Universal locally finite maximally homogeneous semigroups

Robert D. Gray[1]
(joint work with I. Dolinka)

Conference to celebrate the 70th Anniversary of Peter J. Cameron, Lisbon, July 2017

University of East Anglia

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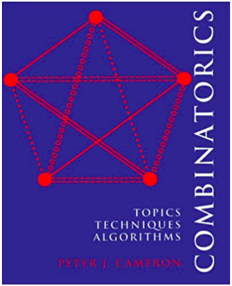
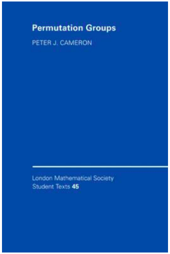
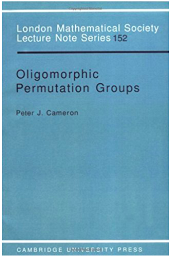
Thank you Peter!







INDEPENDENCE ALGEBRAS

PETER J. CAMERON AND CSABA SZABÓ

ABSTRACT

An independence algebra is an algebra A in which the subalgebras satisfy the exchange axiom, and any map from a basis of A into A extends to an endomorphism. Independence algebras fall into two classes; the first are specified by a set X, a group G, and a G-space C. The second are much more restricted; we show that the subalgebra lattice is a projective or affine geometry, and give a complete classification of the finite algebras.

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INFINITE HIGHLY ARC TRANSITIVE DIGRAPHS AND UNIVERSAL COVERING DIGRAPHS

PETER J. CAMERON, CHERYL E. PRAEGER* and NICHOLAS C. WORMALD†

Received August 18, 1989
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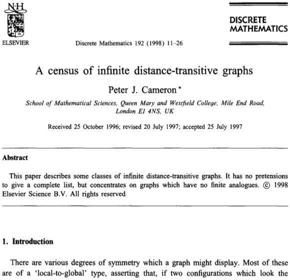
A digraph (that is a directed graph) is said to be highly arc transitive if its automorphism group is transitive on the set of s-arcs for each $s \geq 0$. Several new constructions are given of infinite highly arc transitive digraphs. In particular, for Δ a connected, 1-arc transitive, bipartite digraph, a highly arc transitive digraph $DL(\Delta)$ is constructed and is shown to be a covering digraph for every digraph in a certain class $\mathscr{D}(\Delta)$ of connected digraphs. Moreover, if Δ is locally finite, then $DL(\Delta)$ is a universal covering digraph for $\mathscr{D}(\Delta)$. Further constructions of infinite highly arc transitive digraphs are given.

Thank you Peter!



NH ELSEVIER Discrete Mathematics 192 (1998) 11–26 DISCRETE MATHEMATICS

A census of infinite distance-transitive graphs

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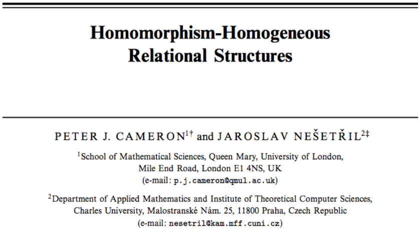
Abstract

This paper describes some classes of infinite distance-transitive graphs. It has no pretensions to give a complete list, but concentrates on graphs which have no finite analogues. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

There are various degrees of symmetry which a graph might display. Most of these are of a 'local-to-global' type, asserting that, if two configurations which look the



Homomorphism-Homogeneous Relational Structures

PETER J. CAMERON[1†] and JAROSLAV NEŠETŘIL[2‡]

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Hall's group

In 1959 Philip Hall constructed a countably infinite group $\mathcal{U}$ with the following properties:

- Universal: contains every finite group as a subgroup
- Locally finite: every finitely generated subgroup is finite
- Homogeneous: every isomorphism $\phi : A \to B$ between finite subgroups A, B of $\mathcal{U}$ extends to an automorphism of $\mathcal{U}$. In fact, any two isomorphic subgroups of $\mathcal{U}$ are conjugate in $\mathcal{U}$.

$\mathcal{U}$ is the unique countable group satisfying these properties.

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AAA83, Novi Sad, 2012, Manfred Droste asked:

"Is there a countable universal locally finite homogeneous semigroup?"

Constructing Hall's group

Example: Let $G = S_4$, the symmetric group, and

$$K = \{(),\ (1\,2)\}, \quad L = \{(),\ (1\,2)(3\,4)\}.$$

Then $K, L \leq G$, with $K \cong L$ but they are not conjugate in G.

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$$g \mapsto \rho_g, \quad x\rho_g = xg \quad \text{for} \quad x \in G.$$

Now $\phi(K)$ and $\phi(L)$ are conjugate in $S_G = S_{S_4}$.

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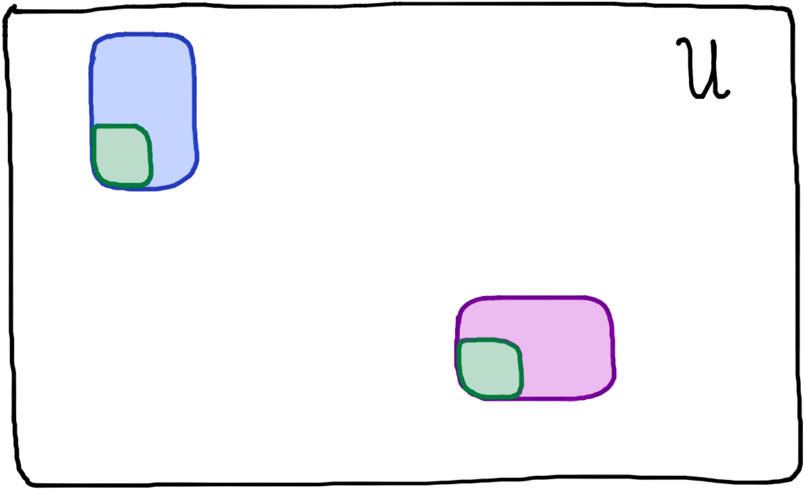
Construct $\mathcal{U}$ by iterating this process

Set $G_0 = S_4$, $G_1 = S_{S_4}$, $G_2 = S_{S_{S_4}}, \ldots$ and let $\phi : G_i \to G_{i+1}$ be given by the right regular representation $g \mapsto \rho_g$, giving

$$G_0 \xrightarrow{\phi_0} G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \ldots$$

Then $\mathcal{U} = \bigcup_{i\geq 0} G_i$ is the direct limit of this chain of symmetric groups.

Amalgamation



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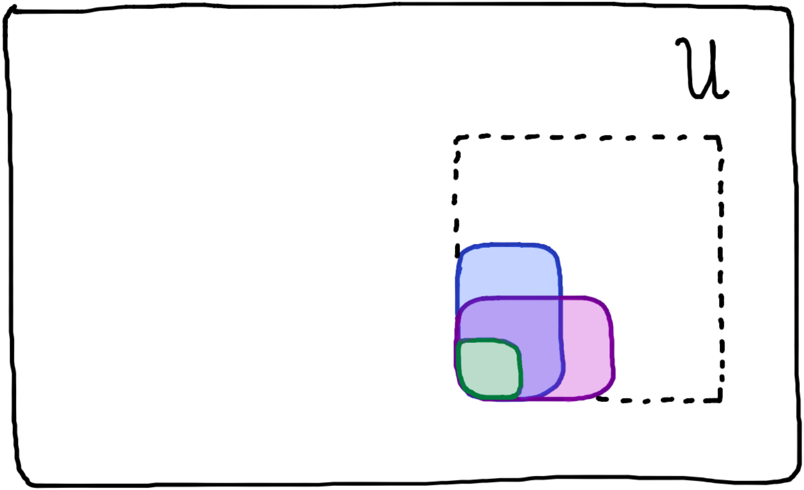

Amalgamation and Fraïssé's Theorem

Definition (Amalgamation property for a class $\mathcal{C}$)

If $S, A, B \in \mathcal{C}$ and $f_1 : S \to A$ and $f_2 : S \to B$ are embeddings then $\exists C \in \mathcal{C}$ and embeddings $g_1 : A \to C$ and $g_2 : B \to C$ such that $f_1 g_1 = f_2 g_2$.

$$\begin{tikzcd}
& A \arrow[dr, dashed, "g_1"] & \\
S \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & C \\
& B \arrow[ur, dashed, "g_2"'] &
\end{tikzcd}$$

- The class of finite groups has the amalgamation property. It is an *amalgamation class* and its Fraïssé limit is $\mathcal{U}$.
- Fraïssé's Theorem implies that a countable homogeneous structure is uniquely determined by its finitely generated substructures (called its *age*).

Conclusion: Hall's group $\mathcal{U}$ is the unique countable homogeneous locally finite group.

Amalgamation bases for finite semigroups

T. E. Hall, C. J. Ash (1975): The class of finite semigroups does *not have* the amalgamation property. Therefore, there is no countable universal locally finite homogeneous semigroup.

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> *"How homogeneous can a countable universal locally finite semigroup be?"*

Definition. A finite semigroup S is an amalgamation base for all finite semigroups if in the class of finite semigroups every

$$\begin{tikzcd} & A \\ S \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$$

can be completed to some

$$\begin{tikzcd} A \arrow[dr] & \\ & C \\ B \arrow[ur] & \end{tikzcd}$$

The class $\mathcal{B}$ of all such semigroups contains all finite:

groups, inverse semigroups whose principal ideals form a chain, full transformation semigroups T_n (K. Shoji (2016))

Maximal homogeneity

$\mathcal{B} = \{S : S$ is an amalgamation base for all finite semigroups$\}$

T – a countable universal locally finite semigroup,
S – a finite semigroup.

Definition

We say $\mathrm{Aut}(T)$ acts homogeneously on copies of S in T if for all $U_1, U_2 \leq T$ with $U_1 \cong S \cong U_2$, every isomorphism $\phi : U_1 \to U_2$ extends to an automorphism of T.

Proposition

$\mathrm{Aut}(T)$ acts homogeneously on copies of S in $T \implies S \in \mathcal{B}$

Definition

We say T is maximally homogeneous if, *for all* $S \in \mathcal{B}$, $\mathrm{Aut}(T)$ acts homogeneously on copies of S in T.

The maximally homogeneous semigroup $\mathcal{T}$

T_n = the full transformation semigroup of all maps from $[n] = \{1, 2, \dots n\}$ to itself under composition.

Definition

If we have a chain

$$M_0 \to M_1 \to M_2 \to \dots$$

of embeddings of semigroups, where each $M_i \cong T_{n_i}$, then the limit $T = \bigcup_{i \geq 0} M_i$ is a full transformation limit semigroup.

Fact: Every infinite full transformation limit semigroup is universal and locally finite.

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Theorem (Dolinka & RDG (2017))

There is a unique maximally homogeneous full transformation limit semigroup $\mathcal{T}$.

Existence and uniqueness of $\mathcal{T}$

Theorem (Dolinka & RDG (2017))

There is a unique maximally homogeneous full transformation limit semigroup $\mathcal{T}$.

- Since $\mathcal{T}$ is not homogeneous it cannot be constructed using Fraïssé's Theorem.
- We instead make use of a well-known generalisation, sometimes called the Hrushovski construction.
    - See D. Evans's Lecture notes from his talks at the Hausdorff Institute for Mathematics, Bonn, September 2013.
- $\mathcal{T}$ is not obtainable by iterating Cayley's theorem for semigroups

$$T_n \to T_{T_n} \to T_{T_{T_n}} \to \dots$$

Structure of T_n

$$\alpha \mathscr{J} \beta \quad \Leftrightarrow \quad \alpha \;\&\; \beta \text{ generate the same ideal}$$
$$\Leftrightarrow \quad |\mathrm{im}\ \alpha| = |\mathrm{im}\ \beta|.$$

Set $J_r = \{\alpha \in T_n : |\mathrm{im}\ \alpha| = r\}$.

Each idempotent ϵ in J_r is contained in a maximal subgroup H_ϵ of S_r.

Example

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix} \in T_4$$

$$H_\epsilon = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & k \end{pmatrix} : \{i, j, k\} = \{1, 2, 3\} \right\}$$

* S_4

	123	124	134	234
12|3|4			*	*
13|2|4		*		*
14|2|3	*			*
23|1|4		*	*	
24|1|3	*		*	
34|1|2	*	*		

	12	13	14	23	24	34
123|4			*		*	*
124|3		*		*		*
134|2	*			*	*	
234|1	*	*	*			
12|34		*	*	*	*	
13|24	*		*	*		*
14|23	*	*			*	*

	1	2	3	4
1234	*	*	*	*

Structure of the maximally homogeneous semigroup $\mathcal{T}$

Main idea

Even though $\mathcal{T}$ is not homogeneous, it still displays a high degree of symmetry in its combinatorial and algebraic structure.

Theorem (Dolinka & RDG (2017))

1. $\mathcal{T}$ is countable universal and locally finite.

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3. Every maximal subgroup is isomorphic to Hall's group $\mathcal{U}$.
4. $\mathrm{Aut}(\mathcal{T})$ acts transitively on the set of $\mathscr{J}$-classes of $\mathcal{T}$ (so all principal factors $\mathscr{J}^*$ are isomorphic to each other).

Graham–Houghton graphs – local structure

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 3 & 5 & 2 & 3 \end{pmatrix},$$

$\ker \alpha = 1\ 4 \mid 2\ 3\ 6 \mid 5$

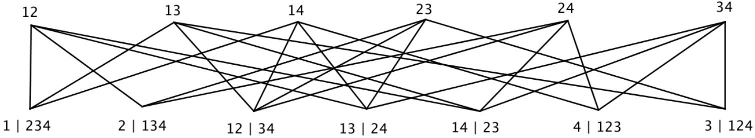
$$\begin{aligned} \alpha \mathscr{R} \beta &\Leftrightarrow \alpha \ \&\ \beta \text{ generate same right ideal} \\ &\Leftrightarrow \ker\ \alpha = \ker\ \beta. \\ \alpha \mathscr{L} \beta &\Leftrightarrow \alpha \ \&\ \beta \text{ generate same left ideal} \\ &\Leftrightarrow \text{im } \alpha = \text{im } \beta. \\ \mathscr{H} &= \mathscr{R} \cap \mathscr{L} \end{aligned}$$

	12	13	14	23	24	34
123|4			*		*	*
124|3		*		*		*
134|2	*			*	*	
234|1	*	*	*			
12|34		*	*	*	*	
13|24	*		*	*		*
14|23	*	*			*	*

I - r-element set, P - partition with r parts

$H_{P,I}$ is a group $\Leftrightarrow$ $H_{P,I}$ contains an idempotent $\Leftrightarrow$ I a transversal of P

$\Gamma(J_2)$

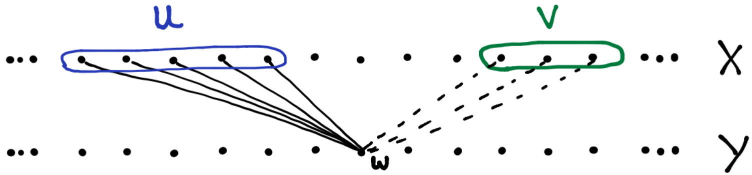


Graham–Houghton graphs in $\mathcal{T}$

Definition (The countable random bipartite graph)

It is the unique countable universal homogeneous bipartite graph. It is characterised as the countably infinite bipartite graph satisfying:

$(*)$ for any two finite disjoint sets U, V from one part of the bipartition, there is a vertex w in the other part with $w \sim U$ but $w \nsim V$.



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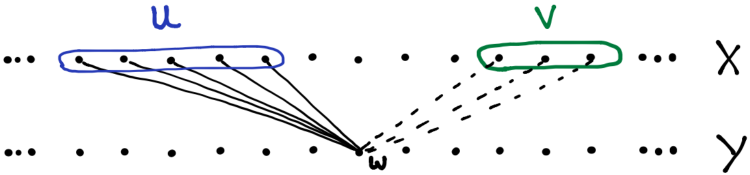
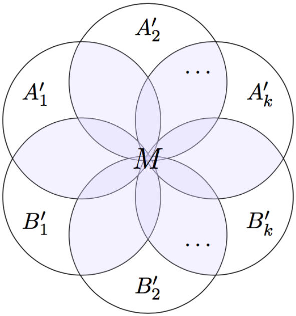
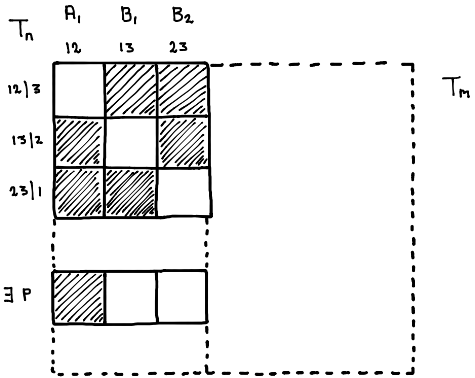


Theorem (Dolinka & RDG (2017))

Every Graham–Houghton graph of $\mathcal{T}$ is isomorphic to the countable random bipartite graph.

The flower lemma





Lemma. Let $A_1, \dots, A_k$, $B_1, \dots, B_l$ be t-element subsets of $\{1, \dots, m\}$. If $|M| < t$ then there exists a partition P of $[m]$ with t parts: $P \perp A_i$ and $P \not\perp B_j$.

Proposition. Let $1 < r < n$. Then $\exists \phi : T_n \to T_m$ such that $\forall a_1, \dots, a_k$, $b_1, \dots, b_l \in J_r \subseteq T_n$ from distinct $\mathscr{L}$-classes $\exists c \in T_m$ such that in T_m

- $R_c \cap L_{a_i \phi}$ are groups
- $R_c \cap L_{b_j \phi}$ are not groups

Inverse semigroups

The symmetric inverse monoid I_n of partial bijections

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & - & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & - \end{pmatrix}$$

T. E. Hall (1975): Amalgamation bases for finite inverse semigroups are precisely the finite $\mathscr{J}$-linear inverse semigroups.

Theorem (Dolinka & RDG (2017))

There is a unique maximally homogeneous symmetric inverse limit semigroup $\mathcal{I}$.

1. $\mathcal{I}$ is locally finite and universal for finite inverse semigroups.
2. $\mathcal{I}/\mathscr{J}$ is a chain isomorphic to $(\mathbb{Q}, \leq)$.
3. Every maximal subgroup if isomorphic to Hall's group $\mathcal{U}$.
4. The semilattice of idempotents $E(\mathcal{I})$ is isomorphic to the universal countable homogeneous semilattice.

Some open problems

We have seen that among T_n-limit semigroups $\mathcal{T}$ is the unique example that is maximally homogeneous.

Problem 1: *Is $\mathcal{T}$ the only countable universal locally finite maximally homogeneous semigroup?*

We know that $\mathcal{T}$ embeds every finite semigroup, but

Problem 2: *Does every countable locally finite semigroup embed into $\mathcal{T}$?*

Problem 3: *Does there exist a countable locally finite semigroup which embeds every countable locally finite semigroup?*

(Note: There exist $2^{\aleph_0}$ non-isomorphic, countable, locally finite, groups, and $\mathcal{U}$ embeds all countable locally finite groups.)