Membership problems for positive one-relator groups and one-relation monoids

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(joint work with Islam Foniqi and Carl-Fredrik Nyberg-Brodda)

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Presentations and the word problem

$$\mathrm{Mon}\langle A \mid R \rangle = \mathrm{Mon}\langle \underbrace{a_1, \ldots, a_n}_{\text{letters / generators}} \mid \underbrace{u_1 = v_1, \ldots, u_m = v_m}_{\text{words / defining relations}} \rangle$$

- Defines the monoid $M = A^*/\rho$ where ρ is the equivalence relation on the free monoid A^* of all words over A where two words are in the same equivalence class (i.e. they represent the same element of M) if one can be transformed into the other by applying the relations R.

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One-relator monoids

Open problem

Is the word problem decidable for one relator monoids $\mathrm{Mon}\langle A \mid u = v\rangle$?

Theorem (Adian & Oganesian, 1978+1987)

The word problem for a given $\mathrm{Mon}\langle A \mid u = v\rangle$ can be reduced to the word problem for a one-relator monoid of the form

$$\mathrm{Mon}\langle a, b \mid bUa = aVa\rangle \quad \text{or} \quad \mathrm{Mon}\langle a, b \mid bUa = a\rangle\,.$$

- Both of these cases remain open!

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Bold claim

I believe there is a one-relator monoid of the form $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ with an undecidable word problem!

Approaching the word problem

$$\boxed{\text{Finitely presented groups}}$$

$$G \cong \mathrm{Gp}\langle A \mid u_i = v_i \ (i \in I)\rangle \quad (u_i, v_i \in (A \cup A^{-1})^*)$$

Elements of G - equivalence classes of words over $A \cup A^{-1}$ where $u = v$ in $G \Leftrightarrow$ we can transform u into v by applying defining relations or relations $aa^{-1} = 1 = a^{-1}a$ from the free group.

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Markov / Post (1947) — Novikov (1955) / Boone (1958) — Turing (1950)

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Theorem (Novikov (1955) & Boone (1958))

There is a finitely presented group $G \cong \mathrm{Gp}\langle A \mid R\rangle$ with $|A| < \infty, |R| < \infty$ with an undecidable word problem.

Word problem for $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$

Reductions We write

$$\boxed{\text{Decision problem } \mathscr{P}} \longrightarrow \boxed{\text{Decision problem } \mathscr{Q}}$$

to mean if $\mathscr{P}$ is decidable then $\mathscr{Q}$ is decidable.

Strategy

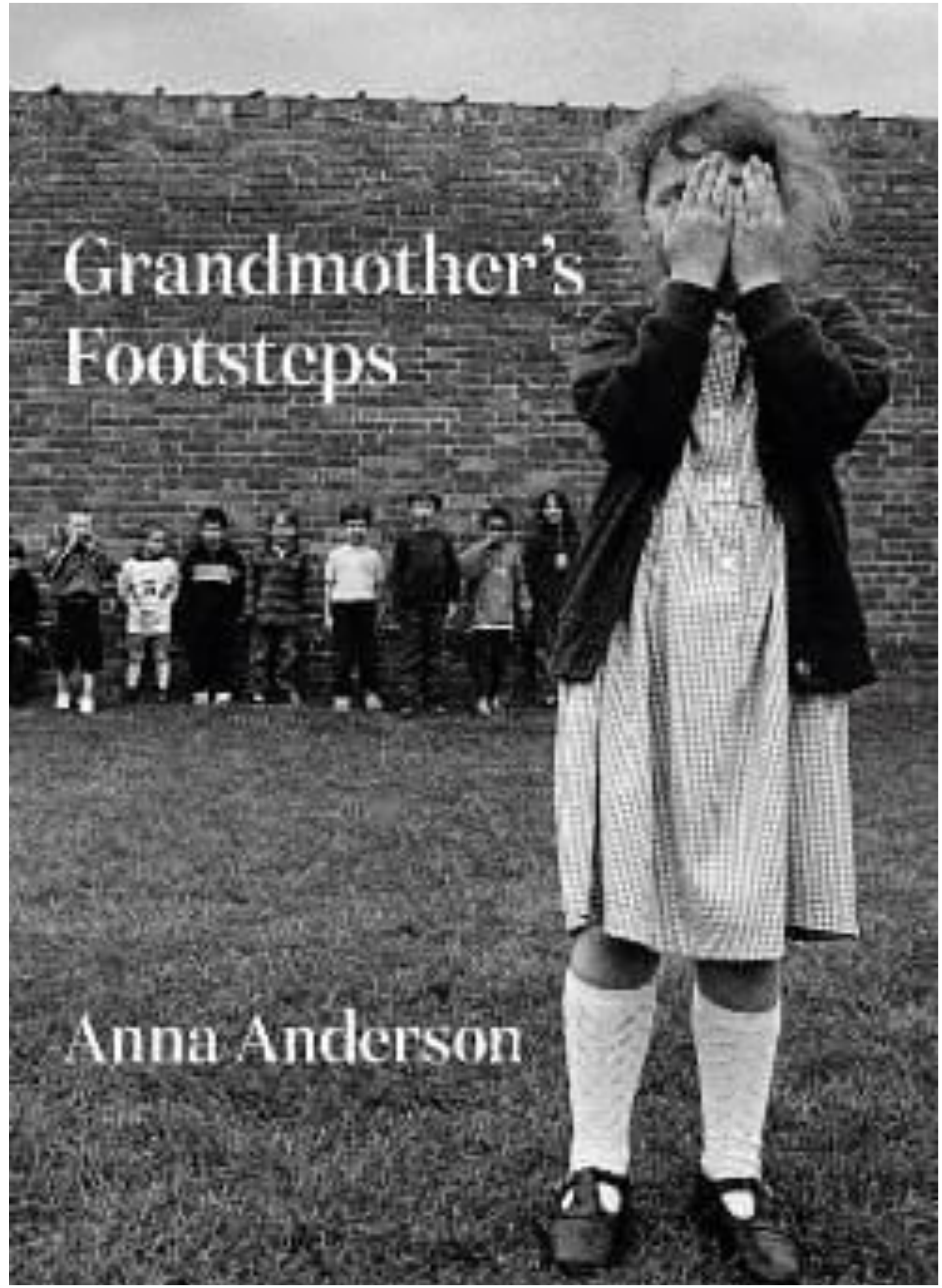
- Identify problems $\mathscr{P}$ such that

$$\boxed{\text{Decision problem } \mathscr{P}} \longrightarrow \boxed{\text{Word problem for all } \mathrm{Mon}\langle a, b \mid bUa = a\rangle \text{ decidable}}$$

- Prove that $\mathscr{P}$ is undecidable.

- Repeat the process finding problems that approximate more closely the problem on the right.

$$\begin{array}{c}\boxed{\text{Decision problem } \mathscr{P}_1}\\ \Big\downarrow \\ \boxed{\text{Decision problem } \mathscr{P}_2}\\ \Big\downarrow \\ \vdots \\ \boxed{\begin{array}{c}\text{Word problem for all}\\ \mathrm{Mon}\langle a, b \mid bUa = a\rangle\\ \text{decidable}\end{array}}\end{array}$$



(I) Positive one-relator groups

Submonoid membership problem for positive one-relator groups $\xrightarrow{\text{Guba (1997)}}$ Word problem for all $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ decidable

Theorem (Guba 1997)

For every $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ there is a positive one-relator group $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ with $w \in A^+$ and a finitely generated submonoid T of G such that if the membership problem for $T \leq G$ is decidable then $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ has decidable word problem.

Submonoid membership problem

G - a finitely generated group with a finite group generating set A.

$\pi : (A \cup A^{-1})^* \to G$ - the canonical homomorphism.

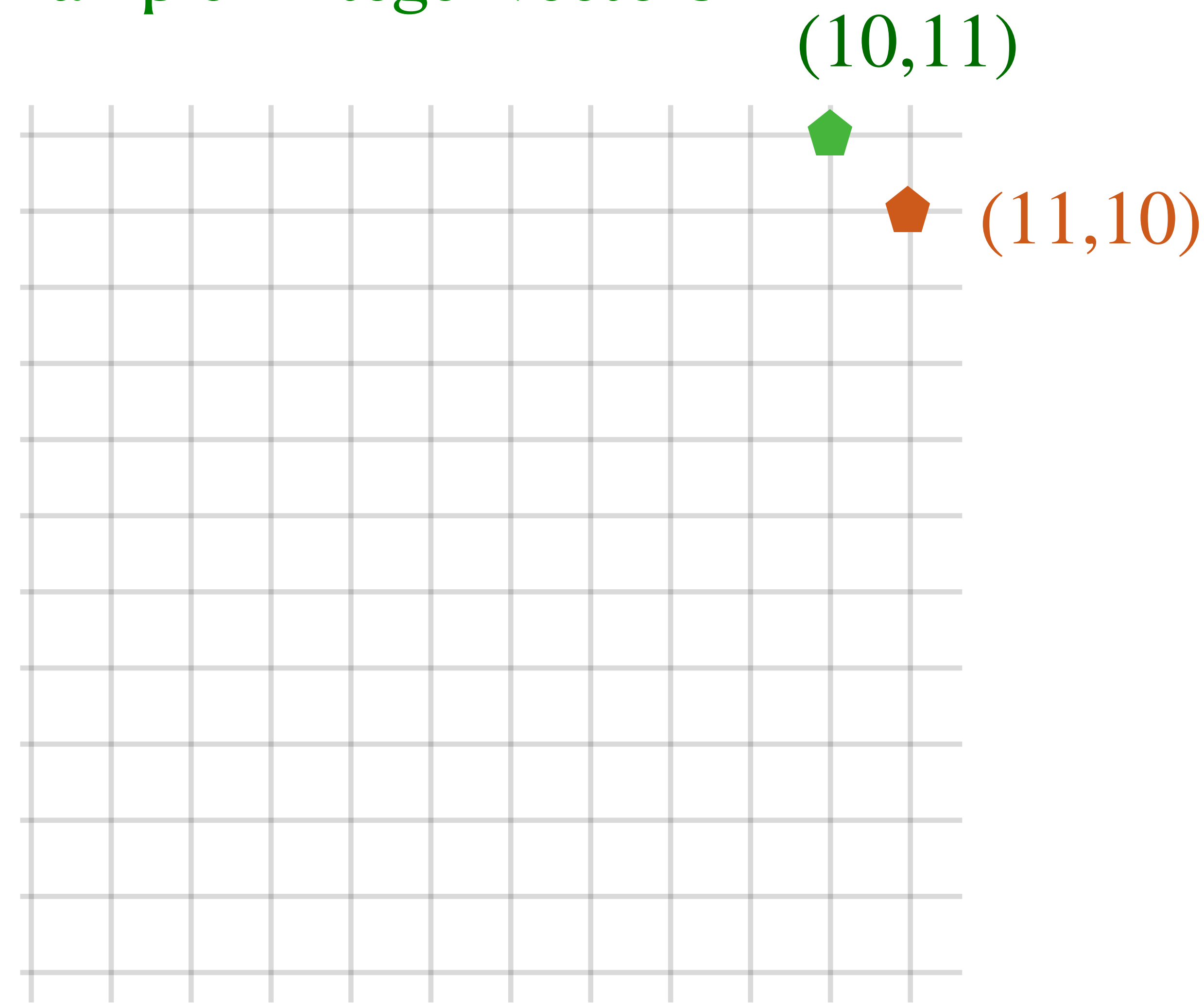
T - a finitely generated submonoid of G.

The membership problem for $T \leq G$ is decidable if there's an algorithm solving:

INPUT: A element $\pi(w) \in G$.

QUESTION: $\pi(w) \in T$?

Example - integer vectors



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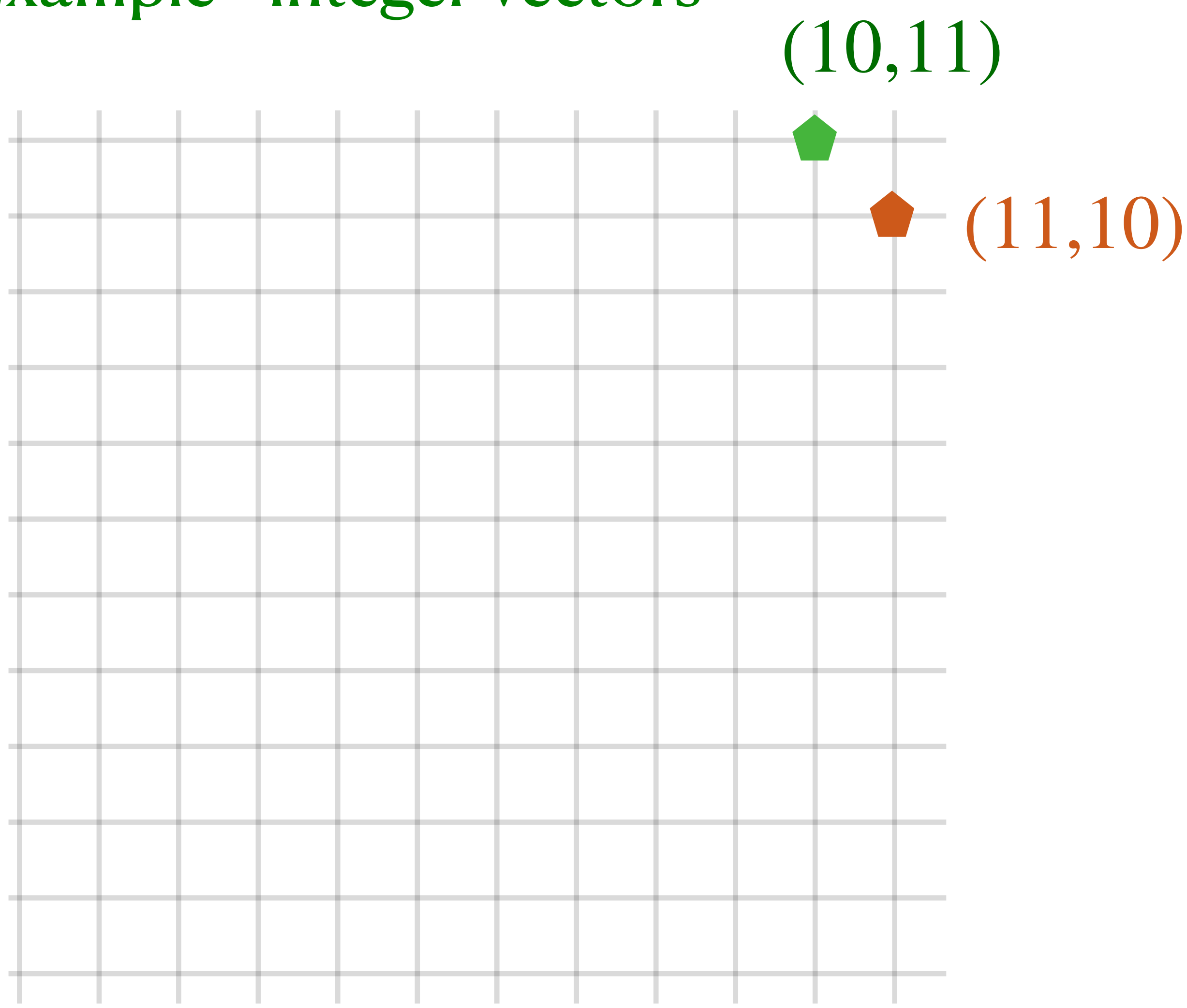

$$T = \langle (2,4), (4,-1) \rangle \leq \mathbb{Z} \times \mathbb{Z} = G$$

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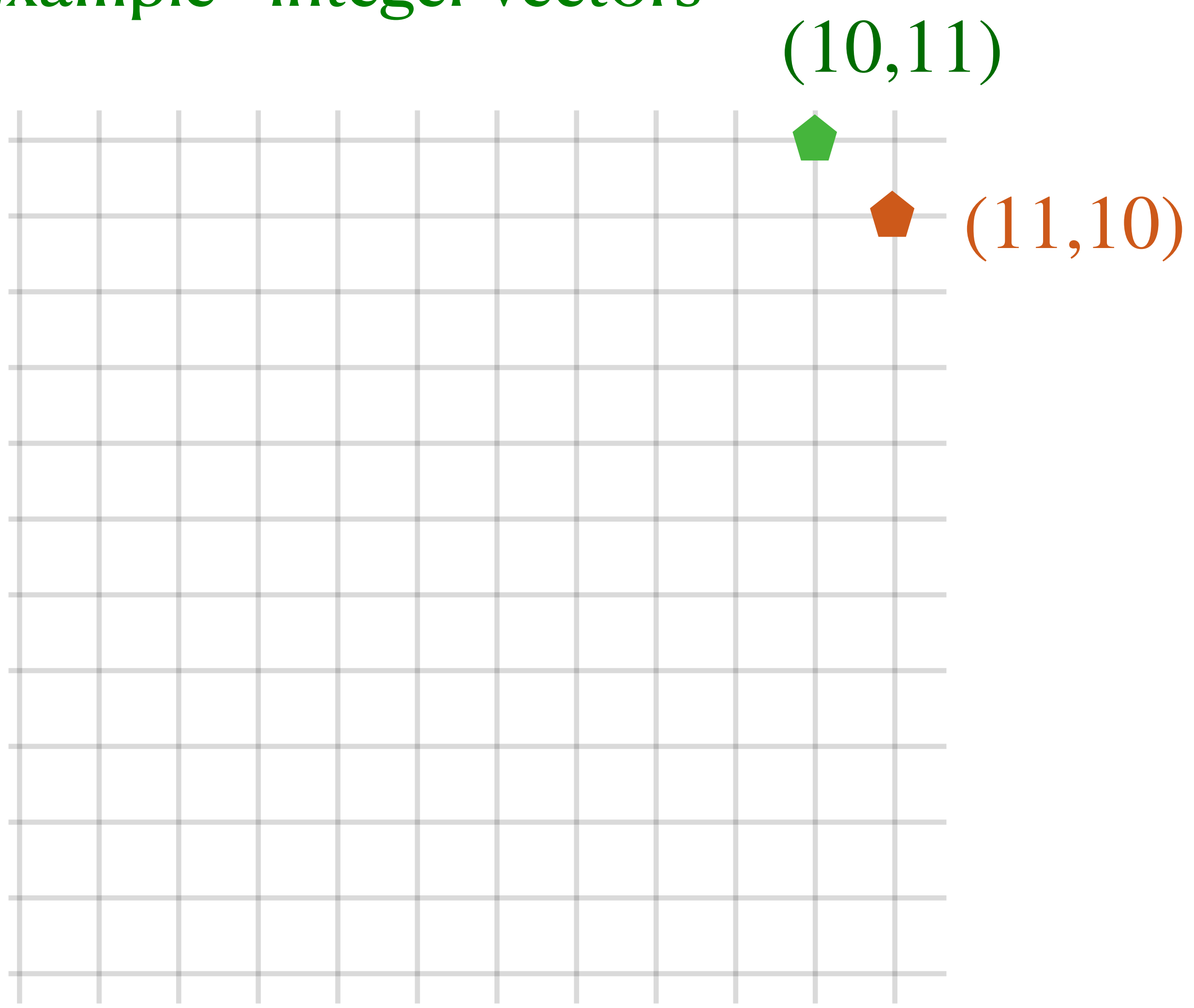

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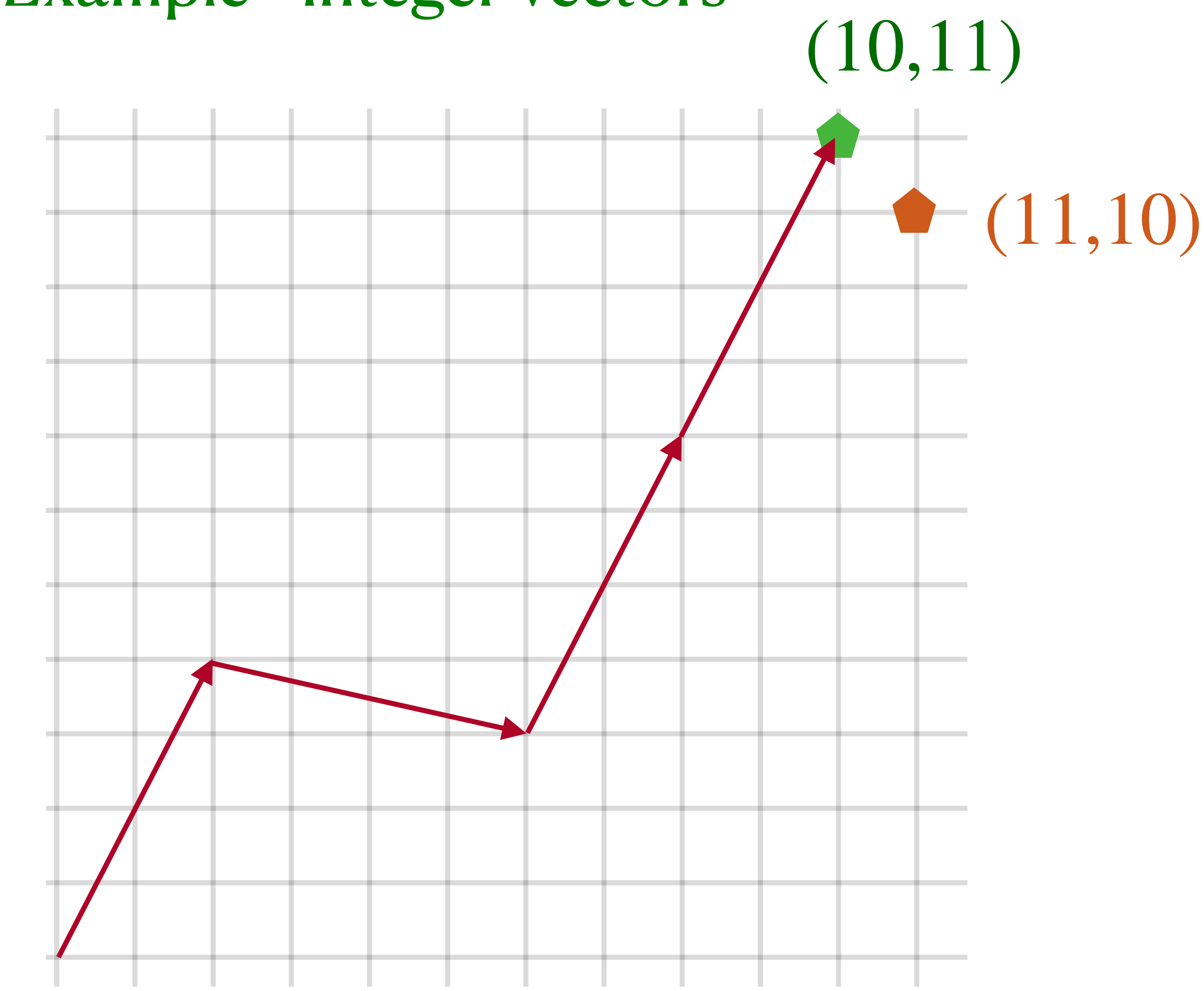

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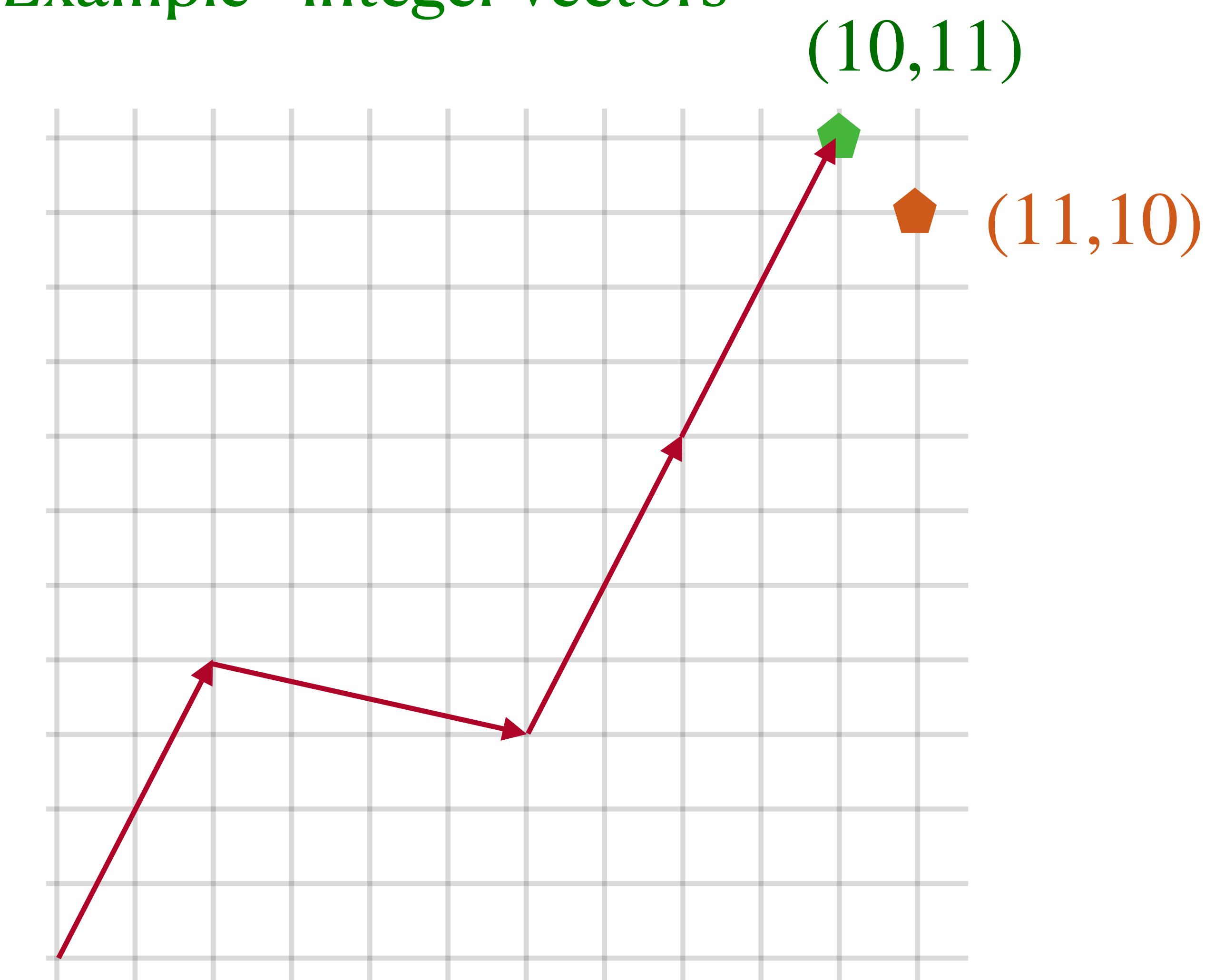

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$(11,10) \notin T$

Positive one-relator groups

Theorem (Perrin & Schupp, 1984) A finitely generated group has a one-relation presentation with a positive relator $\Leftrightarrow$ it has a one-relator monoid presentation.

$\therefore$ A positive one-relator group with undecidable submonoid membership problem would also give an example of a one-relator monoid with the same property.

One-relator groups

$\mathrm{Gp}\langle a, b \mid aba^{-1}b^{-1} = 1\rangle$
$= \mathbb{Z} \times \mathbb{Z}$

One-relator monoids

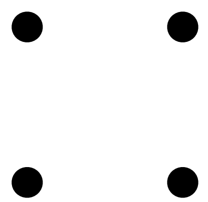
$\mathrm{Mon}\langle a, b \mid baba = a\rangle$ $\quad$ $\mathrm{Mon}\langle b, c \mid bc = 1\rangle$

Positive one-relator groups

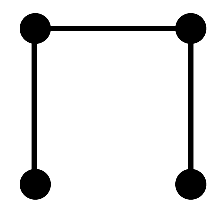
$\mathrm{Gp}\langle A \mid w = 1\rangle$

$\mathrm{Gp}\langle a, b \mid a^2b^2 = 1\rangle$
$\cong \mathrm{Mon}\langle a, x \mid a^2(xa)^2 = 1\rangle$

Submonoid membership problem in Right-angled Artin groups



$\mathrm{Gp}\langle a_i \mid \ \rangle$



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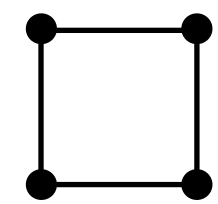
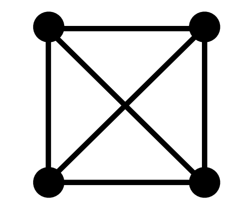


$\mathrm{Gp}\langle a_i \mid a_i a_{i+1} = a_{i+1} a_i \pmod 4 \rangle$



$\mathrm{Gp}\langle a_i \mid a_i a_j = a_j a_i \rangle$

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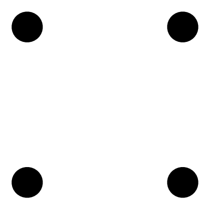

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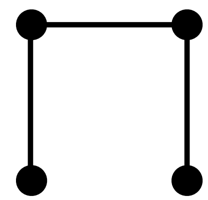

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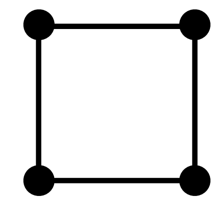

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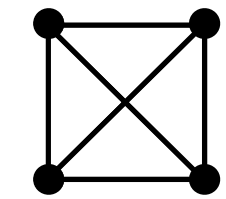

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Benois (1969)

Free groups

DECIDABLE

Grunschlag (1999)

f.g. abelian groups

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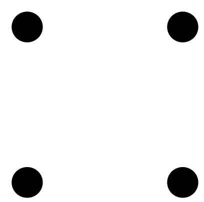

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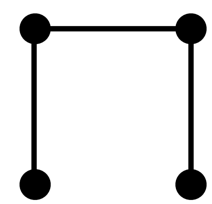

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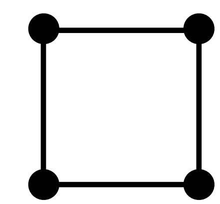
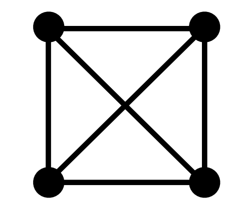


$\mathrm{Gp}\langle a_i \mid a_i a_{i+1} = a_{i+1} a_i \ (\mathrm{mod}\ 4) \rangle$

Mikhailova (1958)

$A(C_4) = F_2 \times F_2$

UNDECIDABLE



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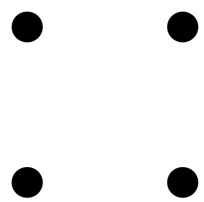

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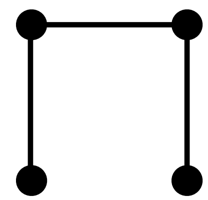

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Lohrey & Steinberg (2008)

$A(P_4)$

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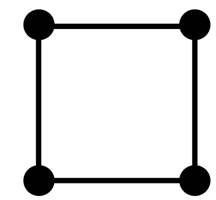

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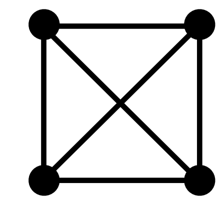

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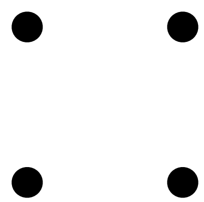

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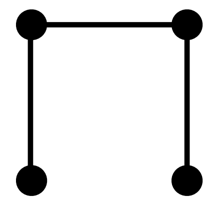
DECIDABLE

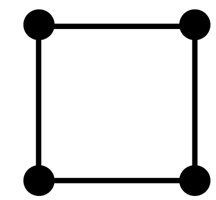


$\mathrm{Gp}\langle a_i \mid a_i a_{i+1} = a_{i+1} a_i \rangle$

Lohrey & Steinberg (2008)

$A(P_4)$

UNDECIDABLE

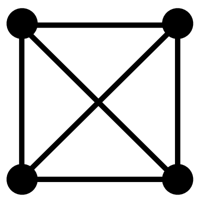


$\mathrm{Gp}\langle a_i \mid a_i a_{i+1} = a_{i+1} a_i \ (\mathrm{mod}\ 4) \rangle$

Mikhailova (1958)

$A(C_4) = F_2 \times F_2$

UNDECIDABLE



$\mathrm{Gp}\langle a_i \mid a_i a_j = a_j a_i \rangle$

Grunschlag (1999)

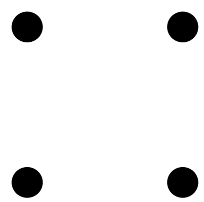
f.g. abelian groups

DECIDABLE

Definition $A(\Gamma) = \mathrm{Gp}\langle V\Gamma \mid uv = vu$ if and only if $\{u, v\} \in E\Gamma \rangle$.

Corollary If a group G embeds $A(C_4)$ or $A(P_4)$ then G has undecidable submonoid membership problem.

Submonoid membership problem in Right-angled Artin groups

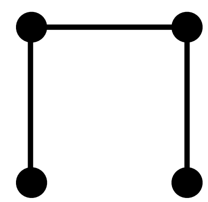


$\mathrm{Gp}\langle a_i \mid \ \rangle$

Benois (1969)

Free groups

DECIDABLE



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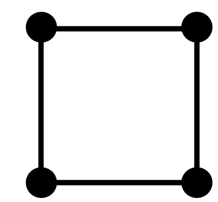

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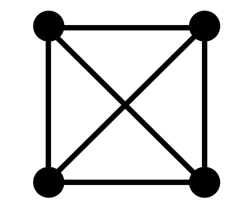

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Corollary If a group G embeds $A(C_4)$ or $A(P_4)$ then G has undecidable submonoid membership problem.

Louder and Wilton (2017): $A(C_4)$ cannot embed in a one-relator group.

Submonoid membership problem in one-relator groups

$\mathrm{Gp}\langle a, b \mid \ \rangle$ $\qquad$ $\mathrm{Gp}\langle a, t \mid a(tat^{-1}) = (tat^{-1})a\rangle$ $\qquad$ $\mathrm{Gp}\langle a, b \mid ab = ba\rangle$

Submonoid membership problem in one-relator groups

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Benois (1969):

$\underbrace{\text{Free groups}}$

DECIDABLE

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RDG (2020)

UNDECIDABLE

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f.g. abelian groups

DECIDABLE

Theorem (RDG (2020)) The one-relator group $\mathrm{Gp}\langle a, t \mid [a, tat^{-1}] = 1 \rangle$ has undecidable submonoid membership problem.

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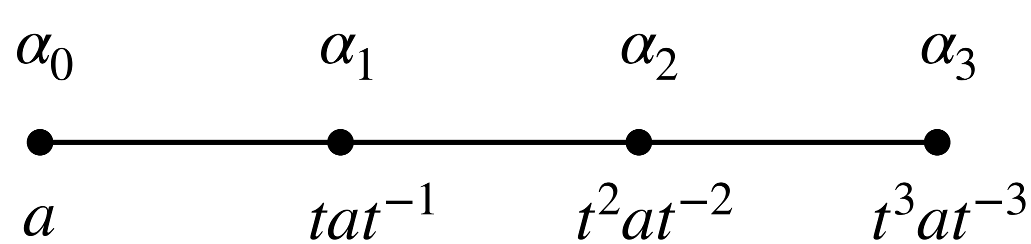
Theorem (RDG (2020)) The one-relator group $\mathrm{Gp}\langle a, t \mid [a, tat^{-1}] = 1 \rangle$ has undecidable submonoid membership problem.

Proof idea

$$A(P_4) = \mathrm{Gp}\langle \alpha_i \mid \alpha_i\alpha_{i+1} = \alpha_{i+1}\alpha_i \rangle \hookrightarrow \mathrm{Gp}\langle a, t \mid a(tat^{-1}) = (tat^{-1})a \rangle = G$$

$\theta : \alpha_i \mapsto t^i a t^{-i}$



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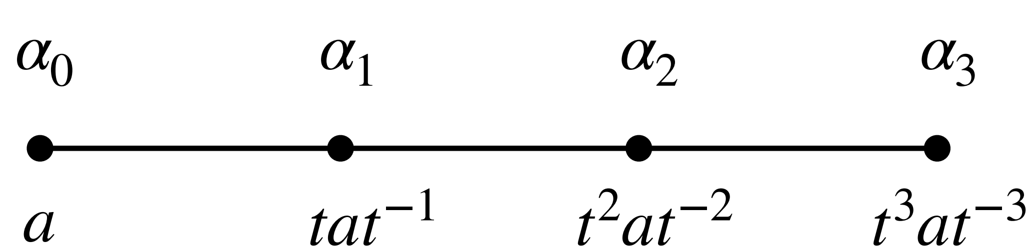


θ is proved an embedding using HNN extensions.

Submonoid membership problem in one-relator groups

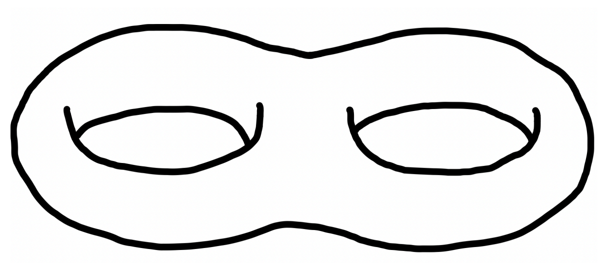
RDG (2020): $\mathrm{Gp}\langle a, t \mid [a, tat^{-1}] = 1 \rangle$

UNDECIDABLE

Surface groups

$\mathrm{Gp}\langle a, b, c, d \mid [a, b][c, d] = 1 \rangle$

$\mathrm{Gp}\langle a, b, c \mid a^2b^2c^2 = 1 \rangle$



UNKNOWN

Baumslag–Solitar groups $BS(m, n)$

$\mathrm{Gp}\langle a, b \mid ba^m = a^nb \rangle, m, n \in \mathbb{Z}$

Cadilhac et al. (2020):

Baumslag–Solitar groups $BS(1,n)$

Benois (1969): Free groups

DECIDABLE

$\mathrm{Gp}\langle a, b \mid [a, b] = 1 \rangle = \mathbb{Z}^2$

Submonoid membership problem in one-relator groups

RDG (2020): $\mathrm{Gp}\langle a, t \mid [a, tat^{-1}] = 1 \rangle$

UNDECIDABLE

Nyberg-Brodda (2022):

$\mathrm{Gp}\langle a, b \mid [a^m, b^n] = 1 \rangle$ for $m, n \geq 2$.

Surface groups

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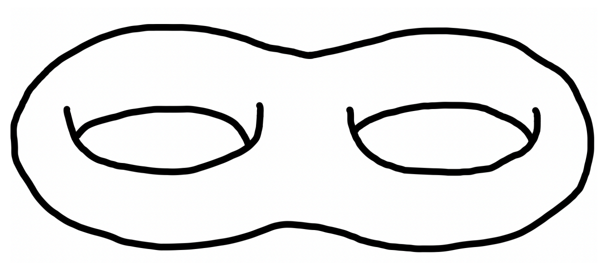

UNKNOWN

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Lyndon's Identity Theorem implies none of these groups are positive.

Surface groups

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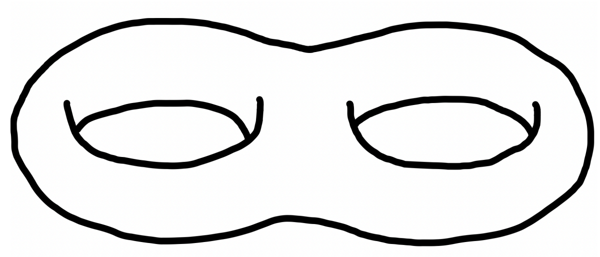

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Benois (1969): Free groups

DECIDABLE

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(I) Positive one-relator groups

Submonoid membership problem for positive one-relator groups	Guba (1997) $\longrightarrow$	Word problem for all $\mathrm{Mon}\langle a, b \mid bUa = a \rangle$ decidable

(I) Positive one-relator groups

Submonoid membership problem for positive one-relator groups

Undecidable

Guba (1997) $\longrightarrow$

Word problem for all $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ decidable

Theorem (Foniqi, RDG, Nyberg-Brodda, 2023) For all $m, n \geq 2$,

$\mathrm{Gp}\langle a, b \mid (ba^n)^m(a^nb)^m = 1\rangle$ has undecidable submonoid membership problem.

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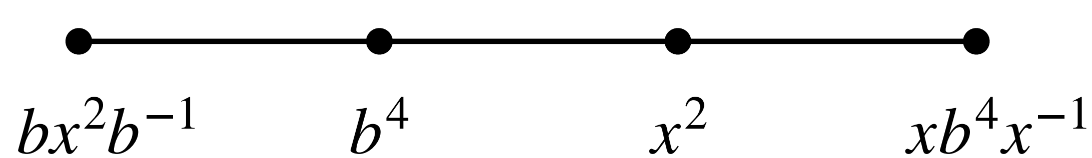
Proof idea: Show $H = \mathrm{Gp}\langle a, b \mid (ab^2)^2(b^2a)^2 = 1 \rangle \cong \mathrm{Gp}\langle x, b \mid b^2x^2b^{-2} = x^{-2} \rangle$.

Observe: $b^4x^2b^{-4} = x^2$.

Then find a copy of $A(P_4)$ in H generated by:



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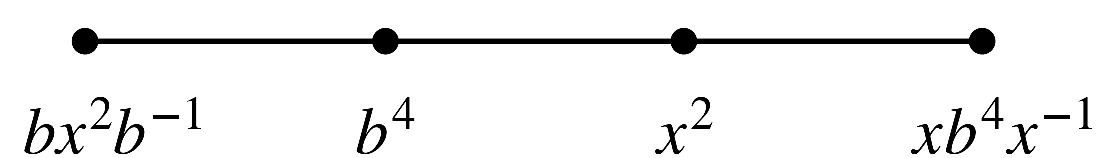

Proved an embedding using Reidemeister–Schreier rewriting, free products with amalgamation + results from Nyberg-Brodda (2022) on RABSAGs.

Membership problem in one-relator monoids

Theorem (Foniqi, RDG, Nyberg-Brodda, 2023)
There are one-relator monoids with undecidable submonoid membership problem, including:

(1) Examples that are groups;

(2) Non-group examples with defining relation $w = 1$;

(3) Examples with relation $u = v$ where $u \neq 1$ & $v \neq 1$.

One-relator groups

RDG (2020):

$\mathrm{Gp}\langle a, t \mid [a, tat^{-1}] = 1\rangle$

Nyberg-Brodda (2022):

$\mathrm{Gp}\langle a, b \mid [a^m, b^n] = 1\rangle$

for $m, n \geq 2$.

One-relator monoids

(3) Non-special

$\mathrm{Mon}\langle A \mid u = v\rangle$

UNDECIDABLE

(2) Special one-relator monoids

$\mathrm{Mon}\langle A \mid w = 1\rangle$

UNDECIDABLE

(1) Positive one-relator groups

$\mathrm{Gp}\langle A \mid w = 1\rangle$

$\mathrm{Gp}\langle a, b \mid (ba^n)^m(a^nb)^m = 1\rangle$

$\parallel$

$\mathrm{Mon}\langle a, b \mid (ba^n)^m(a^nb)^m = 1\rangle$

UNDECIDABLE

(II) Prefix membership problem

Prefix membership problem
for $\mathrm{Gp}\langle A \mid uv^{-1} = 1\rangle$
$u, v \in A^+$, uv^{-1} reduced.

Guba (1997) $\Rightarrow$

Word problem for all
$\mathrm{Mon}\langle a, b \mid bUa = a\rangle$
decidable

(II) Prefix membership problem

Prefix membership problem for $\mathrm{Gp}\langle A \mid uv^{-1} = 1\rangle$ $u, v \in A^+, uv^{-1}$ reduced.	Guba (1997) $\Rightarrow$	Word problem for all $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ decidable

Definition The prefix membership problem for $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ asks if we can decide membership in the submonoid $\mathscr{P}_G$ generated by the prefixes of r.

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Theorem (Guba 1997) For every $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ there is a group $G = \mathrm{Gp}\langle x, y, C \mid xWyx^{-1} = 1\rangle$, with W positive, such that if G has decidable prefix membership problem then M has decidable word problem.

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- Prefix membership problem for $\mathrm{Gp}\langle u, v \mid uv^{-1} = 1\rangle$ solved positively in several cases: Margolis, Meakin, and Sunic (2005); Inam (2017), Dolinka and Gray (2021).

(II) Prefix membership problem

Prefix membership problem for $\mathrm{Gp}\langle A \mid uv^{-1} = 1 \rangle$ $u, v \in A^+$, uv^{-1} reduced.

Undecidable

Guba (1997)

Word problem for all $\mathrm{Mon}\langle a, b \mid bUa = a \rangle$ decidable

Theorem (Foniqi, RDG, Nyberg-Brodda, 2023) There is a one-relator group $\mathrm{Gp}\langle A \mid uv^{-1} = 1 \rangle$, with $u, v \in A^+$ and uv^{-1} reduced, with an undecidable prefix membership problem.

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Theorem (Foniqi, RDG, Nyberg-Brodda, 2023) There is a finite alphabet A and positive words $\alpha, \beta \in A^+$ such that

(1) $\mathrm{Inv}\langle A \mid \alpha = 1, \beta = 1\rangle$ has undecidable word problem; and

(2) $\mathrm{Gp}\langle A \mid \alpha = 1, \beta = 1\rangle$ has undecidable prefix membership problem; and

(3) $\mathrm{Gp}\langle A \mid \underbrace{\alpha\beta}_{u}\,\underbrace{\alpha^{-1}}_{v^{-1}} = 1\rangle$ has undecidable prefix membership problem.

(III) Rational subset membership

Principal right ideal membership problem in $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$

Adian & Oganesian (1987)

Guba 1997

Word problem for $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ decidable

Rational subset membership problem for $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$

Definition (Rational subsets of a monoid M)
Is the smallest collection of subsets containing all finite subsets of M and is closed under the operations: union, product, and Kleene hull (submonoid generation).

Examples

- F.g. submonoids of M.
- Finite subsets of M.
- Principal right ideals sM.
- Finite unions of these.
- ...

(III) Rational subset membership

Rational subset membership problem for $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$

Undecidable

Adian & Oganesian (1987)

Word problem for $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ decidable

Theorem (Foniqi, RDG, Nyberg-Brodda, 2023)

There is a one-relator monoid of the form $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ that contains a fixed rational subset in which membership is undecidable.

(III) Rational subset membership

Rational subset membership problem for Mon$\langle a, b \mid bUa = a\rangle$

Undecidable

Adian & Oganesian (1987) $\Rightarrow$

Word problem for Mon$\langle a, b \mid bUa = a\rangle$ decidable

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Notes on proof:

(III) Rational subset membership

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Rational subset membership problem for $\mathrm{Mon}\langle a, b \mid bUa = a \rangle$

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so 1 is the unique idempotent in M. Also, the group of units of M is trivial.

(III) Rational subset membership

Rational subset membership problem for $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$

Undecidable

Adian & Oganesian (1987)

$\Longrightarrow$

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- Hence the group $A(P_4)$ does not embed in M!

(III) Rational subset membership

Rational subset membership problem for $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$

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Notes on proof:

- Adian (1960) $\Rightarrow M = \mathrm{Mon}\langle a, b \mid bUa = a\rangle$ is left cancellative i.e.

$$\alpha\beta = \alpha\gamma \Rightarrow \beta = \gamma$$

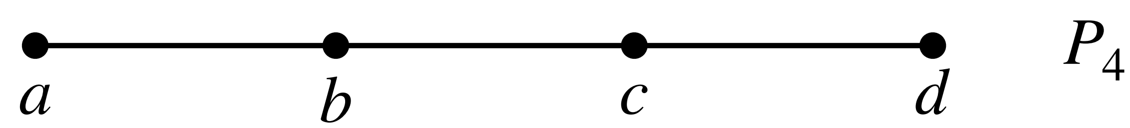
  so 1 is the unique idempotent in M. Also, the group of units of M is trivial.

- Hence the group $A(P_4)$ does not embed in M!

- To prove our theorem we introduce a new condition for undecidability by embedding trace semigroups in $\mathscr{L}$-classes of left-cancellative monoids.

A theorem about trace monoids

Definition



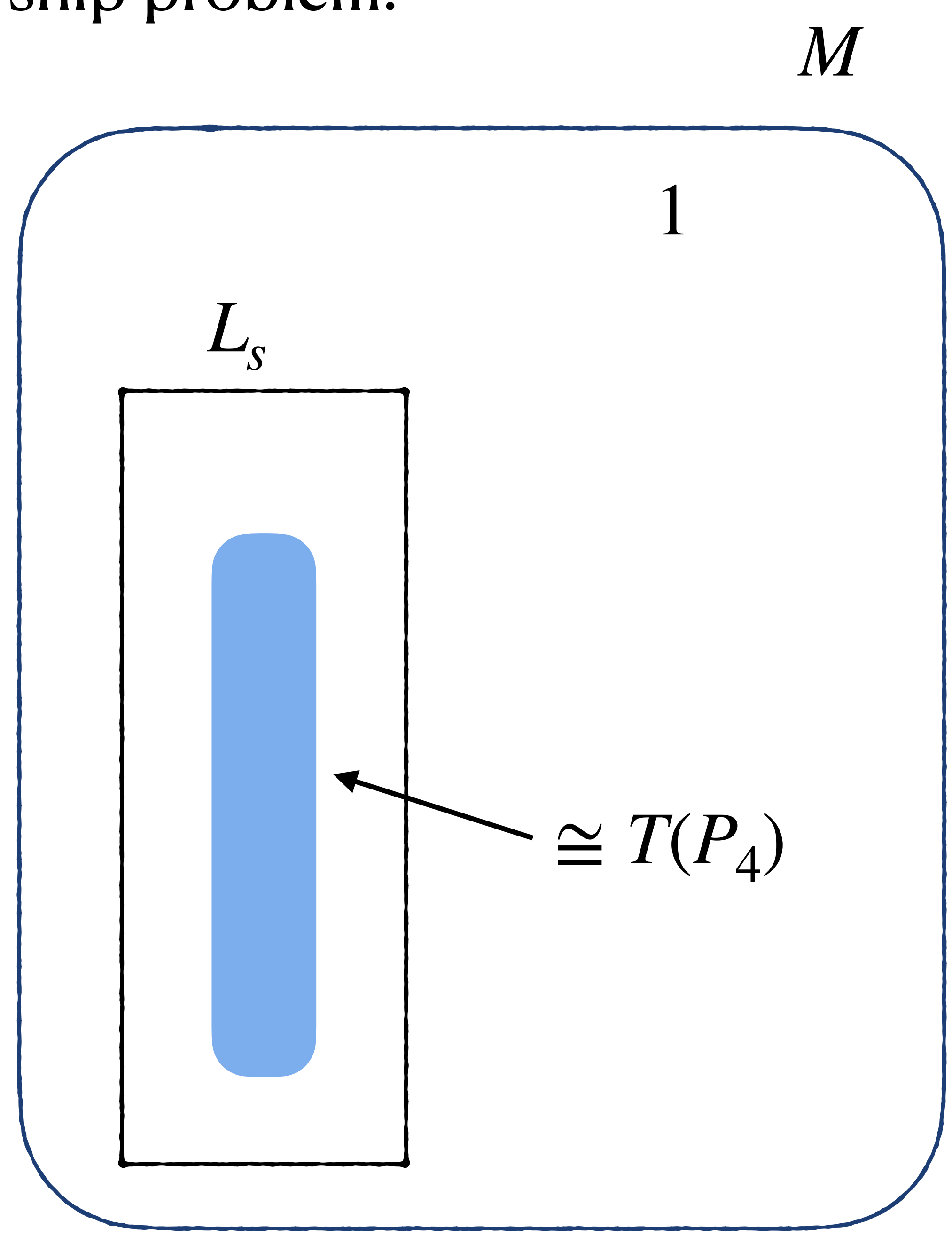
The trace semigroup $T(P_4) = \mathrm{Sgp}\langle a, b, c, d \mid ab = ba,\ bc = cb,\ cd = dc\rangle$.

Fact $T(P_4)$ has decidable rational subset membership problem.

Green's relation: $s\mathscr{L}t \Leftrightarrow Ms = Mt$.

Theorem (Foniqi, RDG, Nyberg-Brodda, 2023)

If a left-cancellative monoid M embeds a copy of $T(P_4)$ that is contained in an $\mathscr{L}$-class of M, then M contains a fixed rational subset in which membership is undecidable.



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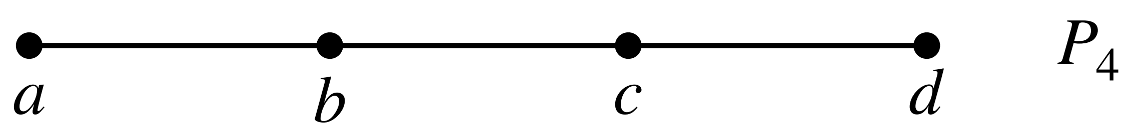

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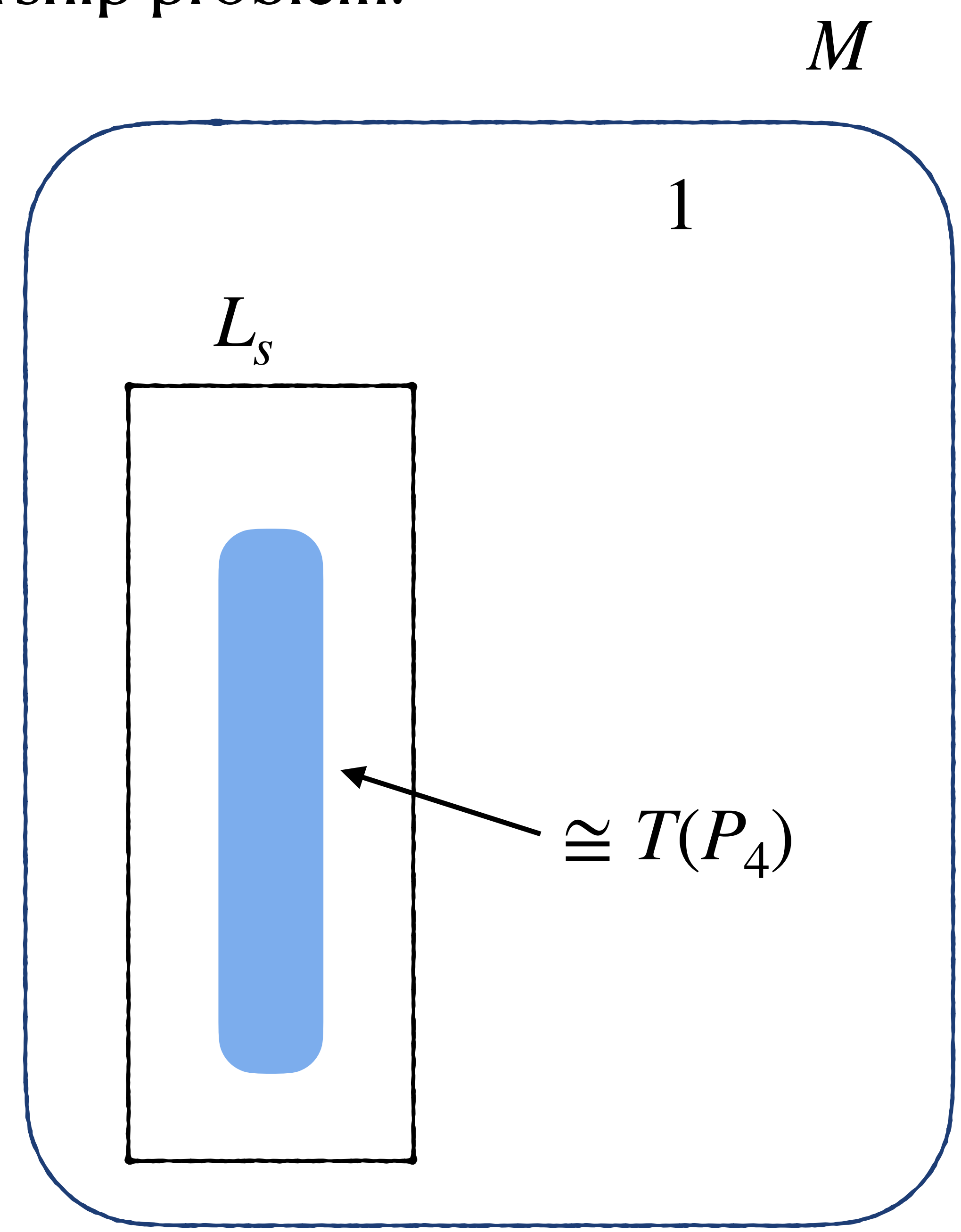
Theorem (Foniqi, RDG, Nyberg-Brodda, 2023)
There is a monoid $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ such that the $\mathscr{L}$-class L_a embeds $T(P_4)$, and this monoid is left-cancellative.



An application to groups

Definition

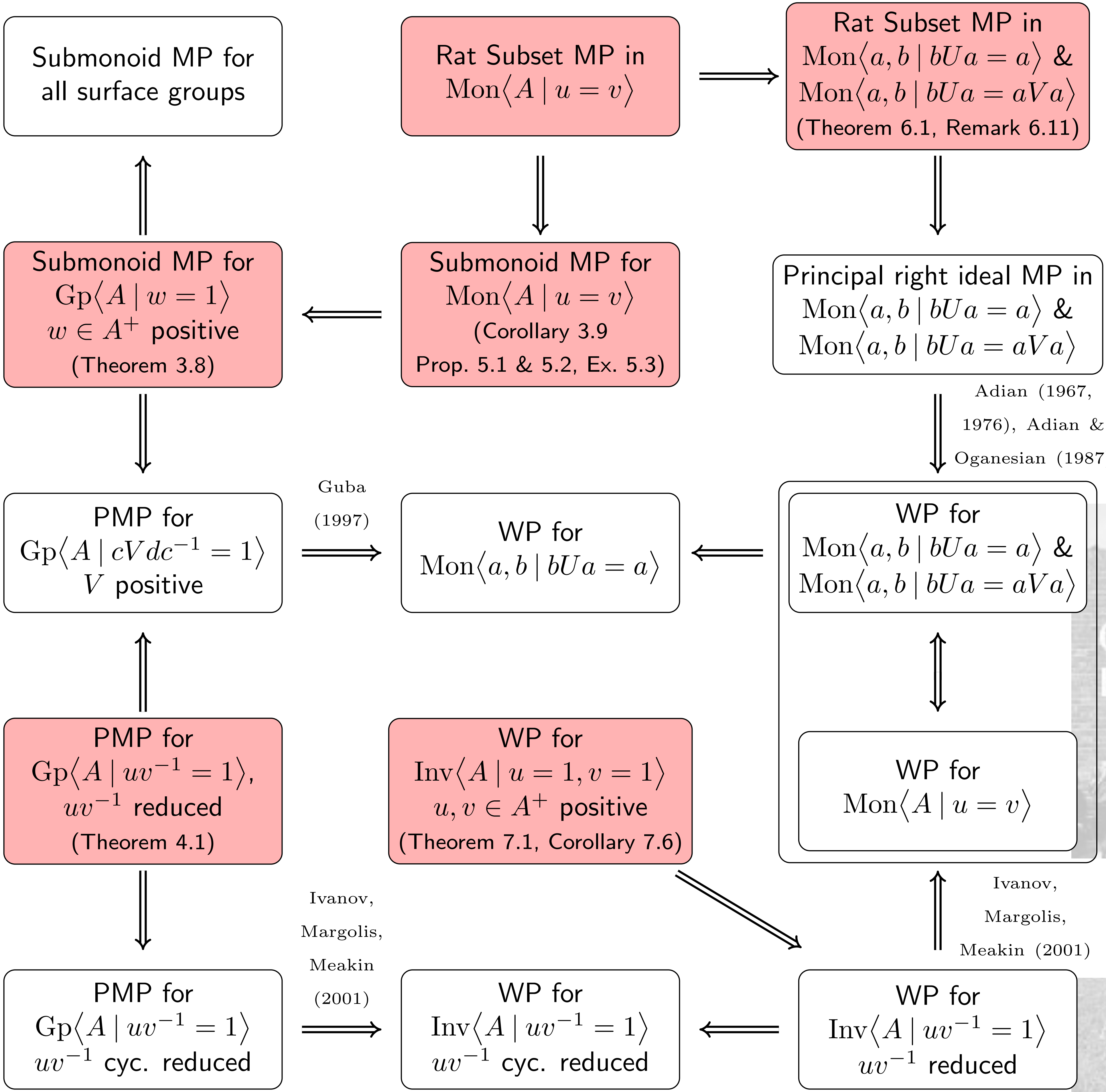
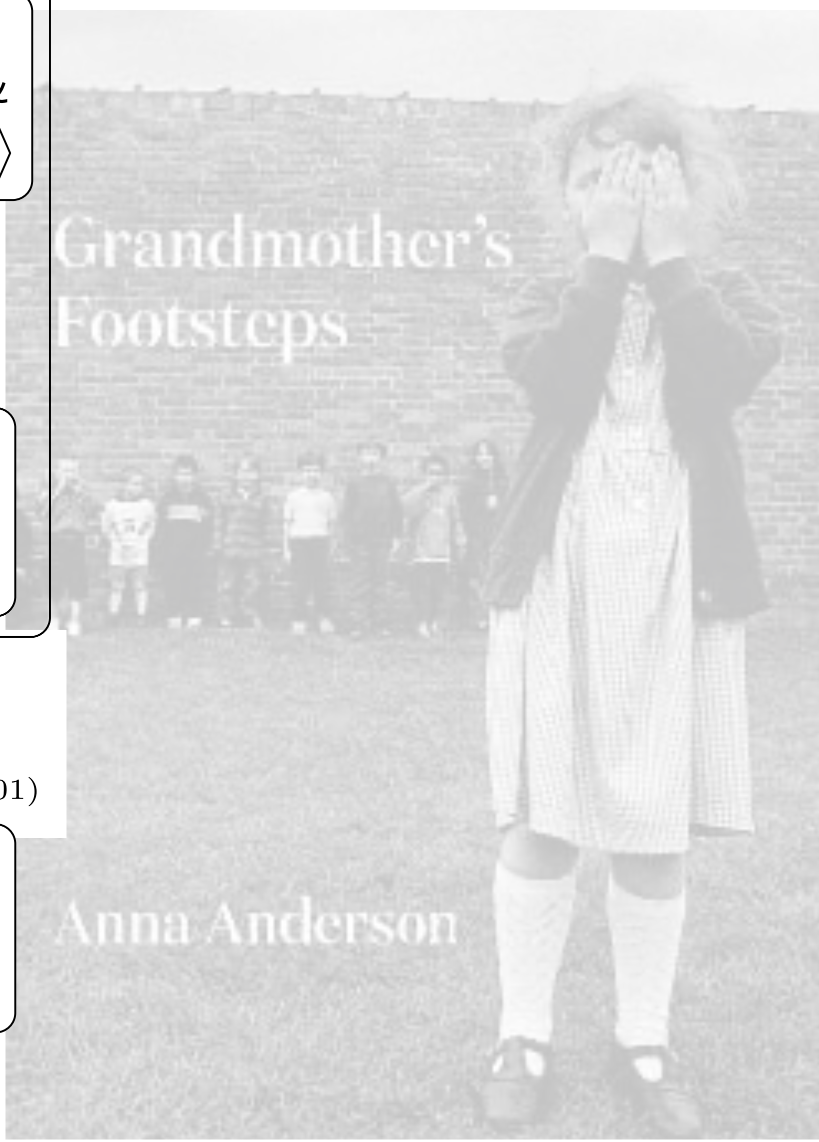
P_4

The trace semigroup $T(P_4) = \mathrm{Sgp}\langle a, b, c, d \mid ab = ba,\ bc = cb,\ cd = dc \rangle$.

Corollary (Foniqi, RDG, Nyberg-Brodda, 2023)

If G is a finitely generated group which embeds $T(P_4)$ then G contains a rational subset in which rational subset membership is undecidable.

- There are examples of $\{a, b, c, d\} \subseteq G$ such that
  - The subsemigroup generated by $\{a, b, c, d\}$ is $T(P_4)$ but
  - The subgroup generated by $\{a, b, c, d\}$ is not $A(P_4)$.


Submonoid MP for all surface groups
Rat Subset MP in Mon⟨A | u = v⟩
Rat Subset MP in Mon⟨a, b | bUa = a⟩ & Mon⟨a, b | bUa = aVa⟩ (Theorem 6.1, Remark 6.11)
Submonoid MP for Gp⟨A | w = 1⟩ w ∈ A+ positive (Theorem 3.8)
Submonoid MP for Mon⟨A | u = v⟩ (Corollary 3.9 Prop. 5.1 & 5.2, Ex. 5.3)
Principal right ideal MP in Mon⟨a, b | bUa = a⟩ & Mon⟨a, b | bUa = aVa⟩
Adian (1967, 1976), Adian & Oganesian (1987)
PMP for Gp⟨A | cVdc−1 = 1⟩ V positive
Guba (1997)
WP for Mon⟨a, b | bUa = a⟩
WP for Mon⟨a, b | bUa = a⟩ & Mon⟨a, b | bUa = aVa⟩
WP for Mon⟨A | u = v⟩
PMP for Gp⟨A | uv−1 = 1⟩, uv−1 reduced (Theorem 4.1)
WP for Inv⟨A | u = 1, v = 1⟩ u, v ∈ A+ positive (Theorem 7.1, Corollary 7.6)
Ivanov, Margolis, Meakin (2001)
PMP for Gp⟨A | uv−1 = 1⟩ uv−1 cyc. reduced
Ivanov, Margolis, Meakin (2001)
WP for Inv⟨A | uv−1 = 1⟩ uv−1 cyc. reduced
WP for Inv⟨A | uv−1 = 1⟩ uv−1 reduced

Grandmother's Footsteps
Anna Anderson

Or are we playing Squid Games...

Open problems

- Is there a monoid $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ such that membership in the $\mathscr{L}$-class L_a is undecidable ? Important because such a monoid would have undecidable word problem by:

Word problem for $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ decidable $\xrightarrow{\text{Guba (1997)}}$ Membership problem for L_a in $\mathrm{Mon}\langle a, b \mid bUa = a\rangle$ decidable

Note: There exist $\mathrm{Inv}\langle A \mid w = 1\rangle$ such that membership in L_1 is undecidable.

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$$\boxed{\begin{array}{c}\text{Word problem for}\\ \mathrm{Mon}\langle a, b \mid bUa = a\rangle\\ \text{decidable}\end{array}} \xrightarrow{\text{Guba (1997)}} \boxed{\begin{array}{c}\text{Membership problem for } L_a\\ \text{in } \mathrm{Mon}\langle a, b \mid bUa = a\rangle\\ \text{decidable}\end{array}}$$

Note: There exist $\mathrm{Inv}\langle A \mid w = 1\rangle$ such that membership in L_1 is undecidable.

- Does every group $G = \mathrm{Gp}\langle x, y, C \mid xWyx^{-1} = 1\rangle$ with W positive positive have decidable prefix membership problem?

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- Let $\mathrm{Gp}\langle A \mid r = 1\rangle$. Is membership in $\langle A\rangle$ decidable? i.e. is there an algorithm that decides if a given word can be written positively?