

# Free idempotent generated semigroups over the full linear monoid

Robert Gray  
(joint work with Igor Dolinka)



Centro de Álgebra  
da Universidade de Lisboa

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UNIVERSIDADE  
DE LISBOA

# Combinatorics and algebra

Combinatorics often lies at the heart of problems in algebra we are interested in solving...

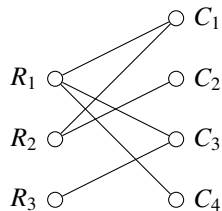
*“[Roger] Lyndon produces elegant mathematics and thinks in terms of broad and deep ideas . . . I once asked him whether there was a common thread to the diverse work in so many different fields of mathematics, he replied that he felt the problems on which he had worked had all been combinatorial in nature.”*

K. I. Appel, in Contributions to Group Theory, 1984.

# Combinatorics: $(0, 1)$ -matrices

$$\begin{array}{c} C_1 \ C_2 \ C_3 \ C_4 \\ \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left( \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

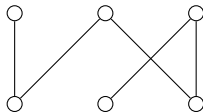
$(0, 1)$ -matrix



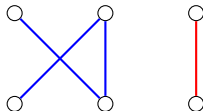
Bipartite graph

## $(0, 1)$ -matrices and connectedness

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



- The 1s in the matrix are **connected** if any pair of entries 1 is connected by a sequence of 1s where adjacent terms in the sequence belong to same row/column.

# Combinatorics

Symbols

$$A = \{\heartsuit, \odot, \otimes, \mathbb{J}\}$$

Table

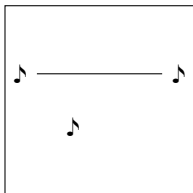
$$M = \begin{pmatrix} \otimes & \heartsuit & \odot & \heartsuit \\ \mathbb{J} & \otimes & \otimes & \mathbb{J} \\ \otimes & \mathbb{J} & \otimes & \otimes \\ \odot & \otimes & \odot & \heartsuit \end{pmatrix}$$

For each symbol  $x$  we can ask whether the  $x$ s are connected in  $M$ .

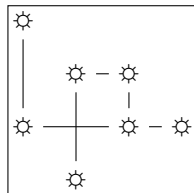
Let  $\Delta(x)$  be a graph with vertices the occurrences of the symbol  $x$  and symbols in the same row/col connected by an edge.

## Connectedness in tables

$$M = \begin{pmatrix} \text{☀} & \text{♥} & \text{😊} & \text{♥} \\ \text{🎵} & \text{☀} & \text{☀} & \text{🎵} \\ \text{☀} & \text{🎵} & \text{☀} & \text{☀} \\ \text{😊} & \text{☀} & \text{😊} & \text{♥} \end{pmatrix}$$



$\Delta(\text{🎵})$  is not connected



$\Delta(\text{☀})$  is connected

# Tables in algebra

## Multiplication tables

### Group multiplication tables

	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$
$a$	$a$	1	$c$	$b$
$b$	$b$	$c$	1	$a$
$c$	$c$	$b$	$a$	1

- ▶ The multiplication table of a group is a Latin square, so..
- ▶ None of the graphs  $\Delta(x)$  will be connected.

# Tables in algebra

## Multiplication tables

Multiplication table of a field.

Field with three elements  $\mathbb{F} = \{0, 1, 2\}$ .

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

- ▶  $\Delta(0)$  is connected
- ▶  $\Delta(f)$  is not connected for every  $f \neq 0$



# Tables in algebra

## Vectors

$\mathbb{F} = \{0, 1\}$ , vectors in  $\mathbb{F}^3$ , entries in table from  $\mathbb{F}$

	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
(0, 0, 0)	0	0	0	0	0	0	0	0
(0, 0, 1)	0	1	0	1	0	1	0	1
(0, 1, 0)	0	0	1	1	0	0	1	1
(0, 1, 1)	0	1	1	0	0	1	1	0
(1, 0, 0)	0	0	0	0	1	1	1	1
(1, 0, 1)	0	1	0	1	1	0	1	0
(1, 1, 0)	0	0	1	1	1	1	0	0
(1, 1, 1)	0	1	1	0	1	0	0	1

- For every symbol  $x$  in the table  $\Delta(x)$  is connected.

# Outline

## Free idempotent generated semigroups

- Background and recent results

- Maximal subgroups of free idempotent generated semigroups

## The full linear monoid

- Basic properties

- The free idempotent generated semigroup over the full linear monoid

- Proof sketch: connectedness properties in tables

## Open problems

# Idempotent generated semigroups

$S$  - semigroup,  $E = E(S)$  - idempotents  $e = e^2$  of  $S$

**Definition.**  $S$  is **idempotent generated** if  $\langle E(S) \rangle = S$

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**Definition.**  $S$  is **idempotent generated** if  $\langle E(S) \rangle = S$

- ▶ Many natural examples
  - ▶ Howie (1966) -  $T_n \setminus S_n$ , the non-invertible transformations;
  - ▶ Erdős (1967) - singular part of  $M_n(\mathbb{F})$ , semigroup of all  $n \times n$  matrices over a field  $\mathbb{F}$ ;
  - ▶ Putcha (2006) - conditions for a reductive linear algebraic monoid to have the same property.

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  - ▶ Putcha (2006) - conditions for a reductive linear algebraic monoid to have the same property.
- ▶ Idempotent generated semigroups are “general”
  - ▶ Every semigroup  $S$  embeds into an idempotent generated semigroup.

# Free idempotent generated semigroups

A problem in algebra

$S$  - semigroup,  $E = E(S)$  - idempotents of  $S$

Let  $IG(E)$  denote the semigroup defined by the following presentation.

$$IG(E) = \langle E \mid e \cdot f = ef \ (e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset) \rangle$$

$IG(E)$  is called the **free idempotent generated semigroup on  $E$** .

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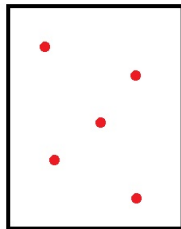
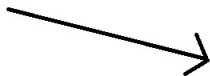
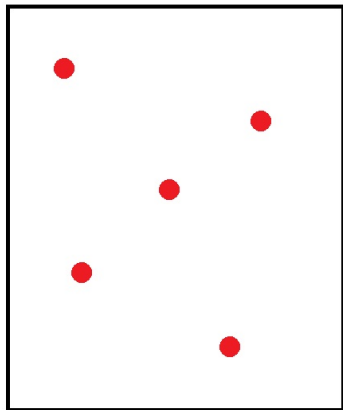
$IG(E)$  is called the **free idempotent generated semigroup on  $E$** .

## Theorem (Easdown (1985))

*Let  $S$  be an idempotent generated semigroup with  $E = E(S)$ . Then  $IG(E)$  is an idempotent generated semigroup and there is a surjective homomorphism  $\phi : IG(E) \rightarrow S$  which is bijective on idempotents.*

$IG(E)$

$S = \langle E(S) \rangle$



$E$



bijection



$E$



## First steps towards understanding $IG(E)$

**Conclusion.** It is important to understand  $IG(E)$  if one is interested in understanding an arbitrary idempotent generated semigroups.

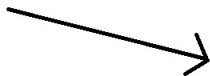
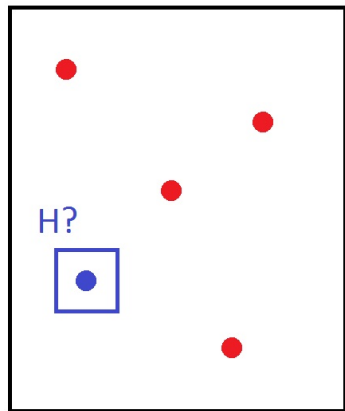
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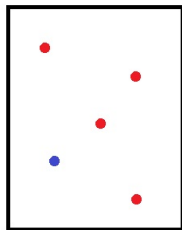
**Question.** Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

$IG(E)$

$E = E(S)$



$S$



$E$



bijection



$E$

# Maximal subgroups of $IG(E)$

**Question.** Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

- ▶ Work of Pastijn (1977, 1980), Nambooripad & Pastijn (1980), McElwee (2002) led to a conjecture that all these groups must be free groups.
- ▶ Brittenham, Margolis & Meakin (2009) - gave the first counterexamples to this conjecture obtaining the groups
  - ▶  $\mathbb{Z} \oplus \mathbb{Z}$  and  $\mathbb{F}^*$  where  $\mathbb{F}$  is an arbitrary field.

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- ▶ Gray & Ruskuc (2012) proved that every group is a maximal subgroup of some free idempotent generated semigroup.

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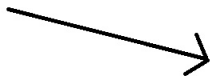
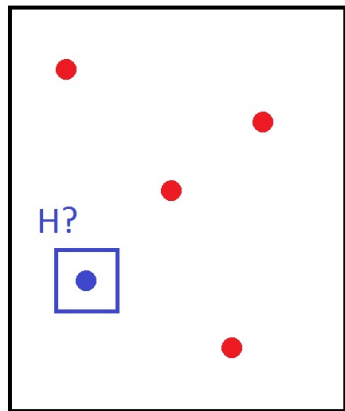
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## New focus

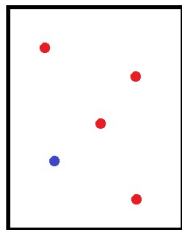
What can be said about maximal subgroups of  $IG(E)$  where  $E = E(S)$  for semigroups  $S$  that arise in nature?

$IG(E)$

$E = E(S)$



$S$



$E$

bijection

$E$

# The full linear monoid

$\mathbb{F}$  - arbitrary field,  $n \in \mathbb{N}$

$$M_n(\mathbb{F}) = \{n \times n \text{ matrices over } \mathbb{F}\}.$$

- ▶ Plays an analogous role in semigroup theory as the general linear group does in group theory.
- ▶ Important in a range of areas:
  - ▶ Representation theory of semigroups
  - ▶ Putcha–Renner theory of linear algebraic monoids and finite monoids of Lie type



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## Aim

Investigate the above problem in the case  $S = M_n(\mathbb{F})$  and  $E = E(S)$ .

# Properties of $M_n(\mathbb{F})$

## Theorem (J.A. Erdős (1967))

$$\langle E(M_n(\mathbb{F})) \rangle = \{\text{identity matrix and all non-invertible matrices}\}.$$

- ▶  $M_n(\mathbb{F})$  may be partitioned into the sets

$$D_r = \{A : \text{rank}(A) = r\}, \quad r \leq n.$$

(these are the  $\mathcal{D}$ -classes).

- ▶ The maximal subgroups in  $D_r$  are isomorphic to  $GL_r(\mathbb{F})$ .

# The problem

By Easdown (1985) we may identify

$$E = E(M_n(\mathbb{F})) = E(IG(E)).$$

Let

$$W = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \in D_r \subseteq M_n(\mathbb{F})$$

where  $I_r$  denotes the  $r \times r$  identity matrix.

$W$  is an idempotent matrix of rank  $r$ .

**Problem:** Identify the maximal subgroup  $H_W$  of

$$IG(E) = \langle E \mid e \cdot f = ef \ (e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset) \rangle$$

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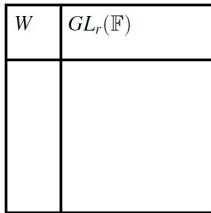
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**General fact:**  $H_W$  is a homomorphic preimage of  $GL_r(\mathbb{F})$ .

$IG(E)$ 
 $S = M_n(\mathbb{F})$ 
 $E = E(S)$ 
 $GL_n(\mathbb{F})$ 
 $D_n$ 
 $H_W$ 

 $D_r$ 
 $\square$ 
 $D_0$

# Results

$n \in \mathbb{N}$ ,  $\mathbb{F}$  - field,  $E = E(M_n(\mathbb{F}))$ ,

$W \in M_n(\mathbb{F})$  - idempotent of rank  $r$

$H_W$  = maximal subgroup of  $IG(E)$

**Theorem (Brittenham, Margolis, Meakin (2009))**

*For  $n \geq 3$  and  $r = 1$  we have  $H_W \cong GL_r(\mathbb{F}) \cong \mathbb{F}^*$ .*

- This result provided the first example of a torsion group that arises as a maximal subgroup of a free idempotent generated semigroup.

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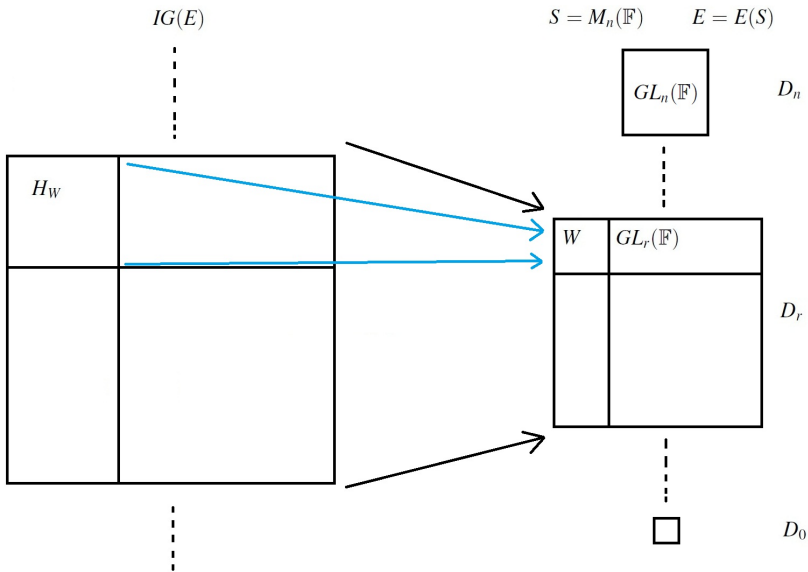
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## Theorem (Dolinka, Gray (2012))

Let  $n$  and  $r$  be positive integers with  $r < n/3$ . Then  $H_W \cong GL_r(\mathbb{F})$ .





# Recent results

Our proof builds on ideas developed in the following recent papers:



M. Brittenham, S. W. Margolis, and J. Meakin,

Subgroups of the free idempotent generated semigroups need not be free.

*J. Algebra* 321 (2009), 3026–3042.



M. Brittenham, S. W. Margolis, and J. Meakin,

Subgroups of free idempotent generated semigroups: full linear monoids.

arXiv: 1009.5683.



R. Gray and N. Ruškuc,

On maximal subgroups of free idempotent generated semigroups.

*Israel J. Math.* (to appear).



R. Gray and N. Ruškuc,

Maximal subgroups of free idempotent generated semigroups over the full transformation monoid.

*Proc. London Math. Soc.* (to appear)

# Step 1: Writing down a presentation for $H_W$

## Definition

A matrix is in **reduced row echelon form** (RRE form) if:

- ▶ rows with at least one nonzero element are above any rows of all zeros
- ▶ the leading coefficient (the first nonzero number from the left) of a nonzero row is always strictly to the right of the leading coefficient of the row above it, and
- ▶ every leading coefficient is 1 and is the only nonzero entry in its column.

## Examples

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

## Step 1: Writing down a presentation for $H_W$

$n, r \in \mathbb{N}$  fixed with  $r < n$

$$\mathcal{Y}_r = \{r \times n \text{ rank } r \text{ matrices in RRE form}\}$$

$$\mathcal{X}_r = \{\text{transposes of elements of } \mathcal{Y}_r\}$$

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- ▶ Matrices in  $\mathcal{Y}_r$  have no rows of zeros, so have  $r$  leading columns.

$$\text{e.g. } n = 4, r = 3, \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{Y}_3.$$

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- ▶ Define a matrix  $P_r = (P_r(Y, X))$  defined for  $Y \in \mathcal{Y}_r, X \in \mathcal{X}_r$  by

$$P_r(Y, X) = YX \in M_r(\mathbb{F}).$$

$$\mathcal{X}_r$$

$$n\begin{pmatrix} r \\ X \end{pmatrix}$$

$$P_r$$

$$\mathcal{Y}_r$$

$$A_j$$

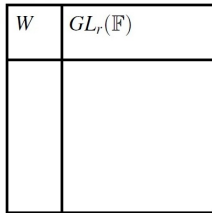
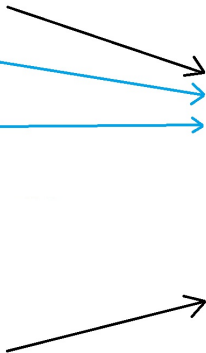
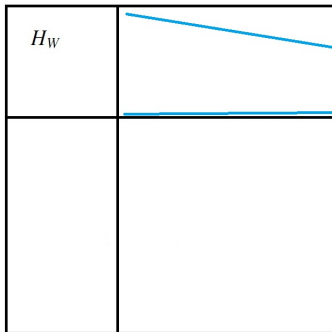
$$A_l$$

$$r\left(\begin{array}{c} n \\ Y \end{array}\right)$$

$$A_k$$

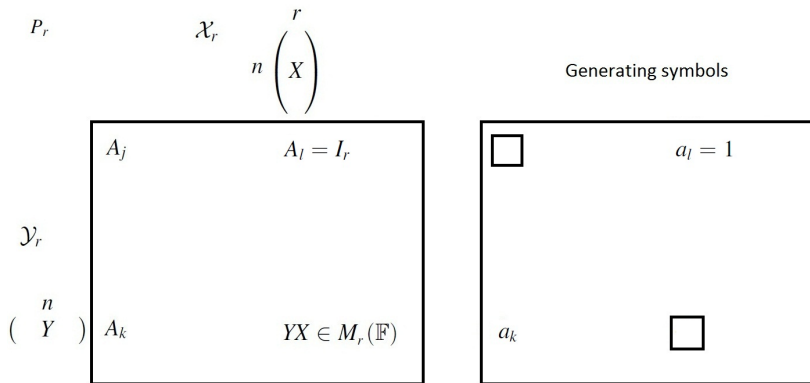
$$YX\in M_r(\mathbb{F})$$

$IG(E)$ 
 $S = M_n(\mathbb{F})$ 
 $E = E(S)$ 

 $D_n$ 

 $D_r$ 

 $D_0$

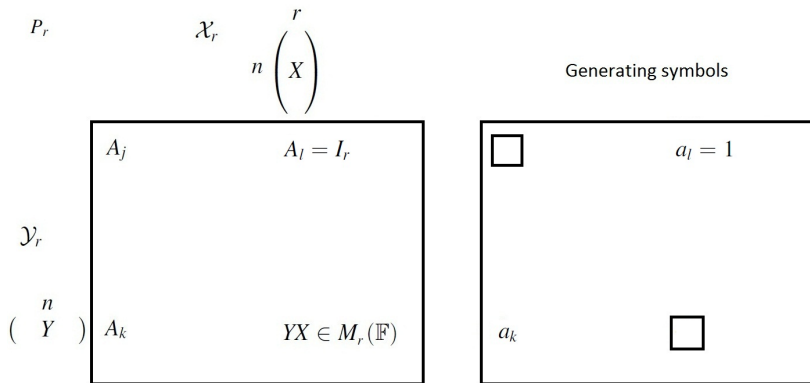
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**Generators:**  $\{a_j \mid A_j \text{ is an entry in } P_r \text{ satisfying } A_j \in GL_r(\mathbb{F})\}$



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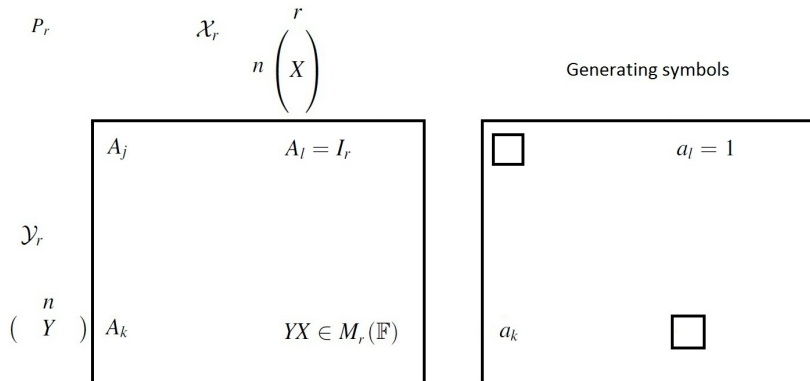


**Generators:**  $\{a_j \mid A_j \text{ is an entry in } P_r \text{ satisfying } A_j \in GL_r(\mathbb{F})\}$

**Relations:**

- (I)  $a_j = 1$  for all entries  $A_j$  in  $P_r$  satisfying  $A_j = I_r$   $A_j$   $A_k$
- (II)  $a_j a_k^{-1} = a_l a_m^{-1} \Leftrightarrow (A_j, A_k, A_l, A_m)$  is a **singular square** of invertible  $r \times r$  matrices from  $P_r$  with  $A_j A_k^{-1} = A_l A_m^{-1}$ .  $A_l$   $A_m$

# Structure of the proof that $H_W \cong GL_r(\mathbb{F})$



**Step 1:** Write down a presentation for  $H_W$ .

**Step 2:** Prove that for any two entries  $A_j, A_k$  in the table  $P_r$ , if  $A_j = A_k \in GL_r(\mathbb{F})$  then  $a_j = a_k$  is deducible from the relations.

**Step 3:** Find defining relations for  $GL_r(\mathbb{F})$  using the singular square relations (II).

## Step 2: Strong edges and relations

### Definition

We say entries  $A_j$  and  $A_k$  with  $A_j = A_k$  are connected by a **strong edge** if

$$\begin{array}{ccc}
 A_j & \text{---} & A_k \\
 I_r & & I_r
 \end{array}
 \quad \text{or} \quad
 \begin{array}{cc}
 A_j & I_r \\
 | & \\
 A_k & I_r
 \end{array}$$

**Lemma:** If  $A_j = A_k \in GL_r(\mathbb{F})$  are connected by a strong edge then  $a_j = a_k$  is a consequence of the relations.

$$\begin{array}{ccc}
 A_j & \text{---} & A_k & a_j & a_k \\
 \Rightarrow & & \Rightarrow & a_j = a_k \text{ can be deduced} \\
 I_r & & I_r & 1 & 1
 \end{array}$$

A singular square    Using relations (I)

## Step 2: Proving $A_i = A_j$ invertible $\Rightarrow a_i = a_j$

### Definition

**Strong path** = path composed of strong edges.

### Aim

Prove that for every pair  $A_j, A_k$  of entries in  $P_r$ , if  $A_j = A_k$  then there is a strong path from  $A_j$  to  $A_k$ .

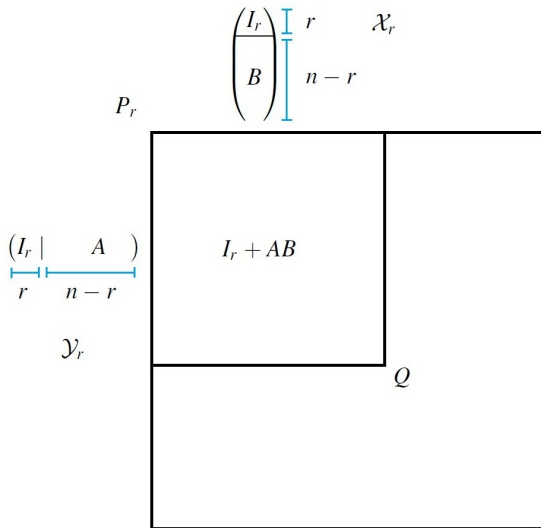
Once proved this will have the following:

### Corollary

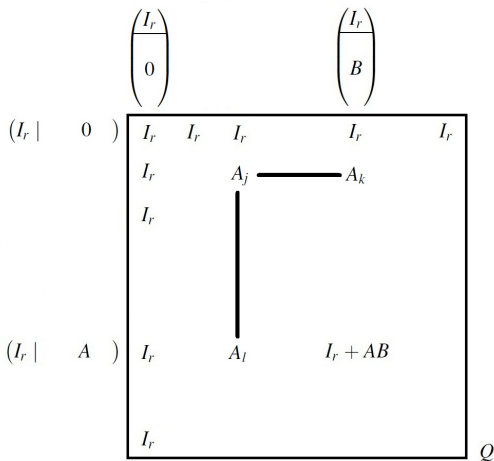
For every pair  $A_j = A_k \in GL_r(\mathbb{F})$  in the table  $P_r$  the relation  $a_j = a_k$  is a consequence of the defining relations in the presentation.

## The small box $Q$

Is the subtable of  $P_r$  containing entries whose row and column are labelled by matrices of the form  $(I_r \mid A)$  and their transposes, where  $A$  is an  $r \times (n - r)$  matrix over  $\mathbb{F}$ .



## Strongly connecting the small box $Q$



**Observation:** In the small box every edge is a strong edge.

$\therefore$  strongly connecting the small box  $\equiv$  connecting the small box.

# An equivalent problem

$T$  = matrix obtained by taking  $Q$  and subtracting  $I_r$  from every entry

$$\begin{array}{c}
 \begin{array}{c} \left( \begin{array}{c} I_r \\ 0 \end{array} \right) \end{array} \quad \begin{array}{c} \left( \begin{array}{c} I_r \\ B \end{array} \right) \end{array} \\
 (I_r \mid 0) \quad \begin{array}{|c|c|c|c|c|} \hline I_r & I_r & I_r & I_r & I_r \\ \hline I_r & & & & \\ I_r & & & & \\ \hline I_r & & & I_r + AB & \\ \hline I_r & & & & \end{array} \quad \begin{array}{c} (0 \mid 0) \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 0_r & 0_r & 0_r & 0_r & 0_r \\ \hline 0_r & & & & \\ 0_r & & & & \\ \hline 0_r & & & AB & \\ \hline 0_r & & & & \end{array} \\
 \begin{array}{c} (I_r \mid A) \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline I_r & & & I_r + AB & \\ \hline I_r & & & & \end{array} \quad \begin{array}{c} r \text{ } \left( \begin{array}{c} A \end{array} \right) \\ \hline n-r \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 0_r & & & AB & \\ \hline 0_r & & & & \end{array} \\
 \begin{array}{c} Q \end{array} \quad \begin{array}{c} T \end{array}
 \end{array}$$

For every symbol  $X$  in the table  $Q$  the graph  $\Delta(X)$  in  $Q$  is connected.

$\Leftrightarrow$

For every symbol  $X$  in the table  $T$  the graph  $\Delta(X)$  in  $T$  is connected.

## Connecting the small box

So, we have reduced the problem of strongly connecting the small box in  $P_r$  to the following:

Let  $m, k \in \mathbb{N}$  with  $k < m$ , and let

$$\mathcal{B} = \{\text{all } k \times m \text{ matrices over } \mathbb{F}\},$$

$$\mathcal{A} = \{\text{all } m \times k \text{ matrices over } \mathbb{F}\}.$$

Define the matrix  $T = T(B, A)$  by

$$T(B, A) = BA \in M_k(\mathbb{F}), \quad B \in \mathcal{B}, \quad A \in \mathcal{A}.$$

**Question:** Is it true that for every symbol  $X \in M_k(\mathbb{F})$  in the table  $T$  the graph  $\Delta(X)$  is connected?



## Déjà vu

$\mathbb{F} = \{0, 1\}$ , vectors in  $\mathbb{F}^3$ , entries in table from  $\mathbb{F}$

	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
(0, 0, 0)	0	0	0	0	0	0	0	0
(0, 0, 1)	0	1	0	1	0	1	0	1
(0, 1, 0)	0	0	1	1	0	0	1	1
(0, 1, 1)	0	1	1	0	0	1	1	0
(1, 0, 0)	0	0	0	0	1	1	1	1
(1, 0, 1)	0	1	0	1	1	0	1	0
(1, 1, 0)	0	0	1	1	1	1	0	0
(1, 1, 1)	0	1	1	0	1	0	0	1

- For every symbol  $x$  in the table  $\Delta(x)$  is connected.

# Combinatorial properties of tables

And it generalises...

## Proposition

Let  $m, k \in \mathbb{N}$  with  $k < m$ , and let

$$\begin{aligned}\mathcal{B} &= \{\text{all } k \times m \text{ matrices over } \mathbb{F}\}, \\ \mathcal{A} &= \{\text{all } m \times k \text{ matrices over } \mathbb{F}\}.\end{aligned}$$

Define the matrix  $T = T(B, A)$  by

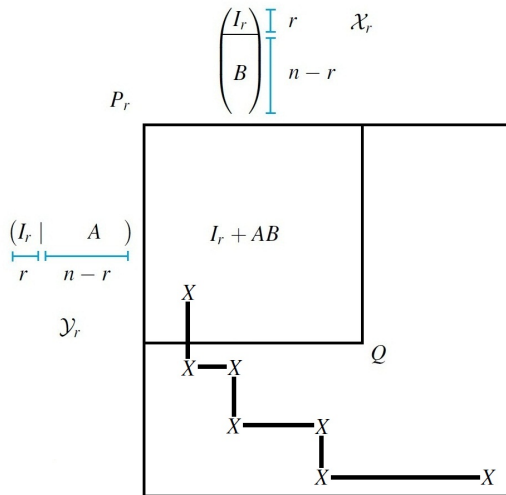
$$T(B, A) = BA \in M_k(\mathbb{F}), \quad B \in \mathcal{B}, \quad A \in \mathcal{A}.$$

Then for every symbol  $X \in M_k(\mathbb{F})$  in the table  $T$  the graph  $\Delta(X)$  is connected.

## Corollary

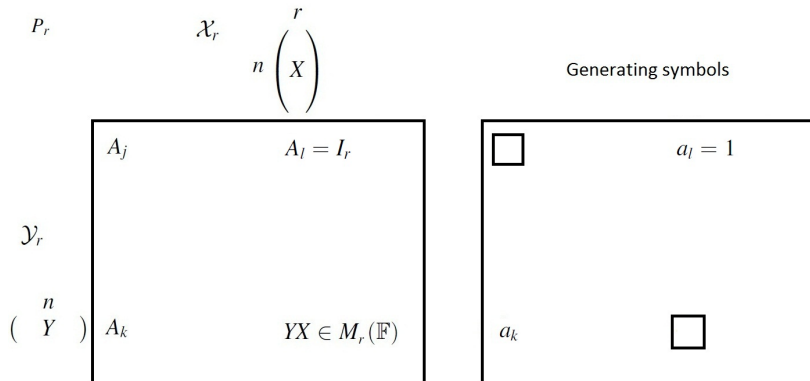
For every pair  $A_j, A_k$  in the small box, if  $A_j = A_k$  then there is a strong path in the small box from  $A_j$  to  $A_k$ .

## Finishing off Step 2



**Proposition:** For every pair  $A_j, A_k$  of entries in  $P_r$ , if  $A_j = A_k$  then there is a strong path between  $A_j$  and  $A_k$ . Thus, for every pair  $A_j = A_k \in GL_r(\mathbb{F})$  in the table  $P_r$  the relation  $a_j = a_k$  is deducible.

# Structure of the proof that $H_W \cong GL_r(\mathbb{F})$



**Step 1:** Write down a presentation for  $H_W$ .

**Step 2:** Prove that for any two entries  $A_j, A_k$  in the table  $P_r$ , if  $A_j = A_k \in GL_r(\mathbb{F})$  then  $a_j = a_k$  is deducible from the relations.

**Step 3:** Find defining relations for  $GL_r(\mathbb{F})$  among the singular square relations (II).

## Finishing off the proof

For any pair of matrices  $A, B \in GL_r(\mathbb{F})$  we can find the following singular square in  $P_r$ :

$$\begin{array}{c|c|c|c|c|c|c}
 & & & & & \left[ \begin{array}{c} \hline 0_{r \times r} \\ \hline 0_{r \times r} \\ \hline I_r \\ \hline 0_{(n-3r) \times r} \end{array} \right] & \left[ \begin{array}{c} \hline I_r \\ \hline 0_{r \times r} \\ \hline B \\ \hline 0_{(n-3r) \times r} \end{array} \right] \\
 \hline
 \left[ \begin{array}{c|c|c|c} 0_{r \times r} & I_r & A & 0_{r \times (n-3r)} \end{array} \right] & & A & AB \\
 \left[ \begin{array}{c|c|c|c} 0_{r \times r} & 0_{r \times r} & I_r & 0_{(n-3r) \times r} \end{array} \right] & & I_r & B
 \end{array}$$

- ▶ Every relation in the presentation holds in  $GL_r(\mathbb{F})$ .
- ▶ Conversely, every relation that holds in  $GL_r(\mathbb{F})$  can be deduced from the multiplication table relations that arise from the squares above.
- ▶ It follows that  $H_W \cong GL_r(\mathbb{F})$  (when  $r < n/3$ ).



# Open problems

- ▶ What happens in higher ranks?

## Conjecture (Brittenham, Margolis, Meakin (2009))

Let  $n$  and  $r$  be positive integers with  $r \leq n/2$ . Then  $H_W \cong GL_r(\mathbb{F})$ .

- ▶ The same result might even be true for  $r < n - 1$ .

What we do know...

- ▶ We will not find the full multiplication inside the presentation in general. Indeed, it can be shown that for certain finite  $M_n(\mathbb{F})$  and  $r < n - 1$  the number of generators  $a_j$  in the presentation will be strictly less than the number of elements in the corresponding general linear group.
- ▶ The analogous result does hold for  $T_n$ , with  $r < n - 1$ , with the symmetric groups  $S_r$  arising as maximal subgroups of  $IG(E)$  (Gray & Ruskuc (2012)).