Cohomology theory of monoids with a single defining relation

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(joint work with B. Steinberg (City College of New York))

From permutation groups to model theory: a workshop inspired by the interests of Dugald Macpherson, on the occasion of his 60th birthday

ICMS, Edinburgh, September 2018

University of East Anglia

[1] Research supported by the EPSRC grant EP/N033353/1 "Special inverse monoids: subgroups, structure, geometry, rewriting systems and the word problem".

One-relator groups and monoids

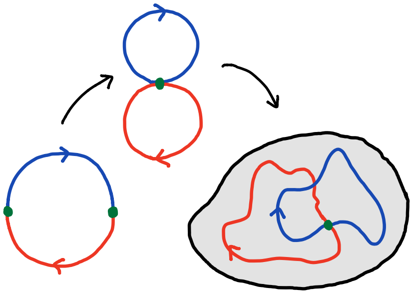
	One-relator groups $\langle A \mid r \rangle = FG(A)/\langle\langle r \rangle\rangle$	One-relator monoids $\langle A \mid u = v \rangle$
Word problem	Magnus 1932	? (Some special cases solved by Adjan, Oganesyan, Lallament...)
Topological & homological properties	Lyndon 1950	RG & Steinberg 2018

Lyndon says:

"Magnus has solved the word problem in the case of a single defining relation. A complementary problem is that of determining *all identities among the relations*. The main theorem of this paper solves this problem."

Groups and topology

X - a space (path connected)



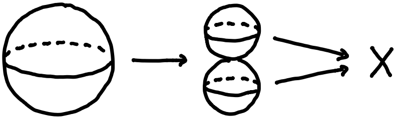
Fundamental group

$\pi_1(X)$ = { homotopy classes of loops }

Higher homotopy groups

$\pi_n(X)$ = { homotopy classes of maps $S^n \to X$ }

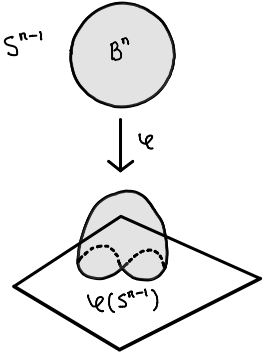
S^n the n-sphere



X is called asphererical if $\pi_n(X)$ is trivial for $n \neq 1$.

Theorem (Hurewicz (1936)) An aspherical space is determined up to homotopy equivalence by its fundamental group.

Classifying spaces of groups



CW complex - a space equipped with a sequence of subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

The n-skeleton X_n is obtained from X_{n-1} by attaching n-cells B^n via maps $\varphi : S^{n-1} \to X_{n-1}$.

Definition

A classifying space Y for a group G is an aspherical CW complex with fundamental group G.

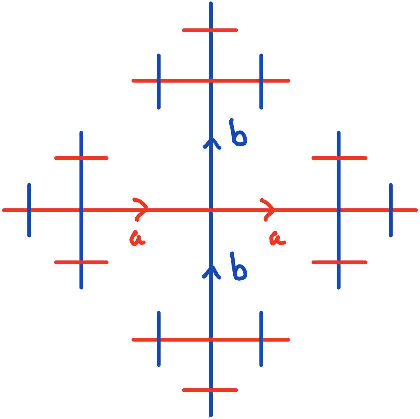
- Classifying spaces exist and are unique up to homotopy equivalence.

Whitehead theorem implies: a CW complex is aspherical $\Leftrightarrow$ its universal cover is contractible.

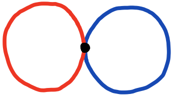
If Y is a classifying space for G then the universal cover of Y is a free G-CW complex which is contractible.

Free group

$$G = \langle a, b \mid \rangle = FG(a,b)$$



Bouquet of circles

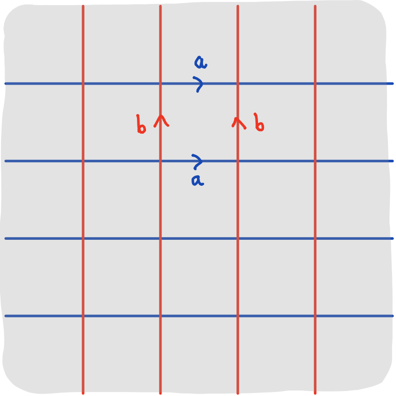


Classifying space Y for G

Universal cover X is a contractible free G-CW complex

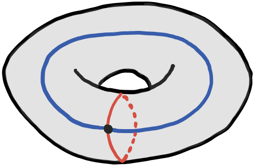
Free abelian group

$$G = \mathrm{Gp}\langle a, b \mid aba^{-1}b^{-1} = 1\rangle \cong \mathbb{Z} \times \mathbb{Z}$$



Universal cover $X = \mathbb{R}^2$ is a contractible free G-CW complex

Torus



Classifying space Y for G

Finiteness properties

Geometric dimension

$\mathrm{gd}\,G$ is the minimum dimension of a classifying space for G

- $\mathrm{gd}\,G = 1 \Leftrightarrow G$ is a non-trivial free group.
- $\mathrm{gd}(\mathbb{Z} \times \mathbb{Z}) = 2$ and more generally $\mathrm{gd}\,\mathbb{Z}^n = n$.

Property F_n (C. T. C. Wall (1965))

G is of type F_n if there is a classifying space with only finitely many k-cells for each $k \leq n$.

- G is of type $\mathrm{F}_1 \Leftrightarrow$ it is finitely generated.
- G is of type $\mathrm{F}_2 \Leftrightarrow$ it is finitely presented.
- $\mathbb{Z} \times \mathbb{Z}$ is of type F_∞ (finitely many cells in every dimension).

Corresponding homological finiteness properties:
cohomological dimension $\mathrm{cd}\,G$ & property FP_n.

The word problem

Definition

A monoid M with a finite generating set A has decidable word problem if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of M.

Example. $M = \langle a, b \mid ba = ab \rangle$ has decidable word problem.
Normal forms = $\{a^i b^j : i, j \geq 0\}$.

Some history

There are finitely presented monoids / groups with undecidable word problem.

- Markov (1947) and Post (1947), Turing (1950), Novikov (1955) and Boone (1958)

Theme. Study interesting classes with decidable word problem e.g.

- hyperbolic groups
- automatic groups
- finite complete presentations

Connection to topology. Groups / monoids in these classes are all type F_∞.

Finiteness properties of monoids

Definition (RG & Steinberg (2017))

An equivariant classifying space for a monoid M is a free M-CW complex which is contractible.

- Equivariant classifying spaces exist and are unique up to M-homotopy equivalence.

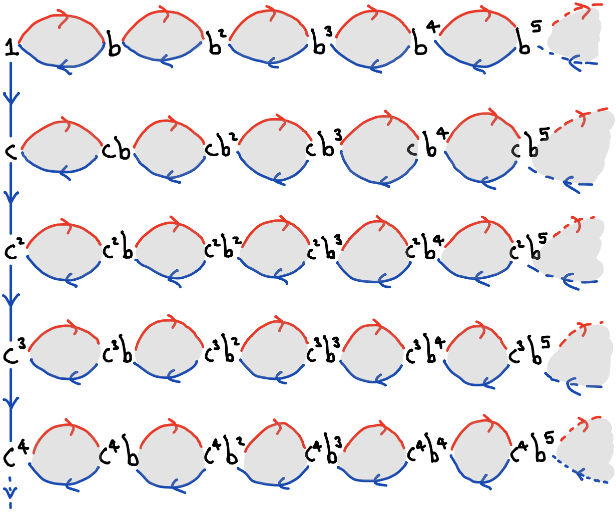
Geometric dimension

$\operatorname{gd} M$ is the minimum dimension of an equivariant classifying space for M.

Property $\mathbf{F}_n$

M is of type F_n if there is an equivariant classifying space X for M such that the set of k-cells is a finitely generated free M-set for all $k \leq n$.

Bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$



One-relator monoids

Longstanding open problem

Is the word problem decidable for one-relation monoids $\langle A \mid u = v \rangle$?

Related open problem

Does every one-relation monoid $\langle A \mid u = v \rangle$ admit a finite complete presentation?

If yes then every one-relation monoid would be of type F_∞.

This motivates the following question of Kobayashi (2000)

Question: Is every one-relator monoid $\langle A \mid u = v \rangle$ of type F_∞?

One relator groups

Magnus (1932): Proved one-relator groups have decidable word problem.

Theorem (Lyndon 1950)

Let $G = \langle A \mid r = 1 \rangle$ be a one-relator group. Then

(a) G is of type F_∞ and

(b) If r is not a proper power then $\operatorname{gd} G \leq 2$, otherwise $\operatorname{gd} G = \infty$.
i.e. torsion free one-relator groups have geometric dimension at most 2.

For (b) Lyndon proved that the presentation 2-complex X of any torsion free one-relator group $G = \langle A \mid r = 1 \rangle$ is aspherical and thus is a classifying space for G.

X has a single 0-cell, a 1-cell for each generator $a \in A$, and a single 2-cell with boundary reading the word r.

One-relator monoids

We obtain a positive answer to the question of Kobayahi (2000).

Theorem (RG & Steinberg 2018)

Every one relator monoid $\langle A \mid u = v \rangle$ is of type F_∞.

Our methods also allow us to prove:

Theorem (RG & Steinberg 2018)

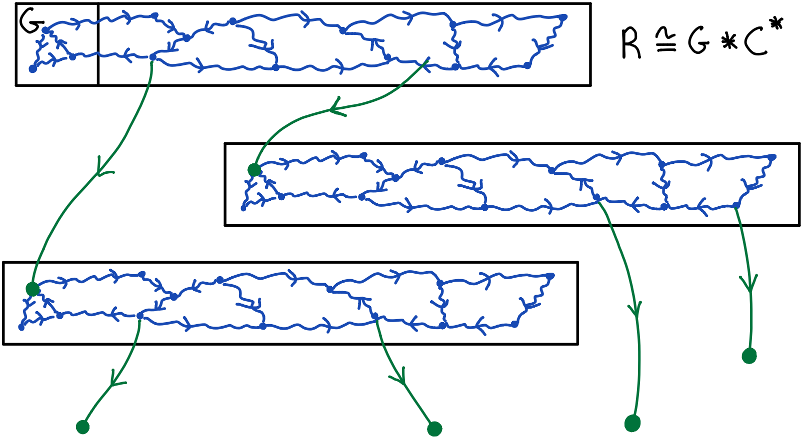
Let M be a monoid defined by a one-relator presentation $\langle A \mid u = v \rangle$. Suppose without loss of generality that $|v| \leq |u|$. Let $z \in A^*$ be the longest word which is a prefix and a suffix of both u and of v. Then

(i) $\mathrm{gd}(M) = \infty$ if M has a maximal subgroup with torsion; and

(ii) $\mathrm{gd}(M) \leq 2$ if and only if M is torsion free and either z is the empty word or $z = v$.

$M = \langle A \mid u = 1 \rangle$ $\quad$ G - group of units

$\Gamma(M, A)$



$R \cong G * C^*$