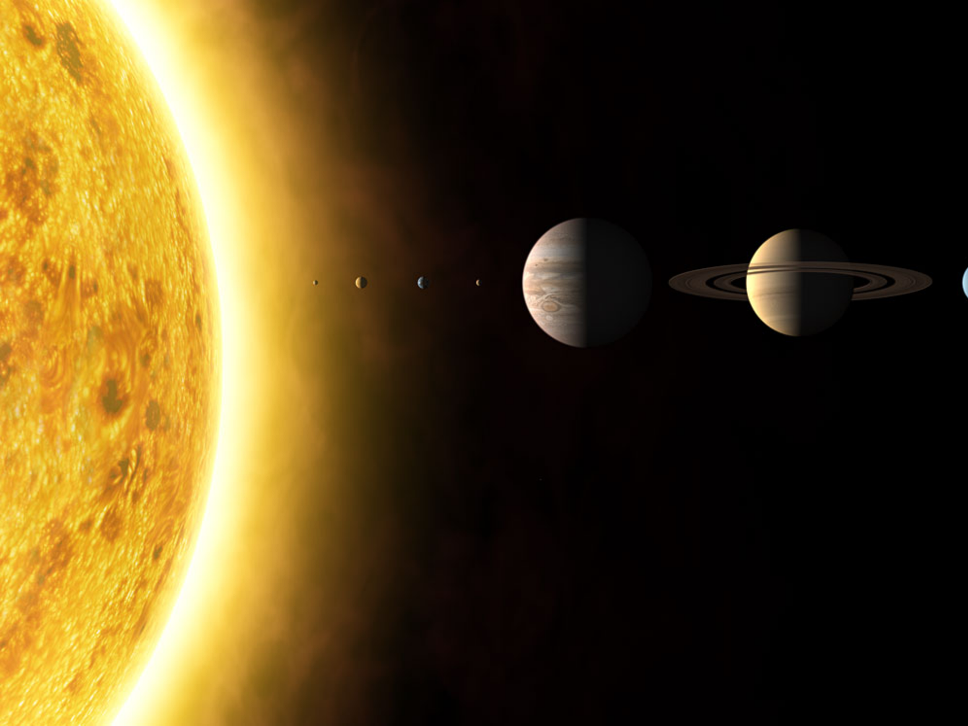
Free Idempotent Generated Semigroups and their Maximal Subgroups

Robert Gray

Centro de Álgebra
da Universidade de Lisboa

University of Lisbon, July 2011

UNIVERSIDADE
DE LISBOA



Thinking about semigroups

- Semigroup $\longleftrightarrow$ Universe
- Idempotents $\longleftrightarrow$ Stars
- Solar system (one for each star) $\longleftrightarrow$ Maximal subgroups (one for each idempotent)

Outline

History and motivation

- Idempotent generated semigroups
- Biordered sets and free idempotent generated semigroups

Maximal subgroups of free idempotent generated semigroups

- Singular squares and presentations

Examples that occur in nature

- Transformation monoids
- Linear semigroups

Idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents $e = e^2$ of S

Definition. S is idempotent generated if $\langle E(S) \rangle = S$

Idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents $e = e^2$ of S

Definition. S is idempotent generated if $\langle E(S) \rangle = S$

- Many natural examples
  - Howie (1966) - $T_n \setminus S_n$, the non-invertible transformations;
  - Erdös (1967) - singular part of $M_n(\mathbb{F})$, semigroup of all $n \times n$ matrices over a field $\mathbb{F}$;
  - Laffey (1983) - singular part of $M_n(Q)$, Q an arbitrary division ring;
  - Putcha (2006) - conditions for a reductive linear algebraic monoid to have the same property.

Idempotent generated semigroups

S - semigroup, $E = E(S)$ - idempotents $e = e^2$ of S

Definition. S is idempotent generated if $\langle E(S) \rangle = S$

- Many natural examples
  - Howie (1966) - $T_n \setminus S_n$, the non-invertible transformations;
  - Erdös (1967) - singular part of $M_n(\mathbb{F})$, semigroup of all $n \times n$ matrices over a field $\mathbb{F}$;
  - Laffey (1983) - singular part of $M_n(Q)$, Q an arbitrary division ring;
  - Putcha (2006) - conditions for a reductive linear algebraic monoid to have the same property.
- Idempotent generated semigroups are "general"
  - Every semigroup S embeds into an idempotent generated semigroup.

The biordered set of a semigroup

Nambooripad (1979)

S - semigroup, $\quad E = E(S)$ - idempotents of S

Definition. The biordered set of a semigroup S is the partial algebra consisting of the set $E = E(S)$ with multiplication restricted to basic pairs.

$(e, f) \in E \times E$ is called a basic if

$$ef = e \;\text{ or }\; ef = f \;\text{ or }\; fe = e \;\text{ or }\; fe = f.$$

i.e. one of the idempotents stabilizes the other under left or right multiplication.

The biordered set of a semigroup

Nambooripad (1979)

S - semigroup, $\quad E = E(S)$ - idempotents of S

Definition. The biordered set of a semigroup S is the partial algebra consisting of the set $E = E(S)$ with multiplication restricted to basic pairs.

$(e, f) \in E \times E$ is called a basic if

$$ef = e \;\text{ or }\; ef = f \;\text{ or }\; fe = e \;\text{ or }\; fe = f.$$

i.e. one of the idempotents stabilizes the other under left or right multiplication.

If (e, f) is basic then both $ef \in E$ and $fe \in E$.
(e.g. if $ef = f$ then $(fe)^2 = f(ef)e = ffe = fe$)

Free idempotent generated semigroups

S - semigroup, $\quad E = E(S)$

Let $IG(E)$ denote the semigroup defined by the following presentation.

$$IG(E) = \langle E \mid e \cdot f = ef \text{ if } (e,f) \text{ is a basic pair} \rangle.$$

$IG(E)$ is called the free idempotent generated semigroup on E.

Free idempotent generated semigroups

S - semigroup, $\quad E = E(S)$

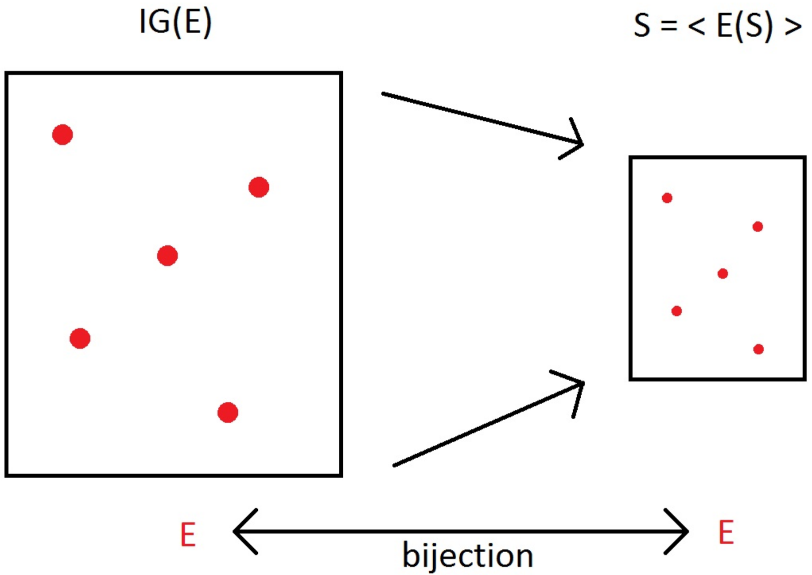
Let $IG(E)$ denote the semigroup defined by the following presentation.

$$IG(E) = \langle E \mid e \cdot f = ef \text{ if } (e,f) \text{ is a basic pair}\rangle.$$

$IG(E)$ is called the free idempotent generated semigroup on E.

Theorem (Easdown (1985))

The biordered set of idempotents of $IG(E)$ is E. If S is any idempotent generated semigroup with biordered set of idempotents isomorphic to E then the natural map $E \to S$ extends uniquely to a homomorphism $\phi : IG(E) \to S$.


IG(E)
S = < E(S) >
E
bijection
E

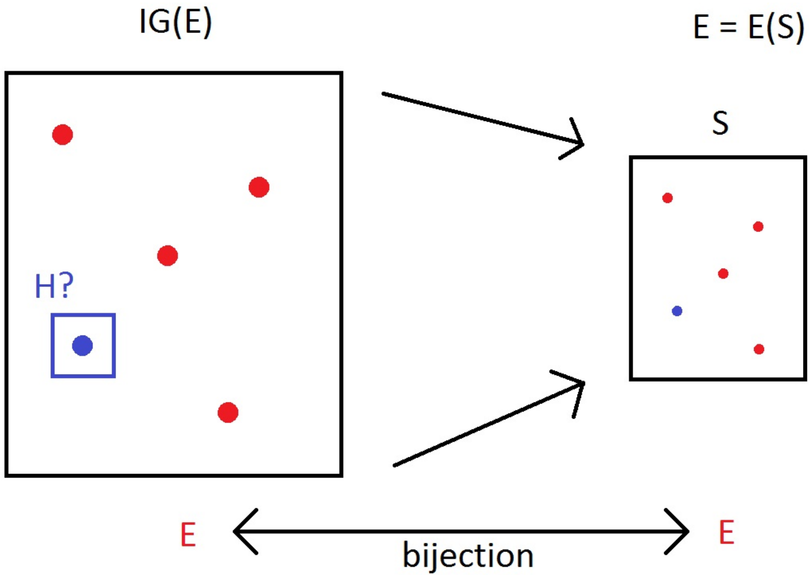
First steps towards understanding $IG(E)$

Conclusion. It is important to understand $IG(E)$ if one is interested in understanding an arbitrary idempotent generated semigroup with biordered set E.

First steps towards understanding $IG(E)$

Conclusion. It is important to understand $IG(E)$ if one is interested in understanding an arbitrary idempotent generated semigroup with biordered set E.

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?


IG(E)
E = E(S)
S
H?
E
E
bijection

First steps towards understanding $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

First steps towards understanding $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

- It was conjectured that maximal subgroups of free idempotent generated semigroups must always be free groups.
- This conjecture was confirmed for several classes of biordered set:
    - Pastijn (1977, 1980), Nambooripad & Pastijn (1980), McElwee (2002).

First steps towards understanding $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

- It was conjectured that maximal subgroups of free idempotent generated semigroups must always be free groups.
- This conjecture was confirmed for several classes of biordered set:
  - Pastijn (1977, 1980), Nambooripad & Pastijn (1980), McElwee (2002).

- Brittenham, Margolis & Meakin (2009) gave the first counterexamples to this conjecture.
  - Give a 72-element semigroup S and prove that $IG(E(S))$ has a maximal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
  - They also reported that the multiplicative group $\mathbb{F}^*$ of a field $\mathbb{F}$ arises as a maximal subgroup of $IG(E(M_3(\mathbb{F})))$, where $M_3(\mathbb{F})$ is the semigroup of all 3×3 matrices over $\mathbb{F}$.

Maximal subgroup of free idempotent generated semigroups

Theorem (RG & Ruskuc (2011))

Every group is a maximal subgroup of some free idempotent generated semigroup.

Green's relations and maximal subgroups

Green's relations: equivalence relations reflecting ideal structure.

Green's relations and maximal subgroups

Green's relations: equivalence relations reflecting ideal structure.

For $u, v \in S$ we define

$$u\mathcal{R}v \iff uS \cup \{u\} = vS \cup \{v\}, \quad u\mathcal{L}v \iff Su \cup \{u\} = Sv \cup \{v\},$$

Green's relations and maximal subgroups

Green's relations: equivalence relations reflecting ideal structure.

For $u, v \in S$ we define

$$u\mathcal{R}v \;\Leftrightarrow\; uS \cup \{u\} = vS \cup \{v\}, \quad u\mathcal{L}v \;\Leftrightarrow\; Su \cup \{u\} = Sv \cup \{v\},$$

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

Green's relations and maximal subgroups

Green's relations: equivalence relations reflecting ideal structure.

For $u, v \in S$ we define

$$u\mathcal{R}v \;\Leftrightarrow\; uS \cup \{u\} = vS \cup \{v\}, \quad u\mathcal{L}v \;\Leftrightarrow\; Su \cup \{u\} = Sv \cup \{v\},$$

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

{ Maximal subgroups of S } = { $\mathcal{H}$-classes that contain idempotents }

Green's relations and maximal subgroups

Green's relations: equivalence relations reflecting ideal structure.

For $u, v \in S$ we define

$$u\mathcal{R}v \;\Leftrightarrow\; uS \cup \{u\} = vS \cup \{v\}, \quad u\mathcal{L}v \;\Leftrightarrow\; Su \cup \{u\} = Sv \cup \{v\},$$

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

{ Maximal subgroups of S } = { $\mathcal{H}$-classes that contain idempotents }

Example. Let $S = T_3$ and $\epsilon = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{smallmatrix}\right) \in E(S)$. Then

$$H_\epsilon = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{smallmatrix}\right) \right\} \cong S_2.$$

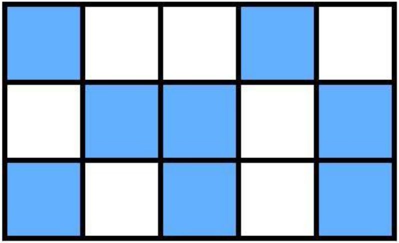
Thinking about semigroups

- Semigroup $\longleftrightarrow$ Universe
- Idempotents $\longleftrightarrow$ Stars
- Solar system (one for each star) $\longleftrightarrow$ Maximal subgroups (one for each idempotent)
- Galaxies (collection of star systems) $\longleftrightarrow$ Regular $\mathcal{D}$-classes (‘collection’ of maximal subgroups)

Regular $\mathcal{D}$-classes

$$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

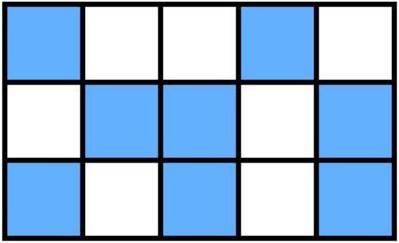
- A $\mathcal{D}$-class is (von Neumann) regular if it contains an idempotent



Regular $\mathcal{D}$-classes

$$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

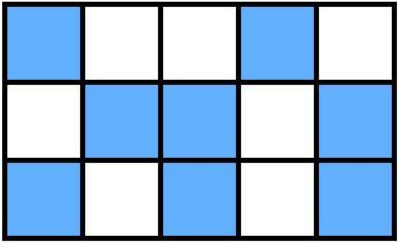
- A $\mathcal{D}$-class is (von Neumann) regular if it contains an idempotent
- A regular $\mathcal{D}$-class has ≥ 1 idempotent in every $\mathcal{R}$- and every $\mathcal{L}$-class.



Regular $\mathcal{D}$-classes

$$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

- A $\mathcal{D}$-class is (von Neumann) regular if it contains an idempotent
- A regular $\mathcal{D}$-class has ≥ 1 idempotent in every $\mathcal{R}$- and every $\mathcal{L}$-class.
- All maximal subgroups in a regular $\mathcal{D}$-class are isomorphic.



Presentations for maximal subgroups of $IG(E)$

S - semigroup, $E = E(S)$, assume $\langle E(S) \rangle = S$

Free idempotent generated semigroup:

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e, f \in E,\ \{e, f\} \cap \{ef, fe\} \neq \emptyset) \ \rangle$$

$\phi : IG(E) \to S$ - the natural homomorphism

- ϕ is bijective on idempotents (Easdown (1985))

Presentations for maximal subgroups of $IG(E)$

S - semigroup, $E = E(S)$, assume $\langle E(S) \rangle = S$

Free idempotent generated semigroup:

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e, f \in E,\ \{e, f\} \cap \{ef, fe\} \neq \emptyset)\ \rangle$$

$\phi : IG(E) \to S$ - the natural homomorphism

- ϕ is bijective on idempotents (Easdown (1985))
- ϕ maps the $\mathcal{R}$-class of $e \in E$ onto the corresponding class of e in S; this induces a bijection between the set of all $\mathcal{R}$-classes in the $\mathcal{D}$-class of e in $IG(E)$ and the corresponding set in S, dually for $\mathcal{L}$-classes (Fitz-Gerald (1972))

Presentations for maximal subgroups of $IG(E)$

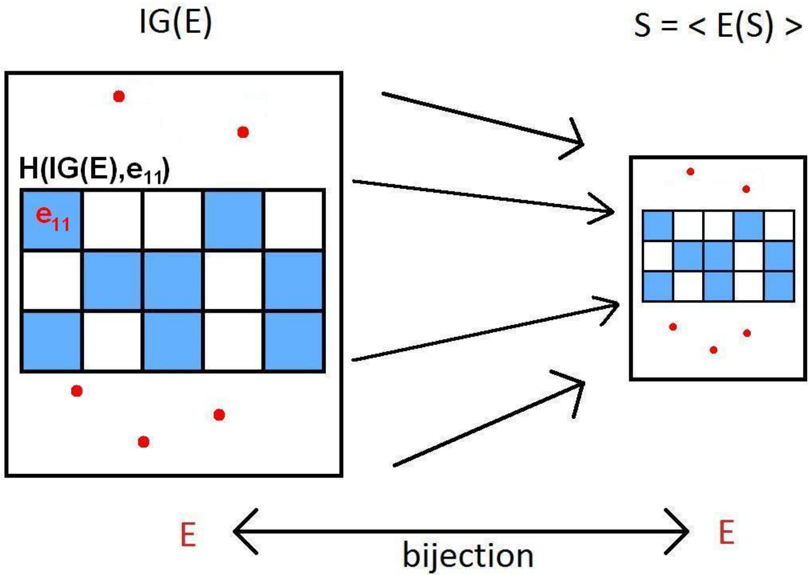
S - semigroup, $E = E(S)$, assume $\langle E(S) \rangle = S$

Free idempotent generated semigroup:

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e,f \in E,\ \{e,f\} \cap \{ef, fe\} \neq \emptyset)\ \rangle$$

$\phi : IG(E) \to S$ - the natural homomorphism

- ϕ is bijective on idempotents (Easdown (1985))
- ϕ maps the $\mathcal{R}$-class of $e \in E$ onto the corresponding class of e in S; this induces a bijection between the set of all $\mathcal{R}$-classes in the $\mathcal{D}$-class of e in $IG(E)$ and the corresponding set in S, dually for $\mathcal{L}$-classes (Fitz-Gerald (1972))
- The restriction of ϕ to the maximal subgroup of $IG(E)$ containing $e \in E$ is a homomorphism onto the maximal subgroup of S containing e.


IG(E)
S = < E(S) >
H(IG(E),e11)
e11
E
E
bijection

Presentations for maximal subgroups of $IG(E)$

S - semigroup, $E = E(S)$, assume $\langle E(S) \rangle = S$

Free idempotent generated semigroup:

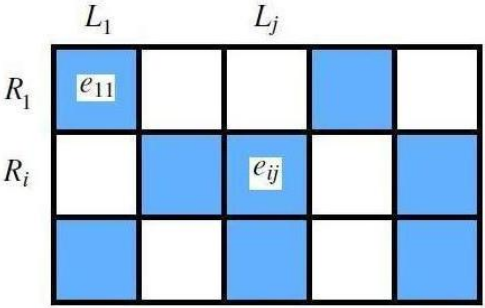
$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e,f \in E,\ \{e,f\} \cap \{ef, fe\} \neq \emptyset) \ \rangle$$

Let $e_{11} \in E$ - arbitrary.

Aim: Write down a presentation for the maximal subgroup H of $IG(E)$ containing e_{11}.

Singular squares

- D - $\mathcal{D}$-class in S of e_{11}
- R_i $(i \in I)$ - $\mathcal{R}$-classes in D
- L_j $(j \in J)$ - $\mathcal{L}$-classes in D
- $H_{ij} = R_i \cap L_j$ for $i \in I, j \in J$
- e_{ij} - identity of H_{ij} when a group



Singular squares

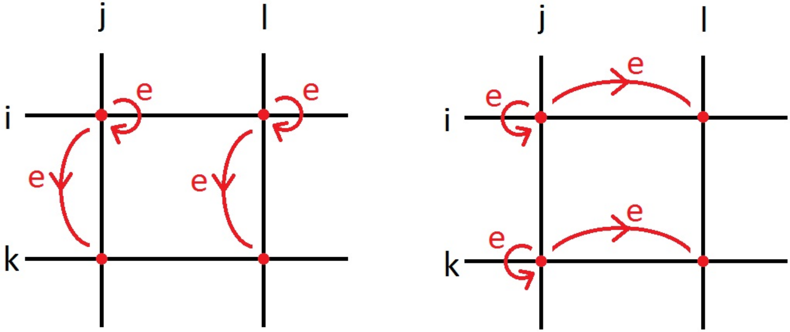
Definition

A quadruple $(i,k;j,l) \in I \times I \times J \times J$ is a singular square if there exists an idempotent $e \in E$ such that one of the following dual conditions holds:

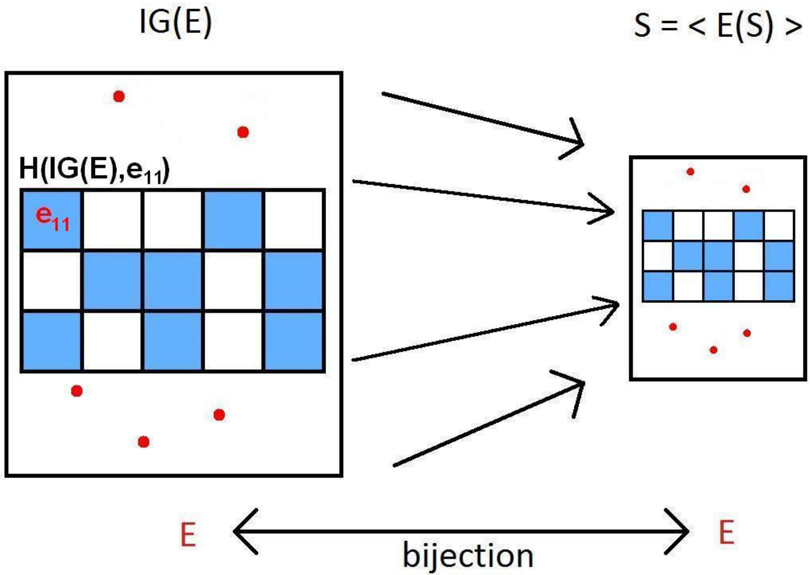
$$ee_{ij} = e_{ij},\ ee_{kj} = e_{kj},\ e_{ij}e = e_{il},\ e_{kj}e = e_{kl},\ \text{or}$$
$$e_{ij}e = e_{ij},\ e_{il}e = e_{il},\ ee_{ij} = e_{kj},\ ee_{il} = e_{kl}.$$

We will say that e singularises the square.

- Determining which squares are singular is something that is computed inside S.



j
l
i
k
e
e
e
e
j
l
i
k
e
e
e
e


IG(E)
S = < E(S) >
H(IG(E),e11)
e11
E
bijection
E

Presentations for maximal subgroups of $IG(E)$

Roughly speaking...

Maximal subgroups of free idempotent generated semigroups are defined by the relations arising from singular squares.

Presentations for maximal subgroups of $IG(E)$

Roughly speaking...

Maximal subgroups of free idempotent generated semigroups are defined by the relations arising from singular squares.

The maximal subgroup H of $IG(E)$ containing e_{11} is given by a presentation with:

Generators: $F = \{f_{ij} : H_{ij} \text{ is a group}\}$

An abstract set of generating symbols in bijective correspondence with the set of idempotents in the $\mathcal{D}$-class D of e_{11}.

Presentations for maximal subgroups of $IG(E)$

Roughly speaking...

Maximal subgroups of free idempotent generated semigroups are defined by the relations arising from singular squares.

The maximal subgroup H of $IG(E)$ containing e_{11} is given by a presentation with:

Generators: $F = \{f_{ij} : H_{ij} \text{ is a group}\}$

An abstract set of generating symbols in bijective correspondence with the set of idempotents in the $\mathcal{D}$-class D of e_{11}.

Relations: $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$ for every singular square $(i,k;j,l)$

(together with two further families of relations, one which identifies certain pairs of generators, and another which sets certain generators equal to 1)

Presentation for maximal subgroup of $IG(E)$

Theorem (RG & Ruskuc (2011))

A presentation for the group $H = H(IG(E), e_{11})$ is given by

$$
\begin{array}{lll}
\langle F \quad | & f_{ij} = f_{il} & (r_j e_{il} \text{ is a Schreier word \& both } H_{ij}, H_{il} \text{ groups}), \\
& f_{i,\pi(i)} = 1 & (i \in I), \\
& f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl} & ((i,k;j,l) \in \Sigma)\rangle
\end{array}
$$

where Σ is the set of all singular squares.

Schreier system of representatives: r_j $(j \in J)$ - words from $(E \cap D_e)^*$ such that $H_{11} r_j = H_{1j}$, for all $j \in J$, and every prefix of an r_j is some r_l.

Presentation for maximal subgroup of $IG(E)$

Theorem (RG & Ruskuc (2011))

A presentation for the group $H = H(IG(E), e_{11})$ is given by

$$
\begin{array}{lll}
\langle F \quad | & f_{ij} = f_{il} & (r_j e_{il} \text{ is a Schreier word \& both } H_{ij},\, H_{il} \text{ groups}),\\
& f_{i,\pi(i)} = 1 & (i \in I),\\
& f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl} & ((i,k;j,l) \in \Sigma)\rangle
\end{array}
$$

where Σ is the set of all singular squares.

Schreier system of representatives: r_j $(j \in J)$ - words from $(E \cap D_e)^*$ such that $H_{11} r_j = H_{1j}$, for all $j \in J$, and every prefix of an r_j is some r_l.

Pivot function: For each $i \in I$, fix $\pi(i) \in J$ such that $H_{i,\pi(i)}$ is a group.

Presentation for maximal subgroup of $IG(E)$

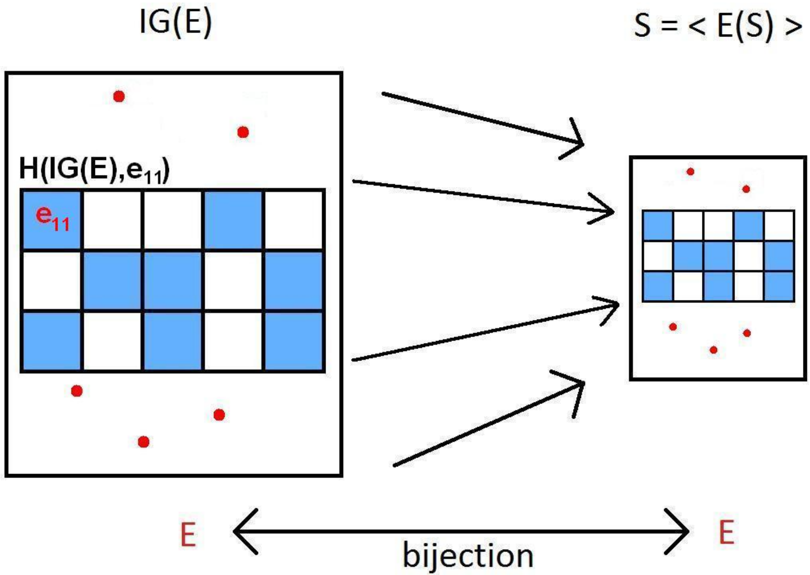
Theorem (RG & Ruskuc (2011))

A presentation for the group $H = H(IG(E), e_{11})$ is given by

$$\begin{aligned} \langle F \quad | \quad & f_{ij} = f_{il} && (r_j e_{il} \text{ is a Schreier word \& both } H_{ij}, H_{il} \text{ groups}), \\ & f_{i,\pi(i)} = 1 && (i \in I), \\ & f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl} && ((i,k;j,l) \in \Sigma) \rangle \end{aligned}$$

where Σ is the set of all singular squares.

- Generalises the corresponding result for (von Neumann) regular semigroups proved by Nambooripad (1979).
- Proof makes use of general Redemeister–Schreier theory for subgroups of semigroups developed in Ruskuc (1999).
- If there are no singular squares then we obtain a free group.
- The presentation has “many generators” and relations that are all short.



IG(E)
S = < E(S) >
H(IG(E),e11)
e11
E
E
bijection

Obtaining any given group

Input: G - arbitrary group

Task: Find a pair S and $e \in E(S)$ such that $G \cong H(IG(E), e)$.

Obtaining any given group

Input: G - arbitrary group

Task: Find a pair S and $e \in E(S)$ such that $G \cong H(IG(E), e)$.

We work inside a certain semigroup $B_{I,J}$. Elements are pairs of mappings.

Obtaining any given group

Input: G - arbitrary group

Task: Find a pair S and $e \in E(S)$ such that $G \cong H(IG(E), e)$.

We work inside a certain semigroup $B_{I,J}$. Elements are pairs of mappings.

$B_{I,J}$ has a minimal ideal $R_{I,J}$, which is a $\mathcal{D}$-class of $B_{I,J}$, $\mathcal{R}$-classes indexed by I and $\mathcal{L}$-classes indexed by J.

Every element of $R_{I,J}$ is idempotent (it is a rectangular band). Fix $\rho_{11} \in R_{I,J}$.

Obtaining any given group

Input: G - arbitrary group

Task: Find a pair S and $e \in E(S)$ such that $G \cong H(IG(E), e)$.

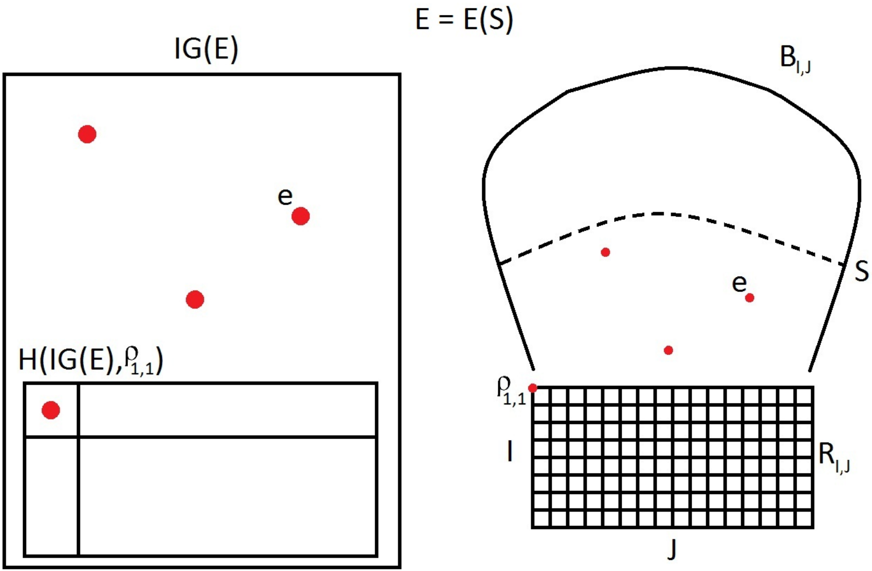
We work inside a certain semigroup $B_{I,J}$. Elements are pairs of mappings.

$B_{I,J}$ has a minimal ideal $R_{I,J}$, which is a $\mathcal{D}$-class of $B_{I,J}$, $\mathcal{R}$-classes indexed by I and $\mathcal{L}$-classes indexed by J.

Every element of $R_{I,J}$ is idempotent (it is a rectangular band). Fix $\rho_{11} \in R_{I,J}$.

We consider semigroups in the range

$$R_{I,J} \leq S \leq B_{I,J}.$$



E = E(S)
IG(E)
e
H(IG(E),ρ1,1)
B_{I,J}
S
e
ρ1,1
I
R_{I,J}
J

Obtaining any given group

Fundamental idea: Careful choice of I, J, and S allow us to create a collection of singular squares in $R_{I,J}$ that encode the full multiplication table of G inside the presentation.

Preserving finiteness properties

Our construction proves:

Theorem (RG & Ruskuc (2011))

Every group is a maximal subgroup of some free idempotent generated semigroup.

- One drawback of the above construction is that if G is infinite, then the semigroup S constructed will necessarily be infinite.

Preserving finiteness properties

Our construction proves:

Theorem (RG & Ruskuc (2011))

Every group is a maximal subgroup of some free idempotent generated semigroup.

- One drawback of the above construction is that if G is infinite, then the semigroup S constructed will necessarily be infinite.
- If G is finitely presented then we can do better than this:

Theorem (RG & Ruskuc (2011))

Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup.

The word problem

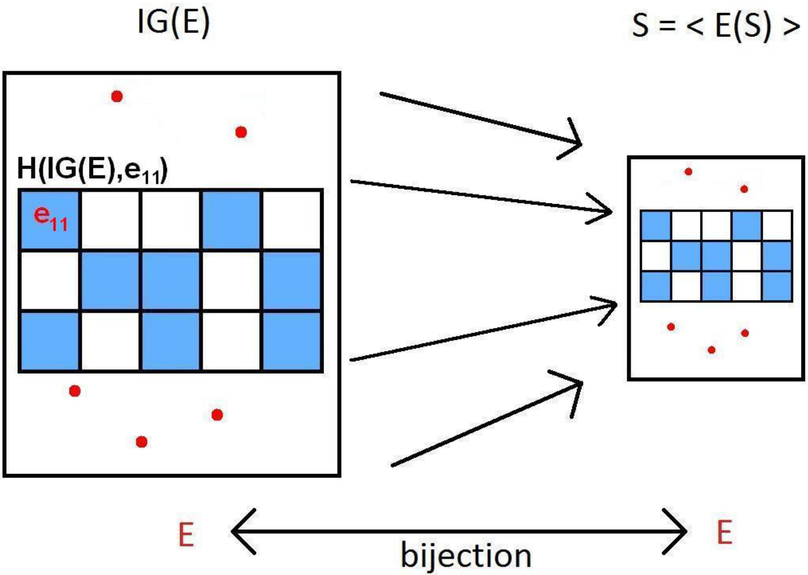
Since there exist finitely presented groups that have unsolvable word problem, combining such a group with the above theorem gives:

Corollary

There exists a free idempotent generated semigroup S arising from a finite semigroup such that the word problem for S is unsolvable.

What next?

Investigate $IG(E)$ for biordered sets of idempotents E of semigroups that occur “in nature”.



IG(E)
S = < E(S) >
H(IG(E),e11)
e11
E
E
bijection

The full transformation monoid T_n

The $\mathcal{D}$-classes of T_n are

$$D_r = \{\alpha \in T_n : |\mathrm{im}(\alpha)| = r\} \ (1 \leq r \leq n).$$

- The maximal subgroups of T_n in D_r are isomorphic to S_r.
- Fix $e \in E(T_n)$ with $|\mathrm{im}(e)| = r$.
- Can we identify the group $H = H(IG(E(T_n), e)$?

The full transformation monoid T_n

The $\mathcal{D}$-classes of T_n are

$$D_r = \{\alpha \in T_n : |\mathrm{im}(\alpha)| = r\} \ (1 \le r \le n).$$

- The maximal subgroups of T_n in D_r are isomorphic to S_r.
- Fix $e \in E(T_n)$ with $|\mathrm{im}(e)| = r$.
- Can we identify the group $H = H(IG(E(T_n), e)$?
- We know that H is a homomorphic preimage of S_r.

Let $e \in E(T_n)$ with $|\mathrm{im}(e)| \le n - 2$ and consider $H = H(IG(E(T_n), e)$.

The full transformation monoid T_n

Using the general theory above, and then applying carefully a set of Tietze transformations we obtain:

$$
\begin{aligned}
\langle g_1, \ldots, g_{r-1} \mid g_i^2 &= 1 && (i \in [1, r-1]), \\
g_i g_j &= g_j g_i && (i, j \in [1, r-1],\ |i-j| > 1), \\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && (i \in [1, r-1])\rangle,
\end{aligned}
$$

The full transformation monoid T_n

Using the general theory above, and then applying carefully a set of Tietze transformations we obtain:

$$\begin{aligned}
\langle g_1, \ldots, g_{r-1} \mid g_i^2 &= 1 && (i \in [1, r-1]),\\
g_i g_j &= g_j g_i && (i, j \in [1, r-1],\ |i-j| > 1),\\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && (i \in [1, r-1])\rangle,
\end{aligned}$$

the standard *Coxeter presentation* defining the group S_r.

The full transformation monoid T_n

Using the general theory above, and then applying carefully a set of Tietze transformations we obtain:

$$
\begin{aligned}
\langle g_1, \ldots, g_{r-1} \mid g_i^2 &= 1 && (i \in [1, r-1]),\\
g_i g_j &= g_j g_i && (i, j \in [1, r-1],\ |i-j| > 1),\\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && (i \in [1, r-1])\rangle,
\end{aligned}
$$

the standard *Coxeter presentation* defining the group S_r.

Theorem (RG & Ruškuc (2011))

Let T_n be the full transformation semigroup, let E be its set of idempotents, and let $e \in E$ be an arbitrary idempotent with image size r ($1 \leq r \leq n-2$). Then the maximal subgroup H_e of the free idempotent generated semigroup $IG(E)$ containing e is isomorphic to the symmetric group S_r.

Conclusion: The structure of the idempotents of T_n naturally “encode” the standard Coxeter presentations for the symmetric groups.

Semigroups of matrices

Theorem (Brittenham, Margolis & Meakin (2010))

Let E be the biordered set of idempotents of $M_n(Q)$, for Q a division ring, and let e be an idempotent matrix of rank 1 in $M_n(Q)$. For $n \geq 3$, the maximal subgroup of $IG(E)$ containing e is isomorphic to Q^*, the multiplicative group of units of Q.

Semigroups of matrices

Theorem (Brittenham, Margolis & Meakin (2010))

Let E be the biordered set of idempotents of $M_n(Q)$, for Q a division ring, and let e be an idempotent matrix of rank 1 in $M_n(Q)$. For $n \geq 3$, the maximal subgroup of $IG(E)$ containing e is isomorphic to Q^*, the multiplicative group of units of Q.

Theorem (Dolinka & RG (2011))

Let E be the biordered set of idempotents of $M_n(Q)$, for Q a division ring, and let e be an idempotent matrix of $M_n(Q)$ with rank $r < n/3$. For $n \geq 3$, the maximal subgroup of $IG(E)$ with identity e is isomorphic to the general linear group $GL_r(Q)$.