

Linear Analysis

Lecturer:

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Course schedule

Normed and Banach spaces. Linear mappings, continuity, boundedness, and norms. Finite-dimensional normed spaces. [4]

The Baire category theorem. The principle of uniform boundedness, the closed graph theorem and the inversion theorem; other applications. [5]

The normality of compact Hausdorff spaces. Urysohn's lemma and Tietze's extension theorem. Spaces of continuous functions. The Stone–Weierstrass theorem and applications. Equicontinuity: the Ascoli–Arzelá theorem. [5]

Inner product spaces and Hilbert spaces; examples and elementary properties. Orthonormal systems, and the orthogonalization process. Bessel's inequality, the Parseval equation, and the Riesz–Fischer theorem. Duality; the self duality of Hilbert space. [5]

Bounded linear operations, invariant subspaces, eigenvectors; the spectrum and resolvent set. Compact operators on Hilbert space; discreteness of spectrum. Spectral theorem for compact Hermitian operators. [5]

Recommended books

B. Bollobas *Linear Analysis*. Cambridge University Press 1999.

G.J.O. Jameson *Topology and Normed Spaces*. Chapman and Hall 1974.

G. Allan *Introduction to Banach Spaces and Algebras*. Oxford University Press 2010.

W. Rudin *Real and Complex Analysis*. McGraw–Hill International Edition 1987.

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1 Normed spaces and bounded linear maps

1.1 Definitions and examples

Let X be a vector space over \mathbb{R} or \mathbb{C} . For ease of notation and discussion, we will sometimes just take our scalars to be in \mathbb{R} , although the statement may be easily generalised to \mathbb{C} -vector spaces.

Definition Norm

A norm on X is a function $\|\cdot\|: X \rightarrow \mathbb{R}$ such that

- (i) $\|x\| \geq 0$ for all $x \in X$, with $\|x\| = 0$ iff $x = 0$
- (ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and any scalar λ
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

Definition Normed space

A normed space is a pair $(X, \|\cdot\|)$ where X is a vector space and $\|\cdot\|$ is a norm on X .

Example Some finite-dimensional normed spaces

- (1) $\ell_2^n = (\mathbb{R}^n, \|\cdot\|_2)$ or $(\mathbb{C}^n, \|\cdot\|_2)$, where the norm is given by

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

This is called the ℓ_2 -norm or euclidean norm.

(i),(ii) are easy to check, whereas (iii) follows from Cauchy-Schwarz.

- (2) $\ell_1^n = (\mathbb{R}^n, \|\cdot\|_1)$ where $\|x\|_1 = \sum_{i=1}^n |x_i|$ (called the ℓ_1 -norm)

- (3) $\ell_\infty^n = (\mathbb{R}^n, \|\cdot\|_\infty)$ where $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ (called the ℓ_∞ -norm or the sup-norm)

Given a normed space X , its norm $\|\cdot\|$ induces a metric on X :

$$d(x, y) = \|x - y\|$$

Indeed, d is a metric:

- $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0 \iff x - y = 0 \iff x = y$
- $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$
- $d(x, z) = \|x - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$

This metric, in turn, induces a topology on X , called the *norm topology*. This allows us talk about open/closed sets, convergence, and continuity, as we illustrate in the following examples.

Example

The algebraic operations are continuous:

- if $x_n \rightarrow x$ and $y_n \rightarrow y$ in X , then $x_n + y_n \rightarrow x + y$
- if $x_n \rightarrow x$ in X and $\lambda_n \rightarrow \lambda$ in \mathbb{R} , then $\lambda_n x_n \rightarrow \lambda x$

Example

The norm $\|\cdot\|: X \rightarrow \mathbb{R}$ is continuous: by the triangle inequality, we have

$$|||x| - |y|| \leq \|x - y\|$$

so $\|\cdot\|$ is, in fact, Lipschitz.

Definition Banach space

A Banach space is a complete normed space, i.e., a normed space that is complete in its norm topology.

Example

$\ell_2^n, \ell_1^n, \ell_\infty^n$ are complete: for any of these spaces,

- $x^{(k)} \rightarrow x \iff x_i^{(k)} \rightarrow x_i$ for all $1 \leq i \leq n$
- $(x^{(k)})_{k \in \mathbb{N}}$ is Cauchy $\iff (x_i^{(k)})_{k \in \mathbb{N}}$ is Cauchy for all $1 \leq i \leq n$

In a normed space, a useful object is the *unit ball*

$$B_X := \{x \in X : \|x\| \leq 1\}$$

Remarks

- B_X defines a norm on X :

$$\|x\| = \inf\{t \geq 0 : x \in tB_X\}$$

- B_X is symmetric ($x \in B_X \implies -x \in B_X$), convex, and closed
- If $B \subset \mathbb{R}^n$ is a closed, convex, symmetric, bounded neighbourhood of 0, then B is the unit ball of $(\mathbb{R}^n, \|\cdot\|)$ for some norm $\|\cdot\|$
- ‘Geometry of Banach spaces’

Previously, we gave $\ell_2, \ell_1, \ell_\infty$ as examples of finite-dimensional normed spaces. More generally, we have the following family of examples

Example

- (4) $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ for $1 \leq p < \infty$, where $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ (called the ℓ_p -norm)

Again, (i) and (ii) are easy to check, whereas (iii) is not obvious.¹

Now, let S denote the set of all scalar sequences. This is a vector spaces under the coordinate operations $(x_n) + (y_n) = (x_n + y_n)$ and $\lambda(x_n) = (\lambda x_n)$.

Example Sequence spaces

$$(5) \ell_1 = \left\{ (x_n) \in S : \sum_{n=1}^{\infty} |x_n| < \infty \right\}, \quad \|(x_n)\|_1 = \sum_{n=1}^{\infty} |x_n| \quad (\ell_1\text{-norm})$$

(i) and (ii): easy to check.

(iii): Given $(x_n), (y_n) \in \ell_1$, we have $|x_n + y_n| \leq |x_n| + |y_n|$ for all $n \in \mathbb{N}$. Summing over all $n \in \mathbb{N}$, we deduce that $(x_n) + (y_n) \in \ell_1$ and $\|(x_n) + (y_n)\|_1 \leq \|(x_n)\|_1 + \|(y_n)\|_1$.

Hence, ℓ_1 is a subspace of S and $\|\cdot\|_1$ is a norm on ℓ_1 .

$$(6) \ell_2 = \left\{ (x_n) \in S : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}, \quad \|(x_n)\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} \quad (\ell_2\text{-norm})$$

(i) and (ii): easy to check.

(iii): Given $(x_n), (y_n) \in \ell_2$, the triangle inequality in ℓ_2^N gives us

$$\left(\sum_{k=1}^N |x_k + y_k|^2 \right)^{1/2} \leq \left(\sum_{k=1}^N |x_k|^2 \right)^{1/2} + \left(\sum_{k=1}^N |y_k|^2 \right)^{1/2}.$$

Taking $N \rightarrow \infty$, we get $(x_n) + (y_n) \in \ell_2$ and $\|(x_n) + (y_n)\|_2 \leq \|(x_n)\|_2 + \|(y_n)\|_2$

¹We will return to this later in the next subsection.

More generally, for $1 \leq p < \infty$, the set

$$\ell_p = \left\{ (x_n) \in S : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

is a subspace of S , and

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \quad (\ell_p\text{-norm})$$

is a norm on ℓ_p . [(iii) follows from the triangle inequality on ℓ_p^n , which we will see later.]

Example More sequence spaces

$$(7) \ell_{\infty} = \{(x_n) \in S : \exists M \geq 0 \forall n \in \mathbb{N} |x_n| \leq M\}, \quad \|(x_n)\|_{\infty} = \sup_{n \in \mathbb{N}} |x_n| \quad (\ell_{\infty}\text{-norm})$$

(i) and (ii): easy to check.

(iii): Given $x = (x_n), y = (y_n) \in \ell_{\infty}$,

$$|x_n + y_n| \leq |x_n| + |y_n| \leq \|x\|_{\infty} + \|y\|_{\infty} \quad \forall n \in \mathbb{N}$$

so $x + y \in \ell_{\infty}$ and $\|x + y\|_{\infty} \leq \|x\|_{\infty} + \|y\|_{\infty}$.

$$(8) c_0 = \{(x_n) \in S : x_n \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

$$c = \{(x_n) \in S : \lim_{n \rightarrow \infty} x_n \text{ exists}\}$$

Both c_0 and c are subspaces of ℓ_{∞} and are hence normed spaces in the ℓ_{∞} -norm.

1.2 Inequalities of Minkowski and Hölder

Recall that a function $f: (0, \infty) \rightarrow \mathbb{R}$ is *convex* if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) \quad \forall x, y \in (0, \infty) \forall t \in [0, 1]$$

and *concave* if the above holds with \leq replaced by \geq .

Lemma 1.1

Let $1 \leq p < \infty$. Then the map

$$(0, \infty) \rightarrow \mathbb{R}$$

$$x \mapsto x^p$$

is *convex*.

Proof. Fix $y > 0, t \in [0, 1]$, and define

$$g(x) = [(1-t)x + ty]^p - [(1-t)x^p + ty^p], \quad x > 0.$$

Differentiating, we get

$$g'(x) = p(1-t)[(1-t)x + ty]^{p-1} - p(1-t)x^{p-1}.$$

Observe that $0 < x < y \implies g'(x) \geq 0$ and that $x > y \implies g'(x) \leq 0$. By the MVT, we deduce that $g(x) \leq g(y) = 0$ for all $x \in (0, \infty)$. ■

Theorem 1.2 Minkowski's inequality

Let $1 \leq p < \infty, n \in \mathbb{N}$. For $x, y \in \mathbb{R}^n$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Remark. This shows that ℓ_p^n and ℓ_p are normed spaces.

Exercise. Show that $\ell_p, 1 \leq p \leq \infty$, is complete.²

Proof of Theorem 1.2. Let $B = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$. We first show that B is convex. Let $x, y \in B$ and $t \in [0, 1]$. For $1 \leq k \leq n$,

$$|(1-t)x_k + ty_k|^p \leq ((1-t)|x_k| + t|y_k|)^p \leq (1-t)|x_k|^p + t|y_k|^p$$

by Lemma 1.1 for $x_k \neq 0, y_k \neq 0$; the inequality holds trivially if $x_k = 0$ or $y_k = 0$. Summing over k , we then get

$$\|(1-t)x + ty\|_p^p \leq (1-t)\|x\|_p^p + t\|y\|_p^p \leq 1,$$

so $(1-t)x + ty \in B$.

We then complete the proof as follows. Let $x, y \in \mathbb{R}^n$. WLOG, $x, y, x+y$ are nonzero. By convexity of B , we have

$$\frac{x+y}{\|x\|_p + \|y\|_p} = \frac{\|x\|_p}{\|x\|_p + \|y\|_p} \cdot \underbrace{\frac{x}{\|x\|_p}}_{\in B} + \frac{\|y\|_p}{\|x\|_p + \|y\|_p} \cdot \underbrace{\frac{y}{\|y\|_p}}_{\in B} \in B.$$

Thus, it follows that

$$\left\| \frac{x+y}{\|x\|_p + \|y\|_p} \right\| \leq 1 \implies \|x+y\|_p \leq \|x\|_p + \|y\|_p,$$

as required. ■

Let $x = (x_n) \in \ell_1$ and $y = (y_n) \in \ell_\infty$. We then write $x \cdot y = (x_n y_n)$. Note that, for all $n \in \mathbb{N}$, $|x_n y_n| = |x_n| |y_n| \leq |x_n| \|y\|_\infty$. Thus, $x \cdot y \in \ell_1$ and $\|x \cdot y\|_1 \leq \|x\|_1 \|y\|_\infty$.

Definition Conjugate index

Let $p \in (1, \infty)$. The conjugate index of p is the unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.3

Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $a, b \geq 0$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. The inequality holds trivially if $a = 0$ or $b = 0$, so it remains to consider the case $a, b > 0$. A proof similar to that of Lemma 1.1 shows that $\log: (0, \infty) \rightarrow \mathbb{R}$ is concave. Hence,

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p} \log(a^p) + \frac{1}{q} \log(b^q) = \log(ab).$$

We then apply exp to get the required result. ■

Theorem 1.4 Hölder's inequality

Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in \ell_p$ and $y \in \ell_q$, then $x \cdot y \in \ell_1$ and

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q.$$

Remark. As discussed above, $p = 1, q = \infty$ also works. Moreover, setting $p = q = 2$, we recover Cauchy-Schwarz.

Exercise. Deduce Minkowski's inequality from Hölder's inequality.

²A slick proof of this will be provided later.

Proof of Theorem 1.4. WLOG, $x \neq 0$ and $y \neq 0$. By homogeneity, we may also take $\|x\|_p = \|y\|_q = 1$ WLOG. Now, by Lemma 1.3, we have $|x_n y_n| \leq |x_n|^p/p + |y_n|^q/q$ for all $n \in \mathbb{N}$. Summing over n , we have

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \frac{\|x\|_p^p}{p} + \frac{\|y\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 = \|x\|_p \|y\|_q,$$

as required. ■

1.3 More examples: function spaces

Example

- (9) $C[0, 1] = \{f: [0, 1] \rightarrow \mathbb{R} \mid f \text{ cts}\}$, $\|f\|_{\infty} = \sup_{[0,1]} |f|$ (sup norm or uniform norm)

By the uniform limit theorem, this is a Banach space.

- (10) More generally, given a compact, Hausdorff topological space K ,

$$C(K) = \{f: K \rightarrow \mathbb{R} \mid f \text{ cts}\}$$

is a Banach space in the sup norm $\|f\|_{\infty} = \sup_K |f|$.

- (11) $(C[0, 1], \|\cdot\|_1)$, $\|f\|_1 = \int_0^1 |f(t)| dt$ (L_1 -norm)

This is an *incomplete* normed space — see Example Sheet 1.

More generally, $C[0, 1]$ is incomplete in the L_p -norm, $1 \leq p < \infty$, given by

$$\|f\|_p = \left(\int_0^1 |f(t)|^p dt \right)^{1/p}.$$

In II Probability and Measure, you will encounter the completion of $(C[0, 1], \|\cdot\|_p)$, which is the Lebesgue space $L_p[0, 1]$.

- (12) $C^1[0, 1] = \{f \in C[0, 1] \mid f \text{ continuously differentiable}\}$ is a subspace of $C[0, 1]$, so it is a normed space in $\|\cdot\|_{\infty}$ but incomplete, i.e. not closed in $C[0, 1]$. However, it is complete in the norm $\|f\| = \|f\|_{\infty} + \|f'\|_{\infty}$ — see Example Sheet 1.

- (13) Let $\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$. The set

$$A(\Delta) = \{f \in C(\Delta) \mid f \text{ analytic on int } \Delta\}$$

is a subspace of $C(\Delta)$. In fact, it is closed in $C(\Delta)$ and hence a Banach space in $\|\cdot\|_{\infty}$.

1.4 More on the normed topology

Let X be a normed space and $A \subset X$. Recall that the *closure* of A in X is

$$\overline{A} = \{x \in X \mid \exists (a_n) \text{ in } A \text{ s.t. } a_n \rightarrow x \text{ as } n \rightarrow \infty\}.$$

We then say that A is *dense* in X if $\overline{A} = X$. Moreover, A is *separable* if it has a countable dense subset.

If $Y \subset X$ is a subspace, then so is \overline{Y} : if $x, y \in \overline{Y}$, then there exists $(x_n), (y_n)$ in Y such that $x_n \rightarrow x$ and $y_n \rightarrow y$. So $\lambda x_n + \mu y_n \rightarrow \lambda x + \mu y \in \overline{Y}$. Similarly, if $A \subset X$ is convex, then so is \overline{A} .

For a subset $A \subset X$, the *closed linear span* of A , denoted by $\overline{\text{span}} A$, is the closure of $\text{span } A$.

Remarks

- If A is countable, then $\overline{\text{span}} A$ is separable.
- The set of all rational linear combinations of elements of A is countable and dense in $\overline{\text{span}} A$.

Example

- $\overline{\mathbb{Q}} = \mathbb{R}$, so \mathbb{R} is separable.
- $\ell_p, 1 \leq p < \infty$, is separable.

Let $e_n = (0, \dots, 0, \underset{n}{1}, 0, \dots)$, $n \in \mathbb{N}$ (unit vector basis)

Let $c_{00} = \text{span}\{e_n : n \in \mathbb{N}\} = \{(x_n) \in S : \exists N \in \mathbb{N} \forall n > N x_n = 0\}$

We then show that $\ell_p = \overline{\text{span}}\{e_n : n \in \mathbb{N}\}$: if $x = (x_n) \in \ell_p$, then

$$\left\| x - \sum_{i=1}^N x_i e_i \right\|_p = \left(\sum_{i>N} |x_i|^p \right)^{1/p} \rightarrow 0 \text{ as } N \rightarrow \infty$$

- Similarly, in ℓ_∞ , we have $\overline{\text{span}}\{e_n : n \in \mathbb{N}\} = c_0$. Moreover, c is separable, whereas ℓ_∞ is not.

Exercise. Prove the claims in the last example above.

1.5 Bounded linear maps

Theorem 1.5

Let X, Y be normed spaces and $T : X \rightarrow Y$ be a linear map. The following are equivalent:

- T is continuous at 0
- T is continuous
- T is Lipschitz
- T is bounded, i.e., $\exists C \geq 0 \forall x \in X \|Tx\| \leq C\|x\|$.

Proof. (iv) \implies (iii): Observe that

$$d(Tx, Ty) = \|Tx - Ty\| = \|T(x - y)\| \leq C\|x - y\| = Cd(x, y)$$

iii) \implies (ii): Given $\varepsilon > 0$ take $\delta = \varepsilon/(C + 1)$.

(ii) \implies (i): Trivial.

(i) \implies (iv): $\exists \delta > 0 \forall x \in X d(x, 0) = \|x\| \leq \delta \implies d(Tx, T0) = \|Tx\| \leq 1$. For $x \neq 0$, $\|\delta x / \|x\|\| = \delta$, so $\|T(\delta x / \|x\|)\| \leq 1$. Hence, $\|Tx\| \leq \delta^{-1}\|x\|$. ■

For normed spaces X, Y , let $\mathcal{B}(X, Y) = \{T : X \rightarrow Y \mid T \text{ linear and bounded}\}$. For $T \in \mathcal{B}(X, Y)$, its *operator norm* is

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

Remark. Since $T \in \mathcal{B}(X, Y)$, we have $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$. So if $\|x\| \leq 1$, then $\|Tx\| \leq C$. Thus, by definition, $\|T\| \leq C$. Conversely, for all $x \in B_X$, we have $\|Tx\| \leq \|T\|$, so by homogeneity, $\|Tx\| \leq \|T\|\|x\|$. Hence, $\|T\|$ is the least C such that (iv) in Theorem 1.5 above holds.

The operator norm is a norm on $\mathcal{B}(X, Y)$: given $S, T \in \mathcal{B}(X, Y)$, we have, for all $x \in X$,

$$\|(S + T)x\| = \|Sx + Tx\| \leq \|Sx\| + \|Tx\| \leq \|S\|\|x\| + \|T\|\|x\| \leq (\|S\| + \|T\|)\|x\|,$$

from which it follows that $S + T \in \mathcal{B}(X, Y)$ and $\|S + T\| \leq \|S\| + \|T\|$.

Notation. We write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$.

Proposition 1.6

Let X, Y, Z be normed spaces, $S \in \mathcal{B}(X, Y)$, $T \in \mathcal{B}(Y, Z)$. Then $TS \in \mathcal{B}(X, Z)$ and $\|TS\| \leq \|T\|\|S\|$.

Proof. For all $x \in X$, we have $\|TSx\| \leq \|T\|\|Sx\| \leq \|T\|\|S\|\|x\|$. ■

Example

- (1) $T: \ell_2^n \rightarrow \ell_2^n, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$

$$\|Tx\|_2 = \left(\sum_{i=1}^r |x_i|^2 \right)^{1/2} \leq \|x\|_2 \implies \|T\| \leq 1$$

But $Te_1 = e_1$ so $\|T\| = 1$.

More generally, if T is represented by a matrix A wrt the standard basis, then Cauchy-Schwarz gives us

$$\|T\| \leq \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}$$

- (2) Let $1 \leq p < \infty$; $R: \ell_p \rightarrow \ell_p, (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$ (right shift)

For all $x \in \ell_p$, $\|Rx\|_p = \|x\|_p$, so R is isometric and $\|R\| = 1$. Note that R is injective but not surjective.

- (3) Let $1 \leq p < \infty$; $L: \ell_p \rightarrow \ell_p, (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$ (left shift)

For all $x \in \ell_p$, $\|Lx\|_p \leq \|x\|_p$, so $L \in \mathcal{B}(\ell_p)$ with $\|L\| \leq 1$. Since $Le_2 = e_1$ and $\|e_1\|_p = \|e_2\|_p = 1$, we in fact have $\|L\| = 1$. Note that L is surjective but not injective.

- (4) $T: \ell_1 \rightarrow \ell_2, x \mapsto x$

► **Claim.** $\ell_1 \subset \ell_2$, and $\forall x \in \ell_2$ $\|x\|_2 \leq \|x\|_1$

Proof. WLOG assume $\|x\|_1 = 1$ by homogeneity. Since $\sum_{n=1}^{\infty} |x_i| = 1$, we have $|x_i| \leq 1$ for all i . Thus,

$$|x_i|^2 \leq |x_i| \quad \forall i \implies \|x\|_2^2 \leq \|x\|_1 = 1 \implies \|x\|_2 = 1 = \|x\|_1$$

as claimed. ■

Using the above claim, we have $T \in \mathcal{B}(\ell_1, \ell_2)$ and $\|T\| = 1$.

- (5) $T: \ell_2 \rightarrow \ell_1, (x_n) \mapsto (x_n/n)$

By Cauchy-Schwarz,

$$\sum_{n=1}^{\infty} \left| \frac{x_n}{n} \right| \leq \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2}$$

so $T \in \mathcal{B}(\ell_2, \ell_1)$ with $\|T\| \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2}$. In fact, we can replace \leq with $=$.

- (6) $D: (C^1[0, 1], \|\cdot\|) \rightarrow (C[0, 1], \|\cdot\|_{\infty}), f \mapsto f'$

Note that $\|Df\|_{\infty} = \|f'\|_{\infty} \leq \|f\|_{\infty} + \|f'\|_{\infty} = \|f\|$, so $\|D\| \leq 1$. But taking $f(x) = \sin(n\pi x)$, we have

$$\|Df\|_{\infty} = n\pi, \quad \|f\| = n\pi + 1,$$

so in fact $\|D\| = 1$. Note also that, for $f \neq 0$, $\|Df\|_\infty < \|f\|$, so $\|D\|$ is not attained.

(7) On a normed space X , the identity $x \mapsto x$ is denoted by Id , I , Id_X or I_X . This map is isometric, i.e., $\|\text{Id}(x)\| = \|x\| \forall x \in X$.

(8) For normed spaces X, Y , we let

$$X \oplus Y = \{(x, y) : x \in X, y \in Y\}$$

with norm $\|(x, y)\|_1 = \|x\| + \|y\|$. The corresponding norm topology is the product topology.

Define $P: X \oplus Y \rightarrow X$, $(x, y) \mapsto x$ (projection onto X). Note that $P \in \mathcal{B}(X \oplus Y, X)$ with $\|P\| = 1$.

Let X, Y be normed spaces. We introduce some terminology:

- An *isomorphism* $X \rightarrow Y$ is a linear homeomorphism $T: X \rightarrow Y$, i.e., T is a linear bijection such that T and T^{-1} are bounded. Equivalently, T is a linear bijection³ such that

$$\exists a, b > 0 \forall x \in X \ a\|x\| \leq \|Tx\| \leq b\|x\|$$

If such T exists, we say that X and Y are *isomorphic*, and we write $X \sim Y$.

- An *isometric isomorphism* is a linear bijection $T: X \rightarrow Y$ such that

$$\forall x \in X \ \|Tx\| = \|x\|$$

If such T exists, we say that X and Y are *isometrically isomorphic*, and we write $X \cong Y$.

The Banach-Mazur distance is defined as

$$d(X, Y) = \begin{cases} \infty, & \text{if } X \not\sim Y \\ \inf\{\|T\|\|T^{-1}\| \mid T: X \rightarrow Y \text{ is an isomorphism}\}, & \text{otherwise} \end{cases}$$

Note that $\|T\|\|T^{-1}\| \geq \|TT^{-1}\| = 1$. If $X \cong Y$, then $d(X, Y) = 1$. Does the converse hold?

- An *isomorphic embedding* $X \rightarrow Y$ is a linear map $T: X \rightarrow Y$ such that $T: X \rightarrow TX = \text{im } T$ is an isomorphism. If such T exists, we say that X (*isomorphically*) *embeds into* Y , and we write $X \hookrightarrow Y$.

Definition Equivalent norms

Let X be a normed space. Two norms $\|\cdot\|, \|\cdot\|'$ are equivalent if

$$\text{Id}: (X, \|\cdot\|) \rightarrow (X, \|\cdot\|') \text{ is an isomorphism}$$

$$\iff \|\cdot\|, \|\cdot\|' \text{ induce the same norm topology on } X$$

$$\iff \exists a, b > 0 \forall x \in X \ a\|x\| \leq \|x\|' \leq b\|x\|$$

$$\iff \exists a, b > 0 \ aB'_X \subset B_X \subset bB'_X$$

Remarks

- If $X \sim Y$, then X is complete iff Y is complete.

If $\|\cdot\|, \|\cdot\|'$ are equivalent norms on a vector space X , then $(X, \|\cdot\|)$ is complete iff $(X, \|\cdot\|')$ is complete.

³We can actually replace ‘bijection’ with ‘surjection’.

- Let X and Y be normed spaces. On $X \oplus Y$, the norm $\|(x, y)\|_1 = \|x\| + \|y\|$ is equivalent to $\|(x, y)\|_p = (\|x\|^p + \|y\|^p)^{1/p}$ for all $1 \leq p < \infty$ and to $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$.
- $(C[0, 1], \|\cdot\|_\infty)$ is complete whereas $(C[0, 1], \|\cdot\|_1)$ is incomplete. Thus, we can use the first remark above to deduce that $\|\cdot\|_\infty \not\sim \|\cdot\|_1$ (but this can easily be proven directly as well). However, $\|f\|_1 = \int_0^1 |f(t)| dt \leq \|f\|_\infty$, so

$$\text{Id}: (C[0, 1], \|\cdot\|_\infty) \rightarrow (C[0, 1], \|\cdot\|_1)$$

is a continuous linear bijection but its inverse is not continuous.

- On c_{00} , $\|\cdot\|_1 \not\sim \|\cdot\|_2$. To see why, consider $x = (\underbrace{1, \dots, 1}_n, 0, 0, \dots)$ and note that $\|x\|_1 = n$, $\|x\|_2 = \sqrt{n}$.

Finally, we discuss convergence and completeness. Let X, Y be normed spaces. In $\mathcal{B}(X, Y)$, convergence implies pointwise convergence, i.e., if $T_n \rightarrow T$ in $\mathcal{B}(X, Y)$, then, for all $x \in X$, $T_n x \rightarrow T x$ in Y . To see why, note that, for fixed $x \in X$, we have $\|T_n x - T x\| \leq \|T_n - T\| \|x\| \rightarrow 0$. However, the converse is false in general, e.g., $T_n: \ell_1 \rightarrow \mathbb{R}, x \mapsto x_n$. We have $T_n \rightarrow 0$ pointwise, but $\|T_n\| = 1$ for all $n \in \mathbb{N}$.

Theorem 1.7

Let X, Y be normed spaces. If Y is complete, then $\mathcal{B}(X, Y)$ is complete.

Proof. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X, Y)$. Fix $x \in X$. Then

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \|x\| \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

So $(T_n x)$ is Cauchy in Y and thus convergent. Now, define $T: X \rightarrow Y$ by $x \mapsto \lim_{n \rightarrow \infty} T_n x$. Observe that

- T is linear

$$T(\lambda x + \mu y) = \lim_{n \rightarrow \infty} T_n(\lambda x + \mu y) = \lim_{n \rightarrow \infty} [\lambda T_n x + \mu T_n y] = \lambda T x + \mu T y$$

- T is bounded

(T_n) is Cauchy implies (T_n) is bounded, i.e., there exists $M \geq 0$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. Fix $x \in X$. Then, for all $n \in \mathbb{N}$, we have $\|T_n x\| \leq \|T_n\| \|x\| \leq M \|x\|$. Letting $n \rightarrow \infty$, we obtain $\|T x\| \leq M \|x\|$.

- $T_n \rightarrow T$ in $\mathcal{B}(X, Y)$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|T_m - T_n\| \leq \varepsilon$ for all $m, n \geq N$. Fix $x \in X$. Note that, for all $m, n \geq N$, we have

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \|x\| \leq \varepsilon \|x\|$$

Letting $n \rightarrow \infty$ with $m \geq N$ fixed yields $\|T_m x - T x\| \leq \varepsilon \|x\|$. Hence, $\|T_m - T\| \leq \varepsilon$ for all $m \geq N$. ■

2 Dual spaces

2.1 Basics

Let X be a normed space. A *functional* on X is a map $X \rightarrow \mathbb{R}$. The *dual space* X^* of X is the space of all bounded linear functionals on X , i.e., $X^* = \mathcal{B}(X, \mathbb{R})$ equipped with the operator norm. Since \mathbb{R} is complete, Theorem 1.7 gives us the following result.

Theorem 2.1

For any normed space X , its dual X^* is a Banach space.

Notation. For $x \in X$ and $f \in X^*$, we let $\langle x, f \rangle = f(x)$.

Now, we know that $0 \in X^*$. Are there other elements?

Theorem 2.2 Hahn-Banach theorem

Let X be a normed space, $Y \subset X$ be a subspace and $g \in Y^*$. Then $f \in X^*$ such that $f|_Y = g$ and $\|f\| = \|g\|$.

Proof. See II Analysis of Functions. ■

Corollary 2.3

Let X be a normed space, $x_0 \in X \setminus \{0\}$. Then there exists $f \in S_{X^*} = \{f \in X^* : \|f\| = 1\}$ such that $f(x_0) = \|x_0\|$.

Remarks

- For any $g \in B_{X^*}$, $|g(x_0)| \leq \|g\| \|x_0\| \leq \|x_0\|$. Corollary 2.3 says that there exists $f \in B_{X^*}$ such that $f(x_0) = \|x_0\|$, so

$$\|x_0\| = \sup\{g(x_0) : g \in B_{X^*}\} = \max\{g(x_0) : g \in B_{X^*}\}.$$

We call f a *norming functional* at x_0 .

- Given $x \neq y$ in X , we can set $x_0 = x - y$ and Corollary 2.3 implies that there exists $f \in X^*$ such that $f(x) \neq f(y)$. Thus, X^* separates the points of X .

Proof of Corollary 2.3. Set $Y = \text{span}\{x_0\}$ and define $g(\lambda x_0) = \lambda \|x_0\|$. Then $g \in S_{Y^*}$ with $g(x_0) = \|x_0\|$. Finally, apply Theorem 2.2. ■

2.2 Dual space of ℓ_p

Motivation: Recall that, for $1 \leq p < \infty$, we have $\ell_p = \overline{\text{span}\{e_n : n \in \mathbb{N}\}} = \overline{c_{00}}$. Given $\varphi \in \ell_p^*$ and $x = (x_n) \in \ell_p$,

$$\varphi(x) = \varphi\left(\lim_{n \rightarrow \infty} \sum_{k=1}^n x_k e_k\right) = \sum_{k=1}^{\infty} x_k \varphi(e_k)$$

so φ corresponds to the sequence $y = (\varphi(e_n))_{n \in \mathbb{N}}$. We may then ask: is $\ell_p^* \cong \ell_q$ for some q ?

Fix $1 < p < \infty$, and let q be the conjugate index of p . Fix $y = (y_n) \in \ell_q$. Define

$$\begin{aligned} \varphi_y : \ell_p &\rightarrow \mathbb{R} \\ x &\mapsto \sum_{n=1}^{\infty} x_n y_n \end{aligned}$$

By Holder's inequality (Theorem 1.4), this is well-defined and $|\varphi_y(x)| \leq \|x\|_p \|y\|_q$. So φ_y is linear and bounded: $\|\varphi_y\| \leq \|y\|_q$. Thus, $\varphi_y \in \ell_p^*$, which means that we have a map

$$\begin{aligned}\varphi: \ell_q &\rightarrow \ell_p^* \\ y &\mapsto \varphi_y\end{aligned}$$

Note that φ is linear and bounded with $\|\varphi\| \leq 1$.

Theorem 2.4

Let p, q, φ be as above. Then φ is an isometric isomorphism $\ell_q \rightarrow \ell_p^*$.

Proof. It remains to check that φ is isometric and surjective:

- φ is isometric

Fix $y \in \ell_q$. WLOG $y \neq 0$. Define

$$x_n = \begin{cases} \frac{|y_n|^q}{y_n}, & y_n \neq 0 \\ 0, & y_n = 0 \end{cases}$$

Observe that $\sum_{n=1}^{\infty} |x_n|^p = \sum_{n=1}^{\infty} |y_n|^{(q-1)p} = \sum_{n=1}^{\infty} |y_n|^q = \|y\|_q^q < \infty$, so $x \in \ell_p$ with $\|x\|_p^p = \|y\|_q^q$.

Since $y \neq 0$, we have $x \neq 0$, so $x/\|x\|_p \in B_{\ell_p}$. Note that

$$\|\varphi_y\| \geq \varphi_y \left(\frac{x}{\|x\|_p} \right) = \frac{1}{\|x\|_p} \sum_{n=1}^{\infty} x_n y_n = \frac{\|y\|_q^q}{\|y\|_q^{q/p}} = \|y\|_q.$$

Hence, $\|\varphi_y\| = \|y\|_q$.

- φ is surjective

Fix $f \in \ell_p^*$. Define $y_n = f(e_n)$, $n \in \mathbb{N}$. Let $y = (y_n)$. For some fixed $N \in \mathbb{N}$, set

$$x_n = \begin{cases} \frac{|y_n|^q}{y_n}, & y_n \neq 0, n \leq N \\ 0, & \text{otherwise} \end{cases}$$

Then $x = (x_n) \in \ell_p$, so

$$\begin{aligned}f(x) &= \sum_{n=1}^N x_n f(e_n) = \sum_{n=1}^N x_n y_n = \sum_{n=1}^N |y_n|^q \leq \|f\| \|x\|_p \\ \|x\|_p &= \left(\sum_{n=1}^N |x_n|^p \right)^{1/p} = \left(\sum_{n=1}^N |y_n|^{(q-1)p} \right)^{1/p} = \left(\sum_{n=1}^N |y_n|^q \right)^{1/p}\end{aligned}$$

Hence, $\sum_{n=1}^N |y_n|^q \leq \|f\| \left(\sum_{n=1}^N |y_n|^q \right)^{1/p}$, i.e.

$$\left(\sum_{n=1}^N |y_n|^q \right)^{1/q} \leq \|f\|$$

Let $N \rightarrow \infty$ to deduce that $y \in \ell_q$. Finally, observe that

$$\begin{aligned}f(e_n) &= y_n = \varphi_y(e_n) \quad \forall n \in \mathbb{N} \\ \implies f(x) &= \varphi_y(x) \quad \forall x \in \text{span}\{e_n : n \in \mathbb{N}\} = c_{00} \quad \text{by linearity}\end{aligned}$$

$$\implies f(x) = \varphi_y(x) \quad \forall x \in \overline{\text{span}}\{e_n : n \in \mathbb{N}\} = \ell_p \quad \text{by continuity}$$

Thus, $f = \varphi_y$, so φ is surjective. ■

Remarks

- We also have $\ell_1^* \cong \ell_\infty$ and $c_0^* \cong \ell_1$. The proof also shows that $\ell_1 \hookrightarrow \ell_\infty^*$ isometrically. However, the proof of surjectivity breaks down since $\overline{\text{span}}\{e_n : n \in \mathbb{N}\}$ in ℓ_∞ is $c_0 \subsetneq \ell_\infty$.
- From the proof, we can show Corollary 2.3 holds for ℓ_p .
- We've shown that $\ell_p, 1 \leq p \leq \infty$, is complete as they are dual spaces. For c_0 , one simply has to show that c_0 is closed in ℓ_∞ .

2.3 Bidual

Let X be a normed space. Then $X^{**} = (X^*)^* = \mathcal{B}(X^*, \mathbb{R})$ is the *bidual* or *second dual* of X .

For each $x \in X$, define the map

$$\begin{aligned} \hat{x} : X^* &\rightarrow \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

Note that \hat{x} is linear and bounded: $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$. So $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leq \|x\|$. Thus, we have

$$\begin{aligned} \hat{\cdot} : X &\rightarrow X^{**} \\ x &\mapsto \hat{x} \end{aligned}$$

This is linear: $\widehat{\lambda x + \mu y}(f) = f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = (\lambda \hat{x} + \mu \hat{y})(f)$.

For $x \neq 0$, let $f \in X^*$ be a norming functional at x . Then

$$\hat{x}(f) = f(x) = \|x\| \implies \|\hat{x}\| = \|x\|$$

so the canonical map $X \rightarrow X^{**}, x \mapsto \hat{x}$ is an isometric embedding into X^{**} . If f is surjective, we say that X is *reflexive*.

2.4 Dual operators

Let X, Y be normed spaces and $T \in \mathcal{B}(X, Y)$. The *dual operator* T^* of T is the map

$$\begin{aligned} T^* : Y^* &\rightarrow X^* \\ g &\mapsto g \circ T \end{aligned}$$

By Proposition 1.6, $T^*(g) = g \circ T \in X^*$ and $\|T^*(g)\| \leq \|g\| \|T\|$, so T^* is well-defined. Moreover, it is clearly linear and bounded with $\|T^*\| \leq \|T\|$.

Remark. Note that $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{R}$ is bilinear. Moreover, for $x \in X$ and $g \in Y^*$, we have $\langle x, T^*(g) \rangle = \langle T(x), g \rangle$.

It turns out that $\|T^*\| = \|T\|$:

$$\|T^*\| = \sup_{g \in B_{Y^*}} \|T^*g\| = \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^*(g) \rangle| = \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g \rangle| = \sup_{x \in B_X} \|Tx\| = \|T\|,$$

where the penultimate equality follows from Corollary 2.3.

Example

Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider the right-shift map $R: \ell_p \rightarrow \ell_p$. What is $R^*: \ell_p^* \rightarrow \ell_p^*$? Recall that $\ell_p^* \cong \ell_q$. Thought of as a map $\ell_q \rightarrow \ell_q$, it turns out that $R^* = L$, the left-shift map.

Now, let's note some properties of dual operators:

- (1) $(\text{Id}_X)^* = \text{Id}_{X^*}$
- (2) $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for all $S, T \in \mathcal{B}(X, Y)$ and all scalars λ, μ
Indeed, for $g \in Y^*$, $x \in X$,

$$\begin{aligned} \langle x, (\lambda S + \mu T)^* g \rangle &= \langle (\lambda S + \mu T)x, g \rangle \\ &= \langle \lambda Sx + \mu Tx, g \rangle \\ &= \lambda \langle Sx, g \rangle + \mu \langle Tx, g \rangle \\ &= \lambda \langle x, S^* g \rangle + \mu \langle x, T^* g \rangle \\ &= \langle x, (\lambda S^* + \mu T^*) g \rangle \end{aligned}$$

Since x is arbitrary, $(\lambda S + \mu T)^* g = (\lambda S^* + \mu T^*) g$ for all $g \in Y^*$, and we are done.

- (3) $(ST)^* = T^* S^*$ for all $T \in \mathcal{B}(X, Y)$ and all $S \in \mathcal{B}(Y, Z)$

$$\langle x, (ST)^* g \rangle = \langle STx, g \rangle = \langle S(Tx), g \rangle = \langle Tx, S^* g \rangle = \langle x, T^* S^* g \rangle$$

- (4) Let $T \in \mathcal{B}(X, Y)$. We have $T^* \in \mathcal{B}(Y^*, X^*)$ and $T^{**} \in \mathcal{B}(X^{**}, Y^{**})$. The diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow \hat{\cdot} & & \downarrow \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array}$$

commutes, i.e., $\hat{T}x = T^{**}\hat{x}$ for all $x \in X$. For $x \in X, g \in Y^*$,

$$\langle g, T^{**}\hat{x} \rangle = \langle T^*g, \hat{x} \rangle = \langle x, T^*g \rangle = \langle Tx, g \rangle = \langle g, \hat{Tx} \rangle$$

Remark. Properties (1) and (3) imply that $X \sim Y \implies X^* \sim Y^*$.

3 Finite-dimensional normed spaces

Recall that norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space X are equivalent if $\text{Id}: (X, \|\cdot\|) \rightarrow (X, \|\cdot\|')$ is an isomorphism or, equivalently, if $\exists a, b > 0 \forall x \in X \ a\|x\| \leq \|x\|' \leq b\|x\|$.

Example

On \mathbb{R}^n , the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. We've already seen that $\|x\|_2 \leq \|x\|_1$ for all $x \in \mathbb{R}^n$. Moreover, by Cauchy-Schwarz, we have

$$\|x\|_1 = \sum_{i=1}^n |x_i| \leq \sqrt{n} \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} = \sqrt{n} \|x\|_2.$$

Theorem 3.1

Any two norms on a finite-dimensional vector space are equivalent.

Proof. Let X be a f.d. vector space. Fix a basis (e_1, \dots, e_n) of X . For $x = \sum_{i=1}^n x_i e_i \in X$, define $\|x\|_1 = \sum_{i=1}^n |x_i|$. Let $\|\cdot\|$ be an arbitrary norm on X .

We show that $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. For $x = \sum_{k=1}^n x_k e_k \in X$, we have

$$\|x\| \leq \sum_{k=1}^n |x_k| \|e_k\| \leq M \|x\|_1$$

where $M = \max_{1 \leq k \leq n} \|e_k\|$.

Now, let $S = \{x \in X : \|x\|_1 = 1\}$, the unit sphere of $(X, \|\cdot\|_1)$. We have the following result:

► **Claim.** S is compact.

Proof. Let $(x^{(r)})_{r \in \mathbb{N}}$ be a sequence in S . Write $x^{(r)} = \sum_{k=1}^n x_k^{(r)} e_k$. For each $1 \leq k \leq n$, $|x_k^{(r)}| \leq \|x^{(r)}\|_1 = 1$ for all $r \in \mathbb{N}$. By repeated application of Bolzano-Weierstrass, there exists $r_1 < r_2 < r_3 < \dots$ in \mathbb{N} such that $(x_k^{(r_\ell)})_{\ell \in \mathbb{N}}$ is convergent for each $1 \leq k \leq n$. Let $x_k = \lim_{\ell \rightarrow \infty} x_k^{(r_\ell)}$ and $x = \sum_{k=1}^n x_k e_k$. Note that

$$\|x\|_1 = \sum_{k=1}^n |x_k| = \lim_{\ell \rightarrow \infty} \sum_{k=1}^n |x_k^{(r_\ell)}| = 1$$

so $x \in S$. Moreover,

$$\|x^{(r_\ell)} - x\|_1 = \sum_{k=1}^n |x_k^{(r_\ell)} - x_k| \rightarrow 0 \quad \text{as } \ell \rightarrow \infty$$

so $x^{(r_\ell)} \rightarrow x$ in S . Thus, S is sequentially compact and hence compact. ■

For any $x, y \in S$, $||x| - |y|| \leq \|x - y\| \leq M \|x - y\|_1$. So $\|\cdot\|$ is continuous on S with respect to $\|\cdot\|_1$. So $c = \inf\{\|x\| : x \in S\}$ is achieved: $\exists x \in S \ \|x\| = c$. Since $0 \notin S$ and $c > 0$, we have $\|y\| \geq c = c\|y\|_1$ for all $y \in S$. By homogeneity, $\|y\| \geq c\|y\|_1$ for all $y \in X$. ■

Corollary 3.2

Let $T: X \rightarrow Y$ be a linear map between two normed spaces. If X is f.d., then T is continuous.

Proof. Let $\|\cdot\|$ denote the norm on X and Y . Define $\|x\|' = \|Tx\| + \|x\|$ for all $x \in X$. It is easy to check that this is a norm on X . By Theorem 3.1, there exists $b > 0$ such that, for all $x \in X$, $\|x\|' \leq b\|x\|$. In particular, $\|Tx\| \leq b\|x\|$ for all $x \in X$. ■

Corollary 3.3

If $\dim X = \dim Y < \infty$, then $X \sim Y$.

Proof. We have a linear bijection $T: X \rightarrow Y$. By Corollary 3.2, T and T^{-1} are continuous. ■

Remark. Corollary 3.3 does *not* imply that the theory of f.d. normed spaces is uninteresting.

Recall that, for X a metric space and $Y \subset X$, we have

- Y complete $\implies Y$ is closed in X
- Y closed in X and X complete $\implies Y$ complete

Corollary 3.4

- (i) A f.d. normed space X is complete.
- (ii) A f.d. subspace X of a normed space Y is closed in Y .

Proof. (i) Let $n = \dim X$. By Corollary 3.3, $X \sim \ell_2^n$ which is complete. (ii) follows from above properties of metric spaces. ■

Corollary 3.5

Let X be a f.d. normed space and $A \subset X$. Then

$$A \text{ is compact} \iff A \text{ is closed and bounded}$$

Proof. If $X = \ell_2^n$, then this is simply Heine-Borel. For general X , the result follows by invoking Corollary 3.3 to deduce that $X \sim \ell_2^n$ and noting isomorphisms map compact subsets to compact subsets (ditto for closed and bounded subsets). ■

In particular, $B_X = \{x \in X: \|x\| = 1\}$ is compact. How about if $\dim X = \infty$? Note that, in ℓ_p , $1 \leq p < \infty$, $\|e_n\|_p = 1$ for all n and $\|e_m - e_n\| = 2^{1/p}$ for all $m \neq n$, so (e_n) has no convergent subsequence. Hence, B_{ℓ_p} is not compact.

A similar obstruction does, in fact, hold for any infinite-dimensional normed space. To show this, we need the following lemma:

Proposition 3.6 Riesz's lemma

Let Y be a proper, closed subspace of a normed space X . Then

$$\forall \varepsilon > 0 \exists x \in B_X \ d(x, Y) = \inf\{\|x - y\|: y \in Y\} > 1 - \varepsilon.$$

Proof. WLOG, $0 < \varepsilon < 1$. Fix $z \in X \setminus Y$. Since Y is closed, $d = d(z, Y) > 0$. Pick $y \in Y$ such that $d \leq \|z - y\| < d/(1 - \varepsilon)$. Set $x = \frac{z - y}{\|z - y\|}$. Note that $d(x, Y) > 1 - \varepsilon$: for $y' \in Y$,

$$\|x - y'\| = \left\| \frac{z - y - \|z - y\|y'}{\|z - y\|} \right\| \geq \frac{d}{\|z - y\|} > 1 - \varepsilon$$

so $d(x, Y) \geq d/\|z - y\| > 1 - \varepsilon$. ■

Theorem 3.7

Let X be a normed space. Then B_X is compact if and only if $\dim X < \infty$.

Proof. (\Leftarrow) Corollary 3.5

(\Rightarrow) Similar to the ℓ_p case, we construct (x_n) in B_X such that $\|x_m - x_n\| > 1/2$ for all $m \neq n$. As before, we then deduce that (x_n) has no convergent subsequence and so B_X is not compact.

Pick any $x_1 \in B_X$. Suppose we have already picked x_1, \dots, x_n for some $n \in \mathbb{N}$. We then set $Y = \text{span}\{x_1, \dots, x_n\}$. Then Y is a proper ($\dim X = \infty$) and closed (Corollary 3.4) subspace of X . By Proposition 3.6, we can then pick $x_{n+1} \in B_X$ such that $d(x_{n+1}, Y) > 1/2$. In particular, $\|x_{n+1} - x_m\| > 1/2$ for $1 \leq m \leq n$. ■

4 The Baire category theorem and its applications

Let (X, d) be a metric space. In this course, we will denote closed and open balls as

$$\begin{aligned} B_r(x) &= \{y \in X : d(x, y) \leq r\} \\ D_r(x) &= \{y \in X : d(x, y) < r\} \end{aligned}$$

Recall that, for $A \subset X$, the *closure of A in X* is

$$\begin{aligned} \overline{A} &:= \bigcap_{\substack{F \text{ closed in } X \\ A \subset F}} F \\ &= \{x \in X : \forall r > 0 \ D_r(x) \cap A \neq \emptyset\} \\ &= \{x \in X : \exists (a_n) \text{ in } A \text{ s.t. } a_n \rightarrow x\} \end{aligned}$$

Note that $\overline{D_r(x)} \subset B_r(x)$. In general, this inclusion can be strict. But normed spaces are nice:

Exercise. Show that, in a normed space, $\overline{D_r(x)} = B_r(x)$.

Recall also that, for $A \subset X$, we say that A is *dense in X* if

$$\begin{aligned} \overline{A} &= X \\ \iff \forall x \in X \ \forall r > 0 \ D_r(x) \cap A \neq \emptyset \\ \iff \forall \text{ non-empty open } U \subset X \ U \cap A \neq \emptyset \end{aligned}$$

Example

\mathbb{Q} is dense in \mathbb{R} and so is $\sqrt{2} + \mathbb{Q}$. But $\mathbb{Q} \cap (\sqrt{2} + \mathbb{Q}) = \emptyset$.

4.1 Baire category theorem

Theorem 4.1 Baire category theorem

Let (X, d) be a complete metric space and $U_n \subset X$ be open and dense in X for each $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X .

Proof. Fix $x_0 \in X$ and $r_0 > 0$. Since U_1 is dense, $U_1 \cap D_{r_0}(x_0) \neq \emptyset$. Then we can pick $x_1 \in U_1 \cap D_{r_0}(x_0)$. Since $U_1 \cap D_{r_0}(x_0)$ is open, there exists $r_1 > 0$ such that $B_{r_1}(x_1) \subset U_1 \cap D_{r_0}(x_0)$. WLOG, we can pick $r_1 < 1$. We then continue inductively. At the n^{th} stage, density of U_n implies that $U_n \cap D_{r_{n-1}}(x_{n-1}) \neq \emptyset$, so we can pick $x_n \in U_n \cap D_{r_{n-1}}(x_{n-1})$. Since $U_n \cap D_{r_{n-1}}(x_{n-1})$ is open, there exists $r_n > 0$ such that $B_{r_n}(x_n) \subset U_n \cap D_{r_{n-1}}(x_{n-1})$. WLOG, $r_n < 1/n$.

We end up with $(x_n)_{n=0}^\infty$ in X and $(r_n)_{n=0}^\infty$ with $0 < r_n < 1/n$ for all $n \in \mathbb{N}$ and, for all $n > N \geq 0$,

$$\begin{aligned} B_{r_n}(x_n) &\subset U_n \cap D_{r_{n-1}}(x_{n-1}) \\ &\subset U_n \cap U_{n-1} \cap D_{r_{n-2}}(x_{n-2}) \\ &\vdots \\ &\subset U_n \cap U_{n-1} \cap \cdots \cap U_{N+1} \cap D_{r_N}(x_N) \end{aligned}$$

so, for all $m, n \geq N$, we have $d(x_m, x_n) \leq 2r_N < 2/N$. Thus, $(x_n)_{n=0}^\infty$ is Cauchy and thus convergent in X . Write $x = \lim_{n \rightarrow \infty} x_n$. Note that, for $n \geq m$, $x_n \in B_{r_m}(x_m)$ so $x \in B_{r_m}(x_m)$.

By fixing $N = 0$ above and taking $n \rightarrow \infty$, we get

$$x \in \left(\bigcap_{n \in \mathbb{N}} U_n \right) \cap D_{r_0}(x_0)$$

as required. ■

Remark. A countable intersection of open sets is called a G_δ -set. Theorem 4.1 then says that a countable intersection of open dense sets in a complete metric space is a dense G_δ -set.

Application *Uncountability of \mathbb{R}*

Suppose, on the contrary, that \mathbb{R} is countable, so we can write $\mathbb{R} = \{r_1, r_2, r_3, \dots\}$. Let $U_n = \mathbb{R} \setminus \{r_n\}$. Then U_n is open and dense in \mathbb{R} . Since \mathbb{R} is complete, Theorem 4.1 tells us that $\bigcap_{n \in \mathbb{N}} U_n$ is dense in \mathbb{R} . But $\bigcap_{n \in \mathbb{N}} U_n = \emptyset$ — a contradiction!

Observe that, if $U \subset X$ is open and dense in X , then $F = X \setminus U$ is closed in X and $\int F = \emptyset$.

Definition *Nowhere dense*

Let (X, d) be a topological space. We say that $A \subset X$ is nowhere dense in X if $\text{int } \overline{A} = \emptyset$.

Remarks

- For $A \subset Y \subset X$, it is possible that A is nowhere dense in X but not in Y (e.g. take $A = Y \neq \emptyset$)
- A is nowhere dense in X if and only if $U \not\subset \overline{U \cap A}$ for any nonempty open $U \subset X$.
 A is dense in X if and only if $U \subset \overline{U \cap A}$ for every open $U \subset X$.

Example

- In \mathbb{R} , any finite set and the Cantor set are nowhere dense.
- Write $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$ and let $(\delta_n)_{n \in \mathbb{N}}$ in $(0, 1)$. Then $U = \bigcup_{n \in \mathbb{N}} (q_n - \delta_n, q_n + \delta_n)$ is open and dense in \mathbb{R} . So $\mathbb{R} \setminus U$ is closed and nowhere dense in \mathbb{R} .

Theorem 4.1'

Let (X, d) be a non-empty complete metric space. Suppose $X = \bigcup_{n \in \mathbb{N}} A_n$ for some $A_n \subset X$. Then there exists $N \in \mathbb{N}$ such that $\text{int } \overline{A_n} \neq \emptyset$.

Proof. Suppose, on the contrary, that $\text{int } \overline{A_n} = \emptyset$ for all $n \in \mathbb{N}$. Then $\forall x \in X \forall r > 0 \ D_r(x) \not\subset \overline{A_n}$ and thus $D_r(x) \cap U_n = \emptyset$. Thus, $U_n = X \setminus \overline{A_n}$ is open and dense in X . Hence, by Theorem 4.1, $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X . But note that $\bigcap_{n \in \mathbb{N}} U_n = \left(\bigcup_{n \in \mathbb{N}} \overline{A_n} \right)^c = \emptyset$ — a contradiction! ■

Exercise. Deduce Theorem 4.1 from Theorem 4.1'.

Definition *First and second category*

Let X be a topological space and $A \subset X$.

- We say that A is meagre in X or is of first category in X if $A = \bigcup_{n \in \mathbb{N}} A_n$ where A_n is nowhere dense in X for all $n \in \mathbb{N}$.
- We say that A is of second category in X if A is not of first category.

Remarks

- Intuition: Think of meagre sets as ‘small’.
- Typical Baire argument: Theorem 4.1' is useful as, to find some element $x \in X$ (in a non-empty complete metric space) with some property P , we just have to show that $A = \{x \in X : x \text{ fails } P\}$.

Application *Existence of a nowhere differentiable function in $C[0, 1]$*

Note that $(C[0, 1], \|\cdot\|_\infty)$ is a nonempty complete metric space. Let

$$A = \{f \in C[0, 1] : \exists x \in [0, 1] \text{ s.t. } f \text{ differentiable at } x\}$$

Observe that, if $f'(x)$ exists, i.e. $[f(y) - f(x)]/(y - x) \rightarrow f'(x)$ as $y \rightarrow x$, then there exists $N \in \mathbb{N}$ such that, for all $y \in X$,

$$|y - x| < \frac{1}{N} \implies \left| \frac{f(y) - f(x)}{y - x} \right| \leq N$$

Thus, for $n \in \mathbb{N}$, consider the set

$$A_n = \left\{ f \in C[0, 1] : \exists x \in [0, 1] \forall y \in [0, 1] |y - x| < \frac{1}{n} \implies |f(y) - f(x)| \leq n|y - x| \right\}$$

and note that $A \subset \bigcup_{n \in \mathbb{N}} A_n$.

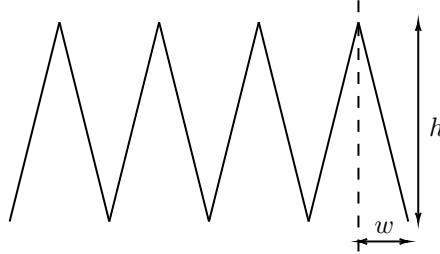
It then remains to show that, for all $n \in \mathbb{N}$, A_n is closed and $\text{int } A_n = \emptyset$.

- A_n is closed: Consider $(f_k)_{k \in \mathbb{N}}$ in A_n with $f_k \rightarrow f$ in $C[0, 1]$. For each $k \in \mathbb{N}$, we can pick $x_k \in [0, 1]$ such that, for all $y \in [0, 1]$, $|y - x_k| < 1/n \implies |f_k(y) - f_k(x_k)| \leq n|y - x_k|$. Passing to a subsequence if necessary, $x_k \rightarrow x$ in $[0, 1]$ WLOG. By IB Analysis and Topology Example Sheet 1 Q5 (2024), $f_k(x_k) \rightarrow f(x)$ and hence

$$\forall y \in [0, 1] |y - x| < \frac{1}{n} \implies |f(y) - f(x)| \leq n|y - x|$$

as required.

- Fix $f \in A_n$ and $r > 0$. To get $D_r(f) \not\subset A_n$, the idea is to consider a small but rapidly oscillating perturbation of f . Let $0 < \varepsilon < r/4$. Pick $\delta > 0$ such that $|y - x| < \delta \implies |f(y) - f(x)| < \varepsilon$. Choose h, w such that $4\varepsilon < h < r$ and $w = \min\{\varepsilon/n, \delta\}$. Set g to be the function



We can check that $f + g \in D_r(f) \setminus A_n$.

Direct proof: Take g_n similar to above with height h_n and width w_n , where $h_n \searrow 0$ fast and $h_n/w_n \rightarrow \infty$ fast. Then $\sum g_n$ is nowhere differentiable.

4.2 Consequences for Banach spaces

Theorem 4.2 Principle of uniform boundedness⁴

Let X be a Banach space, Y a normed space and $\mathcal{T} \subset \mathcal{B}(X, Y)$. If T is pointwise bounded (i.e., $\forall x \in X \sup_{T \in \mathcal{T}} \|Tx\| < \infty$), then T is uniformly bounded (i.e., $\sup_{T \in \mathcal{T}} \|T\| < \infty$).

Proof. Let $A_n = \{x \in X : \sup_{T \in \mathcal{T}} \|Tx\| \leq n\}$. By hypothesis, $X = \bigcup_{n \in \mathbb{N}} A_n$. By Theorem 4.1', there exists $n \in \mathbb{N}$ such that $\text{int } \overline{A_n} \neq \emptyset$. Note that $A_n = \bigcap_{T \in \mathcal{T}} \{x \in X : \|Tx\| \leq n\}$ is closed as

⁴This result is also known as the *Banach-Steinhaus theorem*.

the map $x \mapsto \|Tx\|$ is continuous. Thus, there exists $r > 0$ and $x \in A_n$ such that $B_r(x) \subset A_n$. Given $y \in B_X$, $T \in \mathcal{T}$, we have $x + ry, x - ry \in B_r(x)$ and thus

$$\|Ty\| = \left\| \frac{T(x + ry) - T(x - ry)}{2r} \right\| \leq \frac{2n}{2r} = \frac{n}{r}$$

Hence, $\|T\| \leq n/r$ for all $T \in \mathcal{T}$. ■

Corollary 4.3

Let X be a Banach space, Y a normed space, and $(T_n)_{n \in \mathbb{N}}$ a sequence in $\mathcal{B}(X, Y)$ that pointwise converges to T . Then T is linear and bounded. Moreover, $\sup_n \|T_n\| < \infty$.

Proof. For all $x \in X$, $(T_n x)_{n \in \mathbb{N}}$ is convergent and thus bounded. So $\{T_n : n \in \mathbb{N}\}$ is pointwise bounded. Hence, by Theorem 4.2, there exists $M \geq 0$ such that, for all $n \in \mathbb{N}$, we have $\|T_n\| \leq M$.

- T linear: $T(\lambda x + \mu y) = \lim_{n \rightarrow \infty} T_n(\lambda x + \mu y) = \lim_{n \rightarrow \infty} [\lambda T_n(x) + \mu T_n(y)] = \lambda T(x) + \mu T(y)$
- T bounded: $\forall x \in B_X \forall n \in \mathbb{N} \|T_n x\| \leq \|T_n\| \leq M$, so $\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| \leq M$ for all $x \in B_X$. Hence, T is bounded with $\|T\| \leq M$. ■

Exercise. Show that $\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$.

Definition δ -dense

Let A, B be subsets of a metric space (X, d) and $\delta > 0$. We say that A is δ -dense in B if $\forall b \in B \exists a \in A d(a, b) \leq \delta$.

Remark. If $\bar{A} \supset B$, then A is δ -dense in B for all $\delta > 0$.

Lemma 4.4 Open mapping lemma

Let X be a Banach space, Y a normed space, $T \in \mathcal{B}(X, Y)$. Suppose that $T(MB_X)$ is δ -dense in B_Y for some $M \geq 0$ and $0 \leq \delta < 1$. Then $T(\frac{M}{1-\delta}B_X) \supset B_Y$.

Remarks

- Another way to think of the open mapping lemma is as follows.
Condition: for all $y \in B_Y$, $y = Tx$ has a δ -approximate solution in MB_X
Conclusion: for all $y \in B_Y$, $y = Tx$ has an exact solution in $\frac{M}{1-\delta}B_X$
- For any $M \geq 0$, $T(MB_X)$ is 1-dense in B_Y since $0 = T(0) \in T(MB_X)$ and $\forall y \in Y \|y - T(0)\| = \|y\| \leq 1$
- Lemma 4.4 implies that T is surjective
- Lemma 4.4 shows that $\overline{T(B_X)} \supset B_Y \implies T(D_X) \supset D_Y$.

Proof of Lemma 4.4. The strategy is to use ‘successive approximations’. Fix $y \in B_Y$. Pick $x_1 \in MB_X$ such that $\|y - Tx_1\| \leq \delta$. Note that $\frac{y - Tx_1}{\delta} \in B_Y$, so we can pick $x_2 \in MB_X$ such that $\|\frac{y - Tx_1}{\delta} - Tx_2\| \leq \delta$. Note that $\frac{y - T(x_1 + \delta x_2)}{\delta^2} \in B_Y$, so we can pick $x_3 \in MB_X$ such that $\|\frac{y - T(x_1 + \delta x_2)}{\delta^2} - Tx_3\| \leq \delta$. Continuing inductively, we obtain a sequence $(x_n)_{n \in \mathbb{N}}$ in MB_X such that

$$\left\| y - T \left(\sum_{k=1}^n \delta^{k-1} x_k \right) \right\| \leq \delta^n \quad \forall n \in \mathbb{N}$$

Note that $\|\delta^{n-1} x_n\| \leq M \delta^{n-1}$ for all $n \in \mathbb{N}$, so the series $\sum_{n=1}^{\infty} \delta^{n-1} x_n$ converges absolutely and hence converges since X is complete (cf. Example Sheet 1 Q8). Let $x = \sum_{n=1}^{\infty} \delta^{n-1} x_n$. Then $\|x\| \leq \sum_{n=1}^{\infty} \delta^{n-1} M = \frac{M}{1-\delta}$, so $x \in \frac{M}{1-\delta}B_X$. Since T is continuous, $Tx = \sum_{n=1}^{\infty} T(\delta^{n-1} x_n) = \lim_{n \rightarrow \infty} T(\sum_{k=1}^n \delta^{k-1} x_k) = y$. ■

Theorem 4.5 Open mapping theorem

Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is surjective, then T is open.

Remark. In particular, we have $T(B_X) \supset T(D_X) \supset rB_Y$ for some $r > 0$. Equivalently, $T(MB_X) \supset B_Y$ for some $M > 0$. So the conclusion of Theorem 4.5 is that, for all $y \in Y$, $y = Tx$ has a solution such that $\|x\| \leq M\|y\|$.

Proof of Theorem 4.5. Observe that it suffices to show that $T(MB_X) \supset B_Y$ for some $M > 0$. Indeed, it would mean that, given open $U \subset X$ and $y \in T(U)$, we can pick $x \in U$ such that $y = Tx$. As U is open, $B_r(x) \subset U$ for some $r > 0$. Then $T(U) \supset T(B_r(x)) = T(x + rB_X) = T(x) + \frac{r}{M}MB_X \supset y + \frac{r}{M}B_Y = B_{r/M}(y)$.

Note that $Y = T(X) = T(\bigcup_{n \in \mathbb{N}} nB_X) = \bigcup_{n \in \mathbb{N}} T(nB_X)$. As Y is non-empty and complete, Theorem 4.1 implies that there exists $n \in \mathbb{N}$ such that $\text{int } \overline{T(nB_X)} \neq \emptyset$. So there exists $y \in Y$ and $r > 0$ such that $B_r(y) \subset \overline{T(nB_X)}$. Moreover, since B_X is convex and symmetric, so is $\overline{T(nB_X)}$. Thus, given $z \in B_Y$, we have $y \pm rz \in B_r(y) \subset \overline{T(nB_X)}$ and also $-y \pm rz \in \overline{T(nB_X)}$. Now, note that $rz = \frac{1}{2}(y + rz) + \frac{1}{2}(-y + rz) \in \overline{T(nB_X)}$. Thus, $rB_Y \subset \overline{T(nB_X)}$ or $B_Y \subset \overline{T(\frac{n}{r}B_X)}$. So $T(\frac{n}{r}B_X)$ is $\frac{1}{2}$ -dense in B_Y . By Lemma 4.4, $T(\frac{2n}{r}B_X) \supset B_Y$. ■

Theorem 4.6 Inversion theorem

Let X, Y be Banach spaces and $T: X \rightarrow Y$ a continuous linear bijection. Then $T^{-1}: Y \rightarrow X$ is also continuous.

Proof 1. T is surjective so it is open by Theorem 4.5. So for open $U \subset X$, $(T^{-1})^{-1}(U) = T(U)$ is open in Y . So T^{-1} is continuous. ■

Proof 2. By Theorem 4.5, we have $T(MB_X) \supset B_Y$ for some $M > 0$. Given $y \in Y$, there exists $x \in X$ such that $y = Tx$ and $\|x\| \leq M\|y\|$, i.e., $\|T^{-1}y\| \leq M\|y\|$. So T^{-1} is bounded. ■

Corollary 4.7

Let $\|\cdot\|$ and $\|\cdot\|'$ be complete norms on a vector space X . If there exists $b > 0$ such that $\|x\|' \leq b\|x\|$ for all $x \in X$, then $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

Proof. $\text{Id}: (X, \|\cdot\|) \rightarrow (X, \|\cdot\|')$ is a linear bijection and bounded by hypothesis. ■

Remark. This gives us another proof that the L_1 -norm on $C[0, 1]$ is complete.

Recall that, for a function $f: X \rightarrow Y$ between sets, the *graph* of f is the set

$$\Gamma(f) = \{(x, y) \in X \times Y : y = f(x)\}.$$

If X, Y are topological spaces with Y Hausdorff and f continuous, then $\Gamma(f)$ is closed.⁵

Theorem 4.8 Closed graph theorem

Let X, Y be Banach spaces and $T: X \rightarrow Y$ a linear map. If $\Gamma(T)$ is closed, then T is continuous.

Remark. Note that continuity of T means:

$$x_n \rightarrow x \text{ in } X \implies (Tx_n) \text{ converges in } Y \text{ and } \lim_{n \rightarrow \infty} Tx_n = Tx$$

But by Theorem 4.8, to prove continuity of T , it suffices to show that

$$x_n \rightarrow x \text{ in } X \text{ and } Tx_n \rightarrow y \text{ in } Y \implies y = Tx$$

⁵You may recognise this from [Tripos 2025 Paper 2 Section II Question 10G](#)

Proof of Theorem 4.8. Consider the map $S: X \rightarrow \Gamma(T), x \mapsto (x, Tx)$. This is a linear bijection with $S^{-1}: \Gamma(T) \rightarrow X, (x, y) \mapsto x$. So $S^{-1} = P_X|_{\Gamma(T)}$ is continuous. Since $\Gamma(T)$ is a closed subspace of the Banach space $X \oplus Y$, $\Gamma(T)$ is complete. By Theorem 4.6, $S = (S^{-1})^{-1}$ is continuous. Hence, $T = P_Y \circ S$ is continuous. ■

Remark. Note that the condition ‘ $x_n \rightarrow x$ in X and $Tx_n \rightarrow y$ in $Y \implies y = Tx$ ’ is equivalent to ‘ $x_n \rightarrow 0$ in X and $Tx_n \rightarrow y$ in $Y \implies y = 0$ ’. The latter version will be quite useful for applications.

Exercise. Deduce Theorem 4.6 from Theorem 4.8.

4.3 Applications

Example

Let X be a closed subspace of ℓ_2 . We further assume that $X \subset \ell_1$.

► **Claim.** There exists $C > 0$ such that, for all $x \in X$, $\|x\|_1 \leq C\|x\|_2$.

Remark. We know that $\ell_1 \subset \ell_2$ with $\|x\|_2 \leq \|x\|_1$ for all $x \in \ell_1$. We might want to use Corollary 4.7, but we don’t know if $\|\cdot\|_1$ is complete on X . Similarly, $\text{Id}: (X, \|\cdot\|_1) \rightarrow (X, \|\cdot\|_2)$ is a continuous linear bijection. However, we cannot use Theorem 4.6 to deduce that Id^{-1} is continuous since we don’t know if $(X, \|\cdot\|_1)$ is complete.

Proof of Claim. Consider $T: (X, \|\cdot\|_1) \rightarrow \ell_1, x \mapsto x$. Since X is closed in ℓ_2 , $(X, \|\cdot\|_2)$ is complete and so is ℓ_1 . Moreover, T is linear, so by Theorem 4.8, it suffices to show that $\Gamma(T)$ is closed. To do this, suppose $x^n \rightarrow 0$ in $(X, \|\cdot\|_2)$ and $Tx^n = x^n \rightarrow y \in \ell_1$. Write $x^n = (x_k^n)_{k \in \mathbb{N}}$ and $y = (y_k)_{k \in \mathbb{N}}$. For $k \in \mathbb{N}$, $|x_k^n| \leq \|x^n\|_2 \rightarrow 0$ and $|x_k^n - y_k| \leq \|x^n - y\|_1 \rightarrow 0$ as $n \rightarrow \infty$. So $y = 0$ which means that $\Gamma(T)$ is closed. ■

Example

Let X be a normed space and $(x_n)_{n \in \mathbb{N}}$ a sequence in X such that $\forall f \in X^* \sum_{n \in \mathbb{N}} |f(x_n)| < \infty$.

► **Claim.** There exists $C \geq 0$ such that, for all $f \in X^*$, $\sum_{n \in \mathbb{N}} |f(x_n)| \leq C\|f\|$.

Proof of Claim. Define $T: X^* \rightarrow \ell_1, f \mapsto (f(x_n))_{n \in \mathbb{N}}$. This is well-defined by hypothesis and also linear. Moreover, X^* and ℓ_1 are complete, so we can again use Theorem 4.8. Suppose $f_n \rightarrow 0$ in X^* and $Tf_n \rightarrow y = (y_k)_{k \in \mathbb{N}}$ in ℓ_1 . As before, for $k \in \mathbb{N}$, $f_n(x_k) \rightarrow y_k$ as $n \rightarrow \infty$. But $|f_n(x_k)| \leq \|f_n\| \|x_k\| \rightarrow 0$ as $n \rightarrow \infty$. Hence, $y = 0$ as required. ■

Exercise. Prove the above claims using Theorem 4.2 and Corollary 4.3.

Example

► **Claim.** An infinite-dimensional Banach space X has no countable (algebraic) basis.

Proof. Suppose $B = \{e_1, e_2, e_3, \dots\}$ is a countable basis of X . Note that B is countably infinite since $\dim X = \infty$. For each $n \in \mathbb{N}$, let $F_n = \text{span}\{e_1, e_2, \dots, e_n\}$. Since $\dim F_n < \infty$, we know from Corollary 3.2 that F_n is closed in X . Let $x \in F_n, r > 0$. Then

$$x + \frac{r}{\|e_{n+1}\| + 1} e_{n+1} \in D_r(x) \setminus F_n$$

So $D_r(x) \not\subset F_n$ and thus $\text{int } F_n \neq \emptyset$, i.e., F_n is nowhere dense. Since $X = \text{span } B$, we have $X = \bigcup_{n \in \mathbb{N}} F_n$ — a contradiction to Theorem 4.1’. ■

Example

Let X be a vector space. Suppose Y and Z are subspaces of X such that $X = Y + Z$ and $Y \cap Z = \{0\}$, i.e., X is an algebraic direct sum of Y and Z . Then $T: Y \times Z \rightarrow X, (y, z) \mapsto y + z$ is a linear bijection. Note that $P: X \rightarrow Y, x = y + z \mapsto y$ is linear with $\text{im } P = Y$ and $\ker P = Z$. This is called the *projection of X onto Y along Z* . On the other hand, $I - P$ is the projection of X onto Z along Y .

Now, suppose X is a normed space. Recall that $Y \times Z$ is a normed space when equipped with the norm $\|(y, z)\| = \|y\| + \|z\|$. We denote this normed space by $Y \oplus Z$. The norm topology on $Y \oplus Z$ is the product topology. Note that $T: Y \oplus Z \rightarrow X$ is a continuous linear bijection:

$$\|T(y, z)\| = \|y + z\| \leq \|y\| + \|z\| = \|(y, z)\|$$

If T^{-1} is also continuous (i.e., $\|T^{-1}(y + z)\| = \|(y, z)\| = \|y\| + \|z\| \leq C\|y + z\|$ for some $C > 0$), then $X \sim Y \oplus Z$ and we write $X = Y \oplus Z$ and say that X is the (*topological*) *direct sum of Y and Z* .

► **Claim.** Suppose X is a Banach space, with subspaces Y and Z such that $X = Y + Z$ and $Y \cap Z = \{0\}$. The following are equivalent:

- (i) Y and Z are closed subspaces
- (ii) $X = Y \oplus Z$
- (iii) P is continuous

Proof. (i) \implies (ii): Y and Z are Banach spaces and hence so is $Y \oplus Z$. Since $T: Y \oplus Z \rightarrow X$ is a continuous linear bijection, T^{-1} is continuous by Theorem 4.6.

(ii) \implies (iii): $\|P(y + z)\| = \|y\| \leq \|y\| + \|z\| \leq C\|y + z\|$

(iii) \implies (i): $Z = \ker P = P^{-1}(\{0\})$ and $Y = \ker(I - P) = (I - P)^{-1}(\{0\})$ ■

Exercise. Show that (i) \implies (iii) directly using the closed graph theorem.

5 $C(K)$ spaces

Let K be a set, Let $\ell_\infty(K) := \{f: K \rightarrow \mathbb{R} \mid f \text{ bdd}\}$. This is a Banach space in the sup-norm $\|f\|_\infty = \sup_{x \in K} |f(x)|$.

Example. For $K = \mathbb{N}$, $\ell_\infty(K) = \ell_\infty$.

Let K be a topological space. Let $C_b(K) := \{f \in \ell_\infty(K) \mid f \text{ cts}\}$. By the uniform limit theorem, this is a closed subspace of $\ell_\infty(K)$ and hence a Banach space. In the case that K is a *compact* topological space, we also consider $C(K) = \{f: K \rightarrow \mathbb{R} \mid f \text{ cts}\} = C_b(K)$.

Remark. We can replace \mathbb{R} by \mathbb{C} in all of the above.