On Maximal Subgroups of Free Idempotent Generated Semigroups

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Outline

History and motivation

- Idempotent generated semigroups
- Biordered sets and free idempotent generated semigroups

Maximal subgroups of free idempotent generated semigroups

- The main result
- Singular squares and presentations

Future work and open problems

Idempotent generated semigroups

S - semigroup, $\quad E = E(S)$ - idempotents of S

Definition. S is idempotent generated if $\langle E(S) \rangle = S$.

- Many natural examples
  - Howie (1966) - $T_n \setminus S_n$, the non-invertible transformations;
  - Erdös (1967) - singular part of $M_n(\mathbb{F})$, semigroup of all $n \times n$ matrices over a field $\mathbb{F}$;
  - Laffey (1983) - singular part of $M_n(Q)$, Q an arbitrary division ring;
  - Putcha (2006) - conditions for a reductive linear algebraic monoid to have the same property.
- Independence algebras
  - Gould (1995), Fountain and Lewin (1992, 1993), Araújo (2002–2007)
- Generating sets of idempotents
  - Gomes and Howie (1987, 1992), Howie and McFadden (1990)
- They are "general"
  - Every semigroup S embeds into an idempotent generated semigroup.

The biordered set of a semigroup

Nambooripad (1979)

S - semigroup, $E = E(S)$ - idempotents of S

Definition. The biordered set of a semigroup S is the partial algebra consisting of the set $E = E(S)$ with multiplication restricted to basic pairs.

$(e, f) \in E \times E$ is called a basic if

$$ef = e \;\text{ or }\; ef = f \;\text{ or }\; fe = e \;\text{ or }\; fe = f.$$

i.e. one of the idempotents stabilizes the other under left or right multiplication.

If (e, f) is basic then both $ef \in E$ and $fe \in E$.
(e.g. if $ef = f$ then $(fe)^2 = f(ef)e = ffe = fe$)

Semigroup presentations

Presentation: $\langle A|R\rangle$

A - alphabet

- a non-empty set giving the abstract generators for the semigroup

R - defining relations

- pairs of words over A, written as $\alpha = \beta$

Defines a semigroup $S = A^+/\rho$ where ρ is congruence on A^+ generated by R.

- The elements of S are equivalence classes of words where two words u and v are equivalent (represent the same element of S) iff u can be transformed into v by applying relations from R.

Semigroup presentations

Examples

- $\langle A | \ \rangle$ defines the free semigroup on A^+. Elements: all words. Multiplication: concatenation.

- $\langle a | a^2 = a \rangle$ defines the trivial semigroup. Elements: $\{a\}$. Multiplication: $aa = a$.

- $\langle a, b | ab = ba \rangle$ defines the free commutative monoid of rank 2. Elements: $\{a^i b^j : i, j \geq 0\}$. Multiplication: $a^i b^j \cdot a^k b^l = a^{i+k} b^{j+l}$.

- $\langle a, a^{-1}, b, b^{-1} | a^\epsilon a^{-\epsilon} = b^\epsilon b^{-\epsilon} = 1 (\epsilon = \pm 1) \rangle$ defines the free group on $\{a, b\}$. Elements: reduced words. Multiplication: concatenation followed by free reduction.

- $\langle a, b | aba = b, bab = a \rangle$ defines the quaternion group.

- Every semigroup is defined by a presentation (multiplication table).

Free idempotent generated semigroups

S - semigroup, $\quad E = E(S)$

Let $IG(E)$ denote the semigroup defined by the following presentation.

$$IG(E) = \langle E \mid e \cdot f = ef \text{ if } (e,f) \text{ is a basic pair} \rangle.$$

$IG(E)$ is called the free idempotent generated semigroup on E.

Free idempotent generated semigroups

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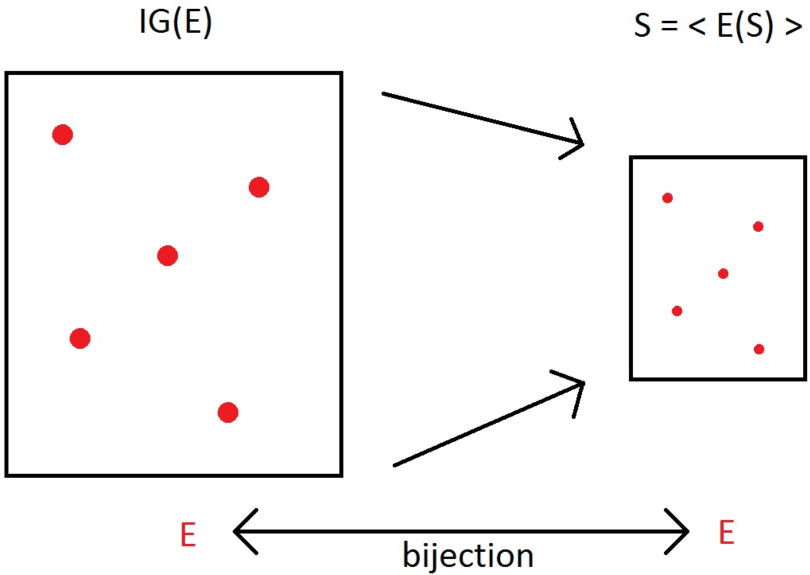
Theorem (Easdown (1985))

The biordered set of idempotents of $IG(E)$ is E. If S is any idempotent generated semigroup with biordered set of idempotents isomorphic to E then the natural map $E \to S$ extends uniquely to a homomorphism $IG(E) \to S$.

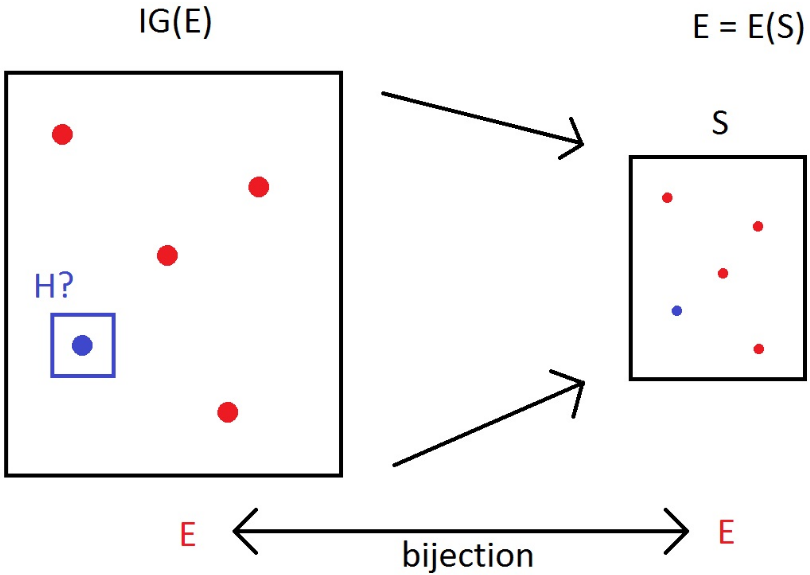
Conclusion. It is important to understand $IG(E)$ if one is interested in understanding an arbitrary idempotent generated semigroup with biordered set E.



IG(E)
S = < E(S) >
E
E
bijection

First steps towards understanding $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?


IG(E)
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bijection
E

First steps towards understanding $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

First steps towards understanding $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

- It was conjectured that maximal subgroups of free idempotent generated semigroups must always be free groups.
- This conjecture was confirmed for several classes of biordered set:
  - Pastijn (1977, 1980), Nambooripad & Pastijn (1980), McElwee (2002).

First steps towards understanding $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?

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  - Pastijn (1977, 1980), Nambooripad & Pastijn (1980), McElwee (2002).

- Brittenham, Margolis & Meakin (2009) gave the first counterexamples to this conjecture.
  - Give a 72-element semigroup S and prove that $IG(E(S))$ has a maximal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
  - They also report that the multiplicative group $\mathbb{F}^*$ of a field $\mathbb{F}$ arises as a maximal subgroup of $IG(E(M_3(\mathbb{F})))$, where $M_3(\mathbb{F})$ is the semigroup of all 3×3 matrices over $\mathbb{F}$.

Main result

Theorem (RG & Ruskuc (2010))

Every group is a maximal subgroup of some free idempotent generated semigroup.

The environment semigroup $B_{I,J}$

Let X be a set.

$T_X^{(r)}$ - full transformation monoid on X, maps composed from left to right.
$T_X^{(l)}$ - full transformation monoid on X, maps composed from right to left.
(If $X = \{1, \ldots, n\}$ we write $T_n^{(r)}$ and $T_n^{(l)}$.)

Define

$$B_{I,J} = T_I^{(l)} \times T_J^{(r)}.$$

A typical element of $\beta \in B_{I,J}$ has the form $\beta = (\beta^{(l)}, \beta^{(r)})$.

Multiplication in $B_{I,J}$

Example

$I = \{1,2,3\}$, $J = \{1,2,3,4\}$

$B_{I,J} = T_3^{(l)} \times T_4^{(r)}$

$$\sigma = (\sigma^{(l)}, \sigma^{(r)}) = \left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \right) \in B_{I,J}$$

$$\tau = (\tau^{(l)}, \tau^{(r)}) = \left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix} \right) \in B_{I,J}$$

$$\sigma\tau = (\sigma^{(l)}\tau^{(l)}, \sigma^{(r)}\tau^{(r)}) = \left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 4 & 4 \end{pmatrix} \right) \in B_{I,J}$$

Multiplying constant mappings

Example

$I = \{1, 2, 3\}, J = \{1, 2, 3, 4\}$

$B_{I,J} = T_3^{(l)} \times T_4^{(r)}$

$$\sigma = (\sigma^{(l)}, \sigma^{(r)}) = \left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \right) \in B_{I,J}$$

$$\tau = (\tau^{(l)}, \tau^{(r)}) = \left(\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right) \in B_{I,J}$$

$$\sigma\tau = (\sigma^{(l)}\tau^{(l)}, \sigma^{(r)}\tau^{(r)}) = \left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right) \in B_{I,J}$$

Minimal ideal of $B_{I,J}$

The semigroup $B_{I,J}$ has a unique minimal ideal

$$R_{I,J} = \{\rho_{ij} = (\rho_i, \rho_j) \ : \ i \in I, \ j \in J\},$$

where

$$\rho_i : I \to I, \ x \mapsto i, \quad \rho_j : J \to J, \ x \mapsto j$$

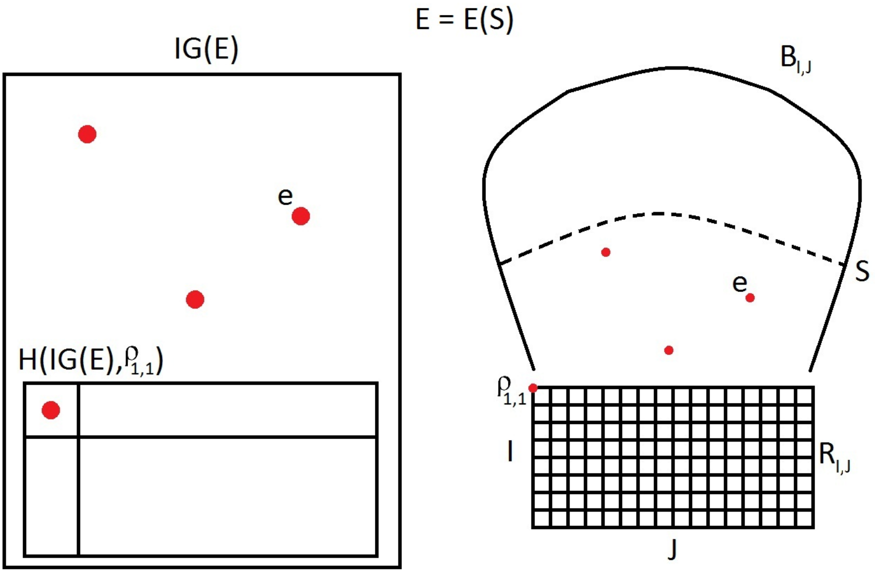
are the constant maps.

The multiplication in $R_{I,J}$ works as follows:

$$\rho_{ij}\rho_{kl} = \rho_{il},$$

i.e. $R_{I,J}$ is an $I \times J$ rectangular band.

Fix a distinguished idempotent ρ_{11} in $R_{I,J}$ (think ‘top left’).


E = E(S)
IG(E)
e
H(IG(E),ρ1,1)
B I,J
S
e
ρ1,1
I
R I,J
J

$IG(E(S))$ where $R_{I,J} \leq S \leq B_{I,J}$

Let S be a semigroup such that $R_{I,J} \leq S \leq B_{I,J}$.

Aim: Describe the maximal subgroup $H = H(IG(E(S)), \rho_{11})$ of $IG(E(S))$ containing $\rho_{11} \in R_{I,J}$.

- Apply Reidemeister–Schreier for subgroups (Ruskuc (1999)) to rewrite the presentation

$$IG(E) = \langle E \mid e \cdot f = ef \;\; (e,f \in E,\ \{e,f\} \cap \{ef,fe\} \neq \emptyset)\ \rangle$$

  to obtain a presentation for the maximal group H.
- The relations in $IG(E)$ arise from basic pairs (e,f) of idempotents.
- The way that basic pairs of idempotents "interact" in S should influence the presentation obtained for H.

Singular squares

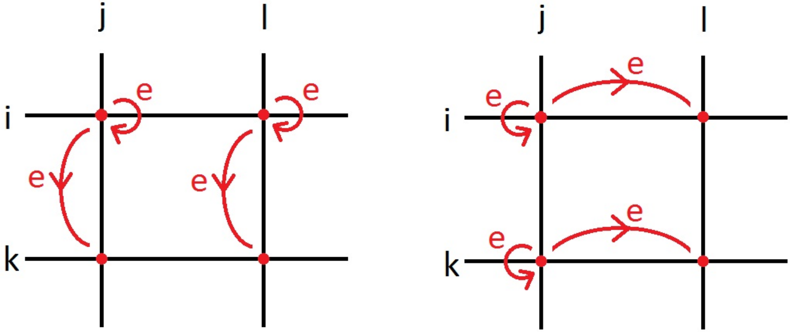
S - a semigroup such that $R_{I,J} \leq S \leq B_{I,J}$, $\ E = E(S)$

Definition

A quadruple $(i,k;j,l) \in I \times I \times J \times J$ is a singular square if there exists an idempotent $e \in E$ such that one of the following dual conditions holds:

$$e\rho_{ij} = \rho_{ij},\ e\rho_{kj} = \rho_{kj},\ \rho_{ij}e = \rho_{il},\ \rho_{kj}e = \rho_{kl}, \text{ or}$$
$$\rho_{ij}e = \rho_{ij},\ \rho_{il}e = \rho_{il},\ e\rho_{ij} = \rho_{kj},\ e\rho_{il} = \rho_{kl}.$$

We will say that e singularises the square.


j
l
i
k
e
e
e
e
e
e
j
l
i
k
e
e
e
e

Singular squares example

Example

$I = \{1, 2, 3\}, J = \{1, 2, 3, 4\}$
$B_{I,J} = T_3^{(l)} \times T_4^{(r)}$

Let $S = \{\sigma\} \cup R_{I,J}$ where:

$$\sigma = (\sigma^{(l)}, \sigma^{(r)}) = \left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \right) \in B_{I,J}$$

Clearly $R_{I,J} \leq S \leq B_{I,J}$.

$(1, 2; 3, 1)$ is a singular square singularised by σ since:

$$\sigma\rho_{13} = \rho_{13},\ \sigma\rho_{23} = \rho_{23},\ \rho_{13}\sigma = \rho_{11},\ \rho_{23}\sigma = \rho_{21}.$$

$(1, 2; 1, 2)$ is not singular

A presentation for the maximal subgroup

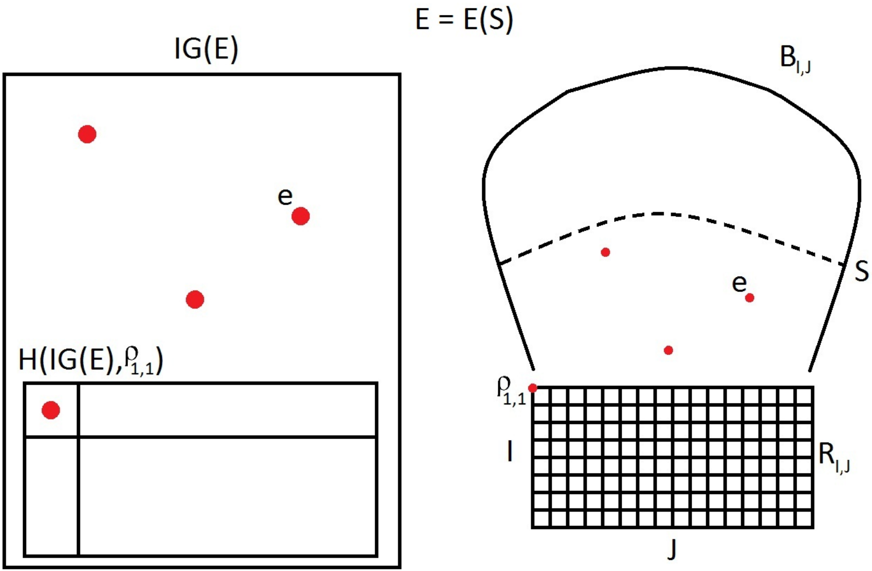
- The abstract generators for the group H are in one-to-one correspondence with the elements of the rectangular band $R_{I,J} \leq S$.
- The singular squares in the rectangular band $R_{I,J} \leq S$ give rise to the relations that define the maximal subgroup H of $IG(E)$.

Theorem

Let S be a semigroup such that $R_{I,J} \leq S \leq B_{I,J}$ and let $\rho_{11} \in R_{I,J}$. Then the group $H = H(IG(E(S)), \rho_{11})$ is defined by the presentation

$$\begin{array}{lclr} \langle\, f_{ij}\ (i \in I,\ j \in J) & \mid & f_{1j} = f_{i1} = 1 & (i \in I,\ j \in J), \\ & & f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl} & ((i,k;j,l) \in \Sigma))\ \rangle \end{array}$$

where Σ is the set of all singular squares.



E = E(S)
IG(E)
e
H(IG(E),ρ1,1)
BI,J
S
e
ρ1,1
I
RI,J
J

Singular squares and the relations they yield

$$\begin{tikzcd}
 & 1 & j \\
1 & 1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "f_{ij} = 1"] & 1 \arrow[d, dash] \\
i & 1 \arrow[r, dash] & f_{ij}
\end{tikzcd}$$

Corner

$$\begin{tikzcd}
 & 1 & j & l \\
1 & 1 & 1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "f_{ij} = f_{il}"] & 1 \arrow[d, dash] \\
i & 1 & f_{ij} \arrow[r, dash] & f_{il}
\end{tikzcd}$$

Flush top

$$\begin{tikzcd}
 & 1 & j \\
1 & 1 & 1 \\
i & 1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "f_{ij} = f_{kj}"] & f_{ij} \arrow[d, dash] \\
k & 1 \arrow[r, dash] & f_{kj}
\end{tikzcd}$$

Flush left

$$\begin{tikzcd}
 & 1 & j & l \\
1 & 1 & 1 & 1 \\
i & 1 & 1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "f_{kj} f_{il} = f_{kl}"] & f_{il} \arrow[d, dash] \\
k & 1 & f_{kj} \arrow[r, dash] & f_{kl}
\end{tikzcd}$$

$3/4$

Goldilocks and the three bears

By varying I, J and S, with $R_{I,J} \leq S \leq B_{I,J}$ we want to see what groups $H(IG(E(S)), \rho_{11})$ we can obtain.

Example

If we set $S = R_{I,J}$ then there are no (non-degenerate) singular squares $(i,k;j,l)$ and so we obtain:

$$\langle\, f_{ij}\ (i \in I,\ j \in J) \qquad | \qquad f_{1j} = f_{i1} = 1 \qquad (i \in I,\ j \in J)\,\rangle.$$

So in this case $H(IG(E(S)), \rho_{11})$ is a free group of rank $(|I|-1)(|J|-1)$.

Example

If we set $S = B_{I,J}$ then every square is singular and from the relations arising from corner squares we obtain:

$$\langle\, f_{ij}\ (i \in I,\ j \in J) \qquad | \qquad f_{ij} = 1 \qquad (i \in I,\ j \in J)\,\rangle.$$

So in this case $H(IG(E(S)), \rho_{11})$ is the trivial group.

Obtaining any given group

G - arbitrary group of order N (possibly infinite), $n = N^2$

We will work in $B_{3,n} = T_3^{(l)} \times T_n^{(r)}$, which has the $3 \times n$ rectangular band $R_{3,n}$ as its minimal ideal.

Aim: Find S with $R_{3,n} \leq S \leq B_{3,n}$ such that $H(IG(E(S)), \rho_{11}) \cong G$.

We must use G somehow to define a collection of idempotents in $B_{3,n} \setminus R_{3,n}$ which, together with $R_{3,n}$, generate the desired semigroup S.

An auxiliary matrix

We define an auxiliary matrix:

$$Y = (y_{ij})_{3\times n} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & y_{22} & y_{23} & \dots & y_{2n} \\ 1 & y_{32} & y_{33} & \dots & y_{3n} \end{pmatrix}.$$

Its entries are the elements of G, arranged arbitrarily subject to the condition that every possible column appears (once and only once):

$$\{(1, y_{2j}, y_{3j}) \ : \ j = 1, \dots, n\} = \{(1, g, h) \ : \ g, h \in G\}.$$

We may identify the index set $J = \{1, \dots, n\}$ and the set $\{(1, g, h) \ : \ g, h \in G\}$ of all columns of Y.

Define six additional idempotents

$$\sigma_u = (\sigma_u^{(l)}, \sigma_u^{(r)}) \in B_{3,n} \; (u = 1, \ldots, 6),$$

given by

$$\sigma_1^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \qquad \sigma_1^{(r)} : (1, g, h) \mapsto (1, g, g)$$

$$\sigma_2^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \qquad \sigma_2^{(r)} : (1, g, h) \mapsto (1, g, 1)$$

$$\sigma_3^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \qquad \sigma_3^{(r)} : (1, g, h) \mapsto (1, 1, h)$$

$$\sigma_4^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix} \qquad \sigma_4^{(r)} : (1, g, h) \mapsto (1, h, h)$$

$$\sigma_5^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \qquad \sigma_5^{(r)} : (1, g, h) \mapsto (1, 1, hg^{-1})$$

$$\sigma_6^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix} \qquad \sigma_6^{(r)} : (1, g, h) \mapsto (1, gh^{-1}, 1).$$

The structure of S

The semigroup

$$S = \langle R_{3,n} \cup \{\sigma_1, \ldots, \sigma_6\} \rangle = R_{3,n} \cup D \leq B_{3,n}$$

has the following properties:

- S is regular;
- S has two $\mathcal{D}$-classes: $R_{3,n}$ and D;
- S has precisely six idempotents $\sigma_1, \ldots, \sigma_6$ outside $R_{3,n}$;
- S has exactly eighteen elements outside $R_{3,n}$;
- S is finite if and only if $R_{3,n}$ is finite, which is the case if and only if G is finite.

Theorem

$H(IG(E(S)), \rho_{11}) \cong G.$

Picture of $S = R_{3,n} \cup D$

σ_1 σ_{14}	σ_2 σ_{13}	σ_7 σ_8
σ_4 σ_{15}	σ_9 σ_{10}	σ_3 σ_{16}
σ_{11} σ_{12}	σ_6 σ_{17}	σ_5 σ_{18}

$R_{3,n}$

Preserving finiteness properties

The above construction proves:

Theorem (RG & Ruskuc (2010))

Every group is a maximal subgroup of some free idempotent generated semigroup.

- One drawback of the above construction is that if G is infinite, then the semigroup S constructed will necessarily be infinite.
- If G is finitely presented then we can do better than this:

Theorem (RG & Ruskuc (2010))

Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup.

The word problem

Since there exist finitely presented groups that have unsolvable word problem, combining such a group with the above theorem gives:

Corollary

There exists a free idempotent generated semigroup F arising from a finite semigroup such that the word problem for F is unsolvable.

Open problems and future directions

- Investigate subgroups of free idempotent generated semigroups $IG(E)$ for biorders E that occur "in nature".

Theorem (Brittenham, Margolis & Meakin (2010))

Let E be the biordered set of $M_n(Q)$, for Q a division ring, and let e be an idempotent matrix of rank 1 in $M_n(Q)$. For $n \geq 3$, the maximal subgroup of $IG(E)$ containing e is isomorphic to Q^*, the multiplicative group of units of Q.

Open problem

Brittenham, Margolis & Meakin conjecture that the maximal subgroup of $IG(E)$ with identity e an idempotent matrix of rank $k < n-1$ is $GL_k(Q)$, if $k < n/2$ and $n \geq 3$.

- We have (very) recently shown that the full transformation monoid analogue of this result does hold (i.e. that the maximal subgroups of $IG(E(T_n))$ are symmetric groups).