Undecidability of the word problem for one-relator inverse monoids via right-angled Artin subgroups of one-relator groups

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Word problem for one-relator groups and monoids

	$\mathrm{Gp}\langle A \mid w = 1\rangle$ $\mathrm{FG}(A)/\langle\!\langle w\rangle\!\rangle$	$\mathrm{Mon}\langle A \mid w = 1\rangle$ $A^*/\langle\!\langle (w, 1)\rangle\!\rangle$	$\mathrm{Inv}\langle A \mid w = 1\rangle$ $\mathrm{FIM}(A)/\langle\!\langle (w, 1)\rangle\!\rangle$
Word problem decidable	Magnus (1932) ✓	Adjan (1966) ✓	?

Theorem (Scheiblich (1973) & Munn (1974))

Free inverse monoids have decidable word problem.

Conjecture (Margolis, Meakin, Stephen (1987))

If $M = \mathrm{Inv}\langle A \mid w = 1\rangle$, then the word problem for M is decidable.

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Theorem (Ivanov, Margolis, Meakin (2001))

If the word problem is decidable for all inverse monoids of the form $\mathrm{Inv}\langle A \mid w = 1\rangle$ then the word problem is also decidable for every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$.

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Proved true in many cases e.g. when w satisfies...

- Idempotent word [Birget, Margolis, Meakin, 1993, 1994]
- w-strictly positive [Ivanov, Margolis, Meakin, 2001]
- Adjan or Baumslag-Solitar type [Margolis, Meakin, Šuniḱ, 2005]
- Sparse word [Hermiller, Lindblad, Meakin, 2010]
- Certain small cancellation conditions [A. Juhász, 2012, 2014]

Word problem for one-relator groups and monoids

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Theorem (RDG (2020))

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid w = 1\rangle$ with undecidable word problem.

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Theorem (RDG (2020))

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid w = 1\rangle$ with undecidable word problem.

Ingredients for the proof:

- Submonoid membership problem for one relator groups.
- HNN-extensions and free products of groups.
- Right-angled Artin groups (RAAGs).
- Right units of special inverse monoids

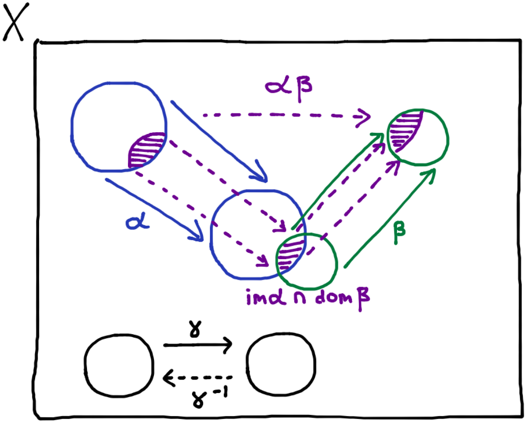
$$\mathrm{Inv}\langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1\rangle$$

  and Stephen's procedure for constructing Schützenberger graphs.
- Properties of E-unitary inverse monoids.

Inverse monoids

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

Example: I_X = monoid of all partial bijections $X \to X$



Examples: In I_3

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & - & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ - & 1 & -\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -\end{pmatrix}^{-1} = \begin{pmatrix}1 & 2 & 3\\ - & 1 & 2\end{pmatrix}$$

Note:
$\gamma\gamma^{-1} = \mathrm{id}_{\mathrm{dom}\gamma}$

Inverse monoid presentations

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

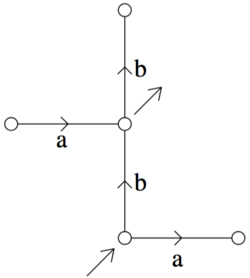
For all $x, y \in M$ we have

$$x = xx^{-1}x, \ (x^{-1})^{-1} = x, \ (xy)^{-1} = y^{-1}x^{-1}, \ xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \qquad (\dagger)$$

$$\mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid u_i = v_i \ (i \in I) \cup (\dagger)\rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and x, y range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\mathrm{FIM}(A) = \mathrm{Inv}\langle A \mid \ \rangle$



Munn (1974)
Elements of $\mathrm{FIM}(A)$ can be represented using Munn trees. e.g. in $\mathrm{FIM}(a, b)$ we have $u = w$ where

$$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$$
$$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$$

The word problem

M - a finitely generated monoid with a finite generating set A.

$\pi : A^* \to M$ – the canonical monoid homomorphism.

The monoid M has decidable word problem if there is an algorithm which solves the following decision problem:

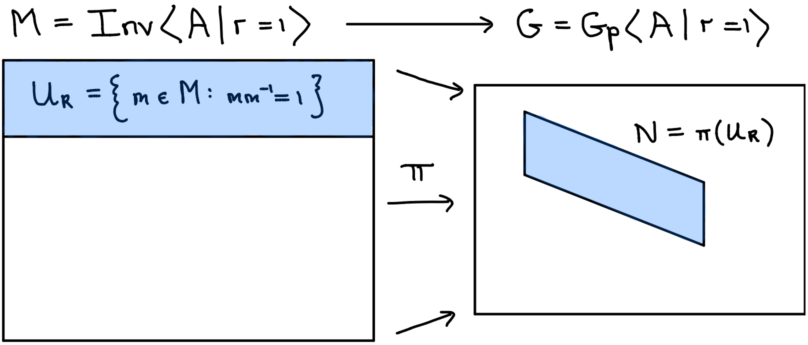
INPUT: Two words $u, v \in A^*$.

QUESTION: $\pi(u) = \pi(v)$? i.e. do u and v represent the same element of the monoid M?

For a group or an inverse monoid with generating set A the word problem is defined in the same way except the input is two words $u, v \in (A \cup A^{-1})^*$.

Example. The bicyclic monoid $\mathrm{Inv}\langle a \mid aa^{-1} = 1 \rangle$ has decidable word problem.

Proof strategy



If M has decidable word problem

$\Longrightarrow$ membership problem for $U_R \leq M$ is decidable

since for $w \in (A \cup A^{-1})^*$ $\boxed{w \in U_R \iff ww^{-1} = 1}$

(sometimes)

$\rightsquigarrow$ membership problem for $N \leq G$ is decidable

RAAGs induced subgraphs and subgroups

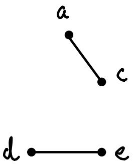
Definition

The right-angled Artin group $A(\Gamma)$ associated with the graph Γ is

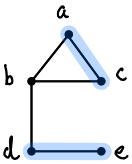
$$\mathrm{Gp}\langle V\Gamma \mid uv = vu \text{ if and only if } \{u, v\} \in E\Gamma\rangle.$$

Fact: If Δ is an induced subgraph of Γ then the embedding $\Delta \to \Gamma$ induces an embedding $A(\Delta) \to A(\Gamma)$.

Example



$= \Delta \hookrightarrow \Gamma =$



$$A(\Delta) = \mathrm{Gp}\langle a, c, d, e \mid ac = ca,\ de = ed \rangle$$

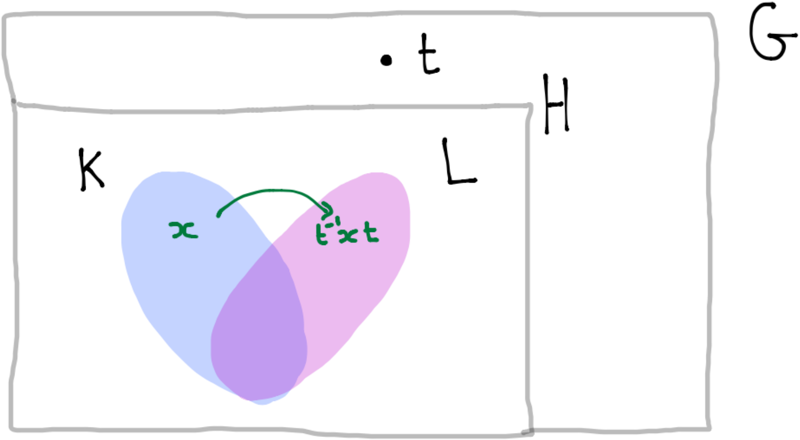
$$\hookrightarrow A(\Gamma) = \mathrm{Gp}\langle a, b, c, d, e \mid ac = ca,\ de = ed,\ ab = ba,\ bc = cb,\ bd = db \rangle$$

HNN-extensions of groups

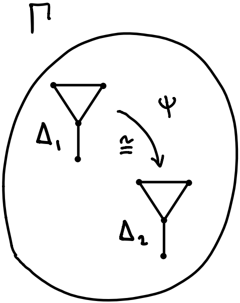
$H \cong \mathrm{Gp}\langle A \mid R \rangle$, $\quad K, L \leq H$ with $K \cong L$. Let $\phi : K \to L$ be an isomorphism.
The HNN-extension of H with respect to ϕ is

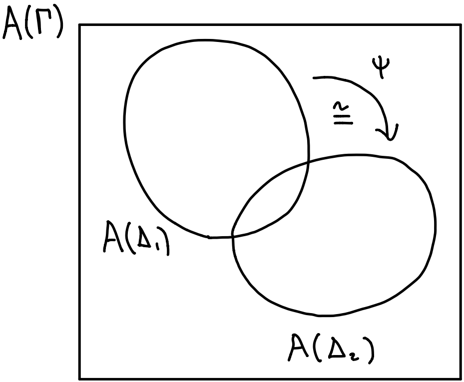
$$G = \mathrm{HNN}(H, \phi) = \mathrm{Gp}\langle A,\ t \mid R,\ t^{-1}kt = \phi(k)\ (k \in K) \rangle$$

Fact: H embeds naturally into the HNN extension $G = HNN(H, \phi)$.



HNN-extensions of RAAGs





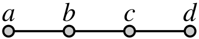
Definition

Γ - finite graph, $\psi : \Delta_1 \to \Delta_2$ an isomorphism between finite induced subgraphs.
$A(\Gamma, \psi)$ is defined to be the HNN-extension of $A(\Gamma)$ with respect to the isomorphism $A(\Delta_1) \to A(\Delta_2)$ induced by ψ.

Fact: $A(\Gamma)$ embeds naturally into $A(\Gamma, \psi)$.

HNN-extension of $A(P_4)$ over $A(P_3)$

Let P_4 be the graph



$$A(P_4) = \mathrm{Gp}\langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle.$$

Δ_1 - subgraph induced by $\{a, b, c\}$, Δ_2 subgraph induced by $\{b, c, d\}$,
$\psi : \Delta_1 \to \Delta_2$ - the isomorphism $a \mapsto b$, $b \mapsto c$, and $c \mapsto d$.

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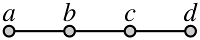

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Then the HNN-extension $A(P_4, \psi)$ of $A(P_4)$ with respect to ψ is

$$\begin{aligned} & A(P_4, \psi) \\ = \quad & \mathrm{Gp}\langle a, b, c, d, t \mid ab = ba, bc = cb, cd = dc, tat^{-1} = b, tbt^{-1} = c, tct^{-1} = d \rangle \end{aligned}$$

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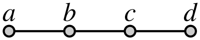

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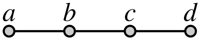

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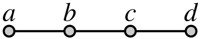

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\end{aligned}$$

Conclusion

$A(P_4)$ embeds into the one-relator group

$$A(P_4, \psi) = \mathrm{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1 \rangle.$$

Submonoid membership problem

G - a finitely generated group with a finite group generating set A.

$\pi : (A \cup A^{-1})^* \to G$ – the canonical monoid homomorphism.

T – a finitely generated submonoid of G.

The membership problem for T within G is decidable if there is an algorithm which solves the following decision problem:

INPUT: A word $w \in (A \cup A^{-1})^*$.

QUESTION: $\pi(w) \in T$?

Theorem B

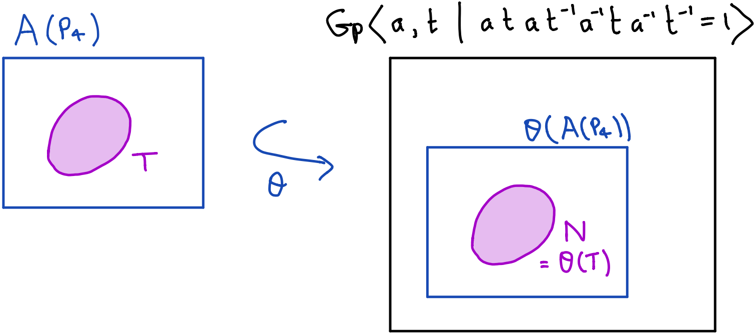
Let G be the one-relator group $\mathrm{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1\rangle$. Then there is a fixed finitely generated submonoid N of G such that the membership problem for N within G is undecidable.

Proof of Theorem B

Theorem B

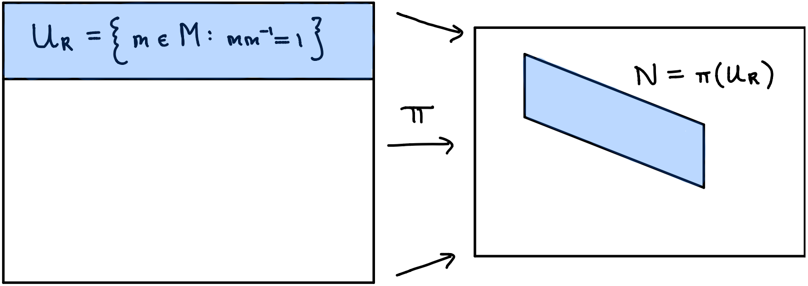
Let G be the one-relator group $\mathrm{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1\rangle$. Then there is a fixed finitely generated submonoid N of G such that the membership problem for N within G is undecidable.

Proof. By [Lohrey & Steinberg, 2008] there is a finitely generated submonoid T of $A(P_4)$ such that the membership problem for T within $A(P_4)$ is undecidable. Let $\theta : A(P_4) \to G$ be an embedding. Then $N = \theta(T)$ is a finitely generated submonoid of G such that the membership problem for N within G is undecidable.



Proof strategy

$$M = \mathrm{Inv}\langle A \mid r = 1\rangle \longrightarrow G = \mathrm{Gp}\langle A \mid r = 1\rangle$$



If M has decidable word problem

$\implies$ membership problem for $U_R \leq M$ is decidable

since for $w \in (A \cup A^{-1})^*$ $\boxed{w \in U_R \iff ww^{-1} = 1}$

(sometimes)

$\rightsquigarrow$ membership problem for $N \leq G$ is decidable

General observations about inverse monoids

S – an inverse monoid generated by A, $E(S)$ – set of idempotents,

$U_R \leq S$ – right units = submonoid if right invertible elements.

- If $e \in E(S)$ and $e \in U_R$ then $e = 1$.
- Two relations for the price of one: If e is an idempotent in $\mathrm{FIM}(A)$ and $r \in (A \cup A^{-1})^*$ then

$$\mathrm{Inv}\langle A \mid er = 1 \rangle = \mathrm{Inv}\langle A \mid e = 1, r = 1 \rangle.$$

- $e \in (A \cup A^{-1})^*$ is an idempotent in $\mathrm{FIM}(A)$ if and only if e freely reduces to 1 in the free group $\mathrm{FG}(A)$. e.g.

$$x^{-1}y^{-1}xx^{-1}yzz^{-1}x \in E(\mathrm{FIM}(x, y, z)).$$

A general construction

For any $r, w_1, \ldots w_k \in (A \cup A^{-1})^*$, with $A = \{a_1, \ldots, a_n\}$, set e equal to

$$a_1 a_1^{-1} \ldots a_n a_n^{-1} (t w_1 t^{-1})(t w_1^{-1} t^{-1})(t w_2 t^{-1})(t w_2^{-1} t^{-1}) \ldots (t w_k t^{-1})(t w_k^{-1} t^{-1}) a_n^{-1} a_n \ldots a_1^{-1} a_1$$

where t is a new symbol. Then

$$\begin{aligned} M &= \mathrm{Inv}\langle A, t \mid er = 1\rangle \\ &= \mathrm{Inv}\langle A, t \mid \ r = 1, aa^{-1} = 1, a^{-1}a = 1 \ (a \in A), (t w_i t^{-1})(t w_i t^{-1})^{-1} = 1 \ (1 \leq i \leq k)\rangle \\ &\cong \mathrm{Gp}\langle A \mid r = 1\rangle * \mathrm{FIM}(t) \ \ / \ \ \{(t w_i t^{-1})(t w_i t^{-1})^{-1} = 1 \ (1 \leq i \leq k)\}. \end{aligned}$$

Key claim

Let T be the submonoid of $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ generated by $\{w_1, w_2, \ldots, w_k\}$. Then for all $u \in (A \cup A^{-1})^*$ we have

$$u \in T \text{ in } G \iff t u t^{-1} \in U_R \text{ in } M.$$

A general construction

For any $r, w_1, \ldots w_k \in (A \cup A^{-1})^*$, with $A = \{a_1, \ldots, a_n\}$, set e equal to
$a_1 a_1^{-1} \ldots a_n a_n^{-1} (tw_1 t^{-1})(tw_1^{-1} t^{-1})(tw_2 t^{-1})(tw_2^{-1} t^{-1}) \ldots (tw_k t^{-1})(tw_k^{-1} t^{-1}) a_n^{-1} a_n \ldots a_1^{-1} a_1$

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Key claim

Let T be the submonoid of $G = \mathrm{Gp}\langle A \mid r = 1 \rangle$ generated by $\{w_1, w_2, \ldots, w_k\}$.
Then for all $u \in (A \cup A^{-1})^*$ we have

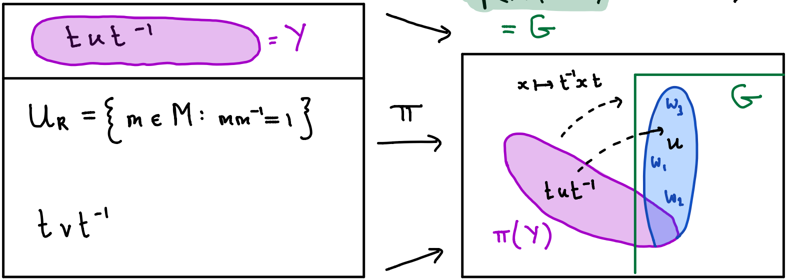
$$u \in T \text{ in } G \iff tut^{-1} \in U_R \text{ in } M.$$

Theorem

If $M = \mathrm{Inv}\langle A, t \mid er = 1 \rangle$ has decidable word problem then the membership problem for T within $G = \mathrm{Gp}\langle A \mid r = 1 \rangle$ is decidable.

Proof strategy refined

$M = \mathrm{Inv}\langle A, t \mid er = 1\rangle \longrightarrow \mathrm{Gp}\langle A, t \mid r = 1\rangle$
$= \mathrm{Gp}\langle A \mid r = 1\rangle * FG(t)$
$= G$



If M has decidable word problem

$\Longrightarrow$ membership problem for $U_R \leq M$ is decidable

$\Longrightarrow$ $\forall\ u \in (A \cup A^{-1})^*$ can decide $\boxed{tut^{-1} \in U_R\ ?}$

$\Longrightarrow$ (by key claim) can decide $\boxed{u \in T = \mathrm{Mon}\langle w_1, \dots, w_k\rangle \leq G}$

Tying things together

Thoerem A

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid w = 1\rangle$ with undecidable word problem.

Proof.

Let $A = \{a, z\}$ and let G be the one-relator group

$$\mathrm{Gp}\langle a, z \mid azaz^{-1}a^{-1}za^{-1}z^{-1} = 1\rangle.$$

Let $W = \{w_1, \ldots, w_k\}$ be a finite subset of $(A \cup A^{-1})^*$ such that the membership problem for $T = \mathrm{Mon}\langle W\rangle$ within G is undecidable. Such a set W exists by Theorem B. Set e to be the idempotent word

$$aa^{-1}zz^{-1}(tw_1t^{-1})(tw_1^{-1}t^{-1})(tw_2t^{-1})(tw_2^{-1}t^{-1})\ldots(tw_kt^{-1})(tw_k^{-1}t^{-1})z^{-1}za^{-1}a.$$

Then by the above theorem the one-relator inverse monoid

$$\mathrm{Inv}\langle a, z, t \mid eazaz^{-1}a^{-1}za^{-1}z^{-1} = 1\rangle$$

has undecidable word problem. This completes the proof. $\square$

What next?

Reduced vs cyclically reduced words

$aba^{-1}ab$ - not reduced,
$abba^{-1}$ - reduced but not cyclically reduced
$aba^{-1}b^{-1}$ - cyclically reduced

Key question

For which words $w \in (A \cup A^{-1})^*$ does $\mathrm{Inv}\langle A \mid w = 1 \rangle$ have decidable word problem? In particular is the word problem always decidable when w is (a) reduced or (b) cyclically reduced?

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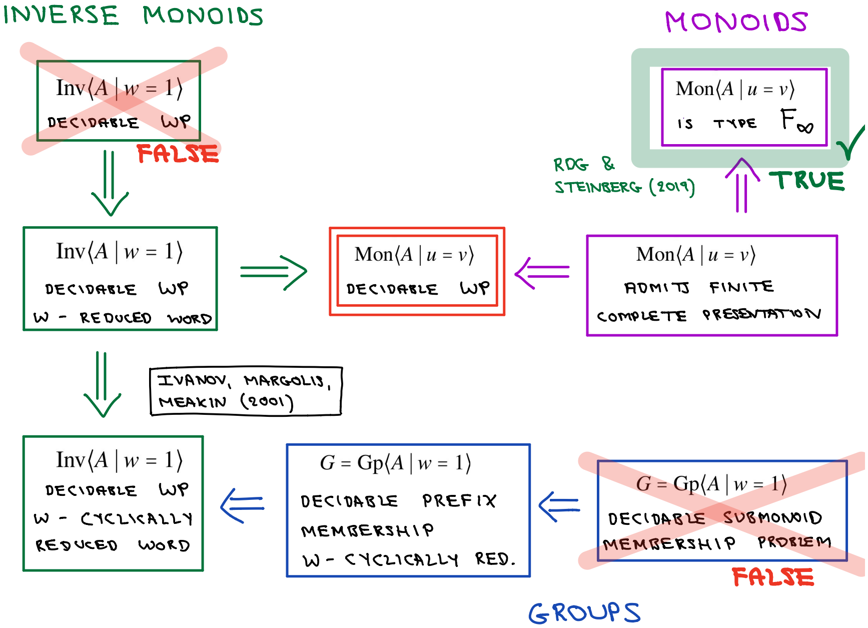
Definition

The prefix submonoid P_w of $\mathrm{Gp}\langle A \mid w = 1\rangle$ is the submonoid generated by all prefixes of the word w.

Theorem (Ivanov, Margolis and Meakin (2001))

Let $w \in (A \cup A^{-1})^*$ be cyclically reduced. If $\mathrm{Gp}\langle A \mid w = 1\rangle$ has decidable prefix membership problem (e.g. can decide membership in P_w) then $\mathrm{Inv}\langle A \mid w = 1\rangle$ has decidable word problem.


INVERSE MONOIDS
MONOIDS
Inv⟨A | w = 1⟩
DECIDABLE WP
FALSE
Mon⟨A | u = v⟩
IS TYPE F∞
TRUE
RDG & STEINBERG (2019)
Inv⟨A | w = 1⟩
DECIDABLE WP
W - REDUCED WORD
Mon⟨A | u = v⟩
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ADMITS FINITE COMPLETE PRESENTATION
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Inv⟨A | w = 1⟩
DECIDABLE WP
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G = Gp⟨A | w = 1⟩
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DECIDABLE SUBMONOID MEMBERSHIP PROBLEM
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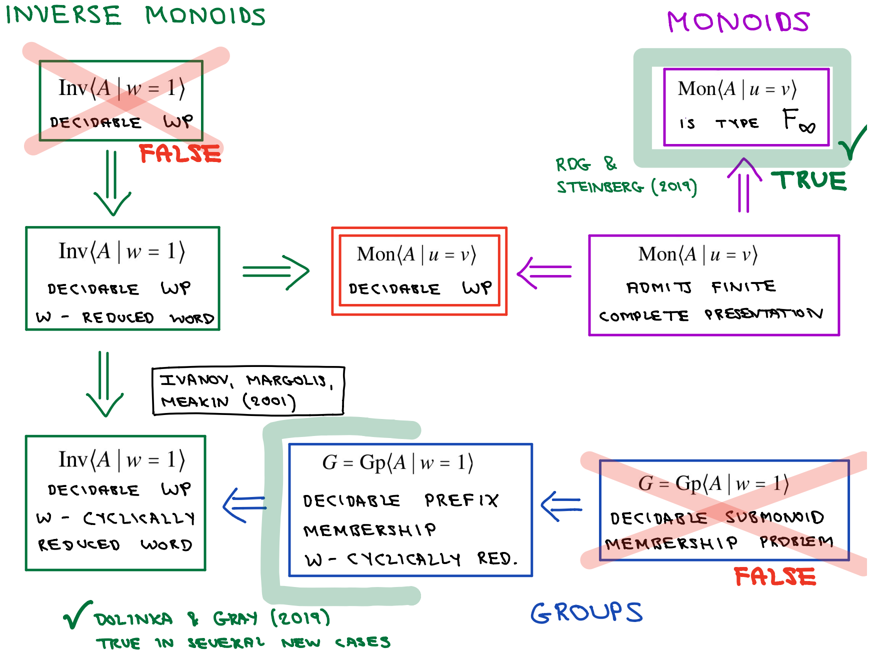
GROUPS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DOLINKA & GRAY (2019)
TRUE IN SEVERAL NEW CASES

More open problems

Problem
Characterise the one-relator groups with decidable submonoid membership problem.

Problem
Characterise the one-relator groups with decidable rational subset membership problem.

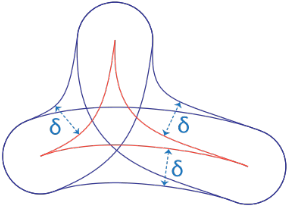
Problem
Is the subgroup membership problem decidable for one-relator groups?

Further applications

- In my paper there is actually a more general construction which shows how to encode the submonoid membership problem for any n-relator group into the word problem of an n-relator inverse monoid.
- This general construction has since been applied in other ways e.g. In joint work with Pedro Silva and Nora Szakacs we have combined that construction with the Rips construction to prove:

Theorem (RDG, Silva & Szakacs (2019))

There is a finitely presented inverse monoid $M = \mathrm{Inv}\langle A \mid w_1 = 1, \ldots, w_k = 1\rangle$ such that all strongly connected components of the directed Cayley graph of M are δ-hyperbolic, but M has undecidable word problem.



- In contrast we prove that if every strongly connected component is quasi-isometric to a tree then the monoid has decidable word problem.