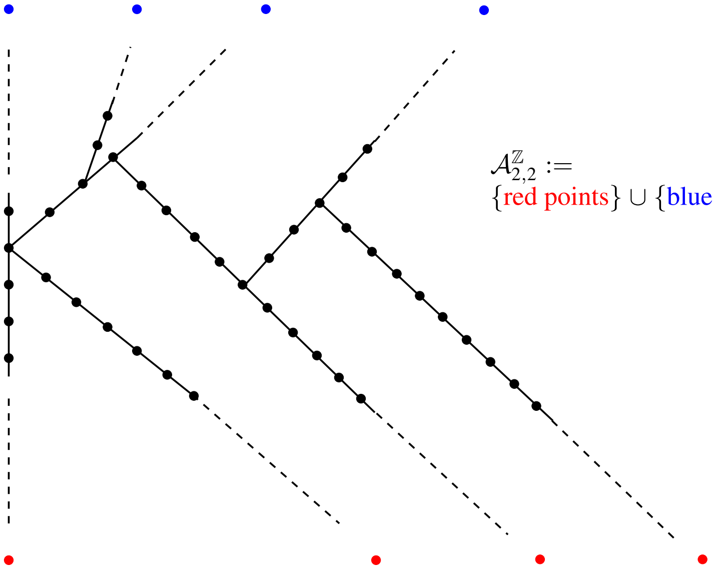
Building *CFPO*s from chains



Repeat the process for each new point.

After a countable number of steps all internal points ramify upwards and downwards.

Building *CFPO*s from chains



$\mathcal{A}^{\mathbb{Z}}_{2,2} :=$
$\{\text{red points}\} \cup \{\text{blue points}\}$

1- and 2-arc-transitive bipartite graphs

- The poset $\mathcal{A}^{\mathbb{Z}}_{2,2}$ is a *CFPO* and it has only has 2 levels (the blue and red points).
- However the completion of $\mathcal{A}^{\mathbb{Z}}_{2,2}$ contains infinite chains (the copies of $\mathbb{Z}$).

Fact 1. The two-level poset $\mathcal{A}^{\mathbb{Z}}_{2,2}$ is 3-CS-homogeneous.

Fact 2. As a bipartite graph $\mathcal{A}^{\mathbb{Z}}_{2,2}$ is 2-arc-transitive.

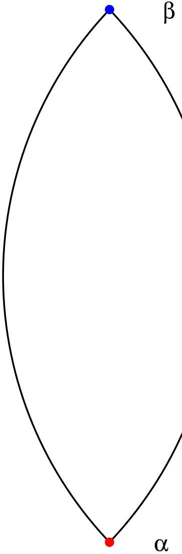
- *CFPO*s give rise to some interesting infinite 1- and 2-arc-transitive bipartite graphs.

Generalised cycle-free partial orders

Begin with P with $\alpha, \beta \in P$:

$\alpha \leq P \leq \beta$,

and two functions assigning upward and downward ramification orders to the points of P.

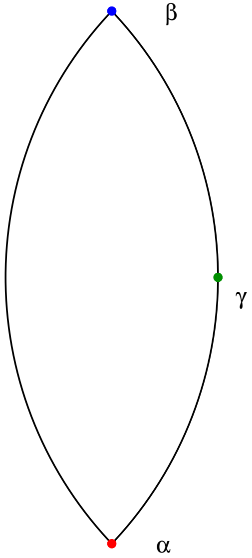


Generalised cycle-free partial orders

For each $\gamma \in P$

we adjoin a number of copies of the interval $[\gamma, \beta]$ above γ.

The number of copies adjoined is determined by the upward ramification order of γ.

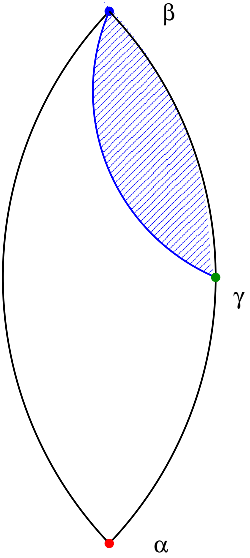


Generalised cycle-free partial orders

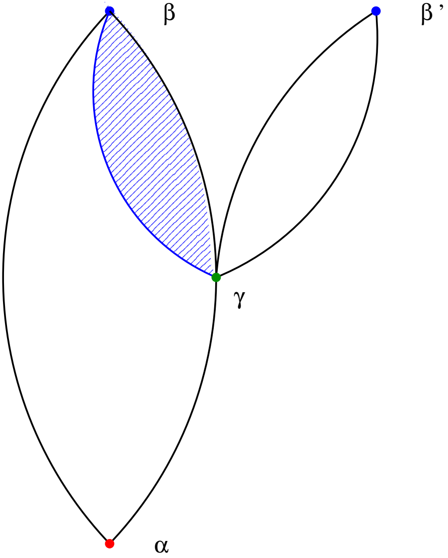
For each $\gamma \in P$

we adjoin a number of copies of the interval $[\gamma, \beta]$ above γ.

The number of copies adjoined is determined by the upward ramification order of γ.



Generalised cycle-free partial orders

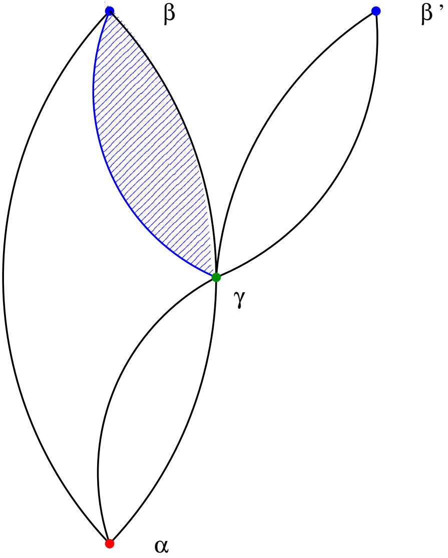


Generalised cycle-free partial orders

For each new maximal point β' we require

$[\alpha, \beta'] \cong [\alpha, \beta] \cong P.$

We introduce new points and relations so that this is the case.



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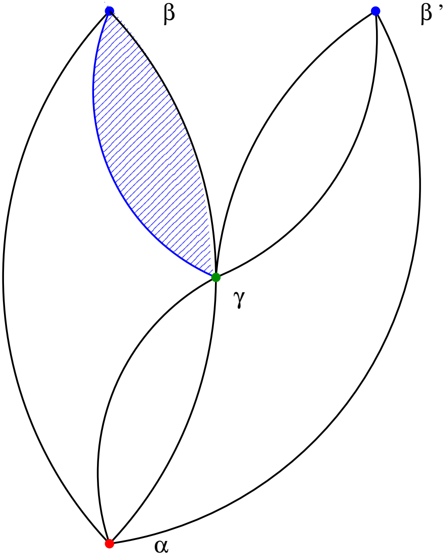
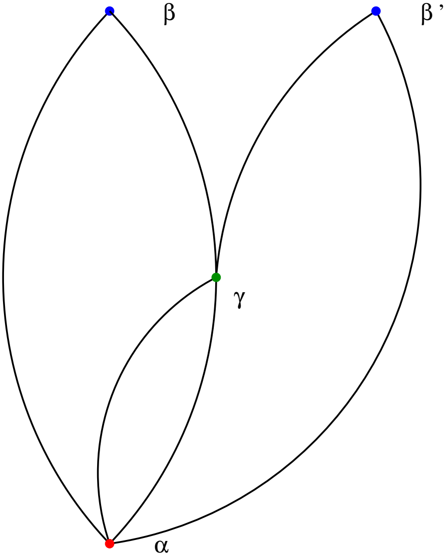


Generalised cycle-free partial orders

This completes the first step of the construction.

The upward ramification order of the point γ has been dealt with.

We repeat the process 'dealing with' each point in turn.



Good news and bad news

Good news

Theorem (RG, Truss (2007))

The infinite 2-level partial orders arising from our construction are all 2-CS-transitive.

Consequently the construction above gives rise to new examples of infinite 1-arc-transitive bipartite graph.

Bad news

It does not give any 'new' examples of 2-arc-transitive graph.

Theorem (RG, Truss (2007))

If Γ is a bipartite graph arising from our construction and is Γ is 2-arc-transitive then Γ is a CFPO.

Cycle-free partial orders and ends of graphs

Let Γ be a locally finite bipartite graph.

Theorem (RG, Truss (2007))

Γ has more than one end if and only if the Hasse graph of its completion Γ^D has more than one end.

Corollary (RG, Truss (2007))

If Γ is cycle-free then Γ has more than one end.