FMN011 Exam — Computational Part I
Solving a Crime with Numerical Methods

Please read the instructions on the course webpage.

Lund, March 2011

Professor Sommar has been found dead in his office. At 8:00 p.m., the county coroner determined the core temperature of the corpse to be 32° C. One hour later, the core temperature had dropped to 29.5° C. Maintenance reported that the building's air conditioning unit broke down at 4:00 p.m. The temperature in the professor's office was 20° C at that time. The computerized climate control system recorded that the office temperature rose at a rate of 0.5° C per hour after the air conditioning stopped working.

Chief Inspector Arewald believes that the infamous Instructor Tögerson killed the professor. Instructor Tögerson, however, claims that he has an alibi. Famous TV0 anchorwoman Helen Johannesdotter was interviewing him at the Student Center Building, just across the street from the professor's office. The receptionist at the Student Center Building checked Instructor Tögerson into the building at 5:55 p.m., and the interview tapes confirm that he was interviewed from 6:00 p.m. until 6:50 p.m. Can Arewald be right?

To answer this question, we need to determine the time of death from the information we have at hand. We will assume the core temperature of the corpse was 37° C at the time of death and began decreasing immediately following death. We will further assume that the decrease in core temperature proceeded according to Newton's Law of Cooling. This principle states that the temperature of an object will change at a rate proportional to the difference between the temperature of the object and that of its surroundings.

To explicitly formulate our model, let $T(t)$ denote the core temperature of the corpse as a function of time, with time measured in hours. Take $t = 0$ to correspond to 8:00 p.m. Using this coordinate system, we know $T(0)$ and $T(1)$. Furthermore, the office temperature is given by $T_{\text{office}}(t) = 22 + 0.5t$. Applying Newton's Law of Cooling, we obtain

$$\frac{dT}{dt} = -k(T - T_{\text{office}})$$

where k is a positive constant of proportionality. To complete our analysis, we must first determine the solution of this equation that satisfies the conditions stated above. Then, using this solution, we must determine the time when the core temperature of the corpse was 37° C.

Task 1. The linear, first–order differential equation may be solved analytically. Show that its general solution is

$$T(t) = 22 + 0.5t - \frac{1}{2k} + ce^{-kt},$$

where c is a constant of integration.

Task 2. Show that using the conditions at $t = 0$ and $t = 1$ we find that

$$T(t) = 22 + 0.5t - \frac{1}{2k} + \left(10 + \frac{1}{2k}\right) e^{-kt}$$

and

$$7 + \frac{1}{2k} = \left(10 + \frac{1}{2k}\right) e^{-k}.$$

This equation cannot be solved explicitly for k, so we will have to obtain an approximate solution.

Task 3. Construct a Matlab algorithm for the bisection method, with the following inputs: function whose zero is to be located, f; left and right endpoints of interval, a, b; convergence tolerance, ε; and maximum number of iterations, $Nmax$. Use your algorithm to approximate k with two significant digits. Justify your input values.

Task 4. Construct a Matlab algorithm for the secant method with the following inputs: function whose zero is to be located, f; two intitial approximations, p_0, p_1; tolerance for the absolute error, δ; tolerance for the residual, ε; and maximum number of iterations, $Nmax$. Refine the first approximation to k to get 6 significant digits using the secant method. Justify your input values.

Task 5. Construct an algorithm for the fixed point method. Check your algorithm by finding an approximation to k with 6 significant figures.

Task 6. Construct an algorithm for the Newton-Raphson method in the same style as described for the above tasks. Use the approximate value of k to compute the approximate time of death t_d using Newton's method and iterate until you get a minimal residual. Justify your input values.

Task 7. Solve the same problem using the fixed point, secant and bisection methods to the same accuracy. Show the work (number of iterations, error estimates) required for each method. Study the speed of convergence of all methods by looking at the quotient e_n/e_{n-1}. Discuss your results.

According to your results, could Instructor Tögerson have murdered Professor Sommar?

Based on a problem in B. Bradie, *A Friendly Introduction to Numerical Analysis,* Pearson International Edition, 2006.

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