Discretizing Earnings Dynamics: A Life-Cycle Non-Stationary AR(1) Process with Gaussian-Mixture Shocks

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April 26, 2022

Abstract

The AR(1) process with gaussian innovations that is standard in life-cycle models of earnings dynamics has been found to be insufficient by recent evidence emphasizing the age-dependent and non-gaussian nature of changes in earnings. Empirical studies find that an AR(1) process with gaussian-mixture innovations, together with age-dependent and lagged-value-dependent parameters, does a much better job. We extend the Farmer-Toda method for discretizings an AR(1) process with gaussian-mixture shocks to allow the dependence of parameters on age and lagged value, and document it's performance. We then solve an otherwise standard OLG model discretizing a variety of process on earnings dynamics to evaluate the impact of the more realistic earnings dynamics, relative to the standard approach of an AR(1) with gaussian innovations. We look at how more realistic earnings dynamics affect measures of inequality, the persistence of poverty, and decisions on savings.

Keywords: Numerical methods, Quadrature, Gaussian Mixture, Life-cycle, OLG model.

JEL Classification: E00; C63; C15; E24.

*Thanks to no-one yet, paper is early stage :) Thanks to Rāpoi at Victoria University of Wellington for the use of their computing facilities. Kirkby: Victoria University of Wellington. Please address all correspondence about this article to Robert Kirkby at <robertdkirkby@gmail.com>, robertdkirkby.com.

Extra About z

This documents just contains a whole bunch of additional appendices about discretizing z. It is based on Models 2 and Model 6 of Guvenen, Karahan, Ozkan, and Song (2021). In Model 2 z is an AR(1) with gaussian innovations. In Model 6 z is an AR(1) with gaussian-mixture innovations.

It looks that discretizations using 31, 51, 75 and 101 points. It looks at discretizations where the max and min grid points are +-3 standard deviations, and at +-4 standard deviations.

For the AR(1) with gaussian innovations the baseline uses Rouwenhorst method. For comparison this is also done using the Farmer-Toda method to target 2 moments (of conditional distributions) and using the Farmer-Toda method to target 4 moments.

References

Fatih Guvenen, Fatih Karahan, Serdar Ozkan, and Jae Song. What do data on millions of u.s. workers say about lifecycle labor income risk? Econometrica, 89(5):2303–2339, 2021. doi: https://doi.org/10.3982/ECTA14603.

A Further Evaluating Discretization of z: 51 points

In this section we report more details on the discretization of z in Models 2 and 6 of Guvenen, Karahan, Ozkan, and Song (2021). The actual processes being discretized are described in full in Appendix of the paper. We here just go into more detail.

In Model 2 z is an AR(1) with gaussian innovations. In Model 6 z is an AR(1) with gaussian-mixture innovations. We also look at how things differ when we set the max/min of the grid to be $+-4*sigma\_z$ versus $+-4*sigma\_z$ (that the range of the grid is +-3 or +-4 standard deviations); in the main results we are using $+-4*sigma\_z$.

We have three 'different' versions of Model 2 for the AR(1) with gaussian innovations. The first (which is also the one used in main paper) used the Fella-Gallipoli-Pan extened-Rouwenhourst method to discretize z. The second uses the extended Farmer-Toda targeting two moments, and the third also uses the extended Farmer-Toda but targets four moments.[1],[2]

[1] A normal distribution is fully defined by the first two moments, but this does not necessarily mean that a discretization that accurately gets the first two moments will be accurate for the higher moments.

[2] These are called Model 21 and Model 22 by the codes.

A.1 Model 2: AR(1) with Gaussian innovations, nSigmas=3

First, just some text to see if the upper end of the grid for z is being used.

How much of z grid is used? (1/4)

41 of 51 points have more than 10^(-9) mass at age j=1

41 of 51 points have more than 10^(-9) mass at age j=2

41 of 51 points have more than 10^(-9) mass at age j=3

41 of 51 points have more than 10^(-9) mass at age j=4

41 of 51 points have more than 10^(-9) mass at age j=5

41 of 51 points have more than 10^(-9) mass at age j=6

41 of 51 points have more than 10^(-9) mass at age j=7

41 of 51 points have more than 10^(-9) mass at age j=8

41 of 51 points have more than 10^(-9) mass at age j=9

41 of 51 points have more than 10^(-9) mass at age j=10

41 of 51 points have more than 10^(-9) mass at age j=11

41 of 51 points have more than 10^(-9) mass at age j=12

41 of 51 points have more than 10^(-9) mass at age j=13

41 of 51 points have more than 10^(-9) mass at age j=14

41 of 51 points have more than 10^(-9) mass at age j=15

41 of 51 points have more than 10^(-9) mass at age j=16

41 of 51 points have more than 10^(-9) mass at age j=17

41 of 51 points have more than 10^(-9) mass at age j=18

41 of 51 points have more than 10^(-9) mass at age j=19

41 of 51 points have more than 10^(-9) mass at age j=20

41 of 51 points have more than 10^(-9) mass at age j=21

41 of 51 points have more than 10^(-9) mass at age j=22

41 of 51 points have more than 10^(-9) mass at age j=23

41 of 51 points have more than 10^(-9) mass at age j=24

41 of 51 points have more than 10^(-9) mass at age j=25

41 of 51 points have more than 10^(-9) mass at age j=26

41 of 51 points have more than 10^(-9) mass at age j=27

41 of 51 points have more than 10^(-9) mass at age j=28

41 of 51 points have more than 10^(-9) mass at age j=29

41 of 51 points have more than 10^(-9) mass at age j=30

41 of 51 points have more than 10^(-9) mass at age j=31

41 of 51 points have more than 10^(-9) mass at age j=32

41 of 51 points have more than 10^(-9) mass at age j=33

41 of 51 points have more than 10^(-9) mass at age j=34

41 of 51 points have more than 10^(-9) mass at age j=35

41 of 51 points have more than 10^(-9) mass at age j=36

41 of 51 points have more than 10^(-9) mass at age j=37

41 of 51 points have more than 10^(-9) mass at age j=38

41 of 51 points have more than 10^(-9) mass at age j=39

41 of 51 points have more than 10^(-9) mass at age j=40

41 of 51 points have more than 10^(-9) mass at age j=41

How much of z grid is used? (2/4)

Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=1

Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=11

Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=21

Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=31

Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=41

How much of z grid is used? (3/4)

Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=1

Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=11

Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=21

Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=31

Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=41

Range of z grid? (4/4)

Min grid point -1.736902, max grid point 1.736902, at age j=1

Min grid point -2.042687, max grid point 2.042687, at age j=11

Min grid point -2.182498, max grid point 2.182498, at age j=21

Min grid point -2.250608, max grid point 2.250608, at age j=31

Min grid point -2.284637, max grid point 2.284637, at age j=41

Now Figure 1 plots the stationary distribution of z for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), to see how much of the grid is being used.

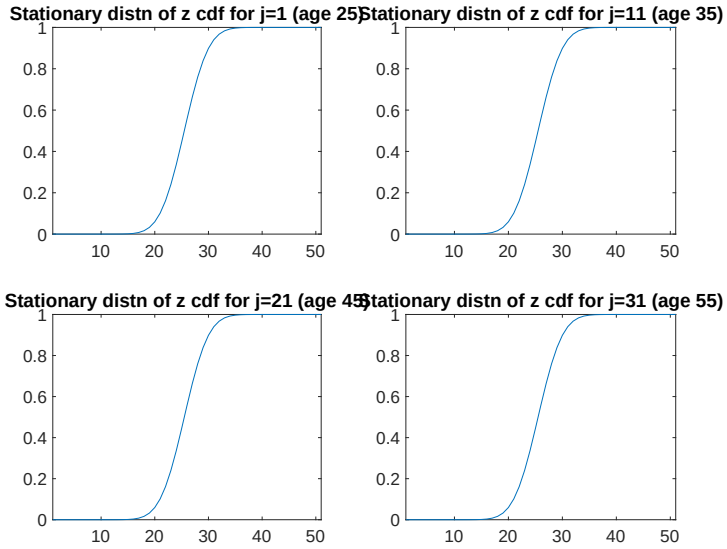

Figure 1: Discretization of Model 2 with nSigma=3: Stationary distribution of z

Now Figure 2 plots the pdf and cdf of the innovations for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), and include the theoretical distribution being approximated.

Now some plots of the transition matrix for just a few ages. Figure 3 shows the transition matrix for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55) as both a pdf and cdf.

Note that because an extended Rouwenhourst method (the Fella-Gallipoli-Pan method) there is no heatmap.

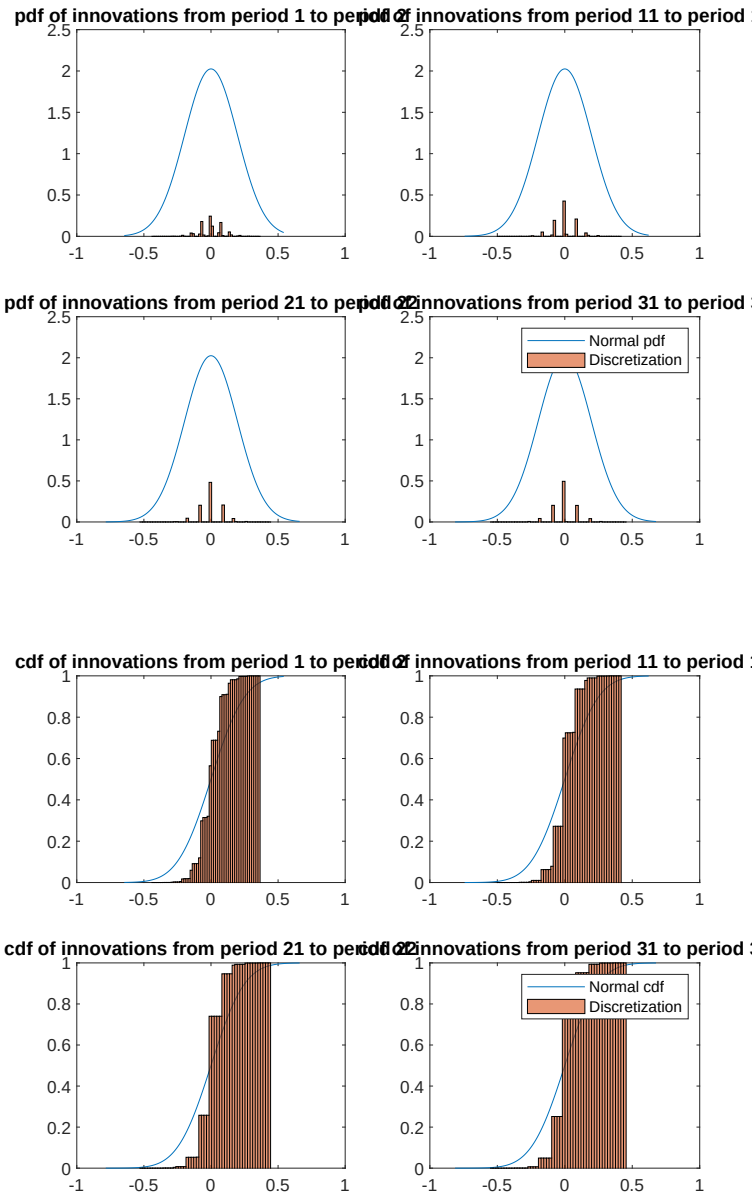

Figure 2: Discretization of Model 2 with nSigma=3: innovations to z

A.1.1 Using Extended Farmer-Toda to target 2 moments

First, just some text to see if the upper end of the grid for z is being used.

How much of z grid is used? (1/4)

51 of 51 points have more than 10^(-9) mass at age j=1

51 of 51 points have more than 10^(-9) mass at age j=2

51 of 51 points have more than 10^(-9) mass at age j=3

51 of 51 points have more than 10^(-9) mass at age j=4

51 of 51 points have more than 10^(-9) mass at age j=5

51 of 51 points have more than 10^(-9) mass at age j=6

51 of 51 points have more than 10^(-9) mass at age j=7

51 of 51 points have more than 10^(-9) mass at age j=8

51 of 51 points have more than 10^(-9) mass at age j=9

51 of 51 points have more than 10^(-9) mass at age j=10

51 of 51 points have more than 10^(-9) mass at age j=11

51 of 51 points have more than 10^(-9) mass at age j=12

51 of 51 points have more than 10^(-9) mass at age j=13

51 of 51 points have more than 10^(-9) mass at age j=14

51 of 51 points have more than 10^(-9) mass at age j=15

51 of 51 points have more than 10^(-9) mass at age j=16

51 of 51 points have more than 10^(-9) mass at age j=17

51 of 51 points have more than 10^(-9) mass at age j=18

51 of 51 points have more than 10^(-9) mass at age j=19

51 of 51 points have more than 10^(-9) mass at age j=20

51 of 51 points have more than 10^(-9) mass at age j=21

51 of 51 points have more than 10^(-9) mass at age j=22

51 of 51 points have more than 10^(-9) mass at age j=23

51 of 51 points have more than 10^(-9) mass at age j=24

51 of 51 points have more than 10^(-9) mass at age j=25

51 of 51 points have more than 10^(-9) mass at age j=26

51 of 51 points have more than 10^(-9) mass at age j=27

51 of 51 points have more than 10^(-9) mass at age j=28

51 of 51 points have more than 10^(-9) mass at age j=29

51 of 51 points have more than 10^(-9) mass at age j=30

51 of 51 points have more than 10^(-9) mass at age j=31

51 of 51 points have more than 10^(-9) mass at age j=32

51 of 51 points have more than 10^(-9) mass at age j=33

51 of 51 points have more than 10^(-9) mass at age j=34

51 of 51 points have more than 10^(-9) mass at age j=35

51 of 51 points have more than 10^(-9) mass at age j=36

51 of 51 points have more than 10^(-9) mass at age j=37

51 of 51 points have more than 10^(-9) mass at age j=38

51 of 51 points have more than 10^(-9) mass at age j=39

51 of 51 points have more than 10^(-9) mass at age j=40

51 of 51 points have more than 10^(-9) mass at age j=41

How much of z grid is used? (2/4)

Bottom 10 points sum to mass 0.031368, top 10 points sum to mass 0.031368, at age j=1

Bottom 10 points sum to mass 0.031717, top 10 points sum to mass 0.031717, at age j=11

Bottom 10 points sum to mass 0.031732, top 10 points sum to mass 0.031732, at age j=21

Bottom 10 points sum to mass 0.031739, top 10 points sum to mass 0.031739, at age j=31

Bottom 10 points sum to mass 0.031743, top 10 points sum to mass 0.031743, at age j=41

How much of z grid is used? (3/4)

Bottom 5 points sum to mass 0.007244, top 5 points sum to mass 0.007244, at age j=1

Bottom 5 points sum to mass 0.007374, top 5 points sum to mass 0.007374, at age j=11

Bottom 5 points sum to mass 0.007361, top 5 points sum to mass 0.007361, at age j=21

Bottom 5 points sum to mass 0.007359, top 5 points sum to mass 0.007359, at age j=31

Bottom 5 points sum to mass 0.007358, top 5 points sum to mass 0.007358, at age j=41

Range of z grid? (4/4)

Min grid point -1.736902, max grid point 1.736902, at age j=1

Min grid point -2.042687, max grid point 2.042687, at age j=11

Min grid point -2.182498, max grid point 2.182498, at age j=21

Min grid point -2.250608, max grid point 2.250608, at age j=31

Min grid point -2.284637, max grid point 2.284637, at age j=41

Now Figure 4 plots the stationary distribution of z for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), to see how much of the grid is being used.

Now Figure 5 plots the pdf and cdf of the innovations for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), and include the theoretical distribution being approximated.

Now some plots of the transition matrix for just a few ages. Figure 6 shows the transition matrix for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55) as both a pdf and cdf.

Now Figure 7 plots a heatmap showing how many moments (of the conditional distributions, i.e., the distribution of the innovation) the extended Farmer-Toda method succeeds in targeting (note that it is trying to target 2).

A.1.2 Using Extended Farmer-Toda to target 4 moments

First, just some text to see if the upper end of the grid for z is being used.

How much of z grid is used? (1/4)

51 of 51 points have more than 10^(-9) mass at age j=1

51 of 51 points have more than 10^(-9) mass at age j=2

51 of 51 points have more than 10^(-9) mass at age j=3

51 of 51 points have more than 10^(-9) mass at age j=4

51 of 51 points have more than 10^(-9) mass at age j=5

51 of 51 points have more than 10^(-9) mass at age j=6

51 of 51 points have more than 10^(-9) mass at age j=7

51 of 51 points have more than 10^(-9) mass at age j=8

51 of 51 points have more than 10^(-9) mass at age j=9

51 of 51 points have more than 10^(-9) mass at age j=10

51 of 51 points have more than 10^(-9) mass at age j=11

51 of 51 points have more than 10^(-9) mass at age j=12

51 of 51 points have more than 10^(-9) mass at age j=13

51 of 51 points have more than 10^(-9) mass at age j=14

51 of 51 points have more than 10^(-9) mass at age j=15

51 of 51 points have more than 10^(-9) mass at age j=16

51 of 51 points have more than 10^(-9) mass at age j=17

51 of 51 points have more than 10^(-9) mass at age j=18

51 of 51 points have more than 10^(-9) mass at age j=19

51 of 51 points have more than 10^(-9) mass at age j=20

51 of 51 points have more than 10^(-9) mass at age j=21

51 of 51 points have more than 10^(-9) mass at age j=22

51 of 51 points have more than 10^(-9) mass at age j=23

51 of 51 points have more than 10^(-9) mass at age j=24

51 of 51 points have more than 10^(-9) mass at age j=25

51 of 51 points have more than 10^(-9) mass at age j=26

51 of 51 points have more than 10^(-9) mass at age j=27

51 of 51 points have more than 10^(-9) mass at age j=28

51 of 51 points have more than 10^(-9) mass at age j=29

51 of 51 points have more than 10^(-9) mass at age j=30

51 of 51 points have more than 10^(-9) mass at age j=31

51 of 51 points have more than 10^(-9) mass at age j=32

51 of 51 points have more than 10^(-9) mass at age j=33

51 of 51 points have more than 10^(-9) mass at age j=34

51 of 51 points have more than 10^(-9) mass at age j=35

51 of 51 points have more than 10^(-9) mass at age j=36

51 of 51 points have more than 10^(-9) mass at age j=37

51 of 51 points have more than 10^(-9) mass at age j=38

51 of 51 points have more than 10^(-9) mass at age j=39

51 of 51 points have more than 10^(-9) mass at age j=40

51 of 51 points have more than 10^(-9) mass at age j=41

How much of z grid is used? (2/4)

Bottom 10 points sum to mass 0.031430, top 10 points sum to mass 0.031430, at age j=1

Bottom 10 points sum to mass 0.031631, top 10 points sum to mass 0.031631, at age j=11

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Bottom 10 points sum to mass 0.031670, top 10 points sum to mass 0.031669, at age j=21
Bottom 10 points sum to mass 0.031685, top 10 points sum to mass 0.031685, at age j=31
Bottom 10 points sum to mass 0.031691, top 10 points sum to mass 0.031691, at age j=41
How much of z grid is used? (3/4)
Bottom 5 points sum to mass 0.006864, top 5 points sum to mass 0.006864, at age j=1
Bottom 5 points sum to mass 0.007114, top 5 points sum to mass 0.007101, at age j=11
Bottom 5 points sum to mass 0.007121, top 5 points sum to mass 0.007128, at age j=21
Bottom 5 points sum to mass 0.007134, top 5 points sum to mass 0.007134, at age j=31
Bottom 5 points sum to mass 0.007142, top 5 points sum to mass 0.007141, at age j=41
Range of z grid? (4/4)
Min grid point -1.736902, max grid point 1.736902, at age j=1
Min grid point -2.042687, max grid point 2.042687, at age j=11
Min grid point -2.182498, max grid point 2.182498, at age j=21
Min grid point -2.250608, max grid point 2.250608, at age j=31
Min grid point -2.284637, max grid point 2.284637, at age j=41
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Now Figure 8 plots the stationary distribution of z for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), to see how much of the grid is being used.

Now Figure 9 plots the pdf and cdf of the innovations for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), and include the theoretical distribution being approximated.

Now some plots of the transition matrix for just a few ages. Figure 10 shows the transition matrix for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55) as both a pdf and cdf.

Now Figure 11 plots a heatmap showing how many moments (of the conditional distributions, i.e., the distribution of the innovation) the extended Farmer-Toda method succeeds in targeting (note that it is here trying to target 4).

A.2 Model 2: AR(1) with Gaussian innovations, nSigmas=4

First, just some text to see if the upper end of the grid for z is being used.

How much of z grid is used? (1/4)

41 of 51 points have more than 10^(-9) mass at age j=1

41 of 51 points have more than 10^(-9) mass at age j=2

41 of 51 points have more than 10^(-9) mass at age j=3

41 of 51 points have more than 10^(-9) mass at age j=4

41 of 51 points have more than 10^(-9) mass at age j=5

41 of 51 points have more than 10^(-9) mass at age j=6

41 of 51 points have more than 10^(-9) mass at age j=7

41 of 51 points have more than 10^(-9) mass at age j=8

41 of 51 points have more than 10^(-9) mass at age j=9

41 of 51 points have more than 10^(-9) mass at age j=10

41 of 51 points have more than 10^(-9) mass at age j=11

41 of 51 points have more than 10^(-9) mass at age j=12

41 of 51 points have more than 10^(-9) mass at age j=13

41 of 51 points have more than 10^(-9) mass at age j=14

41 of 51 points have more than 10^(-9) mass at age j=15

41 of 51 points have more than 10^(-9) mass at age j=16

41 of 51 points have more than 10^(-9) mass at age j=17

41 of 51 points have more than 10^(-9) mass at age j=18

41 of 51 points have more than 10^(-9) mass at age j=19

41 of 51 points have more than 10^(-9) mass at age j=20

41 of 51 points have more than 10^(-9) mass at age j=21

41 of 51 points have more than 10^(-9) mass at age j=22

41 of 51 points have more than 10^(-9) mass at age j=23

41 of 51 points have more than 10^(-9) mass at age j=24

41 of 51 points have more than 10^(-9) mass at age j=25

41 of 51 points have more than 10^(-9) mass at age j=26

41 of 51 points have more than 10^(-9) mass at age j=27

41 of 51 points have more than 10^(-9) mass at age j=28

41 of 51 points have more than 10^(-9) mass at age j=29

41 of 51 points have more than 10^(-9) mass at age j=30

41 of 51 points have more than 10^(-9) mass at age j=31

41 of 51 points have more than 10^(-9) mass at age j=32

41 of 51 points have more than 10^(-9) mass at age j=33

41 of 51 points have more than 10^(-9) mass at age j=34

41 of 51 points have more than 10^(-9) mass at age j=35

41 of 51 points have more than 10^(-9) mass at age j=36

41 of 51 points have more than 10^(-9) mass at age j=37

41 of 51 points have more than 10^(-9) mass at age j=38

41 of 51 points have more than 10^(-9) mass at age j=39

41 of 51 points have more than 10^(-9) mass at age j=40

41 of 51 points have more than 10^(-9) mass at age j=41

How much of z grid is used? (2/4)

Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=1

Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=11

```
Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=21
Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=31
Bottom 10 points sum to mass 0.000003, top 10 points sum to mass 0.000003, at age j=41
How much of z grid is used? (3/4)
Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=1
Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=11
Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=21
Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=31
Bottom 5 points sum to mass 0.000000, top 5 points sum to mass 0.000000, at age j=41
Range of z grid? (4/4)
Min grid point -2.315870, max grid point 2.315870, at age j=1
Min grid point -2.723582, max grid point 2.723582, at age j=11
Min grid point -2.909998, max grid point 2.909998, at age j=21
Min grid point -3.000811, max grid point 3.000811, at age j=31
Min grid point -3.046183, max grid point 3.046183, at age j=41
```

Now Figure 12 plots the stationary distribution of z for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), to see how much of the grid is being used.

Now Figure 13 plots the pdf and cdf of the innovations for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), and include the theoretical distribution being approximated.

Now some plots of the transition matrix for just a few ages. Figure 14 shows the transition matrix for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55) as both a pdf and cdf.

Note that because an extended Rouwenhourst method (the Fella-Gallipoli-Pan method) there is no heatmap.

A.2.1 Using Extended Farmer-Toda to target 2 moments

First, just some text to see if the upper end of the grid for z is being used.

How much of z grid is used? (1/4)

51 of 51 points have more than 10^(-9) mass at age j=1

51 of 51 points have more than 10^(-9) mass at age j=2

51 of 51 points have more than 10^(-9) mass at age j=3

51 of 51 points have more than 10^(-9) mass at age j=4

51 of 51 points have more than 10^(-9) mass at age j=5

51 of 51 points have more than 10^(-9) mass at age j=6

51 of 51 points have more than 10^(-9) mass at age j=7

51 of 51 points have more than 10^(-9) mass at age j=8

51 of 51 points have more than 10^(-9) mass at age j=9

51 of 51 points have more than 10^(-9) mass at age j=10

51 of 51 points have more than 10^(-9) mass at age j=11

51 of 51 points have more than 10^(-9) mass at age j=12

51 of 51 points have more than 10^(-9) mass at age j=13

51 of 51 points have more than 10^(-9) mass at age j=14

51 of 51 points have more than 10^(-9) mass at age j=15

51 of 51 points have more than 10^(-9) mass at age j=16

51 of 51 points have more than 10^(-9) mass at age j=17

51 of 51 points have more than 10^(-9) mass at age j=18

51 of 51 points have more than 10^(-9) mass at age j=19

51 of 51 points have more than 10^(-9) mass at age j=20

51 of 51 points have more than 10^(-9) mass at age j=21

51 of 51 points have more than 10^(-9) mass at age j=22

51 of 51 points have more than 10^(-9) mass at age j=23

51 of 51 points have more than 10^(-9) mass at age j=24

51 of 51 points have more than 10^(-9) mass at age j=25

51 of 51 points have more than 10^(-9) mass at age j=26

51 of 51 points have more than 10^(-9) mass at age j=27

51 of 51 points have more than 10^(-9) mass at age j=28

51 of 51 points have more than 10^(-9) mass at age j=29

51 of 51 points have more than 10^(-9) mass at age j=30

51 of 51 points have more than 10^(-9) mass at age j=31

51 of 51 points have more than 10^(-9) mass at age j=32

51 of 51 points have more than 10^(-9) mass at age j=33

51 of 51 points have more than 10^(-9) mass at age j=34

51 of 51 points have more than 10^(-9) mass at age j=35

51 of 51 points have more than 10^(-9) mass at age j=36

51 of 51 points have more than 10^(-9) mass at age j=37

51 of 51 points have more than 10^(-9) mass at age j=38

51 of 51 points have more than 10^(-9) mass at age j=39

51 of 51 points have more than 10^(-9) mass at age j=40

51 of 51 points have more than 10^(-9) mass at age j=41

How much of z grid is used? (2/4)

Bottom 10 points sum to mass 0.006520, top 10 points sum to mass 0.006520, at age j=1

Bottom 10 points sum to mass 0.006526, top 10 points sum to mass 0.006526, at age j=11

```
Bottom 10 points sum to mass 0.006526, top 10 points sum to mass 0.006526, at age j=21
Bottom 10 points sum to mass 0.006526, top 10 points sum to mass 0.006526, at age j=31
Bottom 10 points sum to mass 0.006526, top 10 points sum to mass 0.006526, at age j=41
How much of z grid is used? (3/4)
Bottom 5 points sum to mass 0.000519, top 5 points sum to mass 0.000519, at age j=1
Bottom 5 points sum to mass 0.000520, top 5 points sum to mass 0.000520, at age j=11
Bottom 5 points sum to mass 0.000519, top 5 points sum to mass 0.000519, at age j=21
Bottom 5 points sum to mass 0.000519, top 5 points sum to mass 0.000519, at age j=31
Bottom 5 points sum to mass 0.000519, top 5 points sum to mass 0.000519, at age j=41
Range of z grid? (4/4)
Min grid point -2.315870, max grid point 2.315870, at age j=1
Min grid point -2.723582, max grid point 2.723582, at age j=11
Min grid point -2.909998, max grid point 2.909998, at age j=21
Min grid point -3.000811, max grid point 3.000811, at age j=31
Min grid point -3.046183, max grid point 3.046183, at age j=41
```

Now Figure 15 plots the stationary distribution of z for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), to see how much of the grid is being used.

Now Figure 16 plots the pdf and cdf of the innovations for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), and include the theoretical distribution being approximated.

Now some plots of the transition matrix for just a few ages. Figure 17 shows the transition matrix for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55) as both a pdf and cdf.

Now Figure 18 plots a heatmap showing how many moments (of the conditional distributions, i.e., the distribution of the innovation) the extended Farmer-Toda method succeeds in targeting (note that it is trying to target 2).

A.2.2 Using Extended Farmer-Toda to target 4 moments

First, just some text to see if the upper end of the grid for z is being used.

How much of z grid is used? (1/4)

51 of 51 points have more than 10^(-9) mass at age j=1

51 of 51 points have more than 10^(-9) mass at age j=2

51 of 51 points have more than 10^(-9) mass at age j=3

51 of 51 points have more than 10^(-9) mass at age j=4

51 of 51 points have more than 10^(-9) mass at age j=5

51 of 51 points have more than 10^(-9) mass at age j=6

51 of 51 points have more than 10^(-9) mass at age j=7

51 of 51 points have more than 10^(-9) mass at age j=8

51 of 51 points have more than 10^(-9) mass at age j=9

51 of 51 points have more than 10^(-9) mass at age j=10

51 of 51 points have more than 10^(-9) mass at age j=11

51 of 51 points have more than 10^(-9) mass at age j=12

51 of 51 points have more than 10^(-9) mass at age j=13

51 of 51 points have more than 10^(-9) mass at age j=14

51 of 51 points have more than 10^(-9) mass at age j=15

51 of 51 points have more than 10^(-9) mass at age j=16

51 of 51 points have more than 10^(-9) mass at age j=17

51 of 51 points have more than 10^(-9) mass at age j=18

51 of 51 points have more than 10^(-9) mass at age j=19

51 of 51 points have more than 10^(-9) mass at age j=20

51 of 51 points have more than 10^(-9) mass at age j=21

51 of 51 points have more than 10^(-9) mass at age j=22

51 of 51 points have more than 10^(-9) mass at age j=23

51 of 51 points have more than 10^(-9) mass at age j=24

51 of 51 points have more than 10^(-9) mass at age j=25

51 of 51 points have more than 10^(-9) mass at age j=26

51 of 51 points have more than 10^(-9) mass at age j=27

51 of 51 points have more than 10^(-9) mass at age j=28

51 of 51 points have more than 10^(-9) mass at age j=29

51 of 51 points have more than 10^(-9) mass at age j=30

51 of 51 points have more than 10^(-9) mass at age j=31

51 of 51 points have more than 10^(-9) mass at age j=32

51 of 51 points have more than 10^(-9) mass at age j=33

51 of 51 points have more than 10^(-9) mass at age j=34

51 of 51 points have more than 10^(-9) mass at age j=35

51 of 51 points have more than 10^(-9) mass at age j=36

51 of 51 points have more than 10^(-9) mass at age j=37

51 of 51 points have more than 10^(-9) mass at age j=38

51 of 51 points have more than 10^(-9) mass at age j=39

51 of 51 points have more than 10^(-9) mass at age j=40

51 of 51 points have more than 10^(-9) mass at age j=41

How much of z grid is used? (2/4)

Bottom 10 points sum to mass 0.006521, top 10 points sum to mass 0.006521, at age j=1

Bottom 10 points sum to mass 0.006523, top 10 points sum to mass 0.006524, at age j=11

Bottom 10 points sum to mass 0.006524, top 10 points sum to mass 0.006524, at age j=21

Bottom 10 points sum to mass 0.006524, top 10 points sum to mass 0.006524, at age j=31

Bottom 10 points sum to mass 0.006524, top 10 points sum to mass 0.006524, at age j=41

How much of z grid is used? (3/4)

Bottom 5 points sum to mass 0.000511, top 5 points sum to mass 0.000511, at age j=1

Bottom 5 points sum to mass 0.000515, top 5 points sum to mass 0.000515, at age j=11

Bottom 5 points sum to mass 0.000516, top 5 points sum to mass 0.000516, at age j=21

Bottom 5 points sum to mass 0.000516, top 5 points sum to mass 0.000516, at age j=31

Bottom 5 points sum to mass 0.000516, top 5 points sum to mass 0.000516, at age j=41

Range of z grid? (4/4)

Min grid point -2.315870, max grid point 2.315870, at age j=1

Min grid point -2.723582, max grid point 2.723582, at age j=11

Min grid point -2.909998, max grid point 2.909998, at age j=21

Min grid point -3.000811, max grid point 3.000811, at age j=31

Min grid point -3.046183, max grid point 3.046183, at age j=41

Now Figure 19 plots the stationary distribution of z for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), to see how much of the grid is being used.

Now Figure 20 plots the pdf and cdf of the innovations for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), and include the theoretical distribution being approximated.

Now some plots of the transition matrix for just a few ages. Figure 21 shows the transition matrix for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55) as both a pdf and cdf.

Now Figure 22 plots a heatmap showing how many moments (of the conditional distributions, i.e., the distribution of the innovation) the extended Farmer-Toda method succeeds in targeting (note that it is here trying to target 4).

A.3 Model 6: AR(1) with Gaussian-Mixture innovations, nSigmas=3

First, just some text to see if the upper end of the grid for z is being used.

How much of z grid is used? (1/4)

51 of 51 points have more than 10^(-9) mass at age j=1

51 of 51 points have more than 10^(-9) mass at age j=2

51 of 51 points have more than 10^(-9) mass at age j=3

51 of 51 points have more than 10^(-9) mass at age j=4

51 of 51 points have more than 10^(-9) mass at age j=5

51 of 51 points have more than 10^(-9) mass at age j=6

51 of 51 points have more than 10^(-9) mass at age j=7

51 of 51 points have more than 10^(-9) mass at age j=8

51 of 51 points have more than 10^(-9) mass at age j=9

51 of 51 points have more than 10^(-9) mass at age j=10

51 of 51 points have more than 10^(-9) mass at age j=11

51 of 51 points have more than 10^(-9) mass at age j=12

51 of 51 points have more than 10^(-9) mass at age j=13

51 of 51 points have more than 10^(-9) mass at age j=14

51 of 51 points have more than 10^(-9) mass at age j=15

51 of 51 points have more than 10^(-9) mass at age j=16

51 of 51 points have more than 10^(-9) mass at age j=17

51 of 51 points have more than 10^(-9) mass at age j=18

51 of 51 points have more than 10^(-9) mass at age j=19

51 of 51 points have more than 10^(-9) mass at age j=20

51 of 51 points have more than 10^(-9) mass at age j=21

51 of 51 points have more than 10^(-9) mass at age j=22

51 of 51 points have more than 10^(-9) mass at age j=23

51 of 51 points have more than 10^(-9) mass at age j=24

51 of 51 points have more than 10^(-9) mass at age j=25

51 of 51 points have more than 10^(-9) mass at age j=26

51 of 51 points have more than 10^(-9) mass at age j=27

51 of 51 points have more than 10^(-9) mass at age j=28

51 of 51 points have more than 10^(-9) mass at age j=29

51 of 51 points have more than 10^(-9) mass at age j=30

51 of 51 points have more than 10^(-9) mass at age j=31

51 of 51 points have more than 10^(-9) mass at age j=32

51 of 51 points have more than 10^(-9) mass at age j=33

51 of 51 points have more than 10^(-9) mass at age j=34

51 of 51 points have more than 10^(-9) mass at age j=35

51 of 51 points have more than 10^(-9) mass at age j=36

51 of 51 points have more than 10^(-9) mass at age j=37

51 of 51 points have more than 10^(-9) mass at age j=38

51 of 51 points have more than 10^(-9) mass at age j=39

51 of 51 points have more than 10^(-9) mass at age j=40

51 of 51 points have more than 10^(-9) mass at age j=41

How much of z grid is used? (2/4)

Bottom 10 points sum to mass 0.033363, top 10 points sum to mass 0.028064, at age j=1

Bottom 10 points sum to mass 0.036487, top 10 points sum to mass 0.027066, at age j=11

```
Bottom 10 points sum to mass 0.036928, top 10 points sum to mass 0.026631, at age j=21
Bottom 10 points sum to mass 0.036924, top 10 points sum to mass 0.026619, at age j=31
Bottom 10 points sum to mass 0.036884, top 10 points sum to mass 0.026656, at age j=41
How much of z grid is used? (3/4)
Bottom 5 points sum to mass 0.008383, top 5 points sum to mass 0.007211, at age j=1
Bottom 5 points sum to mass 0.011246, top 5 points sum to mass 0.005543, at age j=11
Bottom 5 points sum to mass 0.011231, top 5 points sum to mass 0.005226, at age j=21
Bottom 5 points sum to mass 0.011137, top 5 points sum to mass 0.005193, at age j=31
Bottom 5 points sum to mass 0.011081, top 5 points sum to mass 0.005160, at age j=41
Range of z grid? (4/4)
Min grid point -2.185182, max grid point 2.185182, at age j=1
Min grid point -2.447119, max grid point 2.447119, at age j=11
Min grid point -2.552184, max grid point 2.552184, at age j=21
Min grid point -2.596347, max grid point 2.596347, at age j=31
Min grid point -2.615233, max grid point 2.615233, at age j=41
```

Now Figure 23 plots the stationary distribution of z for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), to see how much of the grid is being used.

Now Figure 24 plots the pdf and cdf of the innovations for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), and include the theoretical distribution being approximated.

Now some plots of the transition matrix for just a few ages. Figure 25 shows the transition matrix for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55) as both a pdf and cdf.

Now Figure 26 plots a heatmap showing how many moments (of the conditional distributions, i.e., the distribution of the innovation) the extended Farmer-Toda method succeeds in targeting (note that it is here trying to target 4).

A.4 Model 6: AR(1) with Gaussian-Mixture innovations, nSigmas=4

First, just some text to see if the upper end of the grid for z is being used.

How much of z grid is used? (1/4)

51 of 51 points have more than 10^(-9) mass at age j=1

51 of 51 points have more than 10^(-9) mass at age j=2

51 of 51 points have more than 10^(-9) mass at age j=3

51 of 51 points have more than 10^(-9) mass at age j=4

51 of 51 points have more than 10^(-9) mass at age j=5

51 of 51 points have more than 10^(-9) mass at age j=6

51 of 51 points have more than 10^(-9) mass at age j=7

51 of 51 points have more than 10^(-9) mass at age j=8

51 of 51 points have more than 10^(-9) mass at age j=9

51 of 51 points have more than 10^(-9) mass at age j=10

51 of 51 points have more than 10^(-9) mass at age j=11

51 of 51 points have more than 10^(-9) mass at age j=12

51 of 51 points have more than 10^(-9) mass at age j=13

51 of 51 points have more than 10^(-9) mass at age j=14

51 of 51 points have more than 10^(-9) mass at age j=15

51 of 51 points have more than 10^(-9) mass at age j=16

51 of 51 points have more than 10^(-9) mass at age j=17

51 of 51 points have more than 10^(-9) mass at age j=18

51 of 51 points have more than 10^(-9) mass at age j=19

51 of 51 points have more than 10^(-9) mass at age j=20

51 of 51 points have more than 10^(-9) mass at age j=21

51 of 51 points have more than 10^(-9) mass at age j=22

51 of 51 points have more than 10^(-9) mass at age j=23

51 of 51 points have more than 10^(-9) mass at age j=24

51 of 51 points have more than 10^(-9) mass at age j=25

51 of 51 points have more than 10^(-9) mass at age j=26

51 of 51 points have more than 10^(-9) mass at age j=27

51 of 51 points have more than 10^(-9) mass at age j=28

51 of 51 points have more than 10^(-9) mass at age j=29

51 of 51 points have more than 10^(-9) mass at age j=30

51 of 51 points have more than 10^(-9) mass at age j=31

51 of 51 points have more than 10^(-9) mass at age j=32

51 of 51 points have more than 10^(-9) mass at age j=33

51 of 51 points have more than 10^(-9) mass at age j=34

51 of 51 points have more than 10^(-9) mass at age j=35

51 of 51 points have more than 10^(-9) mass at age j=36

51 of 51 points have more than 10^(-9) mass at age j=37

51 of 51 points have more than 10^(-9) mass at age j=38

51 of 51 points have more than 10^(-9) mass at age j=39

51 of 51 points have more than 10^(-9) mass at age j=40

51 of 51 points have more than 10^(-9) mass at age j=41

How much of z grid is used? (2/4)

Bottom 10 points sum to mass 0.007049, top 10 points sum to mass 0.006844, at age j=1

Bottom 10 points sum to mass 0.009631, top 10 points sum to mass 0.005028, at age j=11

Bottom 10 points sum to mass 0.009811, top 10 points sum to mass 0.004776, at age j=21

Bottom 10 points sum to mass 0.009783, top 10 points sum to mass 0.004723, at age j=31

Bottom 10 points sum to mass 0.009749, top 10 points sum to mass 0.004717, at age j=41

How much of z grid is used? (3/4)

Bottom 5 points sum to mass 0.000682, top 5 points sum to mass 0.000515, at age j=1

Bottom 5 points sum to mass 0.001329, top 5 points sum to mass 0.000372, at age j=11

Bottom 5 points sum to mass 0.001363, top 5 points sum to mass 0.000333, at age j=21

Bottom 5 points sum to mass 0.001348, top 5 points sum to mass 0.000322, at age j=31

Bottom 5 points sum to mass 0.001336, top 5 points sum to mass 0.000320, at age j=41

Range of z grid? (4/4)

Min grid point -2.913576, max grid point 2.913576, at age j=1

Min grid point -3.262826, max grid point 3.262826, at age j=11

Min grid point -3.402912, max grid point 3.402912, at age j=21

Min grid point -3.461796, max grid point 3.461796, at age j=31

Min grid point -3.486977, max grid point 3.486977, at age j=41

Now Figure 27 plots the stationary distribution of z for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), to see how much of the grid is being used.

Now Figure 28 plots the pdf and cdf of the innovations for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55), and include the theoretical distribution being approximated.

Now some plots of the transition matrix for just a few ages. Figure 29 shows the transition matrix for periods 1, 11, 21, and 31 (ages 25, 35, 45, and 55) as both a pdf and cdf.

Now Figure 30 plots a heatmap showing how many moments (of the conditional distributions, i.e., the distribution of the innovation) the extended Farmer-Toda method succeeds in targeting (note that it is here trying to target 4).

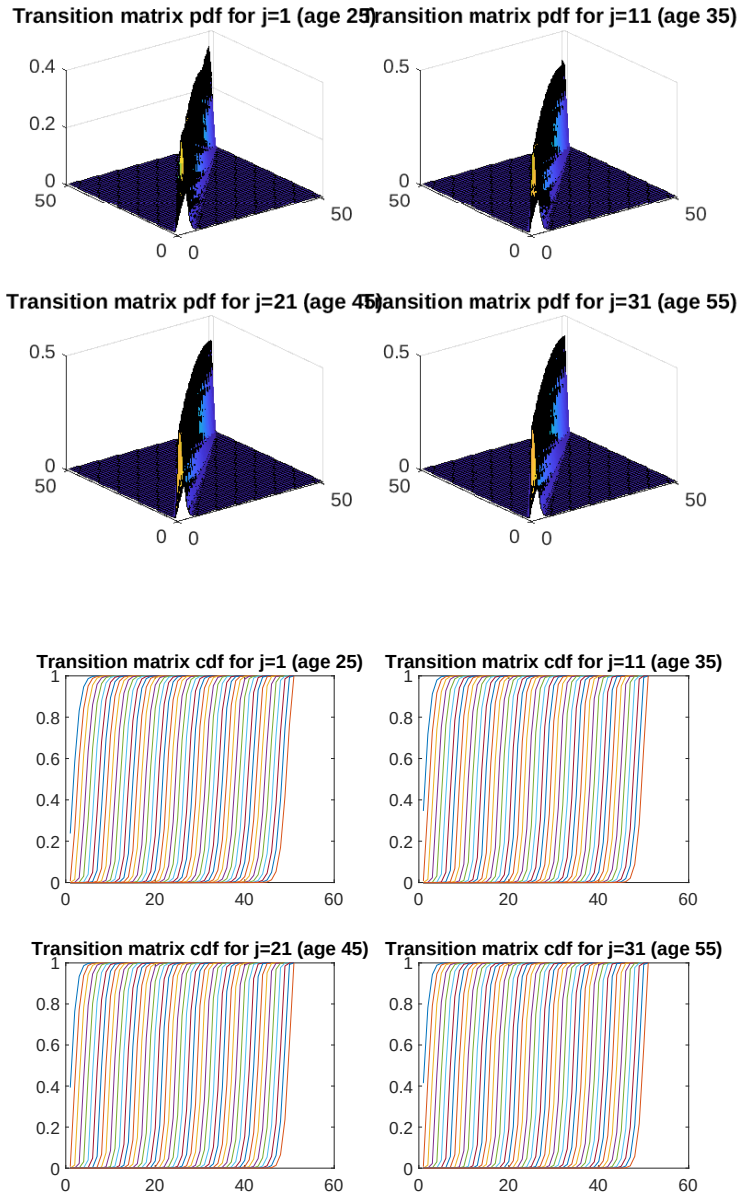

Figure 3: Discretization of Model 2 with nSigma=3: transition matrix of z as pdf and cdf

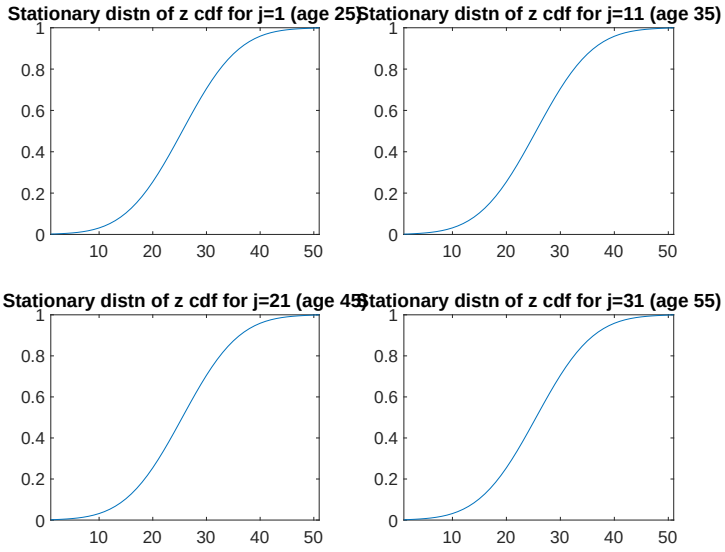

Figure 4: Discretization of Model 2 with nSigma=3: Stationary distribution of z

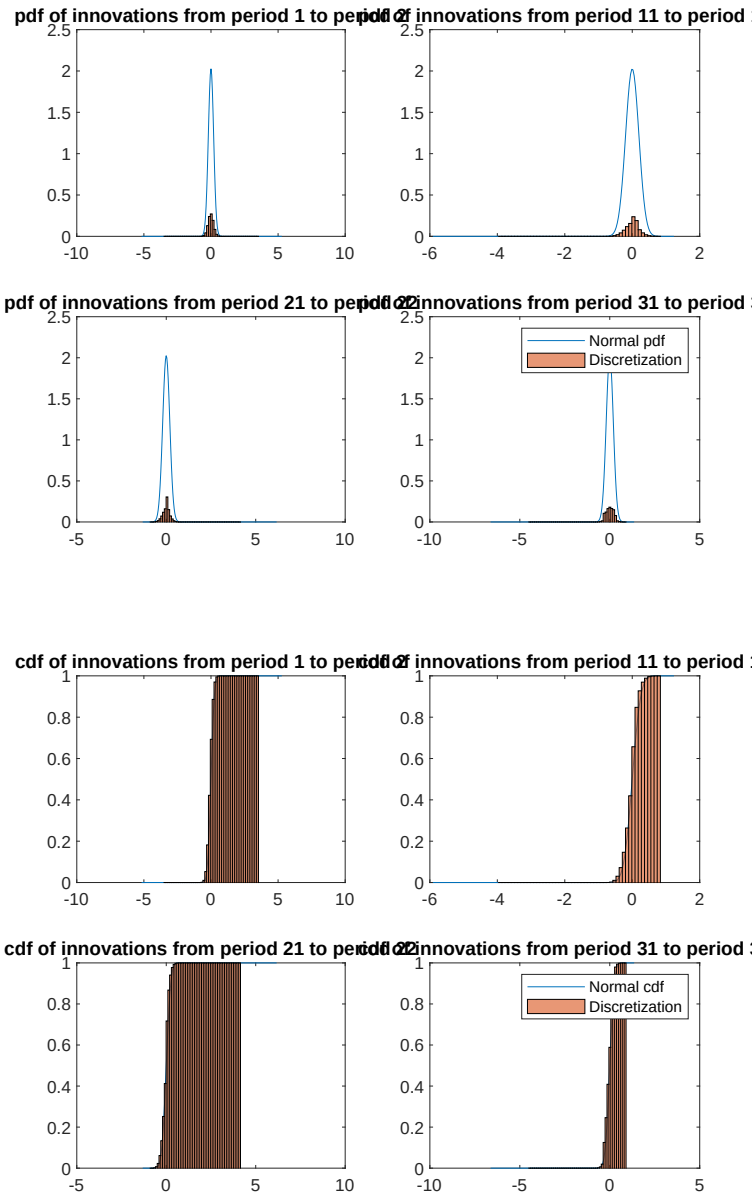

Figure 5: Discretization of Model 2 with nSigma=3: innovations to z

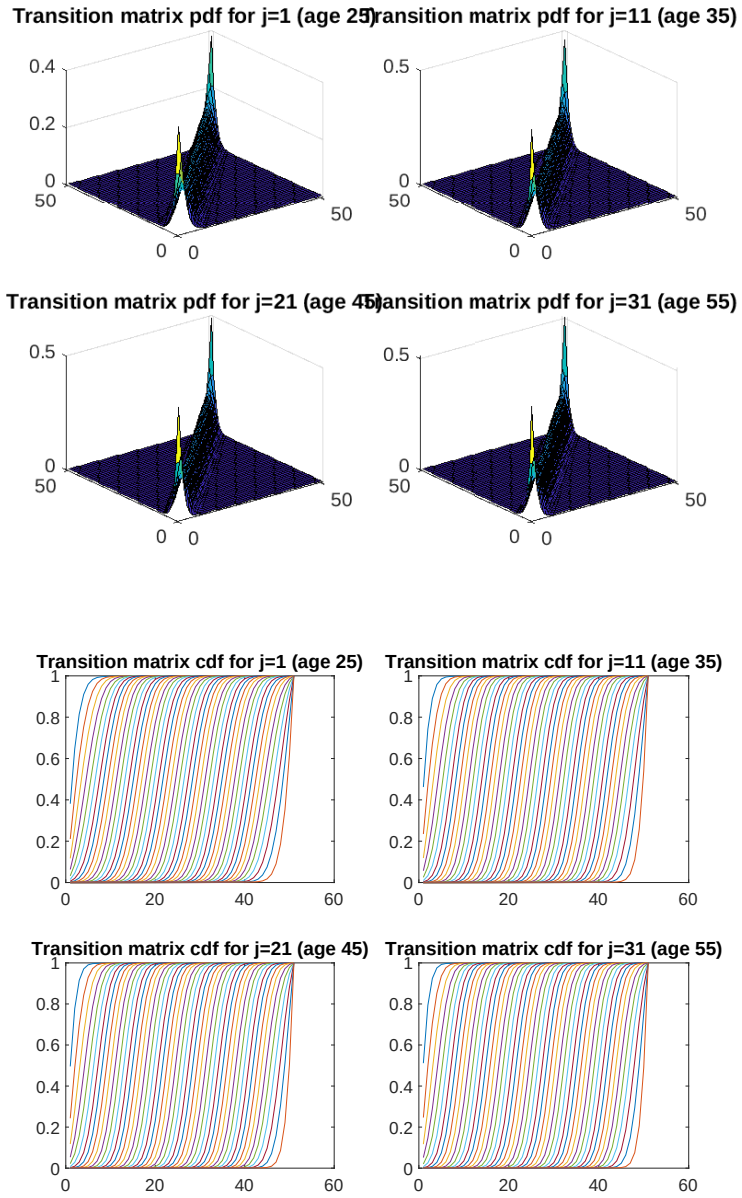

Figure 6: Discretization of Model 2 with nSigma=3: transition matrix of z as pdf and cdf

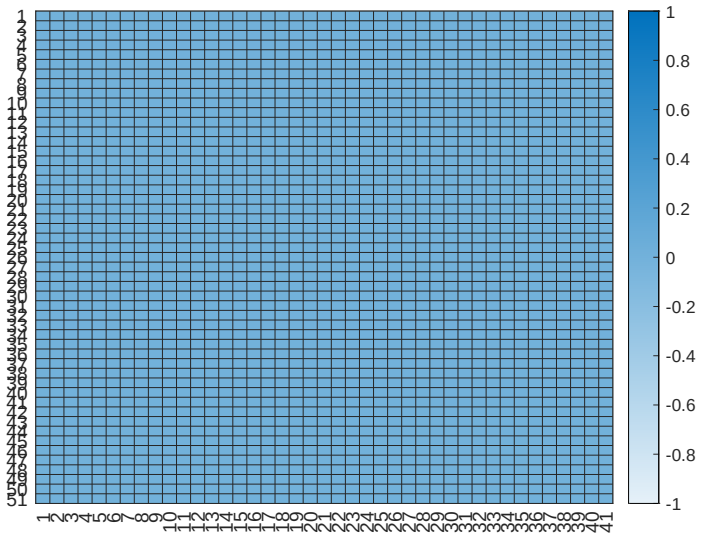

Figure 7: Discretization of Model 2 with nSigma=3: Heatmap of moments targeted

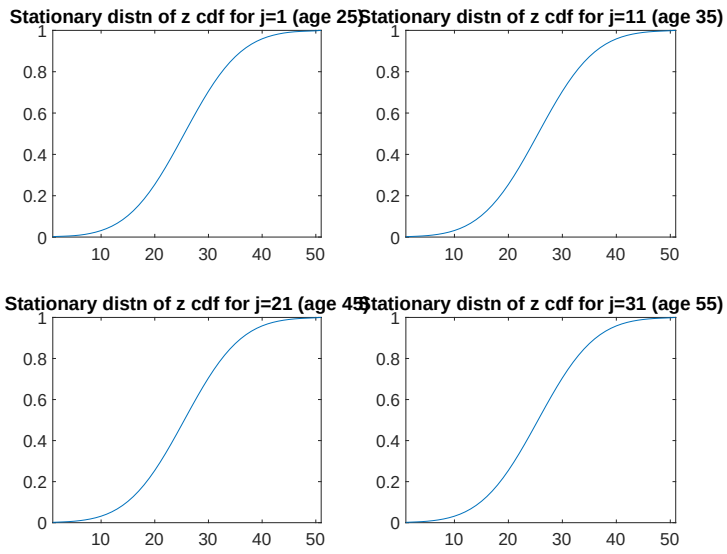

Figure 8: Discretization of Model 2 with nSigma=3: Stationary distribution of z

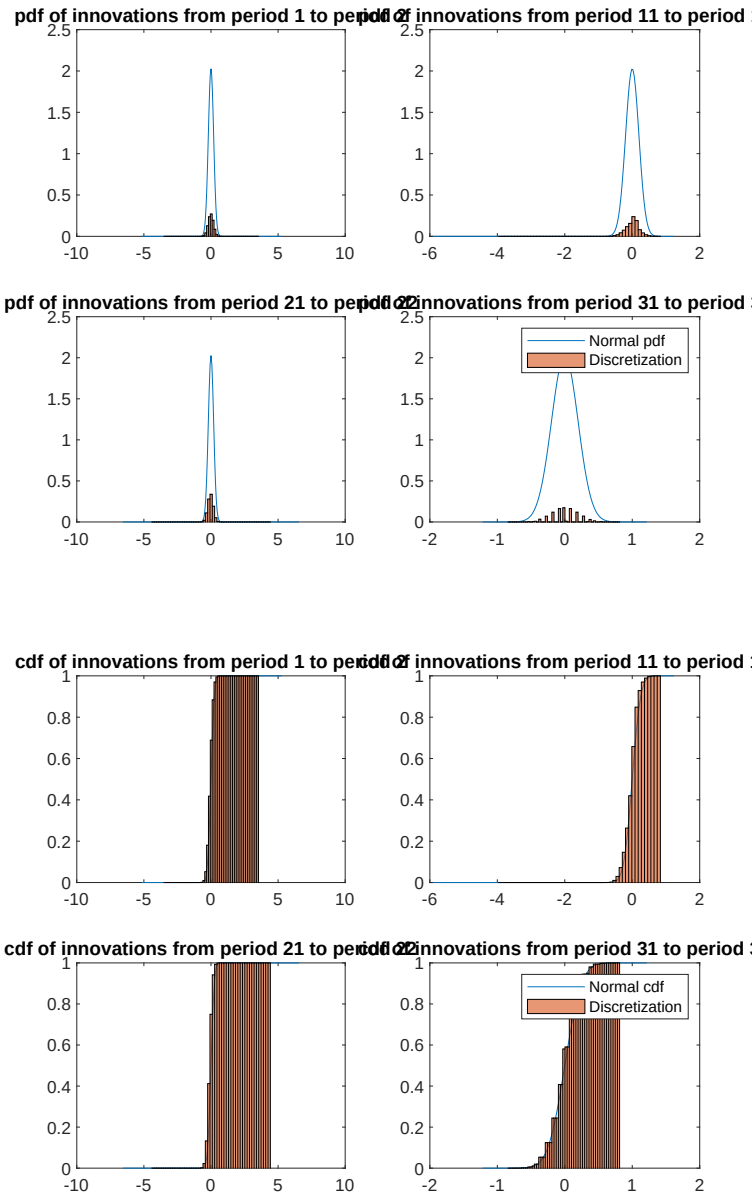

Figure 9: Discretization of Model 2 with nSigma=3: innovations to z

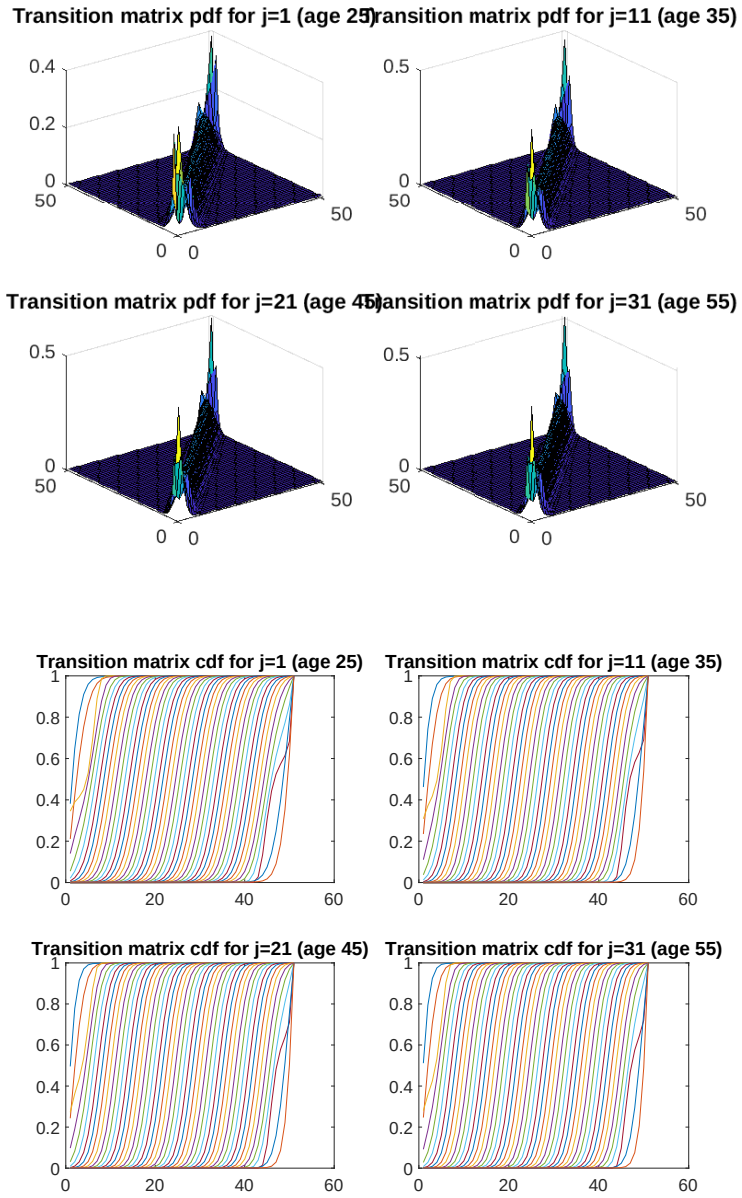

Figure 10: Discretization of Model 2 with nSigma=3: transition matrix of z as pdf and cdf

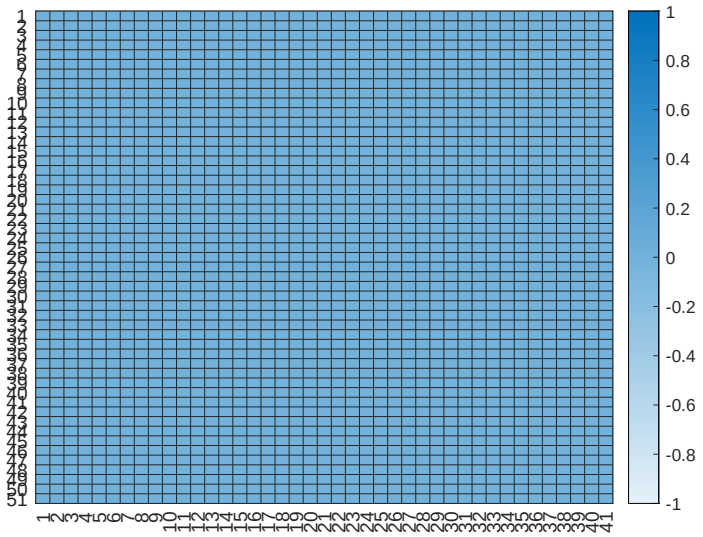

Figure 11: Discretization of Model 2 with nSigma=3: Heatmap of moments targeted

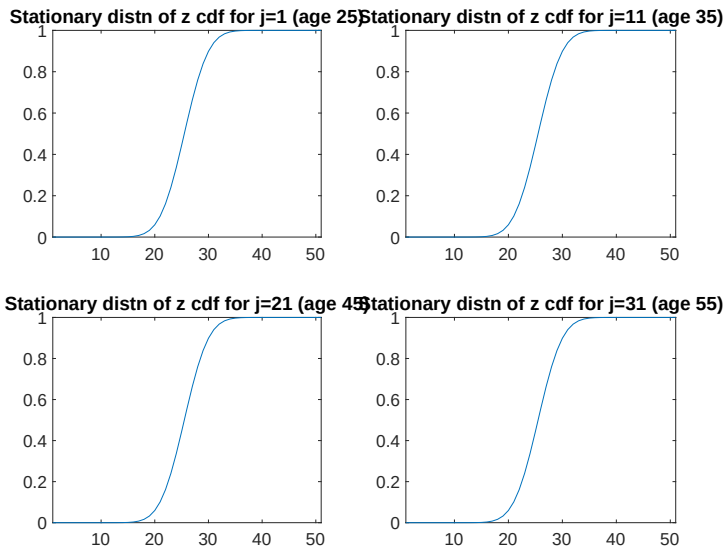

Figure 12: Discretization of Model 2 with nSigma=4: Stationary distribution of z

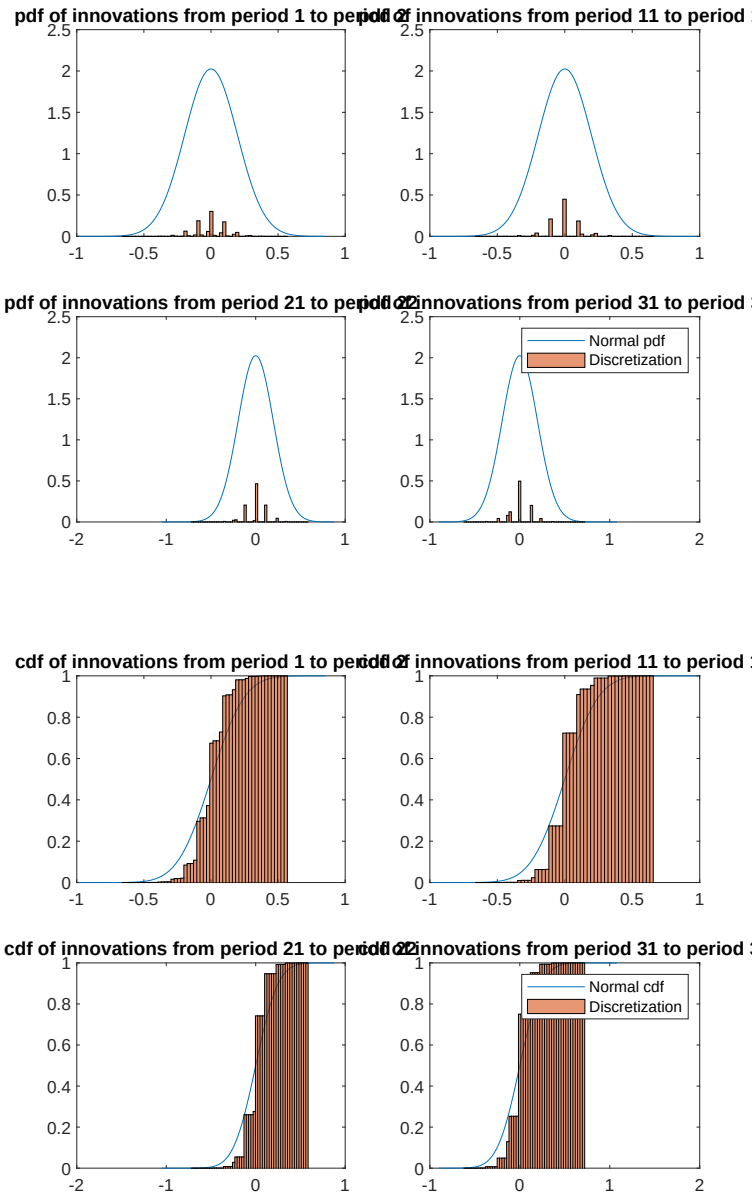

Figure 13: Discretization of Model 2 with nSigma=4: innovations to z

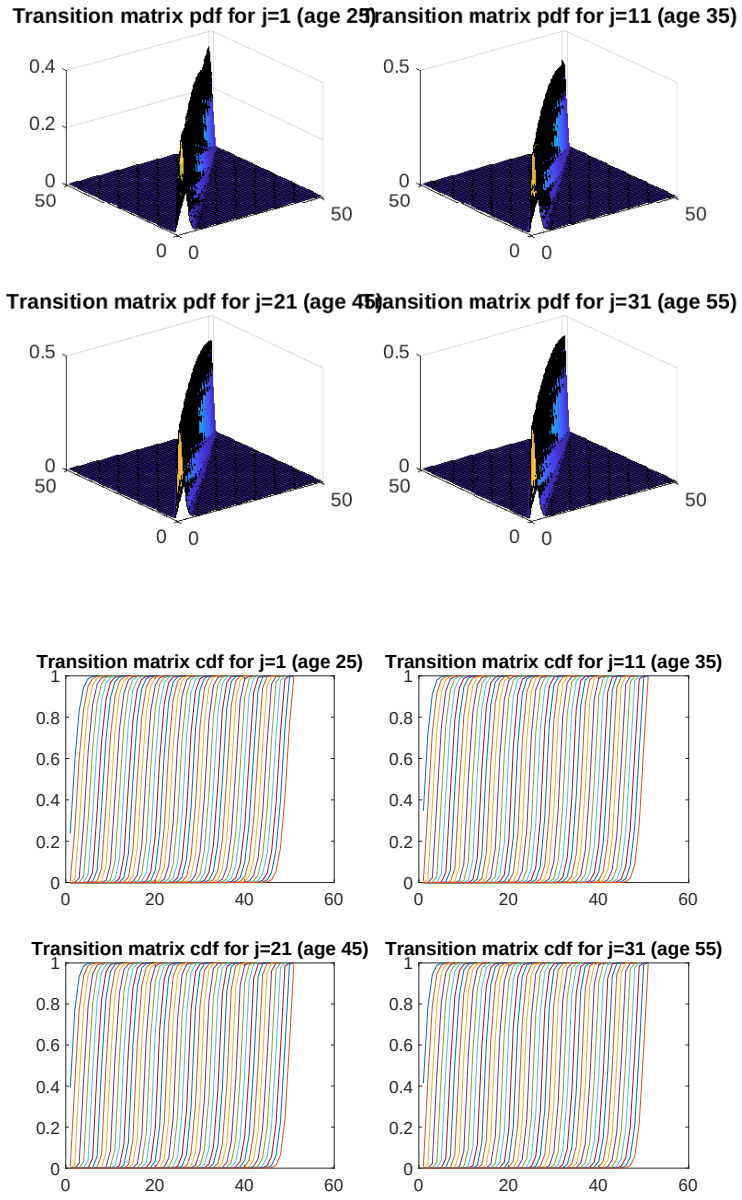

Figure 14: Discretization of Model 2 with nSigma=4: transition matrix of z as pdf and cdf

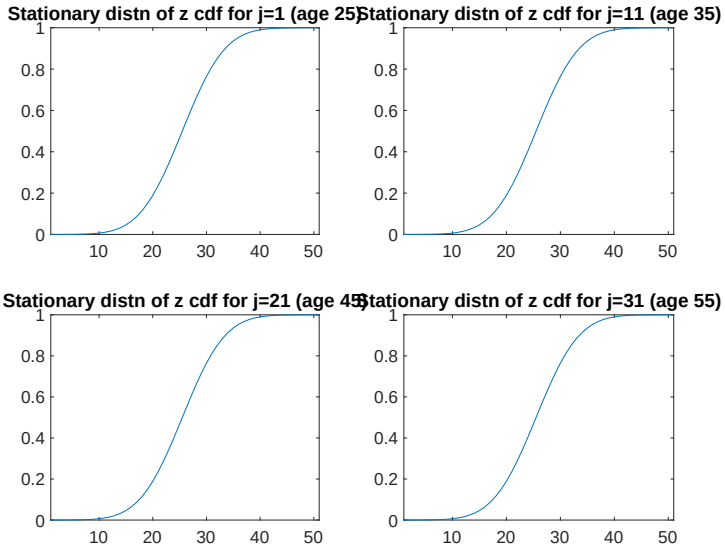

Figure 15: Discretization of Model 2 with nSigma=4: Stationary distribution of z

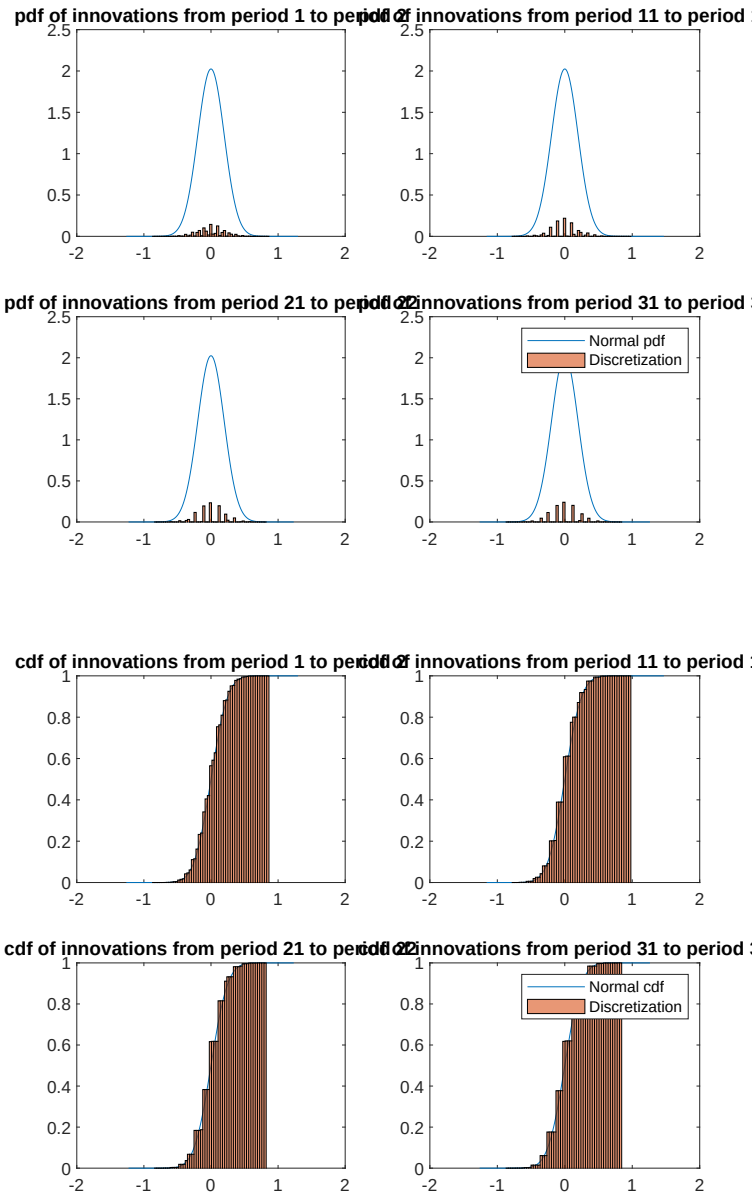

Figure 16: Discretization of Model 2 with nSigma=4: innovations to z

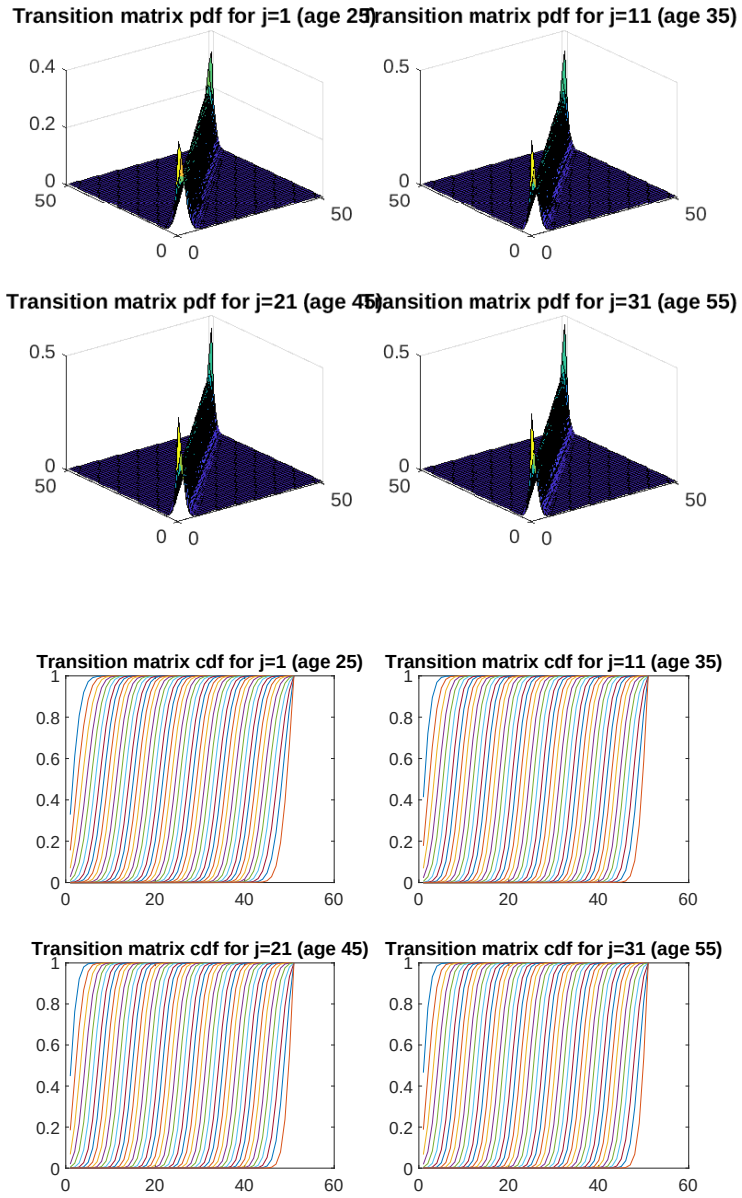

Figure 17: Discretization of Model 2 with nSigma=4: transition matrix of z as pdf and cdf

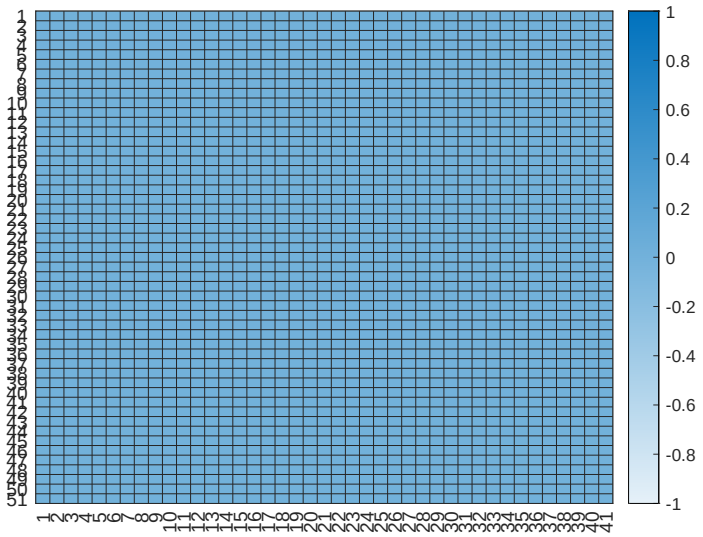

Figure 18: Discretization of Model 2 with nSigma=4: Heatmap of moments targeted

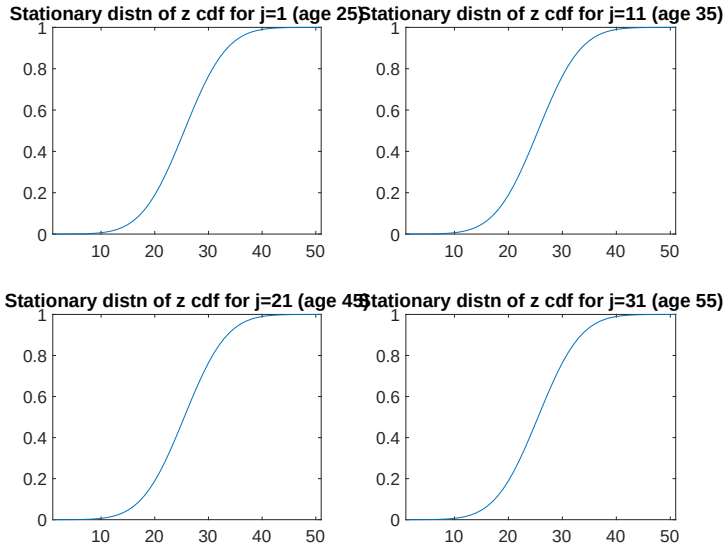

Figure 19: Discretization of Model 2 with nSigma=4: Stationary distribution of z

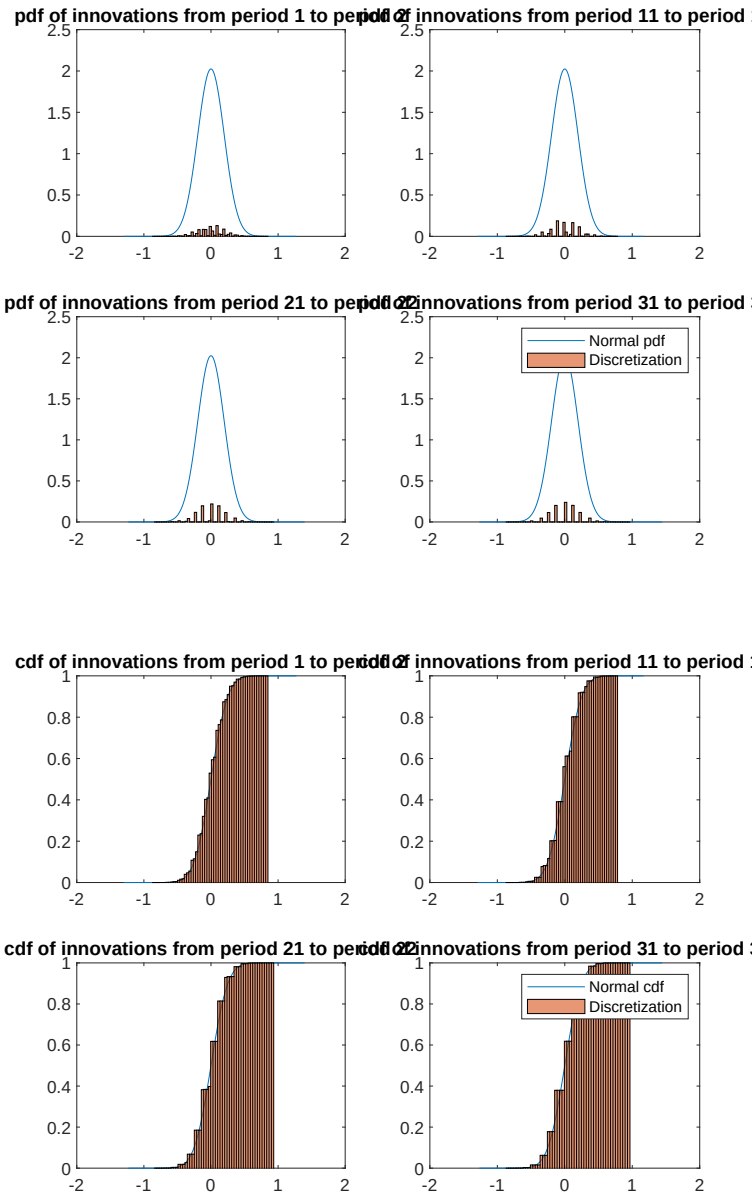

Figure 20: Discretization of Model 2 with nSigma=4: innovations to z

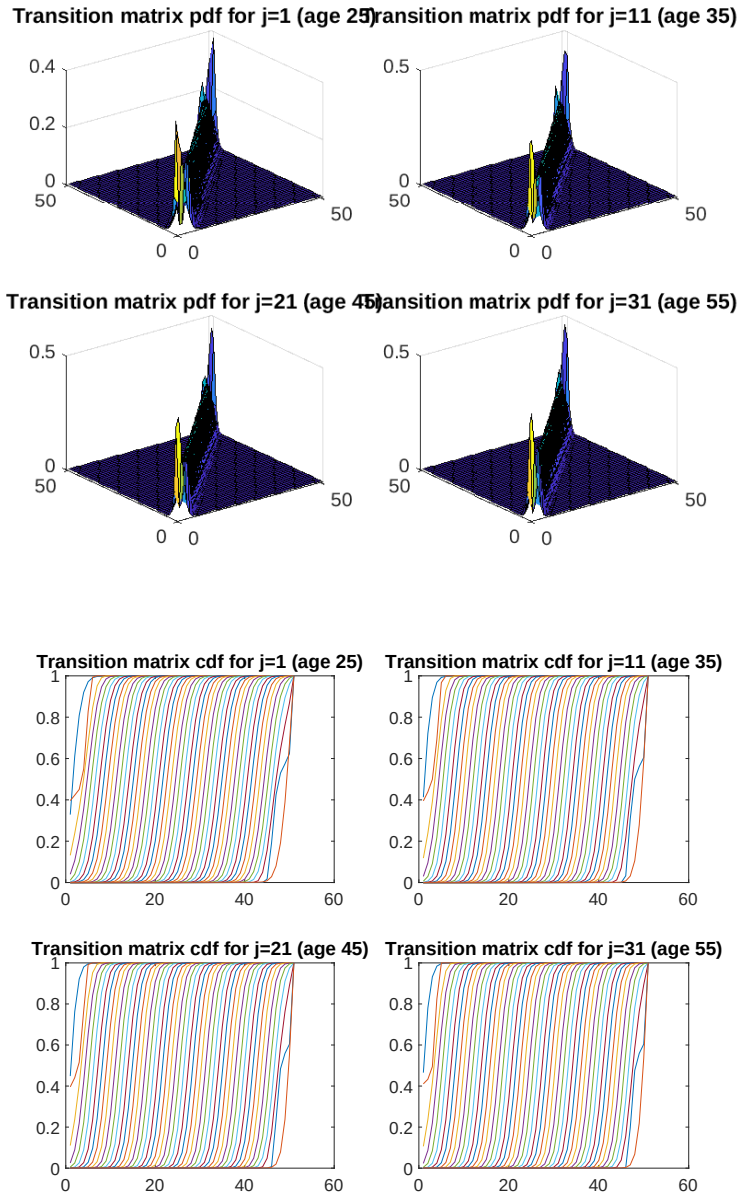

Figure 21: Discretization of Model 2 with nSigma=4: transition matrix of z as pdf and cdf

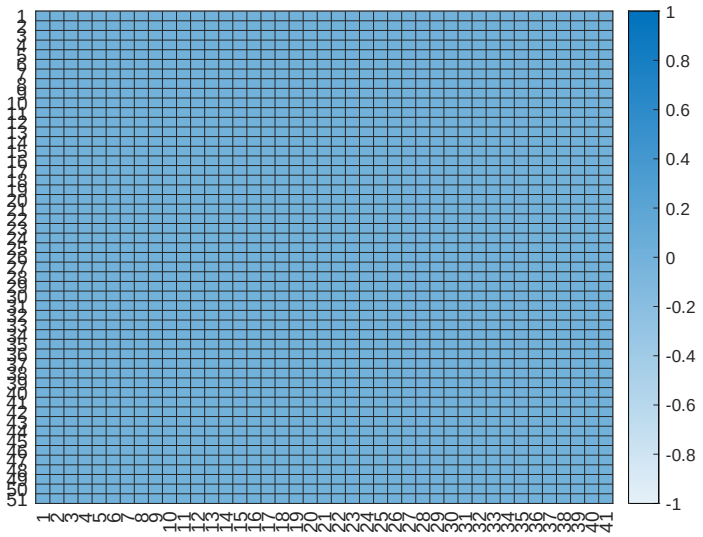

Figure 22: Discretization of Model 2 with nSigma=4: Heatmap of moments targeted

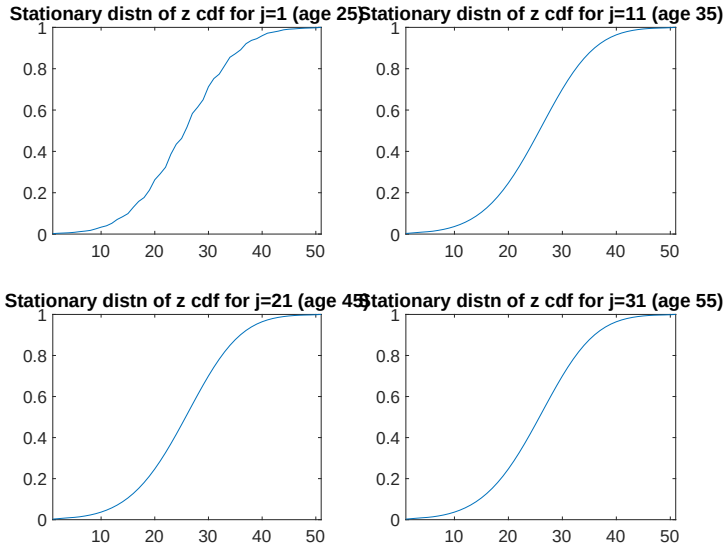

Figure 23: Discretization of Model 2 with nSigma=3: Stationary distribution of z

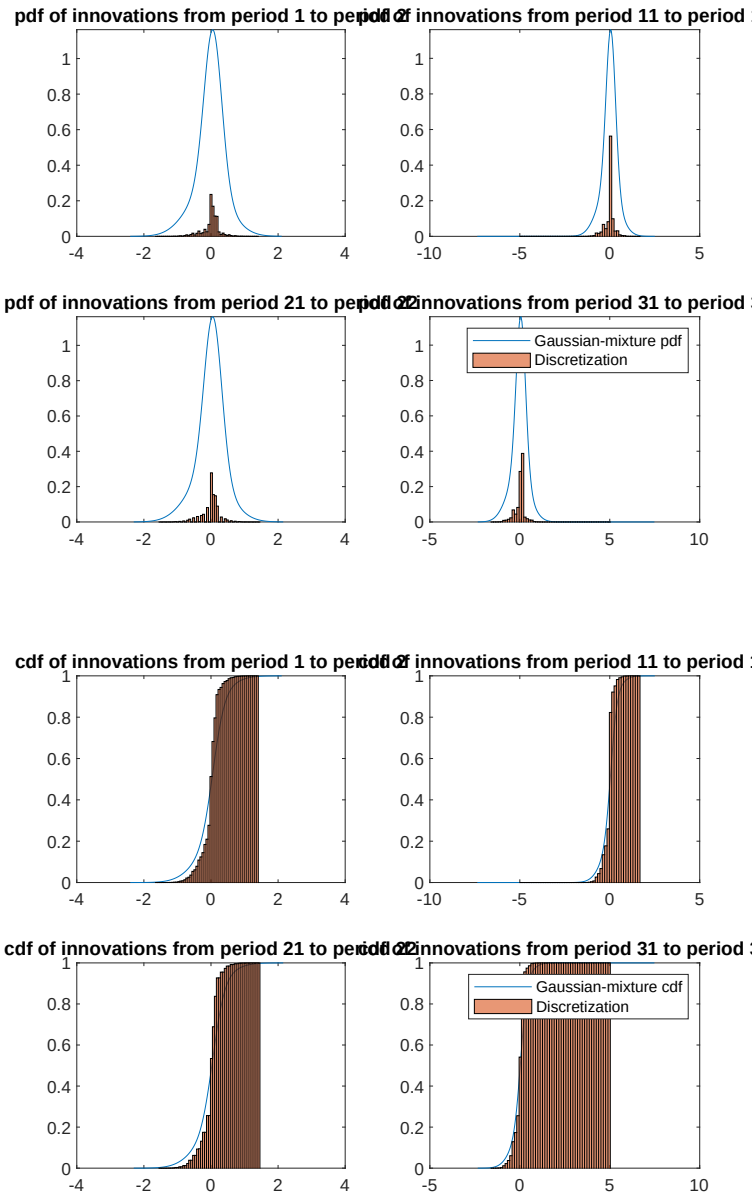

Figure 24: Discretization of Model 2 with nSigma=3: innovations to z

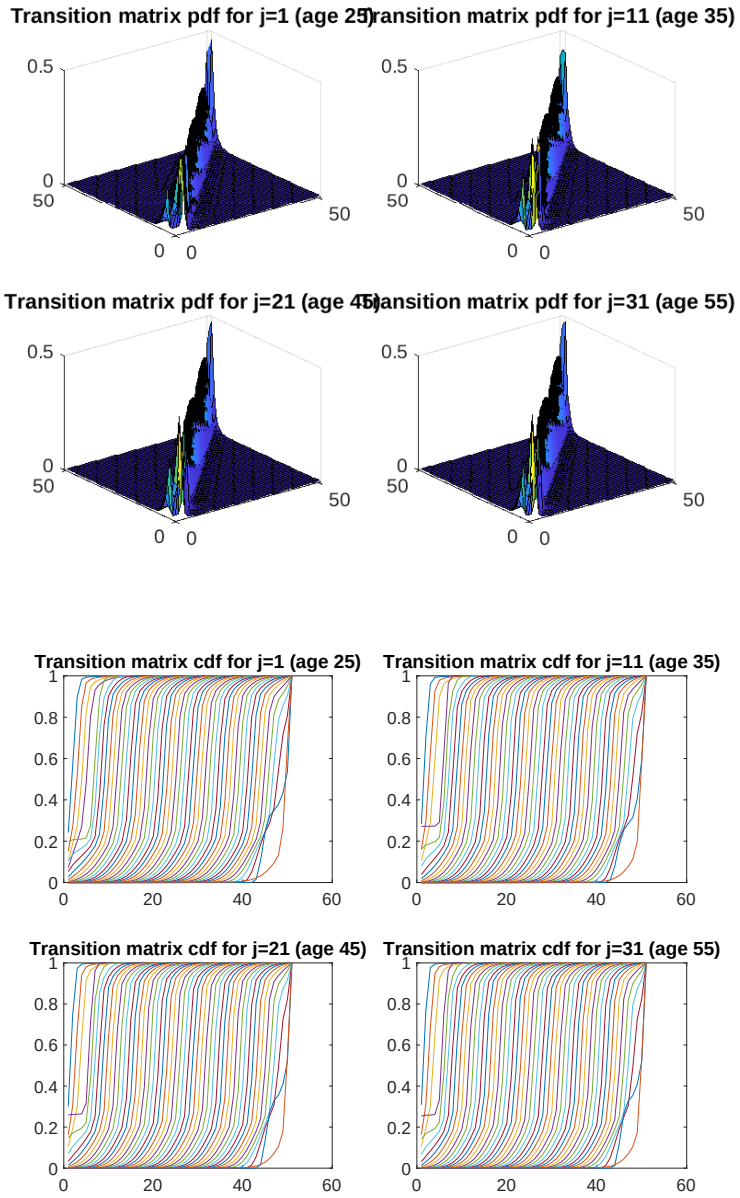

Figure 25: Discretization of Model 2 with nSigma=3: transition matrix of z as pdf and cdf

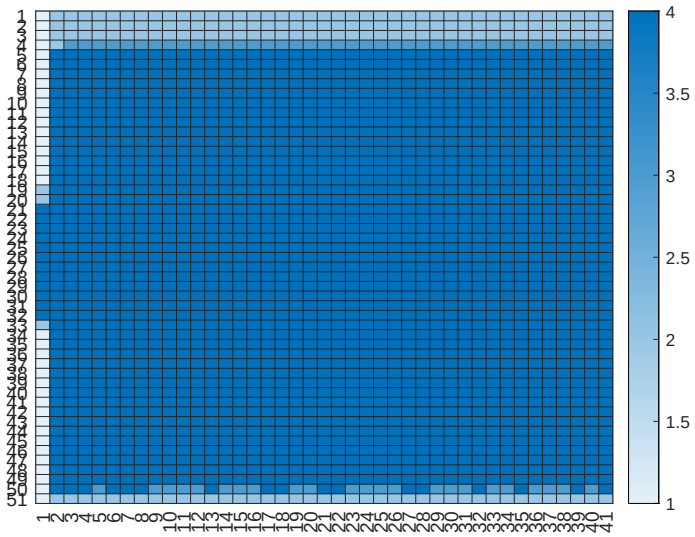

Figure 26: Discretization of Model 6 with nSigma=3: Heatmap of moments targeted

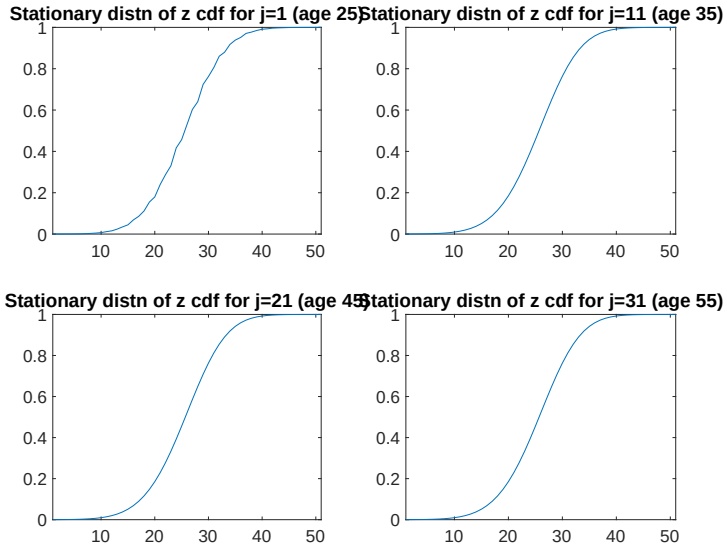

Figure 27: Discretization of Model 2 with nSigma=4: Stationary distribution of z

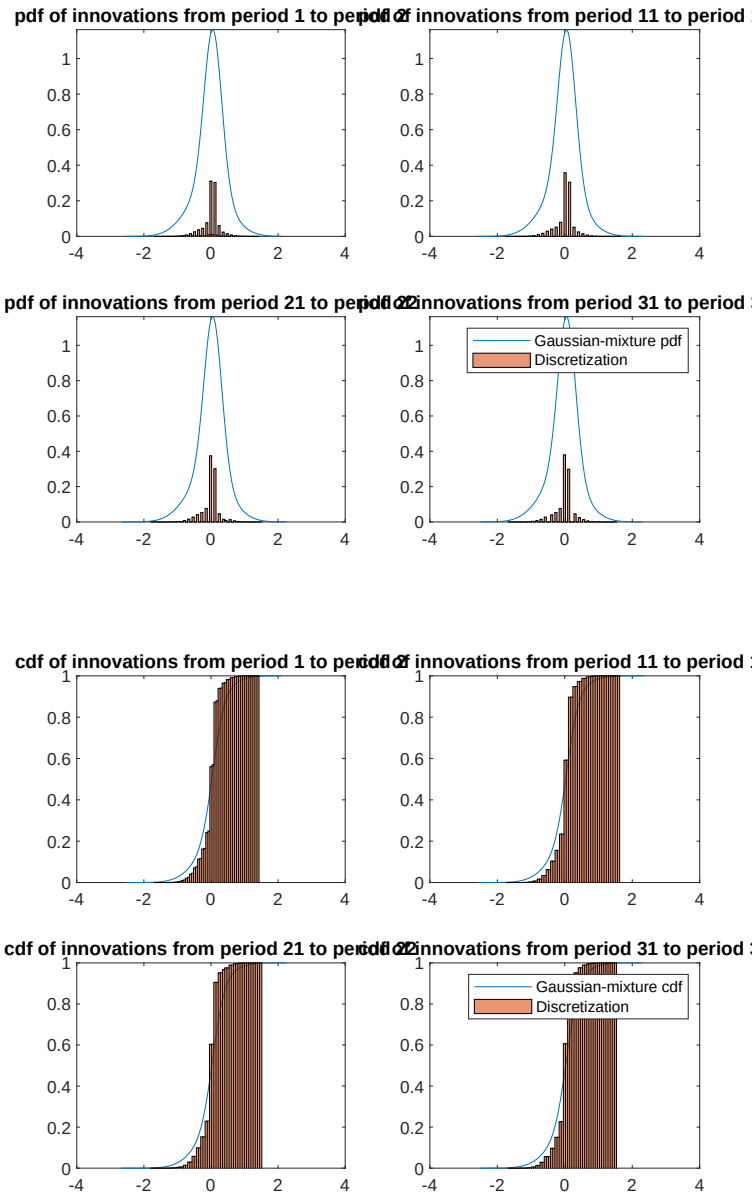

Figure 28: Discretization of Model 2 with nSigma=4: innovations to z

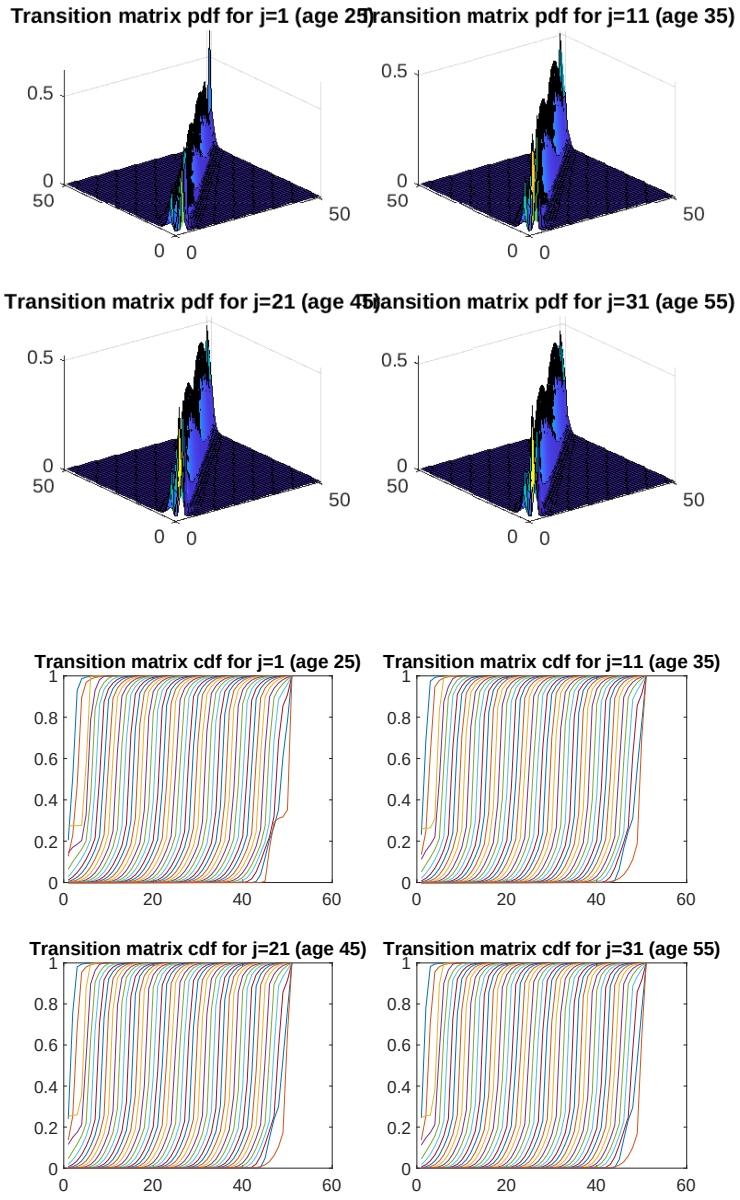

Figure 29: Discretization of Model 2 with nSigma=4: transition matrix of z as pdf and cdf

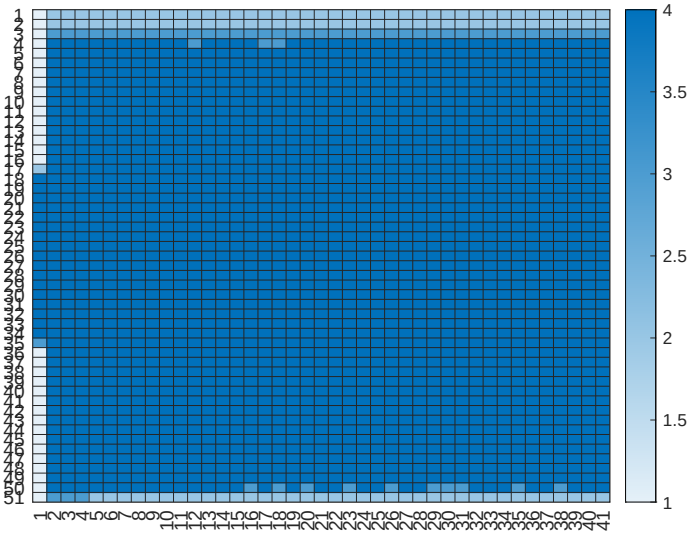

Figure 30: Discretization of Model 6 with nSigma=4: Heatmap of moments targeted