Finiteness conditions and index in semigroup theory

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Outline

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   - Group index vs. Rees index
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   - Boundaries in Cayley graphs
3. General problem 2: A common generalisation
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Semigroups and finiteness conditions

Definition

A *semigroup* is a pair $S = (S, \cdot)$ where S is a set and $\cdot$ is a binary operation satisfying the associative law.

Examples

- Groups
- Subsemigroups of groups (e.g. $(\mathbb{N}, +) \leq (\mathbb{Z}, +)$)
- Semigroups of transformations
- Free semigroup A^+ over (finite) alphabet A

Definition

A property $\mathcal{P}$ of semigroups is a *finiteness condition* if all finite semigroups satisfy $\mathcal{P}$. (e.g. being finite, finitely generated, finitely presented, locally finite, etc.)

Inheritance of properties

Let S be a semigroup with T a subsemigroup of S.

Let $\mathcal{P}$ be a property of semigroups.

- S satisfies $\mathcal{P} \Rightarrow T$ satisfies $\mathcal{P}$?
- T satisfies $\mathcal{P} \Rightarrow S$ satisfies $\mathcal{P}$?

What is index?

Let S be a semigroup and let T be a subsemigroup of S.

Roughly speaking...

Index is a measure of the ‘size’ of T inside S.

A ‘good’ definition of index should have the property that if T is ‘big’ in S then S and T share many properties.

Group index

G - a group, H - subgroup of G

- $[G : H]$ is the number of (right) cosets of H in G.
- A subgroup of finite index "‘differs by a finite amount from the group"’.

Facts

Let G be a group, let H be a subgroup of finite index in G, and let $\mathcal{P}$ be any of the following conditions:

- finitely generated
- periodic
- residually finite
- FP_n, FDT
- finitely presented
- locally finite
- soluble word problem
- automatic.

Then G satisfies $\mathcal{P}$ if and only if H satisfies $\mathcal{P}$.

Rees index

S - a semigroup, T - a subsemigroup of S

Definition (Rees index)

The **Rees** index $[S : T]_R$ is the size of the complement $S \setminus T$.

Facts

Let S be a semigroup, let T be a subsemigroup of finite Rees index in S, and let $\mathcal{P}$ be any of the following conditions:

- finitely generated / presented
- periodic
- residually finite
- locally finite
- soluble word problem
- automatic.

Then S satisfies $\mathcal{P}$ if and only if T satisfies $\mathcal{P}$.

(Jura (1978), Ruškuc (1998), Ruškuc & Thomas (1998), Hoffmann, Thomas, Ruškuc (2002), RG & Ruškuc (2006))

Rees vs. group index

- Rees index behaves very much like group index.
- Rees index does not generalise group index.
- Rees index is a very restrictive condition.

General problem 1

Find a notion of index that is weaker than Rees index but still maintains the “nice” properties of Rees index.

General problem 2

Find a definition of index that:

- generalises both Rees and group index;
- maintains the "‘nice"’ properties of Rees and group index.

Answering problem 1

Weakening Rees index

Boundaries in Cayley graphs

Cayley Graphs

Definition

Let S be a semigroup generated by a finite set A.

The right Cayley graph $\Gamma_r(A, S)$ has:

- Vertices: elements of S.
- Edges: directed and labelled with letters from A.

$$s \xrightarrow{a} t \Leftrightarrow sa = t$$

- Given an edge $e = s \xrightarrow{a} t$ we define

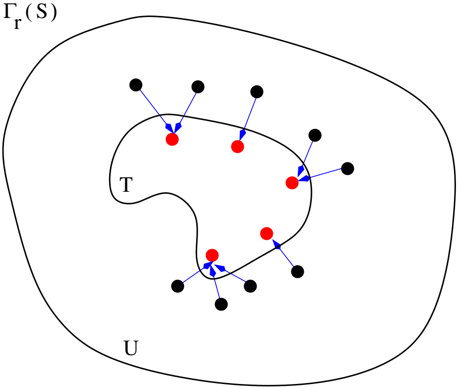
$$\iota(e) = s, \quad \tau(e) = t$$

calling them the initial and terminal vertices of e.

Weakening Rees index

The general idea:

Restrict the number of points where T and $S \setminus T$ "meet each other" in the Cayley graph to be finite.



- Red vertices

The "boundary" of T in S.

Semigroup boundaries

Definition

- Let T be a subsemigroup of S, where $S = \langle A \rangle$.
- The right boundary of T in S is the set of elements of T that *receive an edge from* $S \setminus T$ in the right Cayley graph of S:

$$\mathcal{B}_r(A, T) = (S \setminus T)A \cap T.$$

Semigroup boundaries

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- The left boundary of T in S is the set of elements of T that receive an edge from $S \setminus T$ in the left Cayley graph of S:

$$\mathcal{B}_l(A, T) = A(S \setminus T) \cap T.$$

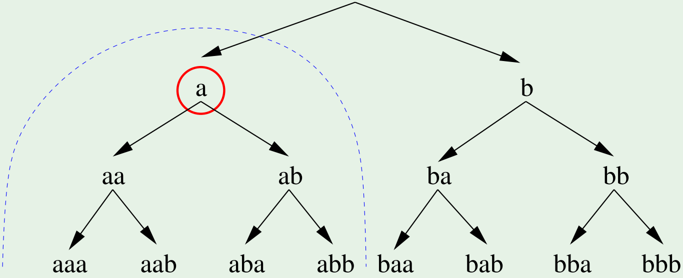
- The (two-sided) boundary is the union of the left and right boundaries:

$$\mathcal{B}(A, T) = \mathcal{B}_l(A, T) \cup \mathcal{B}_r(A, T).$$

A straightforward example

Example (Free monoid on two generators)

- $S = \{a, b\}^*$, $\quad T = \{$words that begin with the letter $a\}$.



Right boundary: $\mathcal{B}_r(\{a, b\}, T) = \{a\}$.

A straightforward example

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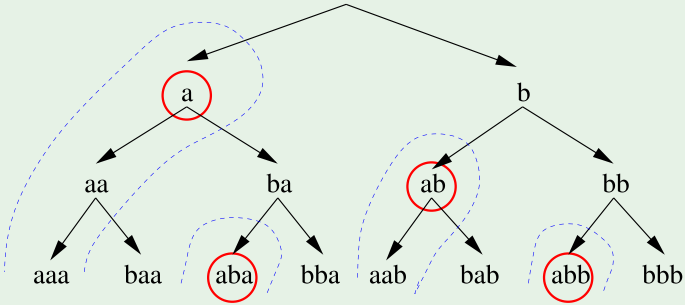

Left boundary: $\mathcal{B}_l(\{a, b\}, T) = \{a\} \cup \{ab\{a, b\}^*\}$.

Generators and relations

Facts

- The finiteness of $\mathcal{B}(A, T)$ is independent of the choice of finite generating set A of S.
- If T has finite Rees index in S then T has a finite boundary in S.

Theorem (RG, Ruškuc (2006))

If S is a finitely generated semigroup and T is a subsemigroup of S with finite boundary then T is finitely generated.

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Theorem (RG, Ruškuc (2006))

Let S be a finitely generated semigroup and T be a subsemigroup of S. If S is a finitely presented and T has a finite boundary in S then T is finitely presented.

Boundaries

Good things

- Much weaker than Rees index (so answers Problem 1)
- Provides common framework for finite Rees index and ideal complement results
- Corrects a mistake in the original finite Rees index proof

Bad things

- Does not provide a common generalisation of group and Rees index (so does not answer Problem 2);
- Is one-directional, only allows results of the form:

$$S \text{ has } \mathcal{P} \Rightarrow T \text{ has } \mathcal{P}.$$

- Not interesting for properties inherited by all substructures (e.g. locally finite, residually finite, soluble word problem etc.)

Higher dimensions - Finite derivation type

- FDT is a property of finitely presented semigroups.
- Can be thought of as a higher dimensional version of the property of being finitely presented, think “relations between relations”
- Originated from the study of finite complete string-rewriting systems

Open problem

Let S be a finitely presented semigroup, $T \leq S$ with finite boundary.

If S has FDT then does T have FDT?

- If T is an ideal with finite Rees index then S has FDT implies T has FDT (Malheiro 2006)
- If S is free then T has FDT (RG, Pride (work in progress))

Answering problem 2

Common generalisation of Rees and group index

Attempt 1: Syntactic index

Syntactic index (Ruškuc & Thomas (1998))

Definition

Let S be a semigroup and let T be a subsemigroup of S.

The (right) syntactic congruence corresponding to T is the largest right congruence ρ on S such that T is a union of congruence classes:

$$\Sigma_r(T) = \{(x, y) \in S \times S : (\forall s \in S^1)(xs \in T \Leftrightarrow ys \in T)\}.$$

The number of $\Sigma_r(T)$-classes is called the (right) syntactic index of T in S, and it denoted $[S : T]_S$.

Syntactic index - examples

Example

- If G is a group and H is a subgroup then

  $$(x, y) \in \Sigma_r(H) \Leftrightarrow x \& y \text{ belong to the same right coset of } H \text{ in } G.$$

  The largest right congruence on G such that H is a union of congruence classes is the one that has the right cosets of H as its congruence classes.

- Let S be a semigroup and let T be a subsemigroup of S. If T has finite Rees index in S then T has finite syntactic index in S.

Conclusion

Syntactic index provides a common generalisation of group and Rees index.

Syntactic index is too weak

Theorem (Ruškuc and Thomas (1998))

- *$\mathcal{P}$ - a non-trivial property of semigroups*
- *There is a semigroup S with a subsemigroup T of finite syntactic index such that either*
  1. *S satisfies $\mathcal{P}$ and T does not satisfy $\mathcal{P}$; or*
  2. *S does not satisfy $\mathcal{P}$ and T satisfies $\mathcal{P}$.*

Corollary. Finiteness, periodicity, local finiteness, and residual finiteness are not inherited by syntactically small extensions.

Fact

Neither of the properties of being finitely generated or being finitely presented are inherited by either syntactically small extensions or syntactically large subsemigroups.

Answering problem 2

Common generalisation of Rees and group index

Attempt 2: Green index

Green’s relations

According to Wikipedia Green’s relations:

- are 5 equivalence relations that characterize the elements of a semigroup in terms of the principal ideals they generate.

- They are...

  “so all-pervading that, on encountering a new semigroup, almost the first question one asks is ‘What are the Green relations like?’ ” (J. M. Howie 2002)

Green's relations

S - semigroup, $a, b \in S$.

- $S^1 a = Sa \cup \{a\}$ is the smallest left ideal of S containing a. It is called the principal left ideal generated by a.
- $aS^1 = aS \cup \{a\}$ - the principal right ideal generated by a.

Definition (Green's $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$ relations)

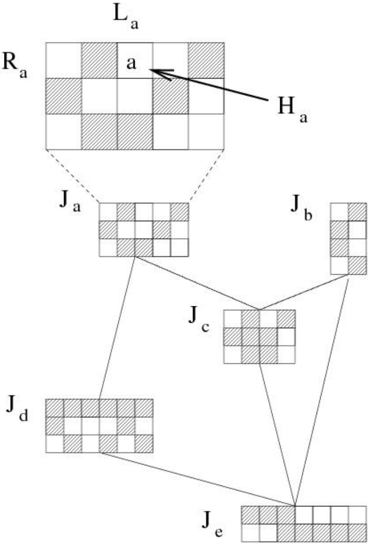
$$a\mathcal{L}b \Leftrightarrow S^1 a = S^1 b, \qquad a\mathcal{R}b \Leftrightarrow aS^1 = bS^1, \qquad \mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

Facts

- $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ are equivalence relations;
- $\mathcal{L}$ is a right congruence, $\mathcal{R}$ is a left congruence;
- The $\mathcal{R}$-classes are the *strongly connected components* of the right Cayley graph (dual for left);
- If S is a group then $\mathcal{L} = \mathcal{R} = \mathcal{H}$.

Green's relations

Important tool for structure theory



S - semigroup, $x, y \in S$

$$
\begin{aligned}
x\mathcal{R}y &\Leftrightarrow xS^1 = yS^1 \\
x\mathcal{L}y &\Leftrightarrow S^1x = S^1y \\
x\mathcal{J}y &\Leftrightarrow S^1xS^1 = S^1yS^1
\end{aligned}
$$

- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $J_x \leq J_y \Leftrightarrow S^1xS^1 \subseteq S^1yS^1$

Relative Green's relations

S - semigroup, T - subsemigroup of S, $a, b \in S$.

- $T^1 a = Ta \cup \{a\}$: T-relative principal left ideal generated by a in S.
- $aT^1 = aT \cup \{a\}$: T-relative principal right ideal generated by a in S.

Definition (Relative Green's relations)

$$a\mathcal{L}^T b \Leftrightarrow T^1 a = T^1 b, \qquad a\mathcal{R}^T b \Leftrightarrow aT^1 = bT^1, \qquad \mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T.$$

Facts

- $\mathcal{L}^T$, $\mathcal{R}^T$, and $\mathcal{H}^T$ are equivalence relations;
- The $\mathcal{R}^T$-classes are the *strong orbits* of T^1 acting on S by right multiplication (dual for left);
- T is a union of $\mathcal{H}^T$-classes.

Wallace (1962) - developed a theory of relative Green's relations.

Definition

Let $\{H_j : j \in J\}$ be the $\mathcal{H}^T$-classes of $S \setminus T$. Then we define:

$$[S : T]_G = |J| + 1$$

and call it the Green index of T in S.

Examples.

- G - group, H - subgroup of G.
  - $u\mathcal{R}^H v \Leftrightarrow uH = vH$ (left cosets of H in G)
  - $u\mathcal{L}^H v \Leftrightarrow Hu = Hv$ (right cosets of H in G)
  - Therefore if $[G : H] < \infty$ then $[G : H]_G < \infty$.
- S - semigroup, T - subsemigroup of S.
  - If T has finite Rees index in S then the number of $\mathcal{H}^T$-classes of $S \setminus T$ is at most $|S \setminus T|$.

Examples and basic properties

- If $U \leq T \leq S$ then $[S : U]_G$ is finite if and only if both $[T : U]_G$ and $[S : T]_G$ are finite.

- If $[S : T]_G$ is finite then S is finite if and only if T is finite.

- If $T \leq S$ is an ideal then T has finite Green index in S if and only if T has finite Rees index in S.

- If S is an inverse semigroup (in particular if S is a group) then the following are equivalent:
  1. T has finite Green index in S;
  2. $S \setminus T$ has finitely many $\mathcal{R}^T$-classes;
  3. $S \setminus T$ has finitely many $\mathcal{L}^T$-classes.

The lemmas of Schreier and Jura

Lemma (Schreier)

- *G - a group generated by A, H - finite index subgroup of G*
- *K be coset representatives (with $1 \in K$)*
- *$\overline{g}$ denotes the coset representative of $g \in G$*

Then H is generated by:

$$X = \{ka(\overline{ka})^{-1} : k \in K, a \in A\}.$$

Lemma (Jura (1978))

- *S - a semigroup generated by A*
- *T - finite Rees index subsemigroup of S*

Then T is generated by:

$$X = \{s_1 a s_2 : s_1, s_2 \in S^1 \setminus T,\ a \in A,\ s_1 a, s_1 a s_2 \in T\}.$$

A generating set for T

S - semigroup, T - subsemigroup with finite Green index.

- $h_i : i \in I$ - representatives of the $\mathcal{H}^T$-classes of $S \setminus T$;
- Define $R_1 = L_1 = H_1 = \{1\}, \quad r_1 = l_1 = h_1 = 1$.
- For $s \in S$ and $i \in I$ define:

$$\rho(s,i) = \begin{cases} j & \text{if } sh_i \in H_j \\ 1 & \text{if } sh_i \in T \end{cases}, \qquad \lambda(i,s) = \begin{cases} j & \text{if } h_i s \in H_j \\ 1 & \text{if } h_i s \in T. \end{cases}$$

- Moreover, let $\sigma(s,i) \in T$ and $\tau(i,s) \in T$ satisfy:

$$sh_i = h_{\rho(s,i)}\sigma(s,i), \qquad h_i s = \tau(i,s)h_{\lambda(i,s)}.$$

Lemma

If $S = \langle A \rangle$ then T is generated by the set:

$$X = \{\tau(i, \sigma(a,j)) \ : \ i,j \in I \cup \{1\}, \ a \in A\}.$$

Finite generation

Corollary (RG and Ruškuc (in preparation))

Let S be a semigroup and let T be a subsemigroup of S with finite Green index. Then S is finitely generated if and only if T is finitely generated.

Fact. There is an example of a semigroup S with subsemigroup T:

- S is finitely generated;
- $S \setminus T$ has finitely many relative $\mathcal{R}$-classes;
- T is not finitely generated.

Other properties

Theorem (RG and Ruškuc (in preparation))

Let S be a semigroup, let T be a subsemigroup of finite Green index in S.

Let $\mathcal{P}$ be any of the following conditions:

- *periodic*
- *locally finite*
- *finitely many left (resp. right) ideals*
- *finitely many idempotents*

Then S satisfies $\mathcal{P}$ if and only if T satisfies $\mathcal{P}$.

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Then S satisfies $\mathcal{P}$ if and only if T satisfies $\mathcal{P}$.

Question

Can we prove the same result for the properties of being finitely presented, residually finite, or having soluble word problem?

Clifford monoid example

Example

- $Y = \{0, 1\}$ with $0 < 1$, a 2 element semilattice
- $G_1 := \langle A \mid \ \rangle$ free group where $|A| = r$
- $G_0 := \langle A \mid R \rangle$ a non-finitely presented homomorphic image of G_1
- $\phi_{1,0} : G_1 \to G_0$: associated homomorphism
- $\phi_{i,i} : G_i \to G_i$: identity maps
- Define multiplication on $S = G_0 \cup G_1$ by:

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}), \quad x \in S_\alpha, y \in S_\beta.$$

$$\begin{tikzcd} \boxed{G_1} \arrow[d, "\phi_{1,0}"'] \\ \boxed{G_0} \end{tikzcd}$$

Clifford monoid example

Proposition

Let $S = G_0 \cup G_1$ (above) and let $T = G_1 \leq S$. Then:

1. *T is finitely presented;*
2. *T has Green index 2 in S;*
3. *S is not finitely presented.*

Proof. G_0 is a retract of S. Since G_0 is not finitely presented it follows that neither is S. □

Clifford monoid example

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Proof. G_0 is a retract of S. Since G_0 is not finitely presented it follows that neither is S. $\square$

Using the same kind of construction we can prove:

1. Residual finiteness is not inherited by finite Green index extensions.
2. Having a soluble word problem is not inherited by finite Green index extensions. (For finitely generated semigroups.)

Generalised Schützenberger groups

Definition

- $T \leq S$, H - any $\mathcal{H}^T$-class of S.
- (Wallace (1962)) TFAE
  - $H^2 \cap H \neq \varnothing$;
  - H contains an idempotent ($e^2 = e$);
  - H is a subgroup of S.

Generalised Schützenberger groups

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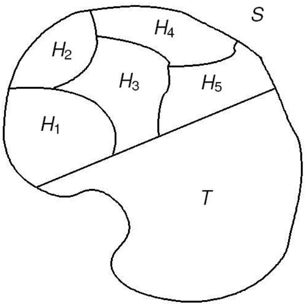
- (Wallace (1962)) TFAE
  - $H^2 \cap H \neq \varnothing$;
  - H contains an idempotent ($e^2 = e$);
  - H is a subgroup of S.
- If H is not a group then we can still associate a group with H:
  - $T(H) = \{t \in T : Ht = H\}$: the stabilizer of H in T.
  - The relation $\sim$ on $T(H) \leq T$ defined by:

    $$x \sim y \Leftrightarrow (\forall h \in H)(hx = hy)$$

    is a congruence.
  - $\Gamma(H) = T(H)/\sim$ is a group, called the generalised Schützenberger group of H.
- $|\Gamma(H)| = |H|$ and if H is a group then $\Gamma(H) \cong H$.

A picture of what is going on

- S - semigroup, T subsemigroup of S
- $H_i : i \in I$: the relative $\mathcal{H}$-classes in the complement
- Γ_i a group associated with the set H_i arising from the action of T on H_i



Question. How are the properties of S related to those of T and the groups $\Gamma_i : i \in I$?

- If T has finite Rees index in S then all the groups Γ_i are finite.
- If $S = G$ and $T = N \trianglelefteq G$ then $\Gamma_i \cong N$ for all $i \in I$.

In both cases T has $\mathcal{P} \Rightarrow \Gamma_i$ has $\mathcal{P}$ for all $i \in I$.

Finite presentability

Theorem (RG and Ruškuc (in preparation))

Let S be a semigroup and let T be a subsemigroup of S with finite Green index.

If T is finitely presented and each group Γ_i is finitely presented then S is finitely presented.

Open problem. Prove that if S is finitely presented then T is finitely presented and each group Γ_i is finitely presented.

Note. We know that if S is finitely generated then T is finitely generated and each group Γ_i is finitely generated.

Syntactic index and residual finiteness

Theorem (RG, Ruškuc (in preparation))

Let S be a semigroup and let T be a subsemigroup of S. If T has finite Green index in S then T has finite syntactic index in S.

Theorem (RG, Ruškuc (in preparation))

Let S be a semigroup and let T a subsemigroup of S with finite Green index.

Then S is residually finite if and only if T is residually finite and all the Schützenberger groups Γ_i are residually finite.

Counting subsemigroups

Theorem (Hall (1949))

Let G be a group and let $n \geq 1$ be an integer. If G is finitely generated then it has only finitely many subgroups of index n.

Theorem (RG, Ruškuc (in preparation))

A finitely generated semigroup has only finitely many subsemigroups of any given Green index n.

Proof. Makes use of the relationship between Green index and syntactic index.

Future work

- Green index gives a framework for providing unifying proofs of group index and Rees index results.
- It gives us a better understanding of why similar results exist for group index and Rees index.

- Open problems
  - Prove the "hard" direction of the finite presentability result
  - Prove the corresponding results for
    - soluble word problem
    - FDT
    - automaticity
    - etc...
- How about inverse semigroup presentations?