On groups of units of special and one-relator inverse monoids

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Adjan's Theorem

Theorem (Adjan (1966))

The group of units G of a one-relator monoid $M = \mathrm{Mon}\langle A \mid r = 1\rangle$ is a one-relator group.

Example

Let $M = \mathrm{Mon}\langle A \mid r = 1\rangle = \mathrm{Mon}\langle a, b, c, d \mid abcdcdab = 1\rangle$. Decompose the relator

$$abcdcdab = (ab)(cd)(cd)(ab)$$

into minimal invertible pieces = subwords of r that are invertible in M and have no proper non-empty invertible prefix. Then

- $X = ab$ and $Y = cd$ together generate the group of units G and satisfy

$$(\underbrace{ab}_{X})(\underbrace{cd}_{Y})(\underbrace{cd}_{Y})(\underbrace{ab}_{X}) = 1$$

- $G = \mathrm{Gp}\langle X, Y \mid XYYX = 1\rangle$.

Makanin's Theorem for special monoids

Theorem (Makanin (1966))

The group of units G of $M = \mathrm{Mon}\langle A \mid r_1 = 1, \ldots, r_k = 1\rangle$ admits a k-relator presentation.

Example

Let $M = \mathrm{Mon}\langle a, b, c, d \mid abab = 1, abcdabcdabcd = 1\rangle$. Decompose the relators into minimal invertible pieces

$$abab = (ab)(ab), \quad abcdabcdabcd = (ab)(cd)(ab)(cd)(ab)(cd).$$

Then

- $X = ab$ and $Y = cd$ together generate the group of units G and satisfy

$$(\underbrace{ab}_{X})(\underbrace{ab}_{X}) = 1, \quad (\underbrace{ab}_{X})(\underbrace{cd}_{Y})(\underbrace{ab}_{X})(\underbrace{cd}_{Y})(\underbrace{ab}_{X})(\underbrace{cd}_{Y}) = 1.$$

- $G = \mathrm{Gp}\langle X, Y \mid X^2 = 1, (XY)^3 = 1\rangle$.

Special inverse monoids

An inverse monoid is a monoid M such that for every $m \in M$ there is a unique $m^{-1} \in M$ such that $mm^{-1}m = m$ and $m^{-1}mm^{-1} = m^{-1}$.

Definition (Special inverse monoid)

$$\mathrm{Inv}\langle A \mid r_i = 1 \ (i \in I)\rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid r_i = 1 \ (i \in I),$$
$$x = xx^{-1}x, \quad xx^{-1}yy^{-1} = yy^{-1}xx^{-1}\rangle$$

where x, y range over all words from $(A \cup A^{-1})^*$.

Example

The bicyclic monoid is defined by $\mathrm{Inv}\langle a \mid aa^{-1} = 1\rangle$.

- Adjan/Makanin results have been applied to prove interesting results about special monoids e.g. word problem for $\mathrm{Mon}\langle A \mid r = 1\rangle$.
- Adjan/Makanin-type theorems for special inverse monoids might via work of Ivanov, Margolis, Meakin (2001) have important applications e.g. word problem for arbitrary one relation monoids $\mathrm{Mon}\langle A \mid u = v\rangle$.

Adjan/Makanin-type theorems for special inverse monoids

Theorem (Ivanov, Margolis, Meakin (2001))

The group of units G of $M = \mathrm{Inv}\langle A \mid r_1 = 1, \ldots, r_k = 1\rangle$ is finitely generated by the minimal invertible pieces of the relators $r_1, \ldots, r_k$.

Example

$M = \mathrm{Inv}\langle a, b, c \mid a(bc^2b^{-1})a(bc^3b^{-1})a(bc^7b^{-1})a(bc^3b^{-1})a(bc^2b^{-1})a = 1\rangle$.

Then the minimal invertible pieces

- $X = a$, $Y = bc^2b^{-1}$, $Z = bc^3b^{-1}$, $T = bc^7b^{-1}$ together generate the group of units G and satisfy

$$\underbrace{a}_{X}\underbrace{(bc^2b^{-1})}_{Y}\underbrace{a}_{X}\underbrace{(bc^3b^{-1})}_{Z}\underbrace{a}_{X}\underbrace{(bc^7b^{-1})}_{T}\underbrace{a}_{X}\underbrace{(bc^3b^{-1})}_{Z}\underbrace{a}_{X}\underbrace{(bc^2b^{-1})}_{Y}\underbrace{a}_{X} = 1$$

Question: Is this relation enough to define G? i.e. do we have

$$G = \mathrm{Gp}\langle X, Y, Z, T \mid XYXZXTXZXYX = 1\rangle \text{ ?}$$

Adjan/Makanin-type theorems for special inverse monoids

Let $M = \mathrm{Inv}\langle A \mid r_1 = 1, \ldots, r_k = 1\rangle$.

Fact: $w_1 a a^{-1} w_2 \in (A \cup A^{-1})^*$ invertible in $M \Longrightarrow w_1 a a^{-1} w_2 = w_1 w_2$ in M.

Example

$$M = \mathrm{Inv}\langle a, b, c \mid \underbrace{a}_{X} \underbrace{bc^2b^{-1}}_{Y} \underbrace{a}_{X} \underbrace{bc^3b^{-1}}_{Z} \underbrace{a}_{X} \underbrace{bc^7b^{-1}}_{T} \underbrace{a}_{X} \underbrace{bc^3b^{-1}}_{Z} \underbrace{a}_{X} \underbrace{bc^2b^{-1}}_{Y} \underbrace{a}_{X} = 1\rangle.$$

Is the group of units G equal to

$$\mathrm{Gp}\langle X, Y, Z, T \mid XYXZX\,T\,XZXYX = 1\rangle = \mathrm{Gp}\langle X, Y, Z \mid \ \rangle = F_{X,Y,Z}?$$

Applying the Fact above in M gives

$$Y^3 = (bc^2b^{-1})^3 = bc^2b^{-1}bc^2b^{-1}bc^2b^{-1} = bc^6b^{-1} = (bc^3b^{-1})^2 = Z^2$$

But $Y^3 \neq X^2$ in $F_{X,Y,Z}$, hence $G \neq \mathrm{Gp}\langle X, Y, Z, T \mid XYXZXTXZXYX = 1\rangle$.

Adjan/Makanin-type theorems for special inverse monoids

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Example

$M = \mathrm{Inv}\langle a, b, c \mid \underbrace{a}_{X} \underbrace{bc^2b^{-1}}_{Y} \underbrace{a}_{X} \underbrace{bc^3b^{-1}}_{Z} \underbrace{a}_{X} \underbrace{bc^7b^{-1}}_{T} \underbrace{a}_{X} \underbrace{bc^3b^{-1}}_{Z} \underbrace{a}_{X} \underbrace{bc^2b^{-1}}_{Y} \underbrace{a}_{X} = 1\rangle$.

Is the group of units G equal to

$$\mathrm{Gp}\langle X, Y, Z, T \mid XYXZX\,T\,XZXYX = 1\rangle = \mathrm{Gp}\langle X, Y, Z \mid \ \rangle = F_{X,Y,Z}?$$

Applying the Fact above in M gives

$$Y^3 = (bc^2b^{-1})^3 = bc^2b^{-1}bc^2b^{-1}bc^2b^{-1} = bc^6b^{-1} = (bc^3b^{-1})^2 = Z^2$$

But $Y^3 \neq X^2$ in $F_{X,Y,Z}$, hence $G \neq \mathrm{Gp}\langle X, Y, Z, T \mid XYXZXTXZXYX = 1\rangle$.

Resolution: Observe $\mathrm{Gp}\langle a, bc^2b^{-1}, bc^3b^{-1}, bc^7b^{-1}\rangle = \mathrm{Gp}\langle a, bcb^{-1}\rangle \leq F_{a,b,c}$.

- $M \cong \mathrm{Inv}\langle a, b, c \mid a(bcb^{-1})^2 a(bcb^{-1})^3 a(bcb^{-1})^7 a(bcb^{-1})^3 a(bcb^{-1})^2 a = 1\rangle$
- Group of units of M is $G = \mathrm{Gp}\langle X, Y \mid XY^2XY^3XY^7XY^3XY^2X = 1\rangle$.

Adjan/Makanin-type theorems for special inverse monoids

Theorem (RDG & Ruškuc (2021))

Let $G = \mathrm{Gp}\langle A \mid r_1 = 1, \ldots, r_k = 1 \rangle$ be a finitely presented k-relator group and let $H \leq G$ be a finitely generated subgroup of G.

Then there is a finitely presented k-relator special inverse monoid

$$M = \mathrm{Inv}\langle B \mid s_1 = 1, \ldots, s_k = 1 \rangle$$

such that the group of units of M is isomorphic to the free product $G * H$.

Strategy: Find pairs $H \leq G$ where

- G has some property $\mathcal{P}$ but
- $G * H$ does not have property $\mathcal{P}$.
- Hence the group of units of M will not have property $\mathcal{P}$.

Non finitely presented group of units

Fact (e.g. Higman (1961))

Finite presentability is not inherited by finitely generated subgroups.

Choose $H \leq G$ such that G is finitely presented and H is finitely generated but not finitely presented. Then $G * H$ is not finitely presented since H is not.

Theorem (RDG & Ruškuc (2021))

There is a finitely presented special inverse monoid $\mathrm{Inv}\langle B \mid s_1 = 1, \ldots, s_k = 1\rangle$ whose group of units is not finitely presented.

Example

The finitely presented special inverse monoid

$$\begin{aligned}
&\mathrm{Inv}\langle c_1, c_2, d_1, d_2, t, C_1, C_2, D_1, D_2, T \mid c_iC_i = 1,\ C_ic_i = 1,\ (i \in \{1,2\}),\\
&d_iD_i = 1,\ D_id_i = 1\ (i \in \{1,2\}),\ tT = 1,\\
&c_id_jC_iD_j = 1, (i,j \in \{1,2\}), \qquad tc_2TtC_2T = 1,\ tC_2Ttc_2T = 1,\\
&td_2TtD_2T = 1,\ tD_2Ttd_2T = 1, \qquad tc_1d_1TtD_1C_1T = 1,\ tD_1C_1Ttc_1d_1T = 1\rangle
\end{aligned}$$

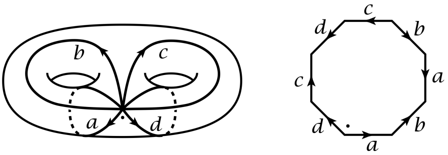
has a group of units that is not finitely presented.

Units of one-relator inverse monoids

Fact

Being one-relator is not preserved by taking free products.

Example. If $K = \mathrm{Gp}\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1\rangle$ then $K * K$ is not one-relator with respect to any finite generating set.



- Proved using Lyndon's Identity Theorem.
- Taking $G = H = K$ in the above theorem then gives $M = \mathrm{Inv}\langle B \mid s = 1\rangle$ with group of units $G * H = K * K$ not one-relator.

Theorem (RDG & Ruškuc (2021))

There exists a one-relator special inverse monoid $M = \mathrm{Inv}\langle B \mid s = 1\rangle$ whose group of units G is not a one-relator group.

Coherence and a question of Baumslag

Question: Is the group of units of $\mathrm{Inv}\langle A \mid r = 1 \rangle$ always finitely presented?

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Definition. A finitely presented group G is said to be coherent if every finitely generated subgroup of G is finitely presented.

Open problem (Baumslag (1973))

Is every one-relator group coherent?

Theorem (RDG & Ruškuc (2021))

If all one-relator special inverse monoids $\mathrm{Inv}\langle A \mid r = 1\rangle$ have finitely presented groups of units then all one-relator groups are coherent.

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Theorem (RDG & Ruškuc (2021))

Let $M = \mathrm{Inv}\langle A \mid r^m = 1\rangle$ where $m > 1$ and r is cyclically reduced. Then the group of units of M is finitely presented.

- Louder and Wilton (2020) & independently Wise (2020) proved that one-relator groups with torsion are coherent.
- Ivanov, Margolis, Meakin (2001) $\Rightarrow$ M is E-unitary in this case.

Open problems

1. Is the group of units of a one-relator inverse monoid $\mathrm{Inv}\langle A \mid r = 1\rangle$ finitely presented?
2. If r is cyclically reduced then is the group of units of $\mathrm{Inv}\langle A \mid r = 1\rangle$ one-relator / finitely presented?
3. Is there an algorithm that given $\mathrm{Inv}\langle A \mid r = 1\rangle$ computes the decomposition $r \equiv r_1 r_2 \ldots r_k$ into minimal invertible pieces?
4. Is the group of units of a one-relator inverse monoid $\mathrm{Inv}\langle A \mid r = 1\rangle$ embeddable into a one-relator group?
5. Does the group of units of a one-relator inverse monoid $\mathrm{Inv}\langle A \mid r = 1\rangle$ have decidable word problem?