University of Cambridge Mathematical Tripos Part II

Linear Analysis

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Course schedule

Normed and Banach spaces. Linear mappings, continuity, boundedness, and norms. Finite-dimensional normed spaces. [4]

The Baire category theorem. The principle of uniform boundedness, the closed graph theorem and the inversion theorem; other applications. [5]

The normality of compact Hausdorff spaces. Urysohn's lemma and Tiezte's extension theorem. Spaces of continuous functions. The Stone–Weierstrass theorem and applications. Equicontinuity: the Ascoli–Arzelá theorem. [5]

Inner product spaces and Hilbert spaces; examples and elementary properties. Orthonormal systems, and the orthogonalization process. Bessel's inequality, the Parseval equation, and the Riesz–Fischer theorem. Duality; the self duality of Hilbert space. [5]

Bounded linear operations, invariant subspaces, eigenvectors; the spectrum and resolvent set. Compact operators on Hilbert space; discreteness of spectrum. Spectral theorem for compact Hermitian operators. [5]

Recommended books

B. Bollobas *Linear Analysis.* Cambridge University Press 1999.
G.J.O. Jameson *Topology and Normed Spaces.* Chapman and Hall 1974.
G. Allan *Introduction to Banach Spaces and Algebras.* Oxford University Press 2010.
W. Rudin *Real and Complex Analysis.* McGraw–Hill International Edition 1987.

Contents

1 Normed spaces and bounded linear maps

1.1 Definitions and examples

Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. For ease of notation and discussion, we will sometimes just take our scalars to be in $\mathbb{R}$, although the statement may be easily generalised to $\mathbb{C}$-vector spaces.

Definition Norm
A norm on X is a function $||\cdot||\colon X \to \mathbb{R}$ such that
(i) $||x|| \geq 0$ *for all* $x \in X$, *with* $||x|| = 0$ *iff* $x = 0$
(ii) $||\lambda x|| = |\lambda| ||x||$ *for all* $x \in X$ *and any scalar* λ
(iii) $||x+y|| \leq ||x|| + ||y||$ *for all* $x, y \in X$

Definition Normed space
A normed space is a pair $(X, ||\cdot||)$ where X is a vector space and $||\cdot||$ is a norm on X.

Example *Some finite-dimensional normed spaces*
(1) $\ell_2^n = (\mathbb{R}^n, ||\cdot||_2)$ or $(\mathbb{C}^n, ||\cdot||_2)$, where the norm is given by

$$||x||_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}, \qquad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$$

This is called the ℓ_2-norm or euclidean norm.

(i),(ii) are easy to check, whereas (iii) follows from Cauchy-Schwarz.

(2) $\ell_1^n = (\mathbb{R}^n, ||\cdot||_1)$ where $||x||_1 = \sum_{i=1}^n |x_i|$ (called the ℓ_1-norm)

(3) $\ell_\infty^n = (\mathbb{R}^n, ||\cdot||_\infty)$ where $||x||_\infty = \max_{1\leq i\leq n} |x_i|$ (called the ℓ_∞-norm or the sup-norm)

Given a normed space X, its norm $||\cdot||$ induces a metric on X:

$$d(x,y) = ||x-y||$$

Indeed, d is a metric:
- $d(x,y) \geq 0$ for all $x, y \in X$, with $d(x,y) = 0 \iff x - y = 0 \iff x = y$
- $d(x,y) = ||x-y|| = ||y-x|| = d(y,x)$
- $d(x,z) = ||x-z|| \leq ||x-y|| + ||y-z|| = d(x,y) + d(y,z)$

This metric, in turn, induces a topology on X, called the *norm topology.* This allows us talk about open/closed sets, convergence, and continuity, as we illustrate in the following examples.

Example
The algebraic operations are continuous:
- if $x_n \to x$ and $y_n \to y$ in X, then $x_n + y_n \to x + y$
- if $x_n \to x$ in X and $\lambda_n \to \lambda$ in $\mathbb{R}$, then $\lambda_n x_n \to \lambda x$

Example
The norm $||\cdot||\colon X \to \mathbb{R}$ is continuous: by the triangle inequality, we have

$$|||x|| - ||y||| \leq ||x-y||$$

so $||\cdot||$ is, in fact, Lipschitz.

Definition Banach space
A Banach space is a complete normed space, i.e., a normed space that is complete in its norm topology.

Example
ℓ_2^n, ℓ_1^n, ℓ_∞^n are complete: for any of these spaces,
- $x^{(k)} \to x \iff x_i^{(k)} \to x_i$ for all $1 \le i \le n$
- $(x^{(k)})_{k\in\mathbb{N}}$ is Cauchy $\iff (x_i^{(k)})_{k\in\mathbb{N}}$ is Cauchy for all $1 \le i \le n$

In a normed space, a useful object is the *unit ball*

$$B_X := \{x \in X\colon ||x|| \le 1\}$$

Remarks
- B_X defines a norm on X:
$$||x|| = \inf\{t \ge 0\colon x \in tB_X\}$$
- B_X is symmetric ($x \in B_X \implies -x \in B_X$), convex, and closed
- If $B \subset \mathbb{R}^n$ is a closed, convex, symmetric, bounded neighbourhood of 0, then B is the unit ball of $(\mathbb{R}^n, ||\cdot||)$ for some norm $||\cdot||$
- 'Geometry of Banach spaces'

Previously, we gave $\ell_2, \ell_1, \ell_\infty$ as examples of finite-dimensional normed spaces. More generally, we have the following family of examples

Example
(4) $\ell_p^n = (\mathbb{R}^n, ||\cdot||_p)$ for $1 \le p < \infty$, where $||x||_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ (called the ℓ_p-norm)
Again, (i) and (ii) are easy to check, whereas (iii) is not obvious.[1]

Now, let S denote the set of all scalar sequences. This is a vector spaces under the coordinate operations $(x_n) + (y_n) = (x_n + y_n)$ and $\lambda(x_n) = (\lambda x_n)$.

Example *Sequence spaces*
(5) $\ell_1 = \left\{(x_n) \in S\colon \sum_{n=1}^\infty |x_n| < \infty\right\}, \quad ||(x_n)||_1 = \sum_{n=1}^\infty |x_n| \quad$ (ℓ_1-norm)
(i) and (ii): easy to check.
(iii): Given $(x_n), (y_n) \in \ell_1$, we have $|x_n + y_n| \le |x_n| + |y_n|$ for all $n \in \mathbb{N}$. Summing over all $n \in \mathbb{N}$, we deduce that $(x_n) + (y_n) \in \ell_1$ and $||(x_n) + (y_n)||_1 \le ||(x_n)||_1 + ||(y_n)||_1$. Hence, ℓ_1 is a subspace of S and $||\cdot||_1$ is a norm on ℓ_1.

(6) $\ell_2 = \left\{(x_n) \in S\colon \sum_{n=1}^\infty |x_n|^2 < \infty\right\}, \quad ||(x_n)||_2 = \left(\sum_{n=1}^\infty |x_n|^2\right)^{1/2} \quad$ (ℓ_2-norm)
(i) and (ii): easy to check.
(iii): Given $(x_n), (y_n) \in \ell_2$, the triangle inequality in ℓ_2^N gives us

$$\left(\sum_{k=1}^N |x_k + y_k|^2\right)^{1/2} \le \left(\sum_{k=1}^N |x_k|^2\right)^{1/2} + \left(\sum_{k=1}^N |y_k|^2\right)^{1/2}.$$

Taking $N \to \infty$, we get $(x_n) + (y_n) \in \ell_2$ and $||(x_n) + (y_n)||_2 \le ||(x_n)||_2 + ||(y_n)||_2$

[1] We will return to this later in the next subsection.

More generally, for $1 \leq p < \infty$, the set

$$\ell_p = \left\{ (x_n) \in S \colon \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

is a subspace of S, and

$$||(x_n)||_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \qquad (\ell_p\text{-norm})$$

is a norm on ℓ_p. [(iii) *follows from the triangle inequality on* ℓ_p^n*, which we will see later.*]

Example *More sequence spaces*

(7) $\ell_\infty = \{(x_n) \in S \colon \exists\, M \geq 0\; \forall\, n \in \mathbb{N}\; |x_n| \leq M\}$, $\quad ||(x_n)||_\infty = \sup_{n\in\mathbb{N}} |x_n| \quad$ (ℓ_∞-norm)

(i) and (ii): easy to check.
(iii): Given $x = (x_n), y = (y_n) \in \ell_\infty$,

$$|x_n + y_n| \leq |x_n| + |y_n| \leq ||x||_\infty + ||y||_\infty \quad \forall\, n \in \mathbb{N}$$

so $x + y \in \ell_\infty$ and $||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$.

(8) $c_0 = \{(x_n) \in S \colon x_n \to 0 \text{ as } n \to \infty\}$
$c = \{(x_n) \in S \colon \lim_{n\to\infty} x_n \text{ exists}\}$

Both c_0 and c are subspaces of ℓ_∞ and are hence normed spaces in the ℓ_∞-norm.

1.2 Inequalities of Minkowski and Hölder

Recall that a function $f \colon (0, \infty) \to \mathbb{R}$ is *convex* if

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y) \qquad \forall\, x, y \in (0, \infty)\; \forall\, t \in [0, 1]$$

and *concave* if the above holds with $\leq$ replaced by $\geq$.

Lemma 1.1
Let $1 \leq p < \infty$*. Then the map*

$$\begin{aligned} (0, \infty) &\to \mathbb{R} \\ x &\mapsto x^p \end{aligned}$$

is convex.

Proof. Fix $y > 0, t \in [0, 1]$, and define

$$g(x) = [(1 - t)x + ty]^p - [(1 - t)x^p + ty^p], \qquad x > 0.$$

Differentiating, we get

$$g'(x) = p(1 - t)[(1 - t)x + ty]^{p-1} - p(1 - t)x^{p-1}.$$

Observe that $0 < x < y \implies g'(x) \geq 0$ and that $x > y \implies g'(x) \leq 0$. By the MVT, we deduce that $g(x) \leq g(y) = 0$ for all $x \in (0, \infty)$. ∎

Theorem 1.2 Minkowski's inequality
Let $1 \leq p < \infty$*,* $n \in \mathbb{N}$*. For* $x, y \in \mathbb{R}^n$*,*

$$||x + y||_p \leq ||x||_p + ||y||_p.$$

Remark. This shows that ℓ_p^n and ℓ_p are normed spaces.

Exercise. Show that $\ell_p, 1 \leq p \leq \infty$, is complete.[2]

Proof of Theorem 1.2. Let $B = \{x \in \mathbb{R}^n \colon ||x||_p \leq 1\}$. We first show that B is convex. Let $x, y \in B$ and $t \in [0,1]$. For $1 \leq k \leq n$,

$$|(1-t)x_k + ty_k|^p \leq ((1-t)|x_k| + t|y_k|)^p \leq (1-t)|x_k|^p + t|y_k|^p$$

by Lemma 1.1 for $x_k \neq 0, y_k \neq 0$; the inequality holds trivially if $x_k = 0$ or $y_k = 0$. Summing over k, we then get

$$||(1-t)x + ty||_p^p \leq (1-t)||x||_p^p + t||y||_p^p \leq 1,$$

so $(1-t)x + ty \in B$.

We then complete the proof as follows. Let $x, y \in \mathbb{R}^n$. WLOG, $x, y, x+y$ are nonzero. By convexity of B, we have

$$\frac{x+y}{||x||_p + ||y||_p} = \frac{||x||_p}{||x||_p + ||y||_p} \cdot \underbrace{\frac{x}{||x||_p}}_{\in B} + \frac{||y||_p}{||x||_p + ||y||_p} \cdot \underbrace{\frac{y}{||y||_p}}_{\in B} \in B.$$

Thus, it follows that

$$\left|\left|\frac{x+y}{||x||_p + ||y||_p}\right|\right| \leq 1 \Longrightarrow ||x+y||_p \leq ||x||_p + ||y||_p,$$

as required. ■

Let $x = (x_n) \in \ell_1$ and $y = (y_n) \in \ell_\infty$. We then write $x \cdot y = (x_n y_n)$. Note that, for all $n \in \mathbb{N}$, $|x_n y_n| = |x_n||y_n| \leq |x_n|||y||_\infty$. Thus, $x \cdot y \in \ell_1$ and $||x \cdot y||_1 \leq ||x||_1 ||y||_\infty$.

Definition Conjugate index
Let $p \in (1, \infty)$. The conjugate index of p is the unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.3
Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $a, b \geq 0$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. The inequality holds trivially if $a = 0$ or $b = 0$, so it remains to consider the case $a, b > 0$. A proof similar to that of Lemma 1.1 shows that $\log \colon (0, \infty) \to \mathbb{R}$ is concave. Hence,

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) = \log(ab).$$

We then apply exp to get the required result. ■

Theorem 1.4 Hölder's inequality
Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in \ell_p$ and $y \in \ell_q$, then $x \cdot y \in \ell_1$ and

$$||x \cdot y||_1 \leq ||x||_p ||y||_q.$$

Remark. As discussed above, $p = 1, q = \infty$ also works. Moreover, setting $p = q = 2$, we recover Cauchy-Schwarz.

Exercise. Deduce Minkowski's inequality from Hölder's inequality.

[2] A slick proof of this will be provided later.

Proof of Theorem 1.4. WLOG, $x \neq 0$ and $y \neq 0$. By homogeneity, we may also take $||x||_p = ||y||_q = 1$ WLOG. Now, by Lemma 1.3, we have $|x_n y_n| \leq |x_n|^p/p + |y_n|^q/q$ for all $n \in \mathbb{N}$. Summing over n, we have

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \frac{||x||_p^p}{p} + \frac{||y||_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 = ||x||_p ||y||_q,$$

as required. ■

1.3 More examples: function spaces

Example

(9) $C[0,1] = \{f \colon [0,1] \to \mathbb{R} \mid f \text{ cts}\}$, $\quad ||f||_\infty = \sup_{[0,1]} |f|$ (sup norm or uniform norm)

By the uniform limit theorem, this is a Banach space.

(10) More generally, given a compact, Hausdorff topological space K,

$$C(K) = \{f \colon K \to \mathbb{R} \mid f \text{ cts}\}$$

is a Banach space in the sup norm $||f||_\infty = \sup_K |f|$.

(11) $(C[0,1], ||\cdot||_1)$, $\quad ||\cdot||_1 = \int_0^1 |f(t)|\, dt$ (L_1-norm)

This is an *incomplete* normed space — see Example Sheet 1.

More generally, $C[0,1]$ is incomplete in the L_p-norm, $1 \leq p < \infty$, given by

$$||f||_p = \left(\int_0^1 |f(t)|^p \, dt \right)^{1/p}.$$

> In II Probability and Measure, you will encounter the completion of $(C[0,1], ||\cdot||_p)$, which is the Lebesgue space $L_p[0,1]$.

(12) $C^1[0,1] = \{f \in C[0,1] \mid f \text{ continuously differentiable}\}$ is a subspace of $C[0,1]$, so it is a normed space in $||\cdot||_\infty$ but incomplete, i.e. not closed in $C[0,1]$. However, it is complete in the norm $||f|| = ||f||_\infty + ||f'||_\infty$ — see Example Sheet 1.

(13) Let $\triangle = \{z \in \mathbb{C} \mid |z| \leq 1\}$. The set

$$A(\triangle) = \{f \in C(\triangle) \mid f \text{ analytic on int} \triangle\}$$

is a subspace of $C(\triangle)$. In fact, it is closed in $C(\triangle)$ and hence a Banach space in $||\cdot||_\infty$.

1.4 More on the normed topology

Let X be a normed space and $A \subset C$. Recall that the *closure* of A in X is

$$\overline{A} = \{x \in X \mid \exists\, (a_n) \text{ in } X \text{ s.t. } a_n \to x \text{ as } n \to \infty\}.$$

We then say that A is *dense* in X if $\overline{A} = X$. Moreover, A is *separable* if it has a countable dense subset.

If $Y \subset X$ is a subspace, then so is $\overline{Y}$: if $x, y \in \overline{Y}$, then there exists (x_n), (y_n) in Y such that $x_n \to x$ and $y_n \to y$. So $\lambda x_n + \mu y_n \to \lambda x + \mu y \in \overline{Y}$. Similarly, if $A \subset X$ is convex, then so is $\overline{A}$.

For a subset $A \subset X$, the *closed linear span* of A, denoted by $\overline{\text{span}}\, A$, is the closure of span A.

Remarks

- If A is countable, then $\overline{\operatorname{span}}\, A$ is separable.
- The set of all rational linear combinations of elements of A is countable and dense in $\overline{\operatorname{span}}\, A$.

Example

- $\overline{\mathbb{Q}} = \mathbb{R}$, so $\mathbb{R}$ is separable.
- $\ell_p, 1 \leq p < \infty$, is separable.
  Let $e_n = (0, \cdots, 0, \underset{n}{1}, 0, \cdots)$, $n \in \mathbb{N}$ (unit vector basis)
  Let $c_{00} = \operatorname{span}\{e_n \colon n \in \mathbb{N}\} = \{(x_n) \in S \colon \exists\, N \in \mathbb{N}\ \forall\, n > N x_n = 0\}$
  We then show that $\ell_p = \overline{\operatorname{span}}\{e_n \colon n \in \mathbb{N}\}$: if $x = (x_n) \in \ell_p$, then
  $$\left|\left| x - \sum_{i=1}^{N} x_i e_i \right|\right|_p = \left(\sum_{i>n} |x_i|^p \right)^{1/p} \to 0 \text{ as } N \to \infty$$
- Similarly, in ℓ_∞, we have $\overline{\operatorname{span}}\{e_n \colon n \in \mathbb{N}\} = c_0$. Moreover, c is separable, whereas ℓ_∞ is not.

Exercise. Prove the claims in the last example above.

1.5 Bounded linear maps

Theorem 1.5

Let X, Y be normed spaces and $T \colon X \to Y$ be a linear map. The following are equivalent:

(i) *T is continuous at 0*

(ii) *T is continuous*

(iii) *T is Lipschitz*

(iv) *T is bounded, i.e., $\exists\, C \geq 0\ \forall\, x \in X\ ||Tx|| \leq C||x||$.*

Proof. (iv) $\implies$ (iii): Observe that

$$d(Tx, Ty) = ||Tx - Ty|| = ||T(x - y)|| \leq C||x - y|| = Cd(x, y)$$

iii) $\implies$ (ii): Given $\varepsilon > 0$ take $\delta = \varepsilon/(C + 1)$.
(ii) $\implies$ (i): Trivial.
(i) $\implies$ (iv): $\exists\, \delta > 0\ \forall\, x \in X\ d(x, 0) = ||x|| \leq \delta \implies d(Tx, T0) = ||Tx|| \leq 1$. For $x \neq 0$, $||\delta x/||x|||| = \delta$, so $||T(\delta x/||x||)|| \leq 1$. Hence, $||Tx|| \leq \delta^{-1}||x||$. ■

For normed spaces X, Y, let $\mathcal{B}(X, Y) = \{T \colon X \to Y \mid T \text{ linear and bounded}\}$. For $T \in \mathcal{B}(X, Y)$, its *operator norm* is

$$||T|| = \sup\{||Tx|| \colon x \in B_X\}.$$

Remark. Since $T \in \mathcal{B}(X, Y)$, we have $C \geq 0$ such that $||Tx|| \leq C||x||$ for all $x \in X$. So if $||x|| \leq 1$, then $||Tx|| \leq C$. Thus, by definition, $||T|| \leq C$. Conversely, for all $x \in B_X$, we have $||Tx|| \leq ||T||$, so by homogeneity, $||Tx|| \leq ||T||||x||$. Hence, $||T||$ is the least C such that (iv) in Theorem 1.5 above holds.

The operator norm is a norm on $\mathcal{B}(X, Y)$: given $S, T \in \mathcal{B}(X, Y)$, we have, for all $x \in X$,

$$||(S + T)x|| = ||Sx + Tx|| \leq ||Sx|| + ||Tx|| \leq ||S||||x|| + ||T||||x|| \leq (||S|| + ||T||)||x||,$$

from which it follows that $S + T \in \mathcal{B}(X, Y)$ and $||S + T|| \leq ||S|| + ||T||$.

Notation. We write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$.

Proposition 1.6
Let X, Y, Z be normed spaces, $S \in \mathcal{B}(X,Y)$, $T \in \mathcal{B}(Y,Z)$. Then $TS \in \mathcal{B}(X,Z)$ and $||TS|| \leq ||T||||S||$.

Proof. For all $x \in X$, we have $||TSx|| \leq ||T||||Sx|| \leq ||T||||S||||x||$. ∎

Example

(1) $T\colon \ell_2^n \to \ell_2^n, (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_r, 0, \cdots, 0)$

$$||Tx||_2 = \left(\sum_{i=1}^{r} |x_i|^2\right)^{1/2} \leq ||x||_2 \implies ||T|| \leq 1$$

But $Te_1 = e_1$ so $||T|| = 1$.

More generally, if T is represented by a matrix A wrt the standard basis, then Cauchy-Schwarz gives us

$$||T|| \leq \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

(2) Let $1 \leq p < \infty$; $R\colon \ell_p \to \ell_p, (x_1, x_2, x_3, \cdots) \mapsto (0, x_1, x_2, \cdots)$ (right shift)

For all $x \in \ell_p$, $||Rx||_p = ||x||_p$, so R is isometric and $||R|| = 1$. Note that R is injective but not surjective.

(3) Let $1 \leq p < \infty$; $L\colon \ell_p \to \ell_p, (x_1, x_2, x_3, \cdots) \mapsto (x_2, x_3, x_4, \cdots)$ (left shift)

For all $x \in \ell_p$, $||Lx||_p \leq ||x||_p$, so $L \in \mathcal{B}(\ell_p)$ with $||L|| \leq 1$. Since $Le_2 = e_1$ and $||e_1||_p = ||e_2||_p = 1$, we in fact have $||L|| = 1$. Note that L is surjective but not injective.

(4) $T\colon \ell_1 \to \ell_2, x \mapsto x$

- **Claim.** $\ell_1 \subset \ell_2$, and $\forall\, x \in \ell_2$ $||x||_2 \leq ||x||_1$

  Proof. WLOG assume $||x||_1 = 1$ by homogeneity. Since $\sum_{n=1}^{\infty} |x_i| = 1$, we have $|x_i| \leq 1$ for all i. Thus,

  $$|x_i|^2 \leq |x_i| \;\forall\, i \implies ||x||_2^2 \leq ||x||_1 = 1 \implies ||x||_2 = 1 = ||x||_1$$

  as claimed. ∎

Using the above claim, we have $T \in \mathcal{B}(\ell_1, \ell_2)$ and $||T|| = 1$.

(5) $T\colon \ell_2 \to \ell_1, (x_n) \mapsto (x_n/n)$

By Cauchy-Schwarz,

$$\sum_{n=1}^{\infty} \left|\frac{x_i}{n}\right| \leq \left(\sum_{n=1}^{\infty} x_i^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2}$$

so $T \in \mathcal{B}(\ell_2, \ell_1)$ with $||T|| \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2}$. In fact, we can replace $\leq$ with $=$.

(6) $D\colon (C^1[0,1], ||\cdot||) \to (C[0,1], ||\cdot||_\infty), f \mapsto f'$

Note that $||Df||_\infty = ||f'||_\infty \leq ||f||_\infty + ||f'||_\infty = ||f||$, so $||D|| \leq 1$. But taking $f(x) = \sin(n\pi x)$, we have

$$||Df||_\infty = n\pi, \qquad ||f|| = n\pi + 1,$$

so in fact $||D|| = 1$. Note also that, for $f \neq 0$, $||Df||_\infty < ||f||$, so $||D||$ is not attained.

(7) On a normed space X, the identity $x \mapsto x$ is denoted by Id, I, Id_X or I_X. This map is isometric, i.e., $||\mathrm{Id}(x)|| = ||x|| \; \forall x \in X$.

(8) For normed spaces X, Y, we let

$$X \oplus Y = \{(x, y) \colon x \in X, y \in Y\}$$

with norm $||(x, y)||_1 = ||x|| + ||y||$. The corresponding norm topology is the product topology.

Define $P \colon X \oplus Y \to X, (x, y) \mapsto x$ (projection onto X). Note that $P \in \mathcal{B}(X \oplus Y, X)$ with $||P|| = 1$.

Let X, Y be normed spaces. We introduce some terminology:

- An *isomorphism* $X \to Y$ is a linear homeomorphism $T \colon X \to Y$, i.e., T is a linear bijection such that T and T^{-1} are bounded. Equivalently, T is a linear bijection[3] such that

$$\exists\, a, b > 0 \; \forall x \in X \;\; a||x|| \leq ||Tx|| \leq b||x||$$

  If such T exists, we say that X and Y are *isomorphic*, and we write $X \sim Y$.
- An *isometric isomorphism* is a linear bijection $T \colon X \to Y$ such that

$$\forall x \in X \;\; ||Tx|| = ||x||$$

  If such T exists, we say that X and Y are *isometrically isomorphic*, and we write $X \cong Y$.

> The Banach-Mazur distance is defined as
>
> $$d(X, Y) = \begin{cases} \infty, & \text{if } X \not\sim Y \\ \inf\{||T||||T^{-1}|| \mid T \colon X \to Y \text{ is an isomorphism}\}, & \text{otherwise} \end{cases}$$
>
> Note that $||T||||T^{-1}|| \geq ||TT^{-1}|| = 1$. If $X \cong Y$, then $d(X, Y) = 1$. Does the converse hold?

- An *isomorphic embedding* $X \to Y$ is a linear map $T \colon X \to Y$ such that $T \colon X \to TX = \operatorname{im} T$ is an isomorphism. If such T exists, we say that X *(isomorphically) embeds into* Y, and we write $X \hookrightarrow Y$.

Definition Equivalent norms

Let X be a normed space. Two norms $|| \cdot ||, || \cdot ||'$ are equivalent if

$$\begin{aligned} & \mathrm{Id} \colon (X, || \cdot ||) \to (X, || \cdot ||') \textit{ is an isomorphism} \\ \iff & || \cdot ||, || \cdot ||' \textit{ induce the same norm topology on } X \\ \iff & \exists\, a, b > 0 \; \forall x \in X a||x|| \leq ||x||' \leq b||x|| \\ \iff & \exists\, a, b > 0 \; aB'_X \subset B_X \subset bB'_X \end{aligned}$$

Remarks

- If $X \sim Y$, then X is complete iff Y is complete.
  If $||\cdot||$, $||\cdot||'$ are equivalent norms on a vector space X, then $(X, ||\cdot||)$ is complete iff $(X, ||\cdot||')$ is complete.

[3] We can actually replace 'bijection' with 'surjection'.

- Let X and Y be normed spaces. On $X \oplus Y$, the norm $||(x,y)||_1 = ||x|| + ||y||$ is equivalent to $||(x,y)||_p = (||x||^p + ||y||^p)^{1/p}$ for all $1 \leq p < \infty$ and to $||(x,y)||_\infty = \max\{||x||, ||y||\}$.
- $(C[0,1], ||\cdot||_\infty)$ is complete whereas $(C[0,1], ||\cdot||_1)$ is incomplete. Thus, we can use the first remark above to deduce that $||\cdot||_\infty \not\sim ||\cdot||_1$ (but this can easily be proven directly as well). However, $||f||_1 = \int_0^1 |f(t)|\, dt \leq ||f||_\infty$, so
$$\mathrm{Id}\colon (C[0,1], ||\cdot||_\infty) \to (C[0,1], ||\cdot||_1)$$
is a continuous linear bijection but its inverse is not continuous.
- On c_{00}, $||\cdot||_1 \not\sim ||\cdot||_2$. To see why, consider $x = (\underbrace{1, \cdots, 1}_{n}, 0, 0, \cdots)$ and note that $||x||_1 = n, ||x||_2 = \sqrt{n}$.

Finally, we discuss convergence and completeness. Let X, Y be normed spaces. In $\mathcal{B}(X,Y)$, convergence implies pointwise convergence, i.e., if $T_n \to T$ in $\mathcal{B}(X,Y)$, then, for all $x \in X$, $T_n x \to Tx$ in Y. To see why, note that, for fixed $x \in X$, we have $||T_n x - Tx|| \leq ||T_n - T||||x|| \to 0$. However, the converse is false in general, e.g., $T_n\colon \ell_1 \to \mathbb{R}, x \mapsto x_n$. We have $T_n \to 0$ pointwise, but $||T_n|| = 1$ for all $n \in \mathbb{N}$.

Theorem 1.7
Let X, Y be normed spaces. If Y is complete, then $\mathcal{B}(X,Y)$ is complete.

Proof. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X,Y)$. Fix $x \in X$. Then
$$||T_m x - T_n x|| \leq ||T_m - T_n||||x|| \to 0 \text{ as } m, n \to \infty$$
So $(T_n x)$ is Cauchy in Y and thus convergent. Now, define $T\colon X \to Y$ by $x \mapsto \lim_{n\to\infty} T_n x$. Observe that
- T is linear
$$T(\lambda x + \mu y) = \lim_{n\to\infty} T_n(\lambda x + \mu y) = \lim_{n\to\infty} [\lambda T_n x + \mu T_n y] = \lambda Tx + \mu Ty$$
- T is bounded
(T_n) is Cauchy implies (T_n) is bounded, i.e., there exists $M \geq 0$ such that $||T_n|| \leq M$ for all $n \in \mathbb{N}$. Fix $x \in X$. Then, for all $n \in \mathbb{N}$, we have $||T_n x|| \leq ||T_n||||x|| \leq M||x||$. Letting $n \to \infty$, we obtain $||Tx|| \leq M||x||$.
- $T_n \to T$ in $\mathcal{B}(X,Y)$
Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $||T_m - T_n|| \leq \varepsilon$ for all $m, n \geq N$. Fix $x \in X$. Note that, for all $m, n \geq N$, we have
$$||T_m x - T_n x|| \leq ||T_m - T_n||||x|| \leq \varepsilon||x||$$
Letting $n \to \infty$ with $m \geq N$ fixed yields $||T_m x - Tx|| \leq \varepsilon||x||$. Hence, $||T_m - T|| \leq \varepsilon$ for all $m \geq N$. ∎

2 Dual spaces

2.1 Basics

Let X be a normed space. A *functional* on X is a map $X \to \mathbb{R}$. The *dual space* X^* of X is the space of all bounded linear functionals on X, i.e., $X^* = \mathcal{B}(X, \mathbb{R})$ equipped with the operator norm. Since $\mathbb{R}$ is complete, Theorem 1.7 gives us the following result.

Theorem 2.1
For any normed space X, its dual X^* is a Banach space.

Notation. For $x \in X$ and $f \in X^*$, we let $\langle x, f \rangle = f(x)$.

Now, we know that $0 \in X^*$. Are there other elements?

Theorem 2.2 Hahn-Banach theorem
Let X be a normed space, $Y \subset X$ be a subspace and $g \in Y^*$. Then $f \in X^*$ such that $f|_Y = g$ and $||f|| = ||g||$.

Proof. See II Analysis of Functions. ■

Corollary 2.3
Let X be a normed space, $x_0 \in X \backslash \{0\}$. Then there exists $f \in S_{X^*} = \{f \in X^* \colon ||f|| = 1\}$ such that $f(x_0) = ||x_0||$.

Remarks
- For any $g \in B_{X^*}$, $|g(x_0)| \leq ||g||||x_0|| \leq ||x_0||$. Corollary 2.3 says that there exists $f \in B_{X^*}$ such that $f(x_0) = ||x_0||$, so
$$||x_0|| = \sup\{g(x_0) \colon g \in B_{X^*}\} = \max\{g(x_0) \colon g \in B_{X^*}\}.$$
We call f a *norming functional* at x_0.
- Given $x \neq y$ in X, we can set $x_0 = x - y$ and Corollary 2.3 implies that there exists $f \in X^*$ such that $f(x) \neq f(y)$. Thus, X^* separates the points of X.

Proof of Corollary 2.3. Set $Y = \text{span}\{x_0\}$ and define $g(\lambda x_0) = \lambda ||x_0||$. Then $g \in S_{Y^*}$ with $g(x_0) = ||x_0||$. Finally, apply Theorem 2.2. ■

2.2 Dual space of ℓ_p

Motivation: Recall that, for $1 \leq p < \infty$, we have $\ell_p = \overline{\text{span}}\{e_n \colon n \in \mathbb{N}\} = \overline{c_{00}}$. Given $\varphi \in \ell_p^*$ and $x = (x_n) \in \ell_p$,
$$\varphi(x) = \varphi\left(\lim_{n\to\infty} \sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{\infty} x_k \varphi(e_k)$$
so φ corresponds to the sequence $y = (\varphi(e_n))_{n\in\mathbb{N}}$. We may then ask: is $\ell_p^* \cong \ell_q$ for some q?

Fix $1 < p < \infty$, and let q be the conjugate index of p. Fix $y = (y_n) \in \ell_q$. Define
$$\begin{aligned} \varphi_y \colon \ell_p &\to \mathbb{R} \\ x &\mapsto \sum_{n=1}^{\infty} x_n y_n \end{aligned}$$

By Holder's inequality (Theorem 1.4), this is well-defined and $|\varphi_y(x)| \leq ||x||_p||y||_q$. So φ_y is linear and bounded: $||\varphi_y|| \leq ||y||_q$. Thus, $\varphi_y \in \ell_p^*$, which means that we have a map

$$\begin{aligned} \varphi\colon \ell_q &\to \ell_p^* \\ y &\mapsto \varphi_y \end{aligned}$$

Note that φ is linear and bounded with $||\varphi|| \leq 1$.

Theorem 2.4

Let p, q, φ be as above. Then φ is an isometric isomorphism $\ell_q \to \ell_p^*$.

Proof. It remains to check that φ is isometric and surjective:

- φ is isometric

  Fix $y \in \ell_q$. WLOG $y \neq 0$. Define

  $$x_n = \begin{cases} \dfrac{|y_n|^q}{y_n}, & y_n \neq 0 \\ 0, & y_n = 0 \end{cases}$$

  Observe that $\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^\infty |y_n|^{(q-1)p} = \sum_{n=1}^\infty |y_n|^q = ||y||_q^q < \infty$, so $x \in \ell_p$ with $||x||_p^p = ||y||_q^q$.

  Since $y \neq 0$, we have $x \neq 0$, so $x/||x||_p \in B_{\ell_p}$. Note that

  $$||\varphi_y|| \geq \varphi_y\left(\frac{x}{||x||_p}\right) = \frac{1}{||x||_p}\sum_{n=1}^\infty x_n y_n = \frac{||y||_q^q}{||y||_q^{q/p}} = ||y||_q.$$

  Hence, $||\varphi_y|| = ||y||_q$.

- φ is surjective

  Fix $f \in \ell_p^*$. Define $y_n = f(e_n), n \in \mathbb{N}$. Let $y = (y_n)$. For some fixed $N \in \mathbb{N}$, set

  $$x_n = \begin{cases} \dfrac{|y_n|^q}{y_n}, & y_n \neq 0, n \leq N \\ 0, & \text{otherwise} \end{cases}$$

  Then $x = (x_n) \in \ell_p$, so

  $$f(x) = \sum_{n=1}^N x_n f(e_n) = \sum_{n=1}^N x_n y_n = \sum_{n=1}^N |y_n|^q \leq ||f||||x||_p$$
  $$||x||_p = \left(\sum_{n=1}^N |x_n|^p\right)^{1/p} = \left(\sum_{n=1}^N |y_n|^{(q-1)p}\right)^{1/p} = \left(\sum_{n=1}^N |y_n|^q\right)^{1/p}$$

  Hence, $\sum_{n=1}^N |y_n|^q \leq ||f|| \left(\sum_{n=1}^N |y_n|^q\right)^{1/p}$, i.e.

  $$\left(\sum_{n=1}^N |y_n|^q\right)^{1/q} \leq ||f||$$

  Let $N \to \infty$ to deduce that $y \in \ell_q$. Finally, observe that

  $$\begin{aligned} & f(e_n) = y_n = \varphi_y(e_n)\ \forall\, n \in \mathbb{N} \\ \implies & f(x) = \varphi_y(x)\ \forall\, x \in \operatorname{span}\{e_n \colon n \in \mathbb{N}\} = c_{00} \qquad \text{by linearity} \end{aligned}$$

$$\implies f(x) = \varphi_y(x)\ \forall\, x \in \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\} = \ell_p \qquad \text{by continuity}$$

Thus, $f = \varphi_y$, so φ is surjective. ∎

Remarks

- We also have $\ell_1^* \cong \ell_\infty$ and $c_0^* \cong \ell_1$. The proof also shows that $\ell_1 \hookrightarrow \ell_\infty^*$ isometrically. However, the proof of surjectivity breaks down since $\overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\}$ in ℓ_∞ in $c_0 \subsetneq \ell_\infty$.
- From the proof, we can show Corollary 2.3 holds for ℓ_p.
- We've shown that $\ell_p, 1 \le p \le \infty$, is complete as they are dual spaces. For c_0, one simply has to show that c_0 is closed in ℓ_∞.

2.3 Bidual

Let X be a normed space. Then $X^{**} = (X^*)^* = \mathcal{B}(X^*, \mathbb{R})$ is the *bidual* or *second dual* of X.

For each $x \in X$, define the map

$$\begin{aligned} \hat{x} \colon X^* &\to \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

Note that $\hat{x}$ is linear and bounded: $|\hat{x}(f)| = |f(x)| \le ||f||||x||$. So $\hat{x} \in X^{**}$ with $||\hat{x}|| \le ||x||$. Thus, we have

$$\begin{aligned} \hat{\ } \colon X &\to X^{**} \\ x &\mapsto \hat{x} \end{aligned}$$

This is linear: $\widehat{\lambda x + \mu y}(f) = f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = (\lambda\hat{x} + \mu\hat{y})(f)$.

For $x \neq 0$, let $f \in X^*$ be a norming functional at x. Then

$$\hat{x}(f) = f(x) = ||x|| \implies ||\hat{x}|| = ||x||$$

so the canonical map $X \to X^{**}, x \mapsto \hat{x}$ is an isometric embedding into X^{**}. If f is surjective, we say that X is *reflexive*.

2.4 Dual operators

Let X, Y be normed spaces and $T \in \mathcal{B}(X, Y)$. The *dual operator* T^* of T is the map

$$\begin{aligned} T^* \colon Y^* &\to X^* \\ g &\mapsto g \circ T \end{aligned}$$

By Proposition 1.6, $T^*(g) = g \circ T \in X^*$ and $||T^*(g)|| \le ||g||||T||$, so T^* is well-defined. Moreover, it is clearly linear and bounded with $||T^*|| \le ||T||$.

Remark. Note that $\langle \cdot, \cdot \rangle \colon X \times X^* \to \mathbb{R}$ is bilinear. Moreover, for $x \in X$ and $g \in Y^*$, we have $\langle x, T^*(g)\rangle = \langle T(x), g\rangle$.

It turns out that $||T^*|| = ||T||$:

$$||T^*|| = \sup_{g \in B_{Y^*}} ||T^* g|| = \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^*(g)\rangle| = \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g\rangle| = \sup_{x \in B_X} ||Tx|| = ||T||,$$

where the penultimate equality follows from Corollary 2.3.

Example

Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider the right-shift map $R\colon \ell_p \to \ell_p$. What is $R^*\colon \ell_p^* \to \ell_p^*$? Recall that $\ell_p^* \cong \ell_q$. Thought of as a map $\ell_q \to \ell_q$, it turns out that $R^* = L$, the left-shift map.

Now, let's note some properties of dual operators:

(1) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$

(2) $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for all $S, T \in \mathcal{B}(X, Y)$ and all scalars λ, μ

Indeed, for $g \in Y^*$, $x \in X$,

$$\begin{aligned} \langle x, (\lambda S + \mu T)^* g\rangle &= \langle (\lambda S + \mu T)x, g\rangle \\ &= \langle \lambda Sx + \mu Tx, g\rangle \\ &= \lambda\langle Sx, g\rangle + \mu\langle Tx, g\rangle \\ &= \lambda\langle x, S^* g\rangle + \mu\langle x, T^* g\rangle \\ &= \langle x, (\lambda S^* + \mu T^*)g\rangle \end{aligned}$$

Since x is arbitrary, $(\lambda S + \mu T)^* g = (\lambda S^* + \mu T^*)g$ for all $g \in Y^*$, and we are done.

(3) $(ST)^* = T^* S^*$ for all $T \in \mathcal{B}(X, Y)$ and all $S \in \mathcal{B}(Y, Z)$

$$\langle x, (ST)^* g\rangle = \langle STx, g\rangle = \langle S(Tx), g\rangle = \langle Tx, S^* g\rangle = \langle x, T^* S^* g\rangle$$

(4) Let $T \in \mathcal{B}(X, Y)$. We have $T^* \in \mathcal{B}(Y^*, X^*)$ and $T^{**} \in \mathcal{B}(X^{**}, Y^{**})$. The diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\hat{}"'] & Y \arrow[d] \\ X^{**} \arrow[r, "T^{**}"'] & Y^{**} \end{tikzcd}$$

commutes, i.e., $\hat{Tx} = T^{**}\hat{x}$ for all $x \in X$. For $x \in X, g \in Y^*$,

$$\langle g, T^{**}\hat{x}\rangle = \langle T^* g, \hat{x}\rangle = \langle x, T^* g\rangle = \langle Tx, g\rangle = \langle g, \widehat{Tx}\rangle$$

Remark. Properties (1) and (3) imply that $X \sim Y \implies X^* \sim Y^*$.

3 Finite-dimensional normed spaces

Recall that norms $||\cdot||$ and $||\cdot||'$ on a vector space X are equivalent if $\mathrm{Id}\colon (X, ||\cdot||) \to (X, ||\cdot||')$ is an isomorphism or, equivalently, if $\exists\, a, b > 0\ \forall\, x \in X\ a||x|| \le ||x||' \le b||x||$.

Example
On $\mathbb{R}^n$, the norms $||\cdot||_1$ and $||\cdot||_2$ are equivalent. We've already seen that $||x||_2 \le ||x||_1$ for all $x \in \mathbb{R}^n$. Moreover, by Cauchy-Schwarz, we have

$$||x||_1 = \sum_{i=1}^n |x_i| \le \sqrt{n}\left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \sqrt{n}||x||_2.$$

Theorem 3.1
Any two norms on a finite-dimensional vector space are equivalent.

Proof. Let X be a f.d. vector space. Fix a basis $(e_1, \cdots, e_n)$ of X. For $x = \sum_{i=1}^n x_k e_k \in X$, define $||x||_1 = \sum_{k=1}^n |x_k|$. Let $||\cdot||$ be an arbitrary norm on X.

We show that $||\cdot||$ is equivalent to $||\cdot||_1$. For $x = \sum_{k=1}^m x_k e_k \in X$, we have

$$||x|| \le \sum_{k=1}^n |x_k|||e_k|| \le M||x||_1$$

where $M = \max_{1\le k\le n} ||e_k||$.

Now, let $S = \{x \in X\colon ||x||_1 = 1\}$, the unit sphere of $(X, ||\cdot||_1)$. We have the following result:

- **Claim.** S is compact.

  Proof. Let $(x^{(r)})_{r\in\mathbb{N}}$ be a sequence in S. Write $x^{(r)} = \sum_{k=1}^n x_k^{(r)} e_k$. For each $1 \le k \le n$, $|x_k^{(r)}| \le ||x^{(r)}|| = 1$ for all $r \in \mathbb{N}$. By repeated application of Bolzano-Weierstrass, there exists $r_1 < r_2 < r_3 < \cdots$ in $\mathbb{N}$ such that $(x_k^{(r_\ell)})_{\ell\in\mathbb{N}}$ is convergent for each $1 \le k \le n$. Let $x_k = \lim_{\ell\to\infty} x_k^{r_\ell}$ and $x = \sum_{k=1}^n x_k e_k$. Note that

  $$||x||_1 = \sum_{k=1}^n |x_k| = \lim_{\ell\to\infty} \sum_{k=1}^n |x_k^{(r_\ell)}| = 1$$

  so $x \in S$. Moreover,

  $$||x^{(r_\ell)} - x||_1 = \sum_{k=1}^n |x_k^{(r_\ell)} - x_k| \to 0 \quad \text{as } \ell \to \infty$$

  so $x^{(r_\ell)} \to x$ in S. Thus, S is sequentially compact and hence compact. ■

For any $x, y \in S$, $\big|||x|| - ||y||\big| \le ||x - y|| \le M||x - y||_1$. So $||\cdot||$ is continuous on S with respect to $||\cdot||_1$. So $c = \inf\{||x||\colon x \in S\}$ is achieved: $\exists\, x \in S\ ||x|| = c$. Since $0 \notin S$ and $c > 0$, we have $||y|| \ge c = c||y||_1$ for all $y \in S$. By homogeneity, $||y|| \ge c||y||_1$ for all $y \in X$. ■

Corollary 3.2
Let $T\colon X \to Y$ be a linear map between two normed spaces. If X is f.d., then T is continuous.

Proof. Let $||\cdot||$ denote the norm on X and Y. Define $||x||' = ||Tx|| + ||x||$ for all $x \in X$. It is easy to check that this is a norm on X. By Theorem 3.1, there exists $b > 0$ such that, for all $x \in X$, $||x||' \le b||x||$. In particular, $||Tx|| \le b||x||$ for all $x \in X$. ■

Corollary 3.3
If $\dim X = \dim Y < \infty$*, then* $X \sim Y$*.*

Proof. We have a linear bijection $T\colon X \to Y$. By Corollary 3.2, T and T^{-1} are continuous. ∎

Remark. Corollary 3.3 does *not* imply that the theory of f.d. normed spaces is uninteresting.

Recall that, for X a metric space and $Y \subset X$, we have
- Y complete $\implies Y$ is closed in X
- Y closed in X and X complete $\implies Y$ complete

Corollary 3.4
(i) *A f.d. normed space* X *is complete.*
(ii) *A f.d. subspace* X *of a normed space* Y *is closed in* Y*.*

Proof. (i) Let $n = \dim X$. By Corollary 3.3, $X \sim \ell_2^n$ which is complete. (ii) follows from above properties of metric spaces. ∎

Corollary 3.5
Let X *be a f.d. normed space and* $A \subset X$*. Then*

$$A \text{ is compact} \iff A \text{ is closed and bounded}$$

Proof. If $X = \ell_2^n$, then this is simply Heine-Borel. For general X, the result follows by invoking Corollary 3.3 to deduce that $X \sim \ell_2^n$ and noting isomorphisms map compact subsets to compact subsets (ditto for closed and bounded subsets). ∎

In particular, $B_X = \{x \in X \colon ||x|| = 1\}$ is compact. How about if $\dim X = \infty$? Note that, in ℓ_p, $1 \le p < \infty$, $||e_n||_p = 1$ for all n and $||e_m - e_n|| = 2^{1/p}$ for all $m \neq n$, so (e_n) has no convergent subsequence. Hence, B_{ℓ_p} is not compact.

A similar obstruction does, in fact, hold for any infinite-dimensional normed space. To show this, we need the following lemma:

Proposition 3.6 Riesz's lemma
Let Y *be a proper, closed subspace of a normed space* X*. Then*

$$\forall \varepsilon > 0 \ \exists\, x \in B_X \ d(x, Y) = \inf\{||x - y|| \colon y \in Y\} > 1 - \varepsilon.$$

Proof. WLOG, $0 < \varepsilon < 1$. Fix $z \in X \backslash Y$. Since Y is closed, $d = d(x, Y) > 0$. Pick $y \in Y$ such that $d \le ||z - y|| < d/(1-\varepsilon)$. Set $x = \frac{(z-y)}{||z-y||}$. Note that $d(x, Y) > 1 - \varepsilon$: for $y' \in Y$,

$$||x - y'|| = \left|\left|\frac{z - y - ||z-y||y'}{||z-y||}\right|\right| \ge \frac{d}{||z-y||} > 1 - \varepsilon$$

so $d(x, Y) \ge d/||z-y|| > 1 - \varepsilon$. ∎

Theorem 3.7
Let X *be a normed space. Then* B_X *is compact if and only if* $\dim X < \infty$*.*

Proof. ($\Longleftarrow$) Corollary 3.5
($\Longrightarrow$) Similar to the ℓ_p case, we construct (x_n) in B_X such that $||x_m - x_n|| > 1/2$ for all $m \neq n$. As before, we then deduce that (x_n) has no convergent subsequence and so B_X is not compact.

Pick any $x_1 \in B_X$. Suppose we have already picked $x_1, \cdots, x_n$ for some $n \in \mathbb{N}$. We then set $Y = \operatorname{span}\{x_1, \cdots, x_n\}$. Then Y is a proper ($\dim X = \infty$) and closed (Corollary 3.4) subspace of X. By Proposition 3.6, we can then pick $x_{n+1} \in B_X$ such that $d(x_{n+1}, Y) > 1/2$. In particular, $||x_{n+1} - x_m|| > 1/2$ for $1 \leq m \leq n$. ■

4 The Baire category theorem and its applications

Let (X, d) be a metric space. In this course, we will denote closed and open balls as

$$B_r(x) = \{y \in X : d(x, y) \le r\}$$
$$D_r(x) = \{y \in X : d(x, y) < r\}$$

Recall that, for $A \subset X$, the *closure of A in X* is

$$\begin{aligned}\overline{A} &:= \bigcap_{\substack{F \text{ closed in } X \\ A \subset F}} F \\ &= \{x \in X : \forall\, r > 0\ D_r(x) \cap A \neq \varnothing\} \\ &= \{x \in X : \exists\, (a_n) \text{ in } A \text{ s.t. } a_n \to x\}\end{aligned}$$

Note that $\overline{D_r(x)} \subset B_r(x)$. In general, this inclusion can be strict. But normed spaces are nice:

Exercise. Show that, in a normed space, $\overline{D_r(x)} = B_r(x)$.

Recall also that, for $A \subset X$, we say that *A is dense in X* if

$$\begin{aligned}&\overline{A} = X \\ \iff\ &\forall\, x \in X\ \forall\, r > 0\ D_r(x) \cap A \neq \varnothing \\ \iff\ &\forall \text{ non-empty open } U \subset X\ U \cap A \neq \varnothing\end{aligned}$$

Example
$\mathbb{Q}$ is dense in $\mathbb{R}$ and so is $\sqrt{2} + \mathbb{Q}$. But $\mathbb{Q} \cap (\sqrt{2} + \mathbb{Q}) = \varnothing$.

4.1 Baire category theorem

Theorem 4.1 Baire category theorem
Let (X, d) be a complete metric space and $U_n \subset X$ be open and dense in X for each $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X.

Proof. Fix $x_0 \in X$ and $r_0 > 0$. Since U_1 is dense, $U_1 \cap D_{r_0}(x_0) \neq \varnothing$. Then we can pick $x_1 \in U_1 \cap D_{r_0}(x_0)$. Since $U_1 \cap D_{r_0}(x_0)$ is open, there exists $r_1 > 0$ such that $B_{r_1}(x_1) \subset U_1 \cap D_{r_0}(x_0)$. WLOG, we can pick $r_1 < 1$. We then continue inductively. At the n^{th} stage, density of U_n implies that $U_n \cap D_{r_{n-1}}(x_{n-1}) \neq \varnothing$, so we can pick $x_n \in U_n \cap D_{r_{n-1}}(x_{n-1})$. Since $U_n \cap D_{r_{n-1}}(x_{n-1})$ is open, there exists $r_n > 0$ such that $B_{r_n}(x_n) \subset U_n \cap D_{r_{n-1}}(x_{n-1})$. WLOG, $r_n < 1/n$.

We end up with $(x_n)_{n=0}^{\infty}$ in X and $(r_n)_{n=0}^{\infty}$ with $0 < r_n < 1/n$ for all $n \in \mathbb{N}$ and, for all $n > N \ge 0$,

$$\begin{aligned}B_{r_n}(x_n) &\subset U_n \cap D_{r_{n-1}}(x_{n-1}) \\ &\subset U_n \cap U_{n-1} \cap D_{r_{n-2}}(x_{n-2}) \\ &\ \ \vdots \\ &\subset U_n \cap U_{n-1} \cap \cdots \cap U_{N+1} \cap D_{r_N}(x_N)\end{aligned}$$

so, for all $m, n \ge N$, we have $d(x_m, x_n) \le 2r_N < 2/N$. Thus, $(x_n)_{n=0}^{\infty}$ is Cauchy and thus convergent in X. Write $x = \lim_{n\to\infty} x_n$. Note that, for $n \ge m$, $x_n \in B_{r_m}(x_m)$ so $x \in B_{r_m}(x_m)$.

By fixing $N = 0$ above and taking $n \to \infty$, we get

$$x \in \left(\bigcap_{n \in \mathbb{N}} U_n \right) \cap D_{r_0}(x_0)$$

as required. ■

Remark. A countable intersection of open sets is called a G_δ*-set*. Theorem 4.1 then says that a countable intersection of open dense sets in a complete metric space is a dense G_δ-set.

Application *Uncountability of* $\mathbb{R}$
Suppose, on the contrary, that $\mathbb{R}$ is countable, so we can write $\mathbb{R} = \{r_1, r_2, r_3, \cdots\}$. Let $U_n = \mathbb{R} \backslash \{r_n\}$. Then U_n is open and dense in $\mathbb{R}$. Since $\mathbb{R}$ is complete, Theorem 4.1 tells us that $\bigcap_{n \in \mathbb{N}} U_n$ is dense in $\mathbb{R}$. But $\bigcap_{n \in \mathbb{N}} U_n = \varnothing$ — a contradiction!

Observe that, if $U \subset X$ is open and dense in X, then $F = X \backslash U$ is closed in X and $\int F = \varnothing$.

Definition Nowhere dense
Let (X, d) be a topological space. We say that $A \subset X$ is nowhere dense in X if $\operatorname{int} \overline{A} = \varnothing$.

Remarks
- For $A \subset Y \subset X$, it is possible that A is nowhere dense in X but not in Y (e.g. take $A = Y \neq \varnothing$)
- A is nowhere dense in X if and only if $U \not\subset \overline{U \cap A}$ for any nonempty open $U \subset X$.
  A is dense in X if and only if $U \subset \overline{U \cap A}$ for every open $U \subset X$.

Example
- In $\mathbb{R}$, any finite set and the Cantor set are nowhere dense.
- Write $\mathbb{Q} = \{q_1, q_2, q_3, \cdots\}$ and let $(\delta_n)_{n \in \mathbb{N}}$ in $(0, 1)$. Then $U = \bigcup_{n \in \mathbb{N}} (q_n - \delta_n, q_n + \delta_n)$ is open and dense in $\mathbb{R}$. So $\mathbb{R} \backslash U$ is closed and nowhere dense in $\mathbb{R}$.

Theorem 4.1'
Let (X, d) be a non-empty complete metric space. Suppose $X = \bigcup_{n \in \mathbb{N}} A_n$ for some $A_n \subset X$. Then there exists $N \in \mathbb{N}$ such that $\operatorname{int} \overline{A_n} \neq \varnothing$.

Proof. Suppose, on the contrary, that $\operatorname{int} \overline{A_n} = \varnothing$ for all $n \in \mathbb{N}$. Then $\forall\, x \in X\ \forall\, r > 0\ D_r(x) \not\subset \overline{A_n}$ and thus $D_r(x) \cap U_n = \varnothing$. Thus, $U_n = X \backslash \overline{A_n}$ is open and dense in X. Hence, by Theorem 4.1, $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X. But note that $\bigcap_{n \in \mathbb{N}} U_n = \left(\bigcup_{n \in \mathbb{N}} \overline{A_n} \right)^c = \varnothing$ — a contradiction! ■

Exercise. Deduce Theorem 4.1 from Theorem 4.1'.

Definition First and second category
Let X be a topological space and $A \subset X$.
- *We say that A is meagre in X or is of first category in X if $A = \bigcup_{n \in \mathbb{N}} A_n$ where A_n is nowhere dense in X for all $n \in \mathbb{N}$.*
- *We say that A is of second category in X if A is not of first category.*

Remarks
- Intuition: Think of meagre sets as 'small'.
- Typical Baire argument: Theorem 4.1' is useful as, to find some element $x \in X$ (in a non-empty complete metric space) with some property P, we just have to show that $A = \{x \in X : x \text{ fails } P\}$ is meagre.

Application *Existence of a nowhere differentiable function in $C[0,1]$*
Note that $(C[0,1], ||\cdot||_\infty)$ is a nonempty complete metric space. Let

$$A = \{f \in C[0,1] \colon \exists\, x \in [0,1] \text{ s.t. } f \text{ differentiable at } x\}$$

Observe that, if $f'(x)$ exists, i.e. $[f(y) - f(x)]/(y-x) \to f'(x)$ as $y \to x$, then there exists $N \in \mathbb{N}$ such that, for all $y \in X$,

$$|y - x| < \frac{1}{N} \Longrightarrow \left|\frac{f(y) - f(x)}{y - x}\right| \leq N$$

Thus, for $n \in \mathbb{N}$, consider the set

$$A_n = \left\{f \in C[0,1] \colon \exists\, x \in [0,1]\ \forall\, y \in [0,1]\ |y - x| < \frac{1}{n} \Longrightarrow |f(y) - f(x)| \leq n|y - x|\right\}$$

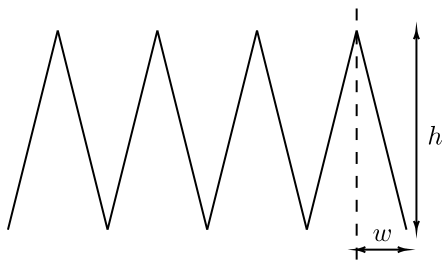
and note that $A \subset \bigcup_{n\in\mathbb{N}} A_n$.

It then remains to show that, for all $n \in \mathbb{N}$, A_n is closed and $\operatorname{int} A_n = \varnothing$.

- A_n is closed: Consider $(f_k)_{k\in\mathbb{N}}$ in A_n with $f_k \to f$ in $C[0,1]$. For each $k \in \mathbb{N}$, we can pick $x_k \in [0,1]$ such that, for all $y \in [0,1]$, $|y - x_k| < 1/n \Longrightarrow |f_k(y) - f_k(x_k)| \leq n|y - x_k|$. Passing to a subsequence if necessary, $x_k \to x$ in $[0,1]$ WLOG. By IB Analysis and Topology Example Sheet 1 Q5 (2024), $f_k(x_k) \to f(x)$ and hence

  $$\forall\, y \in [0,1]\ |y - x| < \frac{1}{n} \Longrightarrow |f(y) - f(x)| \leq n|y - x|$$

  as required.
- Fix $f \in A_n$ and $r > 0$. To get $D_r(f) \not\subset A_n$, the idea is to consider a small but rapidly oscillating perturbation of f. Let $0 < \varepsilon < r/4$. Pick $\delta > 0$ such that $|y - x| < \delta \Longrightarrow |f(y) - f(x)| < \varepsilon$. Choose h, w such that $4\varepsilon < h < r$ and $w = \min\{\varepsilon/n, \delta\}$. Set g to be the function

  We can check that $f + g \in D_r(f) \backslash A_n$.

> *Direct proof:* Take g_n similar to above with height h_n and width w_n, where $h_n \searrow 0$ fast and $h_n/w_n \to \infty$ fast. Then $\sum g_n$ is nowhere differentiable.

4.2 Consequences for Banach spaces

Theorem 4.2 Principle of uniform boundedness[4]
Let X be a Banach space, Y a normed space and $\mathcal{T} \subset \mathcal{B}(X, Y)$. If T is pointwise bounded (i.e., $\forall\, x \in X\ \sup_{T\in\mathcal{T}} ||Tx|| < \infty$), then T is uniformly bounded (i.e., $\sup_{T\in\mathcal{T}} ||T|| < \infty$).

Proof. Let $A_n = \{x \in X \colon \sup_{T\in\mathcal{T}} ||Tx|| \leq n\}$. By hypothesis, $X = \bigcup_{n\in\mathbb{N}} A_n$. By Theorem 4.1', there exists $n \in \mathbb{N}$ such that $\operatorname{int} \overline{A_n} \neq \varnothing$. Note that $A_n = \bigcap_{T\in\mathcal{T}}\{x \in X \colon ||Tx|| \leq n\}$ is closed as

[4]This result is also known as the *Banach-Steinhaus theorem.*

the map $x \mapsto ||Tx||$ is continuous. Thus, there exists $r > 0$ and $x \in A_n$ such that $B_r(x) \subset A_n$. Given $y \in B_X$, $T \in \mathcal{T}$, we have $x + ry, x - ry \in B_r(x)$ and thus

$$||Ty|| = \left|\left|\frac{T(x+ry) - T(x-ry)}{2r}\right|\right| \leq \frac{2n}{2r} = \frac{n}{r}$$

Hence, $||T|| \leq n/r$ for all $T \in \mathcal{T}$. ∎

Corollary 4.3
Let X be a Banach space, Y a normed space, and $(T_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{B}(X,Y)$ that pointwise converges to T. Then T is linear and bounded. Moreover, $\sup_n ||T_n|| < \infty$.

Proof. For all $x \in X$, $(T_n x)_{n\in\mathbb{N}}$ is convergent and thus bounded. So $\{T_n : n \in \mathbb{N}\}$ is pointwise bounded. Hence, by Theorem 4.2, there exists $M \geq 0$ such that, for all $n \in \mathbb{N}$, we have $||T_n|| \leq M$.

- T linear: $T(\lambda x + \mu y) = \lim_{n\to\infty} T_n(\lambda x + \mu y) = \lim_{n\to\infty}[\lambda T_n(x) + \mu T_n(y)] = \lambda T(x) + \mu T(y)$
- T bounded: $\forall\, x \in B_X\ \forall\, n \in \mathbb{N}\ ||T_n x|| \leq ||T_n|| \leq M$, so $||Tx|| = \lim_{n\to\infty} ||T_n x|| \leq M$ for all $x \in B_X$. Hence, T is bounded with $||T|| \leq M$. ∎

Exercise. Show that $||T|| \leq \liminf_{n\to\infty} ||T_n||$.

Definition δ-dense
Let A, B be subsets of a metric space (X,d) and $\delta > 0$. We say that A is δ-dense in B if $\forall\, b \in B\ \exists\, a \in A\ d(a,b) \leq \delta$.

Remark. If $\overline{A} \supset B$, then A is δ-dense in B for all $\delta > 0$.

Lemma 4.4 Open mapping lemma
Let X be a Banach space, Y a normed space, $T \in \mathcal{B}(X,Y)$. Suppose that $T(MB_X)$ is δ-dense in B_Y for some $M \geq 0$ and $0 \leq \delta < 1$. Then $T(\frac{M}{1-\delta}B_X) \supset B_Y$.

Remarks

- Another way to think of the open mapping lemma is as follows.
  Condition: for all $y \in B_Y$, $y = Tx$ has a δ-approximate solution in MB_X
  Conclusion: for all $y \in B_Y$, $y = Tx$ has an exact solution in $\frac{M}{1-\delta}B_X$
- For any $M \geq 0$, $T(MB_X)$ is 1-dense in B_Y since $0 = T(0) \in T(MB_X)$ and $\forall\, y \in Y\ ||y - T(0)|| = ||y|| \leq 1$
- Lemma 4.4 implies that T is surjective
- Lemma 4.4 shows that $\overline{T(B_X)} \supset B_Y \implies T(D_X) \supset D_Y$.

Proof of Lemma 4.4. The strategy is to use 'successive approximations'. Fix $y \in B_Y$. Pick $x_1 \in MB_X$ such that $||y - Tx_1|| \leq \delta$. Note that $\frac{y-Tx_1}{\delta} \in B_X$, so we can pick $x_2 \in MB_X$ such that $||\frac{y-Tx_1}{\delta} - Tx_2|| \leq \delta$. Note that $\frac{y-T(x_1+\delta x_2)}{\delta^2} \in B_Y$, so we can pick $x_3 \in MB_X$ such that $||\frac{y-T(x_1+\delta x_2)}{\delta^2} - Tx_3|| \leq \delta$. Continuing inductively, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in MB_X such that

$$\left|\left| y - T\left(\sum_{k=1}^{n} \delta^{k-1} x_k\right)\right|\right| \leq \delta^n \qquad \forall\, n \in \mathbb{N}$$

Note that $||\delta^{n-1}x_n|| \leq M\delta^{n-1}$ for all $n \in \mathbb{N}$, so the series $\sum_{n=1}^{\infty} \delta^{n-1}x_n$ converges absolutely and hence converges since X is complete (cf. Example Sheet 1 Q8). Let $x = \sum_{n=1}^{\infty} \delta^{n-1}x_n$. Then $||x|| \leq \sum_{n=1}^{\infty} \delta^{n-1}M = \frac{M}{1-\delta}$, so $x \in \frac{M}{1-\delta}B_X$. Since T is continuous, $Tx = \sum_{n=1}^{\infty} T(\delta^{n-1}x_n) = \lim_{n\to\infty} T(\sum_{k=1}^{n} \delta^{k-1}x_k) = y$. ∎

Theorem 4.5 Open mapping theorem
Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is surjective, then T is open.

Remark. In particular, we have $T(B_X) \supset T(D_X) \supset rB_Y$ for some $r > 0$. Equivalently, $T(MB_X) \supset B_Y$ for some $M > 0$. So the conclusion of Theorem 4.5 is that, for all $y \in Y$, $y = Tx$ has a solution such that $||x|| \leq M||y||$.

Proof of Theorem 4.5. Observe that it suffices to show that $T(MB_X) \supset B_Y$ for some $M > 0$. Indeed, it would mean that, given open $U \subset X$ and $y \in T(U)$, we can pick $x \in U$ such that $y = Tx$. As U is open, $B_r(x) \subset U$ for some $r > 0$. Then $T(U) \supset T(B_r(x)) = T(x + rB_X) = T(x + \frac{r}{M}MB_X) \supset y + \frac{r}{M}B_Y = B_{r/M}(y)$.

Note that $Y = T(X) = T(\bigcup_{n\in\mathbb{N}} nB_X) = \bigcup_{n\in\mathbb{N}} T(nB_X)$. As Y is non-empty and complete, Theorem 4.1 implies that there exists $n \in \mathbb{N}$ such that $\operatorname{int}\overline{T(nB_X)} \neq \varnothing$. So there exists $y \in Y$ and $r > 0$ such that $B_r(y) \subset \overline{T(nB_X)}$. Moreover, since B_X is convex and symmetric, so is $\overline{T(nB_X)}$. Thus, given $z \in B_Y$, we have $y \pm rz \in B_r(y) \subset \overline{T(nB_X)}$ and also $-y \pm rz \in \overline{T(nB_X)}$. Now, note that $rz = \frac{1}{2}(y+rz)+\frac{1}{2}(-y+rz) \in \overline{T(nB_X)}$. Thus, $rB_Y \subset \overline{T(nB_X)}$ or $B_Y \subset \overline{T(\frac{n}{r}B_X)}$. So $T(\frac{n}{r}B_X)$ is $\frac{1}{2}$-dense in B_Y. By Lemma 4.4, $T(\frac{2n}{r}B_X) \supset B_Y$. ■

Theorem 4.6 Inversion theorem
Let X, Y be Banach spaces and $T\colon X \to Y$ a continuous linear bijection. Then $T^{-1}\colon Y \to X$ is also continuous.

Proof 1. T is surjective so it is open by Theorem 4.5. So for open $U \subset X$, $(T^{-1})^{-1}(U) = T(U)$ is open in Y. So T^{-1} is continuous. ■

Proof 2. By Theorem 4.5, we have $T(MB_X) \supset B_Y$ for some $M > 0$. Given $y \in Y$, there exists $x \in X$ such that $y = Tx$ and $||x|| \leq M||y||$, i.e., $||T^{-1}y|| \leq M||y||$. So T^{-1} is bounded. ■

Corollary 4.7
Let $||\cdot||$ and $||\cdot||'$ be complete norms on a vector space X. If there exists $b > 0$ such that $||x||' \leq b||x||$ for all $x \in X$, then $||\cdot||$ and $||\cdot||'$ are equivalent.

Proof. $\mathrm{Id}\colon (X, ||\cdot||) \to (X, ||\cdot||')$ is a linear bijection and bounded by hypothesis. ■

Remark. This gives us another proof that the L_1-norm on $C[0,1]$ is complete.

Recall that, for a function $f\colon X \to Y$ between sets, the *graph of f* is the set

$$\Gamma(f) = \{(x, y) \in X \times Y : y = f(x)\}.$$

If X, Y are topological spaces with Y Hausdorff and f continuous, then $\Gamma(f)$ is closed.[5]

Theorem 4.8 Closed graph theorem
Let X, Y be Banach spaces and $T\colon X \to Y$ a linear map. If $\Gamma(T)$ is closed, then T is continuous.

Remark. Note that continuity of T means:

$$x_n \to x \text{ in } X \implies (Tx_n) \text{ converges in } Y \text{ and } \lim_{n\to\infty} Tx_n = Tx$$

But by Theorem 4.8, to prove continuity of T, it suffices to show that

$$x_n \to x \text{ in } X \text{ and } Tx_n \to y \text{ in } Y \implies y = Tx$$

[5] You may recognise this from Tripos 2025 Paper 2 Section II Question 10G

Proof of Theorem 4.8. Consider the map $S\colon X \to \Gamma(T), x \mapsto (x, Tx)$. This is a linear bijection with $S^{-1}\colon \Gamma(T) \to X, (x,y) \mapsto x$. So $S^{-1} = P_X|_{\Gamma(T)}$ is continuous. Since $\Gamma(T)$ is a closed subspace of the Banach space $X \oplus Y$, $\Gamma(T)$ is complete. By Theorem 4.6, $S = (S^{-1})^{-1}$ is continuous. Hence, $T = P_Y \circ S$ is continuous. ■

Remark. Note that the condition '$x_n \to x$ in X and $Tx_n \to y$ in $Y \implies y = Tx$' is equivalent to '$x_n \to 0$ in X and $Tx_n \to y$ in $Y \implies y = 0$'. The latter version will be quite useful for applications.

Exercise. Deduce Theorem 4.6 from Theorem 4.8.

4.3 Applications

Application
Let X be a closed subspace of ℓ_2. We further assume that $X \subset \ell_1$.

- **Claim.** There exists $C > 0$ such that, for all $x \in X$, $||x||_1 \le C||x||_2$.

  Remark. We know that $\ell_1 \subset \ell_2$ with $||x||_2 \le ||x||_1$ for all $x \in \ell_1$. We might want to use Corollary 4.7, but we don't know if $||\cdot||_1$ is complete on X. Similarly, $\mathrm{Id}\colon (X, ||\cdot||_1) \to (X, ||\cdot||_2)$ is a continuous linear bijection. However, we cannot use Theorem 4.6 to deduce that Id^{-1} is continuous since we don't know if $(X, ||\cdot||_1)$ is complete.

  Proof of Claim. Consider $T\colon (X, ||\cdot||_1) \to \ell_1, x \mapsto x$. Since X is closed in ℓ_2, $(X, ||\cdot||_2)$ is complete and so is ℓ_1. Moreover, T is linear, so by Theorem 4.8, it suffices to show that $\Gamma(T)$ is closed. To do this, suppose $x^n \to 0$ in $(X, ||\cdot||_2)$ and $Tx^n = x^n \to y \in \ell_2$. Write $x^n = (x^n_k)_{k\in\mathbb{N}}$ and $y = (y_k)_{k\in\mathbb{N}}$. For $k \in \mathbb{N}$, $|x^n_k| \le ||x^n||_2 \to 0$ and $|x^n_k - y_k| \le ||x^n - y||_1 \to 0$ as $n \to \infty$. So $y = 0$ which means that $\Gamma(T)$ is closed. ■

Application
Let X be a normed space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X such that $\forall f \in X^*$ $\sum_{n\in\mathbb{N}} |f(x_n)| < \infty$.

- **Claim.** There exists $C \ge 0$ such that, for all $f \in X^*$, $\sum_{n\in\mathbb{N}} |f(x_n)| \le C||f||$.

  Proof of Claim. Define $T\colon X^* \to \ell_1, f \mapsto f(x_n)$. This is well-defined by hypothesis and also linear. Moreover, X^* and ℓ_1 are complete, so we can again use Theorem 4.8. Suppose $f_n \to 0$ in X^* and $Tf_n \to y = (y_k)_{k\in\mathbb{N}}$ in ℓ_1. As before, for $k \in \mathbb{N}$, $f_n(x^k) \to y(x^k)$ as $n \to \infty$. But $|f_n(x_k)| \le ||f_n|| ||x^k|| \to 0$ as $n \to \infty$. Hence, $y = 0$ as required. ■

Exercise. Prove the above claims using Theorem 4.2 and Corollary 4.3.

Application

- **Claim.** An infinite-dimensional Banach space X has no countable (algebraic) basis.

  Proof. Suppose $B = \{e_1, e_2, e_3, \cdots\}$ is a countable basis of X. Note that B is countably infinite since $\dim X = \infty$. For each $n \in \mathbb{N}$, let $F_n = \operatorname{span}\{e_1, e_2, \cdots, e_n\}$. Since $\dim F_n < \infty$, we know from Corollary 3.2 that F_n is closed in X. Let $x \in F_n, r > 0$. Then
  $$x + \frac{r}{||e_{n+1}|| + 1} e_{n+1} \in D_r(x) \backslash F_n$$
  So $D_r(x) \not\subset F_n$ and thus $\operatorname{int} F_n \neq \varnothing$, i.e., F_n is nowhere dense. Since $X = \operatorname{span} B$, we have $X = \bigcup_{n\in\mathbb{N}} F_n$ — a contradiction to Theorem 4.1'. ■

Application

Let X be a vector space. Suppose Y and Z are subspaces of X such that $X = Y + Z$ and $Y \cap Z = \{0\}$, i.e., X is an algebraic direct sum of Y and Z. Then $T\colon Y \times Z \to X, (y, z) \mapsto y + z$ is a linear bijection. Note that $P\colon X \to Y, x = y + z \mapsto y$ is linear with $\operatorname{im} P = Y$ and $\ker P = Z$. This is called the *projection of X onto Y along Z*. On the other hand, $I - P$ is the projection of X onto Z along Y.

Now, suppose X is a normed space. Recall that $Y \times Z$ is a normed space when equipped with the norm $||(y, z)|| = ||y|| + ||z||$. We denote this normed space by $Y \oplus Z$. The norm topology on $Y \oplus Z$ is the product topology. Note that $T\colon Y \oplus Z \to X$ is a continuous linear bijection:

$$||T(y, z)|| = ||y + z|| \le ||y|| + ||z|| = ||(y, z)||$$

If T^{-1} is also continuous (i.e., $||T^{-1}(y + z)|| = ||(y, z)|| = ||y|| + ||z|| \le C||y + z||$ for some $C > 0$), then $X \sim Y \oplus Z$ and we write $X = Y \oplus Z$ and say that X is the *(topological) direct sum of Y and Z*.

- **Claim.** Suppose X is a Banach space, with subspaces Y and Z such that $X = Y + Z$ and $Y \cap Z = \{0\}$. The following are equivalent:
  (i) Y and Z are closed subspaces
  (ii) $X = Y \oplus Z$
  (iii) P is continuous

  Proof. (i) $\Longrightarrow$ (ii): Y and Z are Banach spaces and hence so is $Y \oplus Z$. Since $T\colon Y \oplus Z \to X$ is a continuous linear bijection, T^{-1} is continuous by Theorem 4.6.
  (ii) $\Longrightarrow$ (iii): $||P(y + z)|| = ||y|| \le ||y|| + ||z|| \le C||y + z||$
  (iii) $\Longrightarrow$ (i): $Z = \ker P = P^{-1}(\{0\})$ and $Y = \ker(I - P) = (I - P)^{-1}(\{0\})$ ■

Exercise. Show that (i) $\Longrightarrow$ (iii) directly using the closed graph theorem.

5 $C(K)$ spaces

Let K be a set, Let $\ell_\infty(K) := \{f : K \to \mathbb{R} \mid f \text{ bdd}\}$. This is a Banach space in the sup-norm $||f||_\infty = \sup_{x \in K} |f(x)|$.

Example. For $K = \mathbb{N}$, $\ell_\infty(K) = \ell_\infty$.

Let K be a topological space. Let $C_b(K) := \{f \in \ell_\infty(K) \mid f \text{ cts}\}$. By the uniform limit theorem, this is a closed subspace of $\ell_\infty(K)$ and hence a Banach space. In the case that K is a *compact* topological space, we also consider $C(K) = \{f : K \to \mathbb{R} \mid f \text{ cts}\} = C_b(K)$.

Remark. We can replace $\mathbb{R}$ by $\mathbb{C}$ in all of the above.

Interlude: Completions

Let (M, d) be a metric space. A *completion* of (M, d) is a complete metric space $(\tilde{M}, \tilde{d})$ with an isometric map $j : M \to \tilde{M}$ such that $j(M)$ is dense in $\tilde{M}$.

Theorem 5.1
Every metric space (M, d) has a completion.

Proof. WLOG, $M \neq \varnothing$ (result is trivial in that case). For $x \in M$, define $f_x : M \to \mathbb{R}, y \mapsto d(x, y)$. Fix $x_0 \in M$. Define $j : M \to \ell_\infty(M), x \mapsto f_x - f_{x_0}$. Then, for every $y \in M$,

$$|f_x(y) - f_{x_0}(y)| = |d(x, y) - d(x_0, y)| \leq d(x, x_0)$$

so $j(x) \in \ell_\infty(M)$ for all $x \in M$. Moreover, for any $x, z \in M$, we have

$$||j(x) - j(z)||_\infty = \sup_{y \in M} |d(x, y) - d(z, y)| \leq d(x, z)$$

with equality attained when $y = z$, so j is an isometric map. Let $\tilde{M} = \overline{j(M)}$ with $\tilde{d}(f, g) = ||f - g||_\infty$. Since $\tilde{M}$ is a closed subset of the complete space $\ell_\infty(M)$, it is complete. ■

Remark. Completions are unique. Suppose $(\tilde{M}_1, \tilde{d}_1)$ and $(\tilde{M}_2, \tilde{d}_2)$ are completions of (M, d) with isometric maps $j_1 : M \to \tilde{M}_1$ and $j_2 : M \to \tilde{M}_2$. Then there exists an isometry $\theta : \tilde{M}_1 \to \tilde{M}_2$ such that $\theta \circ j_1 = j_2$. Indeed, given $x \in \tilde{M}_1$, pick a sequence $(x_n)_{n \in \mathbb{N}}$ in M such that $j_1(x_n) \to x$ and then define $\theta(x) = \lim_{n \to \infty} j_2(x_n)$.

Theorem 5.2
The completion $(\tilde{X}, \tilde{d})$ of a normed space X is a normed space. Moreover, the isometric map $j : X \to \tilde{X}$ is linear.

Proof. Given $x, y \in \tilde{X}$ and scalar λ, pick sequence $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in X such that $j(x_n) \to x$ and $j(y_n) \to y$ in $\tilde{X}$. Then define $x + y = \lim_{n \to \infty} j(x_n + y_n)$, $\lambda x = \lim_{n \to \infty} j(\lambda x_n)$ and $||x|| = \lim_{n \to \infty} ||x_n||$. It is routine to verify that these are well-defined, that $\tilde{X}$ becomes a normed space with $\tilde{d}$ the metric induced by the norm, and that j is linear. ■

> Note that we have the canonical embedding $X \to X^{**}, x \mapsto \hat{x}$, which is isometric and linear. We can thus take $\tilde{X} = \overline{\{\hat{x} : x \in X\}}$.

5.1 Normal spaces: Urysohn and Tietze

Definition Normal space
Let X be a topological space. We say that X is normal if, for every $E, F \subset X$ such that E, F are closed and $E \cap F = \varnothing$, there exist open sets $U, V \subset X$ such that $E \subset U$, $F \subset V$ and $U \cap V = \varnothing$.

Proposition 5.3
(i) *Metric spaces are normal.*
(ii) *Compact Hausdorff spaces are normal.*

Proof. (i): Let (M, d) be a metric space. For $x, y \in M$, $|d(x, A) - d(y, A)| \leq d(x, y)$. For $x \in M$, $d(x, A) = 0 \iff x \in \overline{A}$. Let E, F be disjoint closed subsets of M, and let

$$u = \{x \in X \colon d(x, E) < d(x, F)\}$$
$$v = \{x \in X \colon d(x, E) > d(x, F)\}$$

Note that $U \cap V = \varnothing$ and that U, V are open since $x \mapsto d(x, E) - d(x, F)$ is continuous. Finally, for $x \in E$, we have $x \notin F = \overline{F}$ so $d(x, E) = 0 < d(x, F)$. Thus $E \subset U$ and, similarly, $F \subset V$.
(ii): Let K be a compact Hausdorff topological space. Let E, F be disjoint closed subsets of K.
STEP 1: We first prove that we can separate $x \in E$ and F.
For all $y \in F$, there exist disjoint open sets $U_y, V_y \subset K$ such that $x \in U_y$ and $y \in V_y$. Note that $\{V_y\}_{y \in F}$ is an open cover for F. As a closed subset of a compact space, F is compact, which implies that there exists $y_1, \cdots, y_n \in F$ such that $F \subset \bigcup_{i=1}^n V_{y_i}$. Now, let $\tilde{U} = \bigcap_{i=1}^n U_{y_i}$ and $\tilde{V} = \bigcup_{i=1}^n V_{y_i}$. Note that $\tilde{U}, \tilde{V}$ are open and disjoint, with $x \in \tilde{U}$ and $F \subset \tilde{V}$.
STEP 2: Separate E and F
Using STEP 1, for each $x \in E$, we can pick disjoint open sets $U_x, V_x \subset K$ such that $x \in U_x$ and $F \subset V_x$. Note that $\{U_x\}_{x \in E}$ is an open cover of E. As a closed subset of a compact space, E is compact, so we can pick $x_1, \cdots, x_m \in E$ such that $E \subset \bigcup_{i=1}^m U_{x_i}$. Let $U = \bigcup_{i=1}^m U_{x_i}$ and $V = \bigcap_{i=1}^m V_{x_i}$. Notably, U and V are open and disjoint, with $E \subset U$ and $F \subset V$. ■

Theorem 5.4 Urysohn's lemma
Let K be a normal topological space. Then, for disjoint closed subsets $E, F \subset K$, there exists a continuous function $f \colon K \to [0, 1]$ such that $f|_E = 0$ and $f|_F = 1$.

Remark. In a metric space, we can take

$$f(x) = \frac{d(x, E)}{d(x, E) + d(x, F)}$$

Of course, for a general topological space, we need a different construction. In that case, we use the following lemma.

Lemma 5.5
Let K be a topological space and $F_t, t \in \mathbb{Q}^+$ be open subsets of K such that
(i) $K = \bigcup_{t \in \mathbb{Q}^+} F_t$
(ii) *for every $s < t$ in $\mathbb{Q}^+$, $\overline{F}_s \subset F_t$*
Then $f \colon K \to [0, \infty), x \mapsto \inf\{t \in \mathbb{Q}^+ \colon x \in F_t\}$ is continuous.

Proof. Note that (i) implies that f is well-defined. It is easy to check that, for $t \in \mathbb{R}$,

$$f(x) < t \iff \exists\, s \in \mathbb{Q}^+ \text{ s.t. } s < t, x \in F_t$$

Moreover, observe that, for $t \in \mathbb{R}$,

$$f(x) \leq t \iff (\forall\, s \in \mathbb{Q}^+ \; t < s \implies x \in \overline{F}_s)$$

Indeed, ($\Longrightarrow$) follows from the definition of inf:

$$t < s \text{ in } \mathbb{Q}^+ \implies \exists\, u \in \mathbb{Q}^+ \cap (t,s) \text{ s.t. } x \in F_u \subset F_s \subset \overline{F}_s$$

To see ($\Longleftarrow$), note that, if $t < s$, we can pick $u \in \mathbb{Q}^+$ such $u \in (t,s)$. Then $x \in \overline{F}_u \subset D_s$, so $f(x) \leq s$. This is true for all $s \in \mathbb{Q}^+_{>t}$, so we have $f(x) \leq t$.

Putting everything together, we have that, for all $a < b$ in $\mathbb{R}$,

$$\{x \in K \colon f(x) \in (a,b)\} = \left(\bigcup_{\substack{t \in \mathbb{Q}^+ \\ t > b}} F_t \right) \setminus \left(\bigcap_{\substack{t \in \mathbb{Q}^+ \\ t > a}} \overline{F}_t \right)$$

is open, so f is continuous. ∎

Remark. Suppose $E \subset W$ in some normal space, with E closed in K and W open in K. Observe that there exists open $U \subset K$ such that $E \subset U \subset \overline{U} \subset W$. Indeed, E and $K \backslash W$ are disjoint closed sets, so there exist disjoint open sets U, V such that $E \subset U$ and $K \backslash W \subset V$. Then $U \subset K \backslash V$, so $\overline{U} \subset K \backslash V$ and thus $E \subset U \subset \overline{U} \subset K \backslash V \subset W$.
With the above lemma established, we can now prove Urysohn's lemma.

Proof of Theorem 5.4. Enumerate $\mathbb{Q} \cap [0,1]$ as $q_0 = 0, q_1 = 1, q_2, q_3, \cdots$. Let $F_0 = E$ and $F_1 = K \backslash F$, $F_t = K$ for all $t \in \mathbb{Q}_{>1}$. We have $\overline{F}_0 = F_0 \subset F_1$ and $\overline{F}_1 \subset F_t$ for all $t \in \mathbb{Q}_{>1}$. We then construct open sets F_{q_n}, $n \in \mathbb{N}$, inductively as follows: Suppose we have $F_{q_0}, F_{q_1}, \cdots, F_{q_n}$ for some $n \geq 2$ such that, letting π be a permutation of $\{0, 1, \cdots, n\}$ such that $q_{\pi(0)} < q_{\pi(1)} < \cdots < q_{\pi(n)}$, we have $\overline{F}_{q_{\pi(i-1)}} \subset F_{q_{\pi(i)}}$ for all $1 \leq i \leq n$. To define $F_{q_{n+1}}$, pick $1 \leq i \leq n$ such that $q_{\pi(i-1)} < q_{n+1} < q_{\pi(i)}$. Since $\overline{F}_{q_{\pi(i-1)}} \subset F_{q_{\pi(i)}}$, the Remark above tells us that we can pick an open set $F_{q_{n+1}}$ such that $\overline{F}_{q_{\pi(i-1)}} \subset F_{q_{n+1}} \subset \overline{F}_{q_{n+1}} \subset F_{q_{\pi(i)}}$, completing the inductive construction.

By Lemma 5.5, $f(x) = \inf\{t \in \mathbb{Q}^+ \colon x \in F_t\}$ is a continuous function on K. On K, $f \geq 0$ and, for $x \in K$, $x \in F_t$ for $t \in \mathbb{Q}_{>1}$ and so $f(x) \leq 1$. Thus, f is a continuous function $K \to [0,1]$. On E, $x \in F_t$ $\forall\, t \in \mathbb{Q}^+$, so $f(x) = 0$. On F, $x \notin F_1$, so $x \in F_t \iff t \in \mathbb{Q}_{>1}$. Thus, $f(x) = 1$ for $x \in F$. ∎

Remark. If K is compact Hausdorff, then $C(K)$ separates points of K: given $x \neq y$ in K, there exists $f \in C(K)$ such that $f(x) \neq f(y)$.

Theorem 5.6 Tietze's extension theorem
Let K be a normal topological space and L a closed subset of K. Then, for all $g \in C_b(L)$, there exists $f \in C_b(K)$ such that $f|_L = g$ and $||f||_\infty = ||g||_\infty$.

Proof. Let $X = C_b(K)$ and $Y = C_b(L)$. Recall that these are Banach spaces in $|| \cdot ||_\infty$. Define $R \colon X \to Y, f \mapsto f|_L$. Then R is a bounded linear map with $||R|| \leq 1$. By homogeneity, it suffices to show that $R(B_X) \supset B_Y$.

Let $G \subset B_Y$. So $g \colon L \to [-1,1]$. Let $E = g^{-1}([-1, -\frac{1}{3}])$ and $F = g^{-1}([\frac{1}{3}, 1])$. Then E, F are disjoint and closed in K. By Theorem 5.4, there exists a continuous function $h \colon K \to [0,1]$ with

$h|_E = 0$ and $h|_F = 1$. Let $f = \frac{2}{3}(h - \frac{1}{2})$. Then $f\colon K \to [-\frac{1}{3}, \frac{1}{3}]$ is continuous and $f|_E = -\frac{1}{3}$ and $f|_F = \frac{1}{3}$. Now, it is easy to check that, for all $x \in L$, $|f(x) - g(x)| \leq \frac{2}{3}$. Then $f \in \frac{1}{3}B_X$ and $||R(f) - g|| \leq \frac{2}{3}$, i.e., $R(\frac{1}{3}B_X)$ is $\frac{2}{3}$-dense in B_Y. By 4.4, we conclude that

$$R\left(\frac{\frac{1}{3}}{1-\frac{2}{3}}B_X\right) = R(B_X) \supset B_Y.$$

as required. ■

Remark. The above proof is for the real case.

Exercise. Deduce the complex case from Theorem 5.6.

5.2 Stone-Weierstrass theorem

In this subsection, we take K to be a compact topological space. We know that $C(K)$ is a Banach space in $||\cdot||_\infty$. As well as being a vector space, $C(K)$ is an algebra: there exists a 'multiplication' binary operation $(f, g) \mapsto fg$ such that, for all $f, g, h \in C(K)$ and scalars λ,

(i) $(fg)h = f(gh)$

(ii) $f(g + h) = fg + fh$ and $(f + g)h = fh + gh$

(iii) $f(\lambda g) = (\lambda f)g = \lambda(fg)$

In fact, it is a commutative unital algebra. Moreover, for all $f, g \in C(K)$, $||fg||_\infty \leq ||f||_\infty||g||_\infty$ since for all $x \in K$, $|(fg)(x)| = |f(x)||g(x)| \leq ||f||_\infty||g||_\infty$.

An algebra A which is also a normed space such that $||ab|| \leq ||a||||b||$ for all $a, b \in A$ is called a *normed algebra*. If A is complete, then it is a *Banach algebra*. For instance, $C(K)$ is a commutative unital Banach algebra.

Notation. In $C(K)$, $|f|$, $f \geq g$, $\max\{f, g\}$ are understood pointwise.

Definition Separates points
We say that $A \subset K$ separates points of K if, for every $x \neq y$ in K, there exists $f \in A$ such that $f(x) \neq f(y)$.

Remarks

- If $C(K)$ separates points of K, then K is Hausdorff.
- By Urysohn's lemma, if K is compact Hausdorff, then $C(K)$ separates points of K.

Definition Strongly separates points
We say that $A \subset K$ strongly separates points of K if A separates points of K and, for all $x \in K$, there exists $f \in A$ such that $f(x) \neq 0$.

Definition Subalgebra
We say that $A \subset C(K)$ is a subalgebra if it is a subspace such that

$$\forall\, f, g \in A \qquad f \cdot g \in A$$

If, in addition, $1 \in A$, then we say that A is a unital subalgebra.

Remark. If $A \subset C(K)$ is a subalgebra, then so is $\overline{A}$. This is because multiplication is continuous.

Thus far, $C(K)$ denotes both the real and complex case. In what follows, we will differentiate between these by letting

$$C^{\mathbb{R}}(K) = \{f\colon K \to \mathbb{R} \mid f \text{ cts}\}, \qquad C^{\mathbb{C}}(K) = \{f\colon K \to \mathbb{C} \mid f \text{ cts}\}$$

Theorem 5.7 Stone-Weierstrass theorem
If A is a subalgebra of $C^{\mathbb{R}}(K)$ that strongly separates points of K, then $\overline{A} = C^{\mathbb{R}}(K)$.

Remark. The condition in the Stone-Weierstrass theorem implies that K is Hausdorff.

Lemma 5.8
For every $\varepsilon > 0$, there exists a real polynomial p such that $p(0) = 0$ and

$$\forall\, t \in [-1, 1] \qquad |p(t) - |t|| \leq \varepsilon$$

To get an idea for how to prove this, note that $|t| = (t^2)^{1/2}$. On $\mathbb{C}$, $z \mapsto z^{1/2}$ has a holomorphic branch, so it has Taylor expansions on sufficiently small neighbourhoods about $z_0 \in (0, \infty)$. We can achieve this by perturbing $t^2 \mapsto t^2 + \varepsilon$.

Proof. Fix $0 < \varepsilon < 1$. Consider $f(t) = (t^2 + \varepsilon^2)^{1/2}$ for $t \in [-1, 1]$. For every $t \in [-1, 1]$, we have

$$|t| \leq \sqrt{t^2 + \varepsilon^2} \leq |t| + \varepsilon$$

so $|f(t) - |t|| \leq \varepsilon$. On $\mathbb{C}$,

$$\forall\, t \in [-1, 1] \quad t^2 + \varepsilon^2 \in [\varepsilon^2, 1 + \varepsilon^2] \subset D_1(1) = 1\{z \in \mathbb{C} \colon |z - 1| < 1\}$$

Let $z^{1/2}$ be the holomorphic branch on $\mathbb{C}\backslash(-\infty, 0]$ such that, for $x \geq 0$, $x^{1/2} \geq 0$. This has Taylor expansion $\sum_{n=0}^{\infty} a_n (z-1)^n$ on $D_1(1)$ (with $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$) which converges uniformly on any compact subset of $D_1(1)$, e.g. on $[\varepsilon^2, 1 + \varepsilon^2]$.

Now, choose $N \in \mathbb{N}$ such that, for all $s \in [\varepsilon^2, 1 + \varepsilon^2]$, we have

$$\left| s^{1/2} - \sum_{n=0}^{N} a_n (s-1)^n \right| < \varepsilon$$

Then, for all $t \in [-1, 1]$,

$$\left| f(t) - \sum_{n=0}^{N} a_n (t^2 + \varepsilon^2 - 1)^n \right| < \varepsilon$$

so q is a real polynomial such that, for all $t \in [-1, 1]$, $||t| - q(t)| \leq 2\varepsilon$. Setting $t = 0$, we see that $|q(t)| \leq 2\varepsilon$. Then $p(t) = q(t) - q(0)$ is a real polynomial such that $p(0) = 0$ and $||t| - p(t)| \leq 4\varepsilon$. ∎

Corollary 5.9
If A is a closed subalgebra of $C^{\mathbb{R}}(K)$, then A is a lattice:

$$\forall\, f, g \in A \qquad \max\{f, g\} \in A, \min\{f, g\} \in A$$

Proof. Note that

$$\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|, \qquad \min\{f, g\} = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$$

It suffices to show that $f \in A \implies |f| \in A$. Fix $f \in A$. WLOG, we may assume that $||f||_\infty \leq 1$. Let $\varepsilon > 0$ and p a real polynomial as in Lemma 5.8. Write $p = \sum_{k=1}^{n} a_k t^k$. Then $p(f) = \sum_{k=1}^{n} a_k f^k \in A$. But for every $x \in K$, $f(x) \in [-1, 1]$, so

$$\left| |f(x)| - \sum_{k=1}^{n} a_k f(x)^k \right| \leq \varepsilon,$$

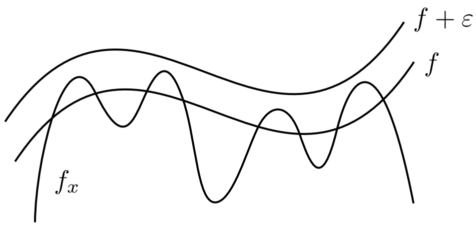
i.e., $|||f| - p(f)||_\infty \leq \varepsilon$. Thus, $|f| \in \overline{A} = A$. ∎

Proof of Theorem 5.7. We observe that, for any $x \neq y$ in K and for any $a, b \in \mathbb{R}$, there exists $f \in A$ such that $f(x) = a$ and $f(y) = b$. To see this, consider the linear map $T\colon A \to \mathbb{R}^2, f \mapsto (f(x), f(y))$. Choose $f \in A$ such that $f(x) \neq f(y)$. If $f(x) \neq 0$ and $f(y) \neq 0$, then Tf and Tf^2 are linearly independent. If WLOG $f(x) = 0$, then choose $g \in A$ such that $g(x) \neq 0$. Then Tf and Tg are linearly independent. In either case, we get that T is surjective as claimed.

Fix $f \in C^{\mathbb{R}}(K)$ and $\varepsilon > 0$. Fix $x \in K$. For any $y \in K$, pick $f_{x,y} \in A$ such that $f_{x,y}(x) = f(x)$ and $f_{x,y}(y) = f(y)$.

STEP 1: Approximate f within ε above

Let $U_y = \{z \in K\colon f_{x,y}(z) < f(z)+\varepsilon\}$. This is open and $y \in U_y$, so $\{U_y\}_{y\in K}$ is an open cover for K. By compactness, there exists $y_1, \cdots, y_n \in K$ such that $K = \bigcup_{j=1}^n U_{y_j}$. Set $f_x = \min_{1\leq j\leq n} f_{x,y_j}$. Then $f_x \in \overline{A}$ (by Corollary 5.9), with $f_x(x) = f(x)$. Given $z \in K$, there exists $1 \leq j \leq n$ so that $z \in U_{y_j}$. Thus, $f_x(z) \leq f_{x,y_j}(z) \leq f(z) + \varepsilon$.



STEP 2: Approximate f within ε below

Let $V_x = \{z \in K\colon f_x(z) > f(z)-\varepsilon\}$. Again this is an open cover with a finite subcover $\{V_{x_i}\}_{i=1}^m$. Let $g = \max_{1\leq i\leq m} f_{x_i}$. Then $g \in \overline{A}$ with

$$\forall\, z \in K \qquad g(z) = \max_{1\leq i\leq m} f_{x_i}(z) < f(z) + \varepsilon$$

Given $z \in K$, there exists $1 \leq i \leq m$ such that $z \in V_{x_i}$. So $g(z) \geq f_{x_i}(z) > f(z) - \varepsilon$. Thus, $||f - g||_\infty < \infty$ and hence $f \in \overline{\overline{A}} = \overline{A}$.

■

Remark. The above result does not hold for the complex case. Consider $C^{\mathbb{C}}(\triangle)$ and $A = A(\triangle) = \{f \in C^{\mathbb{C}}(\triangle)\colon f$ holomorphic on $\int \triangle\}$. A is a subalgebra, strongly separates points of $\triangle$ $(1, z \in A)$. However, $\overline{A} = A \neq C^{\mathbb{C}}(\triangle)$ since $z \mapsto \overline{z}$ is in $C^{\mathbb{C}}(\triangle)\backslash A$.

Theorem 5.10 Complex Stone-Weierstrass theorem
Let A be a subalgebra of $C^{\mathbb{C}}(K)$ that strongly separates points of K and closed under complex conjugation. Then $\overline{A} = C^{\mathbb{C}}(K)$.

Proof. Let $A^{\mathbb{R}} = A \cap C^{\mathbb{R}}(K)$. This is a subalgebra of $C^{\mathbb{R}}(K)$. $A^{\mathbb{R}}$ strongly separates points of K. A is also closed under complex conjugation, so $\operatorname{Re} f \in A^{\mathbb{R}}$ and $\operatorname{Im} f \in A^{\mathbb{R}}$. By Theorem 5.7, $\overline{A^{\mathbb{R}}} = C^{\mathbb{R}}(K)$. Noting that $A \supset A^{\mathbb{R}} + iA^{\mathbb{R}}$, we have

$$\overline{A} \supset \overline{A^{\mathbb{R}} + iA^{\mathbb{R}}} = \overline{A^{\mathbb{R}}} + i\overline{A^{\mathbb{R}}} = C^{\mathbb{R}}(K) + iC^{\mathbb{R}}(K) = C^{\mathbb{C}}(K)$$

as required. ■

Now, let us explore some applications of the Stone-Weierstrass theorem.

Example. The polynomials are dense in $C[a, b]$ for any $a < b$ in $\mathbb{R}$

Example. Let K be a compact space of $\mathbb{R}^d$ $(d \in \mathbb{N})$. Then the polynomials are dense in $C(K)$.

Example. Let $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. Then the trigonometric polynomials, i.e., maps of the form

$$z \mapsto \sum_{k=-n}^{n} a_k z^k, \qquad z \in \mathbb{T},\ a_{-n}, \cdots, a_n \in \mathbb{C}$$

are dense in $C^{\mathbb{C}}(\mathbb{T})$. (A key observation in the proof is that $\overline{z} = z^{-1}$ in $\mathbb{T}$.)

Example. Let K and L be compact Hausdorff. Then the continuous functions on $K \times L$ of the form

$$(x, y) \mapsto \sum_{i=1}^{n} f_i(x) g_i(y)$$

where $f_i \in C(K)$ and $g_i \in C(L)$ are dense in $C(K \times L)$.

Example. If K is a compact metric space, then $C(K)$ is separable.

5.3 Application to Fourier analysis

Let $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. As usual, we sometimes identify $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$. Recall that $C(\mathbb{T}) = \{f \colon \mathbb{T} \to \mathbb{C} \mid f \text{ cts}\}$ is a Banach space in $||\cdot||_\infty$ — in fact, its a Banach algebra. We saw above that the trigonometric polynomials are dense in $C(\mathbb{T})$, which you may recall from IB Methods is exactly what partial sums of Fourier series are.

For $f \in C(\mathbb{T})$ and $n \in \mathbb{Z}$, we call

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta$$

the n^{th} *Fourier coefficient* of f. The series $\sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta}$ is called the *Fourier series* of f. Immediately, it is natural to ask: Does it converge? If so, in what sense?

Defines the *partial sums* by

$$S_N(f)(z) = \sum_{n=-N}^{N} \hat{f}_n z^n = \sum_{n=-N}^{N} \hat{f}_n e^{int}, \qquad (z = e^{it})$$

Note that

$$S_N(f)(e^{-it}) = \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta e^{int} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sum_{n=-N}^{N} e^{in(t-\theta)}\, d\theta$$

Now, define the N^{th} *Dirichlet kernel*

$$D_N(t) = \sum_{n=-N}^{N} e^{int}$$

Observe that $S_N(f) = f * D_N$ and that

$$\begin{aligned}
D_N(t) &= e^{-iNt}\left(1 + e^{it} + e^{2it} + \cdots + e^{2iNt}\right) \\
&= e^{-iNt} \cdot \frac{e^{i(2N+1)t} - 1}{e^{it} - 1} \cdot \frac{e^{-it/2}}{e^{-it/2}} \\
&= \frac{e^{i(N+\frac{1}{2})t} - e^{-i(N+\frac{1}{2})t}}{e^{it/2} - e^{-it/2}} \\
&= \frac{\sin\left[\left(N + \frac{1}{2}\right) t\right]}{\sin\left(\frac{1}{2} t\right)}
\end{aligned}$$

Moreover, $D_N(0) = 2N+1$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1$.

Now, consider the linear functional

$$\begin{aligned} T_N \colon C(\mathbb{T}) &\to \mathbb{C} \\ f &\mapsto (f * D_N)(0) \end{aligned}$$

Note that, by symmetry of D_N, we can write

$$T_N(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) D_N(\theta)\,d\theta$$

Lemma 5.11
$T_N \in C(\mathbb{T})^*$ and $||T_N|| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|\,dt \to \infty$ as $N \to \infty$.

Before we prove this lemma, let us quickly observe a corollary that immediately follows.

Corollary 5.12
There exists $f \in C(\mathbb{T})$ such that $(T_N(f))_{N=0}^{\infty}$ is not convergent. In particular, $S_N(f)(0) \not\to f(0)$.

Proof. If $T_N(f)$ is convergent for all $f \in C(\mathbb{T})$, then $(T_N(f))_{N=0}^{\infty}$ is pointwise bounded and hence uniformly bounded by the principle of uniform boundedness (Theorem 4.2). ■

Proof of Lemma 5.11. For $f \in C(\mathbb{T})$,

$$|T_N(f)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)||D_N(\theta)|\,d\theta \le ||f||_{\theta}\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\,d\theta$$

so T_N is bounded with $||T_N|| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\,d\theta$.

Consider a dissection $-\pi = a_0 < a_1 < \cdots < a_k = \pi$ of $[-\pi, \pi]$ such that $D_N(\theta)$ has constant sign on (a_{i-1}, a_i) for all i. Fix $\delta > 0$. Let $f(\theta) = \operatorname{sign}(D_N(\theta))$ on $[a_{i-1}+\delta, a_i - \delta]$, $f(\pi) = f(-\pi) = 0$, and extend linearly. Then $||f||_\infty = 1$, so

$$\begin{aligned} ||T_N|| \ge T_N(f) &\ge \sum_{i=1}^{k} \frac{1}{2\pi}\int_{a_{i-1}+\delta}^{a_i-\delta} |D_N(\theta)|\,d\theta - \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus\cup_{i=1}^{k}[a_{i-1}+\delta, a_i-\delta]} |D_N(\theta)||f(\theta)|\,d\theta \\ &\ge \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\,d\theta - \frac{1}{\pi}\int_{[-\pi,\pi]\setminus\cup_{i=1}^{k}[a_{i-1}+\delta, a_i-\delta]} |D_N(\theta)|\,d\theta \\ &\ge \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\,d\theta - \frac{1}{\pi}(k+1)(2\delta)(2N+1) \end{aligned}$$

Taking $\delta \to 0$, we get $||T_N|| \ge \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\,d\theta$.

Proving that $||T_N|| \to \infty$ as $N \to \infty$ is left as an exercise. ■

Despite this, it turns out that, thanks to Fejér, we can remedy this by taking averages. Define

$$\sigma_N(f) = \frac{1}{N+1}\sum_{n=0}^{N+1} S_N(f) = f * \left(\frac{1}{N+1}\sum_{n=0}^{N} D_n\right)$$

We call

$$K_N = \frac{1}{N+1}\sum_{n=0}^{N} D_n$$

the N^{th} *Fejér kernel*. Note that

$$K_N(t) = \frac{1}{N+1}\sum_{n=0}^{N} e^{-int}\frac{e^{i(2n+1)t} - 1}{e^{it} - 1}$$

$$
\begin{aligned}
&= \frac{1}{N+1}\frac{1}{e^{it}-1}\sum_{n=0}^{N}[e^{i(n+1)t} - e^{-int}] \\
&= \frac{1}{N+1}\frac{1}{e^{it}-1}\left(e^{it}\cdot\frac{e^{i(N+1)t}-1}{e^{it}-1} - e^{-iNt}\frac{e^{i(N+1)t}-1}{e^{it}-1}\right) \\
&= \frac{1}{N+1}\frac{1}{(e^{it}-1)^2}\left[e^{i(N+2)t} - 2e^{it} + e^{-iNt}\right]\cdot\frac{e^{-it}}{e^{-it}} \\
&= \frac{1}{N+1}\frac{e^{i(N+1)t} - 2 + e^{-i(N+1)t}}{(e^{it/2}-e^{-it/2})^2} \\
&= \frac{1}{N+1}\left[\frac{\sin\left(\frac{N+1}{2}t\right)}{\sin(\frac{1}{2}t)}\right]^2
\end{aligned}
$$

Some other properties of K_N are

(1) $K_N \geq 0$ on $[-\pi, \pi]$

(2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,dt = 1$

(3) for every $\delta > 0$, $K_N \to 0$ uniformly on $[-\pi,\pi]\backslash(-\delta,\delta)$

Theorem 5.13
For every $f \in C(\mathbb{T})$, $f * K_N \to f$ uniformly on $\mathbb{T}$

Proof. Let $f \in C(\mathbb{T})$ and $\varepsilon > 0$. Since f is uniformly continuous, we can pick $\delta > 0$ such that

$$\forall\, s,t \in \mathbb{T} \qquad |s-t| < \delta \Longrightarrow |f(s) - f(t)| < \varepsilon$$

By (3) above, we can pick $N_0 \in \mathbb{N}$ such that

$$\forall\, N \geq N_0\ \forall\, s \in [-\pi,\pi]\backslash(-\delta,\delta) \qquad K_N(s) < \varepsilon \tag{†}$$

For $N \geq N_0$ and $t \in \mathbb{T}$,

$$
\begin{aligned}
|(f*K_N)(t) - f(t)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(s)f(t-s)\,ds - f(t)\right| \\
&\overset{(2)}{=} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(s)[f(t-s)-f(t)]\,ds\right| \\
&\leq \left|\frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(s)[f(t-s)-f(t)]\,ds\right| \\
&\qquad + \left|\frac{1}{2\pi}\int_{[-\pi,\pi]\backslash(-\delta,\delta)} K_N(s)[f(t-s)-f(t)]\,ds\right| \\
&\overset{(1),(\dagger)}{\leq} \frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(s)\,ds + \frac{1}{2\pi}\int_{[-\pi,\pi]\backslash(-\delta,\delta)} \varepsilon\cdot 2||f||_\infty\,ds \\
&\leq (1 + 2||f||_\infty)\varepsilon
\end{aligned}
$$

as required. ■

Remark. This gives us another proof (a direct one) that the trigonometric polynomials are dense in $C(\mathbb{T})$.

Now, recall that $\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$, so $|\hat{f}_n| \leq ||f||_\infty$ for all $n \in \mathbb{Z}$. Thus, $(\hat{f}_n)_{n\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z})$.

Proposition 5.14 Riemann-Lebesgue lemma
For every $f \in C(\mathbb{T})$, $\hat{f}_n \to 0$ as $|n| \to \infty$.

Proof. Let $f \in C(\mathbb{T})$ and $\varepsilon > 0$. By density, we can choose a trigonometric polynomial g such that $||f - g||_\infty < \varepsilon$. Write $g(z) = \sum_{n=-N}^{N} a_n z^n$. Note that

$$\hat{g}_n = \begin{cases} a_n & |n| \leq N \\ 0 & |n| > N \end{cases}$$

This implies that, for $|n| > N$, we have $|\hat{f}_n| \leq |\hat{f}_n - \hat{g}_n| = |(\widehat{f-g})_n| \leq ||f - g||_\infty < \varepsilon$. ■

Thus, $(\hat{f}_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}) = \{(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^\mathbb{Z} \colon a_n \to 0 \text{ as } |n| \to \infty\}$, which is a closed subspace of $\ell_\infty(\mathbb{Z})$.

Next, define the *Fourier transform* as

$$\begin{aligned} \mathcal{F} \colon C(\mathbb{T}) &\to c_0(\mathbb{Z}) \\ f &\mapsto (\hat{f}_n)_{n\in\mathbb{Z}} \end{aligned}$$

This is a linear and bounded ($||\mathcal{F}(f)||_\infty = \sup_{n\in\mathbb{Z}} |\hat{f}_n| \leq ||f||_\infty$). Moreover, $\operatorname{im}\mathcal{F}$ contains all finite sequences and hence dense in $c_0(\mathbb{Z})$.

Theorem 5.15
The Fourier transform $\mathcal{F}$ is injective but not surjective.

Proof. We first prove injectivity. Let $f \in \ker\mathcal{F}$. Then for all $n \in \mathbb{Z}$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt = 0$, so $\int_{-\pi}^{\pi} f(t)g(t)\,dt = 0$ for all trigonometric polynomials g. Given $h \in C(\mathbb{T})$ and $\varepsilon > 0$, pick a trigonometric polynomial g such that $||g - h||_\infty < \varepsilon$. Then

$$\begin{aligned} \left|\int_{-\pi}^{\pi} f(t)h(t)\,dt\right| &\leq \left|\int_{-\pi}^{\pi} f(t)(h(t) - g(t))\,dt\right| \\ &\leq 2\pi||f||_\infty||g - h||_\infty \\ &< 2\pi||f||_\infty\varepsilon \end{aligned}$$

So $\int_{-\pi}^{\pi} f(t)h(t)\,dt = 0$ for all $h \in C(\mathbb{T})$. Take $h = \tilde{f}$ to get $\int_{-\pi}^{\pi} |f|^2\,dt = 0$, which implies that $f = 0$.

Next, suppose, on the contrary, that $\mathcal{F}$ is surjective. Then by Theorem 4.5, $\mathcal{F}^{-1}$ is continuous, i.e., there exists $\delta > 0$ such that $||\mathcal{F}(f)||_\infty \geq \delta||f||_\infty$ for all $f \in C(\mathbb{T})$. Note that

$$D_N(t) = \sum_{n=-N}^{N} e^{int}, \qquad \mathcal{F}(D_N) = \mathbb{1}_{\{-N,-N+1,\cdots,N-1,N\}}$$

so $1 = ||\mathcal{F}(D_N)||_\infty \geq \delta||D_N||_\infty \geq \delta D_N(0) = \delta(2N + 1)$ for all $N \in \mathbb{N}$ — a contradiction! ■

> For those doing II Probability and Measure, you may wonder whether we can get surjectivity by extending the domain to $L^1(\mathbb{T})$. Since $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, the trigonometric polynomials are still dense in $L^1(\mathbb{T})$. However, $\mathcal{F}$ is still not surjective since we still have $||D_N||_1 \to \infty$.

5.4 Arzelà-Ascoli theorem

The aim of this subsection is to characterise compact subsets of $C(K)$ for K a compact topological space.

Before proceeding, let us recall some relevant concepts from IB Analysis and Topology. Let (M, d) be a metric space.

- The *diameter* of a nonempty subset $A \subset M$ is

$$\operatorname{diam} A = \sup\{d(x, y) \colon x, y \in A\}$$

  For instance, $\operatorname{diam} B_r(x) \leq 2r$. If $\operatorname{diam} A \leq r$, then for any $x \in A$, $A \subset B_r(x)$. Moreover, $\operatorname{diam} \overline{A} = \operatorname{diam} A$.
- For $\varepsilon > 0$ and $F \subset M$, we say that F is an *ε-net* for M if

$$\forall\, x \in M \ \exists\, y \in F \qquad d(x, y) \leq \varepsilon$$

  Equivalently, $M = \bigcup_{y \in F} B_\varepsilon(y)$.
- We say that M is *totally bounded* if

$$\begin{aligned} &\forall\, \varepsilon > 0 \ \exists \ \text{finite } \varepsilon\text{-net for } M \\ \iff\ &\forall\, \varepsilon > 0 \ \exists \ \text{nonempty subsets } A_1, \cdots, A_n \text{ of } M \text{ s.t. } M = \bigcup_{i=1}^n A_i \text{ and } \operatorname{diam} A_i \leq \varepsilon \ \forall\, i \\ \iff\ &\forall\, \varepsilon > 0 \ \exists \ \text{closed subsets } A_1, \cdots, A_n \text{ of } M \text{ s.t. } M = \bigcup_{i=1}^n A_i \text{ and } \operatorname{diam} A_i \leq \varepsilon \ \forall\, i \end{aligned}$$

Example. $(0, 1)$ is totally bounded but not compact.

Lemma 5.16
Let (M, d) be a metric space and $N \subset M$.
(i) *If M is totally bounded, then so is N.*
(ii) *If N is totally bounded, then so is $\overline{N}$.*

Proof. (i): Given $\varepsilon > 0$, $M = \bigcup_{i=1}^n A_i$ with $\operatorname{diam} A_i \leq \varepsilon$ for all i. Then $N = \bigcup_{i=1}^n (A_i \cap N)$. Discard $A_i \cap N$ if empty; otherwise, $\operatorname{diam}(A_i \cap N) \leq \varepsilon$.
(ii): Given $\varepsilon > 0$, $N = \bigcup_{i=1}^n A_i$ with $\operatorname{diam} A_i \leq \varepsilon$ for all i. Then $N \subset \bigcup_{i=1}^n \overline{A_i}$ (closure in M) with $\operatorname{diam} \overline{A_i} \leq \varepsilon$ for all i. Finite union of closed sets is closed, so $\overline{N} \subset \bigcup_{i=1}^n \overline{A_i}$. ∎

Theorem 5.17
For a metric space (M, d), the following are equivalent:
(i) *M is compact*
(ii) *M is sequentially compact*
(iii) *M is totally bounded and complete*

Proof. Recall from IB Analysis and Topology. ∎

Now, we introduce some new topological concepts.

Definition Relatively compact
We say that $N \subset M$ is relatively compact in M if $\overline{N}$ is compact.

Corollary 5.18
Let (M, d) be a complete metric space and $N \subset M$. The following are equivalent:
(i) *N is relatively compact in M*
(ii) *Every sequence in N has a subsequence that converges in M*
(iii) *N is totally bounded*

Proof. (i) $\implies$ (ii): $\overline{N}$ is compact and hence sequentially compact.
(ii) $\implies$ (iii): Given a sequence $(x_n)_{n\in\mathbb{N}}$ in $\overline{N}$, for each $n \in \mathbb{N}$, we can choose $y_n \in \mathbb{N}$ such that $d(x_n, y_n) < \frac{1}{n}$. By (ii), there exists $n_1 < n_2 < \cdots$, there exists $z \in M$ such that $y_{n_k} \to z$. Then $x_{n_k} \to z \in \overline{N}$. Thus, $\overline{N}$ is sequentially compact and hence totally bounded.
(iii) $\implies$ (i): Note that $\overline{N}$ is totally bounded and complete, so it is compact. ∎

Definition Equicontinuity and uniform boundedness
Let K be a compact topological space and $\mathcal{F} \subset C(K)$.

- *We say that $\mathcal{F}$ is equicontinuous if, for every $x \in K$ and every $\varepsilon > 0$, there exists an open neighbourhood U of x such that, for every $f \in \mathcal{F}$ and $y \in U$, we have $|f(y) - f(x)| < \varepsilon$.*
- *We say that $\mathcal{F}$ is uniformly bounded if $\mathcal{F}$ is bounded with respect to $||\cdot||_\infty$.*

Theorem 5.19 Arzelà-Ascoli theorem
$\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is uniformly bounded and equicontinuous.

Remark. $\implies$ is trivial for finite $\mathcal{F}$, and compact is 'almost' finite.

Proof. By Corollary 5.18, $\mathcal{F}$ is relatively compact iff $\mathcal{F}$ is totally bounded.

($\implies$) $\mathcal{F}$ equicontinuous: Fix $x \in K$ and $\varepsilon > 0$. Let $f_1, \cdots, f_n$ be an ε-net for K. For $1 \le i \le n$, f_i is continuous (at x), so there exists an open neighbourhood U_i of x such that $y \in U_i \implies |f_i(y) - f_i(x)| < \varepsilon$. Then $U = \bigcap_{i=1}^n U_i$ is an open neighbourhood of x. Given $f \in \mathcal{F}$ and $y \in U$, pick $1 \le i \le n$ such that $||f - f_i|| < \varepsilon$. Then

$$|f(y) - f(x)| \le |f(y) - f_i(y)| + |f_i(y) - f_i(x)| + |f_i(x) - f(x)| < 3\varepsilon$$

$\mathcal{F}$ uniformly bounded: Let $f_1, \cdots, f_n$ be an 1-net for K. Let $M = 1 + \max_{1\le i\le n} ||f_i||_\infty$. For every $f \in \mathcal{F}$, choose $1 \le i \le n$ such that $||f - f_i||_\infty < 1$. Then we have

$$||f||_\infty \le ||f - f_i||_\infty + ||f_i||_\infty \le M$$

($\impliedby$) Fix $\varepsilon > 0$. For each $x \in K$, pick a neighbourhood U_x of x such that, for every $f \in \mathcal{F}$ and every $y \in U_x$, $|f(y) - f(x)| < \varepsilon$. Note that $\{U_x : x \in K\}$ is an open cover for K, so we can pick $x_1, \cdots, x_m \in K$ such that $K = \bigcup_{i=1}^m U_{x_i}$ For $f \in \mathcal{F}$, let $Tf = (f(x_i))_{i=1}^m$ and $S = \{Tf : f \in \mathcal{F}\} \subset \ell_\infty$ (norms of finite-dimensional spaces are equivalent). $\mathcal{F}$ is uniformly bounded, so there exists $M \ge 0$ such that, for every $f \in \mathcal{F}$, $||f||_\infty \le M$. So $||Tf||_\infty \le M$ for all $f \in \mathcal{F}$. Thus, S is bounded in ℓ_∞^m and hence totally bounded. This implies that we can pick $f_1, \cdots, f_n$ in $\mathcal{F}$ such that $\{Tf_1, \cdots, Tf_n\}$ is an ε-net for S.
Finally, we show that $f_1, \cdots, f_n$ is a 3ε-net for $\mathcal{F}$. Fix $f \in \mathcal{F}$. Pick $1 \le j \le n$ such that $||Tf - Tf_j|| < \varepsilon$. Given $y \in K$, choose $1 \le i \le m$ such that $y \in U_{x_i}$. Then

$$|f(y) - f_j(y)| \le |f(y) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(y)| < 3\varepsilon$$

as required. ∎

5.5 Application to compact operators

Let X, Y be Banach spaces, and $T: X \to Y$ be a linear map. We say that T is *compact* if $\overline{T(B_X)}$ is compact. Let $\mathcal{K}(X, Y) = \{T : X \to Y \mid T \text{ linear and compact}\}$.

Remarks

- $\mathcal{K}(X, Y) \subset \mathcal{B}(X, Y)$, i.e., compact $\implies$ bounded
- For linear $T: X \to Y$, the following are equivalent:

  - T is compact
  - $T(B_X)$ is totally bounded
  - For every bounded sequence (x_n) in X, Tx_n has a convergent subsequence in Y
- $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$.
  Indeed, we have
  - $S,T \in \mathcal{K}(X,Y) \implies S+T \in \mathcal{K}(X,Y)$
    Given a bounded sequence (x_n) in X, compactness of S implies that we can pick infinite subset $L \subset \mathbb{N}$ such that $(Sx_n)_{n\in L}$ is convergent in Y. Since T is compact, we can pick an infinite subset $M \subset L$ such that $(Tx_n)_{n\in M}$ is convergent in Y. Hence, $((S+T)x_n)_{n\in M} = (Sx_n + Tx_n)_{n\in M}$ is convergent in Y, implying that $S+T \in \mathcal{K}(X,Y)$.
  - For any sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(X,Y)$ with $T_n \to T$ in $\mathcal{B}(X,Y)$, we have $T \in \mathcal{K}(X,Y)$
    Fix $\varepsilon > 0$. Pick $n \in \mathbb{N}$ such that $||T_n - T|| \le \varepsilon$. Since T_n is compact, $T_n(B_X)$ is totally bounded. Thus, we can pick $x_1, \cdots, x_m \in B_X$ such that $\{T_n x_1, \cdots, T_n x_m\}$ is an ε-net for $T_n(B_X)$.
    Given $x \in B_X$, pick $1 \le i \le m$ such that $||T_n x - T_n x_i|| \le \varepsilon$. Then $||Tx - Tx_i|| \le ||Tx - T_n x|| + ||T_n x - T_n x_i|| + ||T_n x_i - Tx_i|| \le 3\varepsilon$. Thus, $\{Tx_1, \cdots, Tx_m\}$ is a 3ε-net for $T(B_X)$. Thus, $T \in \mathcal{K}(X,Y)$.
- $\mathcal{K}(X,Y)$ has the *ideal property*: given bounded linear maps

  $$W \xrightarrow{B} X \xrightarrow{T} Y \xrightarrow{A} Z$$

  between Banach spaces, $T \in \mathcal{K}(X,Y)$ implies that $ATB \in \mathcal{K}(W,Z)$. Indeed, given a bounded sequence (x_n) in W, (Bx_n) is bounded in X, so there exists an infinite subset $M \subset \mathbb{N}$ such that $(TBx_n)_{n\in M}$ is convergent in Y and thus $(ATBx_n)_{n\in M}$ is convergent in Z.

Example

(1) Every finite-rank operator is compact.
  Recall that $T \in \mathcal{B}(X,Y)$ is *of finite rank* if $\mathrm{rk}(T) = \dim T(X) < \infty$. Let $\mathcal{F}(X,Y) = \{T \in \mathcal{B}(X,Y) \colon \mathrm{rk}(T) < \infty\}$. If $T \in \mathcal{F}(X,Y)$, then $T(B_X)$ lies in the finite-dimensional $T(X)$, so $\overline{T(B_X)}$ is compact (Heine-Borel). Thus, $\mathcal{F}(X,Y) \subset \mathcal{K}(X,Y)$. But $\mathcal{K}(X,Y)$ is closed, so $\overline{\mathcal{F}(X,Y)} \subset \mathcal{K}(X,Y)$.

(2) $T \colon \ell_2 \to \ell_2, (x_n) \mapsto (x_n/n)$ is compact.
  Consider $T_n \colon \ell_2 \to \ell_2, (x_n) \mapsto \sum_{i=1}^{n} \frac{1}{n} x_i e_i$. Then $||T_n - T|| \le \frac{1}{n+1}$, so $T_n \to T$ with $\mathrm{rk}(T_n) = n$. Hence, T is compact.

(3) If $\dim X = \infty$, then Id_X is not compact: $\overline{\mathrm{Id}(B_X)} = B_X$, which is not compact. More generally, if $T \in \mathcal{B}(X,Y)$ is surjective and $\dim Y = \infty$, then T is not compact. Indeed, by Theorem 4.5, there exists $\delta > 0$ such that $T(B_X) \supset \delta B_Y$.

(4) Let $K \colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Define

$$\begin{aligned} T \colon C[0,1] &\to C[0,1] \\ f &\mapsto \left(x \mapsto \int_0^1 f(y) K(x,y)\, dy \right) \end{aligned}$$

  We first show that this is well-defined. Since $[0,1] \times [0,1]$ is compact, K is uniformly continuous. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $||(x_1,y_1) - (x_2,y_2)||_2 < \delta \implies$

$|K(x_1, y_1) - K(x_2, y_2)| < \varepsilon$. Then, for $|x - z| < \delta$, we have

$$(Tf)(x) - (Tf)(z) = \left| \int_0^1 f(y)[K(x,y) - K(z,y)]\, dy \right| \le \int_0^1 |f(y)|\varepsilon\, dy = \varepsilon ||f||_\infty$$

Next, we note that T is manifestly linear. Moreover, for any $x \in [0,1]$,

$$|(Tf)(x)| \le \int_0^1 |f(y)||K(x,y)|\, dy \le ||f||_\infty ||K||_\infty$$

so $||Tf||_\infty \le ||K||_\infty ||f||_\infty$. Hence, T is bounded with $||T||_\infty \le ||K||_\infty$.

Theorem 5.20
Let X, Y be Banach spaces and $T \in \mathcal{B}(X,Y)$. Then T is compact if and only if T^* is compact.

Remark. Note that $T^*(B_{Y^*}) = \{T^* g \colon g \in Y^*, ||g|| \le 1\}$. For $g \in B_{Y^*}$, we have

$$||T^*g|| = \sup_{x \in B_X} |(T^*g)(x)| = \sup_{x \in B_X} |g(Tx)| = \sup_{y \in T(B_X)} |g(y)| = \sup_{y \in \overline{T(B_X)}} |g(y)|$$

Thus, $T^*(B_{Y^*}) \hookrightarrow C(\overline{T(B_X)})$, which motivates the use of Arzelà-Ascoli in the proof of the Theorem.

Proof of Theorem 5.20.
($\Longrightarrow$) Let $K = \overline{TB_X}$, which is a compact metric space with $K \subset Y$. Define $\theta \colon T^*(B_{Y^*}) \to C(K), T^*g \mapsto g|_K$. By the computation in the Remark above, $||T^*g|| = ||g|_K||_\infty$. So θ is well-defined and isometric. Thus, $T^*(B_{Y^*})$ is totally bounded iff $\operatorname{im}\theta$ is totally bounded. Now, we show that $\operatorname{im}\theta$ is uniformly bounded and equicontinuous:

- As K is compact, it is bounded, so we can pick $M \ge 0$ such that $||y|| \le M$ for all $y \in K$. For $g \in B_{Y^*}$ and $y \in K$, $|g(y)| \le ||g||||y|| \le M$, so $||g|_K|| \le M$. So $\operatorname{im}\theta$ is uniformly bounded.
- For $g \in B_{Y^*}$ and $y, z \in K$,

$$|g(y) - g(z)| \le ||g||||y - z|| \le ||y - z||$$

  Given $y \in K$ and $\varepsilon > 0$, we have $|g(y) - g(z)| < \varepsilon$ for every $z \in K$ with $||y - z|| < \varepsilon$ and every $g \in B_{Y^*}$. Thus, $\operatorname{im}\theta$ is equicontinuous.

By Arzelà-Ascoli, $\operatorname{im}\theta$ is totally bounded and hence so is $T^*(B_{Y^*})$.

($\Longleftarrow$) Suppose T^* is compact, By ($\Longrightarrow$), $T^* *$ is compact. But recall that we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\hat{}"'] & Y \arrow[d] \\ X^{**} \arrow[r, "T^{**}"'] & Y^{**} \end{tikzcd}$$

Suitably interpreted via identifications, $T = T^{**}|_X$. Previously, we used Hahn-Banach to deduce that the canonical map $\hat{}\colon X \to X^{**}$ is an isometric embedding, which completes the proof. ■

6 Hilbert spaces

6.1 Basics

Definition Inner product space
Let X be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). An inner product on X is a function $\langle\cdot,\cdot\rangle\colon X \to X \to \mathbb{R}$ (resp. $\mathbb{C}$) such that
(i) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$ (linearity in the first variable)
(ii) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ (conjugate-symmetric)
(iii) $\langle x, x\rangle \geq 0$ with equality iff $x = 0$ (positive-definiteness)
The pair $(X, \langle\cdot,\cdot\rangle)$ is called an inner product space.

Remark. Of course (i) and (ii) above imply conjugate linearity in the second variable.

Example
The following are inner product spaces:
- ℓ_2^n with $\langle x, y\rangle = \sum_{i=1}^n x_i \overline{y_i}$
- ℓ_2 with $\langle x, y\rangle = \sum_{i=1}^\infty x_i \overline{y_i}$
- $C[0,1]$ with $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$
- $C(\mathbb{T})$ with $\langle f, g\rangle = \frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}\,dt$
- $L^2(\Omega)$ with $\langle f, g\rangle = \int_\Omega f\overline{g}\,d\mu$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space

Definition Norm
Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. The norm of $x \in X$ is $||x|| = \langle x, x\rangle^{1/2}$.

Theorem 6.1
Let X be an inner product space and $x, y \in X$.
(i) $|\langle x, y\rangle| \leq ||x||||y||$ (Cauchy-Schwarz)
(ii) $||x + y|| \leq ||x|| + ||y||$ (triangle)

Remarks
- The above result tells us that the object we called 'norm' is indeed a norm.
- $\langle\cdot,\cdot\rangle$ is continuous: if $x_n \to x$ and $y_n \to y$ in X, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$

Proof of Theorem 6.1. (i): For $t \in \mathbb{R}$,

$$0 \leq \langle x + ty, x + ty\rangle = ||x||^2 + t^2||y||^2 + 2t\,\mathrm{Re}\langle x, y\rangle$$

Choose θ such that $e^{i\theta}\langle x, y\rangle = |\langle x, y\rangle|$. Apply above to $x, e^{-i\theta}y$:

$$0 \leq ||x||^2 + t^2||y||^2 + 2t|\langle x, y\rangle|$$

We must have discriminant ≥ 0, which gives us the result.
(ii): We have

$$||x + y||^2 \leq ||x||^2 + 2\,\mathrm{Re}\langle x, y\rangle + ||y||^2 \leq ||x||^2 + 2|\langle x, y\rangle| + ||y||^2 \leq ||x||^2 + 2||x||||y|| + ||y||^2$$

as required. ■

Definition Hilbert space
A Hilbert space is a complete inner product space.

Example
ℓ_2^n, ℓ_2 and L^2 are Hilbert spaces, whereas $C[0,1]$ and $C(\mathbb{T})$ are not.

Proposition 6.2 Polarisation identity
Let X be an inner product space and $x, y \in X$. Then

$$\langle x, y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2) \qquad (\mathbb{R})$$

$$\langle x, y\rangle = \frac{1}{4}(||x+y||^2 + i||x+iy||^2 - ||x-y||^2 - i||x-iy||^2) \qquad (\mathbb{C})$$

Proof. Expand the RHS. ■

Remark. The polarisation identity implies that, if a norm $||\cdot||$ arises from an inner product, then that is the unique inner product that gives rise to $||\cdot||$. This, for instance, allows us to talk about *the* inner product space $(C[0,1], ||\cdot||_2)$.

Theorem 6.3
The completion $\tilde{X}$ of an inner product space X is a Hilbert space.

Proof. Given $x, y \in \tilde{X}$, choose sequences (x_n), (y_n) in X such that $x_n \to x$ and $y_n \to y$. Define the inner product $\langle x, y\rangle = \lim_{n\to\infty}\langle x_n, y_n\rangle$. It is routine to verify that this is well-defined and defines an inner product on $\tilde{X}$ that extends the one on X. Given $x \in \tilde{X}$, choose (x_n) in X such that $x_n \to x$. Then

$$||x|| = \lim_{n\to\infty} ||x_n|| = \lim_{n\to\infty} \sqrt{\langle x_n, x_n\rangle} = \sqrt{\langle x, x\rangle}$$

as required. ■

Proposition 6.4 Parallelogram law
Let X be an inner product space and $x, y \in X$. Then

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$$

Proof. $||x+y||^2 + ||x-y||^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = 2||x||^2 + 2||y||^2$ ■

Remarks

- There is a converse: if a normed space X satisfies the parallelogram law, then X is an inner product space. This gives another proof of Theorem 6.3.
- We also have the *generalised parallelogram law*: Given $x_1, \cdots, x_n$ in an inner product space X, we have

$$\frac{1}{2^n}\sum_{\varepsilon\in\{\pm 1\}^n}\left|\left|\sum_{i=1}^n \varepsilon_i x_i\right|\right|^2 = \sum_{i=1}^n ||x_i||^2$$

Definition Orthogonal
Let X be an inner product space and $x, y \in X$. We say that x and y are orthogonal and write $x \perp y$ if $\langle x, y\rangle = 0$.

Proposition 6.5 Pythagoras' theorem
$x \perp y \implies ||x+y||^2 = ||x||^2 + ||y||^2$

Proof. $||x+y||^2 = \langle x+y, x+y\rangle = ||x||^2 + ||y||^2 + 2\,\mathrm{Re}\langle x, y\rangle = ||x||^2 + ||y||^2$ ■

Remark. We can generalise the above result. For $x_1, \cdots, x_n \in X$ pairwise orthogonal,

$$\left|\left|\sum_{i=1}^n x_i\right|\right|^2 = \sum_{i=1}^n ||x_i||^2$$

Note that, for $S = \{(1+\frac{1}{n})e_n : n \in \mathbb{N}\} \subset \ell_2$, $d(0,S) = \inf_{s \in S} ||0 - s|| = 1$ is not attained by any $s \in S$.

Theorem 6.6 Closed point theorem
Let Y be a closed subspace of a Hilbert space $\mathcal{H}$, and let $x \in \mathcal{H}$. Then there is a unique $y \in Y$ for which $||x-y|| = d(x,Y)$.

Proof. Pick a sequence (y_n) in Y such that $||x - y_n|| \to d(x,Y)$. Applying Proposition 6.4 to $x - y_m$ and $x - y_n$, we get

$$||2x - y_m - y_n||^2 + ||y_m - y_n||^2 = 2||x - y_m||^2 + 2||x - y_n||^2$$

Rearranging, we get

$$\begin{aligned} ||y_m - y_n||^2 &= 2||x - y_m||^2 + 2||x - y_n||^2 - 4\left|\left|x - \underbrace{\frac{y_m + y_n}{2}}_{\in Y}\right|\right|^2 \\ &\leq 2||x - y_m||^2 + 2||x - y_n||^2 - 4d(x,Y)^2 \end{aligned}$$

which implies that (y_n) is Cauchy. Since Y is a closed subspace of a complete metric space, it is complete and thus (y_n) converges to some $y \in Y$. Finally, note that $||x-y|| = \lim_{n\to\infty} ||x-y_n|| = d(x,Y)$.

Now, suppose $z \in Y$ alos satisfies $||x - z|| = d(x,Y)$. By Proposition 6.4, we have

$$\begin{aligned} &||2x - y - z||^2 + ||y - z||^2 = 2||x - y||^2 + 2||x - z||^2 \\ \implies &||y - z||^2 = 4d(x,Y)^2 - 4\left|\left|x - \frac{y+z}{2}\right|\right|^2 \leq 0 \\ \implies &y = z \end{aligned}$$

as required. ■

6.2 Riesz representation theorem

In this subsection, we are concerned with identifying $\mathcal{H}^*$ given a Hilbert space $\mathcal{H}$.

Let X be an inner product space and $y \in X$. Define $\theta_y : X \to \{\text{scalars}\}, x \mapsto \langle x, y\rangle$. Note that θ_y is linear, and by Cauchy-Schwarz, $|\theta_y(x)| \leq ||x|| ||y||$, so $\theta_y \in X^*$ and $||\theta_y|| \leq ||y||$. In fact, taking $x = y$ shows that $||\theta_y|| = ||y||$.

Definition
Let X be an inner product space. For $y \in X$, let

$$y^\perp = \{x \in X : \langle x, y\rangle = 0\} = \ker \theta_y$$

For $S \subset X$, let

$$S^\perp = \{x \in X : \ \forall\, y \in S \ \langle x, y\rangle = 0\} = \bigcap_{s \in S} s^\perp$$

Remarks

- We read the above as 'y perp' and 'S perp'.
- Both are closed subspaces of X.

Theorem 6.7
Let Y be a closed subspace of a Hilbert space $\mathcal{H}$. Then $\mathcal{H} = Y \oplus Y^\perp$.

Remark. We say that $\mathcal{H}$ is the *orthogonal direct sum* of Y and $Y^\perp$ and that $Y^\perp$ is the *orthogonal complement* of Y.

Proof of Theorem 6.7. Since Y and $Y^\perp$ are closed subspaces and $\mathcal{H}$ is complete, it suffices (by one of our applications of the Baire category theorem) to show that $\mathcal{H}$ is the algebraic direct sum of Y and $Y^\perp$:

- $Y \cap Y^\perp = \{0\}$
  $x \in Y \cap Y^\perp \implies ||x||^2 = \langle x, x\rangle = 0 \implies x = 0$
- $Y + Y^\perp = \mathcal{H}$
  Fix $x \in \mathcal{H}$. Let $y \in Y$ be the unique closest point of Y to x. Let $z = x - y$. It remains to show that $z \in Y^\perp$.
  Suppose, on the contrary, that $z \notin Y^\perp$. Then we can pick $w \in Y$ such that $\langle z, w\rangle \neq 0$. By rotating $w \mapsto e^{i\theta}w$ if necessary, we may assume WLOG that $\langle z, w\rangle \in \mathbb{R}_{>0}$. Observe that

$$\begin{aligned}
||x - y - tw||^2 &= ||z - tw||^2 \\
&= \langle z - tw, z - tw\rangle \\
&= ||z||^2 + t^2||w||^2 - 2t\langle z, w\rangle \\
&= ||z||^2 - t(2\langle z, w\rangle - t||w||^2) \\
&< ||z||^2
\end{aligned}$$

  for small $t > 0$, contradicting the fact that y is the unique closest point. ■

Remark. The projection onto Y along $Y^\perp$

$$\begin{aligned}
P &\colon \mathcal{H} \to \mathcal{H} \\
y + z &\mapsto y \qquad (y \in Y, z \in Y^\perp)
\end{aligned}$$

is called the *orthogonal projection onto* Y. For $x \in \mathcal{H}$, we have

$$x = \underbrace{Px}_{\in Y} + \underbrace{(x - Px)}_{\in Y^\perp}$$

so Px is the unique closest point of Y to x.

Theorem 6.8 Riesz representation theorem
Let $\mathcal{H}$ be a Hilbert space and $f \in \mathcal{H}^*$. Then there exists $y \in \mathcal{H}$ such that $f = \theta_y$.

Proof. The result is trivial when $f = 0$, so it remains to carefully consider the case $f \neq 0$. Let $Y = \ker f$, which is a closed subspace of $\mathcal{H}$. By Theorem 6.7, $\mathcal{H} = Y \oplus Y^\perp$ and $Y^\perp \neq \{0\}$ as $f \neq 0$. For $z, w \in Y^\perp$, $f(w)z - f(z)w \in Y$, so $\dim Y^\perp = 1$. Fix $y \in Y^\perp$ such that $f(y) = ||y||^2$. Given $x \in \mathcal{H}$, $x = u + \lambda y$ for some $u \in Y$ and scalar λ. Then $f(x) = f(u) + \lambda f(y) = \lambda||y||^2$ and $\theta_y(x) = \langle x, y\rangle = \lambda||y||^2 = f(x)$. ■

Corollary 6.9
For a Hilbert space $\mathcal{H}$, the map

$$\begin{aligned}
\theta &\colon \mathcal{H} \to \mathcal{H}^* \\
y &\mapsto \theta_y
\end{aligned}$$

is an isometric, conjugate-linear isomorphism.

Proof. From Theorem 6.8 and our computation at the start of this subsection, it remains to show that θ is conjugate-linear: $\theta_{y+\lambda z}(x) = \langle x, y + \lambda z\rangle = \langle x, y\rangle + \overline{\lambda}\langle x, z\rangle = (\theta_y + \overline{\lambda}\theta_z)(x)$. ∎

Remark. In Chapter 2, we had $\varphi\colon \ell_2 \to \ell_2^*$ is an isometric isomorphism, given by $\varphi_y(x) = \sum x_i y_i$. On the other hand, $\theta_y(x) = \langle x, y\rangle = \sum x_i \overline{y_i}$.