University of Cambridge Mathematical Tripos Part II

Differential Geometry

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Course schedule

Smooth manifolds in $\mathbb{R}^n$, tangent spaces, smooth maps and the inverse function theorem. Examples, regular values, Sard's theorem (statement only). Transverse intersection of submanifolds. [4]

Manifolds with boundary, degree mod 2 of smooth maps, applications. [3]

Curves in 2-space and 3-space, arc-length, curvature, torsion. The isoperimetric inequality. [2]

Smooth surfaces in 3-space, first fundamental form, area. [1]

The Gauss map, second fundamental form, principal curvatures and Gaussian curvature. Theorema Egregium. [3]

Minimal surfaces. Normal variations and characterization of minimal surfaces as critical points of the area functional. Isothermal coordinates and relation with harmonic functions. The Weierstrass representation. Examples. [3]

Parallel transport and geodesics for surfaces in 3-space. Geodesic curvature. [2]

The exponential map and geodesic polar coordinates. The Gauss-Bonnet theorem (including the statement about classification of compact surfaces). [4]

Global theorems on curves: Fenchel's theorem (the total curvature of a simple closed curve is greater than or equal to 2π); the Fary-Milnor theorem (the total curvature of a simple knotted closed curve is greater than 4π). [2]

Recommended books

J. Milnor *Topology from the differentiable viewpoint.* Princeton University Press, 1997.
M. Do Carmo *Differential Geometry of Curves and Surfaces.* Pearson Higher Education, 1976

Contents

1 Differential topology

Definition Smooth map on an open subset
Let $U \subset \mathbb{R}^n$. We say that $f\colon U \to \mathbb{R}^m$ is smooth if all partial derivatives to all orders exist and are continuous.

Definition Smooth map
Let $X \subset \mathbb{R}^n$. We say that $f\colon X \to \mathbb{R}^m$ is smooth if, for each $x \in X$, there exists (i) an open neighbourhood $U \subset \mathbb{R}^n$ of x and (ii) a smooth map $\tilde{f}\colon U \to \mathbb{R}^m$ such that $\tilde{f}|_{X\cap U} = f|_{X\cap U}$.

Definition Diffeomorphism
Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$. We say that $f\colon X \to Y$ is a diffeomorphism if f is a smooth bijection with a smooth inverse. If such a map exists, we say that X and Y are diffeomorphic.

Exercise. Give an example of a smooth bijection that is not a diffeomorphism.

Definition k-dimensional manifold
We say that $X \subset \mathbb{R}^N$ is a k-dimensional manifold if, for each $x \in X$, there exists an open neighbourhood $V \subset X$ of x such that V is diffeomorphic to an open subset $U \subset \mathbb{R}^k$. A diffeomorphism $\varphi\colon U \to V$ is called a local parametrisation of V, whereas its inverse $\psi := \varphi^{-1}\colon V \to U$ is called a coordinate system or a chart on V.

Remarks

- By composing φ^{-1} with the projections $\pi_i\colon \mathbb{R}^k \to \mathbb{R}, (x_1, \cdots, x_k) \mapsto x_i$, we get smooth maps $x_i := \pi_i \circ \varphi^{-1}$ which we call *coordinate functions.*
- WLOG, we can replace ‘diffeomorphic to an open subset $U \subset \mathbb{R}^k$” with “diffeomorphic to an open ball in R^k”.
- It is easy to see that, if $X \subset \mathbb{R}^N$ is both a k-dimensional manifold and a $\tilde{k}$-dimensional manifold, then $k = \tilde{k}$.

Definition Dimension
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold. The dimension of X is k, and it is denoted by $\dim X$.

Example *Some trivial examples*

- $X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W
- $X = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^N$

Example S^n
$S^n := \{x \in \mathbb{R}^{n+1} : ||x||_2 = 1\}$ is an n-dimensional manifold. To see this, consider the projection $\Pi_k\colon \mathbb{R}^{n+1} \to \mathbb{R}^n, (x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1})$. It is easy to verify that maps of the form $\psi_k^{\pm} = \Pi_k|_{S^n\cap\{\text{sign}(x_k)=\pm1\}}$ are diffeomorphisms $S^n \cap \{\text{sign}(x_k) = \pm1\} \to B_1(0)$.

Remark. It is easy to show that X is a 0-dimensional manifold iff X is a discrete subset of $\mathbb{R}^N$.

Exercise. Show that, if X and Y are manifolds, then $X \times Y$ is also a manifold, with $\dim X \times Y = \dim X + \dim Y$.

Definition Submanifold
Let $X, Y \subset \mathbb{R}^N$ be manifolds. If $Y \subset X$, then we say that Y is a submanifold of X. The codimension of Y in X is defined as

$$\operatorname*{codim}_X Y := \dim X - \dim Y$$

1.1 Tangent spaces

We first recall some basic facts from our youth. Let $U \subset \mathbb{R}^k$ be open. The *differential* of a smooth map $f\colon U \to \mathbb{R}^m$ at $x \in U$ is defined by

$$\begin{aligned} df_x\colon \mathbb{R}^k &\to \mathbb{R}^N \\ h &\mapsto \lim_{t\to 0} \frac{f(x+th)-f(x)}{t} \end{aligned}$$

This is a linear map, with matrix representation

$$df_x = \left(\frac{\partial f^i}{\partial x^j}\right)_{i,j}$$

Moreover, differentials satisfy the chain rule: given (i) two smooth maps $f\colon U \to \mathbb{R}^l$ and $g\colon V \to \mathbb{R}^m$ with $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$ open and (ii) a point $x \in U$ with $f(x) \in V$, we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Definition Tangent space
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold and $x \in X$. Choose a local parametrisation $\varphi\colon U \to V$ around x. We then define the tangent space T_xX of X at x to be

$$T_xX := \operatorname{im} d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k)$$

Of course, before we can safely proceed, we must show that T_xX is well-defined:

Lemma 1.1
Let X be as above. T_xX is independent of φ, and $\dim T_xX = k$.

Proof. Let $\tilde{\varphi}\colon \tilde{U} \to \tilde{V}$ be another local parametrisation near x. WLOG, by restricting if necessary, we may assume $\tilde{V} = V$. By the chain rule, we have

$$d\varphi_{\varphi^{-1}(x)} = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)} \circ d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}$$

Since $\tilde{\varphi}^{-1} \circ \varphi$ is a diffeomorphism of open subsets of $\mathbb{R}^n$, the corresponding differential $d(\tilde{\varphi}^{-1} \circ \varphi)$ is a linear isomorphism. Thus,

$$d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}(\mathbb{R}^k)) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(\mathbb{R}^k)$$

as claimed.

Now, it remains to show that $\dim T_xX = k$. By definition, there exists an open set $\hat{V} \subset \mathbb{R}^N$ and a smooth map $\Psi\colon \hat{V} \to \mathbb{R}^k$ that extends the chart $\psi := \varphi^{-1}$. Note that $\Psi \circ \varphi = \mathrm{id}_U$, so by the chain rule,

$$d\Psi_x \circ d\varphi_{\varphi^{-1}(x)} = \mathrm{id}_{\mathbb{R}^k}$$

Then, $d\varphi_{\varphi^{-1}(x)}$ must be an isomorphism $\mathbb{R}^k \to T_xX$, and hence $\dim T_xX = k$. ■

Example *Tangent spaces for our trivial examples*
Returning to the trivial examples we previously gave, we now state the corresponding tangent space for an arbitrary point x on each manifold.

- $X = \mathbb{R}^N$: $T_xX = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W: $T_xX = \mathbb{R}^N$
- $X = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^N$: $T_xX = X$

Example *Tangent spaces for S^n*
From any given chart, we can compute (φ and) $d\varphi$:

$$\frac{\partial \varphi}{\partial x^1} = (1, 0, \cdots, 0, -x_1/x_{n+1})$$

and similarly for $\partial\varphi/\partial x^i$. Manifestly, each partial derivative is perpendicular to x. Thus, $T_x X \subset x^\perp := \{v \in \mathbb{R}^{n+1} \colon \langle v, x\rangle = 0\}$. Since we know from the above lemma that $\dim T_x X = n$, we conclude that $T_x X = x^\perp$.

Definition Differential map for manifolds
Let $f\colon X \to Y$ be a smooth map between manifolds and $x \in X$. Choose a local parametrisation φ_1 around x and φ_2 around $f(x) \in Y$. We define the differential $df_x\colon T_x X \to T_{f(x)} Y$ of f at x by

$$df_x = d{\varphi_2}_{\varphi_2^{-1}(f(x))} \circ d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)} \circ (d{\varphi_1}_{\varphi_1^{-1}(x)})^{-1}$$

Lemma 1.2
df_x is independent of the choice of local parametrisations.

Proof. Trivial exercise. ■

Proposition 1.3 Chain rule for manifolds
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be smooth maps between manifolds. For any $x \in X$,

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Proof. Trivial exercise. ■

Theorem 1.4 Inverse function theorem
Let $f\colon X \to Y$ be a smooth map between manifolds and $x \in X$. Suppose $df_x\colon T_x X \to T_{f(x)} Y$ is an isomorphism. Then f is a local diffeomorphism*, i.e., each $x \in X$ has an open neighbourhood $V \subset X$ such that $f|_V \colon V \to f(V)$ is a diffeomorphism.*

Proof. Since df_x is an isomorphism, it follows that $d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)}$ is also an isomorphism. We can then use the usual inverse function theorem to deduce the result. ■

1.2 Regular values and Sard's theorem

Definition Critical and regular points
Let $f\colon X \to Y$ be a smooth map between manifolds. We say that $x \in X$ is a critical point of f if $df_x\colon T_x X \to T_{f(x)} Y$ is not surjective. Otherwise, it is a regular point.

Notation. We denote by C the set of all critical points of f.

Remark. If $\dim Y > \dim X$, then $C = X$ and the pre-image of any regular value is $\varnothing$.

Definition Critical and regular values
Let $f\colon X \to Y$ be a smooth map between manifolds. We say that $y \in Y$ is a critical value of f if $y = f(x)$ for some $x \in C$. Otherwise, we say that y is a regular value of f.

Theorem 1.5 Pre-image theorem
Let $f\colon X \to Y$ be a smooth map between manifolds. Suppose $y \in Y$ is a regular value of f. If $f^{-1}(y) \neq \varnothing$, then $f^{-1}(y) \subset X$ is a submanifold of X with $\dim f^{-1}(y) = \operatorname{codim}_X Y$.

Proof. Fix $x \in f^{-1}(y)$. Since y is a regular value, we know that $df_x \colon T_xX \to T_yY$ is surjective. By the rank-nullity theorem, $\dim \ker df_x = \operatorname{codim}_X Y$. Suppose $X \subset \mathbb{R}^N$, and pick a linear map $T \colon \mathbb{R}^N \to \mathbb{R}^{\operatorname{codim}_X Y}$ such that $\ker T \cap \ker df_x = \{0\}$.[1]

Now, extend f to $F \colon X \to Y \times \mathbb{R}^{\operatorname{codim}_X Y}$ given by $z \mapsto (f(z), T(z))$. Note that the differential of F at x is given by

$$dF_x = (df_x, dT_x) = (df_x, T)$$

Since $\ker T \cap \ker df_x = \{0\}$, we have $\ker dF_x = \{0\}$, i.e., dF_x is injective. By the inverse function theorem for manifolds, there exists an open neighbourhood $U \subset X$ of x such that $F|_U \colon U \to f(U) \times T(U)$ is a diffeomorphism. Hence, $F|_{f^{-1}(y)\cap U}$ is a local parametrisation of $(\{y\} \times \mathbb{R}^{\operatorname{codim}_X Y}) \cap V$, proving that $f^{-1}(y)$ is a manifold of dimension $\operatorname{codim}_X Y$. ■

Corollary 1.6
Let $f \colon X \to Y$ be a smooth map between manifolds of the same dimension, with X compact. If y is a regular value of f, then $f^{-1}(y)$ is finite.

Proof. By the pre-image theorem, $f^{-1}(y)$ is a 0-dimensional manifold, i.e., a collection of points. Since X is compact, such a collection must be finite. ■

With just a bit more analysis and topology, we can actually say more than just finiteness:

Theorem 1.7 Stack of records theorem
Let $f \colon X \to Y$ be a smooth map between manifolds of the same dimension, with X compact. Let y be a regular value of f, and list the elements of $f^{-1}(y)$ as $x_1, \cdots, x_n$. There exists an open neighbourhood $V \subset Y$ of y and a collection of open neighbourhoods $W_i \subset X$ of each x_i such that

$$f^{-1}(V) = \bigsqcup_{i=1}^{n} W_i$$

and each $f|_{W_i} \colon W_i \to V$ is a diffeomorphism.

Proof. By the inverse function theorem for manifolds, we can pick open neighbourhoods W_i of x such that each $f|_{W_i}$ is a diffeomorphism to an open neighbourhood of y. By shrinking neighbourhoods if necessary, W_i can be taken WLOG to be pairwise disjoint. Now, set

$$V = \left[\bigcap_{i=1}^{n} f(W_i)\right] \setminus f\left(X \setminus \bigcup_{i=1}^{n} W_i\right)$$

Note that $f(X \setminus \bigcup_{i=1}^{n} W_i)$ is a compact set that does not contain y, so V is an open neighbourhood of y. Finally, note that $f^{-1}(V) = \bigsqcup_{i=1}^{n} W_i$ by construction. ■

Now, the pre-image theorem can be a powerful tool for generating manifolds or showing that a certain set is one.

Application *S^n is a manifold*
Consider the map $f \colon \mathbb{R}^{n+1} \to \mathbb{R}, (x_1, \cdots, x_{n+1}) \mapsto x_1^2 + \cdots + x_{n+1}^2$. Note that $f^{-1}(1) = S^n$, so to show that S^n is a manifold, it suffices to show that 1 is a regular point. Indeed, note that $df_x = (2x_1, \cdots, 2x_{n+1})$, which is not surjective only if $x = 0 \notin f^{-1}(1)$.

[1] It is easy to constructively show using IB Linear Algebra that such a map exists. [*Exercise!*]

Application *Orthogonal group as a manifold*
Denote by $M(n)$ [resp. $S(n)$] the space of all [resp. symmetric] $n \times n$ matrices with entries in $\mathbb{R}$. Consider the orthogonal group $O(n) = \{A \in M(n) \colon AA^t = I\} \subset M(n) = \mathbb{R}^{n^2}$.

Let $f \colon M(n) \to O(n)$ be the map $A \mapsto AA^t$. This is smooth since multiplication and addition in $\mathbb{R}$ are smooth. Since $O(n) = f^{-1}(I)$, it suffices to show that I is a regular value of f. Note that

$$df_A(H) = \lim_{t \to 0} \frac{f(A + tH) - f(A)}{t} = AH^t + HA^t$$

Now, fix $A \in M(n)$. Given $B \in S(n)$, observe that

$$df_A\left(\frac{1}{2}CA\right) = \frac{1}{2}AA^tC^t + \frac{1}{2}CAA^t = \frac{1}{2}C + \frac{1}{2}C = C$$

completing the proof that I is a regular value of f.

Remark. Recall that, besides being a manifold as we've just shown, $O(n)$ is also a *group*. In fact, the group operations $(A, B) \mapsto AB$ and $A \mapsto A^{-1} = A^t$ are smooth. Hence, we see that $O(n)$ is a *Lie group*.