University of Cambridge Mathematical Tripos Part II

Analysis of Functions

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Course schedule

Lebesgue integration theory
Review of integration: simple functions, monotone and dominated convergence; existence of Lebesgue measure; definition of L^p spaces and their completeness. The Lebesgue differentiation theorem. Egorov's theorem, Lusin's theorem. Mollification by convolution, continuity of translation and separability of L^p when $p \neq \infty$. [5]

Banach and Hilbert space analysis
Strong, weak and weak-* topologies; reflexive spaces. Review of the Riesz representation theorem for Hilbert spaces; the Radon–Nikodym theorem; the dual of L^p. Compactness: review of the Ascoli–Arzelà theorem; weak-* compactnesss of the unit ball for separable Banach spaces. The Riesz representation theorem for spaces of continuous functions. The Hahn–Banach theorem and its consequences: separation theorems; Mazur's theorem. [7]

Fourier analysis
Definition of Fourier transform in L^1; the Riemann–Lebesgue lemma. Fourier inversion theorem. Extension to L^2 by density and Plancherel's isometry. Duality between regularity in real variable and decay in Fourier variable. [3]

Generalized derivatives and function spaces
Definition of generalized derivatives and of the basic spaces in the theory of distributions: $\mathcal{D}/\mathcal{D}'$ and $\mathcal{S}/\mathcal{S}'$. The Fourier transform on $\mathcal{S}'$. Periodic distributions; Fourier series; the Poisson summation formula. Definition of the Sobolev spaces H^s in $\mathbb{R}^d$. Sobolev embedding. The Rellich–Kondrashov theorem. The trace theorem. [5]

Applications
Construction and regularity of solutions for elliptic PDEs with constant coefficients on $\mathbb{R}^n$. Construction and regularity of solutions for the Dirichlet problem of Laplace's equation. The spectral theorem for the Laplacian on a bounded domain. *The direct method of the Calculus of Variations.* [4]

Contents

1 Lebesgue integration theory

1.1 Recap of measure theory

We recall some basic notions from II Probability and Measure whilst establishing the notation for this course.

Let E be a set. A family $\mathcal{B} \subset \mathcal{P}(E)$ is a σ-algebra if $\varnothing \in \mathcal{B}$ and it is closed under countable unions and complements. A map $\mu \colon \mathcal{B} \to [0, \infty]$ is a measure if it is σ-additive, i.e., $\mu(\bigsqcup_n A_n) = \sum_n \mu(A_n)$ for a countable disjoint collection (A_n). A pair $(E, \mathcal{B})$ is called a measurable space, whereas a triple $(E, \mathcal{B}, \mu)$ is called a measure space. If E is a topological space, we can (and, in this course, will) consider $\mathcal{B}$ to be the Borel σ-algebra.

A map $f \colon E \to \mathbb{C}$ is measurable if $f^{-1}(A) \in \mathcal{B}$ for all Borel $A \subset \mathbb{C}$. If $f \colon E \to [0, \infty]$ is measurable, then $\int f \, d\mu$ is well-defined (in $[0, \infty]$). We say that $f \colon E \to \mathbb{C}$ is integrable if it is measurable and $\int |f| \, d\mu < \infty$. We then denote by $\mathcal{L}^1(E, \mathcal{B}, \mu)$ the set of all integrable functions $E \to \mathbb{C}$. Writing $f \sim g \iff f = g$ a.e., we then define $L^1(E, \mathcal{B}, \mu) := \mathcal{L}^1(E, \mathcal{B}, \mu) / \sim$. However, it is of course standard to refer to a concrete function f from some equivalence class $[f] \in L^1(E)$.

Theorem 1.1 Dominated convergence theorem
Let $(E, \mathcal{B}, \mu)$ be a measure space. Let $g, f, f_1, f_2 \in L^1(E, \mathcal{B}, \mu)$. Suppose $f_n(x) \to f(x)$ and $|f_n(x)| \leq g(x)$ for a.e. $x \in E$. Then

$$\int_E f_n \, d\mu \to \int_E f \, d\mu$$

Recall that a measure space $(E, \mathcal{B}, \mu)$ is σ-finite if there exists $(A_n)_{n \in \mathbb{N}}$ such that $E = \bigcup_{n \in \mathbb{N}} A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

Theorem 1.2 Fubini's theorem
Let $(E, \mathcal{A}, \mu)$ and $(F, \mathcal{B}, \mu)$ be two σ-finite measure spaces. Let $f \colon E \times F \to \mathbb{C}$ be $\mathcal{A} \otimes \mathcal{B}$-measurable.
(i) *If the map $x \mapsto \int_F f(x, y) \, d\nu$ in $L^1(E, \mathcal{A}, \mu)$, then $f \in L^1(E \times F)$.*
(ii) *If $f \in L^1(E \times F)$, then*

$$\int_{E \times F} f \, d\mu \otimes \nu = \int_E \int_F f(x, y) \, d\nu(y) \, d\mu(x) = \int_F \int_E f(x, y) \, d\mu(x) \, d\nu(y)$$

1.2 Signed and complex measures

Definition Complex and signed measures
Let $(E, \mathcal{B})$ be a measurable space. A set function $\mu \colon \mathcal{B} \to \mathbb{C}$ is a complex measure if it is σ-additive. We then say that $(E, \mathcal{B}, \mu)$ is a complex measure space. If $\mu(\mathcal{B}) \subset \mathbb{R}$, we call μ a signed measure and $(E, \mathcal{B}, \mu)$ a signed measure space.

Remark. Previously, we really have been considering *positive* measures, though we will always just refer to such maps as *measures*. Note that not every (positive) measure is a complex measure.

Given a complex measure μ, define the *real* and *imaginary parts* $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$ as

$$\forall A \in \mathcal{B} \qquad \operatorname{Re} \mu(A) = \operatorname{Re}(\mu(A)), \qquad \operatorname{Im} \mu(A) = \operatorname{Im}(\mu(A))$$

It is easy to verify that these are signed measures.

Definition Positive and negative sets
Let $(E, \mathcal{B}, \mu)$ be a signed measure space. We say that A is a positive (resp. negative) set if $\mu(B) \geq 0$ (resp. $\mu(B) \leq 0$) for all $B \subset A$.

Theorem 1.3 Hahn decomposition
Let $(E, \mathcal{B}, \mu)$ be a signed measure space. Then there exists a positive set $P \in \mathcal{B}$ and a negative set $N \in \mathcal{B}$ such that $E = P \sqcup N$.

In proving the Hahn decomposition theorem, we will ultimately be defining P be a positive set of the largest possible measure. However, we first show the existence of non-trivial positive sets via the following lemma.

Lemma 1.4
For all $A \subset \mathcal{B}$, there is a positive set $D \subset A$ such that $\mu(D) \geq \mu(A)$.

Proof. If A is positive, then we can simply take $D = A$ and we are done. It remains to carefully consider the case A is not positive. Pick negative set $B_1 \subset A$ such that $\mu(B_1)$ is "as negative as possible" in the approximate sense that there is no $B \subset A$ and $k \in \mathbb{N}$ for which $\mu(B_1) > -1/k \geq \mu(B)$. Let $A_1 = A \backslash B_1$. Continue inductively to define $B_2, A_2, B_3, A_3, \cdots$ in such a way that $B_{j+1} \subset A_j$ and $\mu(B_{j+1}) > -1/k$ for some $k \in \mathbb{N}$ only if this is so for all subsets of A_i; we then take $A_{j+1} = A_j \backslash B_{j+1}$.

Now, take $D = \bigcap A_j$. Then $A = D \sqcup B_1 \sqcup B_2 \sqcup \cdots$, so $\mu(A) = \mu(D) + \sum_j \mu(B_j)$. Since $\mu(B_j) \geq 0$ for each j, we have $\mu(D) \geq \mu(A)$. It then remains to show that D is positive. Note that $\mu(B_j) \to 0$ by convergence of $\sum_j \mu(B_j)$. Fix $k \in \mathbb{N}$. Then $\mu(B_i) > -1/k$ for some i. Then $\mu(B) > -1/k$ for all $B \subset A_{i-1}$ and thus for all $B \subset D$. Hence, D must be positive. ■

Proof of Theorem 1.3. To be done next lecture. ■

Remarks

- A key corollary of the Hahn decomposition theorem is the Jordan decomposition of signed measures.
- The decomposition $E = P \sqcup N$ is manifestly non-unique as we can, for instance, 'move' a negligible subset from P to N. However, this makes no difference in the Jordan decomposition of μ.

Corollary 1.5 Jordan decomposition of signed measures
Let $(E, \mathcal{B}, \mu)$ be a signed measure space, with $E = P \sqcup N$ for positive P and negative N. Then $\mu^+ := \mu|_P$ and $\mu^- := -\mu|_N$ are positive measures satisfying $\mu = \mu^+ - \mu^-$.

Proof. Trivial. ■

Now that we've proven the Hahn and Jordan decompositions, we can now extend Lebesgue integration to signed and complex measures.

Definition Integral with respect to signed measures
Let $(E, \mathcal{B}, \mu)$ be a signed measure space. A measurable function $f\colon E \to \mathbb{C}$ is integrable if it is integrable with respect to μ^+ and μ^-. In that case, we then define its integral to be

$$\int_E f\, d\mu := \int_E f\, d\mu^+ - \int_E f\, d\mu^-$$

Definition Integral with respect to complex measures
Let $(E, \mathcal{B}, \mu)$ be a complex measure space. A measurable function $f\colon E \to \mathbb{C}$ is integrable with respect to $\operatorname{Re}\mu$ and $\operatorname{Im}\mu$. In that case, we then define its integral to be

$$\int_E f\, d\mu := \int_E f\, d\operatorname{Re}\mu + i \int_E f\, d\operatorname{Im}\mu$$

Finally, we end this subsection with a brief discussion on the Banach space structure of the space of complex measures. Given a signed measure μ, we can define its *total variation measure* to be

$$|\mu| := \mu^+ + \mu^-$$

and its *total variation norm* to be

$$||\mu|| := |\mu|(E) = \mu^+(E) + \mu^-(E)$$

On the Example Sheet, you will extend these notions to complex measures. It can then be shown that the space of complex measures on a measurable space $(E, \mathcal{B})$ forms a Banach space with respect to the total variation norm.