On groups encoded by the structure of the idempotents in the full linear monoid

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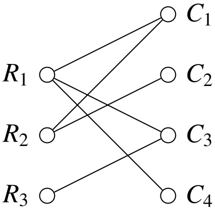
(this is joint work with Igor Dolinka (Novi Sad))

Warwick, 31st October 2013

Combinatorics: (0, 1)-matrices

$$
\begin{array}{cc}
 & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\
\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

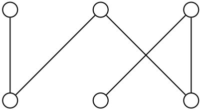
(0, 1)-matrix



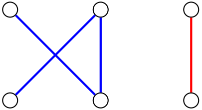
Bipartite graph

$(0,1)$-matrices and connectedness

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



- The 1s in the matrix are connected if any pair of entries 1 is connected by a sequence of 1s where adjacent terms in the sequence belong to same row/column.

Combinatorics

Symbols

$$A = \{\heartsuit, \smiley, \sun, \eighthnote\}$$

Table

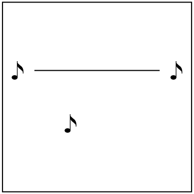
$$M = \begin{pmatrix} \sun & \heartsuit & \smiley & \heartsuit \\ \eighthnote & \sun & \sun & \eighthnote \\ \sun & \eighthnote & \sun & \sun \\ \smiley & \sun & \smiley & \heartsuit \end{pmatrix}$$

For each symbol x we can ask whether the xs are connected in M.

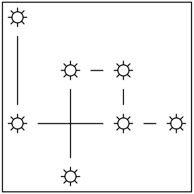
Let $\Delta(x)$ be a graph with vertices the occurrences of the symbol x and symbols in the same row/col connected by an edge.

Connectedness in tables

$$M = \begin{pmatrix} \sun & \heartsuit & \smiley & \heartsuit \\ \eighthnote & \sun & \sun & \eighthnote \\ \sun & \eighthnote & \sun & \sun \\ \smiley & \sun & \smiley & \heartsuit \end{pmatrix}$$



$\Delta(\eighthnote)$ is not connected



$\Delta(\sun)$ is connected

Tables in algebra

Multiplication tables

Group multiplication tables

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

- The multiplication table of a group is a Latin square, so..
- None of the graphs $\Delta(x)$ will be connected.

Tables in algebra

Multiplication tables

Multiplication table of a field.

Field with three elements $\mathbb{F} = \{0, 1, 2\}$.

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

- $\Delta(0)$ is connected
- $\Delta(f)$ is not connected for every $f \neq 0$

Tables in algebra

Vectors

$\mathbb{F} = \{0, 1\}$, vectors in $\mathbb{F}^3$, entries in table from $\mathbb{F}$

	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
(0, 0, 0)	0	0	0	0	0	0	0	0
(0, 0, 1)	0	1	0	1	0	1	0	1
(0, 1, 0)	0	0	1	1	0	0	1	1
(0, 1, 1)	0	1	1	0	0	1	1	0
(1, 0, 0)	0	0	0	0	1	1	1	1
(1, 0, 1)	0	1	0	1	1	0	1	0
(1, 1, 0)	0	0	1	1	1	1	0	0
(1, 1, 1)	0	1	1	0	1	0	0	1

- For every symbol x in the table $\Delta(x)$ is connected.

Outline

Free idempotent generated semigroups

- Background and recent results
- Maximal subgroups of free idempotent generated semigroups

The full linear monoid

- Basic properties
- The free idempotent generated semigroup over the full linear monoid
- Proof sketch: connectedness properties in tables

Open problems

Semigroups, idempotents and maximal subgroups

S - semigroup, $\quad E = E(S)$ - idempotents $e = e^2$ of S

Let $e \in S$ be an idempotent. Then

- eSe is the largest submonoid of S (with respect to inclusion) whose identity element is e.
- The group of units H_e of eSe is the largest subgroup of S with identity e, and is called the maximal subgroup of S containing e.

Example

Let T_n denote the full transformation monoid of all maps from $\{1, 2, \ldots, n\}$ to itself. Let $S = T_3$ and $\epsilon = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{smallmatrix}\right) \in E(S)$. Then

$$H_\epsilon = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{smallmatrix}\right) \right\} \cong S_2.$$

More generally, if $\epsilon \in T_n$ is an idempotent with $|\mathrm{im}(\epsilon)| = r$ then $H_\epsilon \cong S_r$.

- Understanding the maximal subgroups of a semigroup S is often an important first step towards understanding S.

Idempotent generated semigroups

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- Many natural examples
  - Howie (1966) - $T_n \setminus S_n$, the non-invertible transformations;
  - Erdös (1967) - singular part of $M_n(\mathbb{F})$, semigroup of all $n \times n$ matrices over a field $\mathbb{F}$;
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  - Putcha (2006) - conditions for a reductive linear algebraic monoid to have the same property.
- Idempotent generated semigroups are "general"
  - Every semigroup S embeds into an idempotent generated semigroup.

Free idempotent generated semigroups

A problem in algebra

S - semigroup, $E = E(S)$ - idempotents of S

E carries a certain abstract structure: that of a biordered set.

Idea: Fix a biorder E and investigate those semigroups whose idempotents carry this fixed biorder structure.

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E carries a certain abstract structure: that of a biordered set.

Idea: Fix a biorder E and investigate those semigroups whose idempotents carry this fixed biorder structure.

Within this family there is a unique free object $IG(E)$ which is the semigroup defined by presentation:

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e, f \in E, \ \{e, f\} \cap \{ef, fe\} \neq \emptyset) \ \rangle$$

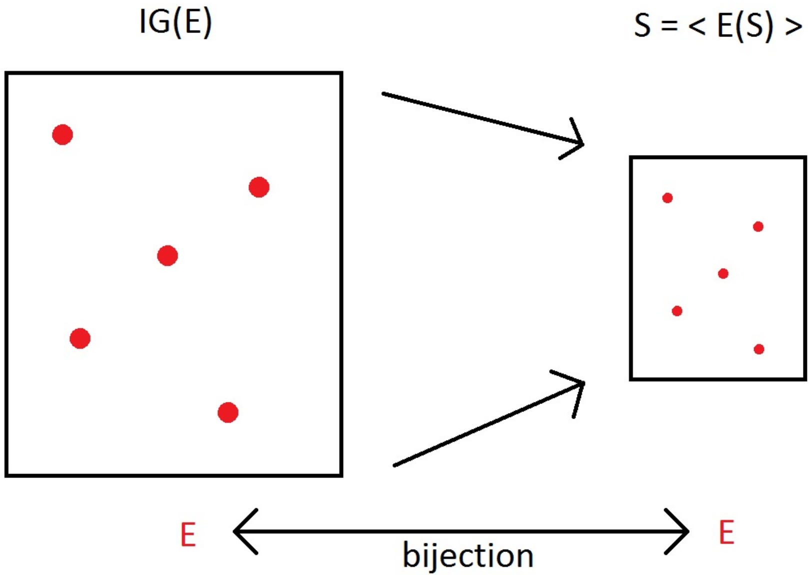
$IG(E)$ is called the free idempotent generated semigroup on E.

First steps towards understanding $IG(E)$

Theorem (Easdown (1985))

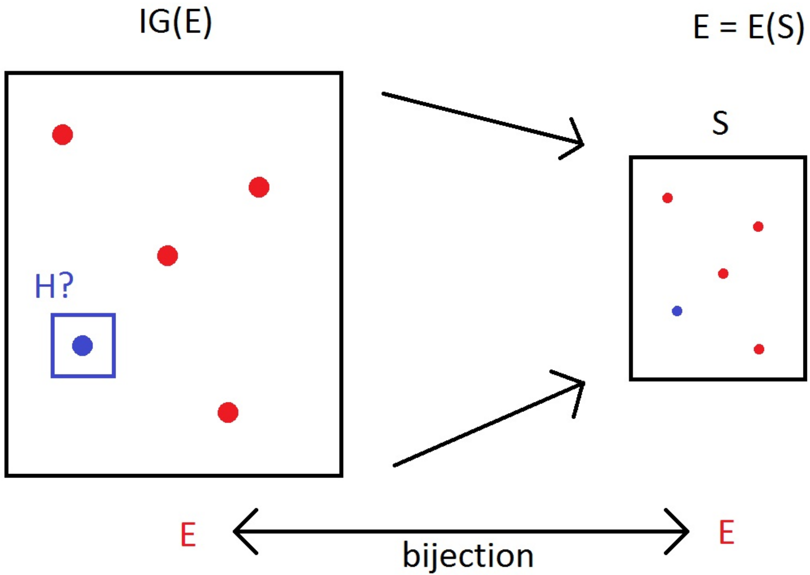
Let S be an idempotent generated semigroup with $E = E(S)$. Then $IG(E)$ is an idempotent generated semigroup and there is a surjective homomorphism $\phi : IG(E) \to S$ which is bijective on idempotents.

Conclusion. It is important to understand $IG(E)$ if one is interested in understanding an arbitrary idempotent generated semigroups.


IG(E)
S = < E(S) >
E
E
bijection

Maximal subgroups of $IG(E)$

Question. Which groups can arise as maximal subgroups of a free idempotent generated semigroups?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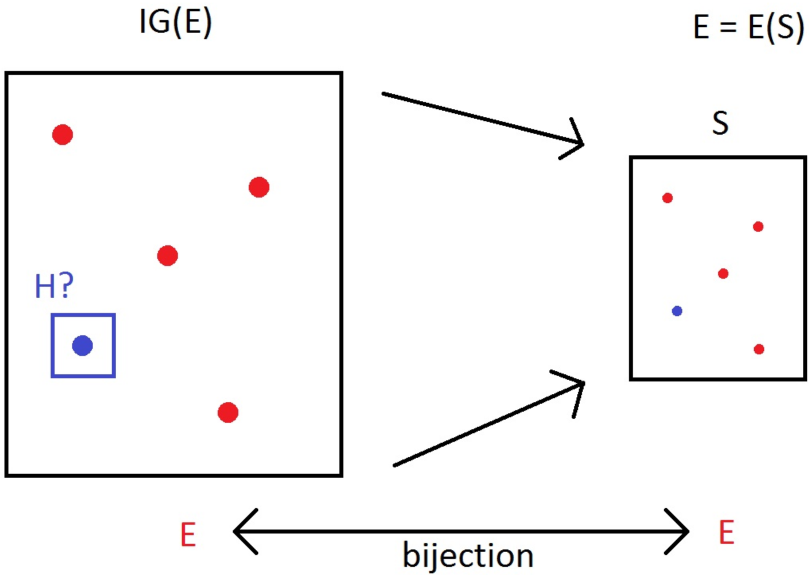
New focus

What can be said about maximal subgroups of $IG(E)$ where $E = E(S)$ for semigroups S that arise in nature?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The full linear monoid

$\mathbb{F}$ - arbitrary field, $\quad n \in \mathbb{N}$

$$M_n(\mathbb{F}) = \{n \times n \text{ matrices over } \mathbb{F}\}.$$

- Plays an analogous role in semigroup theory as the general linear group does in group theory.
- Important in a range of areas:
  - Representation theory of semigroups
  - Putcha–Renner theory of linear algebraic monoids and finite monoids of Lie type.

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Aim

Investigate the above problem in the case $S = M_n(\mathbb{F})$ and $E = E(S)$.

Properties of $M_n(\mathbb{F})$

Theorem (J.A. Erdös (1967))

$$\langle E(M_n(\mathbb{F}))\rangle = \{\textit{identity matrix and all non-invertible matrices}\}.$$

- $M_n(\mathbb{F})$ may be partitioned into the sets

$$D_r = \{A : \operatorname{rank}(A) = r\}, \quad r \leq n,$$

  (these are the $\mathcal{D}$-classes).
- The maximal subgroups in D_r are isomorphic to $GL_r(\mathbb{F})$.

The problem

Let $E = E(M_n(\mathbb{F}))$ the set of idempotent matrices.

By Easdown (1985) we may identify

$$E = E(M_n(\mathbb{F})) = E(IG(E)).$$

Fix an idempotent matrix W of rank r.

Problem: Identify the maximal subgroup H_W of

$$IG(E) = \langle E \mid e \cdot f = ef \ \ (e, f \in E,\ \{e, f\} \cap \{ef, fe\} \neq \emptyset) \ \rangle$$

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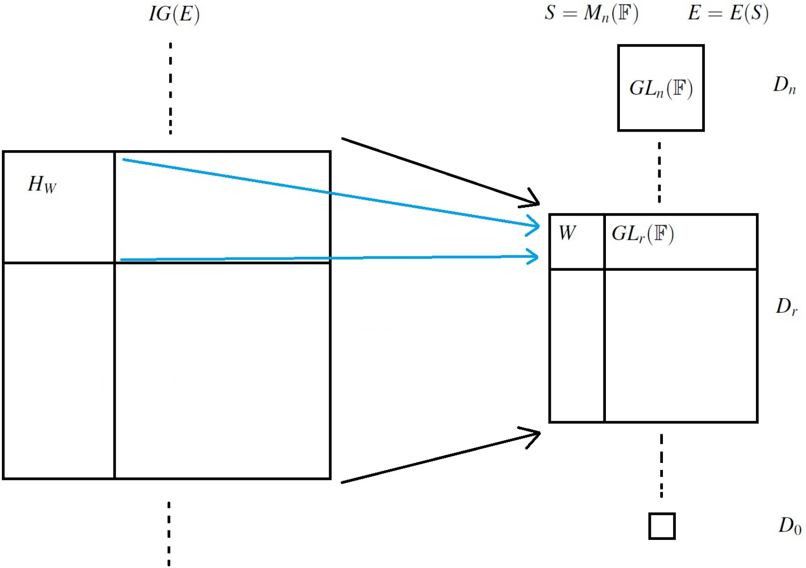
General fact: H_W is a homomorphic preimage of $GL_r(\mathbb{F})$.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D_n
H_W
W
$GL_r(\mathbb{F})$
D_r
D_0

Results

$n \in \mathbb{N}$, $\mathbb{F}$ - field, $E = E(M_n(\mathbb{F}))$,

$W \in M_n(\mathbb{F})$ - idempotent matrix of rank r

$H_W =$ maximal subgroup of

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Theorem (Brittenham, Margolis, Meakin (2009))

For $n \geq 3$ and $r = 1$ we have $H_W \cong GL_r(\mathbb{F}) \cong \mathbb{F}^*$.

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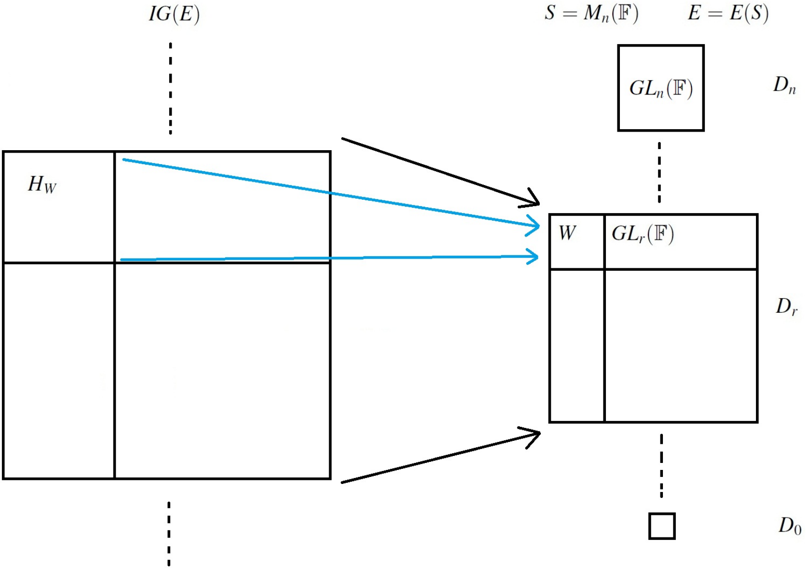
Theorem (Dolinka, Gray (2012))

Let n and r be positive integers with $r < n/3$. Then $H_W \cong GL_r(\mathbb{F})$.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D_r
D_0

Step 1: Writing down a presentation for H_W

Definition

A matrix is in reduced row echelon form (RRE form) if:

- rows with at least one nonzero element are above any rows of all zeros
- the leading coefficient (the first nonzero number from the left) of a nonzero row is always strictly to the right of the leading coefficient of the row above it, and
- every leading coefficient is 1 and is the only nonzero entry in its column.

Examples

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 1: Writing down a presentation for H_W

$n, r \in \mathbb{N}$ fixed with $r < n$

$$\begin{aligned} \mathcal{Y}_r &= \{r \times n \text{ rank } r \text{ matrices over } \mathbb{F} \text{ in RRE form}\} \\ \mathcal{X}_r &= \{\text{transposes of elements of } \mathcal{Y}_r\} \end{aligned}$$

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- Matrices in $\mathcal{Y}_r$ have no rows of zeros, so have r leading columns.

$$\text{e.g. } n = 4,\ r = 3,\ \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{Y}_3.$$

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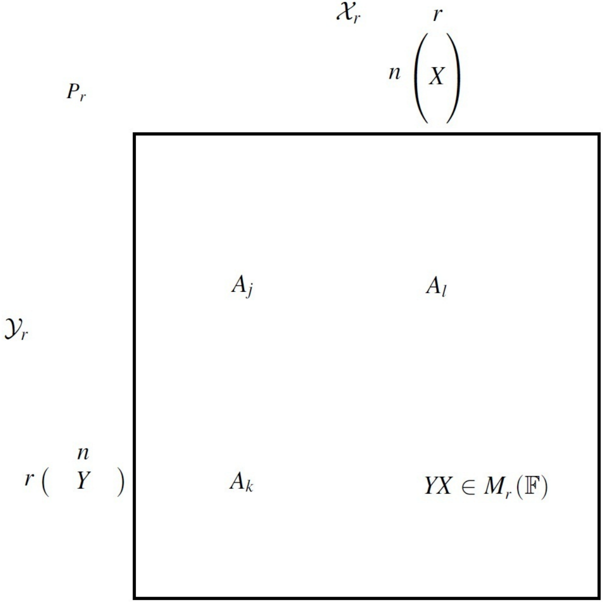
- Define a matrix $P_r = (P_r(Y, X))$ defined for $Y \in \mathcal{Y}_r$, $X \in \mathcal{X}_r$ by

$$P_r(Y, X) = YX \in M_r(\mathbb{F}).$$



$\mathcal{X}_r$
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n
X
P_r
A_j
A_l
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n
r
Y
A_k
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Graham–Houghton 2-complex $\mathcal{GH}$

1-skeleton: a connected bipartite graph $(n, r \in \mathbb{N},\ r < n)$

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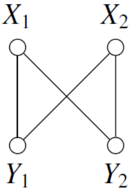
Vertices: $\mathcal{Y}_r \cup \mathcal{X}_r$ (disjoint union), Edges: $Y \sim X \Leftrightarrow YX \in GL_r(\mathbb{F})$

2-cells



$$\Leftrightarrow \quad (Y_1X_1)^{-1}(Y_1X_2) = (Y_2X_1)^{-1}(Y_2X_2).$$

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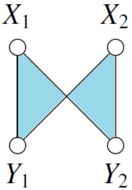

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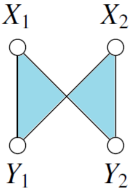

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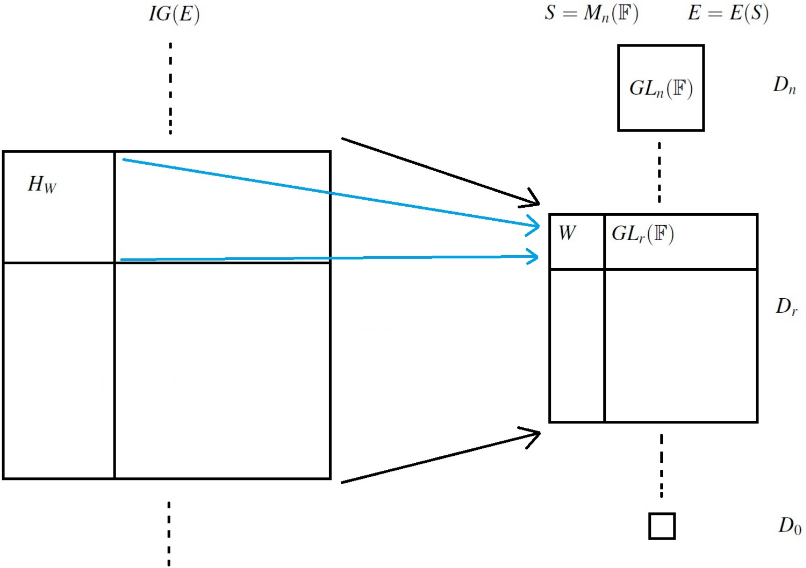
Theorem (Brittenham, Margolis, Meakin (2009))

The maximal subgroup H_W of $IG(E)$ is isomorphic to the fundamental group of the Graham–Houghton complex $\mathcal{GH}$.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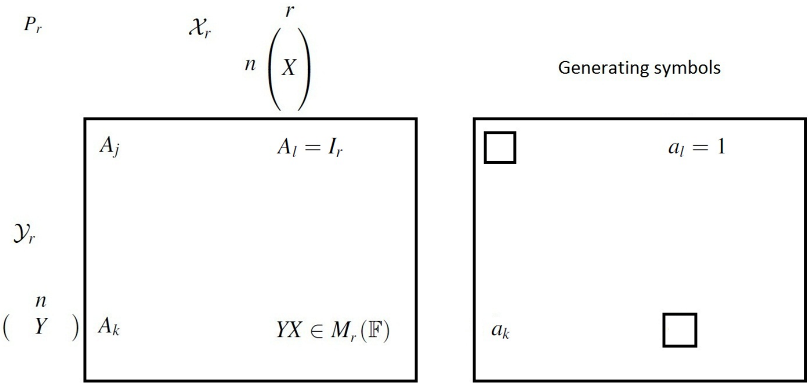

Generators: $\{a_j \ \mid A_j$ is an entry in P_r satisfying $A_j \in GL_r(\mathbb{F})\ \}$

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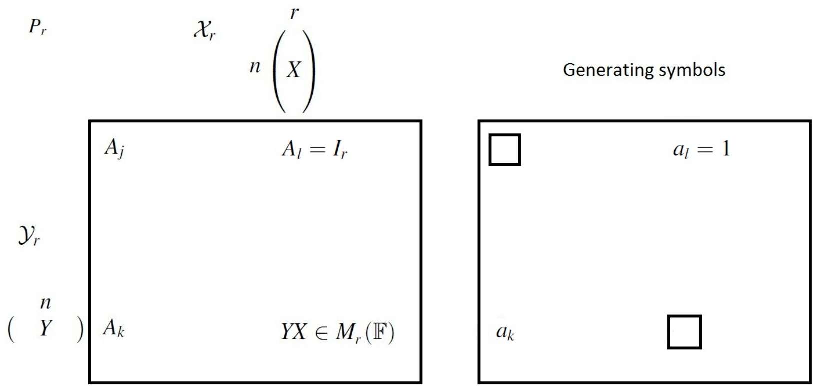

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Relations:

(I) $a_j = 1$ for all entries A_j in P_r satisfying $A_j = I_r$

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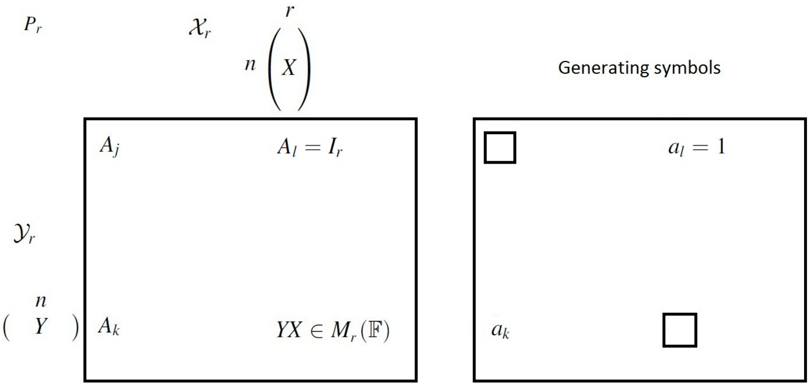

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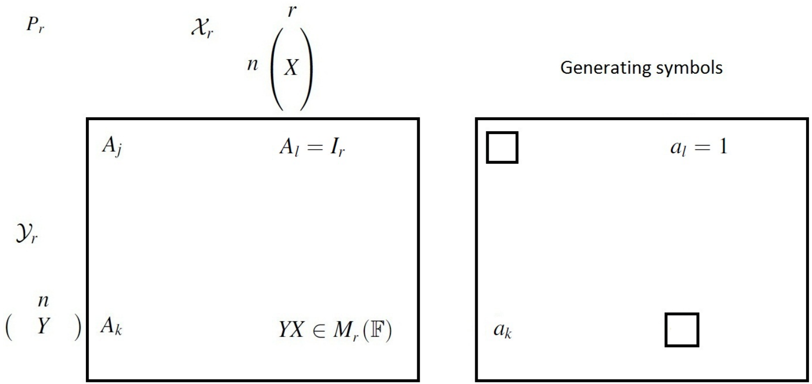
(II) $a_j a_k^{-1} = a_l a_m^{-1} \Leftrightarrow (A_j, A_k, A_l, A_m)$ is a singular square of invertible $r \times r$ matrices from P_r with $A_j^{-1} A_k = A_l^{-1} A_m$.

$$\begin{array}{cc} A_j & A_k \\ A_l & A_m \end{array}$$

Structure of the proof that $H_W \cong GL_r(\mathbb{F})$



Step 1: Write down a presentation for H_W.

Step 2: Prove that for any two entries A_j, A_k in the table P_r, if $A_j = A_k \in GL_r(\mathbb{F})$ then $a_j = a_k$ is deducible from the relations.

Step 3: Find defining relations for $GL_r(\mathbb{F})$ using the singular square relations (II).

Step 2: Strong edges and relations

Definition

We say entries A_j and A_k with $A_j = A_k$ are connected by a strong edge if

$$\begin{tikzcd} A_j \arrow[r, dash, thick] & A_k \\ I_r & I_r \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} A_j \arrow[d, dash, thick] & I_r \\ A_k & I_r \end{tikzcd}$$

Lemma: If $A_j = A_k \in GL_r(\mathbb{F})$ are connected by a strong edge then $a_j = a_k$ is a consequence of the relations.

$$\begin{tikzcd} A_j \arrow[r, dash, thick] & A_k \\ I_r & I_r \end{tikzcd} \quad \Rightarrow \quad \begin{array}{cc} a_j & a_k \\ & \\ 1 & 1 \end{array} \quad \Rightarrow \quad a_j = a_k \text{ can be deduced}$$

A singular square   Using relations (I)

Step 2: Proving $A_i = A_j$ invertible $\Rightarrow a_i = a_j$

Definition

Strong path = path composed of strong edges.

Aim

Prove that for every pair A_j, A_k of entries in P_r, if $A_j = A_k$ then there is a strong path from A_j to A_k.

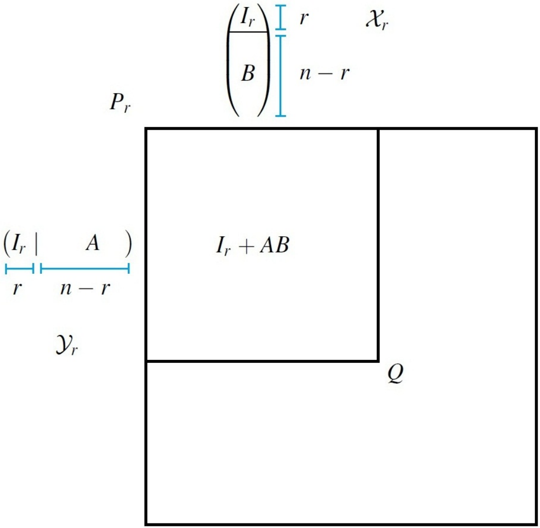
Once proved this will have the following:

Corollary

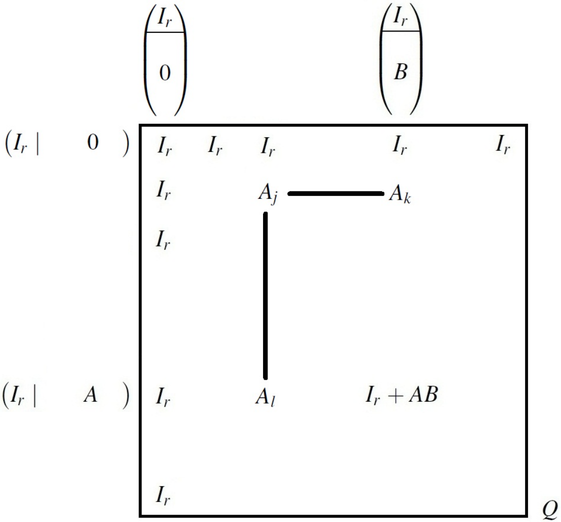
For every pair $A_j = A_k \in GL_r(\mathbb{F})$ in the table P_r the relation $a_j = a_k$ is a consequence of the defining relations in the presentation.

The small box Q

Is the subtable of P_r containing entries whose row and column are labelled by matrices of the form $\left(I_r \mid \quad A \quad\right)$ and their transposes, where A is an $r \times (n-r)$ matrix over $\mathbb{F}$.



Strongly connecting the small box Q

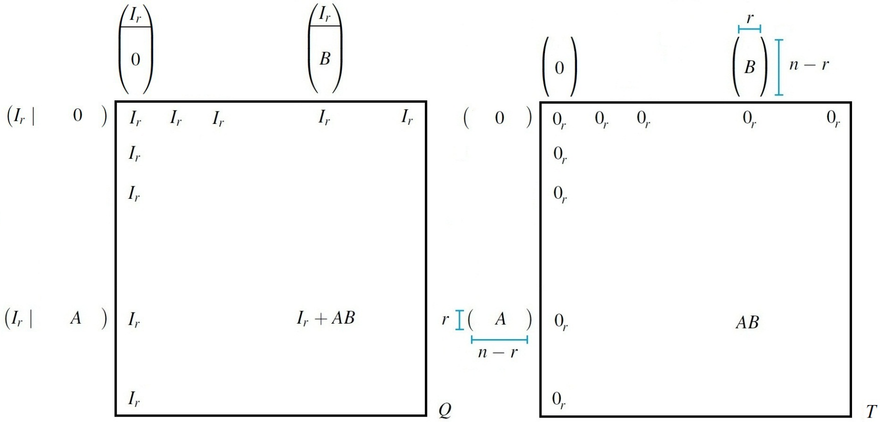


Observation: In the small box every edge is a strong edge.

$\therefore$ strongly connecting the small box $\equiv$ connecting the small box.

An equivalent problem

T = matrix obtained by taking Q and subtracting I_r from every entry



$$
\begin{array}{ll}
 & \text{For every symbol } X \text{ in the table } Q \text{ the graph } \Delta(X) \text{ in } Q \text{ is connected.} \\
\Leftrightarrow & \text{For every symbol } X \text{ in the table } T \text{ the graph } \Delta(X) \text{ in } T \text{ is connected.}
\end{array}
$$

Connecting the small box

So, we have reduced the problem of strongly connecting the small box in P_r to the following:

Let $m, k \in \mathbb{N}$ with $k < m$, and let

$$\begin{aligned} \mathcal{B} &= \{\text{all } k \times m \text{ matrices over } \mathbb{F}\}, \\ \mathcal{A} &= \{\text{all } m \times k \text{ matrices over } \mathbb{F}\}. \end{aligned}$$

Define the matrix $T = T(B, A)$ by

$$T(B, A) = BA \in M_k(\mathbb{F}), \ B \in \mathcal{B}, \ A \in \mathcal{A}.$$

Question: Is it true that for every symbol $X \in M_k(\mathbb{F})$ in the table T the graph $\Delta(X)$ is connected?

Déjà vu

$\mathbb{F} = \{0, 1\}$, vectors in $\mathbb{F}^3$, entries in table from $\mathbb{F}$

	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
$(0, 0, 0)$	0	0	0	0	0	0	0	0
$(0, 0, 1)$	0	1	0	1	0	1	0	1
$(0, 1, 0)$	0	0	1	1	0	0	1	1
$(0, 1, 1)$	0	1	1	0	0	1	1	0
$(1, 0, 0)$	0	0	0	0	1	1	1	1
$(1, 0, 1)$	0	1	0	1	1	0	1	0
$(1, 1, 0)$	0	0	1	1	1	1	0	0
$(1, 1, 1)$	0	1	1	0	1	0	0	1

- For every symbol x in the table $\Delta(x)$ is connected.

Combinatorial properties of tables

And it generalises...

Proposition

Let $m, k \in \mathbb{N}$ with $k < m$, and let

$$\begin{aligned} \mathcal{B} &= \{\text{all } k \times m \text{ matrices over } \mathbb{F}\}, \\ \mathcal{A} &= \{\text{all } m \times k \text{ matrices over } \mathbb{F}\}. \end{aligned}$$

Define the matrix $T = T(B, A)$ by

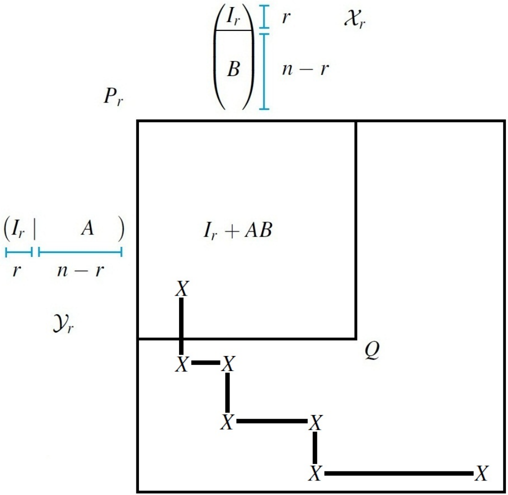
$$T(B, A) = BA \in M_k(\mathbb{F}),\ B \in \mathcal{B},\ A \in \mathcal{A}.$$

Then for every symbol $X \in M_k(\mathbb{F})$ in the table T the graph $\Delta(X)$ is connected.

Corollary

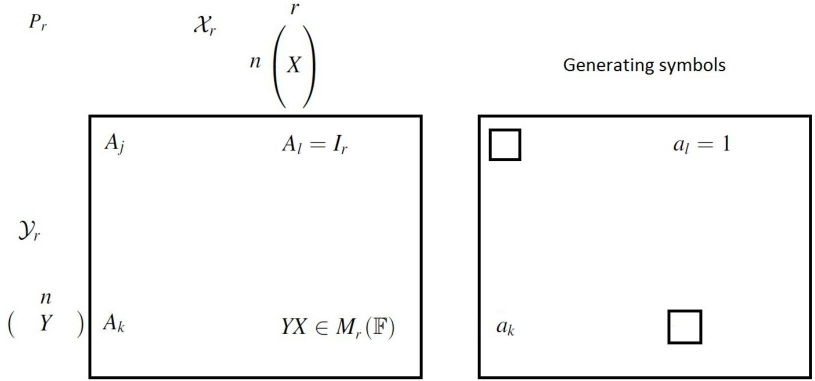
For every pair A_j, A_k in the small box, if $A_j = A_k$ then there is a strong path in the small box from A_j to A_k.

Finishing off Step 2



Proposition: For every pair A_j, A_k of entries in P_r, if $A_j = A_k$ then there is a strong path between A_j and A_k. Thus, for every pair $A_j = A_k \in GL_r(\mathbb{F})$ in the table P_r the relation $a_j = a_k$ is deducible.

Structure of the proof that $H_W \cong GL_r(\mathbb{F})$



Step 1: Write down a presentation for H_W.

Step 2: Prove that for any two entries A_j, A_k in the table P_r, if $A_j = A_k \in GL_r(\mathbb{F})$ then $a_j = a_k$ is deducible from the relations.

Step 3: Find defining relations for $GL_r(\mathbb{F})$ among the singular square relations (II).

Finishing off the proof

For any pair of matrices $A, B \in GL_r(\mathbb{F})$ we can find the following singular square in P_r:

$$
\begin{array}{c||cc}
 & \left[\begin{array}{c} 0_{r\times r} \\ \hline 0_{r\times r} \\ \hline I_r \\ \hline 0_{(n-3r)\times r} \end{array}\right] & \left[\begin{array}{c} I_r \\ \hline 0_{r\times r} \\ \hline B \\ \hline 0_{(n-3r)\times r} \end{array}\right] \\
\hline\hline
\left[\begin{array}{c|c|c|c} 0_{r\times r} & I_r & A & 0_{r\times(n-3r)} \end{array}\right] & A & AB \\
\left[\begin{array}{c|c|c|c} 0_{r\times r} & 0_{r\times r} & I_r & 0_{(n-3r)\times r} \end{array}\right] & I_r & B
\end{array}
$$

- Every relation in the presentation holds in $GL_r(\mathbb{F})$.
- Conversely, every relation that holds in $GL_r(\mathbb{F})$ can be deduced from the multiplication table relations that arise from the squares above.
- It follows that $H_W \cong GL_r(\mathbb{F})$ (when $r < n/3$).

$\square$

Open problems

- What happens in higher ranks?

Conjecture (Brittenham, Margolis, Meakin (2009))

Let n and r be positive integers with $r \leq n/2$. Then $H_W \cong GL_r(\mathbb{F})$.

- The same result might even be true for $r < n - 1$.
- The analogous result does hold for T_n, with $r < n - 1$, with the symmetric groups S_r arising as maximal subgroups of $IG(E)$ (Gray & Ruskuc (2012)).

Further reading

I. Dolinka and R. Gray,
Maximal subgroups of free idempotent generated semigroups over the full linear monoid.
Trans. Amer. Math. Soc. 366(1) (2014), 419–455.

Our paper builds on ideas developed in the following papers:

M. Brittenham, S. W. Margolis, and J. Meakin,
Subgroups of the free idempotent generated semigroups need not be free.
J. Algebra 321 (2009), 3026–3042.

M. Brittenham, S. W. Margolis, and J. Meakin,
Subgroups of free idempotent generated semigroups: full linear monoids.
arXiv: 1009.5683.

R. Gray and N. Ruškuc,
On maximal subgroups of free idempotent generated semigroups.
Israel J. Math. 189 (2012), 147–176.

R. Gray and N. Ruškuc,
Maximal subgroups of free idempotent generated semigroups over the full transformation monoid.
Proc. London Math. Soc. 104 (2012) 997–1018.