Fluisim

Smoothed Particle Hydrodynamics

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Introduction

Smoothed Particle Hydrodynamics (SPH) is a computational method used for simulating fluid flows. Unlike traditional grid-based methods, SPH is a meshless Lagrangian particle method where the fluid is represented by a set of discrete particles that carry physical properties such as mass, density, and velocity.

The fundamental principle of SPH lies in its interpolation method, where any physical quantity of a particle is approximated by summing the contributions from neighboring particles weighted by a smoothing kernel function. This kernel function determines how strongly neighboring particles influence each other based on their distance.

1 Kernel of Interpolation

The kernel is used to determine how particles interact with each other, and how physical quantities of a continuous field are interpolated from discrete particle data.

Definition 1.1. A *smoothing kernel* is a continuous non-negative compactly-supported function

$$\omega : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ : (h, r) \mapsto \omega(h, r)$$

where h is the radius of support, and r is the distance between influencing particle and field quantity.

While a third property, normalisation, is commonly required, we defer this requirement to the derivation of a field kernel.

Definition 1.2. A smoothing kernel is normalized if the volume of the kernel over its support is unital.

$$\int_0^h \omega(h, r)\,\mathrm{d}r = 1$$

Example 1.3 The reference example of a smoothing kernel is the "poly6" kernel.

$$\omega_{\text{poly6}}(h, r) = \begin{cases} (h^2 - r^2)^3 & \text{if} \quad 0 \le r \le h \\ 0 & \text{otherwise} \end{cases}$$

The work by **[muller03]** evaluates this kernel and proposes a new kernel capable of simulating viscosity forces.

2 Field Influence Kernel

For the purposes of making the implementation of smoothing kernels easier, we now define a field influence function. The purpose of a field influence function is to symmetrically rotate and normalise the smoothing kernel over N dimensions of space.

Definition 2.1. The volume of an N-ball of radius r is calculated as:

$$B(N, r) \coloneqq \frac{\pi^{N/2}}{\Gamma\left(1 + \frac{N}{2}\right)} \cdot r^N$$

Definition 2.2. The volume of the kernel rotated symmetrically through N dimensions of space is calculated as:

$$V(N, h) \coloneqq \int_0^h B(N, r)\, \omega(h, r)\ \mathrm{d}r$$

Definition 2.3. The *field influence function* is an N-dimensional rotationally symmetric normalisation of the 1-dimensional smoothing kernel

$$\Omega(N, h)(\vec{x}) \coloneqq \frac{1}{V(N, h)} \cdot \omega(h, |\vec{x}|) \qquad (1)$$

where $\vec{x}$ is the displacement vector.

Theorem 2.4. We calculate the vector gradient of the influence function

$$\nabla \Omega(N,h)(\vec{x}) = \frac{\hat{x}}{V(N,h)} \cdot \frac{\partial}{\partial r}\omega(h,r) \qquad (2)$$

where $r = |\vec{x}|$ is the magnitude of displacement

3 Sample Field

A sample field represents a continuous physical quantity that is discretized into a countable set of samples (or particles). Each particle carries physical quantities such as mass, density, pressure, and velocity. The field quantity at any point in space is computed through a weighted sum of neighboring particles within the kernel's support radius. The weights at each displacement is calculated using the field influence function.

Definition 3.1. The interpolated field quantity A at a position $\vec{x}$ is given by

$$A(N,h)(\vec{x}) = \sum_{i \in \mathbb{N}} A_i \frac{m_i}{\rho_i} \Omega(N,h)(\vec{x} - \vec{x}_i) \qquad (3)$$

where, in addition to the parameters already discussed, A_i is the quantity of the ith particle, $\vec{x}_i$ is the position of the ith particle, m_i is the mass of the ith particle, and ρ_i is the field density at $\vec{x}_i$.

Theorem 3.2. We calculate the initial density field by cancellation of density arguments in the field interpolation equation.

$$\rho(N,h)(\vec{x}) = \sum_{i \in \mathbb{N}} m_i \Omega(N,h)(\vec{x} - \vec{x}_i) \qquad (4)$$

Though the field interpolation method is applicable to all points of the field, we only need to use it at the positions of each particle.

Theorem 3.3. We calculate the vector gradient of an interpolated field.

$$\nabla A(N,h)(\vec{x}) = \sum_{i \in \mathbb{N}} A_i \frac{m_i}{\rho_i} \nabla \Omega(N,h)(\vec{x} - \vec{x}_i) \qquad (5)$$

4 Numeric Gradient

Theorem 4.1. Given a function $f(x)$, we calculate the n-th gradient as

$$f^{[n]}(x) = \lim_{h \mapsto 0} \frac{1}{(2h)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\big(x + (n-2k)h\big) \qquad (6)$$