Graphs with a high degree of symmetry

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Outline

Outline

Graphs and automorphisms

Definition

- A graph Γ is a pair $(V\Gamma, E\Gamma)$
  - $V\Gamma$ - vertex set
  - $E\Gamma$ - set of 2-element subsets of $V\Gamma$, the edge set.
- If $\{u, v\} \in E\Gamma$ we say that u and v are adjacent writing $u \sim v$.
- The neighbourhood of u is $\Gamma(u) = \{v \in V\Gamma : v \sim u\}$, and the degree (or valency) of u is $|\Gamma(u)|$.
- A graph Γ is finite if $V\Gamma$ is finite, and is locally-finite if all of its vertices have finite degree.
- An automorphism of Γ is a bijection $\alpha : V\Gamma \to V\Gamma$ sending edges to edges and non-edges to non-edges. We write $G = \operatorname{Aut}\Gamma$ for the full automorphism group of Γ.

Graphs with symmetry

Roughly speaking, the 'more' symmetry a graph has the 'larger' its automorphism group will be (and vice versa).

Aim. To obtain classifications of families of graphs with a high degree of symmetry.

In each case we impose a symmetry condition $\mathcal{P}$ and then attempt to describe all (countable) graphs with property $\mathcal{P}$.

For each class, this naturally divides into three cases:

- finite graphs;
- infinite locally-finite graphs;
- infinite non-locally-finite graphs.

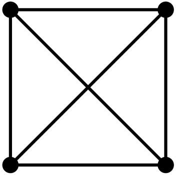
Vertex-transitive graphs

Definition

Γ is vertex transitive if G acts transitively on $V\Gamma$. That is, for all $u, v \in V\Gamma$ there is an automorphism $\alpha \in G$ such that $u^\alpha = v$.

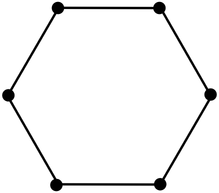
This is the weakest possible condition and there are many examples.

Complete graph K_r has r vertices and every pair of vertices is joined by an edge.



K_4

Cycle C_r has vertex set $\{1, \ldots, r\}$ and edge set $\{\{1,2\}, \{2,3\}, \ldots, \{r,1\}\}$.



C_6

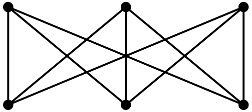
Empty graph I_r is the *complement* of the complete graph K_r. (The complement $\overline{\Gamma}$ of Γ is defined by $V\overline{\Gamma} = V\Gamma$, $E\overline{\Gamma} = \{\{i,j\} : \{i,j\} \notin E\Gamma\}$).

Some vertex transitive bipartite graphs

Definition

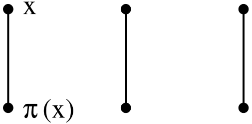
A graph is called bipartite if the vertex set may be partitioned into two disjoint sets X and Y such that no two vertices in X are adjacent, and no two vertices of Y are adjacent.

- Complete bipartite every vertex in X is adjacent to every vertex of Y (written $K_{a,b}$ if $|X| = a$, $|Y| = b$).

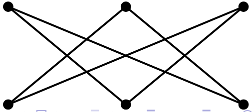


$K_{3,3}$

- Perfect matching there is a bijection $\pi : X \to Y$ such that $E\Gamma = \{\{x, \pi(x)\} : x \in X\}$



- Complement of perfect matching $\{x, y\} \in E\Gamma \Leftrightarrow y \neq \pi(x)$



Cayley graphs of groups

Definition

G - group, $A \subseteq G$ a generating set for G such that $1_G \notin A$ and A is closed under taking inverses (so $x \in A \Rightarrow x^{-1} \in A$).

The (right) Cayley graph $\Gamma = \Gamma(G, A)$ is given by

$$V\Gamma = G; \quad E\Gamma = \{\{g, h\} : g^{-1}h \in A\}.$$

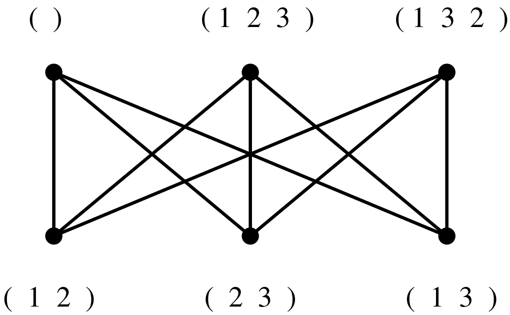
Thus two vertices are adjacent if they differ in G by right multiplication by a generator.

Fact. The Cayley graph of a group is always vertex transitive.

Cayley graph

Example (Cayley graph of S_3)

G = the symmetric group S_3, $\quad A = \{(\ 1\ 2\ ), (\ 2\ 3\ ), (\ 1\ 3\ )\}$



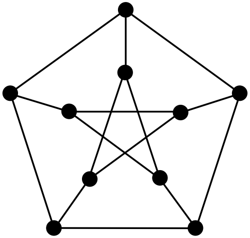
$\Gamma(G, A) \cong K_{3,3}$ a complete bipartite graph.

Vertex-transitive graphs

On the other hand, not every vertex transitive graph arises in this way.

Example (Petersen graph)

The *Petersen graph* is vertex transitive but is not a Cayley graph.



There are ‘far too many’ vertex transitive graphs for us to stand a chance of achieving a classification.

Outline

Distance-transitive graphs

Definition

In a connected graph Γ we define the distance $d(u,v)$ between u and v to be the length of a shortest path from u to v.

Definition

A graph is distance-transitive if for any two pairs (u,v) and (u',v') with $d(u,v) = d(u',v')$, there is an automorphism taking u to u' and v to v'.

$$\text{distance-transitive} \Rightarrow \text{vertex-transitive}$$

Example

A connected finite distance-transitive graph of valency 2 is simply a cycle C_n.

Hamming graphs and hypercubes

A family of distance-transitive graphs

Definition

The Hamming graph $H(d,n)$. Let $\mathbb{Z}_n = \{0,1,2,\dots,n-1\}$. Then the vertex set of $H(d,n)$ is

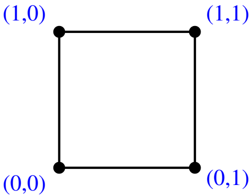
$$\mathbb{Z}_n^d = \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{d \text{ times}}$$

and two vertices u and v are adjacent if and only if they differ in exactly one coordinate.

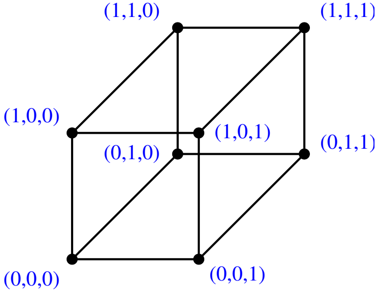
The d-dimensional hypercube is defined to be $Q_d := H(d,2)$. Its vertices are d-dimensional vectors over $\mathbb{Z}_2 = \{0,1\}$.

Fact. $H(d,n)$ is distance transitive

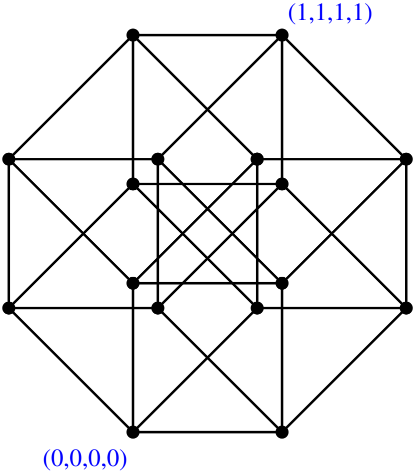
Hypercubes Q_i ($i = 2, 3, 4$)



Q_2



Q_3



Q_4

Finite distance-transitive graphs

The classification of the finite distance-transitive graphs is still incomplete, but a lot of progress has been made.

Definition

A graph is imprimitive if there is an equivalence relation on its vertex set which is preserved by all automorphisms.

Imprimitive distance-transitive graphs

The cube is imprimitive in two different ways.

1. **Bipartite** The bipartition relation

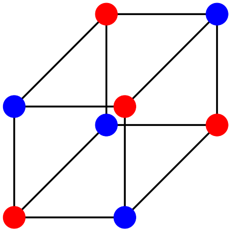
$$u \equiv v \Leftrightarrow d(u,v) \text{ is even}$$

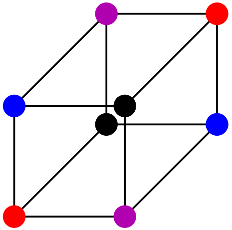
is preserved (2 equivalence classes: red and blue)

2. **Antipodal** The relation

$$u \approx v \Leftrightarrow u = v \text{ or } d(u,v) = 3$$

is preserved (4 equivalence classes: black, blue, purple and red)





Smith's reduction

Smith (1971) showed that the *only* way in which a finite distance-transitive graph (of valency > 2) can be imprimitive is as a result of being bipartite or antipodal (as in the cube example above).

This reduces the classification of finite distance-transitive graphs to:

1. classify the finite primitive distance-transitive graphs (this is close to being complete, using the classification of finite simple groups; see recent survey by John van Bon in *European J. Combin.*);
2. find all 'bipartite doubles' and 'antipodal covers' of these graphs (still far from complete).

Definition (Tree)

A tree is a connected graph without cycles. A tree is regular if all vertices have the same degree. We use T_r to denote a regular tree of valency r.

Fact. A regular tree T_r ($r \in \mathbb{N}$) is an example of an infinite locally-finite distance-transitive graph.

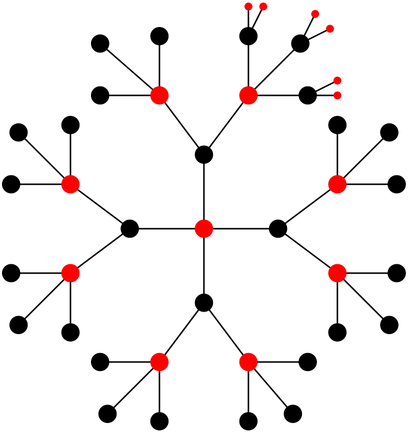
Definition (Semiregular tree)

$T_{a,b}$: A tree $T = X \cup Y$ where $X \cup Y$ is a bipartition, all vertices in X have degree a, and all in Y have degree b.

A semiregular tree will not in general be distance transitive.

Locally finite infinite distance-transitive graphs
A family of examples

- Let $r \geq 1$ and $l \geq 2$ be integers.
- Take a bipartite semiregular tree $T_{r+1,l}$
  - one block A with vertices of degree $r+1$
  - the other B with vertices of degree l
- Define $X_{r,l}$
  - Vertex set = B
  - $b_1, b_2 \in B$ joined iff they are at distance 2 in $T_{r+1,l}$.



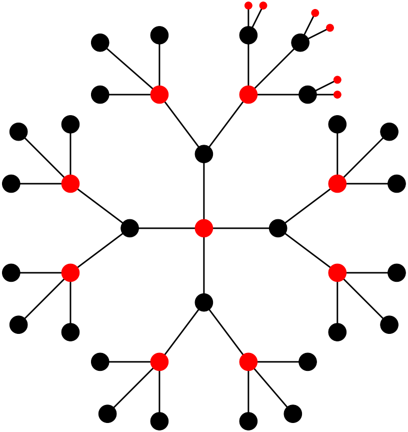
$T_{3,4}$

Locally finite infinite distance-transitive graphs

A family of examples

Example $X_{r,l} = X_{2,4}$.

- Let $r = 2$ and $l = 4$.
- So $T_{r+1,l} = T_{3,4}$
  - A = vertices of degree 3 (in black)
  - B = vertices of degree 4 (in red)
- $X_{2,4}$
  - Vertex set = B = red vertices
  - $b_1, b_2 \in B$ joined iff they are at distance 2 in $T_{3,4}$.



$T_{3,4}$

Locally finite infinite distance-transitive graphs
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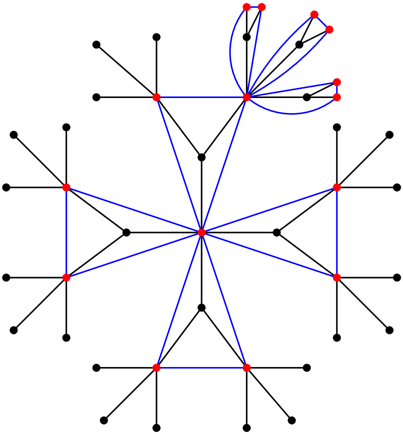

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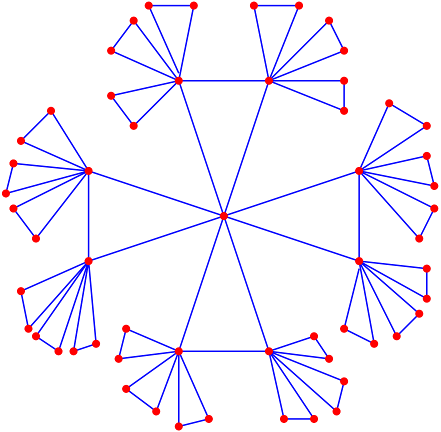


$X_{2,4}$

Macpherson's theorem

The graphs X_{κ_1,κ_2} $(\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\})$ are distance transitive.

The neighbourhood of a vertex consists of κ_2 copies of the complete graph K_{κ_1}.

Theorem (Macpherson (1982))

A locally-finite infinite graph is distance transitive if and only if it is isomorphic to $X_{k,r}$ for some $k, r \in \mathbb{N}$.

The key steps in Macpherson's proof are to take an infinite locally finite distance-transitive graph Γ and

1. prove that Γ is "tree-like" (i.e. show Γ has infinitely many *ends*)
2. apply a powerful theorem of Dunwoody (1982) about graphs with more than one end

Non-locally-finite infinite distance-transitive graphs

On the other hand, for infinite non-locally-finite distance-transitive graphs far less is known.

The following result is due to Evans.

Theorem

There exist $2^{\aleph_0}$ non-isomorphic countable distance-transitive graphs.

Proof. Makes use of a construction originally due to Hrushovski (which is itself a powerful strengthening of Fraïssé's method for constructing countable structures by amalgamation).

Outline

Homogeneous graphs

Definition

A graph Γ is called homogeneous if any isomorphism between finite induced subgraphs extends to an automorphism of the graph.

Homogeneity is the *strongest* possible symmetry condition we can impose.

$$\text{homogeneous} \Rightarrow \text{distance-transitive} \Rightarrow \text{vertex-transitive}$$

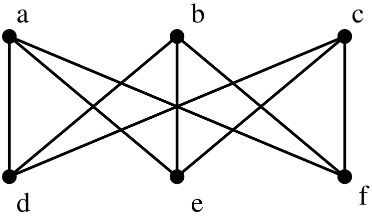
A finite homogeneous graph

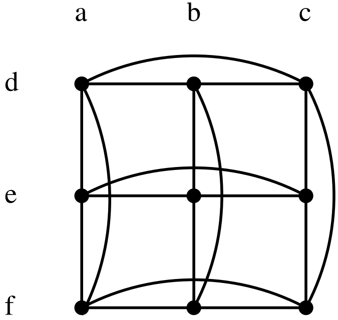
Definition (Line graph)

The line graph $L(\Gamma)$ of a graph Γ has vertex set the edge set of Γ, and two vertices e_1 and e_2 joined in $L(\Gamma)$ iff the edges e_1, e_2 share a common vertex in Γ.

Example

$L(K_{3,3})$ is a finite homogeneous graph





Classification of finite homogeneous graphs

Gardiner classified the finite homogeneous graphs.

Theorem (Gardiner (1976))

A finite graph is homogeneous if and only if it is isomorphic to one of the following:

1. *finitely many disjoint copies of* K_r $(r \geq 1)$ *(or its complement);*
2. *The pentagon* C_5*;*
3. *Line graph* $L(K_{3,3})$ *of the complete bipartite graph* $K_{3,3}$*.*

Infinite homogeneous graphs

Definition (The random graph R)

Constructed by Rado in 1964. The vertex set is the natural numbers (including zero).

For $i, j \in \mathbb{N}$, $i < j$, then i and j are joined if and only if the ith digit in j (in base 2) is 1.

Example

Since $88 = 8 + 16 + 64 = 2^3 + 2^4 + 2^6$ the numbers less that 88 that are adjacent to 88 are just $\{3, 4, 6\}$. Of course, many numbers greater than 88 will also be adjacent to 88 (for example 2^{88} will be).

The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets U and V of vertices, there exists a vertex w adjacent to every vertex in U and to no vertex in V.

Theorem

There exists a countably infinite graph R satisfying property (*), and it is unique up to isomorphism. The graph R is homogeneous.

Existence. The graph R defined above satisfies property (*).

Uniqueness and homogeneity. Both follow from a back-and-forth argument. Property (*) is used to extend the domain (or range) of any isomorphism between finite substructures one vertex at a time.

Fraïssé's theorem

Definition

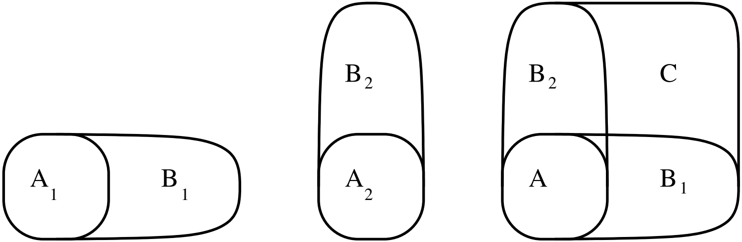
A relational structure M is homogeneous if any isomorphism between finite induced substructures of M extends to an automorphism of M. The age of M is the class of isomorphism types of its finite substructures.

Fraïssé (1953) showed how to recognise the existence of homogeneous structures from their ages.

A class $\mathcal{C}$ is the age of a countable homogeneous structure M if and only if $\mathcal{C}$ is closed under isomorphism, closed under taking substructures, contains only countably many structures up to isomorphism, and satisfies the amalgamation property. If these conditions hold, then M is unique, $\mathcal{C}$ is called a Fraïssé class, and M is called the Fraïssé limit of the class $\mathcal{C}$.

Picture of amalgamation

The amalgamation property says that two structures in $\mathcal{C}$ with isomorphic substructures can be 'glued together', inside a larger structure of $\mathcal{C}$, in such a way that the substructures are identified.



(AP) Given $B_1, B_2 \in \mathcal{C}$ and isomorphism $f : A_1 \to A_2$ with $A_i \subseteq B_i$ $(i = 1, 2)$, $\exists C \in \mathcal{C}$ in which B_1 and B_2 are embedded so that A_1 and A_2 are identified according to f.

Countable homogeneous graphs

Examples

- The class of all finite graphs is a Fraïssé class. Its Fraïssé limit is the random graph R.
- The class of all finite graphs not embedding K_n (for some fixed n) is a Fraïssé class. We call the Fraïssé limit the *countable generic K_n-free graph*.

Theorem (Lachlan and Woodrow (1980))

Let Γ be a countably infinite homogeneous graph. Then Γ is isomorphic to one of: the random graph, a disjoint union of complete graphs (or its complement), the generic K_n-free graph (or its complement).

Outline

Connected-homogeneous graphs

Distance-transitive graphs - classification incomplete

Homogeneous graph - classified

Question. Do there exist natural classes between homogeneous and distance-transitive that can be classified?

Definition

A graph Γ is *connected-homogeneous* if any isomorphism between connected finite induced subgraphs extends to an automorphism.

$$\text{homogeneous} \Rightarrow \text{connected-homogeneous} \Rightarrow \text{distance-transitive}$$

Finite connected-homogeneous graphs

Gardiner classified the finite connected-homogeneous graphs.

Theorem (Gardiner (1978))

A finite graph is connected-homogeneous if and only if it is isomorphic to a disjoint union of copies of one of the following:

1. *a finite homogeneous graph*
2. *complement of a perfect matching*
3. *cycle* C_n $(n \geq 5)$
4. *the line graph* $L(K_{s,s})$ *of a complete bipartite graph* $K_{s,s}$ $(s \geq 3)$
5. *Petersen's graph*
6. *the graph obtained by identifying antipodal vertices of the 5-dimensional cube* Q_5

Infinite connected-homogeneous graphs

Theorem (RG, Macpherson (in preparation))

Any countable connected-homogeneous graph is isomorphic to the disjoint union of a finite or countable number of copies of one of the following:

1. *a finite connected-homogeneous graph;*
2. *a homogeneous graph;*
3. *the random bipartite graph;*
4. *the complement of a perfect matching;*
5. *the line graph of a complete bipartite graph* $K_{\aleph_0,\aleph_0}$*;*
6. *a graph* X_{κ_1,κ_2} *with* $\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$.

(The proof relies on the Lachlan-Woodrow classification of fully homogeneous graphs.)

Possible future work

Consider connected-homogeneity for other kinds of relational structure.

Schmerl (1979) classified the countable homogeneous posets. It turns out that weakening homogeneity to connected-homogeneity here essentially gives rise to no new examples.

Theorem (RG, Macpherson (in preparation))

A countable poset is connected-homogeneous if and only if it is isomorphic to a disjoint union of a countable number of isomorphic copies of some homogeneous countable poset.

The corresponding result for digraphs seems to be difficult.