Countable 2-arc-transitive bipartite graphs via partially ordered sets

Robert Gray

Centro de Álgebra
da Universidade de Lisboa

Workshop on Homogeneous Structures
University of Leeds, Friday 22nd July 2011

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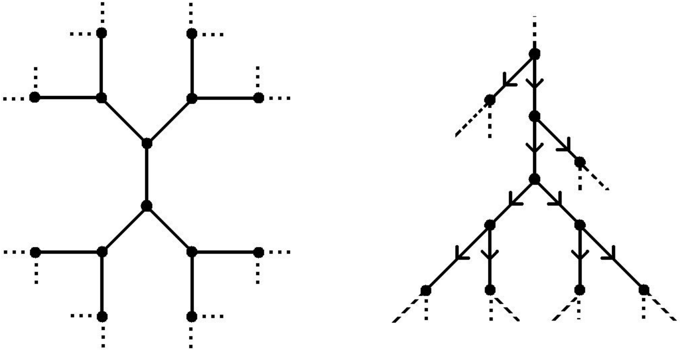


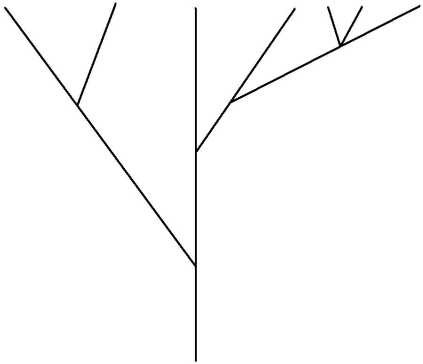
Partially ordered sets

Question: What is a treelike partially ordered set?

Semilinear orders

Definition

A semilinear order (sometimes called a ‘tree’) is a partial order with the property that each principal ideal is totally ordered and any two elements have a common lower bound.



Semilinear order classification

Droste (1985): Classification of countable 2-homogeneous semilinear orders.

- There are $\aleph_0$ many.

Definition

A semilinear order T is 2-CS-homogeneous if its automorphism group is transitive on pairs α, β, with $\alpha < \beta$.

Droste, Holland, Macpherson (1989): Classification of countable 2-CS-homogeneous trees.

- There are $2^{\aleph_0}$ many.

Cycle-free partial orders (*CFPO*s)

Intuition...

- Semilinear order: a poset which may branch as we move upwards through it, but never on going downwards.
- *CFPO*: we may branch upwards and downwards, and repeatedly, but no return to one's starting point is allowed.

Cycle-free partial orders (*CFPO*s)

Some properties one would expect of *CFPO*s:

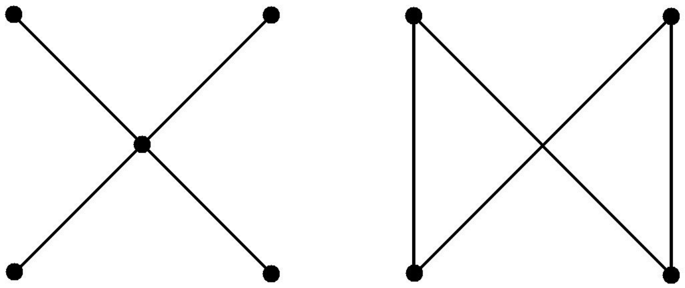
(C1) substructures of cycle-free partial orders should be cycle-free;

(C2) semilinear orders should be cycle-free;

(C3) the diagram representing such an order should be ‘treelike’.

Definition (first attempt)

A poset is cycle-free if its Hasse diagram does ‘not contain any cycles’.



The Dedekind–MacNeille completion

- A poset is called Dedekind–MacNeille complete (DM-complete) if each non-empty upper bounded subset has a supremum (equivalently, each non-empty lower bounded subset has an infimum).

The Dedekind–MacNeille completion

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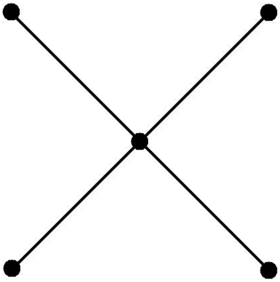
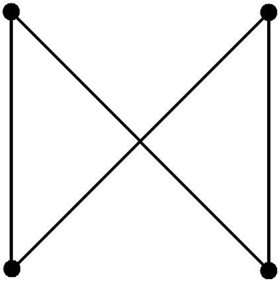
- For any poset M there is a unique (up to isomorphism fixing M pointwise) extension M^D of M which is Dedekind–MacNeille complete, called the Dedekind–MacNeille completion M^D of M.

The Dedekind–MacNeille completion

- A poset is called Dedekind–MacNeille complete (DM-complete) if each non-empty upper bounded subset has a supremum (equivalently, each non-empty lower bounded subset has an infimum).
- For any poset M there is a unique (up to isomorphism fixing M pointwise) extension M^D of M which is Dedekind–MacNeille complete, called the Dedekind–MacNeille completion M^D of M.
- M^D may be realised as the family of all (Dedekind) ideals of M ordered by inclusion, with $M \hookrightarrow M^D$ the principal ideals.
  - (an ideal $J \subseteq M$ is an upper bounded subset that is equal to the set of lower bounds of its set of upper bounds)



M^D



M

Cycle-free partial orders (*CFPO*s)

Warren (1997)

This motivates...

Definition (second attempt)

A poset M is cycle-free if the Hasse diagram of its Dedekind–MacNeille completion M^D does 'not contain any cycles'.

Cycle-free partial orders (*CFPO*s)

Warren (1997)

P - poset, $a, b \in P$
A connecting set from a to b in P is a sequence $(a_0, a_1, \ldots, a_n)$:

$$a = a_0, \;\; b = a_n, \;\; a_r \;\&\; a_{r+1} \text{ comparable for all } r.$$

Then a path from a to b is a set $A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ such that:

- A_r is a maximal chain in $[a_r, a_{r+1}]$;
- $A_r \cap A_{r+1} = \{a_{r+1}\}$ are the only non-empty intersections.

P is connected if there is a connecting set between any two points.

Definition

A poset M is a cycle-free partial order if in M^D there is exactly one path between any two points.

Classification of k-CS-homogeneous *CFPO*s

Definition

A poset $(P, \leq)$ is k-CS-homogeneous if any isomorphism between two connected substructures of P of size k extends to an automorphism of P.

k-CS-homogeneous *CFPO*s

- In non-trivial cases $k \leq 4$ - Warren (1997).
- Classification of countable k-CS-tansitive *CFPO*s for $k \geq 2$:
  - Warren (1997), Truss (1998), Creed, Truss, Warren (1999), RG, Truss (2008)
- Natural division into finite chain and infinite chain cases.

R. Warren. The structure of k-CS-transitive cycle-free partial orders. *Mem. Amer. Math. Soc.*, 129(614):x+166, 1997.

Finite chain *CFPO*s

Connection between *CFPO*s and graphs:

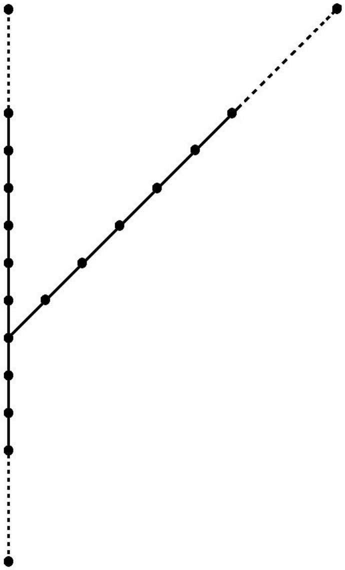
Theorem (The bipartite theorem, Warren (1997))

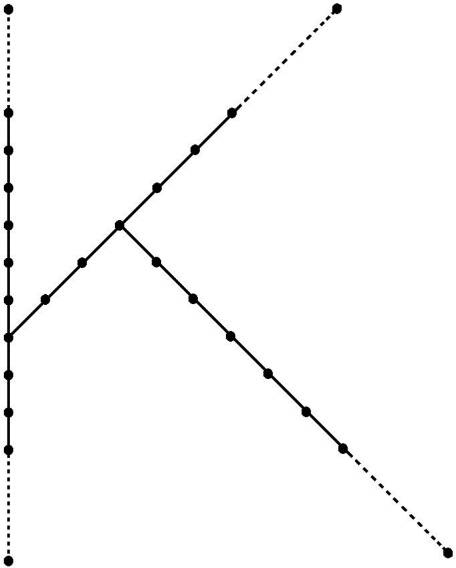
Let M be an infinite CFPO all of whose chains are finite. If M is k-CS-transitive for some $k \geq 2$ and C is a maximal chain in M, then $|C| = 2$.

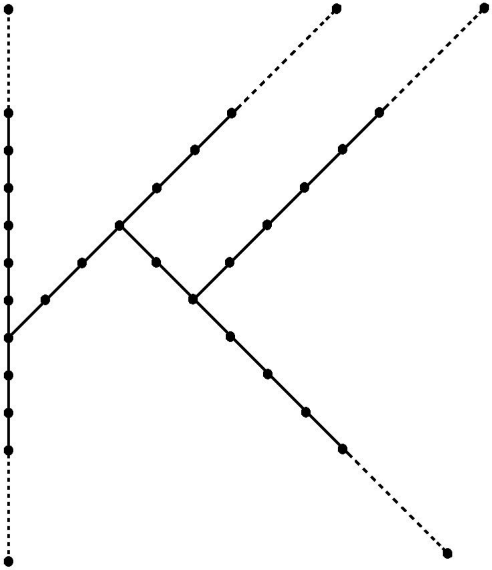
- It follows that finite chain *CFPO*s can be thought of both as partial orders and as bipartite graphs.
- Cycle-free $\Rightarrow$ intervals in M^D are all chains.
- M has finite chains $\not\Rightarrow$ M^D has finite chains.

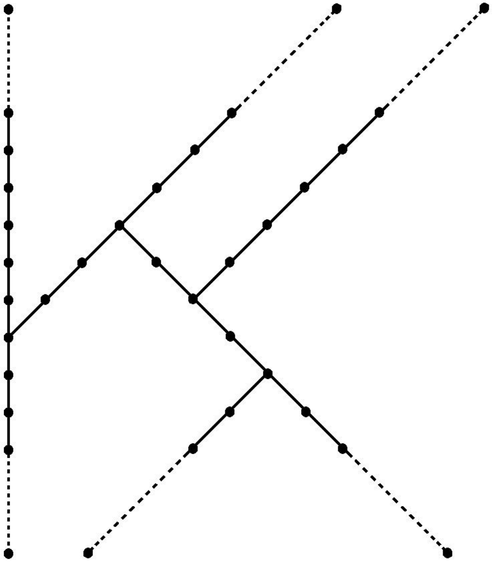
There are many interesting 3-CS-homogeneous examples.

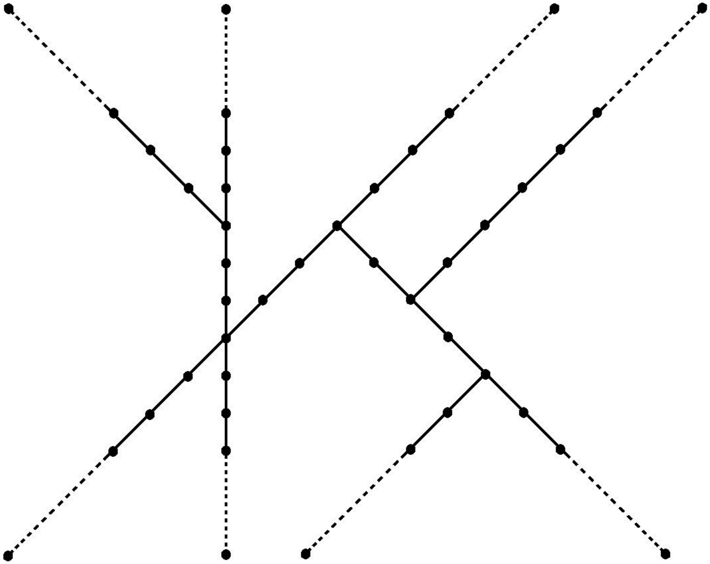
$1 + \mathbb{Z} + 1$

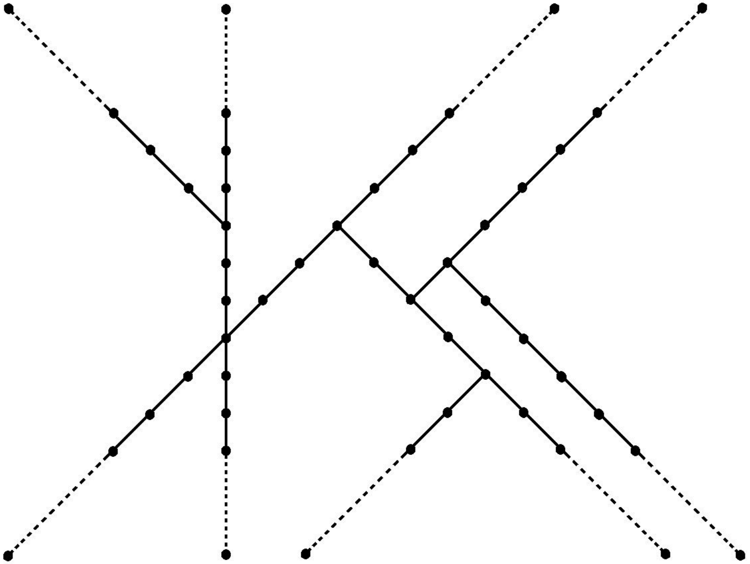


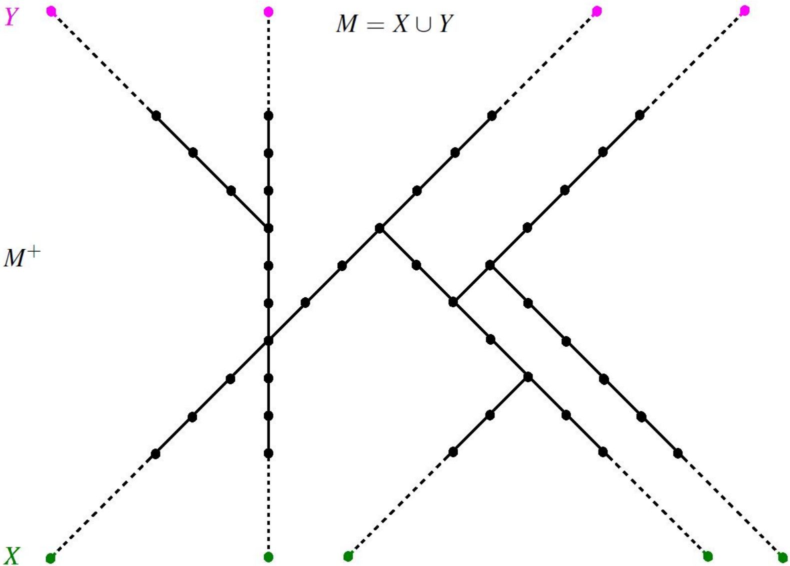




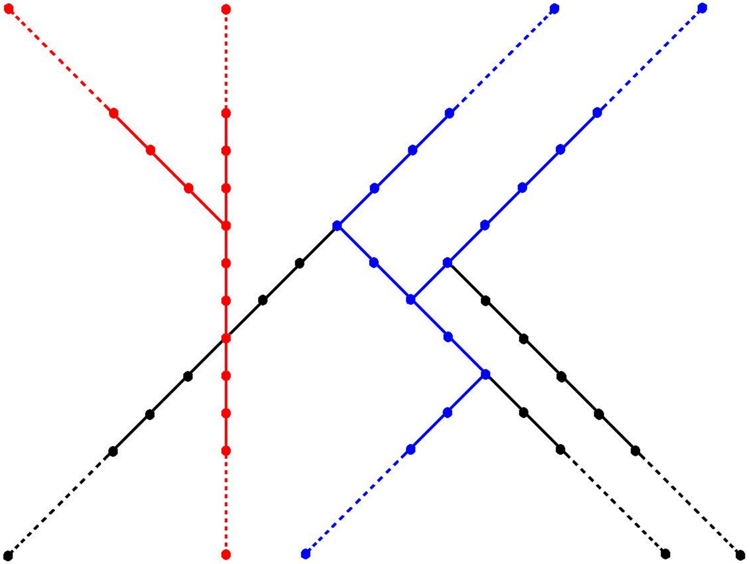




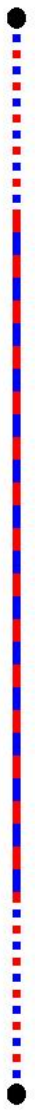


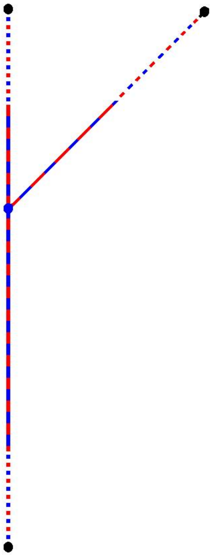


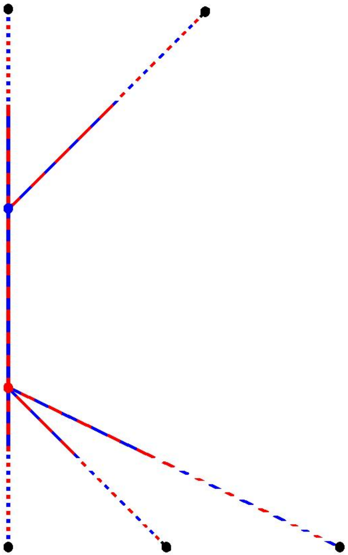
Y
$M = X \cup Y$
M^+
X

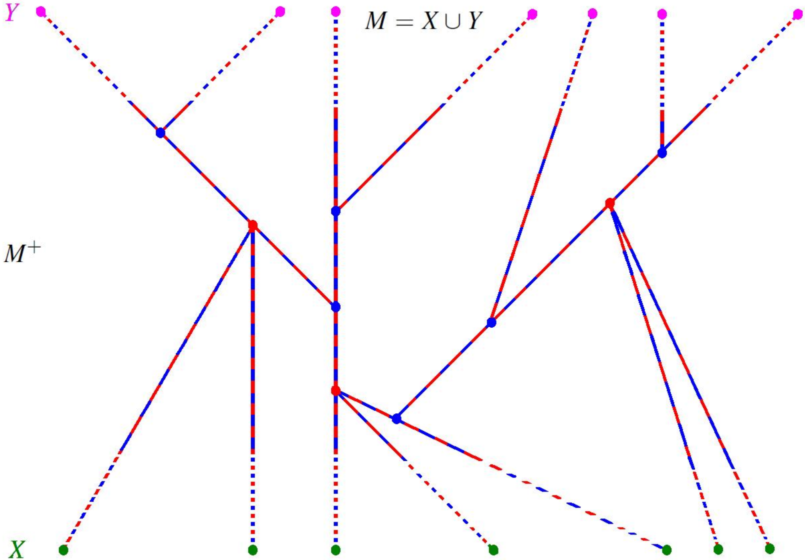


$1 + \mathbb{Q}_2 + 1$









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X

Locally 2-arc-transitive bipartite graphs from CFPOs

Fact. Let M be a connected 2-level partial order. Then M is

3-CS-homogeneous as a poset $\Leftrightarrow$ M is locally 2-arc-transitive when viewed as a bipartite graph.

- (locally) k-arc-transitive finite graphs have received a lot of attention: e.g. Tutte (1947, 1959), Weiss (1981), Giudici, Li, Praeger (2004).

So..

- The 3-CS-transitive *CFPO* classification gives a new interesting family of countable locally 2-arc-transitive graphs.
  - (observation originally made by P. M. Neumann)
- This family of examples is of particular significance since they have no obvious finite analogues.
  - (i.e. they are not simply examples that arise by taking some known finite family and allowing a parameter to go to infinity)

Let M be a countable connected locally 2-arc-transitive bipartite graph, viewed as a 2-level partial order $M = X \cup Y$ with X the set of minimal points and Y the set of maximal points.

Lemma

- *For all $x, x' \in X$ and $y, y' \in Y$ with $x < y$ and $x' < y'$ we have $[x, y]^{M^D} \cong [x', y']^{M^D}$.*
- *Denote this interval by $I(M)$.*

General approach:

(I) Identify the possible partial orders that can arise as intervals $I(M)$ where M is a countable locally 2-arc-transitive bipartite graph.

(II) For each possible interval I (or class of intervals) investigate (classify) those countable locally 2-arc-transitive bipartite graphs such that $I(M) \cong I$.

The case of chain intervals

The 3-CS-homogeneous *CFPO*s are our motivating examples. They all have the property that $I(M)$ is a chain. A natural next step:

- Investigate locally 2-arc-transitive bipartite graphs M such that $I(M)$ is a chain.

Let M be a countable connected locally 2-arc-transitive bipartite graph, viewed as a 2-level partial order $M = X \cup Y$ with X the set of minimal points and Y the set of maximal points.

(I) Identify the possible chains that can arise as intervals $I(M)$.

(II) For each possible such chain I investigate (classify) those countable locally 2-arc-transitive bipartite graphs such that $I(M) \cong I$.

Definition

A partial linear space is a pair (P, L) consisting of a set P of points and a set L of lines such that

(i) every line contains at least two points, and

(ii) any two distinct points are on at most one line.

Lemma

Let M be a countable connected bipartite graph, viewed as a 2-level partial order. Then

M is DM-complete $\Leftrightarrow$ M is the incidence graph of a partial linear space.

(I) Identifying possible chain intervals

Finite chains

Proposition

Let M be a countable connected locally 2-arc-transitive bipartite graph. If $I = I(M)$ is a finite chain then $|I| = 2$ or $|I| = 3$. Moreover:

(i) If $|I| = 3$ then M is complete bipartite.

(ii) If $|I| = 2$ then M is the incidence graph of a partial linear space S such that $\mathrm{Aut}(S)$ is transitive on configurations of the form (p_1, l, p_2), where l is a line incident with $p_1\ p_2$, and on configurations (l_1, p, l_2) where p is a point incident with l_1 and l_2.

- There are $2^{\aleph_0}$ countable examples (e.g. projective planes over countable fields).
- No classification is known. Related work:
  - Classification of finite connected 4-homogeneous partial linear spaces Devillers (2002), (2005), (2008).
  - Some countably infinite examples of partial linear spaces satisfying these conditions arise from the work of Tent (2000) on 'generalized n-gons'.

(I) Identifying possible chain intervals

Infinite chains

Let M be a countable connected locally 2-arc-transitive bipartite graph, viewed as a 2-level partial order $M = X \cup Y$ with X the set of minimal points and Y the set of maximal points.

$$\uparrow\mathrm{Ram}(M) = \{a \wedge b : a, b \in M, a \parallel b\}, \quad \downarrow\mathrm{Ram}(M) = \{a \vee b : a, b \in M, a \parallel b\}$$

$$M^+ = M \cup \uparrow\mathrm{Ram}(M) \cup \downarrow\mathrm{Ram}(M)$$

Fact. $\mathsf{Aut}(M)$ acts transitively on each of $\uparrow\mathrm{Ram}(M)$ and $\downarrow\mathrm{Ram}(M)$.

Fact. M countable $\Rightarrow M^+$ countable.

Lemma

$I(M)$ is a chain if and only if the maximal chains $I^+(M)$ of M^+ are chains, in which case $I(M)$ is the DM-completion of $I^+(M)$.

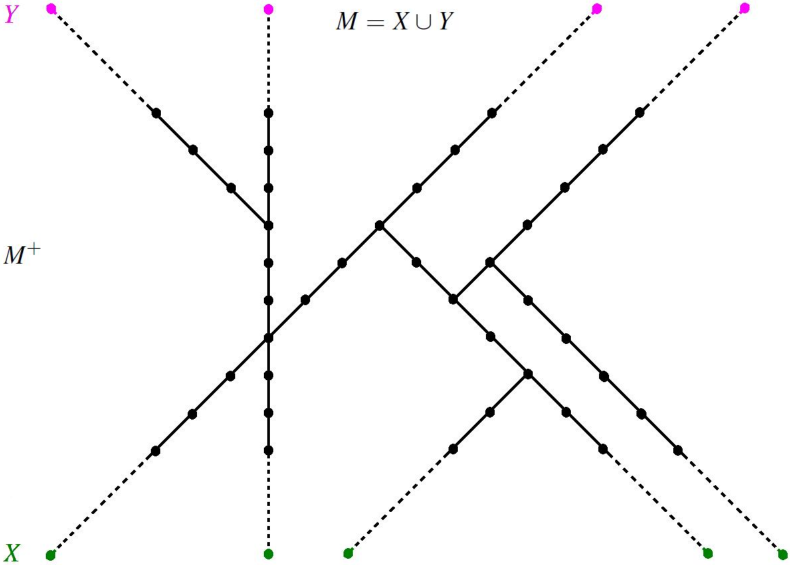
- These facts together with Morel (1965) classification of countable transitive linear orders allow us to list the possible chain types $I^+(M)$.

The special case of $\mathbb{Z}$ intervals

Let M be a connected countable locally 2-arc-transitive bipartite graph such that the intervals of $I^+(M)$ is isomorphic to $1 + \mathbb{Z} + 1$.

Examples?

- *CFPO*s built up by gluing together chains $1 + \mathbb{Z} + 1$.
- Any others?


Y
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M^+
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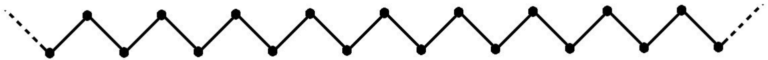

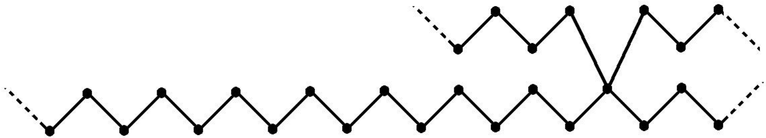

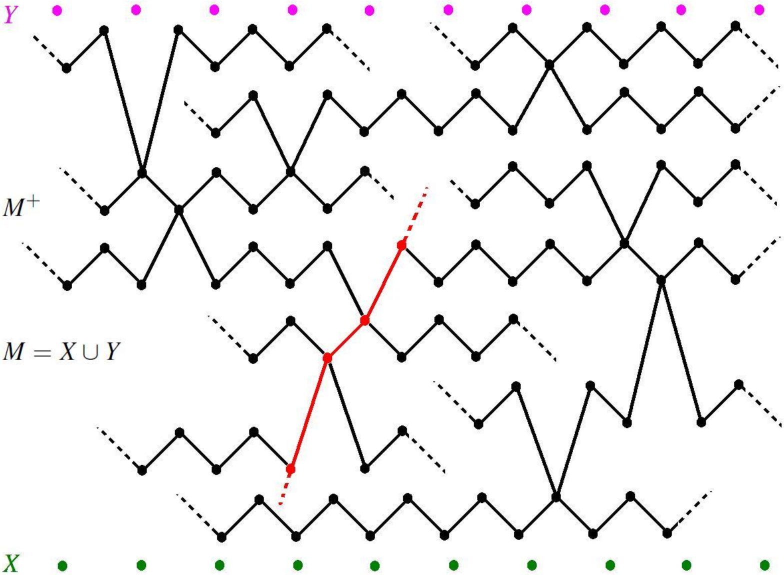

Y
M^+
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X

$1 + \mathbb{Z} + 1$ chain *CFPO*s alternative point of view

- Build the 'scaffolding' T first by gluing together copies of the alternating poset ALT in a treelike way.
- Choose for each point of T a maximal chain passing through that point, and put one point above and below each such chain (i.e. choose a countable 'dense' set of maximal chains of T).

But...

if we view ALT as a directed bipartite graph, then this is the highly arc-transitive digraph given by the universal covering construction $DL(\mathrm{ALT})$ of Cameron, Praeger, Wormald (1992).

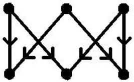
The digraphs $DL(\Delta)$

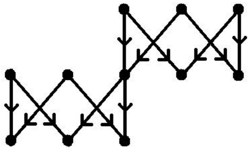
- Examples may be obtained from the universal covering digraphs $DL(\Delta)$ of Cameron, Praeger, Wormald (1992).
- The digraphs $DL(\Delta)$ arise in the study of highly arc-transitive digraphs, meaning the automorphism group is transitive on the set of directed paths of length n for every $n \in \mathbb{N}$.

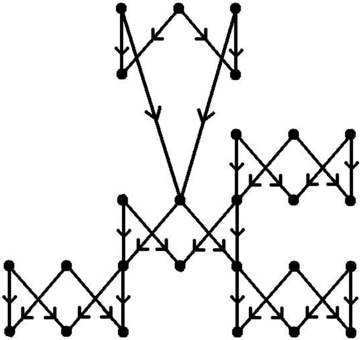
Δ - bipartite digraph

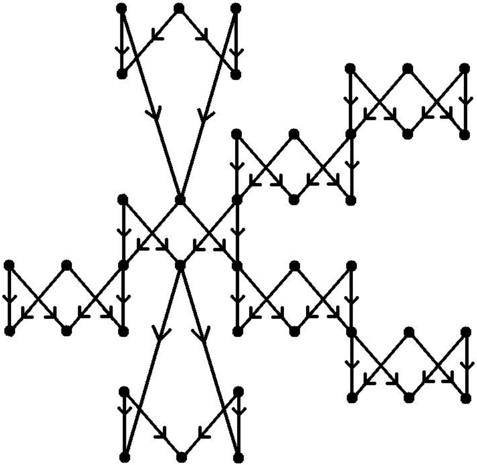
$DL(\Delta)$ - a certain infinite digraph obtained by guling together copies of Δ in a tree-like way.

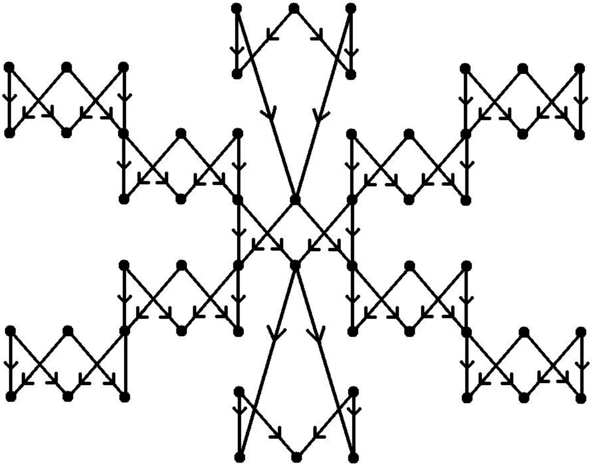
Example. Δ = 6-cycle.







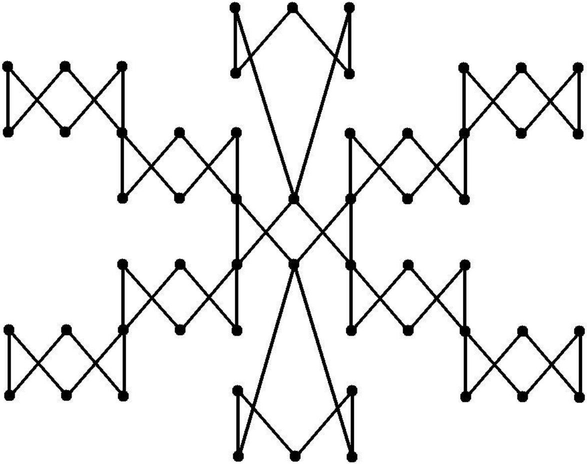


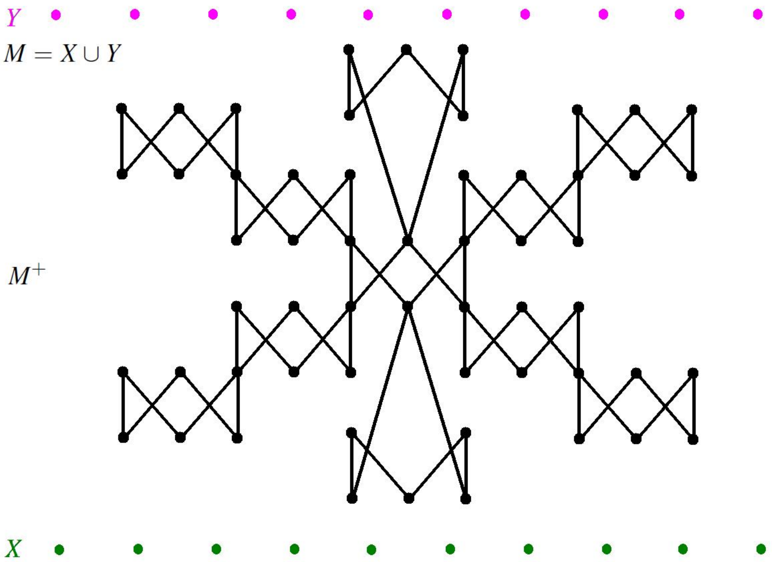


$1 + \mathbb{Z} + 1$ chain *CFPO*s alternative point of view

Example

- Consider $DL(\Delta)$ where Δ is a 6-crown.
- Let $P(DL(\Delta))$ be $DL(\Delta)$ viewed as a poset.
- Choose a countable 'dense' set of maximal chains in $P(DL(\Delta))$ and put one point above and below each such chain.




Y
$M = X \cup Y$
M^+
X

The special case of $\mathbb{Z}$ intervals

What makes this example work?

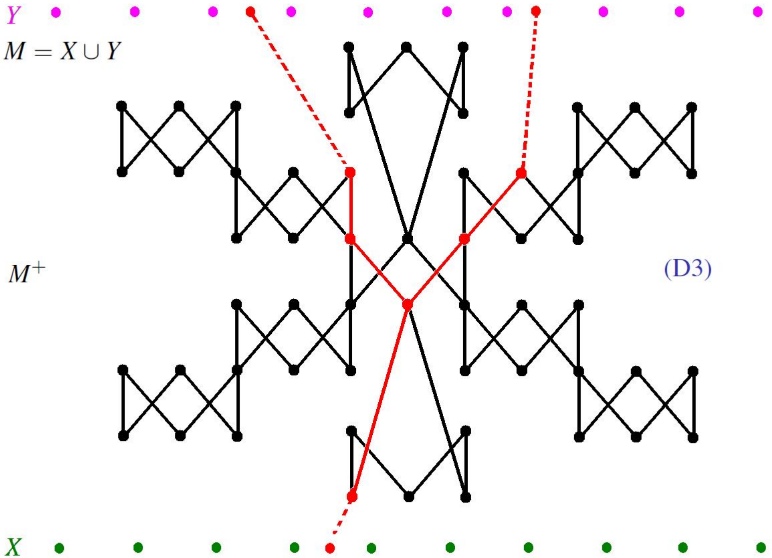
Idea: Given a "sufficiently nice" countable highly arc-transitive digraph D, it can be used to construct a countable locally 2-arc-transitive bipartite graph M, with $I^+(M) \cong 1 + \mathbb{Z} + 1$ intervals, using the above approach.

Sufficiently nice means satisfying the following (and their duals):

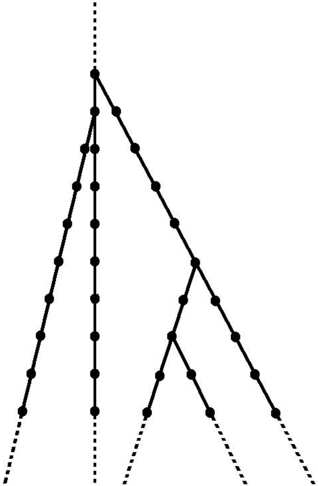
(D1) The substructures induced by the descendants are trees.

(D2) Intersection of any pair of descendant sets is again a descendant set.

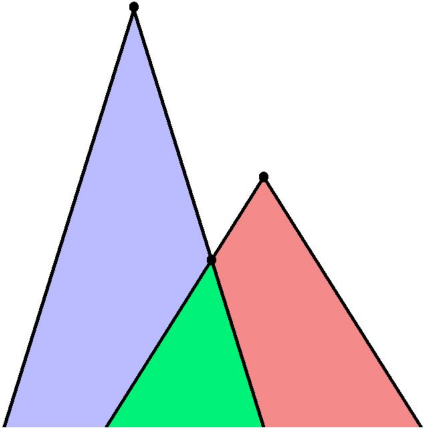
(D3) Automorphism group acts transitively on infinite Y-configurations.


Y
M = X ∪ Y
M+
(D3)
X

(D1)



(D2)



The special case of $\mathbb{Z}$ intervals

Theorem

Let D be a countable connected highly arc-transitive digraph satisfying (D1), (D2) and (D3). Then there is a connected countable locally 2-arc-transitive bipartite graph M such that the digraph $D(M^+ \setminus M)$ naturally defined from the partial order $M^+ \setminus M$ is isomorphic to D.

The special case of $\mathbb{Z}$ intervals

Theorem

Let D be a countable connected highly arc-transitive digraph satisfying (D1), (D2) and (D3). Then there is a connected countable locally 2-arc-transitive bipartite graph M such that the digraph $D(M^+ \setminus M)$ naturally defined from the partial order $M^+ \setminus M$ is isomorphic to D.

Conversely, if Γ is a countable connected locally 2-arc-transitive bipartite graph such that the interval $I^+(M)$ is isomorphic to $1 + \mathbb{Z} + 1$, then the digraph $D(M^+ \setminus M)$ naturally defined from the partial order $M^+ \setminus M$ is a connected highly arc-transitive digraph satisfying properties (D1), (D2), and (D3) for finite Y configurations (and their duals).

Conclusion

The classification problem for intervals $1 + \mathbb{Z} + 1$ comes down to classifying a certain family of countable highly arc-transitive digraphs.

Theorem

There are $2^{\aleph_0}$ connected countable locally 2-arc-transitive bipartite graphs M with $I(M)$ of order-type $1 + \mathbb{Z} + 1$.

Idea of proof

Consider the digraphs $DL(\Delta)$ where Δ ranges through the countable connected *DM*-complete 2-level locally 2-arc-transitive bipartite graphs. Prove there are $2^{\aleph_0}$ such Δ, and distinct Δ give rise to distinct $DL(\Delta)$ which in turn give rise to distinct bipartite graphs $M = X \cup Y$.

- Contrasts with the case of *CFPO*s for which, with the same interval type, there are only $\aleph_0$ many.

Other chain intervals

For the other possible order-types of chain intervals the connection with highly arc-transitive digraphs is lost.

Non-dense cases: $I^+(M)$ has order type $1 + \mathcal{Z} + 1$ where $\mathcal{Z}$ is one of $\mathbb{Z}^\alpha$ or $\mathbb{Q}.\mathbb{Z}^\alpha$ for a countable ordinal $\alpha > 0$, or $\mathbb{Q}.2$.

Dense cases: $I^+(M)$ has order-type $1 + \mathbb{Q} + 1$ or $1 + \mathbb{Q}_2 + 1$.

Theorem

For each of the order-types $\mathcal{I}$ listed above, there are $2^{\aleph_0}$ connected countable locally 2-arc-transitive bipartite graphs M with $I(M)$ of order-type $\mathcal{I}$.

Future directions

- Classify countable locally 2-arc-transitive bipartite graphs whose *DM*-completions have intervals of type $1 + \mathbb{Z} + 1$.
  - Describe the HAT digraphs that can arise as M^+.
  - May only be possible modulo a classification of *DM*-complete locally 2-arc-transitive bipartite graphs (equivalently, partial linear spaces).
- Classify countable locally 2-arc-transitive bipartite graphs whose *DM*-completions have chain intervals.
- Investigate locally 2-arc-transitive bipartite graphs whose *DM*-completions have infinitely many 'ends' (for a suitably defined notion of ends for partially ordered sets).
  - Relationship between ends of graphs and *CFPO*s were investigated in RG, Truss (2009).
- Look at automorphism groups.
  - Automorphism groups of semilinear orders: Droste (1985), Droste, Holland, Macpherson (1989): examples with "many normal subgroups";
  - Automorphism groups of *CFPO*s: Droste, Truss, Warren (1997): many are infinite simple groups.