University of Cambridge Mathematical Tripos Part II

Analysis of Functions

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Course schedule

Lebesgue integration theory

Review of integration: simple functions, monotone and dominated convergence; existence of Lebesgue measure; definition of L^p spaces and their completeness. The Lebesgue differentiation theorem. Egorov's theorem, Lusin's theorem. Mollification by convolution, continuity of translation and separability of L^p when $p \neq \infty$. [5]

Banach and Hilbert space analysis

Strong, weak and weak-* topologies; reflexive spaces. Review of the Riesz representation theorem for Hilbert spaces; the Radon–Nikodym theorem; the dual of L^p. Compactness: review of the Ascoli–Arzelà theorem; weak-* compactnesss of the unit ball for separable Banach spaces. The Riesz representation theorem for spaces of continuous functions. The Hahn–Banach theorem and its consequences: separation theorems; Mazur's theorem. [7]

Fourier analysis

Definition of Fourier transform in L^1; the Riemann–Lebesgue lemma. Fourier inversion theorem. Extension to L^2 by density and Plancherel's isometry. Duality between regularity in real variable and decay in Fourier variable. [3]

Generalized derivatives and function spaces

Definition of generalized derivatives and of the basic spaces in the theory of distributions: $\mathcal{D}/\mathcal{D}'$ and $\mathcal{S}/\mathcal{S}'$. The Fourier transform on $\mathcal{S}'$. Periodic distributions; Fourier series; the Poisson summation formula. Definition of the Sobolev spaces H^s in $\mathbb{R}^d$. Sobolev embedding. The Rellich–Kondrashov theorem. The trace theorem. [5]

Applications

Construction and regularity of solutions for elliptic PDEs with constant coefficients on $\mathbb{R}^n$. Construction and regularity of solutions for the Dirichlet problem of Laplace's equation. The spectral theorem for the Laplacian on a bounded domain. *The direct method of the Calculus of Variations.* [4]

Contents

1 Lebesgue integration theory

1.1 Recap of measure theory

We recall some basic notions from II Probability and Measure whilst establishing the notation for this course.

Let E be a set. A family $\mathcal{B} \subset \mathcal{P}(E)$ is a σ-algebra if $\varnothing \in \mathcal{B}$ and it is closed under countable unions and complements. A map $\mu\colon \mathcal{B} \to [0, \infty]$ is a measure if it is σ-additive, i.e., $\mu(\bigsqcup_n A_n) = \sum_n \mu(A_n)$ for a countable disjoint collection (A_n). A pair $(E, \mathcal{B})$ is called a measurable space, whereas a triple $(E, \mathcal{B}, \mu)$ is called a measure space. If E is a topological space, we can (and, in this course, will) consider $\mathcal{B}$ to be the Borel σ-algebra.

A map $f\colon E \to \mathbb{C}$ is measurable if $f^{-1}(A) \in \mathcal{B}$ for all Borel $A \subset \mathbb{C}$. If $f\colon E \to [0, \infty]$ is measurable, then $\int f \, d\mu$ is well-defined (in $[0, \infty]$). We say that $f\colon E \to \mathbb{C}$ is integrable if it is measurable and $\int |f| \, d\mu < \infty$. We then denote by $\mathcal{L}^1(E, \mathcal{B}, \mu)$ the set of all integrable functions $E \to \mathbb{C}$. Writing $f \sim g \iff f = g$ a.e., we then define $L^1(E, \mathcal{B}, \mu) := \mathcal{L}^1(E, \mathcal{B}, \mu)/\sim$. However, it is of course standard to refer to a concrete function f from some equivalence class $[f] \in L^1(E)$.

Theorem 1.1 Dominated convergence theorem
Let $(E, \mathcal{B}, \mu)$ be a measure space. Let $g, f, f_1, f_2 \in L^1(E, \mathcal{B}, \mu)$. Suppose $f_n(x) \to f(x)$ and $|f_n(x)| \leq g(x)$ for a.e. $x \in E$. Then

$$\int_E f_n \, d\mu \to \int_E f \, d\mu$$

Recall that a measure space $(E, \mathcal{B}, \mu)$ is σ-finite if there exists $(A_n)_{n\in\mathbb{N}}$ such that $E = \bigcup_{n\in\mathbb{N}} A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

Theorem 1.2 Fubini's theorem
Let $(E, \mathcal{A}, \mu)$ and $(F, \mathcal{B}, \mu)$ be two σ-finite measure spaces. Let $f\colon E \times F \to \mathbb{C}$ be $\mathcal{A}\otimes\mathcal{B}$-measurable.
(i) *If the map $x \mapsto \int_F f(x, y) \, d\nu$ in $L^1(E, \mathcal{A}, \mu)$, then $f \in L^1(E \times F)$.*
(ii) *If $f \in L^1(E \times F)$, then*

$$\int_{E\times F} f \, d\mu \otimes \nu = \int_E \int_F f(x, y) \, d\nu(y) \, d\mu(x) = \int_F \int_E f(x, y) \, d\mu(x) \, d\nu(y)$$

1.2 Signed and complex measures

Definition Complex and signed measures
Let $(E, \mathcal{B})$ be a measurable space. A set function $\mu\colon \mathcal{B} \to \mathbb{C}$ is a complex measure if it is σ-additive. We then say that $(E, \mathcal{B}, \mu)$ is a complex measure space. If $\mu(\mathcal{B}) \subset \mathbb{R}$, we call μ a signed measure and $(E, \mathcal{B}, \mu)$ a signed measure space.

Remark. Previously, we really have been considering *positive* measures, though we will always just refer to such maps as *measures*. Note that not every (positive) measure is a complex measure.

Given a complex measure μ, define the *real* and *imaginary parts* $\operatorname{Re}\mu$ and $\operatorname{Im}\mu$ as

$$\forall\, A \in \mathcal{B} \qquad \operatorname{Re}\mu(A) = \operatorname{Re}(\mu(A)), \qquad \operatorname{Im}\mu(A) = \operatorname{Im}(\mu(A))$$

It is easy to verify that these are signed measures.

Definition Positive and negative sets
Let $(E, \mathcal{B}, \mu)$ be a signed measure space. We say that A is a positive (resp. negative) set if $\mu(B) \geq 0$ (resp. $\mu(B) \leq 0$) for all $B \subset A$.

Theorem 1.3 Hahn decomposition
Let $(E, \mathcal{B}, \mu)$ be a signed measure space. Then there exists a positive set $P \in \mathcal{B}$ and a negative set $N \in \mathcal{B}$ such that $E = P \sqcup N$.

In proving the Hahn decomposition theorem, we will ultimately be defining P be a positive set of the largest possible measure. However, we first show the existence of non-trivial positive sets via the following lemma.

Lemma 1.4
For all $A \subset \mathcal{B}$, there is a positive set $D \subset A$ such that $\mu(D) \geq \mu(A)$.

Proof. If A is positive, then we can simply take $D = A$ and we are done. It remains to carefully consider the case A is not positive. Pick negative set $B_1 \subset A$ such that $\mu(B_1)$ is "as negative as possible" in the approximate sense that there is no $B \subset A$ and $k \in \mathbb{N}$ for which $\mu(B_1) > -1/k \geq \mu(B)$. Let $A_1 = A \backslash B_1$. Continue inductively to define $B_2, A_2, B_3, A_3, \cdots$ in such a way that $B_{j+1} \subset A_j$ and $\mu(B_{j+1}) > -1/k$ for some $k \in \mathbb{N}$ only if this is so for all subsets of A_i; we then take $A_{j+1} = A_j \backslash B_{j+1}$.

Now, take $D = \bigcap A_j$. Then $A = D \sqcup B_1 \sqcup B_2 \sqcup \cdots$, so $\mu(A) = \mu(D) + \sum_j \mu(B_j)$. Since $\mu(B_j) \geq 0$ for each j, we have $\mu(D) \geq \mu(A)$. It then remains to show that D is positive. Note that $\mu(B_j) \to 0$ by convergence of $\sum_j \mu(B_j)$. Fix $k \in \mathbb{N}$. Then $\mu(B_i) > -1/k$ for some i. Then $\mu(B) > -1/k$ for all $B \subset A_{i-1}$ and thus for all $B \subset D$. Hence, D must be positive. ■

Proof of Theorem 1.3. Note that a union of positive sets $A = A_1 \cup A_2 \cup \cdots$ is positive; indeed, for any $B \subset A$, we have $B = (B \cap A_1) \sqcup (B \cap A_2 \backslash A_1) \sqcup (B \cap A_3 \backslash (A_1 \cup A_2)) \sqcup \cdots$ which implies that $\mu(B) \geq 0$ by σ-additivity.

Let $s = \sup\{\mu(A) \colon A \subset E \text{ positive}\}$ and let (P_i) be a sequence of positive sets in $\mathcal{B}$ such that $\lim \mu(P_i) = s$. Then by above, we know that $P = \bigcup P_i$ is positive. Since $\mu(P) \geq \mu(P_i)$, we also have $\mu(P) = s$.

Now, suppose, on the contrary, that $N := E \backslash P$ is not a negative set. Then there exists $B \subset N$ such that $\mu(B) > 0$. By Lemma 1.4, there exists $D \subset B$ positive such that $\mu(D) \geq \mu(B)$. It then follows that $\mu(P \cup D) > \mu(P)$ with $P \cup D$ a positive set — a contradiction! ■

Remarks
- A key corollary of the Hahn decomposition theorem is the Jordan decomposition of signed measures.
- The decomposition $E = P \sqcup N$ is manifestly non-unique as we can, for instance, 'move' a negligible subset from P to N. However, this makes no difference in the Jordan decomposition of μ.

Corollary 1.5 Jordan decomposition of signed measures
Let $(E, \mathcal{B}, \mu)$ be a signed measure space, with $E = P \sqcup N$ for positive P and negative N. Then $\mu^+ := \mu|_P$ and $\mu^- := -\mu|_N$ are positive measures satisfying $\mu = \mu^+ - \mu^-$.

Proof. Trivial. ■

Now that we've proven the Hahn and Jordan decompositions, we can now extend Lebesgue integration to signed and complex measures.

Definition Integral with respect to signed measures
Let $(E, \mathcal{B}, \mu)$ be a signed measure space. A measurable function $f \colon E \to \mathbb{C}$ is integrable if it is integrable with respect to μ^+ and μ^-. In that case, we then define its integral to be

$$\int_E f \, d\mu := \int_E f \, d\mu^+ - \int_E f \, d\mu^-$$

Definition Integral with respect to complex measures
Let $(E, \mathcal{B}, \mu)$ be a complex measure space. A measurable function $f \colon E \to \mathbb{C}$ is integrable with respect to $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$. In that case, we then define its integral to be

$$\int_E f \, d\mu := \int_E f \, d \operatorname{Re} \mu + i \int_E f \, d \operatorname{Im} \mu$$

Finally, we end this subsection with a brief discussion on the Banach space structure of the space of complex measures. Given a signed measure μ, we can define its *total variation measure* to be

$$|\mu| := \mu^+ + \mu^-$$

and its *total variation norm* to be

$$||\mu|| := |\mu|(E) = \mu^+(E) + \mu^-(E)$$

On the Example Sheet, you will extend these notions to complex measures. It can then be shown that the space of complex measures on a measurable space $(E, \mathcal{B})$ forms a Banach space with respect to the total variation norm.

1.3 Radon-Nikodym theorem

Motivation: In II Probability and Measure, you met some examples of continuous and discrete random variables. Are there probability distributions that do not belong to either category? How do we know if a random variable has a density?

Let us start off with some definitions. Let μ, ν be (positive) measures on $(E, \mathcal{B})$.

Definition Absolutely continuous measure
We say that ν is absolutely continuous wrt μ if $\mu(A) = 0 \implies \nu(A) = 0$ for all $A \subset B$. We then write $\nu \ll \mu$.

Definition Singular measure
We say that ν is singular wrt μ if there exists a decomposition $E = A \sqcup B$ such that $\nu(A) = 0$ and $\mu(B) = 0$. We then write $\nu \perp \mu$.

Remark. It is easy to see that $\nu \perp \mu \iff \mu \perp \nu$.

Definition Concentrates on a set
We say that μ concentrates on a set $A \in \mathcal{B}$ if $\mu(E \backslash A) = 0$.

Remark. $\mu \perp \nu$ iff there exist disjoint sets $A, B \in \mathcal{B}$ such that μ is concentrated on A and ν is concentrated on B.

We now state the two main theorems of this subsection.

Theorem 1.6 Radon-Nikodym theorem
Let μ, ν be finite measures on $(E, \mathcal{B})$. Suppose that $\nu \ll \mu$. Then there exists $f \in L^1(E, \mathcal{B}, \mu)$ such that

$$\nu(A) = \int_A f \, d\mu$$

If g is another function that satisfies the conclusion of this theorem, then $f = g$ mu-a.e.

The function f in the theorem above is called the *Radon-Nikodym (RN) derivative*, often denoted by

$$\frac{d\nu}{d\mu}$$

Remark. We could extend this theorem by replace finite with σ-finite, but this comes at the expense of having $f \in L^1$.

Theorem 1.7 Lebesgue decomposition
Let μ, ν be finite measures. Then there exist unique measures ν_a, ν_s such that $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu$ and $\nu_s \perp \mu$.

The idea behind the proof of these theorems is as follows. For each $t \in \mathbb{R}_{\geq 0}$, let $P_t \sqcup N_t$ be a Hahn-decomposition of $\nu - t\mu$. Then, for all measurable $A \subset P_t$, we have $(\nu - t\mu)(A) \geq 0$ and so $\nu(A) \geq t\mu(A)$. Similarly, for all measurable $A \subset N_t$, we have $\nu(A) \leq t\mu(A)$. Let $t_1 < t_2$. if $A \in P_{t_1} \cap N_{t_2}$, then

$$t_1 \mu(A) \leq \nu(A) \leq t_2 \mu(A)$$

so it is reasonable to expect that the Radon-Nikodym derivative will be between t_1 and t_2 on $P_{t_1} \cap N_{t_2}$.

The above intuition motivates the following definition: For each $n \in \mathbb{Z}_{\geq 0}$, define

$$f_n(x) := \sup\{t \in 2^{-n}\mathbb{Z}_{\geq 0} \colon x \in P_t\}$$

where $2^{-n}\mathbb{Z}_{\geq 0}$ is the set of all numbers of the form $2^{-n}a$ with $a \in \mathbb{Z}_{\geq 0}$. Set $f(x) := \lim_{n\to\infty} f_n(x)$ for each x.

Lemma 1.8
There exists $A \in \mathcal{B}$ such that $\mu(E \backslash A) = 0$ and

$$\nu(B) = \int_B f \, d\mu$$

for all $B \subset A$.

Proof. For each $n \in \mathbb{Z}_{\geq 0}$, let $A_n := \{x \colon f_n(x) < \infty\}$. We first show that $\mu(E \backslash A_n) = 0$. To do this, fix $t \in 2^{-n}\mathbb{Z}_{\geq 0}$. Then $E \backslash A_n \subset \bigcup_{s \geq t} P_s$, with $\bigcup_{s \geq t} P_s$ positive for $\nu - t\mu$. We thus deduce that $\nu(E) \geq \nu(E \backslash A_n) \geq t\mu(E \backslash A_n)$. Taking $t \to \infty$, we conclude that $\mu(E \backslash A_n) = 0$.

Now, fix $B \subset A_n$. For $s \in 2^{-n}\mathbb{Z}_{\geq 0}$, let

$$B_t = \{x \in B \colon f_n(x) = t\}$$

Note that $B = \bigsqcup_t B_t$ and that, if $x \in B_t$, then $x \in P_t$ but $x \notin P_{t+2^{-n}}$. Thus, $B_t \subset P_t \cap N_{t+2^{-n}}$ and so we have

$$t\mu(B_t) \leq \nu(B_t) \leq (t + 2^{-n})\mu(B_t)$$

This implies that

$$\left|\nu(B_t) - \int_{B_t} f_n \, d\mu\right| \leq 2^{-n}\mu(B_t)$$

from which it follows that

$$\left|\nu(B) - \int_{B} f_n \, d\mu\right| \leq 2^{-n}\mu(B)$$

Finally, set $A := \bigcap_n A_n$. Picking $B \subset A$, the above holds for all n, so by the monotone convergence theorem, we conclude that

$$\int_B f \, d\mu = \lim_{n\to\infty} \int_B f_n \, d\mu = \nu(B)$$

Note also that $\mu(E\backslash A) = \mu(\bigcup(E\backslash A_n)) = 0$. ■

Proof of Theorem 1.6. Since $\nu \ll \mu$, we have $\nu(E\backslash A) = 0$. Then for each $B \in \mathcal{B}$,

$$\nu(B) = \nu(B \cap A) + \nu(B\backslash A) - \int_{B\cap A} f \, d\mu = \int_B f \, d\mu$$

Thus, f is a Radon-Nikodym derivative.

It remains to show uniqueness. Let f, g be two RN derivatives. Suppose that, on the contrary, $f \neq g$ μ-a.e. Then there exists there exists $\varepsilon > 0$ such that either $B = \{x\colon f(x) - g(x) > \varepsilon\}$ or $\{x\colon f(x) - g(x) < \varepsilon\}$ has positive measure. WLOG, let it be the first one. Then we have

$$0 = \nu(B) - \nu(B) = \int_B (f - g) \, d\mu \geq \int_B \varepsilon \, d\mu = \varepsilon\mu(B)$$

as required. ■