

Differential Geometry

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Course schedule

Smooth manifolds in \mathbb{R}^n , tangent spaces, smooth maps and the inverse function theorem. Examples, regular values, Sard's theorem (statement only). Transverse intersection of submanifolds. [4]

Manifolds with boundary, degree mod 2 of smooth maps, applications. [3]

Curves in 2-space and 3-space, arc-length, curvature, torsion. The isoperimetric inequality. [2]

Smooth surfaces in 3-space, first fundamental form, area. [1]

The Gauss map, second fundamental form, principal curvatures and Gaussian curvature. Theorema Egregium. [3]

Minimal surfaces. Normal variations and characterization of minimal surfaces as critical points of the area functional. Isothermal coordinates and relation with harmonic functions. The Weierstrass representation. Examples. [3]

Parallel transport and geodesics for surfaces in 3-space. Geodesic curvature. [2]

The exponential map and geodesic polar coordinates. The Gauss-Bonnet theorem (including the statement about classification of compact surfaces). [4]

Global theorems on curves: Fenchel's theorem (the total curvature of a simple closed curve is greater than or equal to 2π); the Fary-Milnor theorem (the total curvature of a simple knotted closed curve is greater than 4π). [2]

Recommended books

J. Milnor *Topology from the differentiable viewpoint*. Princeton University Press, 1997.

M. Do Carmo *Differential Geometry of Curves and Surfaces*. Pearson Higher Education, 1976

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1 Differential topology

Definition Smooth map on an open subset

Let $U \subset \mathbb{R}^n$. We say that $f: U \rightarrow \mathbb{R}^m$ is smooth if all partial derivatives to all orders exist and are continuous.

Definition Smooth map

Let $X \subset \mathbb{R}^n$. We say that $f: X \rightarrow \mathbb{R}^m$ is smooth if, for each $x \in X$, there exists (i) an open neighbourhood $U \subset \mathbb{R}^n$ of x and (ii) a smooth map $\tilde{f}: U \rightarrow \mathbb{R}^m$ such that $\tilde{f}|_{X \cap U} = f|_{X \cap U}$.

Definition Diffeomorphism

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$. We say that $f: X \rightarrow Y$ is a diffeomorphism if f is a smooth bijection with a smooth inverse. If such a map exists, we say that X and Y are diffeomorphic.

Exercise. Give an example of a smooth bijection that is not a diffeomorphism.

Definition k -dimensional manifold

We say that $X \subset \mathbb{R}^N$ is a k -dimensional manifold if, for each $x \in X$, there exists an open neighbourhood $V \subset X$ of x such that V is diffeomorphic to an open subset $U \subset \mathbb{R}^k$. A diffeomorphism $\varphi: U \rightarrow V$ is called a local parametrisation of V , whereas its inverse $\psi := \varphi^{-1}: V \rightarrow U$ is called a coordinate system or a chart on V .

Remarks

- By composing φ^{-1} with the projections $\pi_i: \mathbb{R}^k \rightarrow \mathbb{R}, (x_1, \dots, x_k) \mapsto x_i$, we get smooth maps $x_i := \pi_i \circ \varphi^{-1}$ which we call *coordinate functions*.
- WLOG, we can replace ‘diffeomorphic to an open subset $U \subset \mathbb{R}^k$ ’ with ‘diffeomorphic to an open ball in \mathbb{R}^k ’.
- It is easy to see that, if $X \subset \mathbb{R}^N$ is both a k -dimensional manifold and a \tilde{k} -dimensional manifold, then $k = \tilde{k}$.

Definition Dimension

Let $X \subset \mathbb{R}^N$ be a k -dimensional manifold. The dimension of X is k , and it is denoted by $\dim X$.

Example Some trivial examples

- $X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W
- $X = \{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^N$

Example S^n

$S^n := \{x \in \mathbb{R}^{n+1}: \|x\|_2 = 1\}$ is an n -dimensional manifold. To see this, consider the projection $\Pi_k: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_k, x_{k+1}, \dots, x_{n+1})$. It is easy to verify that maps of the form $\psi_k^\pm = \Pi_k|_{S^n \cap \{\text{sign}(x_k) = \pm 1\}}$ are diffeomorphisms $S^n \cap \{\text{sign}(x_k) = \pm 1\} \rightarrow B_1(0)$.

Remark. It is easy to show that X is a 0-dimensional manifold iff X is a discrete subset of \mathbb{R}^N .

Exercise. Show that, if X and Y are manifolds, then $X \times Y$ is also a manifold, with $\dim X \times Y = \dim X + \dim Y$.

Definition Submanifold

Let $X, Y \subset \mathbb{R}^N$ be manifolds. If $Y \subset X$, then we say that Y is a submanifold of X . The codimension of Y in X is defined as

$$\text{codim}_X Y := \dim X - \dim Y$$

1.1 Tangent spaces

We first recall some basic facts from our youth. Let $U \subset \mathbb{R}^k$ be open. The *differential* of a smooth map $f: U \rightarrow \mathbb{R}^m$ at $x \in U$ is defined by

$$df_x: \mathbb{R}^k \rightarrow \mathbb{R}^m$$

$$h \mapsto \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t}$$

This is a linear map, with matrix representation

$$df_x = \left(\frac{\partial f^i}{\partial x^j} \right)_{i,j}$$

Moreover, differentials satisfy the chain rule: given (i) two smooth maps $f: U \rightarrow \mathbb{R}^l$ and $g: V \rightarrow \mathbb{R}^m$ with $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$ open and (ii) a point $x \in U$ with $f(x) \in V$, we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Definition Tangent space

Let $X \subset \mathbb{R}^N$ be a k -dimensional manifold and $x \in X$. Choose a local parametrisation $\varphi: U \rightarrow V$ around x . We then define the tangent space $T_x X$ of X at x to be

$$T_x X := \text{im } d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k)$$

Of course, before we can safely proceed, we must show that $T_x X$ is well-defined:

Lemma 1.1

Let X be as above. $T_x X$ is independent of φ , and $\dim T_x X = k$.

Proof. Let $\tilde{\varphi}: \tilde{U} \rightarrow \tilde{V}$ be another local parametrisation near x . WLOG, by restricting if necessary, we may assume $\tilde{V} = V$. By the chain rule, we have

$$d\varphi_{\varphi^{-1}(x)} = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)} \circ d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}$$

Since $\tilde{\varphi}^{-1} \circ \varphi$ is a diffeomorphism of open subsets of \mathbb{R}^n , the corresponding differential $d(\tilde{\varphi}^{-1} \circ \varphi)$ is a linear isomorphism. Thus,

$$d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}(\mathbb{R}^k)) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(\mathbb{R}^k)$$

as claimed.

Now, it remains to show that $\dim T_x X = k$. By definition, there exists an open set $\hat{V} \subset \mathbb{R}^N$ and a smooth map $\Psi: \hat{V} \rightarrow \mathbb{R}^k$ that extends the chart $\psi := \varphi^{-1}$. Note that $\Psi \circ \varphi = \text{id}_U$, so by the chain rule,

$$d\Psi_x \circ d\varphi_{\varphi^{-1}(x)} = \text{id}_{\mathbb{R}^k}$$

Then, $d\varphi_{\varphi^{-1}(x)}$ must be an isomorphism $\mathbb{R}^k \rightarrow T_x X$, and hence $\dim T_x X = k$. ■

Example Tangent spaces for our trivial examples

Returning to the trivial examples we previously gave, we now state the corresponding tangent space for an arbitrary point x on each manifold.

- $X = \mathbb{R}^N$: $T_x X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W : $T_x X = \mathbb{R}^N$
- $X = \{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^N$: $T_x X = X$

Example *Tangent spaces for S^n*

From any given chart, we can compute (φ and) $d\varphi$:

$$\frac{\partial \varphi}{\partial x^1} = (1, 0, \dots, 0, -x_1/x_{n+1})$$

and similarly for $\partial \varphi / \partial x^i$. Manifestly, each partial derivative is perpendicular to x . Thus, $T_x X \subset x^\perp := \{v \in \mathbb{R}^{n+1} : \langle v, x \rangle = 0\}$. Since we know from the above lemma that $\dim T_x X = n$, we conclude that $T_x X = x^\perp$.

Definition *Differential map for manifolds*

Let $f: X \rightarrow Y$ be a smooth map between manifolds and $x \in X$. Choose a local parametrisation φ_1 around x and φ_2 around $f(x) \in Y$. We define the differential $df_x: T_x X \rightarrow T_{f(x)} Y$ of f at x by

$$df_x = d\varphi_{2, \varphi_2^{-1}(f(x))} \circ d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)} \circ (d\varphi_{1, \varphi_1^{-1}(x)})^{-1}$$

Lemma 1.2

df_x is independent of the choice of local parametrisations.

Proof. Trivial exercise. ■

Proposition 1.3 *Chain rule for manifolds*

Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be smooth maps between manifolds. For any $x \in X$,

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Proof. Trivial exercise. ■

Theorem 1.4 *Inverse function theorem*

Let $f: X \rightarrow Y$ be a smooth map between manifolds and $x \in X$. Suppose $df_x: T_x X \rightarrow T_{f(x)} Y$ is an isomorphism. Then f is a local diffeomorphism, i.e., each $x \in X$ has an open neighbourhood $V \subset X$ such that $f|_V: V \rightarrow f(V)$ is a diffeomorphism.

Proof. Since df_x is an isomorphism, it follows that $d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)}$ is also an isomorphism. We can then use the usual inverse function theorem to deduce the result. ■

1.2 Regular values and Sard's theorem

Definition *Critical and regular points*

Let $f: X \rightarrow Y$ be a smooth map between manifolds. We say that $x \in X$ is a critical point of f if $df_x: T_x X \rightarrow T_{f(x)} Y$ is not surjective. Otherwise, it is a regular point.

Notation. We denote by C the set of all critical points of f .

Remark. If $\dim Y > \dim X$, then $C = X$ and the pre-image of any regular value is \emptyset .

Definition *Critical and regular values*

Let $f: X \rightarrow Y$ be a smooth map between manifolds. We say that $y \in Y$ is a critical value of f if $y = f(x)$ for some $x \in C$. Otherwise, we say that y is a regular value of f .

Theorem 1.5 *Pre-image theorem*

Let $f: X \rightarrow Y$ be a smooth map between manifolds. Suppose $y \in Y$ is a regular value of f . If $f^{-1}(y) \neq \emptyset$, then $f^{-1}(y) \subset X$ is a submanifold of X with $\text{codim}_X f^{-1}(y) = \dim Y$.

Proof. Fix $x \in f^{-1}(y)$. Since y is a regular value, we know that $df_x: T_x X \rightarrow T_y Y$ is surjective. By the rank-nullity theorem, $\dim \ker df_x = \text{codim}_X Y$. Suppose $X \subset \mathbb{R}^N$, and pick a linear map $T: \mathbb{R}^N \rightarrow \mathbb{R}^{\text{codim}_X Y}$ such that $\ker T \cap \ker df_x = \{0\}$.¹

Now, extend f to the map $F: X \rightarrow Y \times \mathbb{R}^{\text{codim}_X Y}$ given by $z \mapsto (f(z), T(z))$. Note that the differential of F at x is given by

$$dF_x = (df_x, dT_x) = (df_x, T)$$

Since $\ker T \cap \ker df_x = \{0\}$, we have $\ker dF_x = \{0\}$, i.e., dF_x is injective. By the inverse function theorem for manifolds, there exists an open neighbourhood $U \subset X$ of x such that $F|_U: U \rightarrow V$ is a diffeomorphism to an open neighbourhood V of $(y, T(x))$. Hence, $F|_{f^{-1}(y) \cap U}$ is a local parametrisation of $(\{y\} \times \mathbb{R}^{\text{codim}_X Y}) \cap V$, proving that $f^{-1}(y)$ is a manifold of dimension $\text{codim}_X Y$. ■

Exercise. Show that, under the conditions of the pre-image theorem, $T_x f^{-1}(y) = \ker df_x$.

Corollary 1.6

Let $f: X \rightarrow Y$ be a smooth map between manifolds of the same dimension, with X compact. If y is a regular value of f , then $f^{-1}(y)$ is finite.

Proof. By the pre-image theorem, $f^{-1}(y)$ is a 0-dimensional manifold, i.e., a collection of points. Since X is compact, such a collection must be finite. ■

With just a bit more analysis and topology, we can actually say more than just finiteness:

Theorem 1.7 Stack of records theorem

Let $f: X \rightarrow Y$ be a smooth map between manifolds of the same dimension, with X compact. Let y be a regular value of f , and list the elements of $f^{-1}(y)$ as x_1, \dots, x_n . There exists an open neighbourhood $V \subset Y$ of y and a collection of open neighbourhoods $W_i \subset X$ of each x_i such that

$$f^{-1}(V) = \bigsqcup_{i=1}^n W_i$$

and each $f|_{W_i}: W_i \rightarrow V$ is a diffeomorphism.

Proof. By the inverse function theorem for manifolds, we can pick open neighbourhoods W_i of x_i such that each $f|_{W_i}$ is a diffeomorphism to an open neighbourhood of y . By shrinking neighbourhoods if necessary, W_i can be taken WLOG to be pairwise disjoint. Now, set

$$V = \left[\bigcap_{i=1}^n f(W_i) \right] \setminus f \left(X \setminus \bigcup_{i=1}^n W_i \right)$$

Note that $f(X \setminus \bigcup_{i=1}^n W_i)$ is a compact set that does not contain y , so V is an open neighbourhood of y . Finally, note that $f^{-1}(V) = \bigsqcup_{i=1}^n W_i$ by construction. ■

Now, the pre-image theorem can be a powerful tool for generating manifolds or showing that a certain set is one.

Application S^n is a manifold

Consider the map $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}, (x_1, \dots, x_{n+1}) \mapsto x_1^2 + \dots + x_{n+1}^2$. Note that $f^{-1}(1) = S^n$, so to show that S^n is a manifold, it suffices to show that 1 is a regular point. Indeed, note that $df_x = (2x_1, \dots, 2x_{n+1})$, which is not surjective only if $x = 0 \notin f^{-1}(1)$.

¹It is easy to constructively show using IB Linear Algebra that such a map exists. [Exercise!]

Application *Orthogonal group as a manifold*

Denote by $M(n)$ [resp. $S(n)$] the space of all [resp. symmetric] $n \times n$ matrices with entries in \mathbb{R} . Consider the orthogonal group $O(n) = \{A \in M(n) : AA^t = I\} \subset M(n) = \mathbb{R}^{n^2}$.

Let $f: M(n) \rightarrow O(n)$ be the map $A \mapsto AA^t$. This is smooth since multiplication and addition in \mathbb{R} are smooth. Since $O(n) = f^{-1}(I)$, it suffices to show that I is a regular value of f . Note that

$$df_A(H) = \lim_{t \rightarrow 0} \frac{f(A + tH) - f(A)}{t} = AH^t + HA^t$$

Now, fix $A \in M(n)$. Given $B \in S(n)$, observe that

$$df_A\left(\frac{1}{2}CA\right) = \frac{1}{2}AA^tC^t + \frac{1}{2}CAA^t = \frac{1}{2}C + \frac{1}{2}C = C$$

completing the proof that I is a regular value of f .

Remark. Recall that, besides being a manifold as we've just shown, $O(n)$ is also a group. In fact, the group operations $(A, B) \mapsto AB$ and $A \mapsto A^{-1} = A^t$ are smooth. Hence, we see that $O(n)$ is a *Lie group*.

Now, the pre-image theorem raises the question: how easy is to find regular values? This leads us to Sard's theorem.

Definition *Measure-zero subsets of \mathbb{R}^N*

We say that $S \subset \mathbb{R}^N$ is of measure zero in \mathbb{R}^N if, for each $\varepsilon > 0$, there exists a countable family $\{R_i\}$ of sets of the form $R_i = \prod_{j=1}^N [x_i^{(j)}, y_i^{(j)}]$ such that $S \subset \bigcup_i R_i$ and $\sum_i \text{vol}(R_i) < \varepsilon$.

Definition *Measure zero subsets of manifolds*

Let $X \subset \mathbb{R}^N$ be a k -dimensional manifold. We say that $A \subset X$ is of measure zero in X if, for all local parametrisations $\varphi: U \rightarrow V$ of X , $\varepsilon^{-1}(V \cap A) \subset \mathbb{R}^k$ has measure zero in \mathbb{R}^k .

Exercise. Let $U, \tilde{U} \subset \mathbb{R}^k$ be open and $\psi: U \rightarrow \tilde{U}$ a diffeomorphism. Show that, if $A \subset U$ is of measure zero in \mathbb{R}^k , then $\tilde{T} = \psi(T)$ is of measure zero in \mathbb{R}^k .

Remarks

- In view of the above exercise, $A \subset X$ is of measure zero in X iff $\varphi_i^{-1}(S \cap V_i)$ is of measure zero for all $\varphi_i: U \rightarrow V$ in an atlas of local parametrisations.
- If $\dim Y = 0$, then Y is of measure zero. If $\dim Y > 0$, then every non-empty open subset $V \subset Y$ is not of measure zero in Y .
- If $S \subset X$ is of measure zero in X , then any $\tilde{S} \subset S$ is also of measure zero in X .

Theorem 1.8 *Sard's theorem*

Let $f: X \rightarrow Y$ be a smooth map between manifolds. Then the set of critical values of f is of measure zero in Y .

Proof. Non-examinable — see Milnor's book if interested. ■

Corollary 1.9

The set of regular values of a smooth map $f: X \rightarrow Y$ between manifolds is dense in Y .

Proof. Any open set $V \subset Y$ cannot lie entirely in $f(C)$ since it has measure zero. ■

1.3 Transversality

Definition Transversal

Let $f: X \rightarrow Y$ be smooth and $Z \subset Y$ a submanifold of Y . We say that f is transversal to Z if, for each $x \in f^{-1}(Z)$,

$$T_{f(x)}Y = T_{f(x)}Z + \text{im } df_x$$

We then write $f \pitchfork Z$.

Remarks

- If $f(X) \cap Z = \emptyset$, then f is transversal to Z .
- If $Z = \{y\}$, then f is transversal to Z iff y is a regular value of f . Thus, transversality is really a generalisation of the notion of regular values.

Exercise. Let X also be a submanifold of Y and $\iota: X \hookrightarrow Y$ the inclusion map. Show that $d\iota_x$ is just the inclusion map $T_xX \hookrightarrow T_xY$ of the tangent spaces. Thus, $\iota \pitchfork Z$ iff $T_xX + T_xZ = T_xY$ for all $x \in X \cap Z$.

Now, we state a generalisation of the pre-image theorem for transversal maps:

Theorem 1.10

Let $f: X \rightarrow Y$ be smooth map that is transversal to a submanifold $Z \subset Y$ of Y . If $f^{-1}(Z) \neq \emptyset$, then $f^{-1}(Z) \subset X$ is a submanifold of X , with $\text{codim}_X f^{-1}(Z) = \text{codim}_Y Z$.

Remark. If $Z = \{y\}$, then $\text{codim}_Y Z = \dim Y$ as in the pre-image theorem.

Sketch of proof (non-examinable). Fix $z \in Z$ with $z = f(x)$ for some $x \in X$. Note that, for some open neighbourhood $V \subset Y$ of z , there exists a smooth map $h: V \rightarrow \mathbb{R}^{\text{codim}_Y Z}$ such that $Z \cap V = h^{-1}(0)$ and dh_z is surjective. Locally around $x \in X$, $f^{-1}(Z) = (h \circ f)^{-1}(0)$. Thus, by the pre-image theorem, it suffices to show that 0 is a regular value of $h \circ f$.

Now, since $f \pitchfork Z$, we have $T_zY = T_zZ + \text{im } df_x$. By the exercise after the pre-image theorem, we have $dh_z = T_zZ$. Moreover, $f \pitchfork Z$ gives us

$$T_zY = T_zZ + \text{im } df_x = \ker dh_z + \text{im } df_x$$

This then implies that $\text{im } dh_z = \text{im}(dh_z \circ df_x) = \text{im } d(h \circ f)_x$. Since dh_z is surjective, $d(h \circ f)_z$ is also surjective and hence 0 is a regular value of $h \circ f$. ■

Exercise. Construct the required map h .

Remark. Transversality is both a stable and generic property. It is stable in the sense that small perturbations of f remain transversal to a given submanifold. It is generic in the sense that any given smooth map may be deformed by arbitrarily small amounts into a map that is transversal to Z . See *Differential Topology* by Guillemin and Pollack for more details.

1.4 Manifolds with boundary

Consider the closed upper half plane

$$\mathbb{H}^k := \{(x_1, \dots, x_k) \in \mathbb{R}^n : x_k \geq 0\}$$

We denote its boundary by $\partial\mathbb{H}^k = \{x_k = 0\}$.

Definition Manifold with boundary

We say that $X \subset \mathbb{R}^N$ is a (smooth) k -dimensional manifold with boundary if every $x \in X$ has an open neighbourhood $V \subset X$ that is diffeomorphic to an open subset $U \subset \mathbb{H}^k$.

Remark. Note that a diffeomorphism $\varphi: U \rightarrow V$ has a smooth extension defined on an open subset of \mathbb{R}^k . This allows us to deduce as before that, if X is both k - and \tilde{k} -dimensional, then $k = \tilde{k}$.

Definition Dimension

The dimension of a k -dimensional manifold with boundary X is k .

Definition Boundary of a manifold with boundary

Let X be a k -dimensional manifold with boundary. Its boundary is defined to be

$$\partial X := \{x \in X : \exists \text{ open nhod } V \subset X \text{ and diffeomorphism } \psi: V \rightarrow \psi(V) \text{ s.t. } x \in \psi^{-1}(\partial \mathbb{H}^k)\}$$

Remarks

- In fact, if $x \in \psi^{-1}(\partial \mathbb{H}^k)$ for some diffeomorphism $\psi: V \rightarrow \psi(V) \subset \mathbb{H}^k$ on an open nhod V of x , then it is true for *all* diffeomorphisms on an open nhod of x to its image in \mathbb{H}^k .
- In the definition of manifold with boundary, we may take $U = \mathbb{R}^k$ or $U = \mathbb{H}^k$ WLOG.

Exercise. Prove the first remark.

Definition Interior of manifold with boundary

Let X be a manifold with boundary. We define its interior to be

$$\text{int } X := X \setminus \partial X$$

Proposition 1.11

Let X be a k -dimensional manifold with boundary. Then $\text{int } X$ is a manifold of dimension k and ∂X is a manifold of dimension $k - 1$.

Proof. $\text{int } X$ is a manifold of dimension k because we can always restrict our diffeomorphisms $\varphi: V \rightarrow U$ such that $U \cap \partial \mathbb{H}^k = \emptyset$. See Example Sheet 1 for ∂X . ■

Example

- Trivially, \mathbb{H}^k is a k -dimensional manifold with boundary
- As we will prove later, $B^n := \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ is an n -dimensional manifold with boundary. Note that $\partial B^n = S^{n-1}$ is a manifold of dimension $n - 1$.
- $[0, 1] \times [0, 1]$ is not a manifold with boundary (see Example Sheet 1)
- If X is a manifold with boundary and Y is a manifold, then $X \times Y$ is a manifold with boundary, with $\partial(X \times Y) = (\partial X) \times Y$. (Of course, the previous example is a counterexample to the case that Y is also a manifold with boundary.)

Remark. Note that ∂X and $\text{int } X$ are not the same as the topological notions of ‘boundary’ and ‘interior’ as subsets of \mathbb{R}^N . Indeed, if $\dim X < N$, the topological interior of X is empty, whereas $\text{int } X$ is not.

Definition Tangent space

Let X be a k -dimensional manifold with boundary and $x \in X$. Let $\varphi: U \rightarrow V$ be a diffeomorphism from an open set $U \subset \mathbb{H}^k$ to an open neighbourhood $V \subset X$ of x . Since φ is smooth, there exists

a smooth extension $\tilde{\varphi}$ on an open subset of \mathbb{R}^k , with $d\tilde{\varphi}_{\varphi^{-1}(x)}$ well-defined. We then define the tangent space to be

$$T_x X := \text{im } d\tilde{\varphi}_{\varphi^{-1}(x)}$$

Remark. As before, $T_x X$ is well-defined.

Exercise. Show that, for every $x \in \partial X$, $T_x \partial X \subset T_x X$.

Lemma 1.12

Let X be a manifold of dimension k . Let $f: X \rightarrow \mathbb{R}$ be smooth, with 0 a regular value of f . Then $f^{-1}([0, \infty)) \subset X$ is a k -dimensional manifold with boundary $\partial(f^{-1}([0, \infty))) = f^{-1}(0)$.

Proof. The subset $f^{-1}((0, \infty)) \subset X$ is open in X and thus a submanifold of X . This means that we can restrict a local parametrisation to X such that its image lies in $f^{-1}((0, \infty))$ and get the diffeomorphism we need.

It remains to consider $x \in f^{-1}(0)$. Extend f to a map $\mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}^{k-1}$ as in the proof of the pre-image theorem (Theorem 1.5). We can then proceed as before using the inverse function theorem. ■

Corollary 1.13

B^n is an n -dimensional manifold with boundary.

Proof. This is immediate from the above lemma. ■

Theorem 1.14 Pre-image theorem for manifolds with boundary

Let X be a manifold with boundary and Y a manifold, with $\dim X > \dim Y$. Suppose $f: X \rightarrow Y$ is smooth and $y \in Y$ is a regular value of both f and $f|_{\partial X}$. Then $f^{-1}(y) \subset X$ is a manifold with boundary, with $\text{codim}_X f^{-1}(y) = \dim Y$ and $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial X$.

Proof. WLOG, we may assume that $X = \mathbb{H}^m$ and $Y = \mathbb{R}^n$ since we are always working locally. The easy case $x \in f^{-1}(y) \cap \text{int } \partial \mathbb{H}^m$ is left as an exercise. Now, suppose $x \in f^{-1}(y) \cap \partial \mathbb{H}^m$. Then there exists an open subset $U \subset \mathbb{R}^m$ such that $f|_{U \cap \mathbb{H}^m}$ extends to a smooth map $F: U \rightarrow \mathbb{R}^n$. Since y is a regular value of $f|_{U \cap \mathbb{H}^m}$, dF_x is surjective. Since the map $z \mapsto dF_z$ (defined on U) is smooth, we can shrink U such that dF_z is surjective for all $z \in U$.² Applying the pre-image theorem to F , we have that $F^{-1}(y)$ is a submanifold of U with $\text{codim}_{\mathbb{R}^m} F^{-1}(y) = \dim Y$. Let $\pi: F^{-1}(y) \rightarrow \mathbb{R}, (x_1, \dots, x_m) \mapsto x_m$. Note that

$$(f|_{U \cap \mathbb{H}^m})^{-1}(y) = \pi^{-1}([0, \infty))$$

It then suffices to show that 0 is a regular value of π since, by the previous lemma, it would follow that $\pi^{-1}([0, \infty))$ is a submanifold of $F^{-1}(y)$ with boundary $\pi^{-1}(0) = F^{-1}(y) \cap \partial \mathbb{H}^m = f^{-1}(y) \cap U \cap \partial \mathbb{H}^m$.

Now, to show that, for any $z \in \pi^{-1}(0)$, the map $d\pi_z: T_z F^{-1}(y) \rightarrow \mathbb{R}$ is surjective, it suffices to show that $T_z F^{-1}(y) = \ker dF_z = \ker df_z \not\subset \ker d\pi_z = \mathbb{R}^{m-1} \times \{0\} = T_z \partial \mathbb{H}^m$. Indeed, note that

$$df_z|_{T_z \partial \mathbb{H}^m} = d(f|_{\partial \mathbb{H}^m})_z$$

is surjective. If $\ker df_z \subset T_z \partial \mathbb{H}^m$, then $\ker(df_z|_{T_z \partial \mathbb{H}^m}) = \ker(df_z)$, but these have different dimensions by the rank-nullity theorem — a contradiction! ■

²Indeed, we know that some submatrix of dF_x has nonzero determinant. By continuity of \det , there is some open neighbourhood $\tilde{U} \subset U$ of x on which the determinant of that submatrix remains nonzero and thus $\dim \text{im } dF_z = n$ for all $z \in \tilde{U}$.

Theorem 1.15

Let X be a manifold with boundary and Y a manifold with $Z \subset Y$ a submanifold. Let $f: X \rightarrow Y$ be smooth such that $f \pitchfork Z$ and $f|_{\partial X} \pitchfork Z$. Then $f^{-1}(Z) \subset X$ is a manifold with boundary, with $\text{codim}_X f^{-1}(Z) = \text{codim}_Y Z$ and $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$.

1.5 Degree modulo 2

Definition Smooth homotopy

Let $f, g: X \rightarrow Y$ be smooth maps between manifolds. A smooth homotopy between f and g is a smooth map $F: X \times [0, 1] \rightarrow Y$ such that $F|_{X \times \{0\}} = f$ and $F|_{X \times \{1\}} = g$. If such a map exists, we say that f and g are smoothly homotopic and write $f_0 \simeq f_1$.

Exercise. Show that \simeq is an equivalence relation (cf. Example Sheet 1 Q14).

Definition Smooth isotopy

Let $f, g: X \rightarrow Y$ be diffeomorphisms. A smooth isotopy between f and g is a smooth homotopy $F: X \times [0, 1] \rightarrow Y$ for which $F|_{X \times \{t\}}$ is a diffeomorphism for all $t \in [0, 1]$. If such a map exists, we say that f and g are smoothly isotopic.

Lemma 1.16 Homotopy lemma

Suppose $f, g: X \rightarrow Y$ are smoothly homotopic, with X compact and $\dim X = \dim Y$. If y is a regular value of both f and g , then

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod{2}$$

Proof. Let $F: X \times [0, 1] \rightarrow Y$ be a smooth homotopy between f and g . We first suppose that y is also a regular value of F . By Theorem 1.14, $F^{-1}(y)$ is a 1-dimensional manifold with boundary $\partial F^{-1}(y) = F^{-1}(y) \cap (X \times \{0, 1\}) = f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}$. Thus, $\#\partial F^{-1}(y) = \#f^{-1}(y) + \#g^{-1}(y)$. We then proceed by noting the following result:

► **Theorem.** Let Z be a compact 1-dimensional manifold with boundary. Then Z is diffeomorphic to a disjoint union of finitely many copies of $[0, 1]$ and of S^1 .

In particular, it follows from the above that $\#\partial Z \equiv 0 \pmod{2}$. Hence, we conclude that $\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod{2}$.

Now, suppose y is *not* a regular value of F . By Sard's theorem, the set of regular values of f (resp. g) is dense in Y . Thus, every open set in Y contains a shared regular value of f, g, F . Then, by the stack of records theorem, we can pick an open set $V \subset Y$ such that $f^{-1}(V)$ and $g^{-1}(V)$ are both disjoint union of open sets on which f and g , respectively, are diffeomorphisms to their images. In particular, $z \mapsto \#f^{-1}(z)$ and $z \mapsto \#g^{-1}(z)$ are both constant in V . Since our previous argument holds for the common regular value of f and g in V , it follows that $\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod{2}$. ■

Exercise. Prove the theorem used in the proof of the homotopy lemma (Lemma 1.16).

Exercise. Prove the claim in the proof that f and g have a shared regular value in every open subset of Y .

Lemma 1.17 Homogeneity lemma

Let X be a smooth connected manifold and $y, z \in X$. Then there exists a diffeomorphism $h: X \rightarrow X$ smoothly isotopic to id_X and $h(y) = z$.

Remark. As suggested by the name, this lemma says that each point of X is essentially the same.

Proof sketch. It suffices to show that each $y \in X$ has an open neighbourhood $V \subset X$ such that, for every $z \in V$, there exists a diffeomorphism h with the required properties. Since X locally (i.e. in a neighbourhood of y) is diffeomorphic to \mathbb{R}^k , it suffices to show that there exists $F: \mathbb{R}^k \rightarrow \mathbb{R}^k$ smoothly isotopic to $\text{id}_{\mathbb{R}^k}$, sends $0 \mapsto z$, and $F|_{\mathbb{R}^k \setminus B_1(0)} = \text{id}_{\mathbb{R}^k \setminus B_1(0)}$. To do this, we pick a smooth vector field $V: \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that $V(tz) = z$ for all $t \in [0, 1]$ and $V = 0$ outside $B_1(0)$. By standard ODE theory, there exists a solution $\gamma_x: \mathbb{R} \rightarrow \mathbb{R}^k$ to the ODE

$$\begin{cases} \dot{\gamma}_x(t) = V(\gamma(t)) \\ \gamma_x(0) = x \end{cases}$$

By defining the flow map $\Phi_t: \mathbb{R}^k \rightarrow \mathbb{R}^k, x \mapsto \gamma_x(t)$ for each $x \in \mathbb{R}^k$, we can then construct the smooth map

$$\begin{aligned} \Phi: \mathbb{R}^k \times \mathbb{R} &\rightarrow \mathbb{R}^k \\ (x, t) &\mapsto \Phi_t(x) \end{aligned}$$

which we can restrict to $\mathbb{R}^k \times [0, 1]$ to obtain the required map F . ■

Theorem 1.18

Let $f: X \rightarrow Y$ be a smooth map between manifolds, with X compact, Y connected and $\dim X = \dim Y$. If y, z are regular values of f , then

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \pmod{2}$$

Proof. By the homogeneity lemma (Lemma 1.17), there exists a smooth isotopy $H: X \times [0, 1] \rightarrow X$ such that $H|_{X \times \{0\}} = \text{id}$ and $H|_{X \times \{1\}}$ is a diffeomorphism $X \rightarrow X$ with $h(y) = z$. Observe that $H \circ (f, \text{id}_{[0,1]})$ defines a smooth homotopy between f and $h \circ f$. Note also that z is a regular value of $h \circ f$ since $d(h \circ f) = dh \circ df$, dh is an isomorphism and $y = h^{-1}(z)$ is a regular value of f . By the homotopy lemma (Lemma 1.16), we then have

$$\#f^{-1}(y) \equiv \#(h \circ f)^{-1}(z) \equiv \#f^{-1}(z) \pmod{2}$$

as required. ■

With the above theorem established, we can define the following:

Definition Degree mod 2

For a smooth map $f: X \rightarrow Y$ from a compact manifold to a connected manifold of the same dimension, the degree mod 2 of f is defined to be

$$\deg_2(f) := \#f^{-1}(y) \pmod{2}$$

for any (equivalently, all) regular value of f .

Remark. The theorem, together with the homotopy lemma, implies that $\deg_2(f)$ is a homotopy invariant.

Lemma 1.19

Let X be a compact, connected manifold with $\dim X \geq 1$. The identity map id_X cannot be smoothly homotopic to a constant map on X .

Proof. Suppose, on the contrary, that identity id_X and the constant map $c_{x_0}: X \rightarrow X, x \mapsto x_0$ for some fixed $x_0 \in X$. Note that $\deg_2(\text{id}_X) = 1$ since every $x \in X$ is a regular value of id_X and has exactly one pre-image under id_X . On the other hand, $\deg_2(c_{x_0}) = 0$ since we can simply pick $x \in X \setminus \{x_0\}$ (this set is non-empty since $\dim X \geq 1$ which is a regular value with no pre-image under c_{x_0}). However, this contradicts the fact that degree mod 2 is a homotopy invariant. ■

Corollary 1.20

There does not exist a smooth retraction of B^{n+1} onto its boundary, i.e. there is no smooth map $f: B^{n+1} \rightarrow \partial B^{n+1}$ such that $f|_{\partial B^{n+1}} = \text{id}$.

Proof. Suppose, on the contrary, that there exists such a map f . Define

$$\begin{aligned} F: S^n \times [0, 1] &\rightarrow S^n \\ (x, t) &\mapsto f(xt) \end{aligned}$$

This is smooth since f is smooth. Moreover, we have $F|_{S^n \times \{0\}}$ is the constant map $S^n \rightarrow S^n, x \mapsto f(0)$, whereas $F|_{S^n \times \{1\}} = \text{id}_{S^n}$. However, this contradicts Lemma 1.19. ■

Theorem 1.21 (Smooth) Brouwer's fixed point theorem

Every smooth map $f: B^n \rightarrow B^n$ has a fixed point.

Proof. Suppose, on the contrary, that $f: B^n \rightarrow B^n$ is a smooth map without a fixed point. We will obtain a contradiction by constructing a smooth retraction $g: B^n \rightarrow \partial B^n$.

For each $x \in B^n$, $f(x) \neq x$ so there exists a line joining $f(x)$ to x in B^n . Extending this line to ∂B^n , we define $g(x)$ to be the unique intersection point. It is easy to show that g as we've constructed it is well-defined and smooth. But this contradicts Corollary 1.20! ■

Corollary 1.22 (Topological) Brouwer fixed point theorem

Every continuous map $f: B^n \rightarrow B^n$ has a fixed point.

Proof. Suppose not. Then there exists $\varepsilon > 0$ such that $|f(x) - x| \geq \varepsilon$ for all $x \in B^n$. By convolutions (cf. II Analysis of Functions), we can show that $C^\infty(B^n, B^n)$ is dense in $C(B^n, B^n)$. We can then obtain a contradiction to the smooth Brouwer fixed point theorem. ■

Now, we consider a generalisation of degree mod 2 for regular values to submanifolds via transversality. Let $f: X \rightarrow Y$ be a smooth map, with X compact. Suppose $Z \subset Y$ is a closed submanifold of Y , with $f \pitchfork Z$ and $\dim X + \dim Z = \dim Y$. Note that $f^{-1}(Z)$ is a closed 0-dimensional submanifold of X .

Definition Intersection number modulo 2

We define the intersection number mod 2 $I_2(f, Z)$ of f with Z as

$$I_2(f, Z) := \#f^{-1}(Z) \bmod 2$$

Proposition 1.23

If g is another such map $X \rightarrow Y$ and is smoothly homotopic to f , then $I_2(f, Z) = I_2(g, Z)$.

Remarks

- Recall that we noted the genericity of transversality. This property allows us to define intersection number mod 2 even for maps f that are not transversal to Z . In that case, we define it to be $I_2(\tilde{f}, Z)$ where \tilde{f} is a small perturbation of f with $\tilde{f} \pitchfork Z$.

- For $X \subset Y$, we can define the intersection number mod 2 of X and Z in Y via $I_2(X, Z) = I_2(\iota, Z)$ where $\iota: X \rightarrow Y$ is the inclusion map.
- If $X \subset Y$ and $\dim X = \frac{1}{2} \dim Y$, we can define the self-intersection number mod 2 of X .

1.6 Abstract manifolds

Now, we will briefly discuss the more abstract notion of manifolds instead of the manifolds embedded in \mathbb{R}^N that we've been dealing with.

Definition Abstract smooth k -dimensional manifold

An abstract smooth k -dimensional manifold is a second-countable, Hausdorff topological space X together with a collection $\{\varphi_\alpha\}_{\alpha \in I}$ of maps $\varphi_\alpha: U_\alpha \rightarrow X$ (where $U_\alpha \subset \mathbb{R}^k$ is open) such that

- (i) φ_α is a homeomorphism onto its image
- (ii) $X = \bigcup_\alpha V_\alpha$ where $V_\alpha := \varphi_\alpha(U_\alpha)$
- (iii) $\varphi_\beta^{-1} \circ \varphi_\alpha \Big|_{U_\alpha \cap \varphi_\alpha^{-1}(V_\alpha \cap V_\beta)}$ is a diffeomorphism $U_\alpha \cap \varphi_\alpha^{-1}(V_\alpha \cap V_\beta) \rightarrow U_\beta \cap \varphi_\beta^{-1}(V_\alpha \cap V_\beta)$
- (iv) $\{\varphi_\alpha\}_{\alpha \in I}$ is maximal, i.e. if $\tilde{\varphi}: \tilde{U} \rightarrow X$ is such that $\{\varphi_\alpha\}_{\alpha \in I} \cup \{\tilde{\varphi}\}$ (i) to (iii), then $\tilde{\varphi} = \varphi_\alpha$ for some $\alpha \in I$

The collection $\{\psi_\alpha = \varphi_\alpha^{-1}: V_\alpha \rightarrow U_\alpha\}_{\alpha \in I}$ is called a smooth atlas.

Remark. By changing the regularity conditions on the transition maps, we can define abstract C^k manifolds and C^ω manifolds.

Theorem 1.24 Whitney's embedding theorem

Let X be an abstract smooth k -dimensional manifold. Then there exists a smooth embedding (diffeomorphism to its image) $X \rightarrow \mathbb{R}^{2k+1}$.

2 Curves and surfaces

2.1 Curves in 3-space

Definition Curve

Let $X \subset \mathbb{R}^N$ be a manifold. Let $I \subset \mathbb{R}$ an interval. A curve in X is a smooth map $\alpha: I \rightarrow X$. We say that α is regular if α is an immersion, i.e. $d\alpha_t(1) \neq 0$ for all $t \in I$.

Notation. We will sometimes write $\dot{\alpha}(t)$ to mean $d\alpha_t(1)$.

In what follows, we will for the most part be discussing curves in \mathbb{R}^3 .

Definition Arc length

The arc length of a curve $\alpha: I \rightarrow \mathbb{R}^3$ from a point $t_0 \in I$ to any $t \in I$ is

$$s(t) := \int_{t_0}^t |\dot{\alpha}(\tau)| d\tau$$

where $|\cdot|$ denotes the Euclidean norm. If the interval I has endpoints a, b with $a < b$, then the length of α is

$$\text{length}(\alpha) := \int_a^b |\dot{\alpha}(\tau)| d\tau$$

Remarks

- Arc length is a *Euclidean invariant*. Indeed, consider a general Euclidean transformation T of \mathbb{R}^n of the form $v \mapsto Rv + b$, where $R \in O(n)$ and $b \in \mathbb{R}^n$. Applying T to the curve α , we get the new curve $\tilde{\alpha}(s) = (R \circ \alpha)(s) + b$. Then the arc length parameter s for α is also an arc length parameter for $\tilde{\alpha}$ since $|\dot{\tilde{\alpha}}(s)| = |R\dot{\alpha}(s)| = |\dot{\alpha}(s)| = 1$.
- Note that we can think of s as a smooth map $I \rightarrow \tilde{I} \subset \mathbb{R}$. For a regular curve, $\tau \mapsto s(\tau)$ is a strictly increasing function and thus has a smooth inverse $\tau = \tau(s)$. We can then reparametrise α by s : $s \mapsto \alpha(\tau(s))$. Thus, we may always assume WLOG that regular curves are parametrised by arc length.

In what follows, $\alpha: I \rightarrow \mathbb{R}^3$ is a regular curve in 3-space that is parametrised by arc length.

Definition Tangent vector

The tangent vector to the curve $\alpha: I \rightarrow \mathbb{R}^3$ at $s \in I$ is $t(s) := \dot{\alpha}(s)$.

Remark. We can think of t as a map $I \rightarrow \mathbb{S}^2$.

Definition Curvature and normal vector

The curvature of α at $s \in I$ is

$$\kappa(s) := |\ddot{\alpha}(s)|$$

If $\kappa(s) \neq 0$, we define the normal vector at s to be

$$n(s) := \frac{1}{\kappa(s)} \ddot{\alpha}(s)$$

Definition Binormal vector

The binormal vector to the curve $\alpha: I \rightarrow \mathbb{R}^3$ at $s \in I$ is

$$b(s) := t(s) \times n(s)$$

Remark. For any $s \in I$, $\{t(s), n(s), b(s)\}$ forms an orthonormal basis of \mathbb{R}^3 , which is called the *Frenet frame*.

Now, we compute

$$\dot{b}(s) = \dot{t}(s) \times n(s) + t(s) \times \dot{n}(s) = t(s) \times \dot{n}(s)$$

Differentiating the relation $b(s) \cdot b(s) = 1$, we get $\dot{b}(s) \cdot b(s) = 0$. Putting everything together, we have that \dot{b} is orthogonal to both $t(s)$ and $b(s)$, so it must be parallel to $n(s)$. We may thus write

$$\dot{b}(s) = \tau(s)n(s)$$

Definition Torsion

The quantity $\tau(s)$ defined above is called the torsion of α at $s \in I$.

Remark. Curvature and torsion are proper Euclidean invariants. Fix $R \in SO(3)$ and $b \in \mathbb{R}^3$. Let $\alpha, \tilde{\alpha}: I \rightarrow \mathbb{R}^3$ be regular curves parametrised by arc length s , related by $\tilde{\alpha} = R \circ \alpha + b$. Observe that $\tilde{\kappa}(s) = |\ddot{\tilde{\alpha}}(s)| = |R\ddot{\alpha}(s)| = |\ddot{\alpha}(s)| = \kappa(s)$, so curvature is indeed a proper Euclidean invariant. Note also that $\tilde{t}(s) = Rt(s)$ and

$$\tilde{n}(s) = \frac{\ddot{\tilde{\alpha}}(s)}{\tilde{\kappa}(s)} = \frac{R\ddot{\alpha}(s)}{\kappa(s)} = Rn(s)$$

Since $R \in SO(3)$, we have that

$$\tilde{b}(s) = \tilde{t}(s) \times \tilde{n}(s) = (Rt(s)) \times (Rn(s)) = R(t(s) \times n(s)) = Rb(s)$$

so $\tilde{\tau}(s)(Rn(s)) = \tilde{\tau}(s)\tilde{n}(s) = R(\tau(s)n(s)) = \tau(s)(Rn(s))$ and hence $\tilde{\tau} \equiv \tau$.

Proposition 2.1 Frenet equations

The vectors t, n, b satisfy the system of ODEs

$$\begin{bmatrix} \dot{t} \\ \dot{n} \\ \dot{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

Remark. Remark about antisymmetry of matrix to follow

Theorem 2.2 Fundamental theorem of curves in \mathbb{R}^3

Let $I \subset \mathbb{R}$ be an interval. If $\kappa: I \rightarrow \mathbb{R}_{>0}$ and $\tau: I \rightarrow \mathbb{R}$ are smooth, then there exists a regular curve $\alpha: I \rightarrow \mathbb{R}^3$ parametrised by arc length such that $\kappa(s)$ and $\tau(s)$ are the curvature and torsion at s , respectively. Moreover, if $\tilde{\alpha}: I \rightarrow \mathbb{R}^3$ is another regular curve parametrised by arc length such that $\tilde{\kappa}(s) = \kappa(s)$ and $\tilde{\tau}(s) = \tau(s)$ for every $s \in I$, then $\tilde{\alpha} = R \circ \alpha + a$ for some $R \in SO(3)$ and $a \in \mathbb{R}^3$.

Proof. Pick $t_0, n_0 \in \mathbb{R}^3$ orthonormal vectors. Define $b_0 = t_0 \times n_0$. Fix $s_0 \in I$. By standard ODE theory, there exists a unique solution $t, n, b: I \rightarrow \mathbb{R}^3$ to the Frenet equations (Proposition 2.1). We then define α by

$$\alpha(s) = \int_{s_0}^s t(\tilde{s}) d\tilde{s}$$

Now, let $\tilde{\alpha}$ be another such curve as in the theorem statement. Since $\{t, n, b\}$ and $\{\tilde{t}, \tilde{n}, \tilde{b}\}$ are both orthonormal bases for \mathbb{R}^3 , we can pick $R \in SO(3)$ that sends $(t, n, b) \mapsto (\tilde{t}, \tilde{n}, \tilde{b})$. Define $a := \tilde{\alpha}(s_0) - \alpha(s_0)$. Let $\tilde{\tilde{\alpha}}(s) := (R \circ \alpha)(s) + a$, with $(\tilde{\tilde{t}}, \tilde{\tilde{n}}, \tilde{\tilde{b}})(s_0) = (\tilde{t}, \tilde{n}, \tilde{b})(s_0)$. By uniqueness

of solution to the Frenet equations, $(\tilde{t}, \tilde{n}, \tilde{b}) = (\tilde{t}, \tilde{n}, \tilde{b})$ on I . We then have that

$$\tilde{\alpha}(s) = \tilde{\alpha}(s_0) + \int_{s_0}^s \tilde{t}(\hat{s}) d\hat{s} = \tilde{\alpha}(s_0) + \int_{s_0}^s \tilde{t}(\hat{s}) d\hat{s} = \tilde{\alpha}(s)$$

as required. ■

Lemma 2.3 Plane curves

A regular curve $\alpha: I \rightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$ for all $s \in I$. Then α lies in a plane iff $\tau(s) = 0$ for all $s \in I$.

Proof. (\Leftarrow) If $\tau \equiv 0$, then $\dot{b} = 0$, so $(t \times n)(s) = (t \times n)(s_0)$ for all $s \in I$. Let Π be the plane spanned by $\{t(s), n(s)\}$. Then we have that

$$\alpha(s) = \alpha(s_0) + \int_{s_0}^s t(\hat{s}) d\hat{s} \in \alpha(s_0) + \Pi$$

(\Rightarrow) WLOG set $\alpha(s_0) = 0$, so $\alpha(s) \in \Pi$ for some plane Π through the origin. By differentiation, we see that $\dot{\alpha}, \ddot{\alpha} \in \Pi$. Then $t \times n$ is the unique normal to Π , so $\dot{b} \equiv 0$. Hence, $\tau \equiv 0$. ■

Remark. For plane curves, we can assign a sign to the curvature κ . In particular, we can define the (signed) curvature $\kappa(s)$ via

$$\dot{t}(s) = \kappa(s)n(s)$$

Note that $|\kappa(s)|$ coincides with our previous definition of curvature. The sign essentially arises from demanding that the basis $\{t(s), n(s)\}$ has the same orientation as $\{e_1, e_2\}$.

2.2 Isoperimetric inequality

In this subsection, let $I = [a, b]$ be a closed interval.

Definition Simple, closed, regular curve

We say that a regular curve $\alpha: I \rightarrow \mathbb{R}^3$ is simple, closed if $\alpha^{(n)}(a) = \alpha^{(n)}(b)$ for all $n \in \mathbb{N}_0$.

Remark. Equivalently, we can view α as an injective map $\mathbb{S}^1 \rightarrow \mathbb{R}^3$.

Lemma 2.4 Jordan curve theorem

Let $\alpha: I \rightarrow \mathbb{R}^2$ be a simple, closed, regular curve in \mathbb{R}^2 . Then there exists an open subset $U \subset \mathbb{R}^2$ such that $U \cup \alpha(I)$ is a compact manifold with boundary.

Lemma 2.5 Wirtinger's inequality

Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a smooth periodic function with period L and $\int_0^L f(x) dx = 0$. Then

$$\int_0^L |f(x)|^2 dx \leq \frac{L^2}{4\pi^2} \int_0^L |f'(x)|^2 dx$$

with equality iff $f = ae^{2\pi ix/L} + be^{-2\pi ix/L}$ for some $a, b \in \mathbb{C}$.

Proof. Since f is a smooth periodic function, we can write it as

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x / L}$$

Since $\int_0^L f(x) dx = 0$, then we must have $a_0 = 0$. Since f is smooth, we can differentiate term by term to get

$$f'(x) = \sum_{n \in \mathbb{Z}} \frac{2\pi i n}{L} a_n e^{2\pi i n x / L}$$

By Plancherel,

$$\frac{1}{L} \int_0^L |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |a_n|^2, \quad \frac{1}{L} \int_0^L |f'(x)|^2 dx = \sum_{n \in \mathbb{Z}} \frac{4\pi^2 n^2}{L^2} |a_n|^2$$

It then follows that

$$\int_0^L |f'(x)|^2 dx = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{4\pi^2 n^2}{L^2} |a_n|^2 \geq \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{4\pi^2}{L^2} |a_n|^2 = \frac{4\pi^2}{L^2} \int_0^L |f(x)|^2 dx$$

as required. ■

Remark. The regularity conditions in the above statement are overkill. In fact, the inequality even holds in Sobolev spaces.

Theorem 2.6 Isoperimetric inequality

Let $\alpha: I \rightarrow \mathbb{R}^2$ be a simple, closed, regular curve in \mathbb{R}^2 , with Ω the domain bounded by α given by the Jordan curve theorem. Then

$$\text{length}(\alpha)^2 \geq 4\pi \text{area}(\Omega)$$

with equality iff Ω is a disc.

Proof. WLOG, we may assume that α is parametrised by arc length, $\alpha: [0, \ell] \rightarrow \mathbb{R}^2$, where $\ell = \text{length}(\alpha)$. Write $\alpha(s) = (x(s), y(s))$. Define the vector field $V: \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \mapsto (x, y)$. By the divergence theorem, we have

$$\int_0^\ell \langle V, n \rangle ds = \int_\Omega \text{div } V \, dx dy = 2 \text{area}(\Omega)$$

It then follows that

$$2 \text{area}(\Omega) = \left| \int_0^\ell \langle V, n \rangle ds \right| \leq \int_0^\ell |\langle V, n \rangle| ds \leq \int_0^\ell |V| ds = \int_0^\ell \sqrt{x(s)^2 + y(s)^2} ds$$

By translation, we may assume WLOG that

$$\int_0^\ell x(s) ds = \int_0^\ell y(s) ds = 0$$

This is possible by the intermediate value theorem. By the Cauchy-Schwarz inequality and Wirtinger's inequality (Lemma 2.5), we have

$$\begin{aligned} 2 \text{area}(\Omega) &\leq \int_0^\ell \sqrt{x(s)^2 + y(s)^2} ds \\ &\leq \left(\int_0^\ell x(s)^2 + y(s)^2 ds \right)^{1/2} \left(\int_0^\ell 1 ds \right)^{1/2} \\ &\leq \sqrt{\ell} \left(\int_0^\ell x(s)^2 ds + \int_0^\ell y(s)^2 ds \right)^{1/2} \\ &\leq \sqrt{\ell} \left(\frac{\ell^2}{4\pi^2} \int_0^\ell \dot{x}(s)^2 + \dot{y}(s)^2 ds \right)^{1/2} \\ &= \frac{\ell^2}{2\pi} \end{aligned}$$

Upon rearranging, we get $\text{length}(\alpha)^2 \geq 4\pi \text{area}(\Omega)$. Finally, note that equality in Cauchy-Schwarz is achieved above iff $x(s)^2 + y(s)^2 = \text{const}$, i.e. Ω is a disc. ■

2.3 First fundamental form

Definition Surface

A surface is a 2-dimensional manifold $X \subset \mathbb{R}^N$.

In what follows, we will for the most part be discussing surfaces in \mathbb{R}^3 .

Let $S \subset \mathbb{R}^3$ be a surface and $I = [a, b]$ with $a < b$. Let $\alpha: I \rightarrow \mathbb{R}^3$ be a regular curve with $\alpha(I) \subset S$. As before, the length of α is

$$\text{length}(\alpha) = \int_a^b \langle \dot{\alpha}(\lambda), \dot{\alpha}(\lambda) \rangle^{1/2} d\lambda$$

where $\dot{\alpha}(\lambda) = d\alpha_\lambda(1) \in T_{\alpha(\lambda)}S \subset \mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ is the inner product on \mathbb{R}^3 . Of course, for the surface S , we do not really need the full inner product on \mathbb{R}^3 , leading us to the following definition.

Definition First fundamental form

Let $S \subset \mathbb{R}^3$ be a surface. The first fundamental form of S at $p \in S$ is the map $I_p: T_pS \times T_pS \rightarrow \mathbb{R}$ given by the restriction $I_p = \langle \cdot, \cdot \rangle|_{T_pS \times T_pS}$.

Remark. For any Euclidean transformation Φ that sends $x \mapsto Rx + b$ for some $R \in O(3)$ and $b \in \mathbb{R}^3$, we can define $\phi = \Phi|_S: S \rightarrow \tilde{S} := \Phi(S)$. Let I_p be the FFF of S at $p \in S$ and $\tilde{I}_{\phi(p)}$ the FFF of \tilde{S} at $\phi(p)$. Observe that $I_p(v, w) = \tilde{I}_{\phi(p)}(d\phi_p(v), d\phi_p(w))$ for all $v, w \in T_pS$: indeed, this follows from $d\phi_p = d\Phi_p|_{T_pS} = R|_{T_pS}$ and $\langle Rv, Rw \rangle = \langle v, w \rangle$.

Definition Isometry

Let $S, \tilde{S} \subset \mathbb{R}^3$ be surfaces. We say that a diffeomorphism $\phi: S \rightarrow \tilde{S}$ of surfaces is an isometry if, for every $p \in S$ and $v, w \in T_pS$,

$$I_p(v, w) = \tilde{I}_{\phi(p)}(d\phi_p(v), d\phi_p(w))$$

Remark. By the previous remark, the restriction of a Euclidean transformation is an isometry.

Lemma 2.7

If $\phi: S \rightarrow \tilde{S}$ is a diffeomorphism of surfaces, then

$$\text{length}_{\tilde{S}}(\phi \circ \alpha) = \text{length}_S(\alpha)$$

for any curve $\alpha: [a, b] \rightarrow \mathbb{R}^3$ with $\alpha([a, b]) \subset S$.

Proof. Trivial computation. ■

Exercise. Show that the converse holds as well.

Let $X \subset \mathbb{R}^N$ be a manifold. A (smooth) Riemannian metric on X is a collection $(g_p)_{p \in X}$ of maps $g_p: T_pX \times T_pX \rightarrow \mathbb{R}$ such that

- (i) for every $v, w \in T_pX$, $g_p(v, w) = g_p(w, v)$
 - (ii) for every $v_1, v_2, w \in T_pX$ and $\lambda, \mu \in \mathbb{R}$, $g_p(\lambda v_1 + \mu v_2, w) = \lambda g_p(v_1, w) + \mu g_p(v_2, w)$
 - (iii) for every $v \in T_pX$, $g_p(v, v) \geq 0$, with equality iff $v = 0$
- and $p \mapsto g_p$ is smooth.

This allows us to define length of a curve $\alpha: [a, b] \rightarrow (X, g)$ by

$$\text{length}_g(\alpha) = \int_a^b \sqrt{g_{\alpha(\lambda)}(\dot{\alpha}(\lambda), \dot{\alpha}(\lambda))} d\lambda$$

We then define an isometry to be a diffeomorphism $\phi: X \rightarrow \tilde{X}$ that satisfies

$$g_p(v, w) = \tilde{g}_{\phi(p)}(d\phi_p(v), d\phi_p(w))$$

for all $v, w \in T_p X$. Again, a diffeomorphism is an isometry iff it preserves the lengths of all curves.

Now, let $S \subset \mathbb{R}^3$ be a surface, with $\varphi: U \rightarrow W$ a local parametrisation from $U \subset \mathbb{R}_{(u,v)}^2$ to an open set $W \subset S$. Note that $I_{\varphi(u,v)}$ is completely determined by

$$E(u, v) := I_{\varphi(u,v)}(\varphi_u, \varphi_u) = \langle \varphi_u, \varphi_u \rangle$$

$$F(u, v) := I_{\varphi(u,v)}(\varphi_u, \varphi_v) = \langle \varphi_u, \varphi_v \rangle$$

$$G(u, v) := I_{\varphi(u,v)}(\varphi_v, \varphi_v) = \langle \varphi_v, \varphi_v \rangle$$

In terms of E, F, G , we can write the length of α (viewed as $\alpha(\lambda) = \varphi(u(\lambda), v(\lambda))$) as

$$\text{length}(\alpha) = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} d\lambda$$

Remarks

- If $\phi: S \rightarrow \tilde{S}$ is an isometry and φ is a local parametrisation of S , then $\phi \circ \varphi$ is a local parametrisation of \tilde{S} . Writing $E = \langle \varphi_u, \varphi_u \rangle$ and $\tilde{E} = \langle (\phi \circ \varphi)_u, (\phi \circ \varphi)_u \rangle$ (and similarly for F and G), it can be shown that

$$E = \tilde{E}, \quad F = \tilde{F}, \quad G = \tilde{G}$$

- Under a different local parametrisation $\tilde{\varphi}$, we get different functions $\tilde{E}(\tilde{u}, \tilde{v}), \tilde{F}(\tilde{u}, \tilde{v}), \tilde{G}(\tilde{u}, \tilde{v})$. However, it is easy to verify that the expressions for length are equivalent under the two parametrisations.

Definition Area

Let $S \subset \mathbb{R}^3$ be a surface and $\varphi: U \rightarrow W$ be a local parametrisation of S . Let $\Omega \subset S$ be an open subset such that $\Omega \subset W$. The area of Ω is defined to be

$$\text{area}(\Omega) := \int_{\varphi^{-1}(\Omega)} |\varphi_u \times \varphi_v| du dv$$

Exercise. Show that the area of Ω is well-defined.

Lemma 2.8

The area of open subset $\Omega \subset S$ can be expressed as

$$\text{area}(\Omega) = \int_{\varphi^{-1}(\Omega)} \sqrt{EG - F^2} du dv$$

Exercise. Prove the above lemma.

Example Torus

Consider

$$\begin{aligned} \varphi: [0, 2\pi] \times [0, 2\pi] &\rightarrow \mathbb{R}^3 \\ (u, v) &\mapsto ((a + r \cos u) \cos v, (a + r \cos u) \sin v, r \sin u) \end{aligned}$$

Note that $\text{im } \varphi$ defines a torus T^2 , with $\varphi|_{(0,2\pi)^2}$ a local parametrisation away from a negligible subset. It is easy to compute $E = r^2$, $F = 0$, $G = (r \cos u + a)^2$ and thus $\text{area}(T^2) = 4\pi^2 ra$.

2.4 Second fundamental form

Let $S \subset \mathbb{R}^3$ be a surface and $\varphi: U \rightarrow W$ a local parametrisation of S . Note that

$$\frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}$$

defines a unit vector that is normal to $T_{\varphi(u,v)}S$.

Definition Gauss map

A Gauss map of S is a smooth map $N: S \rightarrow S^2$ such that $N(p) \perp T_p S$.

Remark. If N is a Gauss map of S and $\varphi: U \rightarrow W$ a local parametrisation of S , with U connected, then

$$N|_W(\varphi(u, v)) = \pm \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}(u, v)$$

Definition Orientable and oriented surface

If there exists a Gauss map on S , then we say that S is orientable. In that case, if we equip S with a choice of Gauss map N , then we say that (S, N) is oriented.

Example

- S^2 is an orientable surface since $N = \pm \text{id}$ both define Gauss maps.
- The Möbius strip is not orientable.

Now, let (S, N) be an oriented surface. Note that the differential of N at $p \in S$ is a map $T_p S \rightarrow T_{N(p)} S^2$. But $T_{N(p)} S^2 = T_p S$ since both are planes with normal $N(p)$. Thus, dN_p is a linear map $T_p S \rightarrow T_p S$.

Proposition 2.9

dN_p is self-adjoint with respect to I_p , i.e. for every $v, w \in T_p S$,

$$I_p(dN_p(w_1), w_2) = I_p(w_1, dN_p(w_2))$$

Proof. By linearity, it suffices to show that it holds for $w_1, w_2 \in \{\varphi_u, \varphi_v\}$. The result is trivial if $w_1 = w_2$, so in fact we just have to show that $I_p(dN_p(\varphi_u), \varphi_v) = I_p(\varphi_u, dN_p(\varphi_v))$. To see this, observe that $\langle N, \varphi_u \rangle = 0$ implies that

$$\begin{aligned} \langle N_v, \varphi_u \rangle + \langle N, \varphi_{uv} \rangle &= 0 \\ \langle N_u, \varphi_v \rangle + \langle N, \varphi_{uv} \rangle &= 0 \end{aligned}$$

Thus, $I_p(dN_p(\varphi_u), \varphi_v) = \langle N_u, \varphi_v \rangle = \langle N_v, \varphi_u \rangle = I_p(\varphi_u, dN_p(\varphi_v))$. ■

Definition Second fundamental form

Let (S, N) be an oriented surface. The second fundamental form of (S, N) at $p \in S$ is the map $\text{II}_p: T_p S \times T_p S \rightarrow \mathbb{R}$ defined by $(v, w) \mapsto -I_p(dN_p(v), w)$.

Remarks

- By self-adjointness of dN_p (Proposition 2.9), II_p is symmetric. Moreover, it is bilinear. The associated quadratic form is written as $\text{II}_p(v) := \text{II}_p(v, v)$.

- Π_p is a proper Euclidean invariant: for any proper Euclidean transformation $x \mapsto Rx + b$ for $R \in SO(3)$ and $b \in \mathbb{R}^3$, we can define the set $\tilde{S} := R(S) + b$. It is easy to see that \tilde{S} is a surface with orientation \tilde{N} defined by $\tilde{N} \circ \psi = R \circ N$. Let $\phi: S \rightarrow \tilde{S}$ be the restriction of the proper Euclidean transformation to S . Then

$$\begin{aligned} \tilde{\Pi}_{\phi(p)}(d\phi_p(v), d\phi_p(w)) &= -\tilde{\mathbf{I}}_{\phi(p)}(d\tilde{N}_{\phi(p)}(d\phi_p(v)), d\phi_p(w)) \\ &= -\tilde{\mathbf{I}}_{\phi(p)}(d\phi_p(dN_p(v)), d\phi_p(w)) \\ &= -\mathbf{I}_p(dN_p(v), w) \\ &= \Pi_p(v, w) \end{aligned}$$

where we have used the Euclidean invariance of \mathbf{I}_p in going to the penultimate line.

As before, let (S, N) be an oriented surface and $p \in S$. We define

$$\kappa_1(p) := \max_{\substack{v \in T_p S \\ \mathbf{I}_p(v, v) = 1}} \Pi_p(v, v), \quad \kappa_2(p) := \min_{\substack{v \in T_p S \\ \mathbf{I}_p(v, v) = 1}} \Pi_p(v, v)$$

which exist by compactness of the set $\{v \in T_p S : \mathbf{I}_p(v, v) = 1\}$.

Proposition 2.10

There exists an orthonormal basis $\{e_1, e_2\}$ of $T_p S$ such that, for $i \in \{1, 2\}$,

$$dN_p(e_i) = -\kappa_i e_i$$

Exercise. Prove the above proposition.

Definition Principal curvatures and directions

We call $\kappa_1(p)$ and $\kappa_2(p)$ the principal curvatures of S at p . Moreover, e_1 and e_2 are called the principal directions at p .

Remark. The principal curvatures are proper Euclidean invariants.

Definition Normal curvature

Let (S, N) be an oriented surface and $\alpha: I \rightarrow S$ be a regular curve parametrised by arc length. The normal curvature of α at $s \in I$ is

$$\kappa_n(s) := \langle \ddot{\alpha}(s), (N \circ \alpha)(s) \rangle = \langle \kappa(s)n(s), (N \circ \alpha)(s) \rangle$$

Remark. It is immediate from the definition that $|\kappa_n(s)| \leq \kappa(s)$.

Proposition 2.11 Meusnier's theorem

Let (S, N) and $\alpha: I \rightarrow S$ be as above. Then the normal curvature at $s \in I$ is given by

$$\kappa_n(s) = \Pi_p(\dot{\alpha}(s))$$

Definition Mean and Gauss curvatures

Let (S, N) be an oriented surface. The mean curvature of S at $p \in S$ is

$$H(p) := \frac{1}{2} \text{tr}(-dN_p)$$

The Gauss curvature of S at $p \in S$ is

$$K(p) := \det(-dN_p)$$

Lemma 2.12

The mean and Gauss curvatures are also given by

$$H(p) = \frac{1}{2}(\kappa_1(p) + \kappa_2(p)), \quad K(p) = \kappa_1(p)\kappa_2(p)$$

Proof. This follows immediately from Proposition 2.10. ■

Remark. Note that changing $N \mapsto -N$ will result κ_1, κ_2 changing sign. Thus, H changes sign but K does not.

Example

- For a sphere S^2 , we have $\kappa_1 = \kappa_2 = 1$ so $H = K = 1$.
- For a cylinder, $\kappa_2 = 0$ so $K = 0$. However, $\kappa_1 > 0$ so $H > 0$.
- Consider a torus T^2 . By considering various circles, one can see that K can have any sign depending on where one is on T^2 .