University of Cambridge Mathematical Tripos Part II

Differential Geometry

Lecturer:
Mihalis Dafermos
`md384@cam.ac.uk`

Notes by:
Robert Frederik Uy
`rfu20@cam.ac.uk`

Course schedule

Smooth manifolds in $\mathbb{R}^n$, tangent spaces, smooth maps and the inverse function theorem. Examples, regular values, Sard's theorem (statement only). Transverse intersection of submanifolds. [4]

Manifolds with boundary, degree mod 2 of smooth maps, applications. [3]

Curves in 2-space and 3-space, arc-length, curvature, torsion. The isoperimetric inequality. [2]

Smooth surfaces in 3-space, first fundamental form, area. [1]

The Gauss map, second fundamental form, principal curvatures and Gaussian curvature. Theorema Egregium. [3]

Minimal surfaces. Normal variations and characterization of minimal surfaces as critical points of the area functional. Isothermal coordinates and relation with harmonic functions. The Weierstrass representation. Examples. [3]

Parallel transport and geodesics for surfaces in 3-space. Geodesic curvature. [2]

The exponential map and geodesic polar coordinates. The Gauss-Bonnet theorem (including the statement about classification of compact surfaces). [4]

Global theorems on curves: Fenchel's theorem (the total curvature of a simple closed curve is greater than or equal to 2π); the Fary-Milnor theorem (the total curvature of a simple knotted closed curve is greater than 4π). [2]

Recommended books

J. Milnor *Topology from the differentiable viewpoint.* Princeton University Press, 1997.
M. Do Carmo *Differential Geometry of Curves and Surfaces.* Pearson Higher Education, 1976

Contents

1 Differential topology

Definition Smooth map on an open subset
Let $U \subset \mathbb{R}^n$. We say that $f\colon U \to \mathbb{R}^m$ is smooth if all partial derivatives to all orders exist and are continuous.

Definition Smooth map
Let $X \subset \mathbb{R}^n$. We say that $f\colon X \to \mathbb{R}^m$ is smooth if, for each $x \in X$, there exists (i) an open neighbourhood $U \subset \mathbb{R}^n$ of x and (ii) a smooth map $\tilde{f}\colon U \to \mathbb{R}^m$ such that $\tilde{f}|_{X\cap U} = f|_{X\cap U}$.

Definition Diffeomorphism
Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$. We say that $f\colon X \to Y$ is a diffeomorphism if f is a smooth bijection with a smooth inverse. If such a map exists, we say that X and Y are diffeomorphic.

Exercise. Give an example of a smooth bijection that is not a diffeomorphism.

Definition k-dimensional manifold
We say that $X \subset \mathbb{R}^N$ is a k-dimensional manifold if, for each $x \in X$, there exists an open neighbourhood $V \subset X$ of x such that V is diffeomorphic to an open subset $U \subset \mathbb{R}^k$. A diffeomorphism $\varphi\colon U \to V$ is called a local parametrisation of V, whereas its inverse $\psi := \varphi^{-1}\colon V \to U$ is called a coordinate system or a chart on V.

Remarks

- By composing φ^{-1} with the projections $\pi_i\colon \mathbb{R}^k \to \mathbb{R}, (x_1, \cdots, x_k) \mapsto x_i$, we get smooth maps $x_i := \pi_i \circ \varphi^{-1}$ which we call *coordinate functions.*
- WLOG, we can replace 'diffeomorphic to an open subset $U \subset \mathbb{R}^k$" with "diffeomorphic to an open ball in R^k".
- It is easy to see that, if $X \subset \mathbb{R}^N$ is both a k-dimensional manifold and a $\tilde{k}$-dimensional manifold, then $k = \tilde{k}$.

Definition Dimension
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold. The dimension of X is k, and it is denoted by $\dim X$.

Example *Some trivial examples*

- $X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W
- $X = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^N$

Example S^n
$S^n := \{x \in \mathbb{R}^{n+1}\colon ||x||_2 = 1\}$ is an n-dimensional manifold. To see this, consider the projection $\Pi_k\colon \mathbb{R}^{n+1} \to \mathbb{R}^n, (x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1})$. It is easy to verify that maps of the form $\psi_k^{\pm} = \Pi_k|_{S^n \cap \{\mathrm{sign}(x_k) = \pm 1\}}$ are diffeomorphisms $S^n \cap \{\mathrm{sign}(x_k) = \pm 1\} \to B_1(0)$.

Remark. It is easy to show that X is a 0-dimensional manifold iff X is a discrete subset of $\mathbb{R}^N$.

Exercise. Show that, if X and Y are manifolds, then $X \times Y$ is also a manifold, with $\dim X \times Y = \dim X + \dim Y$.

Definition Submanifold
Let $X, Y \subset \mathbb{R}^N$ be manifolds. If $Y \subset X$, then we say that Y is a submanifold of X. The codimension of Y in X is defined as

$$\operatorname*{codim}_X Y := \dim X - \dim Y$$

1.1 Tangent spaces

We first recall some basic facts from our youth. Let $U \subset \mathbb{R}^k$ be open. The *differential* of a smooth map $f\colon U \to \mathbb{R}^m$ at $x \in U$ is defined by

$$\begin{aligned} df_x\colon \mathbb{R}^k &\to \mathbb{R}^N \\ h &\mapsto \lim_{t\to 0} \frac{f(x+th) - f(x)}{t} \end{aligned}$$

This is a linear map, with matrix representation

$$df_x = \left(\frac{\partial f^i}{\partial x^j}\right)_{i,j}$$

Moreover, differentials satisfy the chain rule: given (i) two smooth maps $f\colon U \to \mathbb{R}^l$ and $g\colon V \to \mathbb{R}^m$ with $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$ open and (ii) a point $x \in U$ with $f(x) \in V$, we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Definition Tangent space
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold and $x \in X$. Choose a local parametrisation $\varphi\colon U \to V$ around x. We then define the tangent space T_xX of X at x to be

$$T_xX := \operatorname{im} d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k)$$

Of course, before we can safely proceed, we must show that T_xX is well-defined:

Lemma 1.1
Let X be as above. T_xX is independent of φ, and $\dim T_xX = k$.

Proof. Let $\tilde{\varphi}\colon \tilde{U} \to \tilde{V}$ be another local parametrisation near x. WLOG, by restricting if necessary, we may assume $\tilde{V} = V$. By the chain rule, we have

$$d\varphi_{\varphi^{-1}(x)} = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)} \circ d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}$$

Since $\tilde{\varphi}^{-1} \circ \varphi$ is a diffeomorphism of open subsets of $\mathbb{R}^n$, the corresponding differential $d(\tilde{\varphi}^{-1} \circ \varphi)$ is a linear isomorphism. Thus,

$$d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}(\mathbb{R}^k)) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(\mathbb{R}^k)$$

as claimed.

Now, it remains to show that $\dim T_xX = k$. By definition, there exists an open set $\hat{V} \subset \mathbb{R}^N$ and a smooth map $\Psi\colon \hat{V} \to \mathbb{R}^k$ that extends the chart $\psi := \varphi^{-1}$. Note that $\Psi \circ \varphi = \mathrm{id}_U$, so by the chain rule,

$$d\Psi_x \circ d\varphi_{\varphi^{-1}(x)} = \mathrm{id}_{\mathbb{R}^k}$$

Then, $d\varphi_{\varphi^{-1}(x)}$ must be an isomorphism $\mathbb{R}^k \to T_xX$, and hence $\dim T_xX = k$. ■

Example *Tangent spaces for our trivial examples*
Returning to the trivial examples we previously gave, we now state the corresponding tangent space for an arbitrary point x on each manifold.

- $X = \mathbb{R}^N$: $T_xX = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W: $T_xX = \mathbb{R}^N$
- $X = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^N$: $T_xX = X$

Example *Tangent spaces for S^n*
From any given chart, we can compute (φ and) $d\varphi$:

$$\frac{\partial \varphi}{\partial x^1} = (1, 0, \cdots, 0, -x_1/x_{n+1})$$

and similarly for $\partial\varphi/\partial x^i$. Manifestly, each partial derivative is perpendicular to x. Thus, $T_xX \subset x^\perp := \{v \in \mathbb{R}^{n+1} \colon \langle v, x\rangle = 0\}$. Since we know from the above lemma that $\dim T_xX = n$, we conclude that $T_xX = x^\perp$.

Definition Differential map for manifolds
Let $f\colon X \to Y$ be a smooth map between manifolds and $x \in X$. Choose a local parametrisation φ_1 around x and φ_2 around $f(x) \in Y$. We define the differential $df_x \colon T_xX \to T_{f(x)}Y$ of f at x by

$$df_x = d{\varphi_2}_{\varphi_2^{-1}(f(x))} \circ d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)} \circ (d{\varphi_1}_{\varphi_1^{-1}(x)})^{-1}$$

Lemma 1.2
df_x is independent of the choice of local parametrisations.

Proof. Trivial exercise. ■

Proposition 1.3 Chain rule for manifolds
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be smooth maps between manifolds. For any $x \in X$,

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Proof. Trivial exercise. ■

Theorem 1.4 Inverse function theorem
Let $f\colon X \to Y$ be a smooth map between manifolds and $x \in X$. Suppose $df_x \colon T_xX \to T_{f(x)}Y$ is an isomorphism. Then f is a local diffeomorphism, *i.e., each $x \in X$ has an open neighbourhood $V \subset X$ such that $f|_V \colon V \to f(V)$ is a diffeomorphism.*

Proof. Since df_x is an isomorphism, it follows that $d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)}$ is also an isomorphism. We can then use the usual inverse function theorem to deduce the result. ■

1.2 Regular values and Sard's theorem

Definition Critical and regular points
Let $f\colon X \to Y$ be a smooth map between manifolds. We say that $x \in X$ is a critical point of f if $df_x \colon T_xX \to T_{f(x)}Y$ is not surjective. Otherwise, it is a regular point.

Notation. We denote by C the set of all critical points of f.

Remark. If $\dim Y > \dim X$, then $C = X$ and the pre-image of any regular value is $\varnothing$.

Definition Critical and regular values
Let $f\colon X \to Y$ be a smooth map between manifolds. We say that $y \in Y$ is a critical value of f if $y = f(x)$ for some $x \in C$. Otherwise, we say that y is a regular value of f.

Theorem 1.5 Pre-image theorem
Let $f\colon X \to Y$ be a smooth map between manifolds. Suppose $y \in Y$ is a regular value of f. If $f^{-1}(y) = \varnothing$, then $f^{-1}(y) \subset X$ is a submanifold of X with $\dim f^{-1}(Y) = \operatorname{codim}_X Y$.

Remark. This theorem can be quite useful in proving that something is a manifold, particularly when we know it is the level set of a certain function (e.g. S^n).