Generating Sets of Ideals of Endomorphism Monoids

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Setting the scene

Given a mathematical structure M the set of endomorphisms of M (written as $\mathrm{End}(M)$) forms a monoid (i.e. a semigroup with identity).

Examples

- When $M = \{1, \ldots, n\}$ then $\mathrm{End}(M) \cong T_n$ the *full transformation semigroup*.
- When $M = V$ an n-dimentional vector space then $\mathrm{End}(M) \cong \mathrm{GLS}(n, F)$ the *general linear semigroup* of <u>all</u> $n \times n$ matrices over the field F.
- When $M = Y_n$ an n-element chain then $\mathrm{End}(M) \cong O_n$ the semigroup of order preserving mappings of $\{1, \ldots, n\}$.

History

Theorem(Howie, 1966) Every singular map of T_n is a product of idempotent maps (maps that satisfy $\alpha^2 = \alpha$):

$$\mathrm{Sing}_n = \{\alpha \in T_n : 1 \leq |\mathrm{Im}(\alpha)| < n\}.$$

Theorem(Erdos, 1967) Every singular $n \times n$ matrix of $\mathrm{GLS}(n, F)$ is a product of idempotent matrices (matrices M satisfying $M^2 = M$):

$$\mathrm{Sing}(V) = \{A \in \mathrm{End}(V) : 1 \leq \dim(\mathrm{Im}(A)) < n\}.$$

Question What is the smallest number of idempotent maps (matrices) that we need in order to generate all the singular maps (matrices) ?

More generally

Given a finite idempotent generated semigroup S:

1. What is the smallest number of elements required to generate S?

$$\mathrm{rank}(S) = \min\{|A| : \langle A \rangle = S\}.$$

2. What is the smallest number of idempotents required to generate S?

$$\mathrm{idrank}(S) = \min\{|A| : A \subseteq E(S), \langle A \rangle = S\}.$$

3. How do these numbers compare i.e. how much more difficult is it to generate S if we restrict our choice of generators to the set of idempotents?

A few more examples

Ideals of T_n and $\mathrm{End}(V)$

- $K(n, r) = \{\alpha \in T_n : |\mathrm{Im}(\alpha)| \leq r\}$;
- $I(r, n, q) = \{A \in \mathrm{End}(V) : \dim(\mathrm{Im}(A)) \leq r\}$;

Order preserving maps

- $O_n = \{\alpha \in \mathrm{Sing}_n : (\forall x, y \in X_n)\ x \leq y \Rightarrow x\alpha \leq y\alpha\}$;

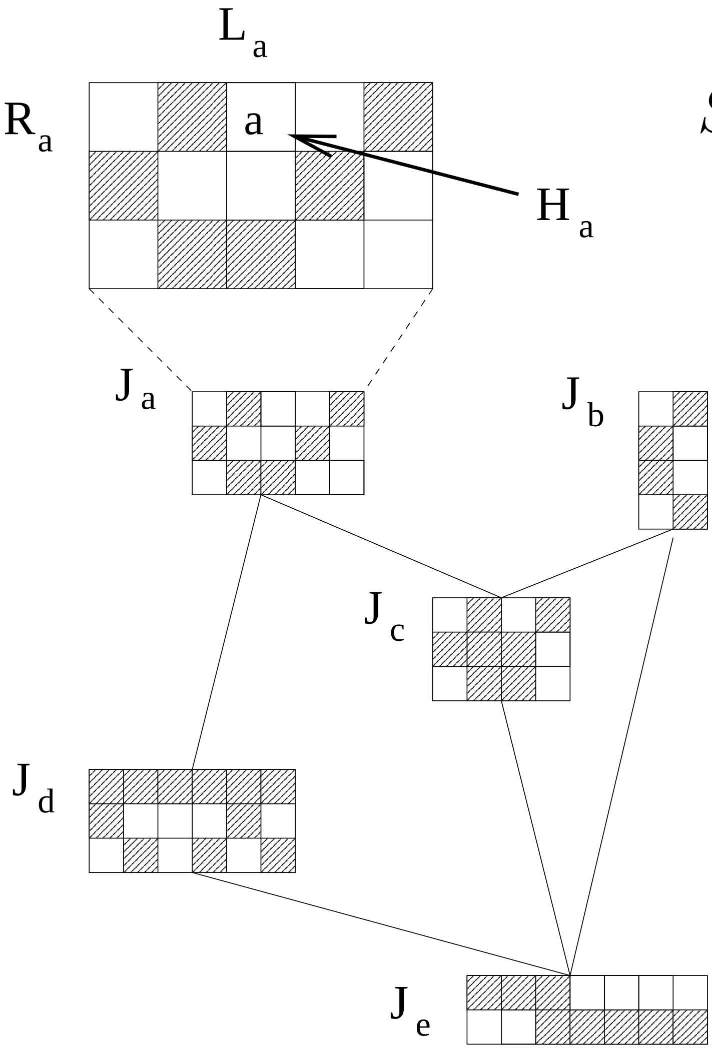
Partial transformations

- $K'(n, r) = \{\alpha \in P_n : |\mathrm{Im}(\alpha)| \leq r\}$;
- PO_n - partial order preserving transformations.

Idempotent ranks

Semigroup	Rank	Idrank
Sing_n	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
$\mathrm{Sing}(V)$	$\frac{q^n-1}{q-1}$	$\frac{q^n-1}{q-1}$ where $q = \lvert F\rvert$
$K(n,r)$	$S(n,r)$	$S(n,r)$
$I(r,n,q)$	$\begin{bmatrix} n \\ r \end{bmatrix}_q$	?
$K'(n,r)$	$S(n+1,r+1)$	$S(n+1,r+1)$
O_n	n	$2n-2$
PO_n	$2n-1$	$3n-2$.

Green's relations

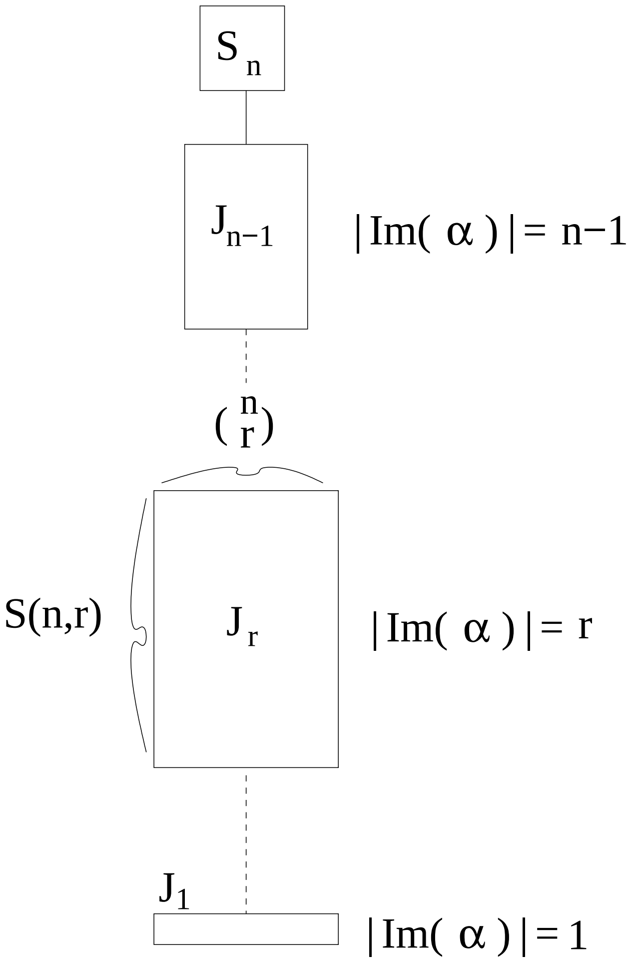


S - semigroup, $x, y \in S$

$$
\begin{aligned}
x\mathcal{R}y &\Leftrightarrow xS^1 = yS^1 \\
x\mathcal{L}y &\Leftrightarrow S^1x = S^1y \\
x\mathcal{J}y &\Leftrightarrow S^1xS^1 = S^1yS^1
\end{aligned}
$$

- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} (= \mathcal{J})$
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $J_x \leq J_y \Leftrightarrow S^1xS^1 \subseteq S^1yS^1$

Green's relations in T_n



T_n - full transformation semigroup, $\alpha, \beta \in T_n$

$$\begin{array}{rcl} \alpha \mathcal{R} \beta & \Leftrightarrow & \mathrm{Im}(\alpha) = \mathrm{Im}(\beta) \\ \alpha \mathcal{L} \beta & \Leftrightarrow & \ker(\alpha) = \ker(\beta) \\ \alpha \mathcal{J} \beta & \Leftrightarrow & |\mathrm{Im}(\alpha)| = |\mathrm{Im}(\beta)| \end{array}$$

$$J_r = \{\alpha \in T_n : |\mathrm{Im}(\alpha)| = r\}$$

$$\begin{array}{rcl} K(n,r) & = & \{\alpha \in T_n : |\mathrm{Im}(\alpha)| \le r\} \\ & = & J_r \cup \ldots \cup J_1. \end{array}$$

$$K(n,r) = \langle J_r \rangle,\ 1 \le r < n.$$

Rees matrix semigroups

Definition

- G - a finite group.
- I, Λ - non-empty finite index sets.
- $P = (p_{\lambda i})$ a *regular* $\Lambda \times I$ matrix over $G \cup \{0\}$.
- $S = (I \times G \times \Lambda) \cup \{0\}$ with multiplication

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, g p_{\lambda j} h, \mu) & : \quad p_{\lambda j} \neq 0 \\ 0 & : \quad \text{otherwise} \end{cases}$$

$$(i, g, \lambda)0 = 0(i, g, \lambda) = 00 = 0.$$

Rectangular 0-bands

Definition A rectangular 0-band is a 0-Rees matrix semigroup over the trivial group written as $\mathcal{M}^0[\{1\}; I, \Lambda; Q]$ where Q is a regular $|\Lambda| \times |I|$ matrix over $\{0, 1\}$ and:

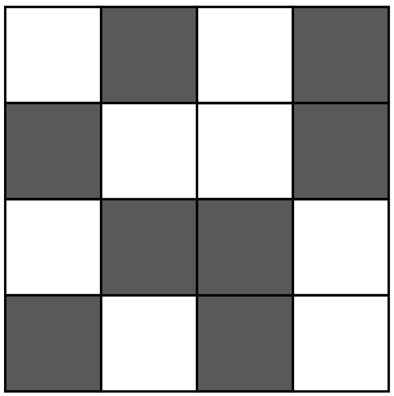
$$(i, \lambda)(j, \mu) = \begin{cases} (i, \mu) & : \quad p_{\lambda j} = 1 \\ 0 & : \quad \text{otherwise} \end{cases}$$

Lemma Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be an idempotent generated completely 0-simple semigroup and T be the natural rectangular 0-band homomorphic image of S ($q_{\lambda i} = 1 \Leftrightarrow p_{\lambda i} \neq 0$). Then

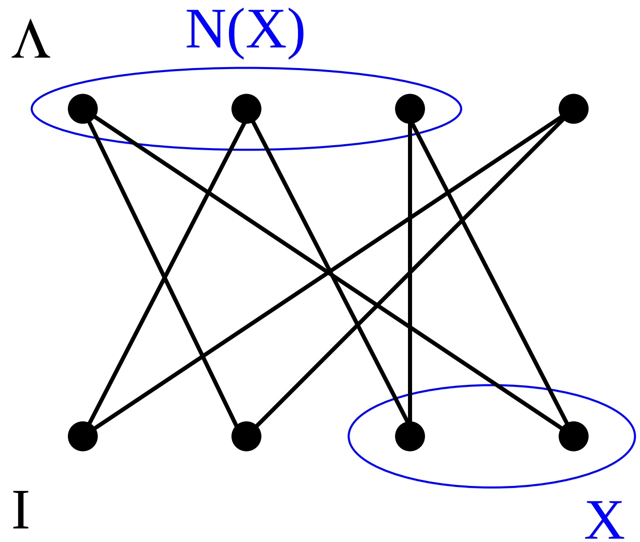
- $\mathrm{rank}(S) = \mathrm{rank}(T) = \max(|I|, |\Lambda|)$;
- $\mathrm{idrank}(S) = \mathrm{idrank}(T)$.

The graph $\Delta(P)$

Definition Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. We let $\Delta(P)$ denote the undirected bipartite graph with set of vertices $I \cup \Lambda$ and an edge between i and λ if and only if $p_{\lambda i} \neq 0$.



$S = \mathcal{M}^0[G; I, \Lambda; P]$



$\Delta(P)$

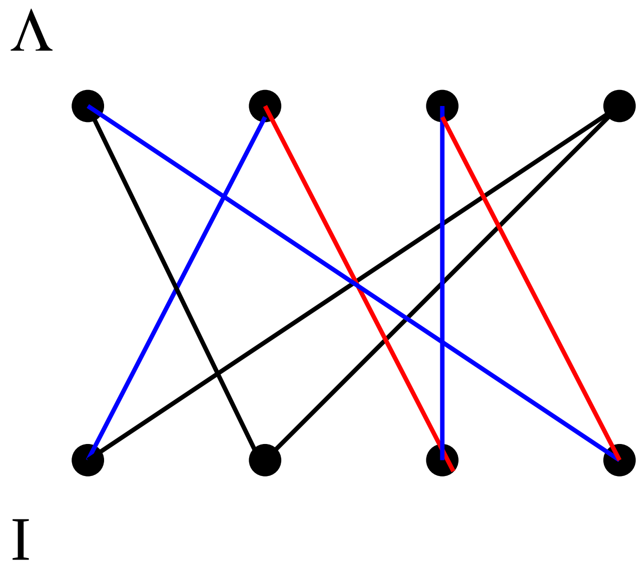
Paths and products

There is a natural correspondence between non-zero products of idempotents in the rectangular 0-band T and paths from I to Λ in $\Delta(Q)$.

Product : $(1,2)(3,3)(4,1) = (1,1)$

since $q_{23} = q_{34} = 1.$

Path :

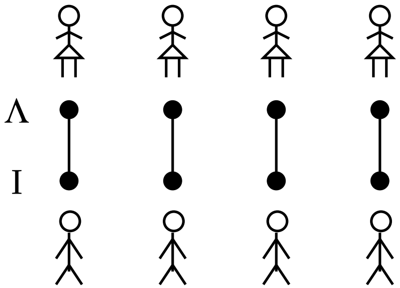


Question

Question When does a square idempotent generated 0-Rees matrix semigroup S satisfy:

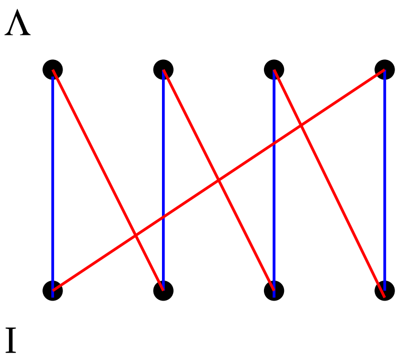
$$\mathrm{idrank}(S) = \mathrm{rank}(S) = \max(|I|, |\Lambda|) = n?$$

Necessary $\Delta(P)$ has a perfect matching.



Hall's Condition: for all $X \subseteq I$, $|N(X)| \geq |X|$.

Sufficient $\Delta(P)$ is hamiltonian.



Every second edge of the hamiltonian circuit constitutes a generating set of idempotents with size n.

Answer

Theorem Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be an idempotent generated completely 0-simple semigroup with $|I| = |\Lambda| = n$. Then the following are equivalent:

1. $\mathrm{rank}(S) = \mathrm{idrank}(S)$;
2. S satisfies SHC. ($\varnothing \subsetneq X \subsetneq I, |N(X)| > |X|$) ;
3. The minimum generating sets of S are precisely the subsets that intersect every (non-zero) $\mathcal{R}$-class in exactly one place and every (non-zero) $\mathcal{L}$-class of S in exactly one place.

Application to ideals of $\mathrm{End}(V)$

Let V be an n dimensional vector space over the finite field F where $|F| = q$. Let $J(r, n, q)$ denote the top $\mathcal{J}$-class of the ideal $I(r, n, q)$ where:

$$I(r, n, q) = \{A \in \mathrm{End}(V) : \dim(\mathrm{Im}(A)) \leq r\};$$

Greens relations in $\mathrm{End}(V)$ are given by:

$$\begin{aligned} A\mathcal{R}B &\Leftrightarrow \ker(A) = \ker(B); \\ A\mathcal{L}B &\Leftrightarrow \mathrm{Im}(A) = \mathrm{Im}(B); \\ A\mathcal{D}B &\Leftrightarrow \dim(\mathrm{Im}(A)) = \dim(\mathrm{Im}(B)). \end{aligned}$$

Theorem (Dawlings, 1980)

$$\mathrm{idrank}(I(n-1, n, q)) = \mathrm{rank}(I(n-1, n, q)) = \frac{q^n - 1}{q - 1}.$$

$I(2,4,2) \subseteq GF(2)^4$

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Uniform distribution of idempotents

Definition We will say that $S = \mathcal{M}^0[G; I, \Lambda; P]$ has a k-uniform distribution of idempotents if the graph $\Delta(P)$ is k-regular.

Corollary Every idempotent generated completely 0-simple semigroup S with $|I| = |\Lambda|$ and a k-uniform distribution of idempotents has an idempotent basis.

$$\begin{array}{ccccc} \text{HC} & \Leftarrow & \text{SHC} & \Leftarrow & \text{hamiltonian} \\ & & \Uparrow & & \\ & & \text{UDI} & & \end{array}$$

Corollaries

Corollary Let V be an n dimensional vector space over the finite field F where $|F| = q$. Then:

$$\operatorname{rank}(I(r,n,q)) = \operatorname{idrank}(I(r,n,q)) = \begin{bmatrix} n \\ r \end{bmatrix}_q .$$

Corollary A subset of $I(r,n,q)$ is a generating set of minimum cardinality for $I(r,n,q)$ if and only if it consists of matrices of rank r no two of which have the same nullspace or the same image space.

Independence Algebras

Definition An *independence algebra* is an algebra (in the sense of universal algebra) that satisfies:

[E] If $z \in \langle X \cup \{y\} \rangle$ and $z \notin \langle X \rangle$ then $y \in \langle X \cup \{z\} \rangle$.

A minimal generating set is called a *basis* for A and its size is the *dimension* of A. (they all have the same size by **[E]**)
[F] Any map from a basis of A into A can be extended to an endomorphism of A.

Examples Both sets and vector spaces are examples of independence algebras. Chains satisfy **[E]** but not **[F]**.

Definition $K(n, r) = \{\alpha \in \mathrm{End}(A) : \dim(\mathrm{Im}(\alpha)) \leq r\}$.

Theorem(Fountain and Lewin, 1990) If A is an independence algebra of finite rank n, then

$$K(n, r) = \langle E(J_r) \rangle \text{ for } r = 1, \ldots, n-1.$$

Independence Algebras

Theorem Let A be a finite independence algebra with dimension $n \geq 3$. Then:

$$\mathrm{idrank}(K(n,r)) = \mathrm{rank}(K(n,r)) \ (r = 1, \ldots, n-1).$$

The above result uses the classification of finite independence algebras given by Cameron and Szabó.

Problem Give a proof of the above result directly from the definition of independence algebra.