Idempotent generating sets for semigroups and combinatorics of bipartite graphs

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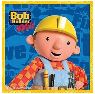
My family

	Leonhard Euler (the “father” of graph theory)	Great great great great great great great great great grandfather
	⋮	⋮
	John Howie	Grandfather
	|	|
	Nik	Father
	|	
	Bob	

A theorem of Howie

T_n - the full transformation monoid

$T_n = S_n \cup \mathrm{Sing}_n$, where $\mathrm{Sing}_n = \{\text{non-bijections}\}$

Theorem (Howie (1966))

The subsemigroup of T_n generated by its idempotents is:

$$\langle E(T_n) \rangle = \mathrm{Sing}_n \cup \{\mathrm{id}\}.$$

Corollary

Every proper two-sided ideal

$$K(n,r) = \{\alpha \in T_n : |\mathrm{im}\,\alpha| \leq r\}$$

of T_n is idempotent generated.

Howie asked:

- How many idempotents are needed to generate $K(n,r)$?

A theorem of Howie and McFadden

S - finite idempotent generated semigroup

Definition

$\mathrm{idrank}(S) =$ smallest size of an idempotent generating set for S.

Theorem (Howie and McFadden (1990))

For $1 < r < n$ we have

$$\mathrm{idrank}(K(n,r)) = S(n,r),$$

where $S(n,r)$ is the Stirling number of the second kind (i.e. the number of partitions of $\{1,\ldots,n\}$ into r non-empty subsets).

Reduction to principal factors

Let $D_r = \{\alpha \in T_n : |\mathrm{im}\,\alpha| = r\}$, so

$$K(n,r) = K(n,r-1) \cup D_r$$

where $K(n,r-1)$ is an ideal of $K(n,r)$.

Lemma

$K(n,r)$ is generated by the idempotents in its top $\mathcal{D}$-class D_r.

Principal factor: $D_r^* = D_r \cup \{0\}$ with multiplication:

$$\alpha\beta = \begin{cases} \alpha\beta & \text{if } \alpha, \beta, \alpha\beta \in D_r \\ 0 & \text{otherwise.} \end{cases}$$

Conclusion: D_r^* is a idempotent generated finite completely 0-simple semigroup satisfying:

$$\mathrm{idrank}(K(n,r)) = \mathrm{idrank}(D_r^*).$$

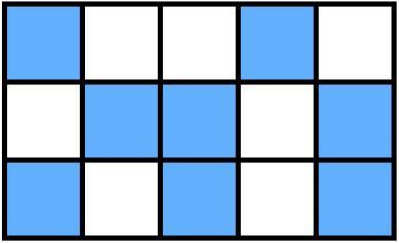
Regular $\mathcal{D}$-classes

For $u, v \in S$ we define

$$uRv \;\Leftrightarrow\; uS \cup \{u\} = vS \cup \{v\}, \quad u\mathcal{L}v \;\Leftrightarrow\; Su \cup \{u\} = Sv \cup \{v\},$$

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L}. \quad \mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

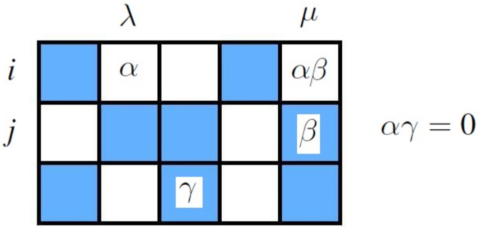
- A $\mathcal{D}$-class is regular if it contains an idempotent
- A regular $\mathcal{D}$-class has ≥ 1 idempotent in every $\mathcal{R}$- and every $\mathcal{L}$-class.



Structure of finite completely 0-simple semigroups

$S = D \cup \{0\}$ - where D is a regular $\mathcal{D}$-class

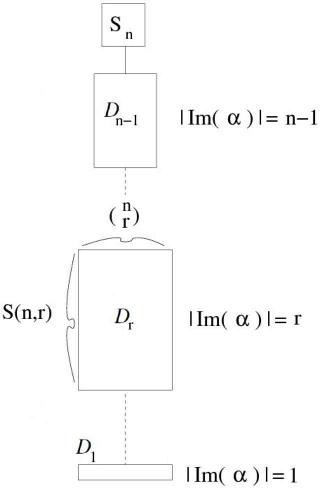
$\mathcal{R}$-classes = $\{R_i : i \in I\}$, $\mathcal{L}$-classes = $\{L_\lambda : \lambda \in \Lambda\}$.
$\mathcal{H}$-classes $= \{H_{i\lambda} : i \in I, \lambda \in \Lambda\}$



Miller & Clifford Theorem. If $\alpha \in H_{i\lambda}$ and $\beta \in H_{j\mu}$ then

$$\alpha\beta \in H_{i\mu} \quad \text{if } H_{j\lambda} \text{ contains an idempotent}$$
$$\alpha\beta = 0 \quad \text{if } H_{j\lambda} \text{ does not contain an idempotent.}$$

Conclusion: Every generating set for S has at least $\max(|I|, |\Lambda|)$ elements.



T_n - full transformation semigroup, $\alpha, \beta \in T_n$

$$
\begin{aligned}
\alpha \mathcal{L} \beta &\Leftrightarrow \operatorname{Im}(\alpha) = \operatorname{Im}(\beta) \\
\alpha \mathcal{R} \beta &\Leftrightarrow \ker(\alpha) = \ker(\beta) \\
\alpha \mathcal{D} \beta &\Leftrightarrow |\operatorname{Im}(\alpha)| = |\operatorname{Im}(\beta)|
\end{aligned}
$$

$$D_r = \{\alpha \in T_n : |\operatorname{Im}(\alpha)| = r\}$$

$$
\begin{aligned}
K(n,r) &= \{\alpha \in T_n : |\operatorname{Im}(\alpha)| \leq r\} \\
&= D_1 \cup \cdots \cup D_r.
\end{aligned}
$$

$$K(n,r) = \langle D_r \rangle \ , \ 1 \leq r < n.$$

A theorem of Howie and McFadden

Theorem (Howie and McFadden (1990))

For $1 < r < n$ *we have*

$$\operatorname{idrank}(K(n,r)) = S(n,r),$$

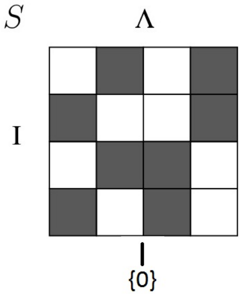
where $S(n,r)$ *is the Stirling number of the second kind (i.e. the number of partitions of* $\{1,\ldots,n\}$ *into* r *non-empty subsets).*

Square completely 0-simple semigroups

$S = D \cup \{0\}$ - an idempotent generated finite completely 0-simple semigroup with:

$\mathcal{R}$-classes - $\{R_i : i \in I\}$, $\mathcal{L}$-classes - $\{L_\lambda : \lambda \in \Lambda\}$.

Suppose D is square, i.e. $|I| = |\Lambda|$.



Clearly every generating set for S must intersect every R_i and every L_λ.

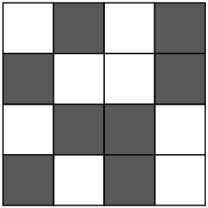
Question: Is there a generating set of idempotents with size $|I| = |\Lambda|$?

Graham–Houghton Graphs

$S = D \cup \{0\}$ - an idempotent generated finite completely 0-simple semigroup with:

$\mathcal{R}$-classes - $\{R_i : i \in I\}$, $\mathcal{L}$-classes - $\{L_\lambda : \lambda \in \Lambda\}$.

Definition. Define a bipartite graph $\Delta(S)$ with
Vertices: $I \cup \Lambda$
Edges: $(i, \lambda) \Leftrightarrow H_{i\lambda} = R_i \cap L_\lambda$ contains an idempotent.



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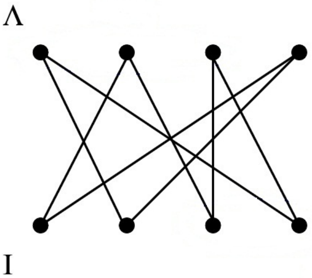

$\Delta(S)$

Graham–Houghton Graphs

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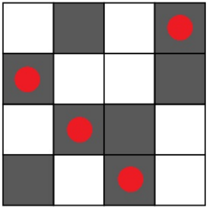

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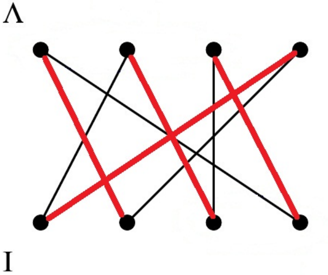


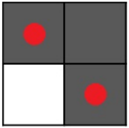
$\Delta(S)$

Close, but no cigar

A necessary condition

If $\mathrm{idrank}(S) = |I| = |\Lambda|$ then $\Delta(S)$ has a perfect matching.

But it is not sufficient



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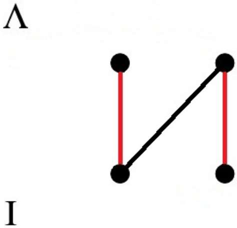


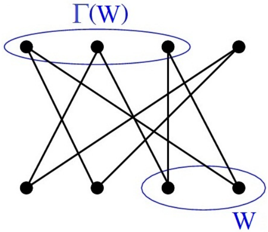
$\Delta(S)$

Hall's marriage theorem

Γ - a graph, $W \subseteq V\Gamma$ a set of vertices

Definition (Neighbourhood)

$$\Gamma(W) = \{v \in V\Gamma : \exists w \in W : v \sim w\}.$$



Theorem (Philip Hall (1935))

A bipartite graph $\Gamma = X \cup Y$ with $|X| = |Y|$ has a perfect matching if and only if the following condition is satisfied:

$$|\Gamma(A)| \geq |A| \text{ for all } A \subseteq X. \tag{HC}$$

Strengthening Hall's condition

Definition

A bipartite graph $\Gamma = X \cup Y$ with $|X| = |Y|$ is said to satisfy the strong Hall condition if it satisfies

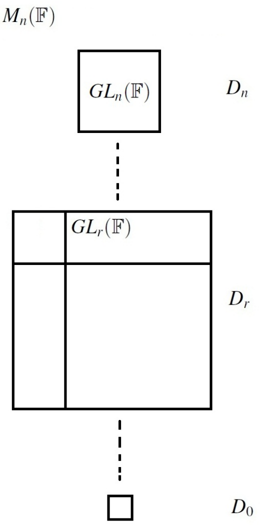
$$|\Gamma(A)| > |A| \text{ for all } A \subsetneq X. \tag{SHC}$$

Theorem (RG (2008))

Let $S = D \cup \{0\}$ be a finite square idempotent generated completely 0-simple with $\mathcal{R}$-classes indexed by I and $\mathcal{L}$-classes by Λ. Then the following are equivalent:

1. $\text{idrank}(S) = |I| = |\Lambda|$;
2. the bipartite graph $\Delta(S)$ satisfies (SHC);
3. $A \subseteq S$ with $|A| = |I| = |\Lambda|$ is a generating set for S if and only if A intersects every non-zero $\mathcal{R}$-class and $\mathcal{L}$-class exactly once.

Application: Full linear monoid



$M_n(\mathbb{F})$ - full linear monoid
$X, Y \in M_n(\mathbb{F})$

Theorem (J.A. Erdös (1967))

$\langle E(M_n(\mathbb{F}))\rangle = \{$*identity matrix and all non-invertible matrices*$\}$.

$$X\mathcal{R}Y \Leftrightarrow \operatorname{Col} X = \operatorname{Col} Y,$$
$$X\mathcal{L}Y \Leftrightarrow \operatorname{Row} X = \operatorname{Row} Y,$$
$$X\mathcal{D}Y \Leftrightarrow \operatorname{rank}(X) = \operatorname{rank}(Y).$$

$$D_r = \{A : \operatorname{rank}(A) = r\}, \quad r \leq n.$$

$$L(n,r) = D_1 \cup \cdots \cup D_r.$$

$$L(n,r) = \langle D_r \rangle$$

$$\operatorname{idrank}(L(n,r)) = \operatorname{idrank}(D_r^*)$$

A $\mathcal{D}$-class picture in $M_4(\mathbb{F}_2)$

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Symmetry implies SHC

Definition

A graph Γ is called regular if all of its vertices have the same degree.

Fact. The Graham-Houghton graphs of the principal factors of the full linear monoid are all connected regular bipartite graphs.

Lemma

Let $\Gamma = X \cup Y$ be a connected bipartite graph with $|X| = |Y|$. If Γ is regular then Γ satisfies (SHC).

Theorem (RG (2008))

Let V be an n-dimensional vector space over the finite field F where $|F| = q$. Then:

$$\mathrm{idrank}(L(n,r)) = \begin{bmatrix} n \\ r \end{bmatrix}_q .$$

Moreover, a subset of $L(n,r)$ is a generating set of minimum cardinality for $L(n,r)$ if and only if it consists of $\begin{bmatrix} n \\ r \end{bmatrix}_q$ matrices of rank r no two of which have the same row space or the same column space.