Set-homogeneous digraphs

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(joint work with C. E. Praeger and D. Macpherson)

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Homogeneity and set-homogeneity

Definition

A relational structure M is homogeneous if every isomorphism between finite substructures of M can be extended to an automorphism of M.

- This notion goes back to the fundamental work of Fraïssé (1953)

Definition

A relational structure M is set-homogeneous if whenever two finite substructures U and V are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $Ug = V$.

- Originally considered in unpublished observations of Fraïssé and Pouzet.

General question

How much stronger is homogeneity than set-homogeneity?

Set-homogeneous finite graphs

Ronse (1978)

...proved that for finite graphs homogeneity and set-homogeneity are equivalent.

- He did this by classifying the finite set-homogeneus graphs and then observing that they are all, in fact, homogeneous.
- This generalised an earlier result of Gardiner, classifying the finite homogeneous graphs.

Enomoto (1981)

...gave a very short direct proof of the fact that for finite graphs set-homogeneous implies homogeneous.

- this avoids the need to classify the set-homogeneous graphs
- the set-homogeneous classification can then be read off from Gardiner's result

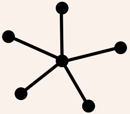
Some graph theoretic terminology and notation

Definition

$\Gamma = (V\Gamma, \sim)$ - a graph

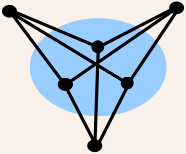
So $\sim$ is a symmetric irreflexive binary relation on $V\Gamma$

- Let v be a vertex of Γ. Then the neighbourhood $\Gamma(v)$ of v is the set of all vertices adjacent to v. So



$$\Gamma(v) = \{w \in V\Gamma : w \sim v\}$$

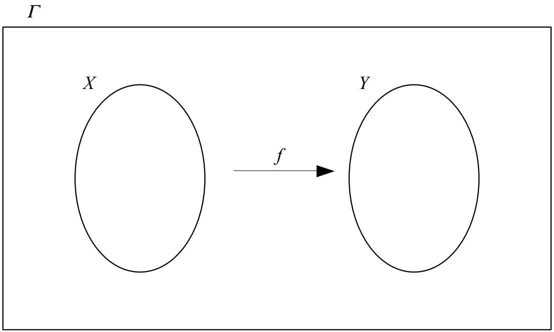
- For $X \subseteq V\Gamma$ we define



$$\Gamma(X) = \{w \in V\Gamma : w \sim x \;\forall x \in X\}$$

Enomoto's argument

Γ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism



Claim: The isomorphism $f : X \to Y$ is either an automorphism, or extends to an isomorphism $f' : X' \to Y'$ where $X' \supsetneq X$ and $Y' \supsetneq Y$.

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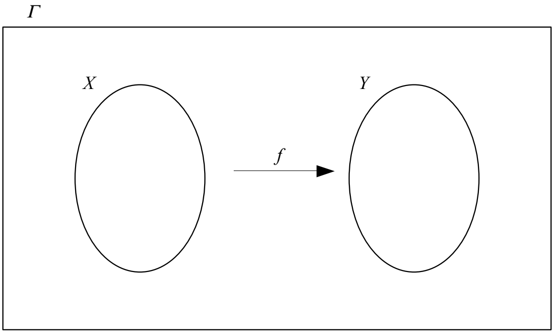

Proof of claim.

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.

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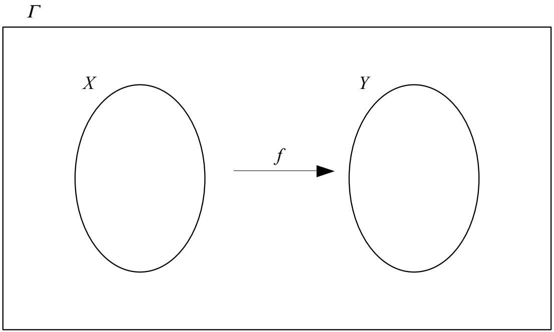

Proof of claim.

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.
- Choose $d \in \Gamma \setminus Y$ with $|\Gamma(d) \cap Y|$ as large as possible.

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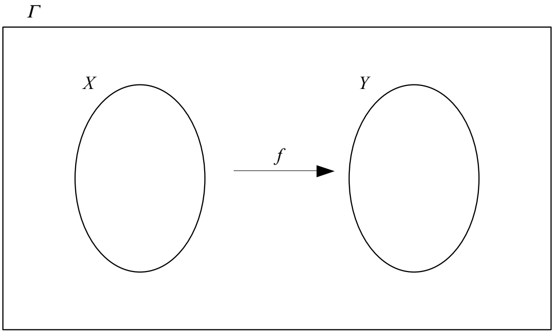

Proof of claim.

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.
- Choose $d \in \Gamma \setminus Y$ with $|\Gamma(d) \cap Y|$ as large as possible.
- Suppose $|\Gamma(a) \cap X| \geq |\Gamma(d) \cap Y|$ (the other possibility is dealt with dually using the isomorphism f^{-1})

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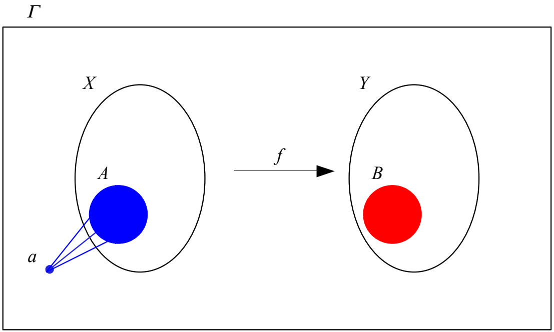

Proof of claim.

- Let $A = \Gamma(a) \cap X$ and define $B = f(A)$.

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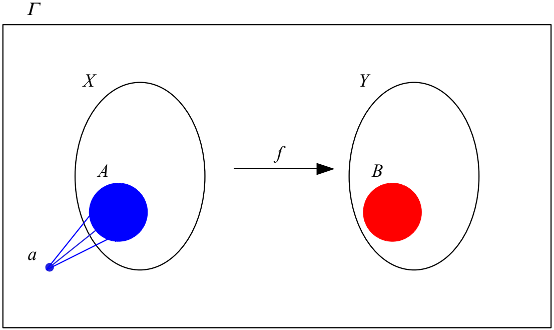

Proof of claim.

- Let $A = \Gamma(a) \cap X$ and define $B = f(A)$.
- $A \cong B$ & Γ is set-homogeneous $\Rightarrow$ $|\Gamma(A)| = |\Gamma(B)|$.

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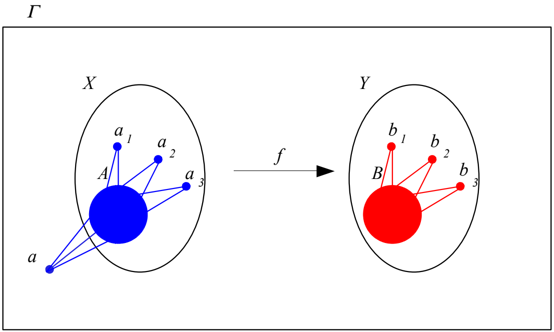

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- $\Gamma(B) \cap Y = f(\Gamma(A) \cap X)$ so $|\Gamma(B) \cap Y| = |\Gamma(A) \cap X|$.

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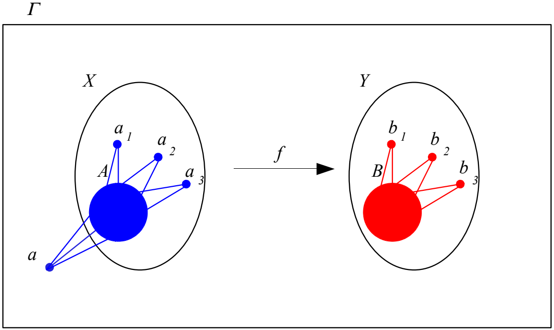

Proof of claim.

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- $A \cong B$ & Γ is set-homogeneous $\Rightarrow$ $|\Gamma(A)| = |\Gamma(B)|$.
- $\Gamma(B) \cap Y = f(\Gamma(A) \cap X)$ so $|\Gamma(B) \cap Y| = |\Gamma(A) \cap X|$.
- $\therefore |\Gamma(B) \setminus Y| = |\Gamma(A) \setminus X)| \geq 1$

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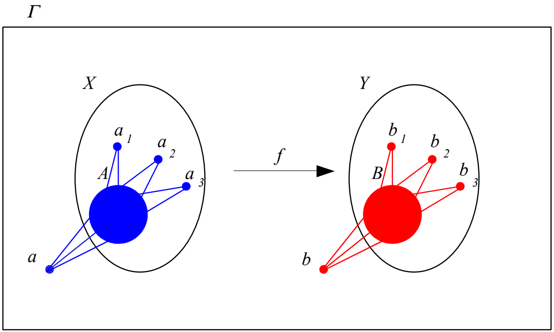

Proof of claim.

- Let $b \in \Gamma(B) \setminus Y$ and extend f to $f' : X \cup \{a\} \to Y \cup \{b\}$ by defining $f'(a) = b$.

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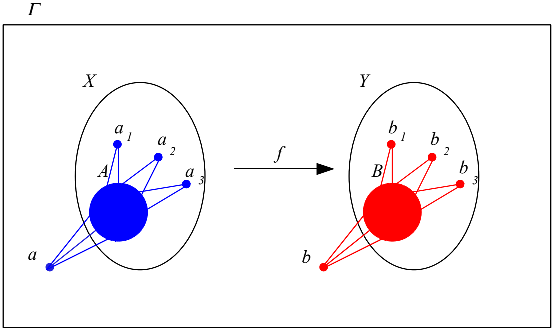

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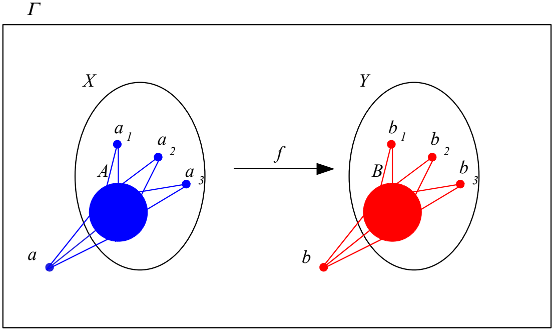

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- $\Gamma(b) \cap Y = B$ by maximality in original definition of a,
- $\therefore$ f' is an isomorphism.

Set-homogeneous digraphs

Question: What about other kinds of relational structure?

Definition (Digraphs)

A digraph D consists of a set VD of vertices together with an irreflexive antisymmetric binary relation $\rightarrow$ on VD.

Enomoto's argument can be adapted for tournaments:

Definition

A tournament is a digraph where for any pair of vertices u, v either $u \rightarrow v$ or $v \rightarrow u$.

Corollary

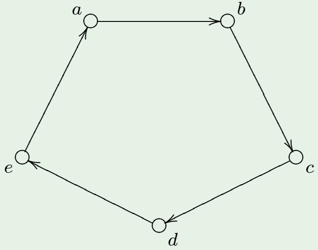
Let T be a finite tournament. Then T is homogeneous if and only if T is set-homogeneous.

A non-homogeneous example

Example

Let D_n denote the digraph with vertex set $\{0, \ldots, n-1\}$ and just with arcs $i \to i+1 \pmod{n}$.

The digraph D_5 is set-homogeneous but is not homogeneous.



- $(a, c) \mapsto (a, d)$ gives an isomorphism between induced subdigraphs that does not extend to an automorphism
- However, there is an automorphism sending $\{a, c\}$ to $\{a, d\}$.

Finite set-homogeneous digraphs

Question

How much bigger is the class of set-homogeneous digraphs than the class of homogeneous digraphs?

Finite set-homogeneous digraphs

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How much bigger is the class of set-homogeneous digraphs than the class of homogeneous digraphs?

Theorem (RG, Macpherson, Praeger (2007))

Let D be a finite set-homogeneous digraph. Then either D is homogeneous or it is isomorphic to D_5.

Proof.

- Carry out the classification of finite set-homogeneous digraphs.
- Proof is inductive using the fact that the subgraph induced by the out-neighbours of a vertex gives a (smaller) set-homogeneous digraph.
- By inspection note that D_5 is the only non-homogeneous example. $\square$

Symmetric-digraphs (s-digraphs)

A common generalisation of graphs and digraphs

Definition (s-digraph)

- An s-digraph is the same as a digraph except that we allow pairs of vertices to have arcs in both directions.
- So for any pair of vertices u, v exactly one of the following holds:

$$u \rightarrow v, \qquad v \rightarrow u, \qquad u \leftrightarrow v, \qquad u \parallel v \ \text{(meaning unrelated).}$$

- Formally we can think of an s-digraph as a structure M with two binary relations $\rightarrow$ and $\sim$ where
  - $\sim$ is irreflexive and symmetric (and corresponds to $\leftrightarrow$ above)
  - $\rightarrow$ is irreflexive and antisymmetric
  - $\sim$ and $\rightarrow$ are disjoint
- A graph is an s-digraph (where there are no $\rightarrow$-related vertices)
- A digraph is an s-digraph (where there are no $\sim$-related vertices)

Classifying the finite homogeneous s-digraphs

- Lachlan (1982) classified the finite homogeneous s-digraphs

To state his result we need the notions of

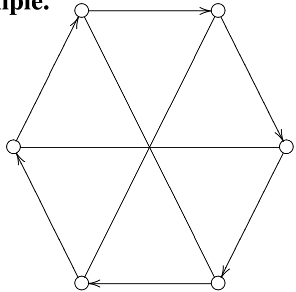
- complement
- compositional product

Taking complements

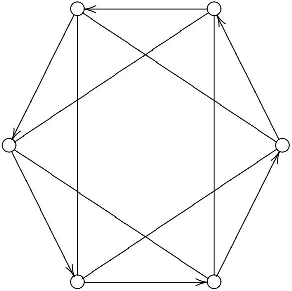
Definition

If M is an s-digraph, then $\bar{M}$, the complement, is the s-digraph with the same vertex set, such that $u \sim v$ in $\bar{M}$ if and only if they are unrelated in M, and $u \to v$ in $\bar{M}$ if and only if $v \to u$ in M.

Example.



M



$\overline{M}$

Compositional product

Definition (Composition)

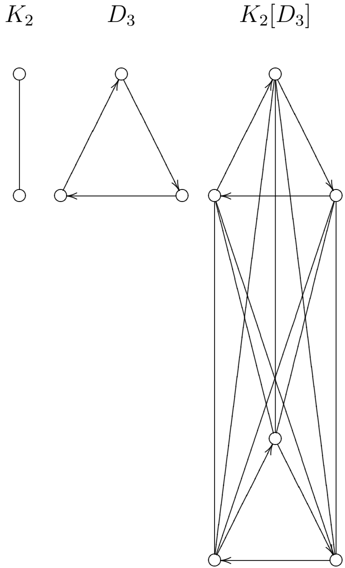
If U and V are s-digraphs, the compositional product $U[V]$ denotes the s-digraph which is

"$|U|$ copies of V"

Vertex set = $U \times V$

$\longrightarrow$ relations are of form $(u, v_1) \longrightarrow (u, v_2)$ where $v_1 \longrightarrow v_2$ in V, or of form $(u_1, v_1) \longrightarrow (u_2, v_2)$ where $u_1 \longrightarrow u_2$ in U,

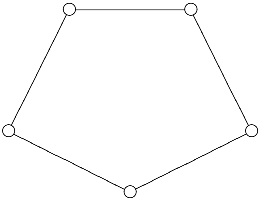
Similarly for $\sim$.



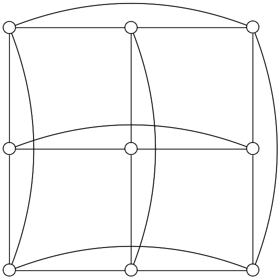
Some finite homogeneous s-digraphs

Sporadic examples

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs



$C_5 \in \mathcal{L}$



$K_3 \times K_3 \in \mathcal{L}$

Some finite homogeneous s-digraphs

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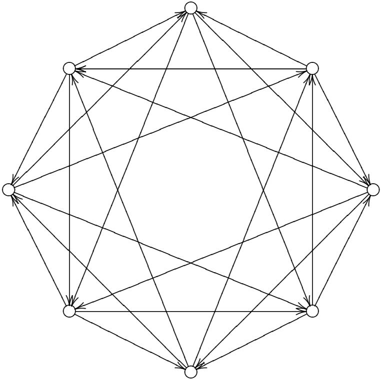
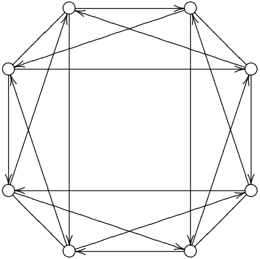


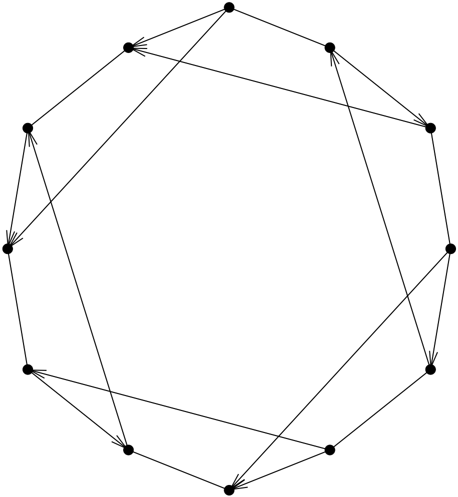
$H_0 \in \mathcal{A}$



$H_1 \in \mathcal{S}$

Some finite homogeneous s-digraphs

Sporadic examples



$H_2 \in \mathcal{S}$

To complete the picture...

In H_2 each vertex v has a unique mate v' to which it is joined by an undirected edge.

Now if $v \to w$ then $w \to v'$ where v' is the mate of v.

Similarly, if $w \to v$ then $v' \to w$.

Lachlan's classification

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs

Theorem (Lachlan (1982))

Let M be a finite s-digraph. Then

Gardiner

(i) $M \in \mathcal{L} \Leftrightarrow M$ or $\bar{M}$ is one of: C_5, $K_3 \times K_3$, $K_m[\bar{K}_n]$ (for $1 \leq m, n \in \mathbb{N}$);

Lachlan

(ii) $M \in \mathcal{A} \Leftrightarrow M$ is one of: D_3, D_4, H_0, $\bar{K}_n$, $\bar{K}_n[D_3]$, or $D_3[\bar{K}_n]$, for some $n \in \mathbb{N}$ with $1 \leq n$;

(iii) $M \in \mathcal{S} \Leftrightarrow M$ or $\bar{M}$ is isomorphic to an s-digraph of one of the following forms. $K_n[A], A[K_n]$, L, $D_3[L]$, $L[D_3]$, H_1, H_2, where $n \in \mathbb{N}$ with $1 \leq n$, $A \in \mathcal{A}$ and $L \in \mathcal{L}$.

Set-homogeneous s-digraphs

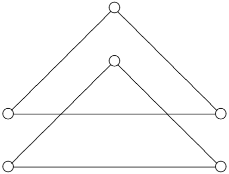
Theorem (RG, Macpherson, Praeger (2007))

The finite s-digraphs that are set-homogeneous but not homogeneous are:

Infinite families (with $n \in \mathbb{N}$)

(i) $K_n[D_5]$ or $D_5[K_n]$

(ii) J_n



J_2

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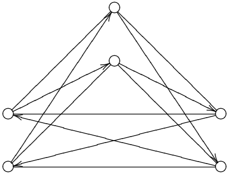

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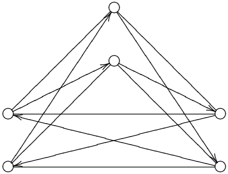

J_2

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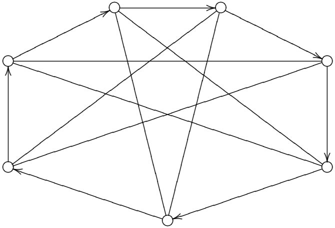
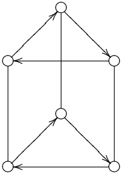


E_7



F_6

Countably infinite graphs

Homogeneous

- Lachlan and Woodrow (1980) - classified the countably infinite homogeneous (undirected) graphs

Set-homogeneous

- Droste, Giraudet, Macpherson, Sauer (1994) - showed that for countably infinite graphs set-homogeneous $\not\Rightarrow$ homogeneous and proved that (up to complementation) there is a unique countable set-homogeneous but not 3-homogeneous graph, called $R(3)$.
- The problem of classifying the countable set-homogeneous graphs is still open.

Countably infinite digraphs

Homogeneous

- Cherlin (1998) - classified the countably infinite homogeneous digraphs

Set-homogeneous

- RG, Macpherson, Praeger (2007) - classified the countably infinite digraphs that are set-homogeneous but not 2-homogeneous:
  - a family R_n built using n-coloured versions of the rationals
  - a sporadic example called $T(4)$ built using a circular construction (originally due to Cameron and Macpherson) similar to that used by Droste, Giraudet, Macpherson, Sauer (1994) to construct $R(3)$

Open problem. Is there a countably infinite tournament that is set-homogeneous but not homogeneous?

Relating to this question, we know:

Proposition (RG, Macpherson, Praeger (2007))

Let T be a set-homogeneous tournament. Then T is 4-homogeneous.