Universal locally finite maximally homogeneous semigroups

Robert D. Gray[1]
(joint work with I. Dolinka)

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University of East Anglia

Hall's group

In 1959 Philip Hall constructed a countably infinite group $\mathcal{U}$ with the following properties:

- Universal: contains every finite group as a subgroup
- Locally finite: every finitely generated subgroup is finite
- Homogeneous: every isomorphism $\phi : A \to B$ between finite subgroups A, B of $\mathcal{U}$ extends to an automorphism of $\mathcal{U}$. In fact, any two isomorphic subgroups of $\mathcal{U}$ are conjugate in $\mathcal{U}$.

$\mathcal{U}$ is the unique countable group satisfying these properties.

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AAA83, Novi Sad, 2012, Manfred Droste asked:

"Is there a countable universal locally finite homogeneous semigroup?"

Constructing Hall's group

Example: Let $G = S_4$, the symmetric group, and

$$K = \{(),\ (1\ 2)\}, \quad L = \{(),\ (1\ 2)(3\ 4)\}.$$

Then $K, L \leq G$, with $K \cong L$ but they are not conjugate in G.

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$$g \mapsto \rho_g, \quad x\rho_g = xg \quad \text{for} \quad x \in G.$$

Now $\phi(K)$ and $\phi(L)$ are conjugate in $S_G = S_{S_4}$.

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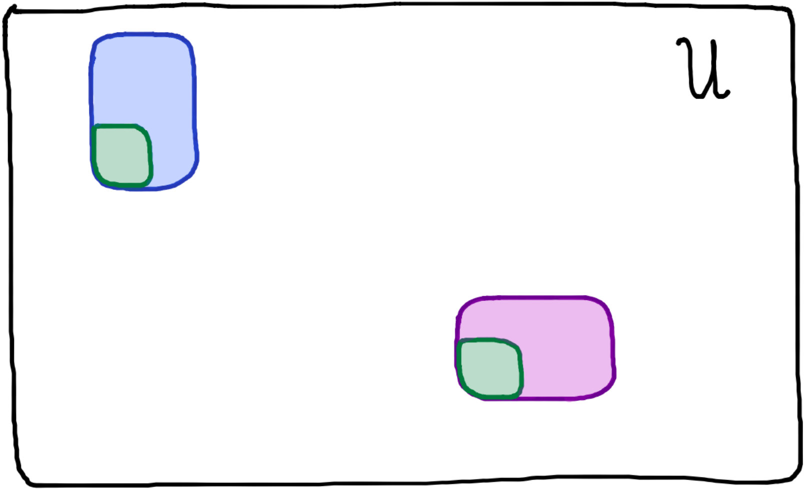
Construct $\mathcal{U}$ by iterating this process

Set $G_0 = S_4$, $G_1 = S_{S_4}$, $G_2 = S_{S_{S_4}}, \ldots$ and let $\phi : G_i \to G_{i+1}$ be given by the right regular representation $g \mapsto \rho_g$, giving

$$G_0 \xrightarrow{\phi_0} G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \ldots$$

Then $\mathcal{U} = \bigcup_{i \geq 0} G_i$ is the direct limit of this chain of symmetric groups.

Amalgamation



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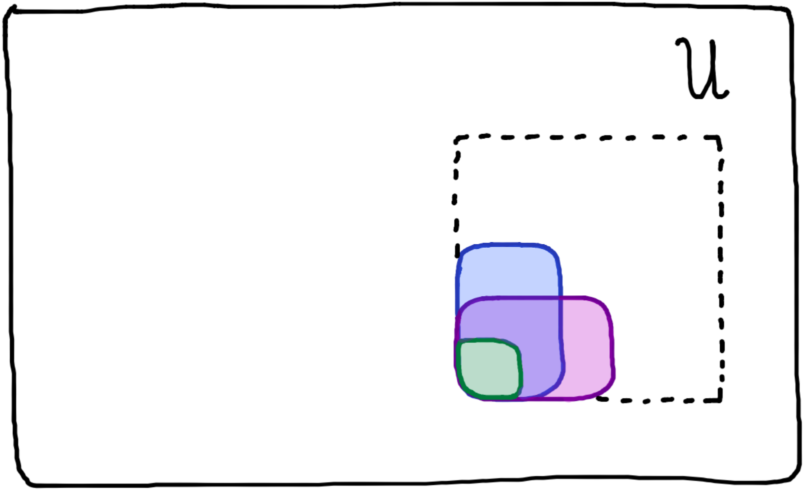
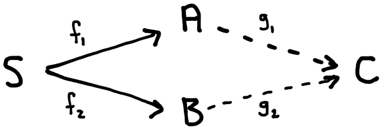


Amalgamation and Fraïssé's Theorem

Definition (Amalgamation property for a class $\mathcal{C}$)

If $S, A, B \in \mathcal{C}$ and $f_1 : S \to A$ and $f_2 : S \to B$ are embeddings then $\exists C \in \mathcal{C}$ and embeddings $g_1 : A \to C$ and $g_2 : B \to C$ such that $f_1 g_1 = f_2 g_2$.



- The class of finite groups has the amalgamation property. It is an *amalgamation class* and its Fraïssé limit is $\mathcal{U}$.
- Fraïssé's Theorem implies that a countable homogeneous structure is uniquely determined by its finitely generated substructures (called its *age*).

Conclusion: Hall's group $\mathcal{U}$ is the unique countable homogeneous locally finite group.

Locally finite structures with maximal symmetry

Groups	Inverse semigroups	Semigroups
Permutations $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$	Partial bijections $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & - & 2 & - \end{pmatrix}$	Transformations $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 2 \end{pmatrix}$
S_n-limit $S_{n_1} \leq S_{n_2} \leq \dots$	I_n-limit $I_{n_1} \leq I_{n_2} \leq \dots$	T_n-limit $T_{n_1} \leq T_{n_2} \leq \dots$
$\mathcal{U}$ (Hall's group)	$\mathcal{I}$	$\mathcal{T}$

General philosophy

Even though neither $\mathcal{T}$ nor $\mathcal{I}$ is homogeneous, they still display a high degree of symmetry in their combinatorial and algebraic structure.

Amalgamation bases for finite semigroups

Kimura (1957): The class of finite semigroups does *not have* the amalgamation property. Therefore, there is no countable universal locally finite homogeneous semigroup.

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> *"How homogeneous can a countable universal locally finite semigroup be?"*

Definition. A finite semigroup S is an amalgamation base for all finite semigroups if in the class of finite semigroups every

$$\begin{tikzcd} & A \\ S \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$$

can be completed to some

$$\begin{tikzcd} A \arrow[dr] & \\ & C \\ B \arrow[ur] & \end{tikzcd}$$

The class $\mathcal{B}$ of all such semigroups contains all finite:

groups, inverse semigroups whose principal ideals form a chain, full transformation semigroups T_n (K. Shoji (2016))

Maximal homogeneity

$\mathcal{B} = \{S : S \text{ is an amalgamation base for all finite semigroups}\}$

T – a countable universal locally finite semigroup,
S – a finite semigroup.

Definition

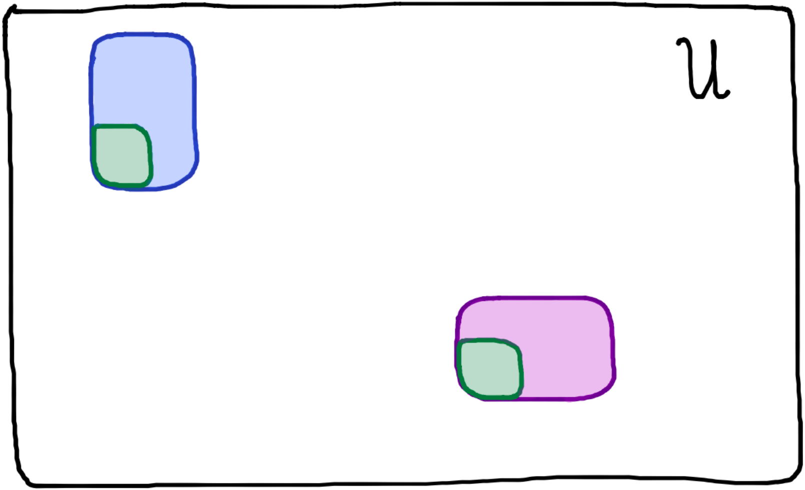
We say $\mathrm{Aut}(T)$ acts homogeneously on copies of S in T if for all $U_1, U_2 \leq T$ with $U_1 \cong S \cong U_2$, every isomorphism $\phi : U_1 \to U_2$ extends to an automorphism of T.

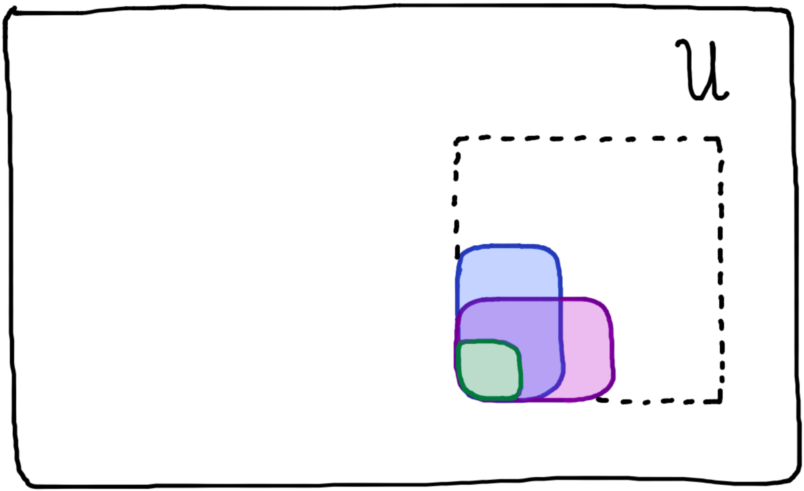
Proposition

$\mathrm{Aut}(T)$ acts homogeneously on copies of S in $T \implies S \in \mathcal{B}$

Definition

We say T is maximally homogeneous if, *for all* $S \in \mathcal{B}$, $\mathrm{Aut}(T)$ acts homogeneously on copies of S in T.


$\mathcal{U}$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The maximally homogeneous semigroup $\mathcal{T}$

T_n = the full transformation semigroup of all maps from $[n] = \{1, 2, \dots n\}$ to itself under composition.

Definition

If we have a chain

$$M_0 \to M_1 \to M_2 \to \dots$$

of embeddings of semigroups, where each $M_i \cong T_{n_i}$, then the limit $T = \bigcup_{i \geq 0} M_i$ is a full transformation limit semigroup.

Fact: Every infinite full transformation limit semigroup is universal and locally finite.

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Theorem (Dolinka & RDG (2017))

There is a unique maximally homogeneous full transformation limit semigroup $\mathcal{T}$.

Existence and uniqueness of $\mathcal{T}$

Theorem (Dolinka & RDG (2017))

There is a unique maximally homogeneous full transformation limit semigroup $\mathcal{T}$.

- Since $\mathcal{T}$ is not homogeneous it cannot be constructed using Fraïssé's Theorem.
- We instead make use of a well-known generalisation, sometimes called the Hrushovski construction.
  - See D. Evans's Lecture notes from his talks at the Hausdorff Institute for Mathematics, Bonn, September 2013.
- $\mathcal{T}$ is not obtainable by iterating Cayley's theorem for semigroups

$$T_n \to T_{T_n} \to T_{T_{T_n}} \to \dots$$

Structure of T_n

$$\alpha \mathscr{J} \beta \quad \Leftrightarrow \quad \alpha \,\&\, \beta \text{ generate the same ideal}$$
$$\Leftrightarrow \quad |\mathrm{im}\ \alpha| = |\mathrm{im}\ \beta|.$$

Set $J_r = \{\alpha \in T_n : |\mathrm{im}\ \alpha| = r\}$.

Each idempotent ϵ in J_r is contained in a maximal subgroup H_ϵ of S_r.

Example

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix} \in T_4$$

$$H_\epsilon = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & k \end{pmatrix} : \{i, j, k\} = \{1, 2, 3\} \right\}$$

lnotesf8.png

Structure of the maximally homogeneous semigroup $\mathcal{T}$

Theorem (Dolinka & RDG (2017))

1. $\mathcal{T}$ is countable universal and locally finite.
2. $\mathcal{T}/\mathscr{J}$ is a chain isomorphic to $(\mathbb{Q}, \leq)$.
3. Every maximal subgroup is isomorphic to Hall's group $\mathcal{U}$.
4. $\mathrm{Aut}(\mathcal{T})$ acts transitively on the set of $\mathscr{J}$-classes of $\mathcal{T}$ (so all principal factors $\mathscr{J}^*$ are isomorphic to each other).

Graham–Houghton graphs – local structure

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 3 & 5 & 2 & 3 \end{pmatrix},$$

$\ker \alpha = 1\ 4 \mid 2\ 3\ 6 \mid 5$

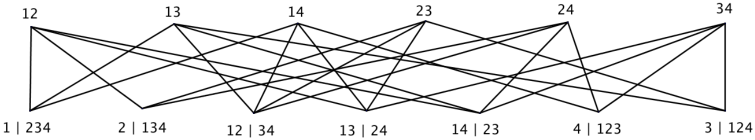
$$\begin{aligned} \alpha \mathscr{R} \beta \quad &\Leftrightarrow \quad \alpha \text{ \& } \beta \text{ generate same right ideal} \\ &\Leftrightarrow \quad \ker\ \alpha = \ker\ \beta. \\ \alpha \mathscr{L} \beta \quad &\Leftrightarrow \quad \alpha \text{ \& } \beta \text{ generate same left ideal} \\ &\Leftrightarrow \quad \operatorname{im} \alpha = \operatorname{im} \beta. \\ \mathscr{H} \quad &= \quad \mathscr{R} \cap \mathscr{L} \end{aligned}$$

	12	13	14	23	24	34
123|4			*		*	*
124|3		*		*		*
134|2	*			*	*	
234|1	*	*	*			
12|34		*	*	*	*	
13|24	*		*	*		*
14|23	*	*			*	*

I - r-element set, P - partition with r parts

$H_{P,I}$ is a group $\Leftrightarrow$ $H_{P,I}$ contains an idempotent $\Leftrightarrow$ I a transversal of P

$\Gamma(J_2)$

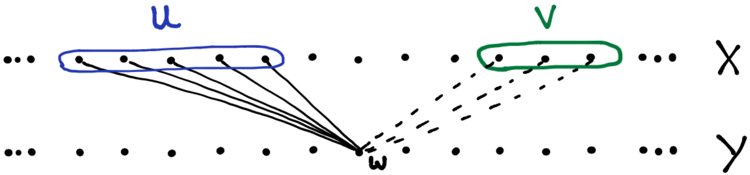


Graham–Houghton graphs in $\mathcal{T}$

Definition (The countable random bipartite graph)

It is the unique countable universal homogeneous bipartite graph. It is characterised as the countably infinite bipartite graph satisfying:

($\ast$) for any two finite disjoint sets U, V from one part of the bipartition, there is a vertex w in the other part with $w \sim U$ but $w \nsim V$.



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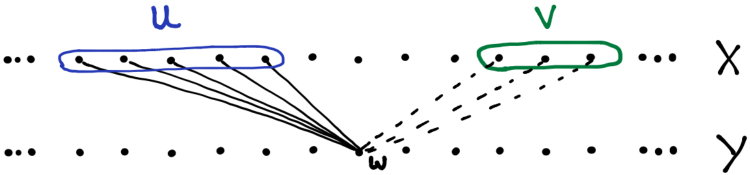
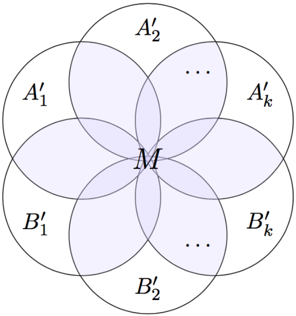
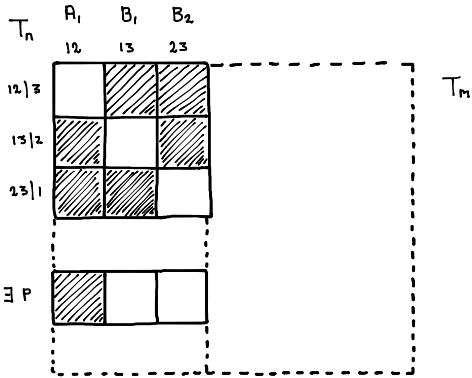


Theorem (Dolinka & RDG (2017))

Every Graham–Houghton graph of $\mathcal{T}$ is isomorphic to the countable random bipartite graph.

The flower lemma





Lemma. Let $A_1, \dots, A_k$, $B_1, \dots, B_l$ be t-element subsets of $\{1, \dots, m\}$. If $|M| < t$ then there exists a partition P of $[m]$ with t parts: $P \perp A_i$ and $P \not\perp B_j$.

Proposition. Let $1 < r < n$. Then $\exists \phi : T_n \to T_m$ such that $\forall a_1, \dots, a_k$, $b_1, \dots, b_l \in J_r \subseteq T_n$ from distinct $\mathscr{L}$-classes $\exists c \in T_m$ such that in T_m

- $R_c \cap L_{a_i\phi}$ are groups
- $R_c \cap L_{b_j\phi}$ are not groups

Locally finite structures with maximal symmetry

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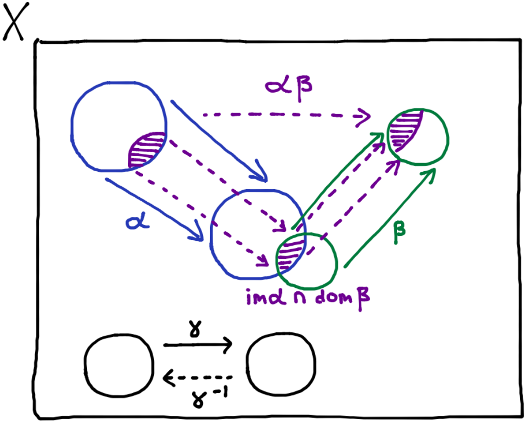
$\mathcal{S}_n$-limit $\mathcal{S}_{n_1} \leq \mathcal{S}_{n_2} \leq \ldots$	$\mathcal{I}_n$-limit $\mathcal{I}_{n_1} \leq \mathcal{I}_{n_2} \leq \ldots$	$\mathcal{T}_n$-limit $\mathcal{T}_{n_1} \leq \mathcal{T}_{n_2} \leq \ldots$
$\mathcal{U}$ (Hall's group)	$\mathcal{I}$	$\mathcal{T}$

General philosophy

Even though neither $\mathcal{T}$ nor $\mathcal{I}$ is homogeneous, they still display a high degree of symmetry in their combinatorial and algebraic structure.

The symmetric inverse semigroup

I_X = the semigroup of all partial bijections $X \to X$



Examples: In I_3

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & - & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & 2 \end{pmatrix}$$

Each element α has a unique inverse α^{-1}. Note that

$$\alpha\alpha^{-1} = \mathrm{id}_{\mathrm{dom}\alpha}, \qquad \alpha\alpha^{-1}\alpha = \alpha \text{ and } \alpha^{-1}\alpha\alpha^{-1} = \alpha^{-1}$$

Inverse semigroups

Definition

An inverse semigroup is a semigroup S such that

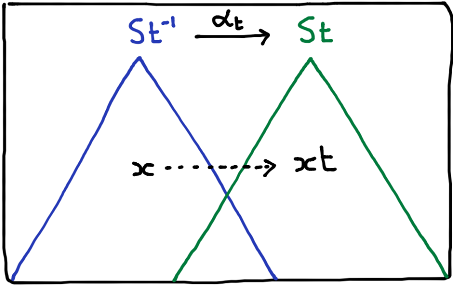
$$(\forall x \in S)(\exists \text{ unique } x^{-1} \in S): \quad xx^{-1}x = x \quad \text{and} \quad x^{-1}xx^{-1} = x^{-1}.$$

Vagner–Preston Theorem

Every inverse semigroup is isomorphic to an inverse subsemigroup of some symmetric inverse semigroup.

For $t \in S$ let
$\alpha_t : St^{-1} \to St,\ x \mapsto xt.$

Then $t \mapsto \alpha_t$
defines an embedding
$S \to I_S$.



Semilattices

Order-theoretic definition

A poset $(P, \leq)$ such that any pair of elements $x, y \in P$ has a greatest lower bound $x \wedge y$.

Algebraic definition

A commutative semigroup $(S, \wedge)$ of idempotents

$$x \wedge y = y \wedge x \quad \text{and}$$
$$x \wedge x = x \quad \text{for all } x, y \in S.$$

- Every semilattice $(E, \wedge)$ is an inverse semigroup were $e^{-1} = e$.
- $E(S) = \{e \in S : e^2 = e\} \leq S$ and is a subsemilattice for any inverse semigroup S.

Roughly speaking: Inverse semigroups = semilattices + groups

– This can be formalised via the notion of inductive groupoid and the Ehresmann-Schein-Nambooripad Theorem.

Amalgamation bases and maximal homogeneity

T. E. Hall, C. J. Ash (1975): The class of finite inverse semigroups does *not have* the amalgamation property.

Theorem (T. E. Hall (1975))

Amalgamation bases for finite inverse semigroups are exactly those whose principal ideals form a chain under inclusion. These are called $\mathscr{J}$-linear inverse semigroups.

T – a countable universal locally finite inverse semigroup,
S – a finite inverse semigroup.

Proposition

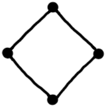
$\mathrm{Aut}(T)$ acts homogeneously on copies of S in $T \implies S$ is $\mathscr{J}$-linear

Definition

We say T is maximally homogeneous if $\mathrm{Aut}(T)$ acts homogeneously on all of its $\mathscr{J}$-linear inverse subsemigroups.

$\mathscr{J}$-linear inverse semigroups

Semilattices

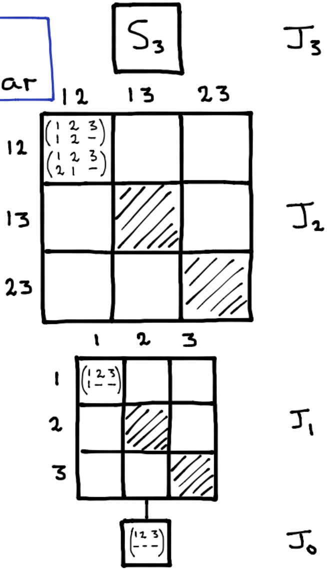


is not $\mathcal{J}$-linear

is $\mathcal{J}$-linear

Groups Every group is $\mathcal{J}$-linear.

I_n is $\mathcal{J}$-linear



The maximally homogeneous semigroup $\mathcal{I}$

$\mathcal{I}_n$ = the symmetric inverse semigroup on $[n] = \{1, 2, \dots n\}$

Definition

If we have a chain

$$M_0 \to M_1 \to M_2 \to \dots$$

of embeddings of inverse semigroups, where each $M_i \cong I_{n_i}$, then the limit $I = \bigcup_{i \geq 0} M_i$ is a symmetric inverse limit semigroup.

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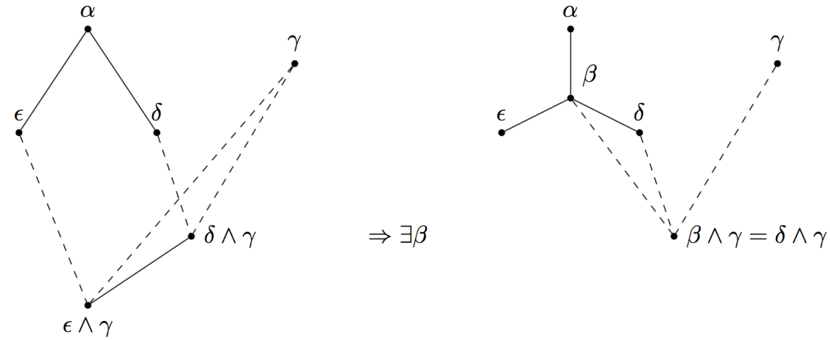
Theorem (Dolinka & RDG (2017))

There is a unique maximally homogeneous symmetric inverse limit semigroup $\mathcal{I}$.

1. $\mathcal{I}$ is locally finite and universal for finite inverse semigroups.
2. $\mathcal{I}/\mathscr{J}$ is a chain isomorphic to $(\mathbb{Q}, \leq)$.
3. Every maximal subgroup if isomorphic to Hall's group $\mathcal{U}$.
4. The semilattice of idempotents $E(\mathcal{I})$ is isomorphic to the universal countable homogeneous semilattice.

The universal countable homogeneous semilattice



Theorem (Albert and Burris (1986), Droste (1992))

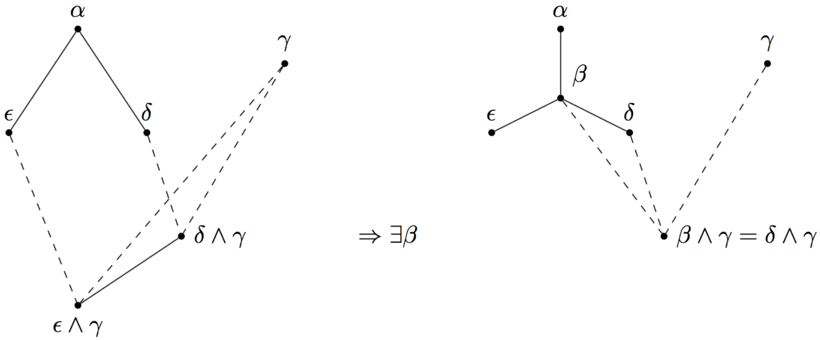
A countable semilattice $(\Omega, \wedge)$ is the universal homogeneous semilattice if and only if the following conditions hold:

(i) no element is maximal or minimal;

(ii) any pair of elements has an upper bound;

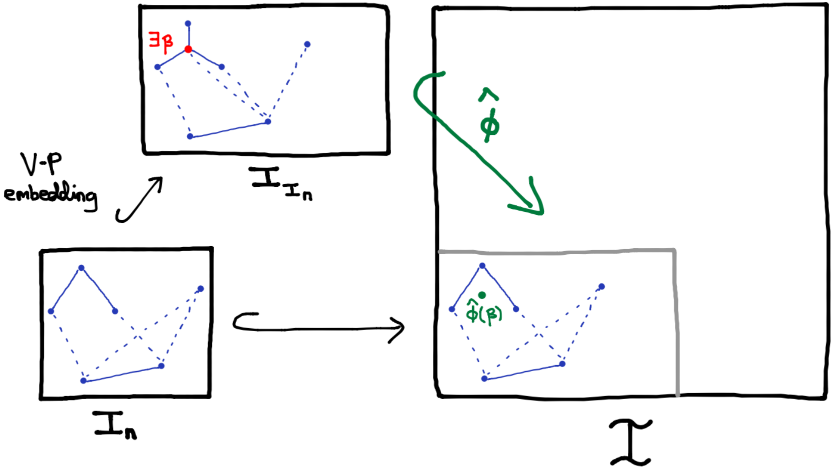
(iii) Ω satisfies axiom $(*)$ illustrated above.

The universal countable homogeneous semilattice



($*$) for any $\alpha, \gamma, \delta, \varepsilon \in \Omega$ such that $\delta, \varepsilon \leq \alpha$, $\gamma \nleq \delta$, $\gamma \nleq \varepsilon$, $\alpha \nleq \gamma$, and either $\delta = \varepsilon$, or $\delta \parallel \varepsilon$ and $\gamma \wedge \varepsilon \leq \gamma \wedge \delta$, there exists $\beta \in \Omega$ such that $\delta, \varepsilon \leq \beta \leq \alpha$ and $\beta \wedge \gamma = \delta \wedge \gamma$ (in particular, $\beta \parallel \gamma$)

$E(\mathcal{I}) \cong$ countable universal homogeneous semilattice



Extension property: Since $\mathrm{Aut}(\mathcal{I})$ acts homogeneously on the finite $\mathscr{J}$-linear substructures of $\mathcal{I}$ any embedding $\phi : I_n \to \mathcal{I}$ extends to an embedding $\hat{\phi} : I_{I_n} \to \mathcal{I}$, where $I_n \leq I_{I_n}$ via Vagner–Preston.

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S_n-limit $S_{n_1} \leq S_{n_2} \leq \dots$	I_n-limit $I_{n_1} \leq I_{n_2} \leq \dots$	T_n-limit $T_{n_1} \leq T_{n_2} \leq \dots$
$\mathcal{U}$ (Hall's group)	$\mathcal{I}$	$\mathcal{T}$

General philosophy

Even though neither $\mathcal{T}$ nor $\mathcal{I}$ is homogeneous, they still display a high degree of symmetry in their combinatorial and algebraic structure.

Open problems about $\mathcal{T}$ and $\mathcal{I}$

We know $\mathcal{T}$ is not obtainable by iterating Cayley's theorem for semigroups

$$T_n \to T_{T_n} \to T_{T_{T_n}} \to \dots$$

Problem 1: *Find a 'nice' description of $\mathcal{T}$ as a T_n-limit semigroup.*

We know that $\mathcal{T}$ embeds every finite semigroup, but

Problem 2: *Does every countable locally finite semigroup embed into $\mathcal{T}$?*

Does there exist a countable locally finite semigroup which embeds every countable locally finite semigroup?

- We ask the analogous questions for the inverse semigroup $\mathcal{I}$.