

# Differential Geometry

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## Course schedule

Smooth manifolds in  $\mathbb{R}^n$ , tangent spaces, smooth maps and the inverse function theorem. Examples, regular values, Sard's theorem (statement only). Transverse intersection of submanifolds. [4]

Manifolds with boundary, degree mod 2 of smooth maps, applications. [3]

Curves in 2-space and 3-space, arc-length, curvature, torsion. The isoperimetric inequality. [2]

Smooth surfaces in 3-space, first fundamental form, area. [1]

The Gauss map, second fundamental form, principal curvatures and Gaussian curvature. Theorema Egregium. [3]

Minimal surfaces. Normal variations and characterization of minimal surfaces as critical points of the area functional. Isothermal coordinates and relation with harmonic functions. The Weierstrass representation. Examples. [3]

Parallel transport and geodesics for surfaces in 3-space. Geodesic curvature. [2]

The exponential map and geodesic polar coordinates. The Gauss-Bonnet theorem (including the statement about classification of compact surfaces). [4]

Global theorems on curves: Fenchel's theorem (the total curvature of a simple closed curve is greater than or equal to  $2\pi$ ); the Fary-Milnor theorem (the total curvature of a simple knotted closed curve is greater than  $4\pi$ ). [2]

## Recommended books

J. Milnor *Topology from the differentiable viewpoint*. Princeton University Press, 1997.

M. Do Carmo *Differential Geometry of Curves and Surfaces*. Pearson Higher Education, 1976

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# 1 Differential topology

**Definition** Smooth map on an open subset

Let  $U \subset \mathbb{R}^n$ . We say that  $f: U \rightarrow \mathbb{R}^m$  is smooth if all partial derivatives to all orders exist and are continuous.

**Definition** Smooth map

Let  $X \subset \mathbb{R}^n$ . We say that  $f: X \rightarrow \mathbb{R}^m$  is smooth if, for each  $x \in X$ , there exists (i) an open neighbourhood  $U \subset \mathbb{R}^n$  of  $x$  and (ii) a smooth map  $\tilde{f}: U \rightarrow \mathbb{R}^m$  such that  $\tilde{f}|_{X \cap U} = f|_{X \cap U}$ .

**Definition** Diffeomorphism

Let  $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ . We say that  $f: X \rightarrow Y$  is a diffeomorphism if  $f$  is a smooth bijection with a smooth inverse. If such a map exists, we say that  $X$  and  $Y$  are diffeomorphic.

**Exercise.** Give an example of a smooth bijection that is not a diffeomorphism.

**Definition**  $k$ -dimensional manifold

We say that  $X \subset \mathbb{R}^N$  is a  $k$ -dimensional manifold if, for each  $x \in X$ , there exists an open neighbourhood  $V \subset X$  of  $x$  such that  $V$  is diffeomorphic to an open subset  $U \subset \mathbb{R}^k$ . A diffeomorphism  $\varphi: U \rightarrow V$  is called a local parametrisation of  $V$ , whereas its inverse  $\psi := \varphi^{-1}: V \rightarrow U$  is called a coordinate system or a chart on  $V$ .

**Remarks**

- By composing  $\varphi^{-1}$  with the projections  $\pi_i: \mathbb{R}^k \rightarrow \mathbb{R}, (x_1, \dots, x_k) \mapsto x_i$ , we get smooth maps  $x_i := \pi_i \circ \varphi^{-1}$  which we call *coordinate functions*.
- WLOG, we can replace ‘diffeomorphic to an open subset  $U \subset \mathbb{R}^k$ ’ with ‘diffeomorphic to an open ball in  $\mathbb{R}^k$ ’.
- It is easy to see that, if  $X \subset \mathbb{R}^N$  is both a  $k$ -dimensional manifold and a  $\tilde{k}$ -dimensional manifold, then  $k = \tilde{k}$ .

**Definition** Dimension

Let  $X \subset \mathbb{R}^N$  be a  $k$ -dimensional manifold. The dimension of  $X$  is  $k$ , and it is denoted by  $\dim X$ .

**Example** Some trivial examples

- $X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$  for any open subset  $W$
- $X = \{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^N$

**Example**  $S^n$

$S^n := \{x \in \mathbb{R}^{n+1}: \|x\|_2 = 1\}$  is an  $n$ -dimensional manifold. To see this, consider the projection  $\Pi_k: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1})$ . It is easy to verify that maps of the form  $\psi_k^\pm = \Pi_k|_{S^n \cap \{\text{sign}(x_k) = \pm 1\}}$  are diffeomorphisms  $S^n \cap \{\text{sign}(x_k) = \pm 1\} \rightarrow B_1(0)$ .

**Remark.** It is easy to show that  $X$  is a 0-dimensional manifold iff  $X$  is a discrete subset of  $\mathbb{R}^N$ .

**Exercise.** Show that, if  $X$  and  $Y$  are manifolds, then  $X \times Y$  is also a manifold, with  $\dim X \times Y = \dim X + \dim Y$ .

**Definition** Submanifold

Let  $X, Y \subset \mathbb{R}^N$  be manifolds. If  $Y \subset X$ , then we say that  $Y$  is a submanifold of  $X$ . The codimension of  $Y$  in  $X$  is defined as

$$\text{codim}_X Y := \dim X - \dim Y$$

## 1.1 Tangent spaces

We first recall some basic facts from our youth. Let  $U \subset \mathbb{R}^k$  be open. The *differential* of a smooth map  $f: U \rightarrow \mathbb{R}^m$  at  $x \in U$  is defined by

$$df_x: \mathbb{R}^k \rightarrow \mathbb{R}^m$$

$$h \mapsto \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t}$$

This is a linear map, with matrix representation

$$df_x = \left( \frac{\partial f^i}{\partial x^j} \right)_{i,j}$$

Moreover, differentials satisfy the chain rule: given (i) two smooth maps  $f: U \rightarrow \mathbb{R}^l$  and  $g: V \rightarrow \mathbb{R}^m$  with  $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$  open and (ii) a point  $x \in U$  with  $f(x) \in V$ , we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

**Definition** Tangent space

Let  $X \subset \mathbb{R}^N$  be a  $k$ -dimensional manifold and  $x \in X$ . Choose a local parametrisation  $\varphi: U \rightarrow V$  around  $x$ . We then define the tangent space  $T_x X$  of  $X$  at  $x$  to be

$$T_x X := \text{im } d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k)$$

Of course, before we can safely proceed, we must show that  $T_x X$  is well-defined:

**Lemma 1.1**

Let  $X$  be as above.  $T_x X$  is independent of  $\varphi$ , and  $\dim T_x X = k$ .

*Proof.* Let  $\tilde{\varphi}: \tilde{U} \rightarrow \tilde{V}$  be another local parametrisation near  $x$ . WLOG, by restricting if necessary, we may assume  $\tilde{V} = V$ . By the chain rule, we have

$$d\varphi_{\varphi^{-1}(x)} = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)} \circ d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}$$

Since  $\tilde{\varphi}^{-1} \circ \varphi$  is a diffeomorphism of open subsets of  $\mathbb{R}^n$ , the corresponding differential  $d(\tilde{\varphi}^{-1} \circ \varphi)$  is a linear isomorphism. Thus,

$$d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}(\mathbb{R}^k)) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(\mathbb{R}^k)$$

as claimed.

Now, it remains to show that  $\dim T_x X = k$ . By definition, there exists an open set  $\hat{V} \subset \mathbb{R}^N$  and a smooth map  $\Psi: \hat{V} \rightarrow \mathbb{R}^k$  that extends the chart  $\psi := \varphi^{-1}$ . Note that  $\Psi \circ \varphi = \text{id}_U$ , so by the chain rule,

$$d\Psi_x \circ d\varphi_{\varphi^{-1}(x)} = \text{id}_{\mathbb{R}^k}$$

Then,  $d\varphi_{\varphi^{-1}(x)}$  must be an isomorphism  $\mathbb{R}^k \rightarrow T_x X$ , and hence  $\dim T_x X = k$ . ■

**Example** Tangent spaces for our trivial examples

Returning to the trivial examples we previously gave, we now state the corresponding tangent space for an arbitrary point  $x$  on each manifold.

- $X = \mathbb{R}^N$ :  $T_x X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$  for any open subset  $W$ :  $T_x X = \mathbb{R}^N$
- $X = \{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^N$ :  $T_x X = X$

**Example** Tangent spaces for  $S^n$ 

From any given chart, we can compute ( $\varphi$  and)  $d\varphi$ :

$$\frac{\partial \varphi}{\partial x^1} = (1, 0, \dots, 0, -x_1/x_{n+1})$$

and similarly for  $\partial \varphi / \partial x^i$ . Manifestly, each partial derivative is perpendicular to  $x$ . Thus,  $T_x X \subset x^\perp := \{v \in \mathbb{R}^{n+1} : \langle v, x \rangle = 0\}$ . Since we know from the above lemma that  $\dim T_x X = n$ , we conclude that  $T_x X = x^\perp$ .

**Definition** Differential map for manifolds

Let  $f: X \rightarrow Y$  be a smooth map between manifolds and  $x \in X$ . Choose a local parametrisation  $\varphi_1$  around  $x$  and  $\varphi_2$  around  $f(x) \in Y$ . We define the differential  $df_x: T_x X \rightarrow T_{f(x)} Y$  of  $f$  at  $x$  by

$$df_x = d\varphi_{2, \varphi_2^{-1}(f(x))} \circ d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)} \circ (d\varphi_{1, \varphi_1^{-1}(x)})^{-1}$$

**Lemma 1.2**

$df_x$  is independent of the choice of local parametrisations.

*Proof.* Trivial exercise. ■

**Proposition 1.3** Chain rule for manifolds

Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  be smooth maps between manifolds. For any  $x \in X$ ,

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

*Proof.* Trivial exercise. ■

**Theorem 1.4** Inverse function theorem

Let  $f: X \rightarrow Y$  be a smooth map between manifolds and  $x \in X$ . Suppose  $df_x: T_x X \rightarrow T_{f(x)} Y$  is an isomorphism. Then  $f$  is a local diffeomorphism, i.e., each  $x \in X$  has an open neighbourhood  $V \subset X$  such that  $f|_V: V \rightarrow f(V)$  is a diffeomorphism.

*Proof.* Since  $df_x$  is an isomorphism, it follows that  $d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)}$  is also an isomorphism. We can then use the usual inverse function theorem to deduce the result. ■

**1.2 Regular values and Sard's theorem****Definition** Critical and regular points

Let  $f: X \rightarrow Y$  be a smooth map between manifolds. We say that  $x \in X$  is a critical point of  $f$  if  $df_x: T_x X \rightarrow T_{f(x)} Y$  is not surjective. Otherwise, it is a regular point.

**Notation.** We denote by  $C$  the set of all critical points of  $f$ .

**Remark.** If  $\dim Y > \dim X$ , then  $C = X$  and the pre-image of any regular value is  $\emptyset$ .

**Definition** Critical and regular values

Let  $f: X \rightarrow Y$  be a smooth map between manifolds. We say that  $y \in Y$  is a critical value of  $f$  if  $y = f(x)$  for some  $x \in C$ . Otherwise, we say that  $y$  is a regular value of  $f$ .

**Theorem 1.5** Pre-image theorem

Let  $f: X \rightarrow Y$  be a smooth map between manifolds. Suppose  $y \in Y$  is a regular value of  $f$ . If  $f^{-1}(y) \neq \emptyset$ , then  $f^{-1}(y) \subset X$  is a submanifold of  $X$  with  $\text{codim}_X f^{-1}(y) = \dim Y$ .

*Proof.* Fix  $x \in f^{-1}(y)$ . Since  $y$  is a regular value, we know that  $df_x: T_x X \rightarrow T_y Y$  is surjective. By the rank-nullity theorem,  $\dim \ker df_x = \text{codim}_X Y$ . Suppose  $X \subset \mathbb{R}^N$ , and pick a linear map  $T: \mathbb{R}^N \rightarrow \mathbb{R}^{\text{codim}_X Y}$  such that  $\ker T \cap \ker df_x = \{0\}$ .<sup>1</sup>

Now, extend  $f$  to the map  $F: X \rightarrow Y \times \mathbb{R}^{\text{codim}_X Y}$  given by  $z \mapsto (f(z), T(z))$ . Note that the differential of  $F$  at  $x$  is given by

$$dF_x = (df_x, dT_x) = (df_x, T)$$

Since  $\ker T \cap \ker df_x = \{0\}$ , we have  $\ker dF_x = \{0\}$ , i.e.,  $dF_x$  is injective. By the inverse function theorem for manifolds, there exists an open neighbourhood  $U \subset X$  of  $x$  such that  $F|_U: U \rightarrow V$  is a diffeomorphism to an open neighbourhood  $V$  of  $(y, T(x))$ . Hence,  $F|_{f^{-1}(y) \cap U}$  is a local parametrisation of  $(\{y\} \times \mathbb{R}^{\text{codim}_X Y}) \cap V$ , proving that  $f^{-1}(y)$  is a manifold of dimension  $\text{codim}_X Y$ . ■

**Exercise.** Show that, under the conditions of the pre-image theorem,  $T_x f^{-1}(y) = \ker df_x$ .

### Corollary 1.6

Let  $f: X \rightarrow Y$  be a smooth map between manifolds of the same dimension, with  $X$  compact. If  $y$  is a regular value of  $f$ , then  $f^{-1}(y)$  is finite.

*Proof.* By the pre-image theorem,  $f^{-1}(y)$  is a 0-dimensional manifold, i.e., a collection of points. Since  $X$  is compact, such a collection must be finite. ■

With just a bit more analysis and topology, we can actually say more than just finiteness:

### Theorem 1.7 Stack of records theorem

Let  $f: X \rightarrow Y$  be a smooth map between manifolds of the same dimension, with  $X$  compact. Let  $y$  be a regular value of  $f$ , and list the elements of  $f^{-1}(y)$  as  $x_1, \dots, x_n$ . There exists an open neighbourhood  $V \subset Y$  of  $y$  and a collection of open neighbourhoods  $W_i \subset X$  of each  $x_i$  such that

$$f^{-1}(V) = \bigsqcup_{i=1}^n W_i$$

and each  $f|_{W_i}: W_i \rightarrow V$  is a diffeomorphism.

*Proof.* By the inverse function theorem for manifolds, we can pick open neighbourhoods  $W_i$  of  $x_i$  such that each  $f|_{W_i}$  is a diffeomorphism to an open neighbourhood of  $y$ . By shrinking neighbourhoods if necessary,  $W_i$  can be taken WLOG to be pairwise disjoint. Now, set

$$V = \left[ \bigcap_{i=1}^n f(W_i) \right] \setminus f \left( X \setminus \bigcup_{i=1}^n W_i \right)$$

Note that  $f(X \setminus \bigcup_{i=1}^n W_i)$  is a compact set that does not contain  $y$ , so  $V$  is an open neighbourhood of  $y$ . Finally, note that  $f^{-1}(V) = \bigsqcup_{i=1}^n W_i$  by construction. ■

Now, the pre-image theorem can be a powerful tool for generating manifolds or showing that a certain set is one.

### Application $S^n$ is a manifold

Consider the map  $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}, (x_1, \dots, x_{n+1}) \mapsto x_1^2 + \dots + x_{n+1}^2$ . Note that  $f^{-1}(1) = S^n$ , so to show that  $S^n$  is a manifold, it suffices to show that 1 is a regular point. Indeed, note that  $df_x = (2x_1, \dots, 2x_{n+1})$ , which is not surjective only if  $x = 0 \notin f^{-1}(1)$ .

<sup>1</sup>It is easy to constructively show using IB Linear Algebra that such a map exists. [Exercise!]

**Application** *Orthogonal group as a manifold*

Denote by  $M(n)$  [resp.  $S(n)$ ] the space of all [resp. symmetric]  $n \times n$  matrices with entries in  $\mathbb{R}$ . Consider the orthogonal group  $O(n) = \{A \in M(n) : AA^t = I\} \subset M(n) = \mathbb{R}^{n^2}$ .

Let  $f: M(n) \rightarrow O(n)$  be the map  $A \mapsto AA^t$ . This is smooth since multiplication and addition in  $\mathbb{R}$  are smooth. Since  $O(n) = f^{-1}(I)$ , it suffices to show that  $I$  is a regular value of  $f$ . Note that

$$df_A(H) = \lim_{t \rightarrow 0} \frac{f(A + tH) - f(A)}{t} = AH^t + HA^t$$

Now, fix  $A \in M(n)$ . Given  $B \in S(n)$ , observe that

$$df_A\left(\frac{1}{2}CA\right) = \frac{1}{2}AA^tC^t + \frac{1}{2}CAA^t = \frac{1}{2}C + \frac{1}{2}C = C$$

completing the proof that  $I$  is a regular value of  $f$ .

**Remark.** Recall that, besides being a manifold as we've just shown,  $O(n)$  is also a group. In fact, the group operations  $(A, B) \mapsto AB$  and  $A \mapsto A^{-1} = A^t$  are smooth. Hence, we see that  $O(n)$  is a *Lie group*.

Now, the pre-image theorem raises the question: how easy is to find regular values? This leads us to Sard's theorem.

**Definition** *Measure-zero subsets of  $\mathbb{R}^N$*

We say that  $S \subset \mathbb{R}^N$  is of measure zero in  $\mathbb{R}^N$  if, for each  $\varepsilon > 0$ , there exists a countable family  $\{R_i\}$  of sets of the form  $R_i = \prod_{j=1}^N [x_i^{(j)}, y_i^{(j)}]$  such that  $S \subset \bigcup_i R_i$  and  $\sum_i \text{vol}(R_i) < \varepsilon$ .

**Definition** *Measure zero subsets of manifolds*

Let  $X \subset \mathbb{R}^N$  be a  $k$ -dimensional manifold. We say that  $A \subset X$  is of measure zero in  $X$  if, for all local parametrisations  $\varphi: U \rightarrow V$  of  $X$ ,  $\varepsilon^{-1}(V \cap A) \subset \mathbb{R}^k$  has measure zero in  $\mathbb{R}^k$ .

**Exercise.** Let  $U, \tilde{U} \subset \mathbb{R}^k$  be open and  $\psi: U \rightarrow \tilde{U}$  a diffeomorphism. Show that, if  $A \subset U$  is of measure zero in  $\mathbb{R}^k$ , then  $\tilde{T} = \psi(T)$  is of measure zero in  $\mathbb{R}^k$ .

**Remarks**

- In view of the above exercise,  $A \subset X$  is of measure zero in  $X$  iff  $\varphi_i^{-1}(S \cap V_i)$  is of measure zero for all  $\varphi_i: U \rightarrow V$  in an atlas of local parametrisations.
- If  $\dim Y = 0$ , then  $Y$  is of measure zero. If  $\dim Y > 0$ , then every non-empty open subset  $V \subset Y$  is not of measure zero in  $Y$ .
- If  $S \subset X$  is of measure zero in  $X$ , then any  $\tilde{S} \subset S$  is also of measure zero in  $X$ .

**Theorem 1.8** *Sard's theorem*

Let  $f: X \rightarrow Y$  be a smooth map between manifolds. Then the set of critical values of  $f$  is of measure zero in  $Y$ .

*Proof.* Non-examinable — see Milnor's book if interested. ■

**Corollary 1.9**

The set of regular values of a smooth map  $f: X \rightarrow Y$  between manifolds is dense in  $Y$ .

*Proof.* Any open set  $V \subset Y$  cannot lie entirely in  $f(C)$  since it has measure zero. ■

### 1.3 Transversality

**Definition** Transversal

Let  $f: X \rightarrow Y$  be smooth and  $Z \subset Y$  a submanifold of  $Y$ . We say that  $f$  is transversal to  $Z$  if, for each  $x \in f^{-1}(Z)$ ,

$$T_{f(x)}Y = T_{f(x)}Z + \text{im } df_x$$

We then write  $f \pitchfork Z$ .

**Remarks**

- If  $f(X) \cap Z = \emptyset$ , then  $f$  is transversal to  $Z$ .
- If  $Z = \{y\}$ , then  $f$  is transversal to  $Z$  iff  $y$  is a regular value of  $f$ . Thus, transversality is really a generalisation of the notion of regular values.

**Exercise.** Let  $X$  also be a submanifold of  $Y$  and  $\iota: X \hookrightarrow Y$  the inclusion map. Show that  $d\iota_x$  is just the inclusion map  $T_xX \hookrightarrow T_xY$  of the tangent spaces. Thus,  $\iota \pitchfork Z$  iff  $T_xX + T_xZ = T_xY$  for all  $x \in X \cap Z$ .

Now, we state a generalisation of the pre-image theorem for transversal maps:

**Theorem 1.10**

Let  $f: X \rightarrow Y$  be smooth map that is transversal to a submanifold  $Z \subset Y$  of  $Y$ . If  $f^{-1}(Z) \neq \emptyset$ , then  $f^{-1}(Z) \subset X$  is a submanifold of  $X$ , with  $\text{codim}_X f^{-1}(Z) = \text{codim}_Y Z$ .

**Remark.** If  $Z = \{y\}$ , then  $\text{codim}_Y Z = \dim Y$  as in the pre-image theorem.

*Sketch of proof (non-examinable).* Fix  $z \in Z$  with  $z = f(x)$  for some  $x \in X$ . Note that, for some open neighbourhood  $V \subset Y$  of  $z$ , there exists a smooth map  $h: V \rightarrow \mathbb{R}^{\text{codim}_Y Z}$  such that  $Z \cap V = h^{-1}(0)$  and  $dh_z$  is surjective. Locally around  $x \in X$ ,  $f^{-1}(Z) = (h \circ f)^{-1}(0)$ . Thus, by the pre-image theorem, it suffices to show that 0 is a regular value of  $h \circ f$ .

Now, since  $f \pitchfork Z$ , we have  $T_zY = T_zZ + \text{im } df_x$ . By the exercise after the pre-image theorem, we have  $dh_z = T_zZ$ . Moreover,  $f \pitchfork Z$  gives us

$$T_zY = T_zZ + \text{im } df_x = \ker dh_z + \text{im } df_x$$

This then implies that  $\text{im } dh_z = \text{im}(dh_z \circ df_x) = \text{im } d(h \circ f)_x$ . Since  $dh_z$  is surjective,  $d(h \circ f)_z$  is also surjective and hence 0 is a regular value of  $h \circ f$ . ■

**Exercise.** Construct the required map  $h$ .

**Remark.** Transversality is both a stable and generic property. It is stable in the sense that small perturbations of  $f$  remain transversal to a given submanifold. It is generic in the sense that any given smooth map may be deformed by arbitrarily small amounts into a map that is transversal to  $Z$ . See *Differential Topology* by Guillemin and Pollack for more details.

### 1.4 Manifolds with boundary

Consider the closed upper half plane

$$\mathbb{H}^k := \{(x_1, \dots, x_k) \in \mathbb{R}^n : x_k \geq 0\}$$

We denote its boundary by  $\partial\mathbb{H}^k = \{x_k = 0\}$ .

**Definition** Manifold with boundary

We say that  $X \subset \mathbb{R}^N$  is a (smooth)  $k$ -dimensional manifold with boundary if every  $x \in X$  has an open neighbourhood  $V \subset X$  that is diffeomorphic to an open subset  $U \subset \mathbb{H}^k$ .

**Remark.** Note that a diffeomorphism  $\varphi: U \rightarrow V$  has a smooth extension defined on an open subset of  $\mathbb{R}^k$ . This allows us to deduce as before that, if  $X$  is both  $k$ - and  $\tilde{k}$ -dimensional, then  $k = \tilde{k}$ .

**Definition** Dimension

The dimension of a  $k$ -dimensional manifold with boundary  $X$  is  $k$ .

**Definition** Boundary of a manifold with boundary

Let  $X$  be a  $k$ -dimensional manifold with boundary. Its boundary is defined to be

$$\partial X := \{x \in X : \exists \text{ open nhod } V \subset X \text{ and diffeomorphism } \psi: V \rightarrow \psi(V) \text{ s.t. } x \in \psi^{-1}(\partial \mathbb{H}^k)\}$$

**Remarks**

- In fact, if  $x \in \psi^{-1}(\partial \mathbb{H}^k)$  for some diffeomorphism  $\psi: V \rightarrow \psi(V) \subset \mathbb{H}^k$  on an open nhod  $V$  of  $x$ , then it is true for *all* diffeomorphisms on an open nhod of  $x$  to its image in  $\mathbb{H}^k$ .
- In the definition of manifold with boundary, we may take  $U = \mathbb{R}^k$  or  $U = \mathbb{H}^k$  WLOG.

**Exercise.** Prove the first remark.

**Definition** Interior of manifold with boundary

Let  $X$  be a manifold with boundary. We define its interior to be

$$\text{int } X := X \setminus \partial X$$

**Proposition 1.11**

Let  $X$  be a  $k$ -dimensional manifold with boundary. Then  $\text{int } X$  is a manifold of dimension  $k$  and  $\partial X$  is a manifold of dimension  $k - 1$ .

*Proof.*  $\text{int } X$  is a manifold of dimension  $k$  because we can always restrict our diffeomorphisms  $\varphi: V \rightarrow U$  such that  $U \cap \partial \mathbb{H}^k = \emptyset$ . See Example Sheet 1 for  $\partial X$ . ■

**Example**

- Trivially,  $\mathbb{H}^k$  is a  $k$ -dimensional manifold with boundary
- As we will prove later,  $B^n := \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$  is an  $n$ -dimensional manifold with boundary. Note that  $\partial B^n = S^{n-1}$  is a manifold of dimension  $n - 1$ .
- $[0, 1] \times [0, 1]$  is not a manifold with boundary (see Example Sheet 1)
- If  $X$  is a manifold with boundary and  $Y$  is a manifold, then  $X \times Y$  is a manifold with boundary, with  $\partial(X \times Y) = (\partial X) \times Y$ . (Of course, the previous example is a counterexample to the case that  $Y$  is also a manifold with boundary.)

**Remark.** Note that  $\partial X$  and  $\text{int } X$  are not the same as the topological notions of ‘boundary’ and ‘interior’ as subsets of  $\mathbb{R}^N$ . Indeed, if  $\dim X < N$ , the topological interior of  $X$  is empty, whereas  $\text{int } X$  is not.

**Definition** Tangent space

Let  $X$  be a  $k$ -dimensional manifold with boundary and  $x \in X$ . Let  $\varphi: U \rightarrow V$  be a diffeomorphism from an open set  $U \subset \mathbb{H}^k$  to an open neighbourhood  $V \subset X$  of  $x$ . Since  $\varphi$  is smooth, there exists

a smooth extension  $\tilde{\varphi}$  on an open subset of  $\mathbb{R}^k$ , with  $d\tilde{\varphi}_{\varphi^{-1}(x)}$  well-defined. We then define the tangent space to be

$$T_x X := \text{im } d\tilde{\varphi}_{\varphi^{-1}(x)}$$

**Remark.** As before,  $T_x X$  is well-defined.

**Exercise.** Show that, for every  $x \in \partial X$ ,  $T_x \partial X \subset T_x X$ .

**Lemma 1.12**

Let  $X$  be a manifold of dimension  $k$ . Let  $f: X \rightarrow \mathbb{R}$  be smooth, with  $0$  a regular value of  $f$ . Then  $f^{-1}([0, \infty)) \subset X$  is a  $k$ -dimensional manifold with boundary  $\partial(f^{-1}([0, \infty))) = f^{-1}(0)$ .

*Proof.* The subset  $f^{-1}((0, \infty)) \subset X$  is open in  $X$  and thus a submanifold of  $X$ . This means that we can restrict a local parametrisation to  $X$  such that its image lies in  $f^{-1}((0, \infty))$  and get the diffeomorphism we need.

It remains to consider  $x \in f^{-1}(0)$ . Extend  $f$  to a map  $\mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}^{k-1}$  as in the proof of the pre-image theorem (Theorem 1.5). We can then proceed as before using the inverse function theorem. ■

**Corollary 1.13**

$B^n$  is an  $n$ -dimensional manifold with boundary.

*Proof.* This is immediate from the above lemma. ■

**Theorem 1.14** Pre-image theorem for manifolds with boundary

Let  $X$  be a manifold with boundary and  $Y$  a manifold, with  $\dim X > \dim Y$ . Suppose  $f: X \rightarrow Y$  is smooth and  $y \in Y$  is a regular value of both  $f$  and  $f|_{\partial X}$ . Then  $f^{-1}(y) \subset X$  is a manifold with boundary, with  $\text{codim}_X f^{-1}(y) = \dim Y$  and  $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial X$ .

*Proof.* WLOG, we may assume that  $X = \mathbb{H}^m$  and  $Y = \mathbb{R}^n$  since we are always working locally. The easy case  $x \in f^{-1}(y) \cap \text{int } \partial \mathbb{H}^m$  is left as an exercise. Now, suppose  $x \in f^{-1}(y) \cap \partial \mathbb{H}^m$ . Then there exists an open subset  $U \subset \mathbb{R}^m$  such that  $f|_{U \cap \mathbb{H}^m}$  extends to a smooth map  $F: U \rightarrow \mathbb{R}^n$ . Since  $y$  is a regular value of  $f|_{U \cap \mathbb{H}^m}$ ,  $dF_x$  is surjective. Since the map  $z \mapsto dF_z$  (defined on  $U$ ) is smooth, we can shrink  $U$  such that  $dF_z$  is surjective for all  $z \in U$ .<sup>2</sup> Applying the pre-image theorem to  $F$ , we have that  $F^{-1}(y)$  is a submanifold of  $U$  with  $\text{codim}_{\mathbb{R}^m} F^{-1}(y) = \dim Y$ . Let  $\pi: F^{-1}(y) \rightarrow \mathbb{R}, (x_1, \dots, x_m) \mapsto x_m$ . Note that

$$(f|_{U \cap \mathbb{H}^m})^{-1}(y) = \pi^{-1}([0, \infty))$$

It then suffices to show that  $0$  is a regular value of  $\pi$  since, by the previous lemma, it would follow that  $\pi^{-1}([0, \infty))$  is a submanifold of  $F^{-1}(y)$  with boundary  $\pi^{-1}(0) = F^{-1}(y) \cap \partial \mathbb{H}^m = f^{-1}(y) \cap U \cap \partial \mathbb{H}^m$ .

Now, to show that, for any  $z \in \pi^{-1}(0)$ , the map  $d\pi_z: T_z F^{-1}(y) \rightarrow \mathbb{R}$  is surjective, it suffices to show that  $T_z F^{-1}(y) = \ker dF_z = \ker df_z \not\subset \ker d\pi_z = \mathbb{R}^{m-1} \times \{0\} = T_z \partial \mathbb{H}^m$ . Indeed, note that

$$df_z|_{T_z \partial \mathbb{H}^m} = d(f|_{\partial \mathbb{H}^m})_z$$

is surjective. If  $\ker df_z \subset T_z \partial \mathbb{H}^m$ , then  $\ker(df_z|_{T_z \partial \mathbb{H}^m}) = \ker(df_z)$ , but these have different dimensions by the rank-nullity theorem — a contradiction! ■

<sup>2</sup>Indeed, we know that some submatrix of  $dF_x$  has nonzero determinant. By continuity of  $\det$ , there is some open neighbourhood  $\tilde{U} \subset U$  of  $x$  on which the determinant of that submatrix remains nonzero and thus  $\dim \text{im } dF_z = n$  for all  $z \in \tilde{U}$ .

**Theorem 1.15**

Let  $X$  be a manifold with boundary and  $Y$  a manifold with  $Z \subset Y$  a submanifold. Let  $f: X \rightarrow Y$  be smooth such that  $f \pitchfork Z$  and  $f|_{\partial X} \pitchfork Z$ . Then  $f^{-1}(Z) \subset X$  is a manifold with boundary, with  $\text{codim}_X f^{-1}(Z) = \text{codim}_Y Z$  and  $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$ .

## 1.5 Degree modulo 2

**Definition** Smooth homotopy

Let  $f, g: X \rightarrow Y$  be smooth maps between manifolds. A smooth homotopy between  $f$  and  $g$  is a smooth map  $F: X \times [0, 1] \rightarrow Y$  such that  $F|_{X \times \{0\}} = f$  and  $F|_{X \times \{1\}} = g$ . If such a map exists, we say that  $f$  and  $g$  are smoothly homotopic and write  $f_0 \simeq f_1$ .

**Exercise.** Show that  $\simeq$  is an equivalence relation (cf. Example Sheet 1 Q14).

**Definition** Smooth isotopy

Let  $f, g: X \rightarrow Y$  be diffeomorphisms. A smooth isotopy between  $f$  and  $g$  is a smooth homotopy  $F: X \times [0, 1] \rightarrow Y$  for which  $F|_{X \times \{t\}}$  is a diffeomorphism for all  $t \in [0, 1]$ . If such a map exists, we say that  $f$  and  $g$  are smoothly isotopic.

**Lemma 1.16** Homotopy lemma

Suppose  $f, g: X \rightarrow Y$  are smoothly homotopic, with  $X$  compact and  $\dim X = \dim Y$ . If  $y$  is a regular value of both  $f$  and  $g$ , then

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod{2}$$

*Proof.* Let  $F: X \times [0, 1] \rightarrow Y$  be a smooth homotopy between  $f$  and  $g$ . We first suppose that  $y$  is also a regular value of  $F$ . By Theorem 1.14,  $F^{-1}(y)$  is a 1-dimensional manifold with boundary  $\partial F^{-1}(y) = F^{-1}(y) \cap (X \times \{0, 1\}) = f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}$ . Thus,  $\#\partial F^{-1}(y) = \#f^{-1}(y) + \#g^{-1}(y)$ . We then proceed by noting the following result:

► **Theorem.** Let  $Z$  be a compact 1-dimensional manifold with boundary. Then  $Z$  is diffeomorphic to a disjoint union of finitely many copies of  $[0, 1]$  and of  $S^1$ .

In particular, it follows from the above that  $\#\partial Z \equiv 0 \pmod{2}$ . Hence, we conclude that  $\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod{2}$ .

Now, suppose  $y$  is *not* a regular value of  $F$ . By Sard's theorem, the set of regular values of  $f$  (resp.  $g$ ) is dense in  $Y$ . Thus, every open set in  $Y$  contains a shared regular value of  $f, g, F$ . Then, by the stack of records theorem, we can pick an open set  $V \subset Y$  such that  $f^{-1}(V)$  and  $g^{-1}(V)$  are both disjoint union of open sets on which  $f$  and  $g$ , respectively, are diffeomorphisms to their images. In particular,  $z \mapsto \#f^{-1}(z)$  and  $z \mapsto \#g^{-1}(z)$  are both constant in  $V$ . Since our previous argument holds for the common regular value of  $f$  and  $g$  in  $V$ , it follows that  $\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod{2}$ . ■

**Exercise.** Prove the theorem used in the proof of the homotopy lemma (Lemma 1.16).

**Exercise.** Prove the claim in the proof that  $f$  and  $g$  have a shared regular value in every open subset of  $Y$ .

**Lemma 1.17** Homogeneity lemma

Let  $X$  be a smooth connected manifold and  $y, z \in X$ . Then there exists a diffeomorphism  $h: X \rightarrow X$  smoothly isotopic to  $\text{id}_X$  and  $h(y) = z$ .

**Remark.** As suggested by the name, this lemma says that each point of  $X$  is essentially the same.

*Proof sketch.* It suffices to show that each  $y \in X$  has an open neighbourhood  $V \subset X$  such that, for every  $z \in V$ , there exists a diffeomorphism  $h$  with the required properties. Since  $X$  locally (i.e. in a neighbourhood of  $y$ ) is diffeomorphic to  $\mathbb{R}^k$ , it suffices to show that there exists  $F: \mathbb{R}^k \rightarrow \mathbb{R}^k$  smoothly isotopic to  $\text{id}_{\mathbb{R}^k}$ , sends  $0 \mapsto z$ , and  $F|_{\mathbb{R}^k \setminus B_1(0)} = \text{id}_{\mathbb{R}^k \setminus B_1(0)}$ . To do this, we pick a smooth vector field  $V: \mathbb{R}^k \rightarrow \mathbb{R}^k$  such that  $V(tz) = z$  for all  $t \in [0, 1]$  and  $V = 0$  outside  $B_1(0)$ . By standard ODE theory, there exists a solution  $\gamma_x: \mathbb{R} \rightarrow \mathbb{R}^k$  to the ODE

$$\begin{cases} \dot{\gamma}_x(t) = V(\gamma(t)) \\ \gamma_x(0) = x \end{cases}$$

By defining the flow map  $\Phi_t: \mathbb{R}^k \rightarrow \mathbb{R}^k, x \mapsto \gamma_x(t)$  for each  $x \in \mathbb{R}^k$ , we can then construct the smooth map

$$\begin{aligned} \Phi: \mathbb{R}^k \times \mathbb{R} &\rightarrow \mathbb{R}^k \\ (x, t) &\mapsto \Phi_t(x) \end{aligned}$$

which we can restrict to  $\mathbb{R}^k \times [0, 1]$  to obtain the required map  $F$ . ■

### Theorem 1.18

Let  $f: X \rightarrow Y$  be a smooth map between manifolds, with  $X$  compact,  $Y$  connected and  $\dim X = \dim Y$ . If  $y, z$  are regular values of  $f$ , then

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \pmod{2}$$

*Proof.* By the homogeneity lemma (Lemma 1.17), there exists a smooth isotopy  $H: X \times [0, 1] \rightarrow X$  such that  $H|_{X \times \{0\}} = \text{id}$  and  $H|_{X \times \{1\}}$  is a diffeomorphism  $X \rightarrow X$  with  $h(y) = z$ . Observe that  $H \circ (f, \text{id}_{[0,1]})$  defines a smooth homotopy between  $f$  and  $h \circ f$ . Note also that  $z$  is a regular value of  $h \circ f$  since  $d(h \circ f) = dh \circ df$ ,  $dh$  is an isomorphism and  $y = h^{-1}(z)$  is a regular value of  $f$ . By the homotopy lemma (Lemma 1.16), we then have

$$\#f^{-1}(y) \equiv \#(h \circ f)^{-1}(z) \equiv \#f^{-1}(z) \pmod{2}$$

as required. ■

With the above theorem established, we can define the following:

### Definition Degree mod 2

For a smooth map  $f: X \rightarrow Y$  from a compact manifold to a connected manifold of the same dimension, the degree mod 2 of  $f$  is defined to be

$$\deg_2(f) := \#f^{-1}(y) \pmod{2}$$

for any (equivalently, all) regular value of  $f$ .

**Remark.** The theorem, together with the homotopy lemma, implies that  $\deg_2(f)$  is a homotopy invariant.

### Lemma 1.19

Let  $X$  be a compact, connected manifold with  $\dim X \geq 1$ . The identity map  $\text{id}_X$  cannot be smoothly homotopic to a constant map on  $X$ .

*Proof.* Suppose, on the contrary, that identity  $\text{id}_X$  and the constant map  $c_{x_0}: X \rightarrow X, x \mapsto x_0$  for some fixed  $x_0 \in X$ . Note that  $\deg_2(\text{id}_X) = 1$  since every  $x \in X$  is a regular value of  $\text{id}_X$  and has exactly one pre-image under  $\text{id}_X$ . On the other hand,  $\deg_2(c_{x_0}) = 0$  since we can simply pick  $x \in X \setminus \{x_0\}$  (this set is non-empty since  $\dim X \geq 1$  which is a regular value with no pre-image under  $c_{x_0}$ ). However, this contradicts the fact that degree mod 2 is a homotopy invariant. ■

**Corollary 1.20**

*There does not exist a smooth retraction of  $B^{n+1}$  onto its boundary, i.e. there is no smooth map  $f: B^{n+1} \rightarrow \partial B^{n+1}$  such that  $f|_{\partial B^{n+1}} = \text{id}$ .*

*Proof.* Suppose, on the contrary, that there exists such a map  $f$ . Define

$$\begin{aligned} F: S^n \times [0, 1] &\rightarrow S^n \\ (x, t) &\mapsto f(xt) \end{aligned}$$

This is smooth since  $f$  is smooth. Moreover, we have  $F|_{S^n \times \{0\}}$  is the constant map  $S^n \rightarrow S^n, x \mapsto f(0)$ , whereas  $F|_{S^n \times \{1\}} = \text{id}_{S^n}$ . However, this contradicts Lemma 1.19. ■

**Theorem 1.21** Brouwer's fixed point theorem

*Every smooth map  $f: B^n \rightarrow B^n$  has a fixed point.*