

Linear Analysis

Lecturer:

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Course schedule

Normed and Banach spaces. Linear mappings, continuity, boundedness, and norms. Finite-dimensional normed spaces. [4]

The Baire category theorem. The principle of uniform boundedness, the closed graph theorem and the inversion theorem; other applications. [5]

The normality of compact Hausdorff spaces. Urysohn's lemma and Tietze's extension theorem. Spaces of continuous functions. The Stone–Weierstrass theorem and applications. Equicontinuity: the Ascoli–Arzelá theorem. [5]

Inner product spaces and Hilbert spaces; examples and elementary properties. Orthonormal systems, and the orthogonalization process. Bessel's inequality, the Parseval equation, and the Riesz–Fischer theorem. Duality; the self duality of Hilbert space. [5]

Bounded linear operations, invariant subspaces, eigenvectors; the spectrum and resolvent set. Compact operators on Hilbert space; discreteness of spectrum. Spectral theorem for compact Hermitian operators. [5]

Recommended books

B. Bollobas *Linear Analysis*. Cambridge University Press 1999.

G.J.O. Jameson *Topology and Normed Spaces*. Chapman and Hall 1974.

G. Allan *Introduction to Banach Spaces and Algebras*. Oxford University Press 2010.

W. Rudin *Real and Complex Analysis*. McGraw–Hill International Edition 1987.

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1 Normed spaces and bounded linear maps

1.1 Definitions and examples

Let X be a vector space over \mathbb{R} or \mathbb{C} . For ease of notation and discussion, we will sometimes just take our scalars to be in \mathbb{R} , although the statement may be easily generalised to \mathbb{C} -vector spaces.

Definition Norm

A norm on X is a function $\|\cdot\|: X \rightarrow \mathbb{R}$ such that

- (i) $\|x\| \geq 0$ for all $x \in X$, with $\|x\| = 0$ iff $x = 0$
- (ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and any scalar λ
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

Definition Normed space

A normed space is a pair $(X, \|\cdot\|)$ where X is a vector space and $\|\cdot\|$ is a norm on X .

Example Some finite-dimensional normed spaces

- (1) $\ell_2^n = (\mathbb{R}^n, \|\cdot\|_2)$ or $(\mathbb{C}^n, \|\cdot\|_2)$, where the norm is given by

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

This is called the ℓ_2 -norm or euclidean norm.

(i),(ii) are easy to check, whereas (iii) follows from Cauchy-Schwarz.

- (2) $\ell_1^n = (\mathbb{R}^n, \|\cdot\|_1)$ where $\|x\|_1 = \sum_{i=1}^n |x_i|$ (called the ℓ_1 -norm)

- (3) $\ell_\infty^n = (\mathbb{R}^n, \|\cdot\|_\infty)$ where $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ (called the ℓ_∞ -norm or the sup-norm)

Given a normed space X , its norm $\|\cdot\|$ induces a metric on X :

$$d(x, y) = \|x - y\|$$

Indeed, d is a metric:

- $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0 \iff x - y = 0 \iff x = y$
- $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$
- $d(x, z) = \|x - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$

This metric, in turn, induces a topology on X , called the *norm topology*. This allows us talk about open/closed sets, convergence, and continuity, as we illustrate in the following examples.

Example

The algebraic operations are continuous:

- if $x_n \rightarrow x$ and $y_n \rightarrow y$ in X , then $x_n + y_n \rightarrow x + y$
- if $x_n \rightarrow x$ in X and $\lambda_n \rightarrow \lambda$ in \mathbb{R} , then $\lambda_n x_n \rightarrow \lambda x$

Example

The norm $\|\cdot\|: X \rightarrow \mathbb{R}$ is continuous: by the triangle inequality, we have

$$|||x| - |y|| \leq \|x - y\|$$

so $\|\cdot\|$ is, in fact, Lipschitz.

Definition Banach space

A Banach space is a complete normed space, i.e., a normed space that is complete in its norm topology.

Example

$\ell_2^n, \ell_1^n, \ell_\infty^n$ are complete: for any of these spaces,

- $x^{(k)} \rightarrow x \iff x_i^{(k)} \rightarrow x_i$ for all $1 \leq i \leq n$
- $(x^{(k)})_{k \in \mathbb{N}}$ is Cauchy $\iff (x_i^{(k)})_{k \in \mathbb{N}}$ is Cauchy for all $1 \leq i \leq n$

In a normed space, a useful object is the *unit ball*

$$B_X := \{x \in X : \|x\| \leq 1\}$$

Remarks

- B_X defines a norm on X :

$$\|x\| = \inf\{t \geq 0 : x \in tB_X\}$$

- B_X is symmetric ($x \in B_X \implies -x \in B_X$), convex, and closed
- If $B \subset \mathbb{R}^n$ is a closed, convex, symmetric, bounded neighbourhood of 0, then B is the unit ball of $(\mathbb{R}^n, \|\cdot\|)$ for some norm $\|\cdot\|$
- ‘Geometry of Banach spaces’

Previously, we gave $\ell_2, \ell_1, \ell_\infty$ as examples of finite-dimensional normed spaces. More generally, we have the following family of examples

Example

- (4) $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ for $1 \leq p < \infty$, where $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ (called the ℓ_p -norm)

Again, (i) and (ii) are easy to check, whereas (iii) is not obvious.¹

Now, let S denote the set of all scalar sequences. This is a vector spaces under the coordinate operations $(x_n) + (y_n) = (x_n + y_n)$ and $\lambda(x_n) = (\lambda x_n)$.

Example Sequence spaces

$$(5) \ell_1 = \left\{ (x_n) \in S : \sum_{n=1}^{\infty} |x_n| < \infty \right\}, \quad \|(x_n)\|_1 = \sum_{n=1}^{\infty} |x_n| \quad (\ell_1\text{-norm})$$

(i) and (ii): easy to check.

(iii): Given $(x_n), (y_n) \in \ell_1$, we have $|x_n + y_n| \leq |x_n| + |y_n|$ for all $n \in \mathbb{N}$. Summing over all $n \in \mathbb{N}$, we deduce that $(x_n) + (y_n) \in \ell_1$ and $\|(x_n) + (y_n)\|_1 \leq \|(x_n)\|_1 + \|(y_n)\|_1$.

Hence, ℓ_1 is a subspace of S and $\|\cdot\|_1$ is a norm on ℓ_1 .

$$(6) \ell_2 = \left\{ (x_n) \in S : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}, \quad \|(x_n)\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} \quad (\ell_2\text{-norm})$$

(i) and (ii): easy to check.

(iii): Given $(x_n), (y_n) \in \ell_2$, the triangle inequality in ℓ_2^N gives us

$$\left(\sum_{k=1}^N |x_k + y_k|^2 \right)^{1/2} \leq \left(\sum_{k=1}^N |x_k|^2 \right)^{1/2} + \left(\sum_{k=1}^N |y_k|^2 \right)^{1/2}.$$

Taking $N \rightarrow \infty$, we get $(x_n) + (y_n) \in \ell_2$ and $\|(x_n) + (y_n)\|_2 \leq \|(x_n)\|_2 + \|(y_n)\|_2$

¹We will return to this later in the next subsection.

More generally, for $1 \leq p < \infty$, the set

$$\ell_p = \left\{ (x_n) \in S : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

is a subspace of S , and

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \quad (\ell_p\text{-norm})$$

is a norm on ℓ_p . [(iii) follows from the triangle inequality on ℓ_p^n , which we will see later.]

Example More sequence spaces

$$(7) \ell_{\infty} = \{(x_n) \in S : \exists M \geq 0 \forall n \in \mathbb{N} |x_n| \leq M\}, \quad \|(x_n)\|_{\infty} = \sup_{n \in \mathbb{N}} |x_n| \quad (\ell_{\infty}\text{-norm})$$

(i) and (ii): easy to check.

(iii): Given $x = (x_n), y = (y_n) \in \ell_{\infty}$,

$$|x_n + y_n| \leq |x_n| + |y_n| \leq \|x\|_{\infty} + \|y\|_{\infty} \quad \forall n \in \mathbb{N}$$

so $x + y \in \ell_{\infty}$ and $\|x + y\|_{\infty} \leq \|x\|_{\infty} + \|y\|_{\infty}$.

$$(8) c_0 = \{(x_n) \in S : x_n \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

$$c = \{(x_n) \in S : \lim_{n \rightarrow \infty} x_n \text{ exists}\}$$

Both c_0 and c are subspaces of ℓ_{∞} and are hence normed spaces in the ℓ_{∞} -norm.

1.2 Inequalities of Minkowski and Hölder

Recall that a function $f: (0, \infty) \rightarrow \mathbb{R}$ is *convex* if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) \quad \forall x, y \in (0, \infty) \forall t \in [0, 1]$$

and *concave* if the above holds with \leq replaced by \geq .

Lemma 1.1

Let $1 \leq p < \infty$. Then the map

$$(0, \infty) \rightarrow \mathbb{R}$$

$$x \mapsto x^p$$

is *convex*.

Proof. Fix $y > 0, t \in [0, 1]$, and define

$$g(x) = [(1-t)x + ty]^p - [(1-t)x^p + ty^p], \quad x > 0.$$

Differentiating, we get

$$g'(x) = p(1-t)[(1-t)x + ty]^{p-1} - p(1-t)x^{p-1}.$$

Observe that $0 < x < y \implies g'(x) \geq 0$ and that $x > y \implies g'(x) \leq 0$. By the MVT, we deduce that $g(x) \leq g(y) = 0$ for all $x \in (0, \infty)$. ■

Theorem 1.2 Minkowski's inequality

Let $1 \leq p < \infty, n \in \mathbb{N}$. For $x, y \in \mathbb{R}^n$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Remark. This shows that ℓ_p^n and ℓ_p are normed spaces.

Exercise. Show that $\ell_p, 1 \leq p \leq \infty$, is complete.²

Proof of Theorem 1.2. Let $B = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$. We first show that B is convex. Let $x, y \in B$ and $t \in [0, 1]$. For $1 \leq k \leq n$,

$$|(1-t)x_k + ty_k|^p \leq ((1-t)|x_k| + t|y_k|)^p \leq (1-t)|x_k|^p + t|y_k|^p$$

by Lemma 1.1 for $x_k \neq 0, y_k \neq 0$; the inequality holds trivially if $x_k = 0$ or $y_k = 0$. Summing over k , we then get

$$\|(1-t)x + ty\|_p^p \leq (1-t)\|x\|_p^p + t\|y\|_p^p \leq 1,$$

so $(1-t)x + ty \in B$.

We then complete the proof as follows. Let $x, y \in \mathbb{R}^n$. WLOG, $x, y, x+y$ are nonzero. By convexity of B , we have

$$\frac{x+y}{\|x\|_p + \|y\|_p} = \frac{\|x\|_p}{\|x\|_p + \|y\|_p} \cdot \underbrace{\frac{x}{\|x\|_p}}_{\in B} + \frac{\|y\|_p}{\|x\|_p + \|y\|_p} \cdot \underbrace{\frac{y}{\|y\|_p}}_{\in B} \in B.$$

Thus, it follows that

$$\left\| \frac{x+y}{\|x\|_p + \|y\|_p} \right\| \leq 1 \implies \|x+y\|_p \leq \|x\|_p + \|y\|_p,$$

as required. ■

Let $x = (x_n) \in \ell_1$ and $y = (y_n) \in \ell_\infty$. We then write $x \cdot y = (x_n y_n)$. Note that, for all $n \in \mathbb{N}$, $|x_n y_n| = |x_n| |y_n| \leq |x_n| \|y\|_\infty$. Thus, $x \cdot y \in \ell_1$ and $\|x \cdot y\|_1 \leq \|x\|_1 \|y\|_\infty$.

Definition Conjugate index

Let $p \in (1, \infty)$. The conjugate index of p is the unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.3

Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $a, b \geq 0$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. The inequality holds trivially if $a = 0$ or $b = 0$, so it remains to consider the case $a, b > 0$. A proof similar to that of Lemma 1.1 shows that $\log: (0, \infty) \rightarrow \mathbb{R}$ is concave. Hence,

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p} \log(a^p) + \frac{1}{q} \log(b^q) = \log(ab).$$

We then apply exp to get the required result. ■

Theorem 1.4 Hölder's inequality

Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in \ell_p$ and $y \in \ell_q$, then $x \cdot y \in \ell_1$ and

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q.$$

Remark. As discussed above, $p = 1, q = \infty$ also works. Moreover, setting $p = q = 2$, we recover Cauchy-Schwarz.

Exercise. Deduce Minkowski's inequality from Hölder's inequality.

²A slick proof of this will be provided later.

Proof of Theorem 1.4. WLOG, $x \neq 0$ and $y \neq 0$. By homogeneity, we may also take $\|x\|_p = \|y\|_q = 1$ WLOG. Now, by Lemma 1.3, we have $|x_n y_n| \leq |x_n|^p/p + |y_n|^q/q$ for all $n \in \mathbb{N}$. Summing over n , we have

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \frac{\|x\|_p^p}{p} + \frac{\|y\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 = \|x\|_p \|y\|_q,$$

as required. ■

1.3 More examples: function spaces

Example

- (9) $C[0, 1] = \{f: [0, 1] \rightarrow \mathbb{R} \mid f \text{ cts}\}$, $\|f\|_{\infty} = \sup_{[0,1]} |f|$ (sup norm or uniform norm)

By the uniform limit theorem, this is a Banach space.

- (10) More generally, given a compact, Hausdorff topological space K ,

$$C(K) = \{f: K \rightarrow \mathbb{R} \mid f \text{ cts}\}$$

is a Banach space in the sup norm $\|f\|_{\infty} = \sup_K |f|$.

- (11) $(C[0, 1], \|\cdot\|_1)$, $\|f\|_1 = \int_0^1 |f(t)| dt$ (L_1 -norm)

This is an *incomplete* normed space — see Example Sheet 1.

More generally, $C[0, 1]$ is incomplete in the L_p -norm, $1 \leq p < \infty$, given by

$$\|f\|_p = \left(\int_0^1 |f(t)|^p dt \right)^{1/p}.$$

In II Probability and Measure, you will encounter the completion of $(C[0, 1], \|\cdot\|_p)$, which is the Lebesgue space $L_p[0, 1]$.

- (12) $C^1[0, 1] = \{f \in C[0, 1] \mid f \text{ continuously differentiable}\}$ is a subspace of $C[0, 1]$, so it is a normed space in $\|\cdot\|_{\infty}$ but incomplete, i.e. not closed in $C[0, 1]$. However, it is complete in the norm $\|f\| = \|f\|_{\infty} + \|f'\|_{\infty}$ — see Example Sheet 1.

- (13) Let $\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$. The set

$$A(\Delta) = \{f \in C(\Delta) \mid f \text{ analytic on int } \Delta\}$$

is a subspace of $C(\Delta)$. In fact, it is closed in $C(\Delta)$ and hence a Banach space in $\|\cdot\|_{\infty}$.

1.4 More on the normed topology

Let X be a normed space and $A \subset X$. Recall that the *closure* of A in X is

$$\overline{A} = \{x \in X \mid \exists (a_n) \text{ in } A \text{ s.t. } a_n \rightarrow x \text{ as } n \rightarrow \infty\}.$$

We then say that A is *dense* in X if $\overline{A} = X$. Moreover, A is *separable* if it has a countable dense subset.

If $Y \subset X$ is a subspace, then so is \overline{Y} : if $x, y \in \overline{Y}$, then there exists $(x_n), (y_n)$ in Y such that $x_n \rightarrow x$ and $y_n \rightarrow y$. So $\lambda x_n + \mu y_n \rightarrow \lambda x + \mu y \in \overline{Y}$. Similarly, if $A \subset X$ is convex, then so is \overline{A} .

For a subset $A \subset X$, the *closed linear span* of A , denoted by $\overline{\text{span}} A$, is the closure of $\text{span } A$.

Remarks

- If A is countable, then $\overline{\text{span}} A$ is separable.
- The set of all rational linear combinations of elements of A is countable and dense in $\overline{\text{span}} A$.

Example

- $\overline{\mathbb{Q}} = \mathbb{R}$, so \mathbb{R} is separable.
- $\ell_p, 1 \leq p < \infty$, is separable.

Let $e_n = (0, \dots, 0, \underset{n}{1}, 0, \dots)$, $n \in \mathbb{N}$ (unit vector basis)

Let $c_{00} = \text{span}\{e_n : n \in \mathbb{N}\} = \{(x_n) \in S : \exists N \in \mathbb{N} \forall n > N x_n = 0\}$

We then show that $\ell_p = \overline{\text{span}}\{e_n : n \in \mathbb{N}\}$: if $x = (x_n) \in \ell_p$, then

$$\left\| x - \sum_{i=1}^N x_i e_i \right\|_p = \left(\sum_{i>N} |x_i|^p \right)^{1/p} \rightarrow 0 \text{ as } N \rightarrow \infty$$

- Similarly, in ℓ_∞ , we have $\overline{\text{span}}\{e_n : n \in \mathbb{N}\} = c_0$. Moreover, c is separable, whereas ℓ_∞ is not.

Exercise. Prove the claims in the last example above.

1.5 Bounded linear maps**Theorem 1.5**

Let X, Y be normed spaces and $T : X \rightarrow Y$ be a linear map. The following are equivalent:

- T is continuous at 0
- T is continuous
- T is Lipschitz
- T is bounded, i.e., $\exists C \geq 0 \forall x \in X \|Tx\| \leq C\|x\|$.

Proof. (iv) \implies (iii): Observe that

$$d(Tx, Ty) = \|Tx - Ty\| = \|T(x - y)\| \leq C\|x - y\| = Cd(x, y)$$

iii) \implies (ii): Given $\varepsilon > 0$ take $\delta = \varepsilon/(C + 1)$.

(ii) \implies (i): Trivial.

(i) \implies (iv): $\exists \delta > 0 \forall x \in X d(x, 0) = \|x\| \leq \delta \implies d(Tx, T0) = \|Tx\| \leq 1$. For $x \neq 0$, $\|\delta x / \|x\|\| = \delta$, so $\|T(\delta x / \|x\|)\| \leq 1$. Hence, $\|Tx\| \leq \delta^{-1}\|x\|$. ■

For normed spaces X, Y , let $\mathcal{B}(X, Y) = \{T : X \rightarrow Y \mid T \text{ linear and bounded}\}$. For $T \in \mathcal{B}(X, Y)$, its *operator norm* is

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

Remark. Since $T \in \mathcal{B}(X, Y)$, we have $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$. So if $\|x\| \leq 1$, then $\|Tx\| \leq C$. Thus, by definition, $\|T\| \leq C$. Conversely, for all $x \in B_X$, we have $\|Tx\| \leq \|T\|$, so by homogeneity, $\|Tx\| \leq \|T\|\|x\|$. Hence, $\|T\|$ is the least C such that (iv) in Theorem 1.5 above holds.

The operator norm is a norm on $\mathcal{B}(X, Y)$: given $S, T \in \mathcal{B}(X, Y)$, we have, for all $x \in X$,

$$\|(S + T)x\| = \|Sx + Tx\| \leq \|Sx\| + \|Tx\| \leq \|S\|\|x\| + \|T\|\|x\| \leq (\|S\| + \|T\|)\|x\|,$$

from which it follows that $S + T \in \mathcal{B}(X, Y)$ and $\|S + T\| \leq \|S\| + \|T\|$.

Notation. We write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$.

Proposition 1.6

Let X, Y, Z be normed spaces, $S \in \mathcal{B}(X, Y)$, $T \in \mathcal{B}(Y, Z)$. Then $TS \in \mathcal{B}(X, Z)$ and $\|TS\| \leq \|T\|\|S\|$.

Proof. For all $x \in X$, we have $\|TSx\| \leq \|T\|\|Sx\| \leq \|T\|\|S\|\|x\|$. ■

Example

- (1) $T: \ell_2^n \rightarrow \ell_2^n, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$

$$\|Tx\|_2 = \left(\sum_{i=1}^r |x_i|^2 \right)^{1/2} \leq \|x\|_2 \implies \|T\| \leq 1$$

But $Te_1 = e_1$ so $\|T\| = 1$.

More generally, if T is represented by a matrix A wrt the standard basis, then Cauchy-Schwarz gives us

$$\|T\| \leq \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}$$

- (2) Let $1 \leq p < \infty$; $R: \ell_p \rightarrow \ell_p, (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$ (right shift)

For all $x \in \ell_p$, $\|Rx\|_p = \|x\|_p$, so R is isometric and $\|R\| = 1$. Note that R is injective but not surjective.

- (3) Let $1 \leq p < \infty$; $L: \ell_p \rightarrow \ell_p, (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$ (left shift)

For all $x \in \ell_p$, $\|Lx\|_p \leq \|x\|_p$, so $L \in \mathcal{B}(\ell_p)$ with $\|L\| \leq 1$. Since $Le_2 = e_1$ and $\|e_1\|_p = \|e_2\|_p = 1$, we in fact have $\|L\| = 1$. Note that L is surjective but not injective.

- (4) $T: \ell_1 \rightarrow \ell_2, x \mapsto x$

► **Claim.** $\ell_1 \subset \ell_2$, and $\forall x \in \ell_2$ $\|x\|_2 \leq \|x\|_1$

Proof. WLOG assume $\|x\|_1 = 1$ by homogeneity. Since $\sum_{n=1}^{\infty} |x_n| = 1$, we have $|x_i| \leq 1$ for all i . Thus,

$$|x_i|^2 \leq |x_i| \quad \forall i \implies \|x\|_2^2 \leq \|x\|_1 = 1 \implies \|x\|_2 = 1 = \|x\|_1$$

as claimed. ■

Using the above claim, we have $T \in \mathcal{B}(\ell_1, \ell_2)$ and $\|T\| = 1$.

- (5) $T: \ell_2 \rightarrow \ell_1, (x_n) \mapsto (x_n/n)$

By Cauchy-Schwarz,

$$\sum_{n=1}^{\infty} \left| \frac{x_n}{n} \right| \leq \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2}$$

so $T \in \mathcal{B}(\ell_2, \ell_1)$ with $\|T\| \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2}$. In fact, we can replace \leq with $=$.

- (6) $D: (C^1[0, 1], \|\cdot\|) \rightarrow (C[0, 1], \|\cdot\|_{\infty}), f \mapsto f'$

Note that $\|Df\|_{\infty} = \|f'\|_{\infty} \leq \|f\|_{\infty} + \|f'\|_{\infty} = \|f\|$, so $\|D\| \leq 1$. But taking $f(x) = \sin(n\pi x)$, we have

$$\|Df\|_{\infty} = n\pi, \quad \|f\| = n\pi + 1,$$

so in fact $\|D\| = 1$. Note also that, for $f \neq 0$, $\|Df\|_\infty < \|f\|$, so $\|D\|$ is not attained.

(7) On a normed space X , the identity $x \mapsto x$ is denoted by Id , I , Id_X or I_X . This map is isometric, i.e., $\|\text{Id}(x)\| = \|x\| \forall x \in X$.

(8) For normed spaces X, Y , we let

$$X \oplus Y = \{(x, y) : x \in X, y \in Y\}$$

with norm $\|(x, y)\|_1 = \|x\| + \|y\|$. The corresponding norm topology is the product topology.

Define $P: X \oplus Y \rightarrow X$, $(x, y) \mapsto x$ (projection onto X). Note that $P \in \mathcal{B}(X \oplus Y, X)$ with $\|P\| = 1$.

Let X, Y be normed spaces. We introduce some terminology:

- An *isomorphism* $X \rightarrow Y$ is a linear homeomorphism $T: X \rightarrow Y$, i.e., T is a linear bijection such that T and T^{-1} are bounded. Equivalently, T is a linear bijection³ such that

$$\exists a, b > 0 \forall x \in X \ a\|x\| \leq \|Tx\| \leq b\|x\|$$

If such T exists, we say that X and Y are *isomorphic*, and we write $X \sim Y$.

- An *isometric isomorphism* is a linear bijection $T: X \rightarrow Y$ such that

$$\forall x \in X \ \|Tx\| = \|x\|$$

If such T exists, we say that X and Y are *isometrically isomorphic*, and we write $X \cong Y$.

The Banach-Mazur distance is defined as

$$d(X, Y) = \begin{cases} \infty, & \text{if } X \not\sim Y \\ \inf\{\|T\|\|T^{-1}\| \mid T: X \rightarrow Y \text{ is an isomorphism}\}, & \text{otherwise} \end{cases}$$

Note that $\|T\|\|T^{-1}\| \geq \|TT^{-1}\| = 1$. If $X \cong Y$, then $d(X, Y) = 1$. Does the converse hold?

- An *isomorphic embedding* $X \rightarrow Y$ is a linear map $T: X \rightarrow Y$ such that $T: X \rightarrow TX = \text{im } T$ is an isomorphism. If such T exists, we say that X (*isomorphically*) *embeds into* Y , and we write $X \hookrightarrow Y$.

Definition Equivalent norms

Let X be a normed space. Two norms $\|\cdot\|, \|\cdot\|'$ are equivalent if

$$\text{Id}: (X, \|\cdot\|) \rightarrow (X, \|\cdot\|') \text{ is an isomorphism}$$

$$\iff \|\cdot\|, \|\cdot\|' \text{ induce the same norm topology on } X$$

$$\iff \exists a, b > 0 \forall x \in X \ a\|x\| \leq \|x\|' \leq b\|x\|$$

$$\iff \exists a, b > 0 \ aB'_X \subset B_X \subset bB'_X$$

Remarks

- If $X \sim Y$, then X is complete iff Y is complete.

If $\|\cdot\|, \|\cdot\|'$ are equivalent norms on a vector space X , then $(X, \|\cdot\|)$ is complete iff $(X, \|\cdot\|')$ is complete.

³We can actually replace ‘bijection’ with ‘surjection’.

- Let X and Y be normed spaces. On $X \oplus Y$, the norm $\|(x, y)\|_1 = \|x\| + \|y\|$ is equivalent to $\|(x, y)\|_p = (\|x\|^p + \|y\|^p)^{1/p}$ for all $1 \leq p < \infty$ and to $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$.
- $(C[0, 1], \|\cdot\|_\infty)$ is complete whereas $(C[0, 1], \|\cdot\|_1)$ is incomplete. Thus, we can use the first remark above to deduce that $\|\cdot\|_\infty \not\sim \|\cdot\|_1$ (but this can easily be proven directly as well). However, $\|f\|_1 = \int_0^1 |f(t)| dt \leq \|f\|_\infty$, so

$$\text{Id}: (C[0, 1], \|\cdot\|_\infty) \rightarrow (C[0, 1], \|\cdot\|_1)$$

is a continuous linear bijection but its inverse is not continuous.

- On c_{00} , $\|\cdot\|_1 \not\sim \|\cdot\|_2$. To see why, consider $x = (\underbrace{1, \dots, 1}_n, 0, 0, \dots)$ and note that $\|x\|_1 = n$, $\|x\|_2 = \sqrt{n}$.

Finally, we discuss convergence and completeness. Let X, Y be normed spaces. In $\mathcal{B}(X, Y)$, convergence implies pointwise convergence, i.e., if $T_n \rightarrow T$ in $\mathcal{B}(X, Y)$, then, for all $x \in X$, $T_n x \rightarrow T x$ in Y . To see why, note that, for fixed $x \in X$, we have $\|T_n x - T x\| \leq \|T_n - T\| \|x\| \rightarrow 0$. However, the converse is false in general, e.g., $T_n: \ell_1 \rightarrow \mathbb{R}, x \mapsto x_n$. We have $T_n \rightarrow 0$ pointwise, but $\|T_n\| = 1$ for all $n \in \mathbb{N}$.

Theorem 1.7

Let X, Y be normed spaces. If Y is complete, then $\mathcal{B}(X, Y)$ is complete.

Proof. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X, Y)$. Fix $x \in X$. Then

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \|x\| \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

So $(T_n x)$ is Cauchy in Y and thus convergent. Now, define $T: X \rightarrow Y$ by $x \mapsto \lim_{n \rightarrow \infty} T_n x$. Observe that

- T is linear

$$T(\lambda x + \mu y) = \lim_{n \rightarrow \infty} T_n(\lambda x + \mu y) = \lim_{n \rightarrow \infty} [\lambda T_n x + \mu T_n y] = \lambda T x + \mu T y$$

- T is bounded

(T_n) is Cauchy implies (T_n) is bounded, i.e., there exists $M \geq 0$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. Fix $x \in X$. Then, for all $n \in \mathbb{N}$, we have $\|T_n x\| \leq \|T_n\| \|x\| \leq M \|x\|$. Letting $n \rightarrow \infty$, we obtain $\|T x\| \leq M \|x\|$.

- $T_n \rightarrow T$ in $\mathcal{B}(X, Y)$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|T_m - T_n\| \leq \varepsilon$ for all $m, n \geq N$. Fix $x \in X$. Note that, for all $m, n \geq N$, we have

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \|x\| \leq \varepsilon \|x\|$$

Letting $n \rightarrow \infty$ with $m \geq N$ fixed yields $\|T_m x - T x\| \leq \varepsilon \|x\|$. Hence, $\|T_m - T\| \leq \varepsilon$ for all $m \geq N$. ■

2 Dual spaces

2.1 Basics

Let X be a normed space. A *functional* on X is a map $X \rightarrow \mathbb{R}$. The *dual space* X^* of X is the space of all bounded linear functionals on X , i.e., $X^* = \mathcal{B}(X, \mathbb{R})$ equipped with the operator norm. Since \mathbb{R} is complete, Theorem 1.7 gives us the following result.

Theorem 2.1

For any normed space X , its dual X^* is a Banach space.

Notation. For $x \in X$ and $f \in X^*$, we let $\langle x, f \rangle = f(x)$.

Now, we know that $0 \in X^*$. Are there other elements?

Theorem 2.2 Hahn-Banach theorem

Let X be a normed space, $Y \subset X$ be a subspace and $g \in Y^*$. Then $f \in X^*$ such that $f|_Y = g$ and $\|f\| = \|g\|$.

Proof. See II Analysis of Functions. ■

Corollary 2.3

Let X be a normed space, $x_0 \in X \setminus \{0\}$. Then there exists $f \in S_{X^*} = \{f \in X^*: \|f\| = 1\}$ such that $f(x_0) = \|x_0\|$.

Remarks

- For any $g \in B_{X^*}$, $|g(x_0)| \leq \|g\| \|x_0\| \leq \|x_0\|$. Corollary 2.3 says that there exists $f \in B_{X^*}$ such that $f(x_0) = \|x_0\|$, so

$$\|x_0\| = \sup\{g(x_0) : g \in B_{X^*}\} = \max\{g(x_0) : g \in B_{X^*}\}.$$

We call f a *norming functional* at x_0 .

- Given $x \neq y$ in X , we can set $x_0 = x - y$ and Corollary 2.3 implies that there exists $f \in X^*$ such that $f(x) \neq f(y)$. Thus, X^* separates the points of X .

Proof of Corollary 2.3. Set $Y = \text{span}\{x_0\}$ and define $g(\lambda x_0) = \lambda \|x_0\|$. Then $g \in S_{Y^*}$ with $g(x_0) = \|x_0\|$. Finally, apply Theorem 2.2. ■

2.2 Dual space of ℓ_p

Motivation: Recall that, for $1 \leq p < \infty$, we have $\ell_p = \overline{\text{span}\{e_n : n \in \mathbb{N}\}} = \overline{c_{00}}$. Given $\varphi \in \ell_p^*$ and $x = (x_n) \in \ell_p$,

$$\varphi(x) = \varphi\left(\lim_{n \rightarrow \infty} \sum_{k=1}^n x_k e_k\right) = \sum_{k=1}^{\infty} x_k \varphi(e_k)$$

so φ corresponds to the sequence $y = (\varphi(e_n))_{n \in \mathbb{N}}$. We may then ask: is $\ell_p^* \cong \ell_q$ for some q ?

Fix $1 < p < \infty$, and let q be the conjugate index of p . Fix $y = (y_n) \in \ell_q$. Define

$$\begin{aligned} \varphi_y : \ell_p &\rightarrow \mathbb{R} \\ x &\mapsto \sum_{n=1}^{\infty} x_n y_n \end{aligned}$$

By Holder's inequality (Theorem 1.4), this is well-defined and $|\varphi_y(x)| \leq \|x\|_p \|y\|_q$. So φ_y is linear and bounded: $\|\varepsilon_y\| \leq \|y\|_q$. Thus, $\varphi_y \in \ell_p^*$, which means that we have a map

$$\begin{aligned}\varphi: \ell_q &\rightarrow \ell_p^* \\ y &\mapsto \varphi_y\end{aligned}$$

Note that φ is linear and bounded with $\|\varphi\| \leq 1$.

Theorem 2.4

Let p, q, φ be as above. Then φ is an isometric isomorphism $\ell_q \rightarrow \ell_p^*$.

Proof. It remains to check that φ is isometric and surjective:

- φ is isometric

Fix $y \in \ell_q$. WLOG $y \neq 0$. Define

$$x_n = \begin{cases} \frac{|y_n|^q}{y_n}, & y_n \neq 0 \\ 0, & y_n = 0 \end{cases}$$

Observe that $\sum_{n=1}^{\infty} |x_n|^p = \sum_{n=1}^{\infty} |y_n|^{(q-1)p} = \sum_{n=1}^{\infty} |y_n|^q = \|y\|_q^q < \infty$, so $x \in \ell_p$ with $\|x\|_p^p = \|y\|_q^q$.

Since $y \neq 0$, we have $x \neq 0$, so $x/\|x\|_p \in B_{\ell_p}$. Note that

$$\|\varphi_y\| \geq \varphi_y \left(\frac{x}{\|x\|_p} \right) = \frac{1}{\|x\|_p} \sum_{n=1}^{\infty} x_n y_n = \frac{\|y\|_q^q}{\|y\|_q^{q/p}} = \|y\|_q.$$

Hence, $\|\varphi_y\| = \|y\|_q$.

- φ is surjective

Fix $f \in \ell_p^*$. Define $y_n = f(e_n)$, $n \in \mathbb{N}$. Let $y = (y_n)$. For some fixed $N \in \mathbb{N}$, set

$$x_n = \begin{cases} \frac{|y_n|^q}{y_n}, & y_n \neq 0, n \leq N \\ 0, & \text{otherwise} \end{cases}$$

Then $x = (x_n) \in \ell_p$, so

$$\begin{aligned}f(x) &= \sum_{n=1}^N x_n f(x_n) = \sum_{n=1}^N x_n y_n = \sum_{n=1}^N |y_n|^2 \leq \|f\| \|x\|_p \\ \|x\|_p &= \left(\sum_{n=1}^N |x_n|^p \right)^{1/p} = \left(\sum_{n=1}^N |y_n|^{(q-1)p} \right)^{1/p} = \left(\sum_{n=1}^N |y_n|^q \right)^{1/p}\end{aligned}$$

Hence, $\sum_{n=1}^N |y_n|^q \leq \|f\| \left(\sum_{n=1}^N |y_n|^q \right)^{1/p}$, i.e.

$$\left(\sum_{n=1}^N |y_n|^q \right)^{1/q} \leq \|f\|$$

Let $N \rightarrow \infty$ to deduce that $y \in \ell_q$. Finally, observe that

$$\begin{aligned}f(e_n) &= y_n = \varphi_y(e_n) \quad \forall n \in \mathbb{N} \\ \implies f(x) &= \varphi_y(x) \quad \forall x \in \text{span}\{e_n : n \in \mathbb{N}\} = c_{00} \quad \text{by linearity}\end{aligned}$$

$$\implies f(x) = \varphi_y(x) \quad \forall x \in \overline{\text{span}}\{e_n : n \in \mathbb{N}\} = \ell_p \quad \text{by continuity}$$

Thus, $f = \varphi_y$, so φ is surjective. ■

Remarks

- We also have $\ell_1^* \cong \ell_\infty$ and $c_0^* \cong \ell_1$. The proof also shows that $\ell_1 \hookrightarrow \ell_\infty^*$ isometrically. However, the proof of surjectivity breaks down since $\overline{\text{span}}\{e_n : n \in \mathbb{N}\}$ in ℓ_∞ is $c_0 \subsetneq \ell_\infty$.
- From the proof, we can show Corollary 2.3 holds for ℓ_p .
- We've shown that $\ell_p, 1 \leq p \leq \infty$, is complete as they are dual spaces. For c_0 , one simply has to show that c_0 is closed in ℓ_∞ .

2.3 Bidual

Let X be a normed space. Then $X^{**} = (X^*)^* = \mathcal{B}(X^*, \mathbb{R})$ is the *bidual* or *second dual* of X .

For each $x \in X$, define the map

$$\begin{aligned} \hat{x} : X^* &\rightarrow \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

Note that \hat{x} is linear and bounded: $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$. So $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leq \|x\|$. Thus, we have

$$\begin{aligned} \hat{} : X &\rightarrow X^{**} \\ x &\mapsto \hat{x} \end{aligned}$$

This is linear: $\widehat{\lambda x + \mu y}(f) = f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = (\lambda \hat{x} + \mu \hat{y})(f)$.

For $x \neq 0$, let $f \in X^*$ be a norming functional at x . Then

$$\hat{x}(f) = f(x) = \|x\| \implies \|\hat{x}\| = \|x\|$$

so the canonical map $X \rightarrow X^{**}, x \mapsto \hat{x}$ is an isometric embedding into X^{**} . If f is surjective, we say that X is *reflexive*.

2.4 Dual operators

Let X, Y be normed spaces and $T \in \mathcal{B}(X, Y)$. The *dual operator* T^* of T is the map

$$\begin{aligned} T^* : Y^* &\rightarrow X^* \\ g &\mapsto g \circ T \end{aligned}$$

By Proposition 1.6, $T^*(g) = g \circ T \in X^*$ and $\|T^*(g)\| \leq \|g\| \|T\|$, so T^* is well-defined. Moreover, it is clearly linear and bounded with $\|T^*\| \leq \|T\|$.

Remark. Note that $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{R}$ is bilinear. Moreover, for $x \in X$ and $g \in Y^*$, we have $\langle x, T^*(g) \rangle = \langle T(x), g \rangle$.

It turns out that $\|T^*\| = \|T\|$:

$$\|T^*\| = \sup_{g \in B_{Y^*}} \|T^*g\| = \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^*(g) \rangle| = \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g \rangle| = \sup_{x \in B_X} \|Tx\| = \|T\|,$$

where the penultimate equality follows from Corollary 2.3.

Example

Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider the right-shift map $R: \ell_p \rightarrow \ell_p$. What is $R^*: \ell_p^* \rightarrow \ell_p^*$? Recall that $\ell_p^* \cong \ell_q$. Thought of as a map $\ell_q \rightarrow \ell_q$, it turns out that $R^* = L$, the left-shift map.

Now, let's note some properties of dual operators:

- (1) $(\text{Id}_X)^* = \text{Id}_{X^*}$
- (2) $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for all $S, T \in \mathcal{B}(X, Y)$ and all scalars λ, μ
Indeed, for $g \in Y^*$, $x \in X$,

$$\begin{aligned} \langle x, (\lambda S + \mu T)^* g \rangle &= \langle (\lambda S + \mu T)x, g \rangle \\ &= \langle \lambda Sx + \mu Tx, g \rangle \\ &= \lambda \langle Sx, g \rangle + \mu \langle Tx, g \rangle \\ &= \lambda \langle x, S^* g \rangle + \mu \langle x, T^* g \rangle \\ &= \langle x, (\lambda S^* + \mu T^*) g \rangle \end{aligned}$$

Since x is arbitrary, $(\lambda S + \mu T)^* g = (\lambda S^* + \mu T^*) g$ for all $g \in Y^*$, and we are done.

- (3) $(ST)^* = T^* S^*$ for all $T \in \mathcal{B}(X, Y)$ and all $S \in \mathcal{B}(Y, Z)$

$$\langle x, (ST)^* g \rangle = \langle STx, g \rangle = \langle S(Tx), g \rangle = \langle Tx, S^* g \rangle = \langle x, T^* S^* g \rangle$$

- (4) Let $T \in \mathcal{B}(X, Y)$. We have $T^* \in \mathcal{B}(Y^*, X^*)$ and $T^{**} \in \mathcal{B}(X^{**}, Y^{**})$. The diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow \hat{\cdot} & & \downarrow \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array}$$

commutes, i.e., $\hat{T}x = T^{**}\hat{x}$ for all $x \in X$. For $x \in X, g \in Y^*$,

$$\langle g, T^{**}\hat{x} \rangle = \langle T^*g, \hat{x} \rangle = \langle x, T^*g \rangle = \langle Tx, g \rangle = \langle g, \hat{Tx} \rangle$$

Remark. Properties (1) and (3) imply that $X \sim Y \implies X^* \sim Y^*$.