Combinatorial structures with lots of symmetry

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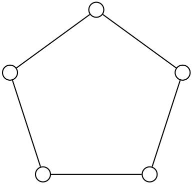
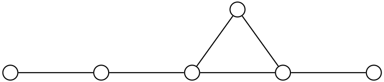
Graphs and symmetry

Definition

A graph Γ is a structure $(V\Gamma, \sim)$ where $V\Gamma$ is a set, and $\sim$ is a symmetric irreflexive binary relation on $V\Gamma$.



Γ_1



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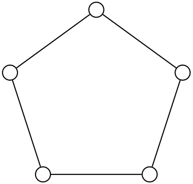

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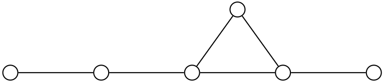

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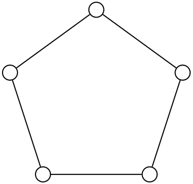

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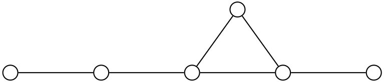

Γ_2

- Γ_1 has "more symmetry" than Γ_2.
- Imagine you are trapped inside the graph:
  - In Γ_1 the world looks the same from every vertex.
  - In Γ_2 the world looks different from each vertex.

Automorphisms

- An isomorphism $\phi : \Gamma_1 \to \Gamma_2$ of graphs is a bijection such that:

$$v \sim w \Leftrightarrow v\phi \sim w\phi \quad (\forall v, w \in \Gamma_1).$$

- An automorphism of a graph Γ is an isomorphism $\phi : \Gamma \to \Gamma$.
- $\mathrm{Aut}(\Gamma)$ - the full automorphism group of the graph Γ.

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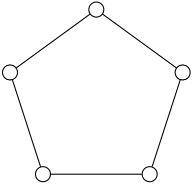
Roughly speaking... the more symmetry a graph has the larger its automorphism group will be, and vice versa.

Vertex transitivity



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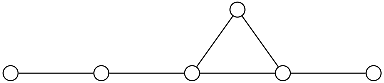

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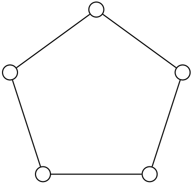
Vertex transitivity



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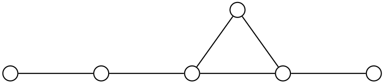

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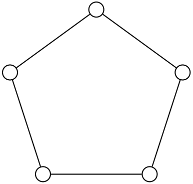

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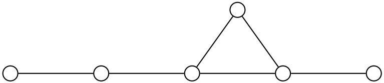

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- Γ_1 is vertex transitive
  - $\mathrm{Aut}(\Gamma)$ = rotations + reflections (Dihedral group)
- Γ_2 is not vertex transitive
  - Even worse, $|\mathrm{Aut}(\Gamma_2)| = 1$

Vertex transitive graphs

Question. Vertex transitive graphs are "nice". Is there any chance we could classify them (i.e. explicitly describe them all)?

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Answer. No.

There are simply "too many" vertex transitive graphs for us to stand a chance of describing them all.

For example:

- G - group, A - generating set for G with $A = A^{-1}$
- the Cayley graph $\Gamma(G,A)$:
  - $V\Gamma(G,A) = G$
  - $x \sim y \Leftrightarrow xy^{-1} \in A$,

  is vertex transitive.

(although not every vertex transitive graph is the Cayley graph of a group)

Symmetry properties for graphs

Examples

- Various symmetry properties have received attention:
  - vertex transitive
  - arc-transitive, k-arc-transitive (Tutte (1947))
  - distance-transitive (Biggs and Smith (1971))
  - homogeneous, k-homogeneous (Fraïssé (1953))

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  - homogeneous, k-homogeneous (Fraïssé (1953))

General problem

For a given symmetry property $\mathcal{P}$, classify those graphs Γ satisfying property $\mathcal{P}$.

Homogeneous graphs

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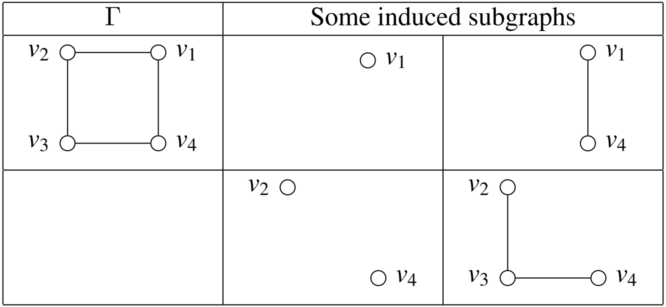
Induced subgraphs.



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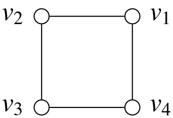
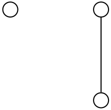
Induced subgraphs.

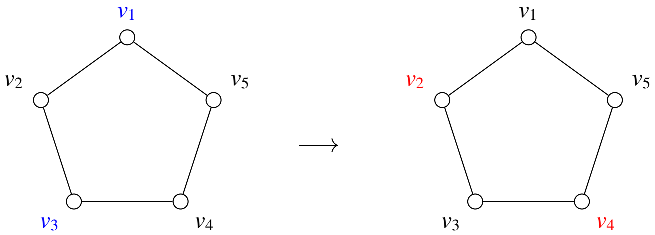
However, the graph Γ:



has no induced subgraph isomorphic to:



Extending isomorphisms to automorphisms



The isomorphism

$$(v_1, v_3) \mapsto (v_4, v_2)$$

between finite induced subgraphs

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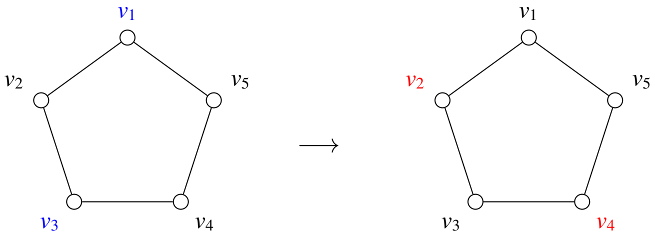

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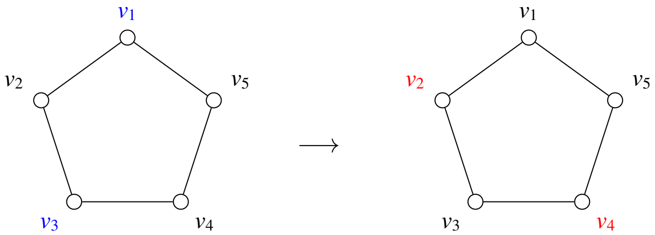

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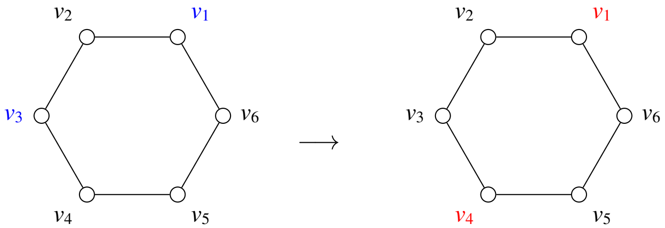
Fact. The pentagon is a homogeneous graph.

Extending isomorphisms to automorphisms



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e.g. There is a path of length 2 from v_1 to v_3, while there is no path of length 2 from v_1 to v_4.

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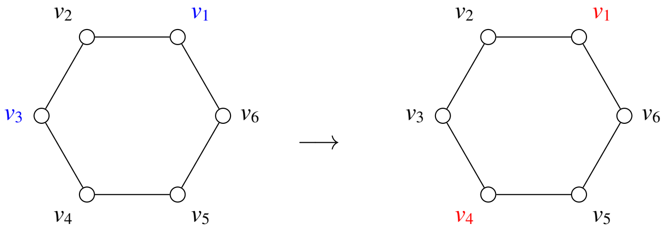

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So the hexagon is not a homogeneous graph.

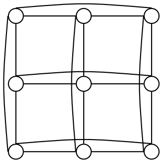
Classification of finite homogeneous graphs

Gardiner classified the finite homogeneous graphs.

Theorem (Gardiner (1976))

A finite graph is homogeneous if and only if it is isomorphic to one of the following:

1. *finitely many disjoint copies of a complete graph K_r (or its complement, complete multipartite graph)*
2. *the pentagon C_5*
3. *the graph $K_3 \times K_3$ drawn below*



The random graph

Definition (The random graph R)

Constructed by Rado (1964). The vertex set is the natural numbers (including zero).

For $i, j \in \mathbb{N}$, $i < j$, then i and j are joined if and only if the ith digit in j in base 2, reading right-to-left, is 1.

Example

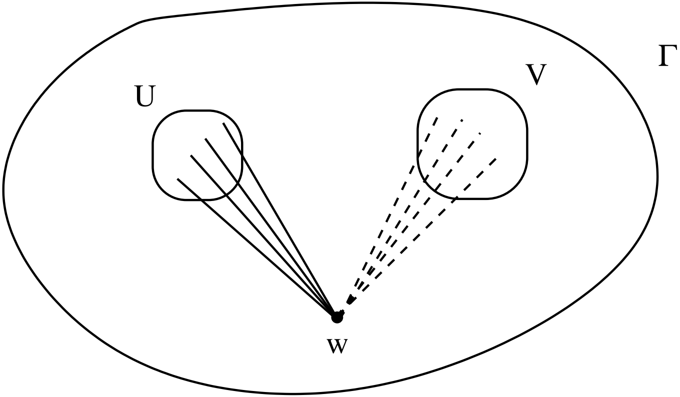
Since $88 = 8 + 16 + 64 = 2^3 + 2^4 + 2^6$ the numbers less that 88 that are adjacent to 88 are just $\{3, 4, 6\}$.

Of course, many numbers greater than 88 will also be adjacent to 88 (for example 2^{88}).

The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets U and V of vertices, there exists a vertex w adjacent to every vertex in U and to no vertex in V.



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Uniqueness and homogeneity. Both follow from a back-and-forth argument. Property (*) is used to extend the domain (or range) of any isomorphism between finite substructures one vertex at a time.

Properties of the random graph

- Universal: it embeds every countable graph as an induced subgraph.
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  - Aside from the complete graph, and empty graph, R is the only countable graph with this property.

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- $\mathrm{Aut}(R)$ is an infinite simple group (Truss, 1985).

Homogeneous relational structures

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Relational structures

- a relational structure consists of a set A, and some relations $R_1, \ldots, R_m$ (can be unary, binary, ternary, ...)
- an (induced) substructure is obtained by taking a subset $B \subseteq A$ and keeping only those relations where all entries in the tuple belong to B
- an isomorphism is a "structure preserving" mapping (i.e. a bijection ϕ such that ϕ and ϕ^{-1} are both homomorphisms)

Examples of homogeneous structures

The countable random graph R

X - a pure set

- automorphism group is the full symmetric group where any partial permutation can be extended to a (full) permutation

$(\mathbb{Q}, \leq)$ - the rationals with their usual ordering

- the automorphisms are the order-preserving permutations
- isomorphisms between finite substructures can be extended to automorphisms that are piecewise-linear

Connection with model theory

Common theme in model theory:

translation between "model theoretic terminology" and "permutation group theoretic terminology"

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(I) A structure M is $\aleph_0$-categorical if all countable models of the first-order theory of M are isomorphic to M.

(II) A permutation group on an infinite set Ω is called oligomorphic, if it has finitely many orbits of n-tuples, for all $n \geq 1$.

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Homogeneous structures provide a rich source of $\aleph_0$-categorical structures.

Homogeneous structures and Fraïssé's theorem

The notion of homogeneous structure goes back to the fundamental work of Fraïssé (1953).

- The age of a relational structure M is the class of isomorphism types of its finite substructures.

Fraïssé proved a theorem which gives a necessary and sufficient condition on a class $\mathcal{C}$ of finite structures for it to be the age of a countable homogeneous structure M.

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- The key condition is the amalgamation property:
  - two structures in $\mathcal{C}$ with isomorphic substructures can be "glued together" so that the substructures are identified, inside a larger structure in $\mathcal{C}$.

If Fraïssé's conditions hold, then M is unique, $\mathcal{C}$ is called a Fraïssé class, and M is called the Fraïssé limit of the class $\mathcal{C}$.

Countable homogeneous graphs

Examples

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Theorem (Lachlan and Woodrow (1980))

Let Γ be a countably infinite homogeneous graph. Then Γ is isomorphic to one of: the random graph, a disjoint union of complete graphs (or its complement), the generic K_n-free graph (or its complement).

Classification results

For various other families, those members that are homogeneous have been completely determined.

Some classification results

	Finite	Countably infinite
Posets	(trivial)	Schmerl (1979)
Tournaments	Woodrow (1976)	Lachlan (1984)
Graphs	Gardiner (1976)	Lachlan & Woodrow (1980)
Digraphs	Lachlan (1982)	Cherlin (1998)

Weakening homogeneity

Symmetry properties

Strong		Weak
homogeneous	$\rightsquigarrow$	vertex transitive
(classification possible)		(classification impossible)

Idea. Consider natural weakenings of homogeneity and the resulting classification problems.

Question. How much can homogeneity be weakened before the corresponding classification problem becomes impossible?

Connected-homogeneous graphs

Definition

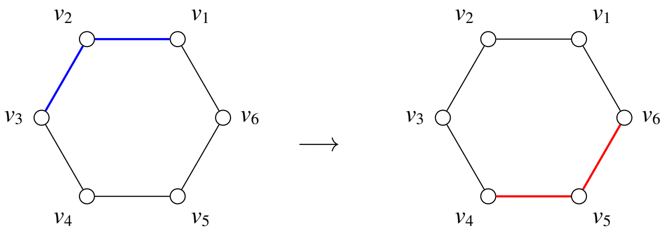
A graph Γ is connected-homogeneous if any isomorphism between *connected* finite induced subgraphs extends to an automorphism.

Connected-homogeneity...

1. is a natural weakening of homogeneity;
2. gives a class of graphs that lie between the (already classified) homogeneous graphs and the (not yet classified) distance-transitive graphs.

$$\text{homogeneous} \Rightarrow \text{connected-homogeneous} \Rightarrow \text{distance-transitive}$$

A finite connected-homogeneous graph



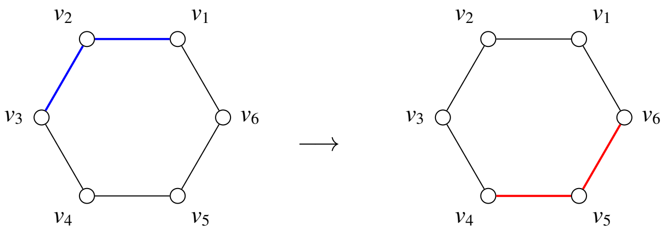
Isomorphisms between finite induced *connected* subgraphs, e.g.

$$(v_1, v_2, v_3) \mapsto (v_6, v_5, v_4),$$

clearly all extend to automorphisms.

So the hexagon is connected-homogeneous.

A finite connected-homogeneous graph



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So the hexagon is connected-homogeneous.

But we have already seen that it is not homogeneous.

Finite connected-homogeneous graphs

Theorem (Weiss (1976), Gardiner (1978))

A finite graph is connected-homogeneous if and only if it is isomorphic to a disjoint union of copies of one of the following:

1. *a finite homogeneous graph*
2. *bipartite "complement of a perfect matching"*
   (obtained by removing a perfect matching from a complete bipartite graph $K_{s,s}$)
3. *cycle C_n*
4. *the graph $K_s \times K_s$*
5. *Petersen's graph*
6. *the graph obtained by identifying antipodal vertices of the 5-dimensional cube Q_5*

Infinite connected-homogeneous graphs

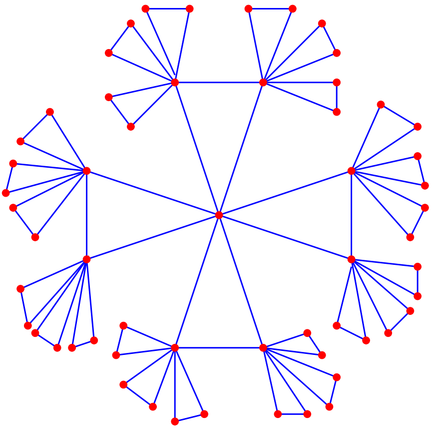
The graph $X_{r,l}$

Take l copies of the complete graph K_r and amalgamate them at a single vertex.

Repeat the process, building a tree-like graph.

These graphs are all connected-homogeneous.

(They arise in Macpherson's (1982) classification of infinite locally-finite distance transitive graphs.)



$X_{3,4}$

Infinite connected-homogeneous graphs

Theorem (RG, Macpherson (2009))

A countable graph is connected-homogeneous if and only if it is isomorphic to the disjoint union of a finite or countable number of copies of one of the following:

1. *a finite connected-homogeneous graph;*
2. *a homogeneous graph;*
3. *the random bipartite graph;*
4. *bipartite infinite complement of a perfect matching;*
5. *the graph $K_{\aleph_0} \times K_{\aleph_0}$;*
6. *a treelike graph X_{κ_1,κ_2} with $\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$.*

Open problems

Digraphs

- Cherlin (1998) classified the countable homogeneous digraphs.
- There are $2^{\aleph_0}$ such graphs.

Problem 1. Classify the countably infinite connected-homogeneous digraphs.

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Problem 1. Classify the countably infinite connected-homogeneous digraphs.

Problem 2. Classify the locally-finite countably infinite connected-homogeneous digraphs.

- In recent joint work with R. Möller we have obtained a partial solution to Problem 2, for digraphs with more than one "end".
- Our work relates the problem to the highly-arc-transitive digraphs of Cameron, Praeger and Wormald (1993).

Alternative ways of weakening homogeneity

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- It is a concept originally due to Fraïssé and Pouzet.
- Ronse (1978) and Enomoto (1981) showed that for finite graphs:

$$\text{set-homogeneous} \equiv \text{homogeneous.}$$

- Droste, Giraudet, Macpherson and Sauer (1994) considered infinite set-homogeneous graphs.
  - **Open problem.** Classify the countable set-homogeneous graphs.

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Definition

A relational structure M is set-homogeneous if whenever two finite substructures U and V are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $Ug = V$.

- It is a concept originally due to Fraïssé and Pouzet.
- Ronse (1978) and Enomoto (1981) showed that for finite graphs:

$$\text{set-homogeneous} \equiv \text{homogeneous}.$$

- Droste, Giraudet, Macpherson and Sauer (1994) considered infinite set-homogeneous graphs.
  - **Open problem.** Classify the countable set-homogeneous graphs.

Recent joint work with D. Macpherson, C. E. Praeger and G. Royle:

- We have classified the finite set-homogeneous digraphs.
  - This generalises work of Lachlan (1981).

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