University of Cambridge Mathematical Tripos Part II

Linear Analysis

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Course schedule

Normed and Banach spaces. Linear mappings, continuity, boundedness, and norms. Finite-dimensional normed spaces. [4]

The Baire category theorem. The principle of uniform boundedness, the closed graph theorem and the inversion theorem; other applications. [5]

The normality of compact Hausdorff spaces. Urysohn's lemma and Tiezte's extension theorem. Spaces of continuous functions. The Stone–Weierstrass theorem and applications. Equicontinuity: the Ascoli–Arzelá theorem. [5]

Inner product spaces and Hilbert spaces; examples and elementary properties. Orthonormal systems, and the orthogonalization process. Bessel's inequality, the Parseval equation, and the Riesz–Fischer theorem. Duality; the self duality of Hilbert space. [5]

Bounded linear operations, invariant subspaces, eigenvectors; the spectrum and resolvent set. Compact operators on Hilbert space; discreteness of spectrum. Spectral theorem for compact Hermitian operators. [5]

Recommended books

B. Bollobas *Linear Analysis.* Cambridge University Press 1999.
G.J.O. Jameson *Topology and Normed Spaces.* Chapman and Hall 1974.
G. Allan *Introduction to Banach Spaces and Algebras.* Oxford University Press 2010.
W. Rudin *Real and Complex Analysis.* McGraw–Hill International Edition 1987.

Contents

1 Normed spaces and bounded linear maps

1.1 Definitions and examples

Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. For ease of notation and discussion, we will sometimes just take our scalars to be in $\mathbb{R}$, although the statement may be easily generalised to $\mathbb{C}$-vector spaces.

Definition Norm
A norm on X is a function $||\cdot||\colon X \to \mathbb{R}$ such that
(i) $||x|| \geq 0$ *for all* $x \in X$, *with* $||x|| = 0$ *iff* $x = 0$
(ii) $||\lambda x|| = |\lambda| ||x||$ *for all* $x \in X$ *and any scalar* λ
(iii) $||x + y|| \leq ||x|| + ||y||$ *for all* $x, y \in X$

Definition Normed space
A normed space is a pair $(X, ||\cdot||)$ where X is a vector space and $||\cdot||$ is a norm on X.

Example *Some finite-dimensional normed spaces*
(1) $\ell_2^n = (\mathbb{R}^n, ||\cdot||_2)$ or $(\mathbb{C}^n, ||\cdot||_2)$, where the norm is given by

$$||x||_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}, \qquad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$$

This is called the ℓ_2-norm or euclidean norm.

(i),(ii) are easy to check, whereas (iii) follows from Cauchy-Schwarz.

(2) $\ell_1^n = (\mathbb{R}^n, ||\cdot||_1)$ where $||x||_1 = \sum_{i=1}^n |x_i|$ (called the ℓ_1-norm)

(3) $\ell_\infty^n = (\mathbb{R}^n, ||\cdot||_\infty)$ where $||x||_\infty = \max_{1\leq i\leq n} |x_i|$ (called the ℓ_∞-norm or the sup-norm)

Given a normed space X, its norm $||\cdot||$ induces a metric on X:

$$d(x, y) = ||x - y||$$

Indeed, d is a metric:

- $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0 \iff x - y = 0 \iff x = y$
- $d(x, y) = ||x - y|| = ||y - x|| = d(y, x)$
- $d(x, z) = ||x - z|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$

This metric, in turn, induces a topology on X, called the *norm topology.* This allows us talk about open/closed sets, convergence, and continuity, as we illustrate in the following examples.

Example
The algebraic operations are continuous:

- if $x_n \to x$ and $y_n \to y$ in X, then $x_n + y_n \to x + y$
- if $x_n \to x$ in X and $\lambda_n \to \lambda$ in $\mathbb{R}$, then $\lambda_n x_n \to \lambda x$

Example
The norm $||\cdot||\colon X \to \mathbb{R}$ is continuous: by the triangle inequality, we have

$$|||x|| - ||y||| \leq ||x - y||$$

so $||\cdot||$ is, in fact, Lipschitz.

Definition Banach space
A Banach space is a complete normed space, i.e., a normed space that is complete in its norm topology.

Example
ℓ_2^n, ℓ_1^n, ℓ_∞^n are complete: for any of these spaces,
- $x^{(k)} \to x \iff x_i^{(k)} \to x_i$ for all $1 \leq i \leq n$
- $(x^{(k)})_{k\in\mathbb{N}}$ is Cauchy $\iff$ $(x_i^{(k)})_{k\in\mathbb{N}}$ is Cauchy for all $1 \leq i \leq n$

In a normed space, a useful object is the *unit ball*

$$B_X := \{x \in X \colon ||x|| \leq 1\}$$

Remarks
- B_X defines a norm on X:
$$||x|| = \inf\{t \geq 0 \colon x \in tB_X\}$$
- B_X is symmetric ($x \in B_X \implies -x \in B_X$), convex, and closed
- If $B \subset \mathbb{R}^n$ is a closed, convex, symmetric, bounded neighbourhood of 0, then B is the unit ball of $(\mathbb{R}^n, ||\cdot||)$ for some norm $||\cdot||$
- 'Geometry of Banach spaces'

Previously, we gave $\ell_2, \ell_1, \ell_\infty$ as examples of finite-dimensional normed spaces. More generally, we have the following family of examples

Example
(4) $\ell_p^n = (\mathbb{R}^n, ||\cdot||_p)$ for $1 \leq p < \infty$, where $||x||_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ (called the ℓ_p-norm)
Again, (i) and (ii) are easy to check, whereas (iii) is not obvious.[1]

Now, let S denote the set of all scalar sequences. This is a vector spaces under the coordinate operations $(x_n) + (y_n) = (x_n + y_n)$ and $\lambda(x_n) = (\lambda x_n)$.

Example *Sequence spaces*
(5) $\ell_1 = \left\{(x_n) \in S \colon \sum_{n=1}^\infty |x_n| < \infty\right\}$, $\quad ||(x_n)||_1 = \sum_{n=1}^\infty |x_n|$ $\quad$ (ℓ_1-norm)
(i) and (ii): easy to check.
(iii): Given $(x_n), (y_n) \in \ell_1$, we have $|x_n + y_n| \leq |x_n| + |y_n|$ for all $n \in \mathbb{N}$. Summing over all $n \in \mathbb{N}$, we deduce that $(x_n) + (y_n) \in \ell_1$ and $||(x_n) + (y_n)||_1 \leq ||(x_n)||_1 + ||(y_n)||_1$. Hence, ℓ_1 is a subspace of S and $||\cdot||_1$ is a norm on ℓ_1.

(6) $\ell_2 = \left\{(x_n) \in S \colon \sum_{n=1}^\infty |x_n|^2 < \infty\right\}$, $\quad ||(x_n)||_2 = \left(\sum_{n=1}^\infty |x_n|^2\right)^{1/2}$ $\quad$ (ℓ_2-norm)
(i) and (ii): easy to check.
(iii): Given $(x_n), (y_n) \in \ell_2$, the triangle inequality in ℓ_2^N gives us

$$\left(\sum_{k=1}^N |x_k + y_k|^2\right)^{1/2} \leq \left(\sum_{k=1}^N |x_k|^2\right)^{1/2} + \left(\sum_{k=1}^N |y_k|^2\right)^{1/2}.$$

Taking $N \to \infty$, we get $(x_n) + (y_n) \in \ell_2$ and $||(x_n) + (y_n)||_2 \leq ||(x_n)||_2 + ||(y_n)||_2$

[1] We will return to this later in the next subsection.

More generally, for $1 \le p < \infty$, the set

$$\ell_p = \left\{ (x_n) \in S \colon \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

is a subspace of S, and

$$||(x_n)||_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \qquad (\ell_p\text{-norm})$$

is a norm on ℓ_p. [(iii) *follows from the triangle inequality on* ℓ_p^n*, which we will see later.*]

Example *More sequence spaces*

(7) $\ell_\infty = \{(x_n) \in S \colon \exists\, M \ge 0\ \forall\, n \in \mathbb{N}\ |x_n| \le M\}$, $\quad ||(x_n)||_\infty = \sup_{n \in \mathbb{N}} |x_n| \qquad (\ell_\infty\text{-norm})$

(i) and (ii): easy to check.
(iii): Given $x = (x_n), y = (y_n) \in \ell_\infty$,

$$|x_n + y_n| \le |x_n| + |y_n| \le ||x||_\infty + ||y||_\infty \quad \forall\, n \in \mathbb{N}$$

so $x + y \in \ell_\infty$ and $||x + y||_\infty \le ||x||_\infty + ||y||_\infty$.

(8) $c_0 = \{(x_n) \in S \colon x_n \to 0 \text{ as } n \to \infty\}$
$c = \{(x_n) \in S \colon \lim_{n\to\infty} x_n \text{ exists}\}$

Both c_0 and c are subspaces of ℓ_∞ and are hence normed spaces in the ℓ_∞-norm.

1.2 Inequalities of Minkowski and Hölder

Recall that a function $f \colon (0, \infty) \to \mathbb{R}$ is *convex* if

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y) \qquad \forall\, x, y \in (0, \infty)\ \forall\, t \in [0,1]$$

and *concave* if the above holds with $\le$ replaced by $\ge$.

Lemma 1.1
Let $1 \le p < \infty$*. Then the map*

$$\begin{aligned} (0, \infty) &\to \mathbb{R} \\ x &\mapsto x^p \end{aligned}$$

is convex.

Proof. Fix $y > 0, t \in [0, 1]$, and define

$$g(x) = [(1-t)x + ty]^p - [(1-t)x^p + ty^p], \qquad x > 0.$$

Differentiating, we get

$$g'(x) = p(1-t)[(1-t)x + ty]^{p-1} - p(1-t)x^{p-1}.$$

Observe that $0 < x < y \implies g'(x) \ge 0$ and that $x > y \implies g'(x) \le 0$. By the MVT, we deduce that $g(x) \le g(y) = 0$ for all $x \in (0, \infty)$. ∎

Theorem 1.2 Minkowski's inequality
Let $1 \le p < \infty$*,* $n \in \mathbb{N}$*. For* $x, y \in \mathbb{R}^n$*,*

$$||x + y||_p \le ||x||_p + ||y||_p.$$

Remark. This shows that ℓ_p^n and ℓ_p are normed spaces.

Exercise. Show that $\ell_p, 1 \le p \le \infty$, is complete.[2]

Proof of Theorem 1.2. Let $B = \{x \in \mathbb{R}^n \colon ||x||_p \le 1\}$. We first show that B is convex. Let $x, y \in B$ and $t \in [0,1]$. For $1 \le k \le n$,

$$|(1-t)x_k + ty_k|^p \le ((1-t)|x_k| + t|y_k|)^p \le (1-t)|x_k|^p + t|y_k|^p$$

by Lemma 1.1 for $x_k \neq 0, y_k \neq 0$; the inequality holds trivially if $x_k = 0$ or $y_k = 0$. Summing over k, we then get

$$||(1-t)x + ty||_p^p \le (1-t)||x||_p^p + t||y||_p^p \le 1,$$

so $(1-t)x + ty \in B$.

We then complete the proof as follows. Let $x, y \in \mathbb{R}^n$. WLOG, $x, y, x+y$ are nonzero. By convexity of B, we have

$$\frac{x+y}{||x||_p + ||y||_p} = \frac{||x||_p}{||x||_p + ||y||_p} \cdot \underbrace{\frac{x}{||x||_p}}_{\in B} + \frac{||y||_p}{||x||_p + ||y||_p} \cdot \underbrace{\frac{y}{||y||_p}}_{\in B} \in B.$$

Thus, it follows that

$$\left|\left| \frac{x+y}{||x||_p + ||y||_p} \right|\right| \le 1 \Longrightarrow ||x+y||_p \le ||x||_p + ||y||_p,$$

as required. ■

Let $x = (x_n) \in \ell_1$ and $y = (y_n) \in \ell_\infty$. We then write $x \cdot y = (x_n y_n)$. Note that, for all $n \in \mathbb{N}$, $|x_n y_n| = |x_n||y_n| \le |x_n|||y||_\infty$. Thus, $x \cdot y \in \ell_1$ and $||x \cdot y||_1 \le ||x||_1 ||y||_\infty$.

Definition Conjugate index
Let $p \in (1, \infty)$. The conjugate index of p is the unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.3
Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $a, b \ge 0$, we have

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. The inequality holds trivially if $a = 0$ or $b = 0$, so it remains to consider the case $a, b > 0$. A proof similar to that of Lemma 1.1 shows that $\log \colon (0, \infty) \to \mathbb{R}$ is concave. Hence,

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \ge \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) = \log(ab).$$

We then apply exp to get the required result. ■

Theorem 1.4 Hölder's inequality
Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in \ell_p$ and $y \in \ell_q$, then $x \cdot y \in \ell_1$ and

$$||x \cdot y||_1 \le ||x||_p ||y||_q.$$

Remark. As discussed above, $p = 1, q = \infty$ also works. Moreover, setting $p = q = 2$, we recover Cauchy-Schwarz.

Exercise. Deduce Minkowski's inequality from Hölder's inequality.

[2] A slick proof of this will be provided later.

Proof of Theorem 1.4. WLOG, $x \neq 0$ and $y \neq 0$. By homogeneity, we may also take $||x||_p = ||y||_q = 1$ WLOG. Now, by Lemma 1.3, we have $|x_n y_n| \leq |x_n|^p/p + |y_n|^q/q$ for all $n \in \mathbb{N}$. Summing over n, we have

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \frac{||x||_p^p}{p} + \frac{||y||_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 = ||x||_p ||y||_q,$$

as required. ■

1.3 More examples: function spaces

Example

(9) $C[0,1] = \{f\colon [0,1] \to \mathbb{R} \mid f \text{ cts}\}$, $\quad ||f||_\infty = \sup_{[0,1]} |f|$ (sup norm or uniform norm)

By the uniform limit theorem, this is a Banach space.

(10) More generally, given a compact, Hausdorff topological space K,

$$C(K) = \{f\colon K \to \mathbb{R} \mid f \text{ cts}\}$$

is a Banach space in the sup norm $||f||_\infty = \sup_K |f|$.

(11) $(C[0,1], ||\cdot||_1)$, $\quad ||\cdot||_1 = \displaystyle\int_0^1 |f(t)|\,dt$ (L_1-norm)

This is an *incomplete* normed space — see Example Sheet 1.

More generally, $C[0,1]$ is incomplete in the L_p-norm, $1 \leq p < \infty$, given by

$$||f||_p = \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}.$$

> In II Probability and Measure, you will encounter the completion of $(C[0,1], ||\cdot||_p)$, which is the Lebesgue space $L_p[0,1]$.

(12) $C^1[0,1] = \{f \in C[0,1] \mid f \text{ continuously differentiable}\}$ is a subspace of $C[0,1]$, so it is a normed space in $||\cdot||_\infty$ but incomplete, i.e. not closed in $C[0,1]$. However, it is complete in the norm $||f|| = ||f||_\infty + ||f'||_\infty$ — see Example Sheet 1.

(13) Let $\triangle = \{z \in \mathbb{C} \mid |z| \leq 1\}$. The set

$$A(\triangle) = \{f \in C(\triangle) \mid f \text{ analytic on int}\,\triangle\}$$

is a subspace of $C(\triangle)$. In fact, it is closed in $C(\triangle)$ and hence a Banach space in $||\cdot||_\infty$.

1.4 More on the normed topology

Let X be a normed space and $A \subset C$. Recall that the *closure* of A in X is

$$\overline{A} = \{x \in X \mid \exists\,(a_n) \text{ in } X \text{ s.t. } a_n \to x \text{ as } n \to \infty\}.$$

We then say that A is *dense* in X if $\overline{A} = X$. Moreover, A is *separable* if it has a countable dense subset.

If $Y \subset X$ is a subspace, then so is $\overline{Y}$: if $x, y \in \overline{Y}$, then there exists (x_n), (y_n) in Y such that $x_n \to x$ and $y_n \to y$. So $\lambda x_n + \mu y_n \to \lambda x + \mu y \in \overline{Y}$. Similarly, if $A \subset X$ is convex, then so is $\overline{A}$.

For a subset $A \subset X$, the *closed linear span* of A, denoted by $\overline{\text{span}}\,A$, is the closure of span A.

Remarks

- If A is countable, then $\overline{\text{span}}\,A$ is separable.
- The set of all rational linear combinations of elements of A is countable and dense in $\overline{\text{span}}\,A$.

Example

- $\overline{\mathbb{Q}} = \mathbb{R}$, so $\mathbb{R}$ is separable.
- $\ell_p, 1 \leq p < \infty$, is separable.
  Let $e_n = (0, \cdots, 0, \underset{n}{1}, 0, \cdots)$, $n \in \mathbb{N}$ (unit vector basis)
  Let $c_{00} = \text{span}\{e_n : n \in \mathbb{N}\} = \{(x_n) \in S : \exists\, N \in \mathbb{N}\ \forall\, n > N x_n = 0\}$
  We then show that $\ell_p = \overline{\text{span}}\{e_n : n \in \mathbb{N}\}$: if $x = (x_n) \in \ell_p$, then
$$\left|\left|x - \sum_{i=1}^{N} x_i e_i\right|\right|_p = \left(\sum_{i>n} |x_i|^p\right)^{1/p} \to 0 \text{ as } N \to \infty$$
- Similarly, in ℓ_∞, we have $\overline{\text{span}}\{e_n : n \in \mathbb{N}\} = c_0$. Moreover, c is separable, whereas ℓ_∞ is not.

Exercise. Prove the claims in the last example above.

1.5 Bounded linear maps

Theorem 1.5

Let X, Y be normed spaces and $T : X \to Y$ be a linear map. The following are equivalent:

(i) *T is continuous at* 0

(ii) *T is continuous*

(iii) *T is Lipschitz*

(iv) *T is bounded, i.e., $\exists\, C \geq 0\ \forall\, x \in X\ ||Tx|| \leq C||x||$.*

Proof. (iv) $\implies$ (iii): Observe that
$$d(Tx, Ty) = ||Tx - Ty|| = ||T(x - y)|| \leq C||x - y|| = Cd(x, y)$$
iii) $\implies$ (ii): Given $\varepsilon > 0$ take $\delta = \varepsilon/(C + 1)$.
(ii) $\implies$ (i): Trivial.
(i) $\implies$ (iv): $\exists\, \delta > 0\ \forall\, x \in X\ d(x, 0) = ||x|| \leq \delta \implies d(Tx, T0) = ||Tx|| \leq 1$. For $x \neq 0$, $||\delta x/||x|||| = \delta$, so $||T(\delta x/||x||)|| \leq 1$. Hence, $||Tx|| \leq \delta^{-1}||x||$. ■

For normed spaces X, Y, let $\mathcal{B}(X, Y) = \{T : X \to Y \mid T \text{ linear and bounded}\}$. For $T \in \mathcal{B}(X, Y)$, its *operator norm* is
$$||T|| = \sup\{||Tx|| : x \in B_X\}.$$

Remark. Since $T \in \mathcal{B}(X, Y)$, we have $C \geq 0$ such that $||Tx|| \leq C||x||$ for all $x \in X$. So if $||x|| \leq 1$, then $||Tx|| \leq C$. Thus, by definition, $||T|| \leq C$. Conversely, for all $x \in B_X$, we have $||Tx|| \leq ||T||$, so by homogeneity, $||Tx|| \leq ||T||||x||$. Hence, $||T||$ is the least C such that (iv) in Theorem 1.5 above holds.

The operator norm is a norm on $\mathcal{B}(X, Y)$: given $S, T \in \mathcal{B}(X, Y)$, we have, for all $x \in X$,
$$||(S + T)x|| = ||Sx + Tx|| \leq ||Sx|| + ||Tx|| \leq ||S||||x|| + ||T||||x|| \leq (||S|| + ||T||)||x||,$$
from which it follows that $S + T \in \mathcal{B}(X, Y)$ and $||S + T|| \leq ||S|| + ||T||$.

Notation. We write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$.

Proposition 1.6
Let X, Y, Z be normed spaces, $S \in \mathcal{B}(X,Y)$, $T \in \mathcal{B}(Y,Z)$. Then $TS \in \mathcal{B}(X,Z)$ and $||TS|| \leq ||T||||S||$.

Proof. For all $x \in X$, we have $||TSx|| \leq ||T||||Sx|| \leq ||T||||S||||x||$. ■

Example

(1) $T\colon \ell_2^n \to \ell_2^n, (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_r, 0, \cdots, 0)$

$$||Tx||_2 = \left(\sum_{i=1}^{r} |x_i|^2\right)^{1/2} \leq ||x||_2 \implies ||T|| \leq 1$$

But $Te_1 = e_1$ so $||T|| = 1$.

More generally, if T is represented by a matrix A wrt the standard basis, then Cauchy-Schwarz gives us

$$||T|| \leq \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

(2) Let $1 \leq p < \infty$; $R\colon \ell_p \to \ell_p, (x_1, x_2, x_3, \cdots) \mapsto (0, x_1, x_2, \cdots)$ (right shift)

For all $x \in \ell_p$, $||Rx||_p = ||x||_p$, so R is isometric and $||R|| = 1$. Note that R is injective but not surjective.

(3) Let $1 \leq p < \infty$; $L\colon \ell_p \to \ell_p, (x_1, x_2, x_3, \cdots) \mapsto (x_2, x_3, x_4, \cdots)$ (left shift)

For all $x \in \ell_p$, $||Lx||_p \leq ||x||_p$, so $L \in \mathcal{B}(\ell_p)$ with $||L|| \leq 1$. Since $Le_2 = e_1$ and $||e_1||_p = ||e_2||_p = 1$, we in fact have $||L|| = 1$. Note that L is surjective but not injective.

(4) $T\colon \ell_1 \to \ell_2, x \mapsto x$

- **Claim.** $\ell_1 \subset \ell_2$, and $\forall\, x \in \ell_2$ $||x||_2 \leq ||x||_1$

  Proof. WLOG assume $||x||_1 = 1$ by homogeneity. Since $\sum_{n=1}^{\infty} |x_i| = 1$, we have $|x_i| \leq 1$ for all i. Thus,

  $$|x_i|^2 \leq |x_i| \;\forall\, i \implies ||x||_2^2 \leq ||x||_1 = 1 \implies ||x||_2 = 1 = ||x||_1$$

  as claimed. ■

Using the above claim, we have $T \in \mathcal{B}(\ell_1, \ell_2)$ and $||T|| = 1$.

(5) $T\colon \ell_2 \to \ell_1, (x_n) \mapsto (x_n/n)$

By Cauchy-Schwarz,

$$\sum_{n=1}^{\infty} \left|\frac{x_i}{n}\right| \leq \left(\sum_{n=1}^{\infty} x_i^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2}$$

so $T \in \mathcal{B}(\ell_2, \ell_1)$ with $||T|| \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2}$. In fact, we can replace $\leq$ with $=$.

(6) $D\colon (C^1[0,1], ||\cdot||) \to (C[0,1], ||\cdot||_\infty), f \mapsto f'$

Note that $||Df||_\infty = ||f'||_\infty \leq ||f||_\infty + ||f'||_\infty = ||f||$, so $||D|| \leq 1$. But taking $f(x) = \sin(n\pi x)$, we have

$$||Df||_\infty = n\pi, \qquad ||f|| = n\pi + 1,$$

so in fact $||D|| = 1$. Note also that, for $f \neq 0$, $||Df||_\infty < ||f||$, so $||D||$ is not attained.

(7) On a normed space X, the identity $x \mapsto x$ is denoted by Id, I, Id_X or I_X. This map is isometric, i.e., $||\mathrm{Id}(x)|| = ||x|| \ \forall x \in X$.

(8) For normed spaces X, Y, we let

$$X \oplus Y = \{(x, y) \colon x \in X, y \in Y\}$$

with norm $||(x, y)||_1 = ||x|| + ||y||$. The corresponding norm topology is the product topology.

Define $P \colon X \oplus Y \to X, (x, y) \mapsto x$ (projection onto X). Note that $P \in \mathcal{B}(X \oplus Y, X)$ with $||P|| = 1$.

Let X, Y be normed spaces. We introduce some terminology:

- An *isomorphism* $X \to Y$ is a linear homeomorphism $T \colon X \to Y$, i.e., T is a linear bijection such that T and T^{-1} are bounded. Equivalently, T is a linear bijection[3] such that

  $$\exists\, a, b > 0 \ \forall x \in X \ a||x|| \leq ||Tx|| \leq b||x||$$

  If such T exists, we say that X and Y are *isomorphic*, and we write $X \sim Y$.
- An *isometric isomorphism* is a linear bijection $T \colon X \to Y$ such that

  $$\forall x \in X \ ||Tx|| = ||x||$$

  If such T exists, we say that X and Y are *isometrically isomorphic*, and we write $X \cong Y$.

> The Banach-Mazur distance is defined as
>
> $$d(X, Y) = \begin{cases} \infty, & \text{if } X \not\sim Y \\ \inf\{||T||||T^{-1}|| \mid T \colon X \to Y \text{ is an isomorphism}\}, & \text{otherwise} \end{cases}$$
>
> Note that $||T||||T^{-1}|| \geq ||TT^{-1}|| = 1$. If $X \cong Y$, then $d(X, Y) = 1$. Does the converse hold?

- An *isomorphic embedding* $X \to Y$ is a linear map $T \colon X \to Y$ such that $T \colon X \to TX = \operatorname{im} T$ is an isomorphism. If such T exists, we say that X *(isomorphically) embeds into* Y, and we write $X \hookrightarrow Y$.

Definition Equivalent norms
Let X be a normed space. Two norms $|| \cdot ||, || \cdot ||'$ are equivalent if

$$\begin{aligned} & \mathrm{Id} \colon (X, || \cdot ||) \to (X, || \cdot ||') \text{ is an isomorphism} \\ \iff & || \cdot ||, || \cdot ||' \text{ induce the same norm topology on } X \\ \iff & \exists\, a, b > 0 \ \forall x \in X a||x|| \leq ||x||' \leq b||x|| \\ \iff & \exists\, a, b > 0 \ aB'_X \subset B_X \subset bB'_X \end{aligned}$$

Remarks

- If $X \sim Y$, then X is complete iff Y is complete.
  If $||\cdot||, ||\cdot||'$ are equivalent norms on a vector space X, then $(X, ||\cdot||)$ is complete iff $(X, ||\cdot||')$ is complete.

[3] We can actually replace 'bijection' with 'surjection'.

- Let X and Y be normed spaces. On $X \oplus Y$, the norm $||(x,y)||_1 = ||x|| + ||y||$ is equivalent to $||(x,y)||_p = (||x||^p + ||y||^p)^{1/p}$ for all $1 \le p < \infty$ and to $||(x,y)||_\infty = \max\{||x||, ||y||\}$.
- $(C[0,1], ||\cdot||_\infty)$ is complete whereas $(C[0,1], ||\cdot||_1)$ is incomplete. Thus, we can use the first remark above to deduce that $||\cdot||_\infty \not\sim ||\cdot||_1$ (but this can easily be proven directly as well). However, $||f||_1 = \int_0^1 |f(t)|\,dt \le ||f||_\infty$, so
$$\mathrm{Id}\colon (C[0,1], ||\cdot||_\infty) \to (C[0,1], ||\cdot||_1)$$
is a continuous linear bijection but its inverse is not continuous.
- On c_{00}, $||\cdot||_1 \not\sim ||\cdot||_2$. To see why, consider $x = (\underbrace{1, \cdots, 1}_{n}, 0, 0, \cdots)$ and note that $||x||_1 = n, ||x||_2 = \sqrt{n}$.

Finally, we discuss convergence and completeness. Let X, Y be normed spaces. In $\mathcal{B}(X,Y)$, convergence implies pointwise convergence, i.e., if $T_n \to T$ in $\mathcal{B}(X,Y)$, then, for all $x \in X$, $T_n x \to Tx$ in Y. To see why, note that, for fixed $x \in X$, we have $||T_n x - Tx|| \le ||T_n - T|| ||x|| \to 0$. However, the converse is false in general, e.g., $T_n\colon \ell_1 \to \mathbb{R}, x \mapsto x_n$. We have $T_n \to 0$ pointwise, but $||T_n|| = 1$ for all $n \in \mathbb{N}$.

Theorem 1.7
Let X, Y be normed spaces. If Y is complete, then $\mathcal{B}(X,Y)$ is complete.

Proof. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X,Y)$. Fix $x \in X$. Then
$$||T_m x - T_n x|| \le ||T_m - T_n|| ||x|| \to 0 \text{ as } m, n \to \infty$$
So $(T_n x)$ is Cauchy in Y and thus convergent. Now, define $T\colon X \to Y$ by $x \mapsto \lim_{n\to\infty} T_n x$. Observe that

- T is linear
$$T(\lambda x + \mu y) = \lim_{n\to\infty} T_n(\lambda x + \mu y) = \lim_{n\to\infty} [\lambda T_n x + \mu T_n y] = \lambda Tx + \mu Ty$$
- T is bounded
(T_n) is Cauchy implies (T_n) is bounded, i.e., there exists $M \ge 0$ such that $||T_n|| \le M$ for all $n \in \mathbb{N}$. Fix $x \in X$. Then, for all $n \in \mathbb{N}$, we have $||T_n x|| \le ||T_n|| ||x|| \le M||x||$. Letting $n \to \infty$, we obtain $||Tx|| \le M||x||$.
- $T_n \to T$ in $\mathcal{B}(X,Y)$
Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $||T_m - T_n|| \le \varepsilon$ for all $m, n \ge N$. Fix $x \in X$. Note that, for all $m, n \ge N$, we have
$$||T_m x - T_n x|| \le ||T_m - T_n|| ||x|| \le \varepsilon ||x||$$
Letting $n \to \infty$ with $m \ge N$ fixed yields $||T_m x - Tx|| \le \varepsilon ||x||$. Hence, $||T_m - T|| \le \varepsilon$ for all $m \ge N$. ■

2 Dual spaces

2.1 Basics

Let X be a normed space. A *functional* on X is a map $X \to \mathbb{R}$. The *dual space* X^* of X is the space of all bounded linear functionals on X, i.e., $X^* = \mathcal{B}(X, \mathbb{R})$ equipped with the operator norm. Since $\mathbb{R}$ is complete, Theorem 1.7 gives us the following result.

Theorem 2.1
For any normed space X, its dual X^* is a Banach space.

Notation. For $x \in X$ and $f \in X^*$, we let $\langle x, f \rangle = f(x)$.

Now, we know that $0 \in X^*$. Are there other elements?

Theorem 2.2 Hanh-Banach theorem
Let X be a normed space, $Y \subset X$ be a subspace and $g \in Y^*$. Then $f \in X^*$ such that $f|_Y = g$ and $||f|| = ||g||$.

Proof. See II Analysis of Functions. ■

Corollary 2.3
Let X be a normed space, $x_0 \in X \backslash \{0\}$. Then there exists $f \in S_{X^*} = \{f \in X^* \colon ||f|| = 1\}$ such that $f(x_0) = ||x_0||$.

Remarks

- For any $g \in B_{X^*}$, $|g(x_0)| \leq ||g||||x_0|| \leq ||x_0||$. Corollary 2.3 says that there exists $f \in B_{X^*}$ such that $f(x_0) = ||x_0||$, so
$$||x_0|| = \sup\{g(x_0) \colon g \in B_{X^*}\} = \max\{g(x_0) \colon g \in B_{X^*}\}.$$
We call f a *norming functional* at x_0.
- Given $x \neq y$ in X, we can set $x_0 = x - y$ and Corollary 2.3 implies that there exists $f \in X^*$ such that $f(x) \neq f(y)$. Thus, X^* separates the points of X.

Proof of Corollary 2.3. Set $Y = \operatorname{span}\{x_0\}$ and define $g(\lambda x_0) = \lambda ||x_0||$. Then $g \in S_{Y^*}$ with $g(x_0) = ||x_0||$. Finally, apply Theorem 2.2. ■

2.2 Dual space of ℓ_p

Motivation: Recall that, for $1 \leq p < \infty$, we have $\ell_p = \overline{\operatorname{span}}\{e_n \colon n \in \mathbb{N}\} = \overline{c_{00}}$. Given $\varphi \in \ell_p^*$ and $x = (x_n) \in \ell_p$,
$$\varphi(x) = \varphi\left(\lim_{n\to\infty} \sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{\infty} x_k \varphi(e_k)$$
so φ corresponds to the sequence $y = (\varphi(e_n))_{n\in\mathbb{N}}$. We may then ask: is $\ell_p^* \cong \ell_q$ for some q?

Fix $1 < p < \infty$, and let q be the conjugate index of p. Fix $y = (y_n) \in \ell_q$. Define
$$\begin{aligned} \varphi_y \colon \ell_p &\to \mathbb{R} \\ x &\mapsto \sum_{n=1}^{\infty} x_n y_n \end{aligned}$$

By Holder's inequality (Theorem 1.4), this is well-defined and $|\varphi_y(x)| \leq ||x||_p ||y||_q$. So φ_y is linear and bounded: $||\varphi_y|| \leq ||y||_q$. Thus, $\varphi_y \in \ell_p^*$, which means that we have a map

$$\begin{aligned} \varphi\colon \ell_q &\to \ell_p^* \\ y &\mapsto \varphi_y \end{aligned}$$

Note that φ is linear and bounded with $||\varphi|| \leq 1$.

Theorem 2.4
Let p, q, φ be as above. Then φ is an isometric isomorphism $\ell_q \to \ell_p^*$.

Proof. It remains to check that φ is isometric and surjective:

- φ is isometric

  Fix $y \in \ell_q$. WLOG $y \neq 0$. Define
  $$x_n = \begin{cases} \dfrac{|y_n|^q}{y_n}, & y_n \neq 0 \\ 0, & y_n = 0 \end{cases}$$
  Observe that $\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^\infty |y_n|^{(q-1)p} = \sum_{n=1}^\infty |y_n|^q = ||y||_q^q < \infty$, so $x \in \ell_p$ with $||x||_p^p = ||y||_q^q$.

  Since $y \neq 0$, we have $x \neq 0$, so $x/||x||_p \in B_{\ell_p}$. Note that
  $$||\varphi_y|| \geq \varphi_y\left(\frac{x}{||x||_p}\right) = \frac{1}{||x||_p}\sum_{n=1}^\infty x_n y_n = \frac{||y||_q^q}{||y||_q^{q/p}} = ||y||_q.$$
  Hence, $||\varphi_y|| = ||y||_q$.

- φ is surjective

  Fix $f \in \ell_p^*$. Define $y_n = f(e_n), n \in \mathbb{N}$. Let $y = (y_n)$. For some fixed $N \in \mathbb{N}$, set
  $$x_n = \begin{cases} \dfrac{|y_n|^q}{y_n}, & y_n \neq 0, n \leq N \\ 0, & \text{otherwise} \end{cases}$$
  Then $x = (x_n) \in \ell_p$, so
  $$f(x) = \sum_{n=1}^N x_n f(e_n) = \sum_{n=1}^N x_n y_n = \sum_{n=1}^N |y_n|^q \leq ||f|| ||x||_p$$
  $$||x||_p = \left(\sum_{n=1}^N |x_n|^p\right)^{1/p} = \left(\sum_{n=1}^N |y_n|^{(q-1)p}\right)^{1/p} = \left(\sum_{n=1}^N |y_n|^q\right)^{1/p}$$
  Hence, $\displaystyle\sum_{n=1}^N |y_n|^q \leq ||f|| \left(\sum_{n=1}^N |y_n|^q\right)^{1/p}$, i.e.
  $$\left(\sum_{n=1}^N |y_n|^q\right)^{1/q} \leq ||f||$$
  Let $N \to \infty$ to deduce that $y \in \ell_q$. Finally, observe that
  $$\begin{aligned} & f(e_n) = y_n = \varphi_y(e_n)\ \forall\, n \in \mathbb{N} \\ \implies\ & f(x) = \varphi_y(x)\ \forall\, x \in \operatorname{span}\{e_n \colon n \in \mathbb{N}\} = c_{00} \qquad \text{by linearity} \end{aligned}$$

$$\implies f(x) = \varphi_y(x)\ \forall\, x \in \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\} = \ell_p \qquad \text{by continuity}$$

Thus, $f = \varphi_y$, so φ is surjective. ■

Remarks

- We also have $\ell_1^* \cong \ell_\infty$ and $c_0^* \cong \ell_1$. The proof also shows that $\ell_1 \hookrightarrow \ell_\infty^*$ isometrically. However, the proof of surjectivity breaks down since $\overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\}$ in ℓ_∞ in $c_0 \subsetneq \ell_\infty$.
- From the proof, we can show Corollary 2.3 holds for ℓ_p.
- We've shown that $\ell_p, 1 \leq p \leq \infty$, is complete as they are dual spaces. For c_0, one simply has to show that c_0 is closed in ℓ_∞.

2.3 Bidual

Let X be a normed space. Then $X^{**} = (X^*)^* = \mathcal{B}(X^*, \mathbb{R})$ is the *bidual* or *second dual* of X.

For each $x \in X$, define the map

$$\begin{aligned} \hat{x} \colon X^* &\to \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

Note that $\hat{x}$ is linear and bounded: $|\hat{x}(f)| = |f(x)| \leq ||f||||x||$. So $\hat{x} \in X^{**}$ with $||\hat{x}|| \leq ||x||$. Thus, we have

$$\begin{aligned} \hat{} \colon X &\to X^{**} \\ x &\mapsto \hat{x} \end{aligned}$$

This is linear: $\widehat{\lambda x + \mu y}(f) = f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = (\lambda \hat{x} + \mu \hat{y})(f)$.

For $x \neq 0$, let $f \in X^*$ be a norming functional at x. Then

$$\hat{x}(f) = f(x) = ||x|| \implies ||\hat{x}|| = ||x||$$

so the canonical map $X \to X^{**}, x \mapsto \hat{x}$ is an isometric embedding into X^{**}. If f is surjective, we say that X is *reflexive*.

2.4 Dual operators

Let X, Y be normed spaces and $T \in \mathcal{B}(X, Y)$. The *dual operator* T^* of T is the map

$$\begin{aligned} T^* \colon Y^* &\to X^* \\ g &\mapsto g \circ T \end{aligned}$$

By Proposition 1.6, $T^*(g) = g \circ T \in X^*$ and $||T^*(g)|| \leq ||g||||T||$, so T^* is well-defined. Moreover, it is clearly linear and bounded with $||T^*|| \leq ||T||$.

Remark. Note that $\langle \cdot, \cdot \rangle \colon X \times X^* \to \mathbb{R}$ is bilinear. Moreover, for $x \in X$ and $g \in Y^*$, we have $\langle x, T^*(g) \rangle = \langle T(x), g \rangle$.

It turns out that $||T^*|| = ||T||$:

$$||T^*|| = \sup_{g \in B_{Y^*}} ||T^* g|| = \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^*(g) \rangle| = \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g \rangle| = \sup_{x \in B_X} ||Tx|| = ||T||,$$

where the penultimate equality follows from Corollary 2.3.

Example

Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider the right-shift map $R \colon \ell_p \to \ell_p$. What is $R^* \colon \ell_p^* \to \ell_p^*$? Recall that $\ell_p^* \cong \ell_q$. Thought of as a map $\ell_q \to \ell_q$, it turns out that $R^* = L$, the left-shift map.

Now, let's note some properties of dual operators:

(1) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$

(2) $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for all $S, T \in \mathcal{B}(X, Y)$ and all scalars λ, μ
Indeed, for $g \in Y^*$, $x \in X$,

$$\begin{aligned} \langle x, (\lambda S + \mu T)^* g \rangle &= \langle (\lambda S + \mu T)x, g \rangle \\ &= \langle \lambda Sx + \mu Tx, g \rangle \\ &= \lambda \langle Sx, g \rangle + \mu \langle Tx, g \rangle \\ &= \lambda \langle x, S^* g \rangle + \mu \langle x, T^* g \rangle \\ &= \langle x, (\lambda S^* + \mu T^*) g \rangle \end{aligned}$$

Since x is arbitrary, $(\lambda S + \mu T)^* g = (\lambda S^* + \mu T^*) g$ for all $g \in Y^*$, and we are done.

(3) $(ST)^* = T^* S^*$ for all $T \in \mathcal{B}(X, Y)$ and all $S \in \mathcal{B}(Y, Z)$

$$\langle x, (ST)^* g \rangle = \langle STx, g \rangle = \langle S(Tx), g \rangle = \langle Tx, S^* g \rangle = \langle x, T^* S^* g \rangle$$

(4) Let $T \in \mathcal{B}(X, Y)$. We have $T^* \in \mathcal{B}(Y^*, X^*)$ and $T^{**} \in \mathcal{B}(X^{**}, Y^{**})$. The diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\hat{}"'] & Y \arrow[d] \\ X^{**} \arrow[r, "T^{**}"'] & Y^{**} \end{tikzcd}$$

commutes, i.e., $\hat{Tx} = T^{**} \hat{x}$ for all $x \in X$. For $x \in X, g \in Y^*$,

$$\langle g, T^{**} \hat{x} \rangle = \langle T^* g, \hat{x} \rangle = \langle x, T^* g \rangle = \langle Tx, g \rangle = \langle g, \widehat{Tx} \rangle$$

Remark. Properties (1) and (3) imply that $X \sim Y \implies X^* \sim Y^*$.

3 Finite-dimensional normed spaces

Recall that norms $||\cdot||$ and $||\cdot||'$ on a vector space X are equivalent if $\mathrm{Id}\colon (X, ||\cdot||) \to (X, ||\cdot||')$ is an isomorphism or, equivalently, if $\exists\, a, b > 0\ \forall\, x \in X\ a||x|| \leq ||x||' \leq b||x||$.

Example
On $\mathbb{R}^n$, the norms $||\cdot||_1$ and $||\cdot||_2$ are equivalent. We've already seen that $||x||_2 \leq ||x||_1$ for all $x \in \mathbb{R}^n$. Moreover, by Cauchy-Schwarz, we have

$$||x||_1 = \sum_{i=1}^{n} |x_i| \leq \sqrt{n}\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \sqrt{n}||x||_2.$$

Theorem 3.1
Any two norms on a finite-dimensional vector space are equivalent.

Proof. Let X be a f.d. vector space. Fix a basis $(e_1, \cdots, e_n)$ of X. For $x = \sum_{i=1}^{n} x_k e_k \in X$, define $||x||_1 = \sum_{k=1}^{n} |x_k|$. Let $||\cdot||$ be an arbitrary norm on X.

We show that $||\cdot||$ is equivalent to $||\cdot||_1$. For $x = \sum_{k=1}^{m} x_k e_k \in X$, we have

$$||x|| \leq \sum_{k=1}^{n} |x_k|||e_k|| \leq M||x||_1$$

where $M = \max_{1 \leq k \leq n} ||e_k||$.

Now, let $S = \{x \in X\colon ||x||_1 = 1\}$, the unit sphere of $(X, ||\cdot||_1)$. We have the following result:

- **Claim.** S is compact.

  Proof. Let $(x^{(r)})_{r\in\mathbb{N}}$ be a sequence in S. Write $x^{(r)} = \sum_{k=1}^{n} x_k^{(r)} e_k$. For each $1 \leq k \leq n$, $|x_k^{(r)}| \leq ||x^{(r)}|| = 1$ for all $r \in \mathbb{N}$. By repeated application of Bolzano-Weierstrass, there exists $r_1 < r_2 < r_3 < \cdots$ in $\mathbb{N}$ such that $(x_k^{(r_\ell)})_{\ell\in\mathbb{N}}$ is convergent for each $1 \leq k \leq n$. Let $x_k = \lim_{\ell\to\infty} x_k^{r_\ell}$ and $x = \sum_{k=1}^{n} x_k e_k$. Note that

  $$||x||_1 = \sum_{k=1}^{n} |x_k| = \lim_{\ell\to\infty} \sum_{k=1}^{n} |x_k^{(r_\ell)}| = 1$$

  so $x \in S$. Moreover,

  $$||x^{(r_\ell)} - x||_1 = \sum_{k=1}^{n} |x_k^{(r_\ell)} - x_k| \to 0 \quad \text{as } \ell \to \infty$$

  so $x^{(r_\ell)} \to x$ in S. Thus, S is sequentially compact and hence compact. ■

For any $x, y \in S$, $\big|||x|| - ||y||\big| \leq ||x - y|| \leq M||x - y||_1$. So $||\cdot||$ is continuous on S with respect to $||\cdot||_1$. So $c = \inf\{||x||\colon x \in S\}$ is achieved: $\exists\, x \in S\ ||x|| = c$. Since $0 \notin S$ and $c > 0$, we have $||y|| \geq c = c||y||_1$ for all $y \in S$. By homogeneity, $||y|| \geq c||y||_1$ for all $y \in X$. ■

Corollary 3.2
Let $T\colon X \to Y$ be a linear map between two normed spaces. If X is f.d., then T is continuous.

Proof. Let $||\cdot||$ denote the norm on X and Y. Define $||x||' = ||Tx|| + ||x||$ for all $x \in X$. It is easy to check that this is a norm on X. By Theorem 3.1, there exists $b > 0$ such that, for all $x \in X$, $||x||' \leq b||x||$. In particular, $||Tx|| \leq b||x||$ for all $x \in X$. ■

Corollary 3.3
If $\dim X = \dim Y < \infty$*, then* $X \sim Y$*.*

Proof. We have a linear bijection $T\colon X \to Y$. By Corollary 3.2, T and T^{-1} are continuous. ∎

Remark. Corollary 3.3 does *not* imply that the theory of f.d. normed spaces is uninteresting.

Recall that, for X a metric space and $Y \subset X$, we have
- Y complete $\Longrightarrow$ Y is closed in X
- Y closed in X and X complete $\Longrightarrow$ Y complete

Corollary 3.4
(i) *A f.d. normed space* X *is complete.*
(ii) *A f.d. subspace* X *of a normed space* Y *is closed in* Y*.*

Proof. (i) Let $n = \dim X$. By Corollary 3.3, $X \sim \ell_2^n$ which is complete. (ii) follows from above properties of metric spaces. ∎

Corollary 3.5
Let X *be a f.d. normed space and* $A \subset X$*. Then*

$$A \text{ is compact} \iff A \text{ is closed and bounded}$$

Proof. If $X = \ell_2^n$, then this is simply Heine-Borel. For general X, the result follows by invoking Corollary 3.3 to deduce that $X \sim \ell_2^n$ and noting isomorphisms map compact subsets to compact subsets (ditto for closed and bounded subsets). ∎

In particular, $B_X = \{x \in X : ||x|| = 1\}$ is compact. How about if $\dim X = \infty$? Note that, in ℓ_p, $1 \le p < \infty$, $||e_n||_p = 1$ for all n and $||e_m - e_n|| = 2^{1/p}$ for all $m \neq n$, so (e_n) has no convergent subsequence. Hence, B_{ℓ_p} is not compact.

A similar obstruction does, in fact, hold for any infinite-dimensional normed space. To show this, we need the following lemma:

Proposition 3.6 Riesz's lemma
Let Y *be a proper, closed subspace of a normed space* X*. Then*

$$\forall \varepsilon > 0 \ \exists\, x \in B_X \ d(x, Y) = \inf\{||x - y|| : y \in Y\} > 1 - \varepsilon.$$

Proof. WLOG, $0 < \varepsilon < 1$. Fix $z \in X \backslash Y$. Since Y is closed, $d = d(x, Y) > 0$. Pick $y \in Y$ such that $d \le ||z - y|| < d/(1 - \varepsilon)$. Set $x = \frac{(z-y)}{||z-y||}$. Note that $d(x, Y) > 1 - \varepsilon$: for $y' \in Y$,

$$||x - y'|| = \left|\left| \frac{z - y - ||z - y||y'}{||z - y||} \right|\right| \ge \frac{d}{||z - y||} > 1 - \varepsilon$$

so $d(x, Y) \ge d/||z - y|| > 1 - \varepsilon$. ∎

Theorem 3.7
Let X *be a normed space. Then* B_X *is compact if and only if* $\dim X < \infty$*.*

Proof. ($\Longleftarrow$) Corollary 3.5
($\Longrightarrow$) Similar to the ℓ_p case, we construct (x_n) in B_X such that $||x_m - x_n|| > 1/2$ for all $m \neq n$. As before, we then deduce that (x_n) has no convergent subsequence and so B_X is not compact.

Pick any $x_1 \in B_X$. Suppose we have already picked $x_1, \cdots, x_n$ for some $n \in \mathbb{N}$. We then set $Y = \operatorname{span}\{x_1, \cdots, x_n\}$. Then Y is a proper ($\dim X = \infty$) and closed (Corollary 3.4) subspace of X. By Proposition 3.6, we can then pick $x_{n+1} \in B_X$ such that $d(x_{n+1}, Y) > 1/2$. In particular, $||x_{n+1} - x_m|| > 1/2$ for $1 \leq m \leq n$. ■

4 The Baire category theorem and its applications

Let (X, d) be a metric space. In this course, we will denote closed and open balls as

$$B_r(x) = \{y \in X : d(x, y) \leq r\}$$
$$D_r(x) = \{y \in X : d(x, y) < r\}$$

Recall that, for $A \subset X$, the *closure of* A *in* X is

$$\begin{aligned}\overline{A} &:= \bigcap_{\substack{F \text{ closed in } X \\ A \subset F}} F \\ &= \{x \in X : \forall\, r > 0\ D_r(x) \cap A \neq \varnothing\} \\ &= \{x \in X : \exists\, (a_n) \text{ in } A \text{ s.t. } a_n \to x\}\end{aligned}$$

Note that $\overline{D_r(x)} \subset B_r(x)$. In general, this inclusion can be strict. But normed spaces are nice:

Exercise. Show that, in a normed space, $\overline{D_r(x)} = B_r(x)$.

Recall also that, for $A \subset X$, we say that A *is dense in* X if

$$\begin{aligned}& \overline{A} = X \\ \iff\ & \forall\, x \in X\ \forall\, r > 0\ D_r(x) \cap A \neq \varnothing \\ \iff\ & \forall \text{ non-empty open } U \subset X\ U \cap A \neq \varnothing\end{aligned}$$

Example
$\mathbb{Q}$ is dense in $\mathbb{R}$ and so is $\sqrt{2} + \mathbb{Q}$. But $\mathbb{Q} \cap (\sqrt{2} + \mathbb{Q}) = \varnothing$.

Theorem 4.1 Baire category theorem
Let (X, d) be a complete metric space and $U_n \subset X$ be open and dense in X for each $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X.

Proof. Fix $x_0 \in X$ and $r_0 > 0$. Since U_1 is dense, $U_1 \cap D_{r_0}(x_0) \neq \varnothing$. Then we can pick $x_1 \in U_1 \cap D_{r_0}(x_0)$. Since $U_1 \cap D_{r_0}(x_0)$ is open, there exists $r_1 > 0$ such that $B_{r_1}(x_1) \subset U_1 \cap D_{r_0}(x_0)$. WLOG, we can pick $r_1 < 1$. We then continue inductively. At the n^{th} stage, density of U_n implies that $U_n \cap D_{r_{n-1}}(x_{n-1}) \neq \varnothing$, so we can pick $x_n \in U_n \cap D_{r_{n-1}}(x_{n-1})$. Since $U_n \cap D_{r_{n-1}}(x_{n-1})$ is open, there exists $r_n > 0$ such that $B_{r_n}(x_n) \subset U_n \cap D_{r_{n-1}}(x_{n-1})$. WLOG, $r_n < 1/n$.

We end up with $(x_n)_{n=0}^{\infty}$ in X and $(r_n)_{n=0}^{\infty}$ with $0 < r_n < 1/n$ for all $n \in \mathbb{N}$ and, for all $n > N \geq 0$,

$$\begin{aligned}B_{r_n}(x_n) &\subset U_n \cap D_{r_{n-1}}(x_{n-1}) \\ &\subset U_n \cap U_{n-1} \cap D_{r_{n-2}}(x_{n-2}) \\ &\ \vdots \\ &\subset U_n \cap U_{n-1} \cap \cdots \cap U_{N+1} \cap D_{r_N}(x_N)\end{aligned}$$

so, for all $m, n \geq N$, we have $d(x_m, x_n) \leq 2r_N < 2/N$. Thus, $(x_n)_{n=0}^{\infty}$ is Cauchy and thus convergent in X. Write $x = \lim_{n\to\infty} x_n$. Note that, for $n \geq m$, $x_n \in B_{r_m}(x_m)$ so $x \in B_{r_m}(x_m)$. By fixing $N = 0$ above and taking $n \to \infty$, we get

$$x \in \left(\bigcap_{n \in \mathbb{N}} U_n \right) \cap D_{r_0}(x_0)$$

as required. ∎

Remark. A countable intersection of open sets is called a G_δ*-set*. Theorem 4.1 then says that a countable intersection of open dense sets in a complete metric space is a dense G_δ-set.

Application *Uncountability of* $\mathbb{R}$
Suppose, on the contrary, that $\mathbb{R}$ is countable, so we can write $\mathbb{R} = \{r_1, r_2, r_3, \cdots\}$. Let $U_n = \mathbb{R}\backslash\{r_n\}$. Then U_n is open and dense in $\mathbb{R}$. Since $\mathbb{R}$ is complete, Theorem 4.1 tells us that $\bigcap_{n\in\mathbb{N}} U_n$ is dense in $\mathbb{R}$. But $\bigcap_{n\in\mathbb{N}} U_n = \varnothing$ — a contradiction!

Observe that, if $U \subset X$ is open and dense in X, then $F = X\backslash U$ is closed in X and $\int F = \varnothing$.

Definition Nowhere dense
Let (X, d) be a topological space. We say that $A \subset X$ is nowhere dense in X if $\operatorname{int}\overline{A} = \varnothing$.

Remarks

- For $A \subset Y \subset X$, it is possible that A is nowhere dense in X but not in Y (e.g. take $A = Y \neq \varnothing$)
- A is nowhere dense in X if and only if $U \not\subset \overline{U \cap A}$ for any nonempty open $U \subset X$.
  A is dense in X if and only if $U \subset \overline{U \cap A}$ for every open $U \subset X$.

Example

- In $\mathbb{R}$, any finite set and the Cantor set are nowhere dense.
- Write $\mathbb{Q} = \{q_1, q_2, q_3, \cdots\}$ and let $(\delta_n)_{n\in\mathbb{N}}$ in $(0, 1)$. Then $U = \bigcup_{n\in\mathbb{N}}(q_n - \delta_n, q_n + \delta_n)$ is open and dense in $\mathbb{R}$. So $\mathbb{R}\backslash U$ is closed and nowhere dense in $\mathbb{R}$.

Theorem 4.1'
Let (X, d) be a non-empty complete metric space. Suppose $X = \bigcup_{n\in\mathbb{N}} A_n$ for some $A_n \subset X$. Then there exists $N \in \mathbb{N}$ such that $\operatorname{int}\overline{A_n} \neq \varnothing$.

Proof. Suppose, on the contrary, that $\operatorname{int}\overline{A_n} = \varnothing$ for all $n \in \mathbb{N}$. Then $\forall\, x \in X\ \forall\, r > 0\ D_r(x) \not\subset \overline{A_n}$ and thus $D_r(x) \cap U_n = \varnothing$. Thus, $U_n = X\backslash\overline{A_n}$ is open and dense in X. Hence, by Theorem 4.1, $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X. But note that $\bigcap_{n\in\mathbb{N}} U_n = \left(\bigcup_{n\in\mathbb{N}} \overline{A_n}\right)^c = \varnothing$ — a contradiction! ■

Exercise. Deduce Theorem 4.1 from Theorem 4.1'.

Definition First and second category
Let X be a topological space and $A \subset X$.

- *We say that A is meagre in X or is of first category in X if $A = \bigcup_{n\in\mathbb{N}} A_n$ where A_n is nowhere dense in X for all $n \in \mathbb{N}$.*
- *We say that A is of second category in X if A is not of first category.*

Remarks

- Intuition: Think of meagre sets as 'small'.
- Typical Baire argument: Theorem 4.1' is useful as, to find some element $x \in X$ (in a non-empty complete metric space) with some property P, we just have to show that $A = \{x \in X : x \text{ fails } P\}$.

Application *Existence of a nowhere differentiable function in* $C[0, 1]$
Note that $(C[0, 1], ||\cdot||_\infty)$ is a nonempty complete metric space. Let

$$A = \{f \in C[0, 1] : \exists\, x \in [0, 1] \text{ s.t. } f \text{ differentiable at } x\}$$

Observe that, if $f'(x)$ exists, i.e. $[f(y)-f(x)]/(y-x) \to f'(x)$ as $y \to x$, then there exists $N \in \mathbb{N}$ such that, for all $y \in X$,

$$|y-x| < \frac{1}{N} \implies \left|\frac{f(y)-f(x)}{y-x}\right| \leq N$$

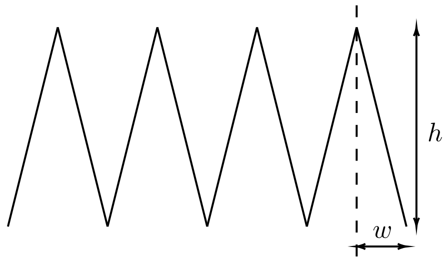
Thus, for $n \in \mathbb{N}$, consider the set

$$A_n = \left\{f \in C[0,1] \colon \exists\, x \in [0,1]\ \forall\, y \in [0,1]\ |y-x| < \frac{1}{n} \implies |f(y)-f(x)| \leq n|y-x|\right\}$$

and note that $A \subset \bigcup_{n\in\mathbb{N}} A_n$.

It then remains to show that, for all $n \in \mathbb{N}$, A_n is closed and $\operatorname{int} A_n = \varnothing$.

- A_n is closed: Consider $(f_k)_{k\in\mathbb{N}}$ in A_n with $f_k \to f$ in $C[0,1]$. For each $k \in \mathbb{N}$, we can pick $x_k \in [0,1]$ such that, for all $y \in [0,1]$, $|y-x_k| < 1/n \implies |f_k(y)-f_k(x_k)| \leq n|y-x_k|$. Passing to a subsequence if necessary, $x_k \to x$ in $[0,1]$ WLOG. By IB Analysis and Topology Example Sheet 1 Q5 (2024), $f_k(x_k) \to f(x)$ and hence
$$\forall\, y \in [0,1]\ |y-x| < \frac{1}{n} \implies |f(y)-f(x)| \leq n|y-x|$$
as required.
- Fix $f \in A_n$ and $r > 0$. To get $D_r(f) \not\subset A_n$, the idea is to consider a small but rapidly oscillating perturbation of f. Let $0 < \varepsilon < r/4$. Pick $\delta > 0$ such that $|y-x| < \delta \implies |f(y)-f(x)| < \varepsilon$. Choose h, w such that $4\varepsilon < h < r$ and $w = \min\{\varepsilon/n, \delta\}$. Set g to be the function



We can check that $f + g \in D_r(f)\backslash A_n$.

> *Direct proof:* Take g_n similar to above with height h_n and width w_n, where $h_n \searrow 0$ fast and $h_n/w_n \to \infty$ fast. Then $\sum g_n$ is nowhere differentiable.

Theorem 4.2 Principle of uniform boundedness[4]
Let X be a Banach space, Y a normed space and $\mathcal{T} \subset \mathcal{B}(X,Y)$. If T is pointwise bounded (i.e., $\forall\, x \in X\ \sup_{T\in\mathcal{T}} ||Tx|| < \infty$), then T is uniformly bounded (i.e., $\sup_{T\in\mathcal{T}} ||T|| < \infty$).

Proof. Let $A_n = \{x \in X \colon \sup_{T\in\mathcal{T}} ||Tx|| \leq n\}$. By hypothesis, $X = \bigcup_{n\in\mathbb{N}} A_n$. By Theorem 4.1', there exists $n \in \mathbb{N}$ such that $\operatorname{int} \overline{A_n} \neq \varnothing$. Note that $A_n = \bigcap_{T\in\mathcal{T}}\{x \in X \colon ||Tx|| \leq n\}$ is closed as the map $x \mapsto ||Tx||$ is continuous. Thus, there exists $r > 0$ and $x \in A_n$ such that $B_r(x) \subset A_n$. Given $y \in B_X$, $T \in \mathcal{T}$, we have $x + ry, x - ry \in B_r(x)$ and thus

$$||Ty|| = \left|\left|\frac{T(x+ry)-T(x-ry)}{2r}\right|\right| \leq \frac{2n}{2r} = \frac{n}{r}$$

Hence, $||T|| \leq n/r$ for all $T \in \mathcal{T}$. ■

[4] This result is also known as the *Banach-Steinhaus theorem.*

Corollary 4.3
Let X be a Banach space, Y a normed space, and $(T_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{B}(X,Y)$ that pointwise converges to T. Then T is linear and bounded. Moreover, $\sup_n ||T_n|| < \infty$.

Proof. For all $x \in X$, $(T_n x)_{n\in\mathbb{N}}$ is convergent and thus bounded. So $\{T_n : n \in \mathbb{N}\}$ is pointwise bounded. Hence, by Theorem 4.2, there exists $M \geq 0$ such that, for all $n \in \mathbb{N}$, we have $||T_n|| \leq M$.

- T linear: $T(\lambda x + \mu y) = \lim_{n\to\infty} T_n(\lambda x + \mu y) = \lim_{n\to\infty}[\lambda T_n(x) + \mu T_n(y)] = \lambda T(x) + \mu T(y)$
- T bounded: $\forall\, x \in B_X\ \forall\, n \in \mathbb{N}\ ||T_n x|| \leq ||T_n|| \leq M$, so $||Tx|| = \lim_{n\to\infty} ||T_n x|| \leq M$ for all $x \in B_X$. Hence, T is bounded with $||T|| \leq M$. ■

Exercise. Show that $||T|| \leq \liminf_{n\to\infty} ||T_n||$.

Definition δ-dense
Let A, B be subsets of a metric space (X,d) and $\delta > 0$. We say that A is δ-dense in B if $\forall\, b \in B\ \exists\, a \in A\ d(a,b) \leq \delta$.

Remark. If $\overline{A} \supset B$, then A is δ-dense in B for all $\delta > 0$.

Lemma 4.4 Open mapping lemma
Let X be a Banach space, Y a normed space, $T \in \mathcal{B}(X,Y)$. Suppose that $T(MB_X)$ is δ-dense in B_Y for some $M \geq 0$ and $0 \leq \delta < 1$. Then $T(\frac{M}{1-\delta}B_X) \supset B_Y$.

Remarks

- Another way to think of the open mapping lemma is as follows.
  Condition: for all $y \in B_Y$, $y = Tx$ has a δ-approximate solution in MB_X
  Conclusion: for all $y \in B_Y$, $y = Tx$ has an exact solution in $\frac{M}{1-\delta}B_X$
- For any $M \geq 0$, $T(MB_X)$ is 1-dense in B_Y since $0 = T(0) \in T(MB_X)$ and $\forall\, y \in Y\ ||y - T(0)|| = ||y|| \leq 1$
- Lemma 4.4 implies that T is surjective
- Lemma 4.4 shows that $\overline{T(B_X)} \supset B_Y \implies T(D_X) \supset D_Y$.

Proof of Lemma 4.4. The strategy is to use 'successive approximations'. Fix $y \in B_Y$. Pick $x_1 \in MB_X$ such that $||y - Tx_1|| \leq \delta$. Note that $\frac{y - Tx_1}{\delta} \in B_X$, so we can pick $x_2 \in MB_X$ such that $||\frac{y-Tx_1}{\delta} - Tx_2|| \leq \delta$. Note that $\frac{y - T(x_1 + \delta x_2)}{\delta^2} \in B_Y$, so we can pick $x_3 \in MB_X$ such that $||\frac{y-T(x_1+\delta x_2)}{\delta^2} - Tx_3|| \leq \delta$. Continuing inductively, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in MB_X such that

$$\left|\left| y - T\left(\sum_{k=1}^{n} \delta^{k-1} x_k\right)\right|\right| \leq \delta^n \qquad \forall\, n \in \mathbb{N}$$

Note that $||\delta^{n-1}x_n|| \leq M\delta^{n-1}$ for all $n \in \mathbb{N}$, so the series $\sum_{n=1}^{\infty} \delta^{n-1}x_n$ converges absolutely and hence converges since X is complete (cf. Example Sheet 1 Q8). Let $x = \sum_{n=1}^{\infty} \delta^{n-1}x_n$. Then $||x|| \leq \sum_{n=1}^{\infty} \delta^{n-1}M = \frac{M}{1-\delta}$, so $x \in \frac{M}{1-\delta}B_X$. Since T is continuous, $Tx = \sum_{n=1}^{\infty} T(\delta^{n-1}x_n) = \lim_{n\to\infty} T(\sum_{k=1}^{n} \delta^{k-1}x_k) = y$. ■

Theorem 4.5 Open mapping theorem
Let X, Y be Banach spaces and $T \in \mathcal{B}(X,Y)$. If T is surjective, then T is open.

Remark. In particular, we have $T(B_X) \supset T(D_X) \supset rB_Y$ for some $r > 0$. Equivalently, $T(MB_X) \supset B_Y$ for some $M > 0$. So the conclusion of Theorem 4.5 is that, for all $y \in Y$, $y = Tx$ has a solution such that $||x|| \leq M||y||$.

Proof of Theorem 4.5. Observe that it suffices to show that $T(MB_X) \supset B_Y$ for some $M > 0$. Indeed, it would mean that, given open $U \subset X$ and $y \in T(U)$, we can pick $x \in U$ such that $y = Tx$. As U is open, $B_r(x) \subset U$ for some $r > 0$. Then $T(U) \supset T(B_r(x)) = T(x + rB_X) = T(x + \frac{r}{M}MB_X) \supset y + \frac{r}{M}B_Y = B_{r/M}(y)$.

Note that $Y = T(X) = T(\bigcup_{n\in\mathbb{N}} nB_X) = \bigcup_{n\in\mathbb{N}} T(nB_X)$. As Y is non-empty and complete, Theorem 4.1 implies that there exists $n \in \mathbb{N}$ such that $\operatorname{int}\overline{T(nB_X)} \neq \varnothing$. So there exists $y \in Y$ and $r > 0$ such that $B_r(y) \subset \overline{T(nB_X)}$. Moreover, since B_X is convex and symmetric, so is $\overline{T(nB_X)}$. Thus, given $z \in B_Y$, we have $y \pm rz \in B_r(y) \subset \overline{T(nB_X)}$ and also $-y \pm rz \in \overline{T(nB_X)}$. Now, note that $rz = \frac{1}{2}(y+rz)+\frac{1}{2}(-y+rz) \in \overline{T(nB_X)}$. Thus, $rB_Y \subset \overline{T(nB_X)}$ or $B_Y \subset \overline{T(\frac{n}{r}B_X)}$. So $T(\frac{n}{r}B_X)$ is $\frac{1}{2}$-dense in B_Y. By Lemma 4.4, $T(\frac{2n}{r}B_X) \supset B_Y$. ■

Theorem 4.6 Inversion theorem
Let X, Y be Banach spaces and $T\colon X \to Y$ a continuous linear bijection. Then $T^{-1}\colon Y \to X$ is also continuous.

Proof 1. T is surjective so it is open by Theorem 4.5. So for open $U \subset X$, $(T^{-1})^{-1}(U) = T(U)$ is open in Y. So T^{-1} is continuous. ■

Proof 2. By Theorem 4.5, we have $T(MB_X) \supset B_Y$ for some $M > 0$. Given $y \in Y$, there exists $x \in X$ such that $y = Tx$ and $||x|| \leq M||y||$, i.e., $||T^{-1}y|| \leq M||y||$. So T^{-1} is bounded. ■

Corollary 4.7
Let $||\cdot||$ and $||\cdot||'$ be complete norms on a vector space X. If there exists $b > 0$ such that $||x||' \leq b||x||$ for all $x \in X$, then $||\cdot||$ and $||\cdot||'$ are equivalent.

Proof. $\mathrm{Id}\colon (X, ||\cdot||) \to (X, ||\cdot||')$ is a linear bijection and bounded by hypothesis. ■

Remark. This gives us another proof that the L_1-norm on $C[0,1]$ is complete.

Recall that, for a function $f\colon X \to Y$ between sets, the *graph of f* is the set

$$\Gamma(f) = \{(x, y) \in X \times Y : y = f(x)\}.$$

If X, Y are topological spaces with Y Hausdorff and f continuous, then $\Gamma(f)$ is closed.[5]

Theorem 4.8 Closed graph theorem
Let X, Y be Banach spaces and $T\colon X \to Y$ be a linear map. If $\Gamma(T)$ is closed, then T is continuous.

Remark. Note that continuity of T means:

$$x_n \to x \text{ in } X \implies (Tx_n) \text{ converges in } Y \text{ and } \lim_{n\to\infty} Tx_n = Tx$$

Now, suppose we only have:

$$x_n \to x \text{ in } X \text{ and } Tx_n \to y \text{ in } Y \implies y = Tx$$

Observe that this means that $\Gamma(T)$ is closed. By Theorem 4.8, this suffices to get continuity of T.

[5] You may recognise this from Tripos 2025 Paper 2 Section II Question 10G

Proof of Theorem 4.8. Consider the map $S\colon X \to \Gamma(T), x \mapsto (x, Tx)$. This is a linear bijection with $S^{-1}\colon \Gamma(T) \to X, (x, y) \mapsto x$. So $S^{-1} = P_X|_{\Gamma(T)}$ is continuous. Since $\Gamma(T)$ is a closed subspace of the Banach space $X \oplus Y$, $\Gamma(T)$ is complete. By Theorem 4.6, $S = (S^{-1})^{-1}$ is continuous. Hence, $T = P_Y \circ S$ is continuous. ■

Remark. Note that the condition '$x_n \to x$ in X and $Tx_n \to y$ in $Y \implies y = Tx$' is equivalent to '$x_n \to 0$ in X and $Tx_n \to y$ in $Y \implies y = 0$'.

Exercise. Deduce Theorem 4.6 from Theorem 4.8.