The prefix membership problem for one-relator groups

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One-relator groups

Definition

A one-relator group is a group defined by a presentation of the form

$$\mathrm{Gp}\langle A \mid w = 1\rangle = \mathrm{FG}(A)/\langle\!\langle w\rangle\!\rangle$$

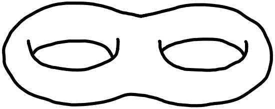
where A is a finite alphabet and $w \in (A \cup A^{-1})^*$.

- Magnus 1932: One-relator groups have decidable word problem.

Example

- $\mathbb{Z} \times \mathbb{Z} = \mathrm{Gp}\langle x, y \mid [x, y] = 1\rangle$
  where $[x, y] = x^{-1}y^{-1}xy$.
- Baumslag–Solitar groups

  $$B(m, n) = \mathrm{Gp}\langle a, b | b^{-1}a^m ba^{-n} = 1\rangle$$



- Surface groups

  $$\mathrm{Gp}\langle a_1, \ldots, a_g, b_1, \ldots, b_g | [a_1, b_1] \ldots [a_g, b_g] = 1\rangle.$$

Submonoid and prefix membership problem

G - a finitely generated group with a finite group generating set A.

$\pi : (A \cup A^{-1})^* \to G$ – the canonical monoid homomorphism.

T – a finitely generated submonoid of G.

The membership problem for T in G is decidable if there is an algorithm which solves the following decision problem:

INPUT: A word $w \in (A \cup A^{-1})^*$.

QUESTION: $\pi(w) \in T$?

Prefix membership problem

Let $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ and set

$$P_w = \mathrm{Mon}\langle \mathrm{pref}(w) \rangle \leq G$$

the submonoid generated by the elements of G represented by prefixes of w. We call P_w the prefix monoid. Then G has decidable prefix membership problem if the membership problem for P_w in G is decidable.

Prefix membership problem

Example

If $G = \mathrm{Gp}\langle A \mid w = 1\rangle = \mathrm{Gp}\langle a, b \mid aba^{-1}b^{-1} = 1\rangle$ then $P_w = \mathrm{Mon}\langle a, ab, aba^{-1} = b\rangle = \mathrm{Mon}\langle a, b\rangle$ - the submonoid of all elements that can be written as positive words. The prefix membership problem is decidable in this case by rewriting to normal form $a^i b^j$ with $i, j \in \mathbb{Z}$ and checking $i, j \geq 0$. For example:

$$a^5 b^{-7} a^{-10} b^8 a^9 \in P_w, \quad \text{but} \quad a^3 b^5 a^{-5} b^2 \notin P_w.$$

Open problem: Does every one-relator group $\mathrm{Gp}\langle A \mid w = 1\rangle$ have decidable prefix membership problem?

Proved true in several cases e.g. when w satisfies...

- Idempotent word [Birget, Margolis, Meakin, 1993, 1994]
- w-strictly positive [Ivanov, Margolis, Meakin, 2001]
- Certain Adjan-type and Baumslag–Solitar type [Margolis, Meakin, Šunić, 2005]
- Certain small cancellation conditions [A. Juhász, 2012, 2014].

GROUPS

INVERSE MONOIDS

MONOIDS

$\mathrm{Gp}\langle A \mid w = 1\rangle$
DECIDABLE PREFIX MEMBERSHIP
W - REDUCED WORD

$\Downarrow$

$\mathrm{Gp}\langle A \mid w = 1\rangle$
DECIDABLE PREFIX MEMBERSHIP
W - CYCLICALLY REDUCED WORD

$\Rightarrow$

$\mathrm{Inv}\langle A \mid w = 1\rangle$
DECIDABLE WORD PROBLEM
W - CYCLICALLY REDUCED WORD

IVANOV, MARGOLIS, MEAKIN (2001)

$\mathrm{Inv}\langle A \mid w = 1\rangle$
DECIDABLE WORD PROBLEM
W - REDUCED WORD

$\Downarrow$

$\Rightarrow$

$\mathrm{Mon}\langle A \mid u = v\rangle$
DECIDABLE WORD PROBLEM

Reduced vs cyclically reduced words

$aba^{-1}ab$ - not reduced,
$abba^{-1}$ - reduced but not cyclically reduced
$aba^{-1}b^{-1}$ - cyclically reduced

Theorem (RDG (2020))

There is a one-relator group $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ with a fixed finitely generated submonoid $N \leq G$ such that the membership problem for N within G is undecidable.

Conservative factorisations

Definition

Let $w \in (A \cup A^{-1})^*$. Then for a factorisation

$$w \equiv w_1 w_2 \dots w_k$$

let $P(w_1, \dots, w_k)$ denote the submonoid of $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ generated by

$$\bigcup_{i=1}^{k} \mathrm{pref}(w_i).$$

- We always have $P_w \subseteq P(w_1, \dots, w_k)$ – since every prefix of w is a product of prefixes of the w_i

$$w_1 w_2 \dots w_{r-1} w_r' w_r'' w_{r+1} \dots w_k.$$

- If $P_w = P(w_1, \dots, w_k)$ then we say that the factorisation $w \equiv w_1 \dots w_k$ is conservative.
- We call the w_i in a conservative factorization pieces of w.

Computing conservative factorisations

Let $G = \mathrm{Gp}\langle A \mid w = 1\rangle$.

Adjan overlap method for computing pieces of w based on the fact that for $\alpha, \beta, \gamma \in (A \cup A^{-1})^*$

$$\alpha\beta \text{ and } \beta\gamma \text{ both pieces} \Longrightarrow \alpha,\ \beta, \text{ and } \gamma \text{ are pieces.}$$

Example

$G = \mathrm{Gp}\langle A \mid abcdabcdcdab = 1\rangle$

$abcdabcdcdab = 1$
$\Rightarrow ab$, $cdabcdcdab$, and $abcdabcdcd$ are all pieces.
$\Rightarrow ab$ and cd are pieces. So
$\mathrm{Gp}\langle A \mid (ab)(cd)(ab)(cd)(cd) = 1\rangle$ is a conservative factorisation.

So the prefix submonoid in example is $P = \mathrm{Mon}\langle a, ab, c, cd\rangle$.

Computing conservative factorisations

Benois method for computing pieces method introduced in [RDG & Ruškuc, 2021]

- An algorithm based for computing pieces applying the following theorem

Theorem (Benois (1969))

Every free group of finite rank has a decidable rational subset[2] membership problem.

Example

The Benois method can be applied to show

$$\mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

is a conservative factorisation. This presentation was first introduced in (Margolis, Meakin and Stephen, 1987).

[2]The class of The *rational subsets* of a group is the smallest set containing all finite subsets and is closed under union, product and submonoid generation.

Prefix membership problem strategy

- Given $G = \mathrm{Gp}\langle A \mid w = 1\rangle$
- Use Adjan overlap and Benois pieces computing algorithms to compute conserviate factorisations

$$\mathrm{Gp}\langle A \mid w = 1\rangle = \mathrm{Gp}\langle A \mid w_1 \ldots w_m = 1\rangle.$$

- Seek conditions on the factors $w_1, \ldots, w_k$ that allow us to solve the membership problem in the prefix monoid

$$P_w = P(w_1, \ldots, w_k) = \mathrm{Mon}\langle \mathrm{pref}(w_1) \cup \mathrm{pref}(w_2) \cup \ldots \cup \mathrm{pref}(w_k)\rangle.$$

Amalgamated free products

Definition

Let $H = \mathrm{Gp}\langle A \mid R\rangle$, $K = \mathrm{Gp}\langle B \mid Q\rangle$ with $A \cap B = \varnothing$. Suppose $f : L \to H$ and $g : L \to K$ are injective group homomorphisms Then

$$H *_L K = \mathrm{Gp}\langle A, B \mid R, Q,\ f(x) = g(x) \text{ for all } x \in L\rangle.$$

Theorem A (Dolinka & RDG (2021))

Let $G = H *_L K$, where L, H, K are finitely generated groups such that both H, K have decidable word problems, and the membership problem for L in both H and K is decidable. Let M be a submonoid of G such that:

(i) $L \subseteq M$;

(ii) both $M \cap H$ and $M \cap K$ are finitely generated and

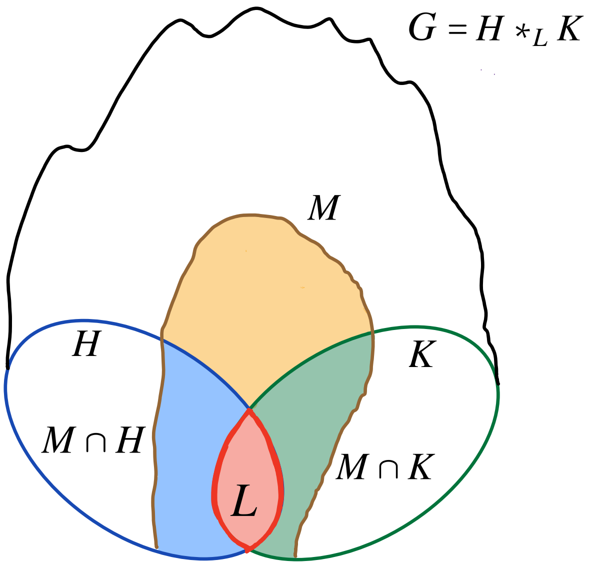
$$M = \mathrm{Mon}\langle (M \cap H) \cup (M \cap K)\rangle;$$

(iii) the membership problem for $M \cap H$ in H is decidable;

(iv) the membership problem for $M \cap K$ in K is decidable.

Then the membership problem for M in G is decidable.

Picture for Theorem A



Unique marker letter theorem

Theorem (Dolinka & RDG (2021))

Let $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ where $w \equiv w_1 \ldots w_m$ is a conservative factorisation. Let $U = \{u_1, \ldots, u_k\}$ be the pieces of this factorisation. Suppose that

- for all $i \in \{1, \ldots, k\}$ there is a letter $a_i \in A$ that appears exactly once in u_i and does not appear in any u_j for $j \neq i$.

Then $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem.

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Then $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ has decidable prefix membership problem.

Notes on proof:

- $G = FG(X_1) * H$ where H is a certain one-relator group.
- P_w = submonoid of $G = FG(X_1) * H$ generated by $Q \cup H$ where Q is a certain finite subset of $FG(X_1)$.
- Apply Magnus's Theorem, Benois' Theorem, and Theorem A.

Unique marker letter theorem example

Example

$$\mathrm{Gp}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle$$

has decidable prefix membership problem, since using the Adjan overlap method

$$(axb)(ayb)(ayb)(axb)(ayb)(axb)$$

is a conservative factorisation, and the factors axb and ayb have the unique marker letter property.

O'Hare example (Margolis, Meakin and Stephen, 1987)

$$\mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

is a conservative factorisation. The factors do not have the unique marker letter property.

O'Hare example (Margolis, Meakin and Stephen, 1987)

$$\mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

is a conservative factorisation. The factors do not have the unique marker letter property. However

$$\begin{aligned} G &= \mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle \\ &= \mathrm{Gp}\langle a, b, c, d \mid (aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad)(ad) \\ &\quad (aba^{-1})(aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad) = 1\rangle. \end{aligned}$$

where this is a conservative factorisation, and aba^{-1}, aca^{-1} and ad do satisfy the unique marker letter property.

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where this is a conservative factorisation, and aba^{-1}, aca^{-1} and ad do satisfy the unique marker letter property.

- It may be shown the prefix monoids of these presentations are equal.

Conclusion:

$$\mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

has decidable prefix membership problem.

Disjoint alphabets theorem

Theorem (Dolinka & RDG (2021))

Let $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ where w is a cyclically reduced word and $w \equiv w_1 \dots w_m$ is a conservative factorisation. Let $U = \{u_1, \dots, u_k\}$ be the pieces of this factorisation. Suppose $k \geq 2$ and that

- for any pair of distinct $i, j \in \{1, \dots, k\}$ the words u_i and u_j have no letters in common.

Then $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem.

Disjoint alphabets theorem

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Then $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem.

Notes on proof:

- $G = FG(X_1) *_{A_1} (FG(X_2) *_{A_2} (\ldots (FG(X_k) *_{A_k} G_k) \ldots))$ a tower of amalgamated free products where G_k is a one-relator group related to the factorisation of w.
- Apply Magnus's Theorem, Benois' Theorem, and Theorem A.

Disjoint alphabets theorem example

Example

$$\mathrm{Gp}\langle a, b, c, d \mid ababcdcdababcdcdcdcdabab = 1\rangle$$

has decidable prefix membership problem, since using the Adjan overlap method

$$(abab)(cdcd)(abab)(cdcd)(cdcd)(abab)$$

is a conservative factorisation, and the factors *abab* and *cdcd* are over disjoint alphabets.

Amalgamated free products and HNN extensions

Amalgamated free products

The results above for the prefix membership problem are obtained by

- Decomposing $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ into smaller groups using amalgamated free products.
- Using the properties of the groups arising in this decomposition and information about the way the prefix monoid P_w sits inside this decomposition to show G has decidable prefix membership problem.

HNN extensions

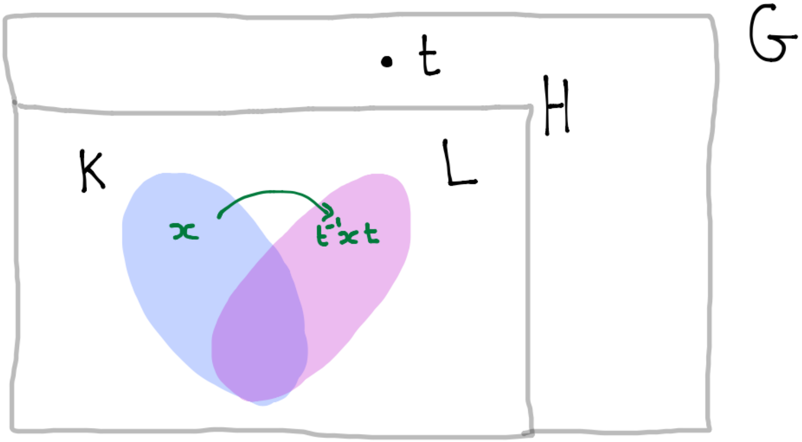
- It is natural to seek a similar approach but using HNN extensions to decompose the group $G = \mathrm{Gp}\langle A \mid w = 1\rangle$.
- In fact, the standard approach to one-relator groups is to decompose them using HNN extensions using a method of McCool and Schupp (1973) and Moldavanskiĭ (1967).

HNN-extensions of groups

$H \cong \mathrm{Gp}\langle A \mid R\rangle$, $\quad K, L \leq H$ with $K \cong L$. Let $\phi : K \to L$ be an isomorphism.
The HNN-extension of H with respect to ϕ is

$$G = H*_{t,\phi:K\to L} = \mathrm{Gp}\langle A,\ t \mid R,\ t^{-1}kt = \phi(k)\ (k \in K)\rangle$$

Fact: H embeds naturally into the HNN extension $G = H*_{t,\phi:K\to L}$.



McCool–Schupp approach to one-relator groups

Based on the following observation of Moldavanskiĭ (1967)

If $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ with $t \in A$ and where w has t-exponent sum zero (e.g. $w = atat^2a^2t^{-3}$). Then the following exist:

- a one-relator group $G' = \mathrm{Gp}\langle A' \mid w' = 1\rangle$ with w' shorter than w.
- sets $C, D \subseteq A'$ that form bases of free subgroups $FG(C), FG(D) \subseteq G'$
- an isomorphism $\phi : FG(C) \to FG(D)$, and
- an isomorphism $G \cong G' *_{t,\phi:FG(C)\to FG(D)}$.

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Theorem (Dolinka & RDG (2021))

With the above notation, if G' is a free group and w is prefix t-positive then $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem.

Example: $G = \mathrm{Gp}\langle a, b, c, t \mid tbcbt^8bbct^{-6}ct^{-3}at^3bt^{-3}at^3ct^{-2}ct^{-1} = 1\rangle$ has decidable prefix membership problem since it is prefix t-positive and $G' = \mathrm{Gp}\langle a_0, b_{-9}, \ldots, b_{-1}, c_{-9}, \ldots, c_{-1} \mid b_{-1}c_{-1}b_{-1}b_{-9}^2c_{-9}c_{-3}a_0b_{-3}a_0c_{-3}c_{-1} = 1\rangle$ is a free group.

Theorem D

Theorem D (Dolinka & RDG (2021))

Let $G = H*_{t,\phi:K\to L}$ be an HNN extension of a finitely generated group H such that K, L are also finitely generated. Assume that the following conditions hold:

(i) the rational subset membership problem is decidable in H;

(ii) $K \leq H$ is effectively closed for rational intersections[3].

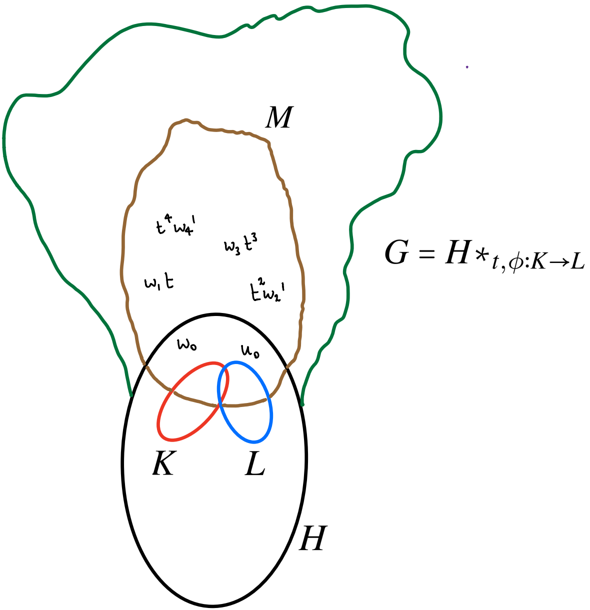
Then for any finite $W_0, W_1, \ldots, W_d, W_1', \ldots, W_d' \subseteq H$, $d \geq 0$, the membership problem for

$$M = \operatorname{Mon}\left\langle W_0 \cup W_1 t \cup W_2 t^2 \cup \cdots \cup W_d t^d \cup tW_1' \cup \cdots \cup t^d W_d' \right\rangle$$

in G is decidable.

[3] K in H is closed for rational intersections if $R \cap K \in \operatorname{Rat}(H)$ for all $R \in \operatorname{Rat}(H)$.

Picture for Theorem D



Further results

Our general results on the submonoid membership in amalgamated free products and HNN extensions can also be used to show the prefix membership problem is decidable for examples including:

- Cyclically pinched groups
  - $\mathrm{Gp}\langle X, Y \mid uv^{-1} = 1\rangle$ where $u \in (X \cup X^{-1})^*$ and $v \in (Y \cup Y^{-1})^*$.

  Including both the orientable surface group

  $$\mathrm{Gp}\langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid [a_1, b_1] \ldots [a_n, b_n] = 1\rangle$$

  and the non-orientable surface group

  $$\mathrm{Gp}\langle a_1, \ldots, a_n \mid a_1^2 \ldots a_n^2 = 1\rangle.$$

- Conjugacy pinched (including Baumslag–Solitar)
  - $\mathrm{Gp}\langle X \cup \{t\} \mid t^{-1}utv^{-1} = 1\rangle$ where $u, v \in (X \cup X^{-1})^*$ are nonempty reduced words.
- Adjan-type (several new cases)
  - $\mathrm{Gp}\langle X \mid uv^{-1} = 1\rangle$ where $u, v \in X^*$ are positive words such that the first letters of u, v are different, and also the last letters of u, v are different.

Limiting expectations

The following result shows that some conditions are needed on the defining relator word w for a positive answer to the prefix membership problem.

Theorem (Dolinka & RDG)

There is a finite alphabet A and a reduced word $w \in (A \cup A^{-1})^*$ such that $\mathrm{Gp}\langle A \mid w = 1 \rangle$ has undecidable prefix membership problem.

- The prefix membership problem for $\mathrm{Gp}\langle A \mid w = 1 \rangle$ remains open for cyclically reduced words w.

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Future directions

Problem

Do one-relator groups $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ have decidable prefix membership problem if w is a cyclically reduced word?

- Find other combinatorial conditions on the pieces of conservative factorisations that suffice to solve the prefix membership problem.
- Extend the HNN-extensions to all one-relator groups that are one step from being free via McCool–Schupp.
- Extend HNN-extension approach to examples higher in the hierarchy.
- Unify the two approaches above e.g. by proving analogous submonoid membership results for graph of groups constructions (i.e. Bass–Serre theory). This relates to:

  M. Kambites, P. V. Silva, B. Steinberg, On the rational subset problem for groups, *J. Algebra* 309 (2007), 622–639.