Boundary index and finite presentability of subsemigroups

Robert Gray

University of St Andrews

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Outline

1. Motivation
   - Subsemigroups: inheritance and extensions
   - The notion of index
2. Boundaries in Cayley graphs
   - Definitions and examples
   - Main results
3. Applications and concluding remarks
   - Applications
   - One sided boundaries and the converse

Subsemigroups and inheritance

Let S be a semigroup with T a subsemigroup of S.

Let $\mathcal{P}$ be a property of semigroups.

- S satisfies $\mathcal{P} \Rightarrow T$ satisfies $\mathcal{P}$?
- T satisfies $\mathcal{P} \Rightarrow S$ satisfies $\mathcal{P}$?

Subsemigroups and inheritance

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- S satisfies $\mathcal{P} \Rightarrow T$ satisfies $\mathcal{P}$?
- T satisfies $\mathcal{P} \Rightarrow S$ satisfies $\mathcal{P}$?

Some properties are passed from S to all of its subsemigroups. (e.g. commutativity, finiteness, solvable word problem, ...)

Others are not. (e.g. being finitely generated / presented, automatic, having finite derivation type, ...)

What is index?

Roughly speaking...

Index is a measure of the 'size' of T inside S.

A 'good' definition of index should have the property that if T is 'big' in S then S and T share many properties.

Established notions of index

Subgroups of groups

Let G be a group with H a subgroup of G.

- $[G : H] =$ number of cosets of H in G

Subsemigroups of semigroups

Let S be a semigroup with T a subsemigroup of S.

- $[S : T]_R = |S \setminus T|$

We call this the Rees index of T in S.

Cayley Graphs

Definition

Let S be a semigroup generated by a finite set A.

The right Cayley graph $\Gamma_r(A, S)$ has:

- Vertices: elements of S.
- Edges: directed and labelled with letters from A.

$$s \xrightarrow{a} t \Leftrightarrow sa = t$$

- Given an edge $e = s \xrightarrow{a} t$ we define

$$\iota(e) = s, \quad \tau(e) = t$$

calling them the initial and terminal vertices of e.

Semigroup boundaries

Definition

- The right boundary edges of T in S:

$$\mathcal{E}_r(A, T) = \{e \in \Gamma_r(A, S) : \iota(e) \notin T \ \& \ \tau(e) \in T\}$$

Semigroup boundaries

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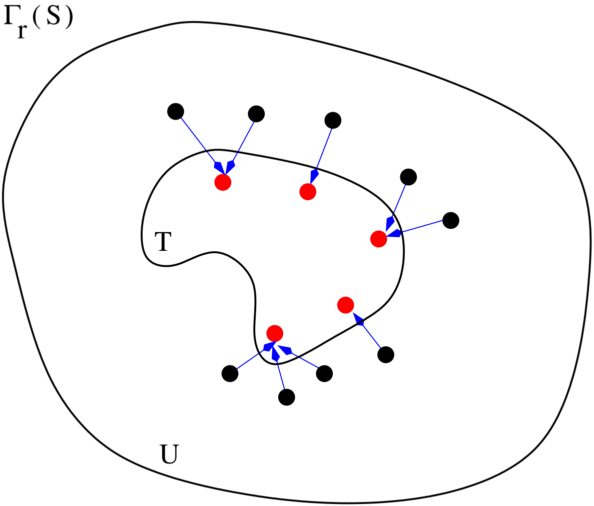
- The right boundary of T in S is the set of terminal vertices of the right boundary edges, together with the generators of A that belong to T. This set is given by:

$$\mathcal{B}_r(A, T) = U^1 A \cap T = \{ua : u \in U^1, a \in A\} \cap T$$

- $S^1 = S$ with an identity adjoined
- $U = S \setminus T$ and $U^1 = S^1 \setminus T$

The picture to have in mind



- Blue edges: right boundary edges
- Red vertices: the right boundary of T in S

Semigroup boundaries

Definition

The left boundary is defined in the analogous way but using the left Cayley graph.

$$\mathcal{B}_l(A, T) = AU^1 \cap T = \{au : u \in U^1, a \in A\} \cap T.$$

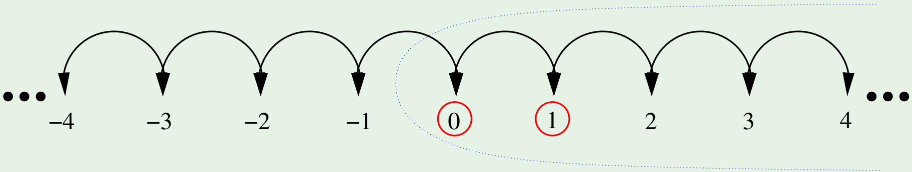
The (two-sided) boundary is the union of the left and right boundaries:

$$\mathcal{B}(A, T) = \mathcal{B}_l(A, T) \cup \mathcal{B}_r(A, T).$$

A (very) straightforward example

Example (Infinite cyclic group)

- $S = \mathbb{Z} = \langle -1, 1 \rangle$ (with operation $+$);
- $T = \mathbb{N} = \{0, 1, 2, \ldots\}$.



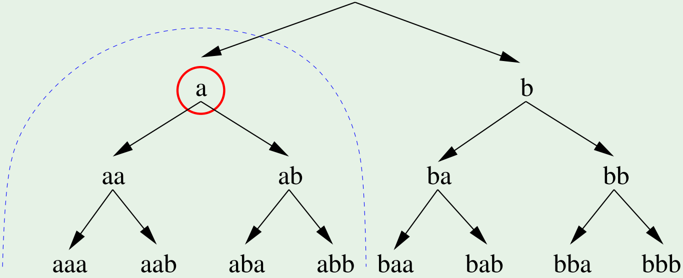
In this example we have:

$$\mathcal{B}_l(\{-1, 1\}, T) = \mathcal{B}_r(\{-1, 1\}, T) = \mathcal{B}(\{-1, 1\}, T) = \{0, 1\}.$$

Another straightforward example

Example (Free monoid on two generators)

- $S = \{a, b\}^*$, $\quad T = \{$words that begin with the letter $a\}$.



Right boundary: $\mathcal{B}_r(\{a, b\}, T) = \{a\}$.

Another straightforward example

Example (Free monoid on two generators)

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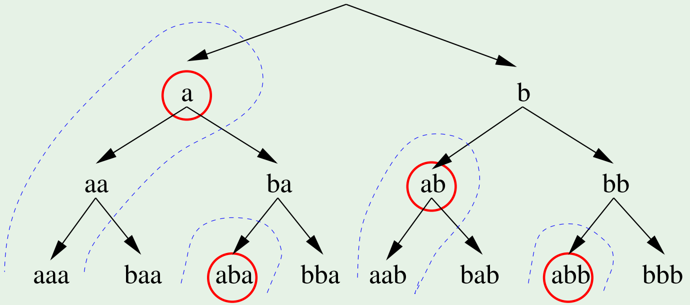


Left boundary: $\mathcal{B}_l(\{a, b\}, T) = \{a\} \cup \{ab\{a, b\}^*\}$.

Changing the generating set

Proposition

S - a finitely generated semigroup

T - subsemigroup of S

$A, B \subseteq S$ - two finite generating sets for S

- $|\mathcal{B}_r(A, T)| < \infty \Leftrightarrow |\mathcal{B}_r(B, T)| < \infty$
- $|\mathcal{B}_l(A, T)| < \infty \Leftrightarrow |\mathcal{B}_l(B, T)| < \infty$

Generating sets

Proposition

Let $S = \langle A \rangle$ where $|A| < \infty$ and let $T \leq S$. Then T is generated by:

$$X = \mathcal{B}_r(A, T)U^1 \cap T.$$

Moreover, the generating set X is finite if $\mathcal{B}(A, T)$ is finite.

Generating sets

Proposition

Let $S = \langle A \rangle$ where $|A| < \infty$ and let $T \leq S$. Then T is generated by:

$$X = \mathcal{B}_r(A, T)U^1 \cap T.$$

Moreover, the generating set X is finite if $\mathcal{B}(A, T)$ is finite.

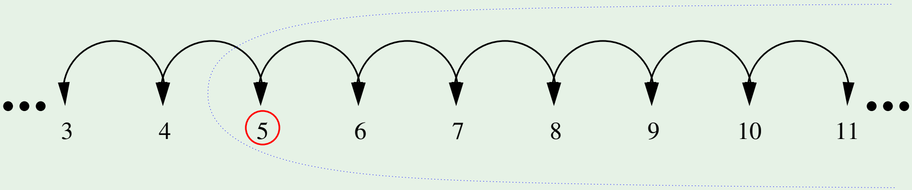
Theorem

If S is finitely generated and T has a finite boundary in S then T is finitely generated.

Generating set example

Example

- $S = \mathbb{Z} = \langle -1, 1 \rangle$ (with operation $+$);
- $T = \{5, 6, 7, \ldots\}$.



$$\mathcal{B}(A, T) = \{5\} \text{ and}$$

$$\langle \mathcal{B}(A, T) \rangle = \{5, 10, 15, \ldots\} \neq T$$

Generating set example

Example

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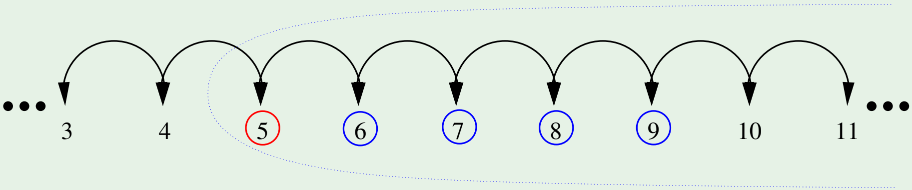

$$\langle X \rangle = \langle \mathcal{B}_r(A, T)U^1 \cap T \rangle = \langle 5, 6, 7, 8, 9 \rangle = T$$

Semigroup presentations

Theorem

If S is finitely presented and T has a finite boundary in S then T is finitely presented.

Semigroup presentations

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Corollary (Ruškuc 1998)

If S is finitely generated (resp. presented) and T has finite Rees index in S (i.e. $|S \setminus T| < \infty$) then T is finitely generated (resp. presented).

Proof. *T has finite Rees index $\Rightarrow$ T has a finite boundary.* $\square$

Applications

Corollary (Folklore)

Let S be a semigroup with T a subsemigroup of S such that $S \setminus T$ is an ideal. If S is finitely generated (resp. presented) then T is finitely generated (resp. presented).

Proof. *$S \setminus T$ is an ideal $\Rightarrow$ T has a finite boundary.* □

Corollary

Let $S = T \cup \mathcal{I}$, a disjoint union, where $\mathcal{I}$ is a two-sided ideal of S and T is a subsemigroup of S. If S is finitely generated (resp. presented) and every orbit of $\mathcal{I}$ under the action of T is finite then $\mathcal{I}$ is finitely generated (resp. presented).

Proof. *Finite orbits $\Rightarrow$ $\mathcal{I}$ has a finite boundary.* □

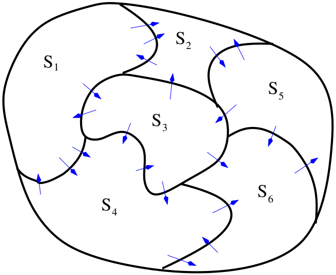
One sided boundaries and the converse

Theorem

Let S be a finitely generated free semigroup and let T be a finitely generated subsemigroup of S. If T has a finite right boundary in S then T is finitely presented.

Theorem

Let $S = \bigcup_{i \in \mathcal{I}} S_i$, a disjoint union, where $\mathcal{I}$ is finite and each S_i is a subsemigroup of S. If each S_i is finitely presented and has a finite boundary in S then S itself is finitely presented.



For the future

- Find other interesting/natural examples of subsemigroups with finite boundaries.

- What other properties are inherited?
  - Being automatic
  - Having a finite complete rewriting system
  - Having finite derivation type