

# Differential Geometry

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**Lecturer:**

Mihalis Dafermos  
md384@cam.ac.uk

**Notes by:**

Robert Frederik Uy  
rfu20@cam.ac.uk

## Course schedule

Smooth manifolds in  $\mathbb{R}^n$ , tangent spaces, smooth maps and the inverse function theorem. Examples, regular values, Sard's theorem (statement only). Transverse intersection of submanifolds. [4]

Manifolds with boundary, degree mod 2 of smooth maps, applications. [3]

Curves in 2-space and 3-space, arc-length, curvature, torsion. The isoperimetric inequality. [2]

Smooth surfaces in 3-space, first fundamental form, area. [1]

The Gauss map, second fundamental form, principal curvatures and Gaussian curvature. Theorema Egregium. [3]

Minimal surfaces. Normal variations and characterization of minimal surfaces as critical points of the area functional. Isothermal coordinates and relation with harmonic functions. The Weierstrass representation. Examples. [3]

Parallel transport and geodesics for surfaces in 3-space. Geodesic curvature. [2]

The exponential map and geodesic polar coordinates. The Gauss-Bonnet theorem (including the statement about classification of compact surfaces). [4]

Global theorems on curves: Fenchel's theorem (the total curvature of a simple closed curve is greater than or equal to  $2\pi$ ); the Fary-Milnor theorem (the total curvature of a simple knotted closed curve is greater than  $4\pi$ ). [2]

## Recommended books

J. Milnor *Topology from the differentiable viewpoint*. Princeton University Press, 1997.

M. Do Carmo *Differential Geometry of Curves and Surfaces*. Pearson Higher Education, 1976

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# 1 Differential topology

**Definition** Smooth map on an open subset

Let  $U \subset \mathbb{R}^n$ . We say that  $f: U \rightarrow \mathbb{R}^m$  is smooth if all partial derivatives to all orders exist and are continuous.

**Definition** Smooth map

Let  $X \subset \mathbb{R}^n$ . We say that  $f: X \rightarrow \mathbb{R}^m$  is smooth if, for each  $x \in X$ , there exists (i) an open neighbourhood  $U \subset \mathbb{R}^n$  of  $x$  and (ii) a smooth map  $\tilde{f}: U \rightarrow \mathbb{R}^m$  such that  $\tilde{f}|_{X \cap U} = f|_{X \cap U}$ .

**Definition** Diffeomorphism

Let  $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ . We say that  $f: X \rightarrow Y$  is a diffeomorphism if  $f$  is a smooth bijection with a smooth inverse. If such a map exists, we say that  $X$  and  $Y$  are diffeomorphic.

**Exercise.** Give an example of a smooth bijection that is not a diffeomorphism.

**Definition**  $k$ -dimensional manifold

We say that  $X \subset \mathbb{R}^N$  is a  $k$ -dimensional manifold if, for each  $x \in X$ , there exists an open neighbourhood  $V \subset X$  of  $x$  such that  $V$  is diffeomorphic to an open subset  $U \subset \mathbb{R}^k$ . A diffeomorphism  $\varphi: U \rightarrow V$  is called a local parametrisation of  $V$ , whereas its inverse  $\psi := \varphi^{-1}: V \rightarrow U$  is called a coordinate system or a chart on  $V$ .

**Remarks**

- By composing  $\varphi^{-1}$  with the projections  $\pi_i: \mathbb{R}^k \rightarrow \mathbb{R}, (x_1, \dots, x_k) \mapsto x_i$ , we get smooth maps  $x_i := \pi_i \circ \varphi^{-1}$  which we call *coordinate functions*.
- WLOG, we can replace ‘diffeomorphic to an open subset  $U \subset \mathbb{R}^k$ ’ with ‘diffeomorphic to an open ball in  $\mathbb{R}^k$ ’.
- It is easy to see that, if  $X \subset \mathbb{R}^N$  is both a  $k$ -dimensional manifold and a  $\tilde{k}$ -dimensional manifold, then  $k = \tilde{k}$ .

**Definition** Dimension

Let  $X \subset \mathbb{R}^N$  be a  $k$ -dimensional manifold. The dimension of  $X$  is  $k$ , and it is denoted by  $\dim X$ .

**Example** Some trivial examples

- $X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$  for any open subset  $W$
- $X = \{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^N$

**Example**  $S^n$

$S^n := \{x \in \mathbb{R}^{n+1}: \|x\|_2 = 1\}$  is an  $n$ -dimensional manifold. To see this, consider the projection  $\Pi_k: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1})$ . It is easy to verify that maps of the form  $\psi_k^\pm = \Pi_k|_{S^n \cap \{\text{sign}(x_k) = \pm 1\}}$  are diffeomorphisms  $S^n \cap \{\text{sign}(x_k) = \pm 1\} \rightarrow B_1(0)$ .

**Remark.** It is easy to show that  $X$  is a 0-dimensional manifold iff  $X$  is a discrete subset of  $\mathbb{R}^N$ .

**Exercise.** Show that, if  $X$  and  $Y$  are manifolds, then  $X \times Y$  is also a manifold, with  $\dim X \times Y = \dim X + \dim Y$ .

**Definition** Submanifold

Let  $X, Y \subset \mathbb{R}^N$  be manifolds. If  $Y \subset X$ , then we say that  $Y$  is a submanifold of  $X$ . The codimension of  $Y$  in  $X$  is defined as

$$\text{codim}_X Y := \dim X - \dim Y$$

## 1.1 Tangent spaces

We first recall some basic facts from our youth. Let  $U \subset \mathbb{R}^k$  be open. The *differential* of a smooth map  $f: U \rightarrow \mathbb{R}^m$  at  $x \in U$  is defined by

$$df_x: \mathbb{R}^k \rightarrow \mathbb{R}^m$$

$$h \mapsto \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t}$$

This is a linear map, with matrix representation

$$df_x = \left( \frac{\partial f^i}{\partial x^j} \right)_{i,j}$$

Moreover, differentials satisfy the chain rule: given (i) two smooth maps  $f: U \rightarrow \mathbb{R}^l$  and  $g: V \rightarrow \mathbb{R}^m$  with  $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$  open and (ii) a point  $x \in U$  with  $f(x) \in V$ , we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

**Definition** Tangent space

Let  $X \subset \mathbb{R}^N$  be a  $k$ -dimensional manifold and  $x \in X$ . Choose a local parametrisation  $\varphi: U \rightarrow V$  around  $x$ . We then define the tangent space  $T_x X$  of  $X$  at  $x$  to be

$$T_x X := \text{im } d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k)$$

Of course, before we can safely proceed, we must show that  $T_x X$  is well-defined:

**Lemma 1.1**

Let  $X$  be as above.  $T_x X$  is independent of  $\varphi$ , and  $\dim T_x X = k$ .

*Proof.* Let  $\tilde{\varphi}: \tilde{U} \rightarrow \tilde{V}$  be another local parametrisation near  $x$ . WLOG, by restricting if necessary, we may assume  $\tilde{V} = V$ . By the chain rule, we have

$$d\varphi_{\varphi^{-1}(x)} = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)} \circ d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}$$

Since  $\tilde{\varphi}^{-1} \circ \varphi$  is a diffeomorphism of open subsets of  $\mathbb{R}^n$ , the corresponding differential  $d(\tilde{\varphi}^{-1} \circ \varphi)$  is a linear isomorphism. Thus,

$$d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}(\mathbb{R}^k)) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(\mathbb{R}^k)$$

as claimed.

Now, it remains to show that  $\dim T_x X = k$ . By definition, there exists an open set  $\hat{V} \subset \mathbb{R}^N$  and a smooth map  $\Psi: \hat{V} \rightarrow \mathbb{R}^k$  that extends the chart  $\psi := \varphi^{-1}$ . Note that  $\Psi \circ \varphi = \text{id}_U$ , so by the chain rule,

$$d\Psi_x \circ d\varphi_{\varphi^{-1}(x)} = \text{id}_{\mathbb{R}^k}$$

Then,  $d\varphi_{\varphi^{-1}(x)}$  must be an isomorphism  $\mathbb{R}^k \rightarrow T_x X$ , and hence  $\dim T_x X = k$ . ■

**Example** Tangent spaces for our trivial examples

Returning to the trivial examples we previously gave, we now state the corresponding tangent space for an arbitrary point  $x$  on each manifold.

- $X = \mathbb{R}^N$ :  $T_x X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$  for any open subset  $W$ :  $T_x X = \mathbb{R}^N$
- $X = \{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^N$ :  $T_x X = X$

**Example** *Tangent spaces for  $S^n$*

From any given chart, we can compute ( $\varphi$  and)  $d\varphi$ :

$$\frac{\partial \varphi}{\partial x^1} = (1, 0, \dots, 0, -x_1/x_{n+1})$$

and similarly for  $\partial \varphi / \partial x^i$ . Manifestly, each partial derivative is perpendicular to  $x$ . Thus,  $T_x X \subset x^\perp := \{v \in \mathbb{R}^{n+1} : \langle v, x \rangle = 0\}$ . Since we know from the above lemma that  $\dim T_x X = n$ , we conclude that  $T_x X = x^\perp$ .

**Definition** *Differential map for manifolds*

Let  $f: X \rightarrow Y$  be a smooth map between manifolds and  $x \in X$ . Choose a local parametrisation  $\varphi_1$  around  $x$  and  $\varphi_2$  around  $f(x) \in Y$ . We define the differential  $df_x: T_x X \rightarrow T_{f(x)} Y$  of  $f$  at  $x$  by

$$df_x = d\varphi_{2, \varphi_2^{-1}(f(x))} \circ d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)} \circ (d\varphi_{1, \varphi_1^{-1}(x)})^{-1}$$

**Lemma 1.2**

$df_x$  is independent of the choice of local parametrisations.

*Proof.* Trivial exercise. ■

**Proposition 1.3** *Chain rule for manifolds*

Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  be smooth maps between manifolds. For any  $x \in X$ ,

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

*Proof.* Trivial exercise. ■

**Theorem 1.4** *Inverse function theorem*

Let  $f: X \rightarrow Y$  be a smooth map between manifolds and  $x \in X$ . Suppose  $df_x: T_x X \rightarrow T_{f(x)} Y$  is an isomorphism. Then  $f$  is a local diffeomorphism, i.e., each  $x \in X$  has an open neighbourhood  $V \subset X$  such that  $f|_V: V \rightarrow f(V)$  is a diffeomorphism.

*Proof.* Since  $df_x$  is an isomorphism, it follows that  $d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)}$  is also an isomorphism. We can then use the usual inverse function theorem to deduce the result. ■

## 1.2 Regular values and Sard's theorem

**Definition** *Critical and regular points*

Let  $f: X \rightarrow Y$  be a smooth map between manifolds. We say that  $x \in X$  is a critical point of  $f$  if  $df_x: T_x X \rightarrow T_{f(x)} Y$  is not surjective. Otherwise, it is a regular point.

**Notation.** We denote by  $C$  the set of all critical points of  $f$ .

**Remark.** If  $\dim Y > \dim X$ , then  $C = X$  and the pre-image of any regular value is  $\emptyset$ .

**Definition** *Critical and regular values*

Let  $f: X \rightarrow Y$  be a smooth map between manifolds. We say that  $y \in Y$  is a critical value of  $f$  if  $y = f(x)$  for some  $x \in C$ . Otherwise, we say that  $y$  is a regular value of  $f$ .

**Theorem 1.5** *Pre-image theorem*

Let  $f: X \rightarrow Y$  be a smooth map between manifolds. Suppose  $y \in Y$  is a regular value of  $f$ . If  $f^{-1}(y) \neq \emptyset$ , then  $f^{-1}(y) \subset X$  is a submanifold of  $X$  with  $\text{codim}_X f^{-1}(y) = \dim Y$ .

*Proof.* Fix  $x \in f^{-1}(y)$ . Since  $y$  is a regular value, we know that  $df_x: T_x X \rightarrow T_y Y$  is surjective. By the rank-nullity theorem,  $\dim \ker df_x = \text{codim}_X Y$ . Suppose  $X \subset \mathbb{R}^N$ , and pick a linear map  $T: \mathbb{R}^N \rightarrow \mathbb{R}^{\text{codim}_X Y}$  such that  $\ker T \cap \ker df_x = \{0\}$ .<sup>1</sup>

Now, extend  $f$  to the map  $F: X \rightarrow Y \times \mathbb{R}^{\text{codim}_X Y}$  given by  $z \mapsto (f(z), T(z))$ . Note that the differential of  $F$  at  $x$  is given by

$$dF_x = (df_x, dT_x) = (df_x, T)$$

Since  $\ker T \cap \ker df_x = \{0\}$ , we have  $\ker dF_x = \{0\}$ , i.e.,  $dF_x$  is injective. By the inverse function theorem for manifolds, there exists an open neighbourhood  $U \subset X$  of  $x$  such that  $F|_U: U \rightarrow V$  is a diffeomorphism to an open neighbourhood  $V$  of  $(y, T(x))$ . Hence,  $F|_{f^{-1}(y) \cap U}$  is a local parametrisation of  $(\{y\} \times \mathbb{R}^{\text{codim}_X Y}) \cap V$ , proving that  $f^{-1}(y)$  is a manifold of dimension  $\text{codim}_X Y$ . ■

**Exercise.** Show that, under the conditions of the pre-image theorem,  $T_x f^{-1}(y) = \ker df_x$ .

### Corollary 1.6

Let  $f: X \rightarrow Y$  be a smooth map between manifolds of the same dimension, with  $X$  compact. If  $y$  is a regular value of  $f$ , then  $f^{-1}(y)$  is finite.

*Proof.* By the pre-image theorem,  $f^{-1}(y)$  is a 0-dimensional manifold, i.e., a collection of points. Since  $X$  is compact, such a collection must be finite. ■

With just a bit more analysis and topology, we can actually say more than just finiteness:

### Theorem 1.7 Stack of records theorem

Let  $f: X \rightarrow Y$  be a smooth map between manifolds of the same dimension, with  $X$  compact. Let  $y$  be a regular value of  $f$ , and list the elements of  $f^{-1}(y)$  as  $x_1, \dots, x_n$ . There exists an open neighbourhood  $V \subset Y$  of  $y$  and a collection of open neighbourhoods  $W_i \subset X$  of each  $x_i$  such that

$$f^{-1}(V) = \bigsqcup_{i=1}^n W_i$$

and each  $f|_{W_i}: W_i \rightarrow V$  is a diffeomorphism.

*Proof.* By the inverse function theorem for manifolds, we can pick open neighbourhoods  $W_i$  of  $x_i$  such that each  $f|_{W_i}$  is a diffeomorphism to an open neighbourhood of  $y$ . By shrinking neighbourhoods if necessary,  $W_i$  can be taken WLOG to be pairwise disjoint. Now, set

$$V = \left[ \bigcap_{i=1}^n f(W_i) \right] \setminus f \left( X \setminus \bigcup_{i=1}^n W_i \right)$$

Note that  $f(X \setminus \bigcup_{i=1}^n W_i)$  is a compact set that does not contain  $y$ , so  $V$  is an open neighbourhood of  $y$ . Finally, note that  $f^{-1}(V) = \bigsqcup_{i=1}^n W_i$  by construction. ■

Now, the pre-image theorem can be a powerful tool for generating manifolds or showing that a certain set is one.

### Application $S^n$ is a manifold

Consider the map  $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}, (x_1, \dots, x_{n+1}) \mapsto x_1^2 + \dots + x_{n+1}^2$ . Note that  $f^{-1}(1) = S^n$ , so to show that  $S^n$  is a manifold, it suffices to show that 1 is a regular point. Indeed, note that  $df_x = (2x_1, \dots, 2x_{n+1})$ , which is not surjective only if  $x = 0 \notin f^{-1}(1)$ .

<sup>1</sup>It is easy to constructively show using IB Linear Algebra that such a map exists. [Exercise!]

**Application** *Orthogonal group as a manifold*

Denote by  $M(n)$  [resp.  $S(n)$ ] the space of all [resp. symmetric]  $n \times n$  matrices with entries in  $\mathbb{R}$ . Consider the orthogonal group  $O(n) = \{A \in M(n) : AA^t = I\} \subset M(n) = \mathbb{R}^{n^2}$ .

Let  $f: M(n) \rightarrow O(n)$  be the map  $A \mapsto AA^t$ . This is smooth since multiplication and addition in  $\mathbb{R}$  are smooth. Since  $O(n) = f^{-1}(I)$ , it suffices to show that  $I$  is a regular value of  $f$ . Note that

$$df_A(H) = \lim_{t \rightarrow 0} \frac{f(A + tH) - f(A)}{t} = AH^t + HA^t$$

Now, fix  $A \in M(n)$ . Given  $B \in S(n)$ , observe that

$$df_A\left(\frac{1}{2}CA\right) = \frac{1}{2}AA^tC^t + \frac{1}{2}CAA^t = \frac{1}{2}C + \frac{1}{2}C = C$$

completing the proof that  $I$  is a regular value of  $f$ .

**Remark.** Recall that, besides being a manifold as we've just shown,  $O(n)$  is also a group. In fact, the group operations  $(A, B) \mapsto AB$  and  $A \mapsto A^{-1} = A^t$  are smooth. Hence, we see that  $O(n)$  is a *Lie group*.

Now, the pre-image theorem raises the question: how easy is to find regular values? This leads us to Sard's theorem.

**Definition** *Measure-zero subsets of  $\mathbb{R}^N$*

We say that  $S \subset \mathbb{R}^N$  is of measure zero in  $\mathbb{R}^N$  if, for each  $\varepsilon > 0$ , there exists a countable family  $\{R_i\}$  of sets of the form  $R_i = \prod_{j=1}^N [x_i^{(j)}, y_i^{(j)}]$  such that  $S \subset \bigcup_i R_i$  and  $\sum_i \text{vol}(R_i) < \varepsilon$ .

**Definition** *Measure zero subsets of manifolds*

Let  $X \subset \mathbb{R}^N$  be a  $k$ -dimensional manifold. We say that  $A \subset X$  is of measure zero in  $X$  if, for all local parametrisations  $\varphi: U \rightarrow V$  of  $X$ ,  $\varepsilon^{-1}(V \cap A) \subset \mathbb{R}^k$  has measure zero in  $\mathbb{R}^k$ .

**Exercise.** Let  $U, \tilde{U} \subset \mathbb{R}^k$  be open and  $\psi: U \rightarrow \tilde{U}$  a diffeomorphism. Show that, if  $A \subset U$  is of measure zero in  $\mathbb{R}^k$ , then  $\tilde{T} = \psi(T)$  is of measure zero in  $\mathbb{R}^k$ .

**Remarks**

- In view of the above exercise,  $A \subset X$  is of measure zero in  $X$  iff  $\varphi_i^{-1}(S \cap V_i)$  is of measure zero for all  $\varphi_i: U \rightarrow V$  in an atlas of local parametrisations.
- If  $\dim Y = 0$ , then  $Y$  is of measure zero. If  $\dim Y > 0$ , then every non-empty open subset  $V \subset Y$  is not of measure zero in  $Y$ .
- If  $S \subset X$  is of measure zero in  $X$ , then any  $\tilde{S} \subset S$  is also of measure zero in  $X$ .

**Theorem 1.8** *Sard's theorem*

Let  $f: X \rightarrow Y$  be a smooth map between manifolds. Then the set of critical values of  $f$  is of measure zero in  $Y$ .

*Proof.* Non-examinable — see Milnor's book if interested. ■

**Corollary 1.9**

The set of regular values of a smooth map  $f: X \rightarrow Y$  between manifolds is dense in  $Y$ .

*Proof.* Any open set  $V \subset Y$  cannot lie entirely in  $f(C)$  since it has measure zero. ■

### 1.3 Transversality

**Definition** Transversal

Let  $f: X \rightarrow Y$  be smooth and  $Z \subset Y$  a submanifold of  $Y$ . We say that  $f$  is transversal to  $Z$  if, for each  $x \in f^{-1}(Z)$ ,

$$T_{f(x)}Y = T_{f(x)}Z + \text{im } df_x$$

We then write  $f \pitchfork Z$ .

**Remarks**

- If  $f(X) \cap Z = \emptyset$ , then  $f$  is transversal to  $Z$ .
- If  $Z = \{y\}$ , then  $f$  is transversal to  $Z$  iff  $y$  is a regular value of  $f$ . Thus, transversality is really a generalisation of the notion of regular values.

**Exercise.** Let  $X$  also be a submanifold of  $Y$  and  $\iota: X \hookrightarrow Y$  the inclusion map. Show that  $d\iota_x$  is just the inclusion map  $T_xX \hookrightarrow T_xY$  of the tangent spaces. Thus,  $\iota \pitchfork Z$  iff  $T_xX + T_xZ = T_xY$  for all  $x \in X \cap Z$ .

Now, we state a generalisation of the pre-image theorem for transversal maps:

**Theorem 1.10**

Let  $f: X \rightarrow Y$  be smooth map that is transversal to a submanifold  $Z \subset Y$  of  $Y$ . If  $f^{-1}(Z) \neq \emptyset$ , then  $f^{-1}(Z) \subset X$  is a submanifold of  $X$ , with  $\text{codim}_X f^{-1}(Z) = \text{codim}_Y Z$ .

**Remark.** If  $Z = \{y\}$ , then  $\text{codim}_Y Z = \dim Y$  as in the pre-image theorem.

*Sketch of proof (non-examinable).* Fix  $z \in Z$  with  $z = f(x)$  for some  $x \in X$ . Note that, for some open neighbourhood  $V \subset Y$  of  $z$ , there exists a smooth map  $h: V \rightarrow \mathbb{R}^{\text{codim}_Y Z}$  such that  $Z \cap V = h^{-1}(0)$  and  $dh_z$  is surjective. Locally around  $x \in X$ ,  $f^{-1}(Z) = (h \circ f)^{-1}(0)$ . Thus, by the pre-image theorem, it suffices to show that 0 is a regular value of  $h \circ f$ .

Now, since  $f \pitchfork Z$ , we have  $T_zY = T_zZ + \text{im } df_x$ . Note that  $\ker dh_z = T_zZ$  since  $T_zZ \subset \ker dh_z$  and they are subspaces of the same dimension. Moreover,  $f \pitchfork Z$  gives us

$$T_zY = T_zZ + \text{im } df_x = \ker dh_z + \text{im } df_x$$

This then implies that  $\text{im } dh_z = \text{im}(dh_z \circ df_x) = \text{im } d(h \circ f)_x$ . Since  $dh_z$  is surjective,  $d(h \circ f)_z$  is also surjective and hence 0 is a regular value of  $h \circ f$ . ■