Topological finiteness properties of monoids

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The word problem for monoids and groups

Definition

A monoid M with a finite generating set A has decidable word problem if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of M.

Example. $M = \langle a, b \mid ab = ba \rangle$ has decidable word problem.

Some history

There are finitely presented monoids / groups with undecidable word problem.

- Markov (1947) and Post (1947), Turing (1950), Novikov (1955) and Boone (1958)

Theme. Identify and study interesting classes with decidable word problem:

- hyperbolic groups (Gromov) / word hyperbolic monoids
- automatic groups / monoids
- finite complete presentations

Finite complete presentations

$$M = \langle A \mid u_1 = v_1,\ u_2 = v_2,\ \ldots\ , u_k = v_k \rangle$$

- $w \in A^*$ is irreducible if it contains no u_i.
- The presentation is complete if there is no infinite sequence

$$w_1 \to w_2 \to w_3 \to \ldots$$

  with w_{i+1} obtained from w_i by applying a relation $u_r \to v_r$, and each element of the monoid M is represented by a unique irreducible word.

Example (Free group)

$\langle a, a^{-1}, b, b^{-1} \mid aa^{-1} = 1,\ a^{-1}a = 1,\ bb^{-1} = 1,\ b^{-1}b = 1 \rangle$.

Normal forms (irreducibles) = $\{$ freely reduced words $\}$.

Important basic fact

If a monoid M admits a finite complete presentation, then M has decidable word problem.

The homological finiteness property FP_n

$\mathbb{Z}M$ - integral monoid ring, e.g. $4m_1 - 2m_2 + 3m_3 \in \mathbb{Z}M$

Definition

A monoid M is of type left-FP_n if there is a resolution:

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

of the trivial left $\mathbb{Z}M$-module $\mathbb{Z}$ such that $F_0, F_1, \ldots, F_n$ are finitely generated free left $\mathbb{Z}M$-modules. A monoid is of type left-FP_∞ if it is left-FP_n for all $n \in \mathbb{N}$.

For any monoid:

- finitely generated $\Rightarrow$ left-FP_1, finitely presented $\Rightarrow$ left-FP_2
- **Anick (1986):** If a monoid M admits a finite complete presentation then M is of type left-FP_∞.

One-relator monoids

Longstanding open problem

Is the word problem decidable for one-relation monoids $\langle A \mid u = v \rangle$?

Related open problem

Does every one-relation monoid $\langle A \mid u = v \rangle$ admit a finite complete presentation?

If yes then every one-relation monoid would be of type left-FP_∞.

This motivates the following question of Kobayashi (2000)

Question: Is every one-relator monoid $\langle A \mid u = v \rangle$ of type left-FP_∞?

Magnus (1932): Proved one-relator groups have decidable word problem.

Lyndon (1950): Shows every one-relator group is of type FP_∞.

The topological finiteness property F_n

Definition (C. T. C. Wall (1965))

A $K(G,1)$-complex is a CW complex with fundamental group G and all other homotopy groups trivial (i.e. the space is aspherical). A group G is of type F_n $(0 \leqslant n < \infty)$ if there is a $K(G,1)$-complex with finite n-skeleton.

For any group:

(i) $\mathrm{F}_1 \equiv$ finitely generated, $\mathrm{F}_2 \equiv$ finite presented.

(ii) $\mathrm{F}_n \Rightarrow \mathrm{FP}_n$

(iii) For finitely presented groups $\mathrm{F}_n \equiv \mathrm{FP}_n$.

Aim

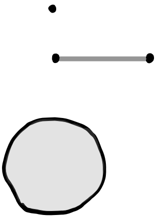
Develop a theory of topological finiteness properties of monoids. A good definition of F_n for monoids should satisfy (ii), so that it can be used to study FP_n.

CW complexes

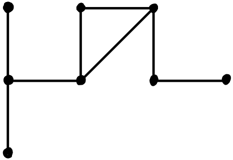
0 - cell

1 - cell

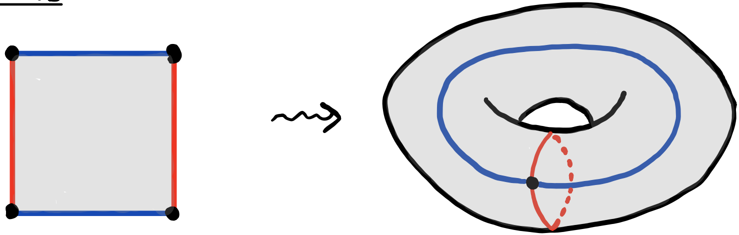
2 - cell



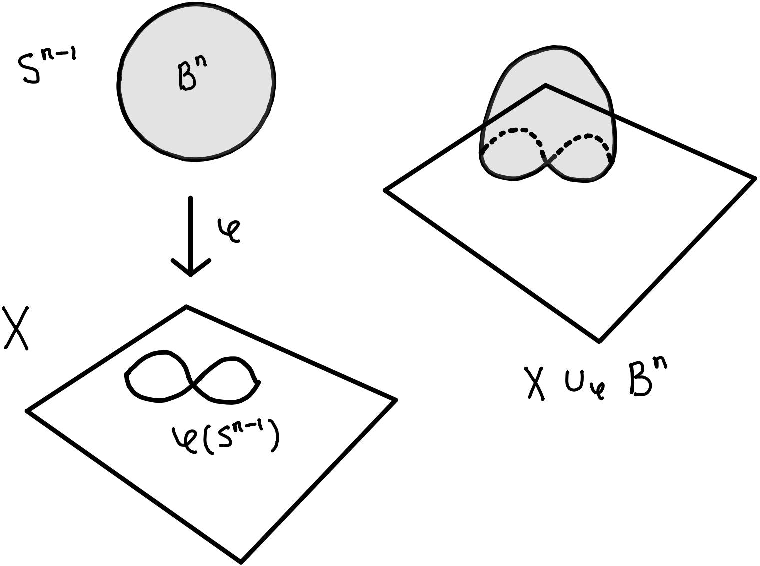
Graphs



Torus



Attaching an n-cell



CW complex definition

Definition

A CW decomposition of a topological space X is a sequence of subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

such that

- X_0 is discrete set, whose points are regarded as 0 cells
- The n-skeleton X_n is obtained from X_{n-1} by attaching a (possibly) infinite number of n-cells via maps $\varphi_\alpha : S^{n-1} \to X_{n-1}$.
- We have $X = \cup X_n$ with the *weak* topology (this means that a set $U \subseteq X$ is open if and only if $U \cap X_n$ is open in X_n for each n).

A CW complex[2] is a space X equipped with a CW decomposition.

[2] C stands for 'closure-finite', and the W for 'weak topology'.

$K(G, 1)$ of a group

Definition

A $K(G, 1)$-complex is a CW complex with fundamental group G and all other homotopy groups trivial (i.e. the space is aspherical).

Existence[3]: Every group G admits a $K(G, 1)$-complex Y.

Uniqueness: If X and Y are CW complexes both of which are $K(G, 1)$-complexes then X and Y are homotopy equivalent **(Hurewicz, 1936)**.

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[3] e.g. start with a connected 2-complex with fundamental group G and attach cells to kill higher homotopy. Alternatively one case use the classifying space $|BG|$.

Free G-CW complexes

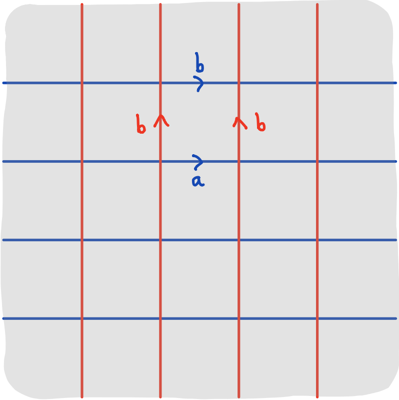
Definition

A $K(G,1)$-complex is a CW complex with fundamental group G and all other homotopy groups trivial (i.e. the space is aspherical).

A G-CW complex is a CW complex X together with an action of G on X which permutes the cells.

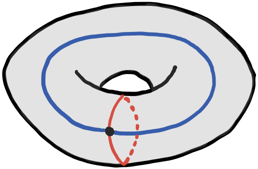
If Y is a $K(G,1)$ then its universal cover X is a contractible, free G-CW complex.

$G = \mathrm{Gp}\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$



Universal cover $X = \mathbb{R}^2$ is a contractible free G-CW complex

Torus



$Y = K(G, 1)$

Observe: X is the Cayley graph of G with a G-orbit of cells for the defining relator.

Free M-CW complex

A left M-space is a topological space X with a continuous left action $M \times X \to X$ where M has the discrete topology.

Definition (free M-CW-complex)

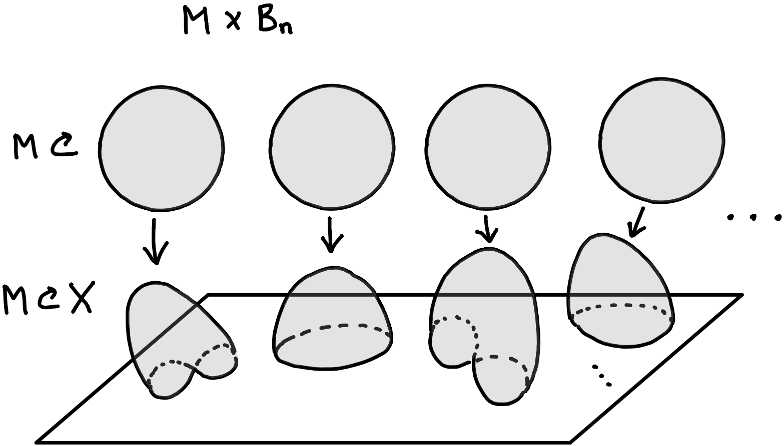
A free M-cell of dimension n is an M-space of the form $M \times B^n$ where B^n has the trivial action.

A free M-CW complex is built up by attaching M-cells $M \times B^n$ via M-equivariant maps from $M \times S^{n-1}$ to the $(n-1)$-skeleton.

X_n is obtained from X_{n-1} as a pushout of M-spaces, with M-equivariant maps, where P_n is a free left M-set.

$$\begin{tikzcd} P_n \times S^{n-1} \arrow[r] \arrow[d, hook] & X_{n-1} \arrow[d, hook] \\ P_n \times B^n \arrow[r] & X_n \end{tikzcd}$$

Attaching an orbit of cells



Equivariant classifying spaces

Definition

We say that a free M-CW complex X is a left equivariant classifying space for M if it is contractible.

Existence[4]: Every monoid has a left equivariant classifying space.

Uniqueness: Let X, Y be equivariant classifying spaces for M. Then X and Y are M-homotopy equivalent.

Definition (Property F_n for monoids)

A monoid M is of type left-F_n if there is an equivariant classifying space X for M such that the set of k-cells is a finitely generated free left M-set for all $k \leqslant n$.

[4] e.g. one can take the geometric realisation $|\overrightarrow{EM}|$ of the nerve of the right Cayley graph category.

Relationship with other finiteness properties

Theorem (RDG & Steinberg)

Let M be a monoid.

1. A group is of type left-F_n if and only if it is of type F_n in the usual sense.
2. If M is of type left-F_n, then[5] it is of type left-FP_n.

[5] The augmented cellular chain complex of an equivariant classifying space for M provides a free resolution of the trivial (left) $\mathbb{Z}M$-module $\mathbb{Z}$.

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4. If M is finitely presented then M is left-F_2.
5. For finitely presented monoids left-$\mathrm{F}_n \equiv$ left-FP_n.

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The (right) Cayley graph $\Gamma(M, A)$ of a monoid M generated by a finite set A is the digraph with
Vertices: M Directed edges: $x \xrightarrow{a} y$ iff $y = xa$ where $x, y \in M$, $a \in A$.

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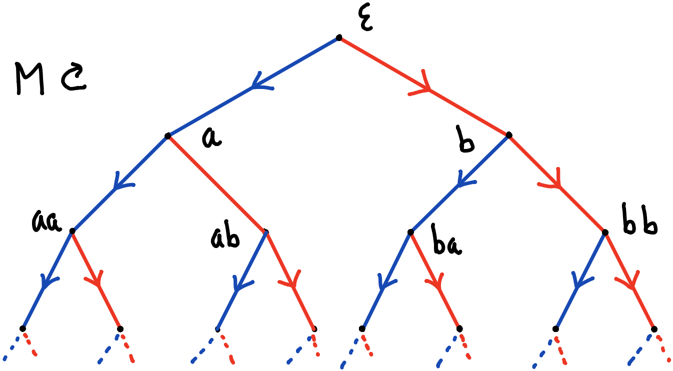
Free monoids are left-F_∞

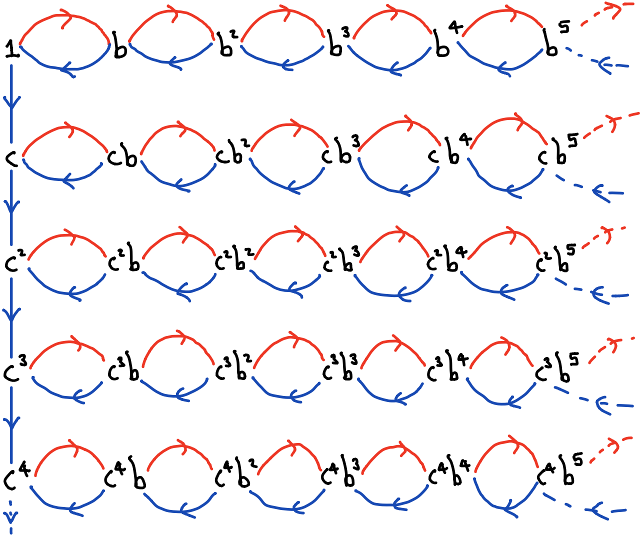
Free monoid $M = \{a, b\}^*$

Cayley graph $\Gamma(M, \{a, b\})$

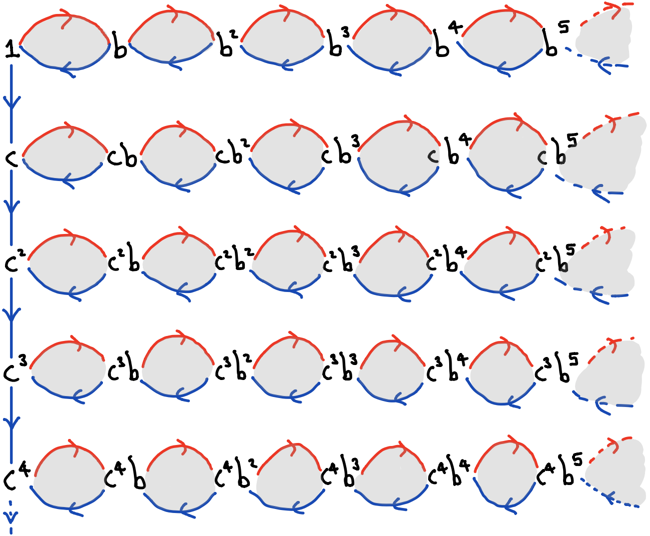
$M \circlearrowright$



Bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$



Bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$ is left-F_∞



One-relation monoids

Question: Is every one-relator monoid $\langle A \mid u = v \rangle$ of type left-FP$_\infty$?

Theorem (RDG & Steinberg)

Let $M = \langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1 \rangle$ and let G be the group of units of M. If G is left-F$_n$ then M is left-F$_n$.

Corollary (RDG & Steinberg)

Every one-relator monoid $M = \langle A \mid w = 1 \rangle$ is of type left-F$_\infty$.

Proof. The group of units G of the one-relator monoid M is a one-relator group by Adjan (1966), which is of type left-F$_\infty$ by Lyndon (1950).

Geometry of special monoids

Let $M = \langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1 \rangle$.

Two elements of M are $\mathcal{R}$-related if they are in the same strongly connected component of the right Cayley graph $\Gamma(M, A)$. The strongly connected components of $\Gamma(M, A)$ are called the Schützenberger graphs.

Ideas used in the proof

- Makanin (1966): The group of units G of M is a k-relator group, and the word problem of the monoid reduces to that of G.

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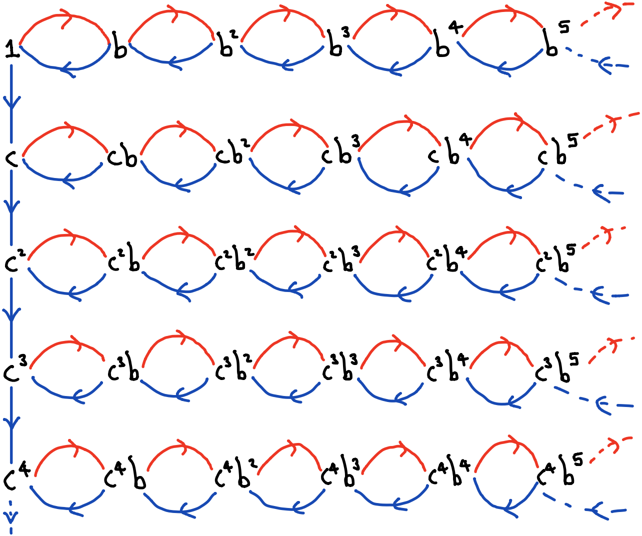
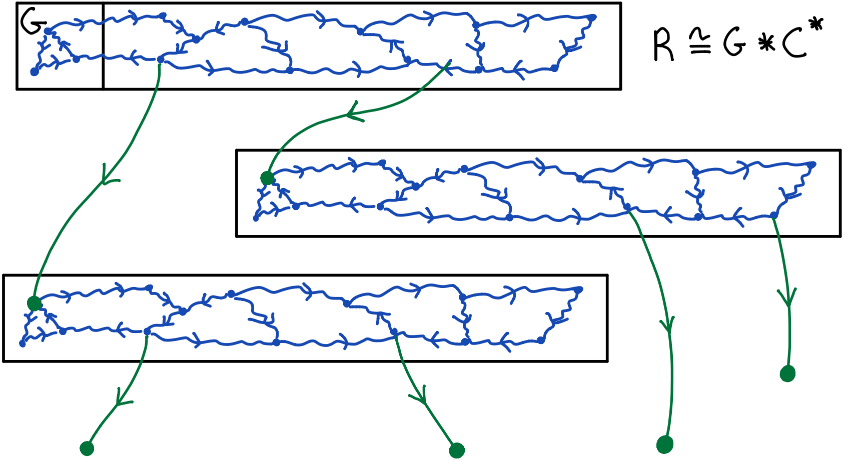
- Each Schützenberger graph has at most one arc entering it from outside.
- $M \circlearrowleft R$ is free with basis $\mathcal{T}$-the terminal vertices of the arcs entering the Schützenberger graphs. The orbits of $M \circlearrowleft R$ are the $\mathcal{R}$-classes of M.
- All Schützenberger graphs are isomorphic.

Bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$



$$M = \langle A \mid w_1 = 1,\ w_2 = 1,\ \ldots,\ w_k = 1 \rangle$$

$\Gamma(M, A)$



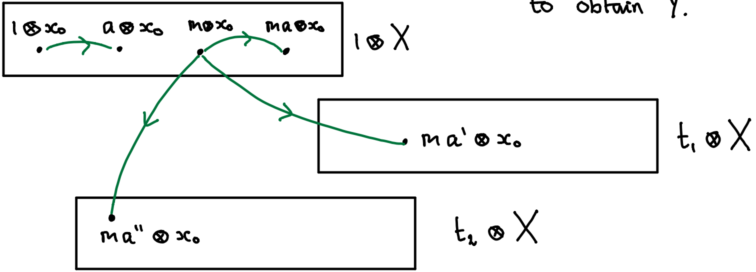
$R \cong G * C^*$

Eq. classifying space for $M = \langle A \mid w_1 = 1, \ldots, w_k = 1 \rangle$

X = equivariant classifying space for $R \cong G * C^*$

<u>Tensor</u> $M \otimes_R X = \coprod_{t \in \mathcal{T}} t \otimes X$. $\boxed{\text{Fix } x_0 \in X}$

<u>Connect</u> Add edges $m \otimes x_0 \to ma \otimes x_0$ $(a \in A)$ to obtain Y.



<u>Kill loops</u> $Z := Y$ with cells added to deal with loops created when collapsing the $t_i \otimes X$.

Free products with amalgamation

A monoid amalgam is a triple $[M_1, M_2; W]$ where M_1, M_2 are monoids with a common submonoid W. The amalgamated free product is the pushout

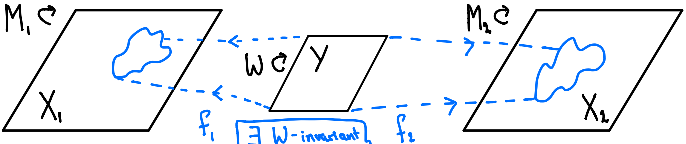
$$\begin{array}{ccc} W & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M_1 *_W M_2 \end{array}$$

Theorem (RDG & Steinberg)

Let $[M_1, M_2; W]$ be an amalgam of monoids such that M_1, M_2 are free as right W-sets. If M_1, M_2 are of type left-F_n and W is of type left-F_{n-1}, then $M_1 *_W M_2$ is of type left-F_n.

- Improves on results of **Cremanns and Otto (1998)**.
- To prove the result we use a homotopy pushout construction to build an equivariant classifying space for M from equivariant classifying spaces for M_1, M_2 and W.

Eq. classifying space for $L = M_1 *_W M_2$



$\boxed{\exists \text{ W-invariant cellular maps}}$

$\boxed{\text{Set } L = M_1 *_W M_2}$

Tensor

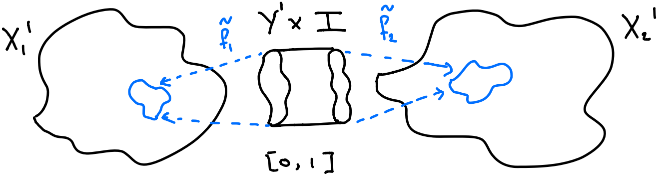
$$X_1' = L \otimes_{M_1} X_1 \xleftarrow{\tilde{f}_1} Y' = L \otimes_W Y \xrightarrow{\tilde{f}_2} X_2' = L \otimes_{M_2} X_2$$

Are free L-CW complexes

$\boxed{\text{McDuff '79: } L \text{ is a free right } M_1\text{-}, M_2\text{-} \ \& \ W\text{-set}}$

Homotopy Pushout

is an eq. class space for L.



Other topological finiteness properties

Definition

The left geometric dimension of M to be the minimum dimension of an equivariant classifying space for M.

Note. For this we actually need to work with projective M-CW complexes and the corresponding definition of equivariant classifying space.

Any projective M-set P is isomorphic to an M-set of the form $\coprod_{a \in A} Me_a$ where the e_a are idempotents.

- The geometric dimension is an upper bound on the cohomological dimension $\operatorname{cd} M$ of M, where $\operatorname{cd} M$ is the shortest length of a projective resolution of the trivial left $\mathbb{Z}M$-module $\mathbb{Z}$.

Some results on geometric dimension

Amalgamated free products

Theorem (RDG & Steinberg)

Let $[M_1, M_2; W]$ be an amalgam of monoids such that M_1, M_2 are free as right W-sets. Suppose that d_i is the left geometric dimension of M_i, for $i = 1, 2$, and d is the left geometric dimension of W. Then the left geometric dimension of $M_1 *_W M_2$ is bounded above by $\max\{d_1, d_2, d + 1\}$.

Special and one-relator monoids

Theorem (RDG & Steinberg)

Let $M = \langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1 \rangle$ and let G be the group of units of M. Then $\operatorname{gd} G \leqslant \text{left } \operatorname{gd} M \leqslant \max\{2, \operatorname{gd} G\}$.

Corollary (RDG & Steinberg)

If $M = \langle A \mid w = 1 \rangle$ with w not a proper power, then left $\operatorname{gd} M \leqslant 2$, otherwise left $\operatorname{gd} M = \infty$.

HNN-like extensions after Otto and Pride

M - monoids, A - submonoid, $\varphi\colon A \to M$ a homomorphism
Then the corresponding Otto-Pride extension is the monoid

$$L = \langle M, t \mid at = t\varphi(a), a \in A \rangle$$

Theorem (RDG & Steinberg)

Let M be a monoid, A a submonoid and $\varphi\colon A \to M$ be a homomorphism. Let L be the Otto-Pride extension. Suppose that M is free as a right A-set. If M is of type left-F_n and A is of type left-F_{n-1}, then L is of type left-F_n.

Notes:

- Is a higher dimensional topological analogue of some results of Otto and Pride (2004).
- Can be used to recover some of their results on homological finiteness properties.

HNN extensions in the sense of Howie (1963)

M - monoids, A, B - submonoids isomorphic via $\varphi\colon A \to B$
$C =$ infinite cyclic group generated by t

The HNN extension of M with base monoids A, B is defined to be

$$L = \langle M, t, t^{-1} \mid tt^{-1} = 1 = t^{-1}t, at = t\varphi(a), \forall a \in A \rangle$$

Theorem (RDG & Steinberg)

Let L be an HNN extension of M with base monoids A, B. Suppose that, furthermore, M is free as both a right A-set and a right B-set. If M is of type left-F_n and A is of type left-F_{n-1}, then L is of type left-F_n.

Notes:

- This result recovers the usual topological finiteness result for HNN extensions of groups.
- It also applies if M is left cancellative and A is a group.

Brown's theory of collapsing schemes

In his 1989 paper "The geometry of rewriting systems: a proof of the Anick–Groves–Squier Theorem", Brown shows:

If a monoid M admits a finite complete presentation then the space $|BM|$ has the homotopy type of a CW-complex with only finitely many cells in each dimension.

- To prove this he introduces the notion of a collapsing scheme.
- This idea has its roots in earlier work of Brown and Geoghegan (1984).
- Collapsing schemes were rediscovered again later on as Morse matchings in Forman's Discrete Morse theory.

Brown's result provides a topological proof that if a *group* G is presentable by a finite complete rewriting system then G is of type F_∞.

Topological proof of Anick's Theorem

We have developed a theory of M-equivariant collapsing schemes which can be used to give a topological proof of

Theorem (RDG & Steinberg)

Let M be a monoid. If M admits a presentation by a finite complete rewriting system then M is of type left-F_∞.

Notes:

- We recover Anick's theorem for monoids as a corollary.
- Our results also apply in the 2-sided case and thus we also recover a theorem of Kobayashi (2005) on bi-FP_n as a corollary.

Other topics and future work

Two-sided theory

- We define the bilateral notion of a classifying space in order to introduce a stronger property, bi-F_n. The property bi-F_n implies bi-FP_n which is of interest from the point of view of Hochschild cohomology.
- Two-sided classifying spaces can be constructed using the two-sided Cayley graph category of the monoid, which is the kernel category of the identity map in the sense of Rhodes and Tilson (1989).

Ongoing & future work

- Complete the proof that *all* one-relator monoids $\langle A \mid u = v \rangle$ are of type left-FP_∞.
  - $\langle A \mid u = 1 \rangle$ ✓
  - $\langle A \mid u = v \rangle$ where u and v have distinct initial / terminal letters (and some related examples) ✓ Kobayashi (1998)
  - We have dealt with some other cases very recently e.g. subspecial (details to be checked...)
- In future it would be good to develop a better understanding of bi-F_n.