
University of Cambridge Mathematical Tripos Part II

Differential Geometry

Lecturer:

Mihalis Dafermos
md384@cam.ac.uk

Notes by:

Robert Frederik Uy
rfu20@cam.ac.uk

Course schedule

Smooth manifolds in \mathbb{R}^n , tangent spaces, smooth maps and the inverse function theorem. Examples, regular values, Sard's theorem (statement only). Transverse intersection of submanifolds. [4]

Manifolds with boundary, degree mod 2 of smooth maps, applications. [3]

Curves in 2-space and 3-space, arc-length, curvature, torsion. The isoperimetric inequality. [2]

Smooth surfaces in 3-space, first fundamental form, area. [1]

The Gauss map, second fundamental form, principal curvatures and Gaussian curvature. Theorema Egregium. [3]

Minimal surfaces. Normal variations and characterization of minimal surfaces as critical points of the area functional. Isothermal coordinates and relation with harmonic functions. The Weierstrass representation. Examples. [3]

Parallel transport and geodesics for surfaces in 3-space. Geodesic curvature. [2]

The exponential map and geodesic polar coordinates. The Gauss-Bonnet theorem (including the statement about classification of compact surfaces). [4]

Global theorems on curves: Fenchel's theorem (the total curvature of a simple closed curve is greater than or equal to 2π); the Fary-Milnor theorem (the total curvature of a simple knotted closed curve is greater than 4π). [2]

Recommended books

J. Milnor *Topology from the differentiable viewpoint*. Princeton University Press, 1997.

M. Do Carmo *Differential Geometry of Curves and Surfaces*. Pearson Higher Education, 1976

Contents

1	Differential topology	3
1.1	Tangent spaces	4

1 Differential topology

Definition Smooth map on an open subset

Let $U \subset \mathbb{R}^n$. We say that $f: U \rightarrow \mathbb{R}^m$ is smooth if all partial derivatives to all orders exist and are continuous.

Definition Smooth map

Let $X \subset \mathbb{R}^n$. We say that $f: X \rightarrow \mathbb{R}^m$ is smooth if, for each $x \in X$, there exists (i) an open neighbourhood $U \subset \mathbb{R}^n$ of x and (ii) a smooth map $\tilde{f}: U \rightarrow \mathbb{R}^m$ such that $\tilde{f}|_{X \cap U} = f|_{X \cap U}$.

Definition Diffeomorphism

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$. We say that $f: X \rightarrow Y$ is a diffeomorphism if f is a smooth bijection with a smooth inverse. If such a map exists, we say that X and Y are diffeomorphic.

Exercise. Give an example of a smooth bijection that is not a diffeomorphism.

Definition k -dimensional manifold

We say that $X \subset \mathbb{R}^N$ is a k -dimensional manifold if, for each $x \in X$, there exists an open neighbourhood $V \subset X$ of x such that V is diffeomorphic to an open subset $U \subset \mathbb{R}^k$. A diffeomorphism $\varphi: U \rightarrow V$ is called a local parametrisation of V , whereas its inverse $\psi := \varphi^{-1}: V \rightarrow U$ is called a coordinate system or a chart on V .

Remarks

- By composing φ^{-1} with the projections $\pi_i: \mathbb{R}^k \rightarrow \mathbb{R}, (x_1, \dots, x_k) \mapsto x_i$, we get smooth maps $x_i := \pi_i \circ \varphi^{-1}$ which we call *coordinate functions*.
- WLOG, we can replace ‘diffeomorphic to an open subset $U \subset \mathbb{R}^k$ ’ with ‘diffeomorphic to an open ball in \mathbb{R}^k ’.
- It is easy to see that, if $X \subset \mathbb{R}^N$ is both a k -dimensional manifold and a \tilde{k} -dimensional manifold, then $k = \tilde{k}$.

Definition Dimension

Let $X \subset \mathbb{R}^N$ be a k -dimensional manifold. The dimension of X is k , and it is denoted by $\dim X$.

Example Some trivial examples

- $X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W
- $X = \{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^N$

Example S^n

$S^n := \{x \in \mathbb{R}^{n+1}: \|x\|_2 = 1\}$ is an n -dimensional manifold. To see this, consider the projection $\Pi_k: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_k, x_{k+1}, \dots, x_{n+1})$. It is easy to verify that maps of the form $\psi_k^\pm = \Pi_k|_{S^n \cap \{\text{sign}(x_k) = \pm 1\}}$ are diffeomorphisms $S^n \cap \{\text{sign}(x_k) = \pm 1\} \rightarrow B_1(0)$.

Remark. It is easy to show that X is a 0-dimensional manifold iff X is a discrete subset of \mathbb{R}^N .

Exercise. Show that, if X and Y are manifolds, then $X \times Y$ is also a manifold, with $\dim X \times Y = \dim X + \dim Y$.

Definition Submanifold

Let $X, Y \subset \mathbb{R}^N$ be manifolds. If $Y \subset X$, then we say that Y is a submanifold of X . The codimension of Y in X is defined as

$$\text{codim}_X Y := \dim X - \dim Y$$

1.1 Tangent spaces

We first recall some basic facts from our youth. Let $U \subset \mathbb{R}^k$ be open. The *differential* of a smooth map $f: U \rightarrow \mathbb{R}^m$ at $x \in U$ is defined by

$$df_x: \mathbb{R}^k \rightarrow \mathbb{R}^m$$

$$h \mapsto \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t}$$

This is a linear map, with matrix representation

$$df_x = \left(\frac{\partial f^i}{\partial x^j} \right)_{i,j}$$

Moreover, differentials satisfy the chain rule: given (i) two smooth maps $f: U \rightarrow \mathbb{R}^l$ and $g: V \rightarrow \mathbb{R}^m$ with $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$ open and (ii) a point $x \in U$ with $f(x) \in V$, we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Definition Tangent space

Let $X \subset \mathbb{R}^N$ be a k -dimensional manifold and $x \in X$. Choose a local parametrisation $\varphi: U \rightarrow V$ around x . We then define the tangent space $T_x X$ of X at x to be

$$T_x X := \text{im } d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k)$$

Of course, before we can safely proceed, we must show that $T_x X$ is well-defined:

Lemma 1.1

Let X be as above. $T_x X$ is independent of φ , and $\dim T_x X = k$.

Proof. Let $\tilde{\varphi}: \tilde{U} \rightarrow \tilde{V}$ be another local parametrisation near x . WLOG, by restricting if necessary, we may assume $\tilde{V} = V$. By the chain rule, we have

$$d\varphi_{\varphi^{-1}(x)} = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)} \circ d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}$$

Since $\tilde{\varphi}^{-1} \circ \varphi$ is a diffeomorphism of open subsets of \mathbb{R}^n , the corresponding differential $d(\tilde{\varphi}^{-1} \circ \varphi)$ is a linear isomorphism. Thus,

$$d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}(\mathbb{R}^k)) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(\mathbb{R}^k)$$

as claimed.

Now, it remains to show that $\dim T_x X = k$. By definition, there exists an open set $\hat{V} \subset \mathbb{R}^N$ and a smooth map $\Psi: \hat{V} \rightarrow \mathbb{R}^k$ that extends the chart $\psi := \varphi^{-1}$. Note that $\Psi \circ \varphi = \text{id}_U$, so by the chain rule,

$$d\Psi_x \circ d\varphi_{\varphi^{-1}(x)} = \text{id}_{\mathbb{R}^k}$$

Then, $d\varphi_{\varphi^{-1}(x)}$ must be an isomorphism $\mathbb{R}^k \rightarrow T_x X$, and hence $\dim T_x X = k$. ■

Example Tangent spaces for our trivial examples

Returning to the trivial examples we previously gave, we now state the corresponding tangent space for an arbitrary point x on each manifold.

- $X = \mathbb{R}^N$: $T_x X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W : $T_x X = \mathbb{R}^N$
- $X = \{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^N$: $T_x X = X$

Example *Tangent spaces for S^n*

From any given chart, we can compute (φ and) $d\varphi$:

$$\frac{\partial \varphi}{\partial x^1} = (1, 0, \dots, 0, -x_1/x_{n+1})$$

and similarly for $\partial \varphi / \partial x^i$. Manifestly, each partial derivative is perpendicular to x . Thus, $T_x X \subset x^\perp := \{v \in \mathbb{R}^{n+1} : \langle v, x \rangle = 0\}$. Since we know from the above lemma that $\dim T_x X = n$, we conclude that $T_x X = x^\perp$.