University of Cambridge Mathematical Tripos Part II

Differential Geometry

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Course schedule

Smooth manifolds in $\mathbb{R}^n$, tangent spaces, smooth maps and the inverse function theorem. Examples, regular values, Sard's theorem (statement only). Transverse intersection of submanifolds. [4]

Manifolds with boundary, degree mod 2 of smooth maps, applications. [3]

Curves in 2-space and 3-space, arc-length, curvature, torsion. The isoperimetric inequality. [2]

Smooth surfaces in 3-space, first fundamental form, area. [1]

The Gauss map, second fundamental form, principal curvatures and Gaussian curvature. Theorema Egregium. [3]

Minimal surfaces. Normal variations and characterization of minimal surfaces as critical points of the area functional. Isothermal coordinates and relation with harmonic functions. The Weierstrass representation. Examples. [3]

Parallel transport and geodesics for surfaces in 3-space. Geodesic curvature. [2]

The exponential map and geodesic polar coordinates. The Gauss-Bonnet theorem (including the statement about classification of compact surfaces). [4]

Global theorems on curves: Fenchel's theorem (the total curvature of a simple closed curve is greater than or equal to 2π); the Fary-Milnor theorem (the total curvature of a simple knotted closed curve is greater than 4π). [2]

Recommended books

J. Milnor *Topology from the differentiable viewpoint.* Princeton University Press, 1997.
M. Do Carmo *Differential Geometry of Curves and Surfaces.* Pearson Higher Education, 1976

Contents

1 Differential topology

Definition Smooth map on an open subset
Let $U \subset \mathbb{R}^n$. We say that $f\colon U \to \mathbb{R}^m$ is smooth if all partial derivatives to all orders exist and are continuous.

Definition Smooth map
Let $X \subset \mathbb{R}^n$. We say that $f\colon X \to \mathbb{R}^m$ is smooth if, for each $x \in X$, there exists (i) an open neighbourhood $U \subset \mathbb{R}^n$ of x and (ii) a smooth map $\tilde{f}\colon U \to \mathbb{R}^m$ such that $\tilde{f}|_{X\cap U} = f|_{X\cap U}$.

Definition Diffeomorphism
Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$. We say that $f\colon X \to Y$ is a diffeomorphism if f is a smooth bijection with a smooth inverse. If such a map exists, we say that X and Y are diffeomorphic.

Exercise. Give an example of a smooth bijection that is not a diffeomorphism.

Definition k-dimensional manifold
We say that $X \subset \mathbb{R}^N$ is a k-dimensional manifold if, for each $x \in X$, there exists an open neighbourhood $V \subset X$ of x such that V is diffeomorphic to an open subset $U \subset \mathbb{R}^k$. A diffeomorphism $\varphi\colon U \to V$ is called a local parametrisation of V, whereas its inverse $\psi := \varphi^{-1}\colon V \to U$ is called a coordinate system or a chart on V.

Remarks

- By composing φ^{-1} with the projections $\pi_i\colon \mathbb{R}^k \to \mathbb{R}, (x_1, \cdots, x_k) \mapsto x_i$, we get smooth maps $x_i := \pi_i \circ \varphi^{-1}$ which we call *coordinate functions.*
- WLOG, we can replace 'diffeomorphic to an open subset $U \subset \mathbb{R}^k$" with "diffeomorphic to an open ball in R^k".
- It is easy to see that, if $X \subset \mathbb{R}^N$ is both a k-dimensional manifold and a $\tilde{k}$-dimensional manifold, then $k = \tilde{k}$.

Definition Dimension
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold. The dimension of X is k, and it is denoted by $\dim X$.

Example *Some trivial examples*

- $X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W
- $X = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^N$

Example S^n
$S^n := \{x \in \mathbb{R}^{n+1}\colon ||x||_2 = 1\}$ is an n-dimensional manifold. To see this, consider the projection $\Pi_k\colon \mathbb{R}^{n+1} \to \mathbb{R}^n, (x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1})$. It is easy to verify that maps of the form $\psi_k^{\pm} = \Pi_k|_{S^n\cap\{\text{sign}(x_k)=\pm 1\}}$ are diffeomorphisms $S^n \cap \{\text{sign}(x_k) = \pm 1\} \to B_1(0)$.

Remark. It is easy to show that X is a 0-dimensional manifold iff X is a discrete subset of $\mathbb{R}^N$.

Exercise. Show that, if X and Y are manifolds, then $X \times Y$ is also a manifold, with $\dim X \times Y = \dim X + \dim Y$.

Definition Submanifold
Let $X, Y \subset \mathbb{R}^N$ be manifolds. If $Y \subset X$, then we say that Y is a submanifold of X. The codimension of Y in X is defined as

$$\operatorname*{codim}_X Y := \dim X - \dim Y$$

1.1 Tangent spaces

We first recall some basic facts from our youth. Let $U \subset \mathbb{R}^k$ be open. The *differential* of a smooth map $f\colon U \to \mathbb{R}^m$ at $x \in U$ is defined by

$$\begin{aligned} df_x\colon \mathbb{R}^k &\to \mathbb{R}^N \\ h &\mapsto \lim_{t\to 0} \frac{f(x+th)-f(x)}{t} \end{aligned}$$

This is a linear map, with matrix representation

$$df_x = \left(\frac{\partial f^i}{\partial x^j}\right)_{i,j}$$

Moreover, differentials satisfy the chain rule: given (i) two smooth maps $f\colon U \to \mathbb{R}^l$ and $g\colon V \to \mathbb{R}^m$ with $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$ open and (ii) a point $x \in U$ with $f(x) \in V$, we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Definition Tangent space
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold and $x \in X$. Choose a local parametrisation $\varphi\colon U \to V$ around x. We then define the tangent space T_xX of X at x to be

$$T_xX := \operatorname{im} d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k)$$

Of course, before we can safely proceed, we must show that T_xX is well-defined:

Lemma 1.1
Let X be as above. T_xX is independent of φ, and $\dim T_xX = k$.

Proof. Let $\tilde{\varphi}\colon \tilde{U} \to \tilde{V}$ be another local parametrisation near x. WLOG, by restricting if necessary, we may assume $\tilde{V} = V$. By the chain rule, we have

$$d\varphi_{\varphi^{-1}(x)} = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)} \circ d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}$$

Since $\tilde{\varphi}^{-1} \circ \varphi$ is a diffeomorphism of open subsets of $\mathbb{R}^n$, the corresponding differential $d(\tilde{\varphi}^{-1} \circ \varphi)$ is a linear isomorphism. Thus,

$$d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}(\mathbb{R}^k)) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(\mathbb{R}^k)$$

as claimed.

Now, it remains to show that $\dim T_xX = k$. By definition, there exists an open set $\hat{V} \subset \mathbb{R}^N$ and a smooth map $\Psi\colon \hat{V} \to \mathbb{R}^k$ that extends the chart $\psi := \varphi^{-1}$. Note that $\Psi \circ \varphi = \mathrm{id}_U$, so by the chain rule,

$$d\Psi_x \circ d\varphi_{\varphi^{-1}(x)} = \mathrm{id}_{\mathbb{R}^k}$$

Then, $d\varphi_{\varphi^{-1}(x)}$ must be an isomorphism $\mathbb{R}^k \to T_xX$, and hence $\dim T_xX = k$. ■

Example *Tangent spaces for our trivial examples*
Returning to the trivial examples we previously gave, we now state the corresponding tangent space for an arbitrary point x on each manifold.

- $X = \mathbb{R}^N$: $T_xX = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W: $T_xX = \mathbb{R}^N$
- $X = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^N$: $T_xX = X$

Example *Tangent spaces for S^n*
From any given chart, we can compute (φ and) $d\varphi$:

$$\frac{\partial \varphi}{\partial x^1} = (1, 0, \cdots, 0, -x_1/x_{n+1})$$

and similarly for $\partial\varphi/\partial x^i$. Manifestly, each partial derivative is perpendicular to x. Thus, $T_xX \subset x^\perp := \{v \in \mathbb{R}^{n+1} : \langle v, x\rangle = 0\}$. Since we know from the above lemma that $\dim T_xX = n$, we conclude that $T_xX = x^\perp$.

Definition Differential map for manifolds
Let $f: X \to Y$ be a smooth map between manifolds and $x \in X$. Choose a local parametrisation φ_1 around x and φ_2 around $f(x) \in Y$. We define the differential $df_x : T_xX \to T_{f(x)}Y$ of f at x by

$$df_x = d{\varphi_2}_{\varphi_2^{-1}(f(x))} \circ d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)} \circ (d{\varphi_1}_{\varphi_1^{-1}(x)})^{-1}$$

Lemma 1.2
df_x is independent of the choice of local parametrisations.

Proof. Trivial exercise. ■

Proposition 1.3 Chain rule for manifolds
Let $f: X \to Y$ and $g: Y \to Z$ be smooth maps between manifolds. For any $x \in X$,

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Proof. Trivial exercise. ■

Theorem 1.4 Inverse function theorem
Let $f: X \to Y$ be a smooth map between manifolds and $x \in X$. Suppose $df_x : T_xX \to T_{f(x)}Y$ is an isomorphism. Then f is a local diffeomorphism*, i.e., each $x \in X$ has an open neighbourhood $V \subset X$ such that $f|_V : V \to f(V)$ is a diffeomorphism.*

Proof. Since df_x is an isomorphism, it follows that $d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)}$ is also an isomorphism. We can then use the usual inverse function theorem to deduce the result. ■

1.2 Regular values and Sard's theorem

Definition Critical and regular points
Let $f: X \to Y$ be a smooth map between manifolds. We say that $x \in X$ is a critical point of f if $df_x : T_xX \to T_{f(x)}Y$ is not surjective. Otherwise, it is a regular point.

Notation. We denote by C the set of all critical points of f.

Remark. If $\dim Y > \dim X$, then $C = X$ and the pre-image of any regular value is $\varnothing$.

Definition Critical and regular values
Let $f: X \to Y$ be a smooth map between manifolds. We say that $y \in Y$ is a critical value of f if $y = f(x)$ for some $x \in C$. Otherwise, we say that y is a regular value of f.

Theorem 1.5 Pre-image theorem
Let $f: X \to Y$ be a smooth map between manifolds. Suppose $y \in Y$ is a regular value of f. If $f^{-1}(y) \neq \varnothing$, then $f^{-1}(y) \subset X$ is a submanifold of X with $\operatorname{codim}_X f^{-1}(y) = \dim Y$.

Proof. Fix $x \in f^{-1}(y)$. Since y is a regular value, we know that $df_x\colon T_xX \to T_yY$ is surjective. By the rank-nullity theorem, $\dim\ker df_x = \operatorname{codim}_X Y$. Suppose $X \subset \mathbb{R}^N$, and pick a linear map $T\colon \mathbb{R}^N \to \mathbb{R}^{\operatorname{codim}_X Y}$ such that $\ker T \cap \ker df_x = \{0\}$.[1]

Now, extend f to the map $F\colon X \to Y \times \mathbb{R}^{\operatorname{codim}_X Y}$ given by $z \mapsto (f(z), T(z))$. Note that the differential of F at x is given by

$$dF_x = (df_x, dT_x) = (df_x, T)$$

Since $\ker T \cap \ker df_x = \{0\}$, we have $\ker dF_x = \{0\}$, i.e., dF_x is injective. By the inverse function theorem for manifolds, there exists an open neighbourhood $U \subset X$ of x such that $F|_U\colon U \to V$ is a diffeomorphism to an open neighbourhood V of $(y, T(x))$. Hence, $F|_{f^{-1}(y)\cap U}$ is a local parametrisation of $(\{y\}\times\mathbb{R}^{\operatorname{codim}_X Y})\cap V$, proving that $f^{-1}(y)$ is a manifold of dimension $\operatorname{codim}_X Y$. ∎

Exercise. Show that, under the conditions of the pre-image theorem, $T_x f^{-1}(y) = \ker df_x$.

Corollary 1.6
Let $f\colon X \to Y$ be a smooth map between manifolds of the same dimension, with X compact. If y is a regular value of f, then $f^{-1}(y)$ is finite.

Proof. By the pre-image theorem, $f^{-1}(y)$ is a 0-dimensional manifold, i.e., a collection of points. Since X is compact, such a collection must be finite. ∎

With just a bit more analysis and topology, we can actually say more than just finiteness:

Theorem 1.7 Stack of records theorem
Let $f\colon X \to Y$ be a smooth map between manifolds of the same dimension, with X compact. Let y be a regular value of f, and list the elements of $f^{-1}(y)$ as $x_1, \cdots, x_n$. There exists an open neighbourhood $V \subset Y$ of y and a collection of open neighbourhoods $W_i \subset X$ of each x_i such that

$$f^{-1}(V) = \bigsqcup_{i=1}^{n} W_i$$

and each $f|_{W_i}\colon W_i \to V$ is a diffeomorphism.

Proof. By the inverse function theorem for manifolds, we can pick open neighbourhoods W_i of x such that each $f|_{W_i}$ is a diffeomorphism to an open neighbourhood of y. By shrinking neighbourhoods if necessary, W_i can be taken WLOG to be pairwise disjoint. Now, set

$$V = \left[\bigcap_{i=1}^{n} f(W_i)\right] \setminus f\left(X \setminus \bigcup_{i=1}^{n} W_i\right)$$

Note that $f(X\setminus\bigcup_{i=1}^n W_i)$ is a compact set that does not contain y, so V is an open neighbourhood of y. Finally, note that $f^{-1}(V) = \bigsqcup_{i=1}^n W_i$ by construction. ∎

Now, the pre-image theorem can be a powerful tool for generating manifolds or showing that a certain set is one.

Application *S^n is a manifold*
Consider the map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}, (x_1, \cdots, x_{n+1}) \mapsto x_1^2 + \cdots + x_{n+1}^2$. Note that $f^{-1}(1) = S^n$, so to show that S^n is a manifold, it suffices to show that 1 is a regular point. Indeed, note that $df_x = (2x_1, \cdots, 2x_{n+1})$, which is not surjective only if $x = 0 \notin f^{-1}(1)$.

[1] It is easy to constructively show using IB Linear Algebra that such a map exists. [*Exercise!*]

Application *Orthogonal group as a manifold*
Denote by $M(n)$ [resp. $S(n)$] the space of all [resp. symmetric] $n \times n$ matrices with entries in $\mathbb{R}$. Consider the orthogonal group $O(n) = \{A \in M(n) \colon AA^t = I\} \subset M(n) = \mathbb{R}^{n^2}$.

Let $f \colon M(n) \to O(n)$ be the map $A \mapsto AA^t$. This is smooth since multiplication and addition in $\mathbb{R}$ are smooth. Since $O(n) = f^{-1}(I)$, it suffices to show that I is a regular value of f. Note that

$$df_A(H) = \lim_{t \to 0} \frac{f(A + tH) - f(A)}{t} = AH^t + HA^t$$

Now, fix $A \in M(n)$. Given $B \in S(n)$, observe that

$$df_A\left(\frac{1}{2}CA\right) = \frac{1}{2}AA^tC^t + \frac{1}{2}CAA^t = \frac{1}{2}C + \frac{1}{2}C = C$$

completing the proof that I is a regular value of f.

Remark. Recall that, besides being a manifold as we've just shown, $O(n)$ is also a group. In fact, the group operations $(A, B) \mapsto AB$ and $A \mapsto A^{-1} = A^t$ are smooth. Hence, we see that $O(n)$ is a *Lie group.*

Now, the pre-image theorem raises the question: how easy is to find regular values? This leads us to Sard's theorem.

Definition Measure-zero subsets of $\mathbb{R}^N$
We say that $S \subset \mathbb{R}^N$ is of measure zero in $\mathbb{R}^N$ if, for each $\varepsilon > 0$, there exists a countable family $\{R_i\}$ of sets of the form $R_i = \prod_{j=1}^N [x_i^{(j)}, y_i^{(j)}]$ such that $S \subset \bigcup_i R_i$ and $\sum_i \operatorname{vol}(R_i) < \varepsilon$.

Definition Measure zero subsets of manifolds
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold. We say that $A \subset X$ is of measure zero in X if, for all local parametrisations $\varphi \colon U \to V$ of X, $\varepsilon^{-1}(V \cap A) \subset \mathbb{R}^k$ has measure zero in $\mathbb{R}^k$.

Exercise. Let $U, \tilde{U} \subset \mathbb{R}^k$ be open and $\psi \colon U \to \tilde{U}$ a diffeomorphism. Show that, if $A \subset U$ is of measure zero in $\mathbb{R}^k$, then $\tilde{T} = \psi(T)$ is of measure zero in $\mathbb{R}^k$.

Remarks

- In view of the above exercise, $A \subset X$ is of measure zero in X iff $\varphi_i^{-1}(S \cap V_i)$ is of measure zero for all $\varphi_i \colon U \to V$ in an atlas of local parametrisations.
- If $\dim Y = 0$, then Y is of measure zero. If $\dim Y > 0$, then every non-empty open subset $V \subset Y$ is not of measure zero in Y.
- If $S \subset X$ is of measure zero in X, then any $\tilde{S} \subset S$ is also of measure zero in X.

Theorem 1.8 Sard's theorem
Let $f \colon X \to Y$ be a smooth map between manifolds. Then the set of critical values of f is of measure zero in Y.

Proof. Non-examinable — see Milnor's book if interested. ∎

Corollary 1.9
The set of regular values of a smooth map $f \colon X \to Y$ between manifolds is dense in Y.

Proof. Any open set $V \subset Y$ cannot lie entirely in $f(C)$ since it has measure zero. ∎

1.3 Transversality

Definition Transversal
Let $f\colon X \to Y$ be smooth and $Z \subset Y$ a submanifold of Y. We say that f is transversal to Z if, for each $x \in f^{-1}(Z)$,

$$T_{f(x)}Y = T_{f(x)}Z + \operatorname{im} df_x$$

We then write $f \pitchfork Z$.

Remarks

- If $f(X) \cap Z = \varnothing$, then f is transversal to Z.
- If $Z = \{y\}$, then f is transversal to Z iff y is a regular value of f. Thus, transversality is really a generalisation of the notion of regular values.

Exercise. Let X also be a submanifold of Y and $\iota\colon X \hookrightarrow Y$ the inclusion map. Show that $d\iota_x$ is just the inclusion map $T_xX \hookrightarrow T_xY$ of the tangent spaces. Thus, $\iota \pitchfork Z$ iff $T_xX + T_xZ = T_xY$ for all $x \in X \cap Z$.

Now, we state a generalisation of the pre-image theorem for transversal maps:

Theorem 1.10
Let $f\colon X \to Y$ be smooth map that is transversal to a submanifold $Z \subset Y$ of Y. If $f^{-1}(Z) \neq \varnothing$, then $f^{-1}(Z) \subset X$ is a submanifold of X, with $\operatorname{codim}_X f^{-1}(Z) = \operatorname{codim}_Y Z$.

Remark. If $Z = \{y\}$, then $\operatorname{codim}_Y Z = \dim Y$ as in the pre-image theorem.

Sketch of proof (non-examinable). Fix $z \in Z$ with $z = f(x)$ for some $x \in X$. Note that, for some open neighbourhood $V \subset Y$ of z, there exists a smooth map $h\colon V \to \mathbb{R}^{\operatorname{codim}_Y Z}$ such that $Z \cap V = h^{-1}(0)$ and dh_z is surjective. Locally around $x \in X$, $f^{-1}(Z) = (h \circ f)^{-1}(0)$. Thus, by the pre-image theorem, it suffices to show that 0 is a regular value of $h \circ f$.

Now, since $f \pitchfork Z$, we have $T_zY = T_zZ + \operatorname{im} df_x$. By the exercise after the pre-image theorem, we have $dh_z = T_zZ$. Moreover, $f \pitchfork Z$ gives us

$$T_zY = T_zZ + \operatorname{im} df_x = \ker dh_z + \operatorname{im} df_x$$

This then implies that $\operatorname{im} dh_z = \operatorname{im}(dh_z \circ df_z) = \operatorname{im} d(h \circ f)_x$. Since dh_z is surjective, $d(h \circ f)_z$ is also surjective and hence 0 is a regular value of $h \circ f$. ■

Exercise. Construct the required map h.

Remark. Transversality is both a stable and generic property. It is stable in the sense that small perturbations of f remain transversal to a given submanifold. It is generic in the sense that any given smooth map may be deformed by arbitrarily small amounts into a map that is transversal to Z. See *Differential Topology* by Guillemin and Pollack for more details.

1.4 Manifolds with boundary

Consider the closed upper half plane

$$\mathbb{H}^k := \{(x_1, \cdots, x_k) \in \mathbb{R}^n : x_k \geq 0\}$$

We denote its boundary by $\partial\mathbb{H}^k = \{x_k = 0\}$.

Definition Manifold with boundary
We say that $X \subset \mathbb{R}^N$ is a (smooth) k-dimensional manifold with boundary if every $x \in X$ has an open neighbourhood $V \subset X$ that is diffeomorphic to an open subset $U \subset \mathbb{H}^k$.

Remark. Note that a diffeomorphism $\varphi\colon U \to V$ has a smooth extension defined on an open subset of $\mathbb{R}^k$. This allows us to deduce as before that, if X is both k- and $\tilde{k}$-dimensional, then $k = \tilde{k}$.

Definition Dimension
The dimension of a k-dimensional manifold with boundary X is k.

Definition Boundary of a manifold with boundary
Let X be a k-dimensional manifold with boundary. Its boundary is defined to be

$$\partial X := \{x \in X \colon \exists \text{ open nhood } V \subset X \text{ and diffeomorphism } \psi\colon V \to \psi(V) \text{ s.t. } x \in \psi^{-1}(\partial \mathbb{H}^k)\}$$

Remarks

- In fact, if $x \in \psi^{-1}(\partial\mathbb{H}^k)$ for some diffeomorphism $\psi\colon V \to \psi(V) \subset \mathbb{H}^k$ on an open nhood V of x, then it is true for *all* diffeomorphisms on an open nhood of x to its image in $\mathbb{H}^k$.
- In the definition of manifold with boundary, we may take $U = \mathbb{R}^k$ or $U = \mathbb{H}^k$ WLOG.

Exercise. Prove the first remark.

Definition Interior of manifold with boundary
Let X be a manifold with boundary. We define its interior to be

$$\operatorname{int} X := X \backslash \partial X$$

Proposition 1.11
Let X be a k-dimensional manifold with boundary. Then $\operatorname{int} X$ is a manifold of dimension k and ∂X is a manifold of dimension $k-1$.

Proof. $\operatorname{int} X$ is a manifold of dimension k because we can always restrict our diffeomorphisms $\varphi\colon V \to U$ such that $U \cap \partial\mathbb{H}^k = \varnothing$. See Example Sheet 1 for ∂X. ∎

Example

- Trivially, $\mathbb{H}^k$ is a k-dimensional manifold with boundary
- As we will prove later, $B^n := \{x \in \mathbb{R}^n \colon ||x||_2 \leq 1\}$ is an n-dimensional manifold with boundary. Note that $\partial B^n = S^{n-1}$ is a manifold of dimension $n-1$.
- $[0,1] \times [0,1]$ is not a manifold with boundary (see Example Sheet 1)
- If X is a manifold with boundary and Y is a manifold, then $X \times Y$ is a manifold with boundary, with $\partial(X \times Y) = (\partial X) \times Y$. (Of course, the previous example is a counterexample to the case that Y is also a manifold with boundary.)

Remark. Note that ∂X and $\operatorname{int} X$ are note the same as the topological notions of 'boundary' and 'interior' as subsets of $\mathbb{R}^N$. Indeed, if $\dim X < N$, the topological interior of X is empty, whereas $\operatorname{int} X$ is not.

Definition Tangent space
Let X be a k-dimensional manifold with boundary and $x \in X$. Let $\varphi\colon U \to V$ be a diffeomorphism from an open set $U \subset \mathbb{H}^k$ to an open neighbourhood $V \subset X$ of x. Since φ is smooth, there exists

a smooth extension $\tilde{\varphi}$ on an open subset of $\mathbb{R}^k$, with $d\tilde{\varphi}_{\varphi^{-1}(x)}$ well-defined. We then define the tangent space to be

$$T_xX := \operatorname{im} d\tilde{\varphi}_{\varphi^{-1}(x)}$$

Remark. As before, T_xX is well-defined.

Exercise. Show that, for every $x \in \partial X$, $T_x\partial X \subset T_xX$.

Lemma 1.12
Let X be a manifold of dimension k. Let $f\colon X \to \mathbb{R}$ be smooth, with 0 a regular value of f. Then $f^{-1}([0,\infty)) \subset X$ is a k-dimensional manifold with boundary $\partial(f^{-1}([0,\infty))) = f^{-1}(0)$.

Proof. The subset $f^{-1}((0,\infty)) \subset X$ is open in X and thus a submanifold of X. This means that we can restrict a local parametrisation to X such that its image lies in $f^{-1}((0,\infty))$ and get the diffeomorphism we need.

It remains to consider $x \in f^{-1}(0)$. Extend f to a map $\mathbb{R} \to \mathbb{R} \times \mathbb{R}^{k-1}$ as in the proof of the pre-image theorem (Theorem 1.5). We can then proceed as before using the inverse function theorem. ∎

Corollary 1.13
B^n is an n-dimensional manifold with boundary.

Proof. This is immediate from the above lemma. ∎

Theorem 1.14 Pre-image theorem for manifolds with boundary
Let X be a manifold with boundary and Y a manifold, with $\dim X > \dim Y$. Suppose $f\colon X \to Y$ is smooth and $y \in Y$ is a regular value of both f and $f|_{\partial X}$. Then $f^{-1}(y) \subset X$ is a manifold with boundary, with $\operatorname{codim}_X f^{-1}(y) = \dim Y$ and $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial X$.

Proof. WLOG, we may assume that $X = \mathbb{H}^m$ and $Y = \mathbb{R}^n$ since we are always working locally. The easy case $x \in f^{-1}(y) \cap \operatorname{int} \partial\mathbb{H}^m$ is left as an exercise. Now, suppose $x \in f^{-1}(y) \cap \partial\mathbb{H}^m$. Then there exists an open subset $U \subset \mathbb{R}^m$ such that $f|_{U\cap\mathbb{H}^m}$ extends to a smooth map $F\colon U \to \mathbb{R}^n$. Since y is a regular value of $f|_{U\cap\mathbb{H}^m}$, dF_x is surjective. Since the map $z \mapsto dF_z$ (defined on U) is smooth, we can shrink U such that dF_z is surjective for all $z \in U$.[2] Applying the pre-image theorem to F, we have that $F^{-1}(y)$ is a submanifold of U with $\operatorname{codim}_{\mathbb{R}^m} F^{-1}(y) = \dim Y$. Let $\pi\colon F^{-1}(y) \to \mathbb{R}, (x_1, \cdots, x_m) \mapsto x_m$. Note that

$$\left(f|_{U\cap\mathbb{H}^m}\right)^{-1}(y) = \pi^{-1}([0,\infty))$$

It then suffices to show that 0 is a regular value of π since, by the previous lemma, it would follow that $\pi^{-1}([0,\infty))$ is a submanifold of $F^{-1}(y)$ with boundary $\pi^{-1}(0) = F^{-1}(y) \cap \partial\mathbb{H}^m = f^{-1}(y) \cap U \cap \partial\mathbb{H}^m$.

Now, to show that, for any $z \in \pi^{-1}(0)$, the map $d\pi_z\colon T_zF^{-1}(y) \to \mathbb{R}$ is surjective, it suffices to show that $T_zF^{-1}(y) = \ker dF_z = \ker df_z \not\subset \ker d\pi_z = \mathbb{R}^{m-1} \times \{0\} = T_z\partial\mathbb{H}^m$. Indeed, note that

$$df_z|_{T_z\partial\mathbb{H}^m} = d\left(f|_{\partial\mathbb{H}^m}\right)_z$$

is surjective. If $\ker df_z \subset T_z\partial\mathbb{H}^m$, then $\ker(df_z|_{T_z\partial\mathbb{H}^m}) = \ker(df_z)$, but these have different dimensions by the rank-nullity theorem — a contradiction! ∎

[2] Indeed, we know that some submatrix of dF_x has nonzero determinant. By continuity of det, there is some open neighbourhood $\tilde{U} \subset U$ of x on which the determinant of that submatrix remains nonzero and thus $\dim \operatorname{im} dF_z = n$ for all $z \in \tilde{U}$.

Theorem 1.15
Let X be a manifold with boundary and Y a manifold with $Z \subset Y$ a submanifold. Let $f\colon X \to Y$ be smooth such that $f \pitchfork Z$ and $f|_{\partial X} \pitchfork Z$. Then $f^{-1}(z) \subset X$ is a manifold with boundary, with $\operatorname{codim}_X f^{-1}(Z) = \operatorname{codim}_Y Z$ and $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$.

1.5 Degree modulo 2

Definition Smooth homotopy
Let $f, g\colon X \to Y$ be smooth maps between manifolds. A smooth homotopy between f and g is a smooth map $F\colon X \times [0,1] \to Y$ such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$. If such a map exists, we say that f and g are smoothly homotopic and write $f_0 \simeq f_1$.

Exercise. Show that $\simeq$ is an equivalence relation (cf. Example Sheet 1 Q14).

Definition Smooth isotopy
Let $f, g\colon X \to Y$ be diffeomorphisms. A smooth isotopy between f and g is a smooth homotopy $F\colon X \times [0,1] \to Y$ for which $F|_{X\times\{t\}}$ is a diffeomorphism for all $t \in [0,1]$. If such a map exists, we say that f and g are smoothly isotopic.

Lemma 1.16 Homotopy lemma
Suppose $f, g\colon X \to Y$ are smoothly homotopic, with X compact and $\dim X = \dim Y$. If y is a regular value of both f and g, then

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \bmod 2$$

Proof. Let $F\colon X \times [0,1] \to Y$ be a smooth homotopy between f and g. We first suppose that y is also a regular value of F. By Theorem 1.14, $F^{-1}(y)$ is a 1-dimensional manifold with boundary $\partial F^{-1}(y) = F^{-1}(y) \cap (X \times \{0,1\}) = f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}$. Thus, $\#\partial F^{-1}(y) = \#f^{-1}(y) + \#g^{-1}(y)$. We then proceed by noting the following result:

- **Theorem.** Let Z be a compact 1-dimensional manifold with boundary. Then Z is diffeomorphic to a disjoint union of finitely many copies of $[0,1]$ and of S^1.

In particular, it follows from the above that $\#\partial Z \equiv 0 \bmod 2$. Hence, we conclude that $\#f^{-1}(y) \equiv \#g^{-1}(y) \bmod 2$.

Now, suppose y is *not* a regular value of F. By Sard's theorem, the set of regular values of f (resp. g) is dense in Y. Thus, every open set in Y contains a shared regular value of f, g, F. Then, by the stack of records theorem, we can pick an open set $V \subset Y$ such that $f^{-1}(V)$ and $g^{-1}(V)$ are both disjoint union of open sets on which f and g, respectively, are diffeomorphisms to their images. In particular, $z \mapsto \#f^{-1}(z)$ and $z \mapsto \#g^{-1}(z)$ are both constant in V. Since our previous argument holds for the common regular value of f and g in V, it follows that $\#f^{-1}(y) \equiv \#g^{-1}(y) \bmod 2$. ∎

Exercise. Prove the theorem used in the proof of the homotopy lemma (Lemma 1.16).

Exercise. Prove the claim in the proof that f and g have a shared regular value in every open subset of Y.

Lemma 1.17 Homogeneity lemma
Let X be a smooth connected manifold and $y, z \in X$. Then there exists a diffeomorphism $h\colon X \to X$ smoothly isotopic to id_X and $h(y) = z$.

Remark. As suggested by the name, this lemma says that each point of X is essentially the same.

Proof sketch. It suffices to show that each $y \in X$ has an open neighbourhood $V \subset X$ such that, for every $z \in V$, there exists a diffeomorphism h with the required properties. Since X locally (i.e. in a neighbourhood of y) is diffeomorphic to $\mathbb{R}^k$, it suffices to show that there exists $F \colon \mathbb{R}^k \to \mathbb{R}^k$ smoothly isotopic to $\mathrm{id}_{\mathbb{R}^k}$, sends $0 \mapsto z$, and $F|_{\mathbb{R}^k \setminus B_1(0)} = \mathrm{id}_{\mathbb{R}^k \setminus B_1(0)}$. To do this, we pick a smooth vector field $V \colon \mathbb{R}^k \to \mathbb{R}^k$ such that $V(tz) = z$ for all $t \in [0,1]$ and $V = 0$ outsides $B_1(0)$. By standard ODE theory, there exists a solution $\gamma_x \colon \mathbb{R} \to \mathbb{R}^k$ to the ODE

$$\begin{cases} \dot{\gamma}_x(t) = V(\gamma(t)) \\ \gamma_x(0) = x \end{cases}$$

By defining the flow map $\Phi_t \colon \mathbb{R}^k \to \mathbb{R}^k, x \mapsto \gamma_x(t)$ for each $x \in \mathbb{R}^k$, we can then construct the smooth map

$$\begin{aligned} \Phi \colon \mathbb{R}^k \times \mathbb{R} &\to \mathbb{R}^k \\ (x,t) &\mapsto \Phi_t(x) \end{aligned}$$

which we can restrict to $\mathbb{R}^k \times [0,1]$ to obtain the required map F. ■

Theorem 1.18
Let $f \colon X \to Y$ be a smooth map between manifolds, with X compact, Y connected and $\dim X = \dim Y$. If y, z are regular values of f, then

$$\# f^{-1}(y) \equiv \# f^{-1}(z) \bmod 2$$

Proof. By the homogeneity lemma (Lemma 1.17), there exists a smooth isotopy $H \colon X \times [0,1] \to X$ such that $H|_{X \times \{0\}} = \mathrm{id}$ and $H|_{X \times \{1\}}$ is a diffeomorphism $X \to X$ with $h(y) = z$. Observe that $H \circ (f, \mathrm{id}_{[0,1]})$ defines a smooth homotopy between f and $h \circ f$. Note also that z is a regular value of $h \circ f$ since $d(h \circ f) = dh \circ df$, dh is an isomorphism and $y = h^{-1}(z)$ is a regular value of f. By the homotopy lemma (Lemma 1.16), we then have

$$\# f^{-1}(y) \equiv \#(h \circ f)^{-1}(z) \equiv \# f^{-1}(z) \bmod 2$$

as required. ■

With the above theorem established, we can define the following:

Definition Degree mod 2
For a smooth map $f \colon X \to Y$ from a compact manifold to a connected manifold of the same dimension, the degree mod 2 of f is defined to be

$$\deg_2(f) := \# f^{-1}(y) \bmod 2$$

for any (equivalently, all) regular value of f.

Remark. The theorem, together with the homotopy lemma, implies that $\deg_2(f)$ is a homotopy invariant.

Lemma 1.19
Let X be a compact, connected manifold with $\dim X \geq 1$. The identity map id_X cannot be smoothly homotopic to a constant map on X.

Proof. Suppose, on the contrary, that identity id_X and the constant map $c_{x_0}: X \to X, x \mapsto x_0$ for some fixed $x_0 \in X$. Note that $\deg_2(\mathrm{id}_X) = 1$ since every $x \in X$ is a regular value of id_X and has exactly one pre-image under id_X. On the other hand, $\deg_2(c_{x_0}) = 0$ since we can simply pick $x \in X \backslash \{x_0\}$ (this set is non-empty since $\dim X \geq 1$ which is a regular value with no pre-image under c_{x_0}. However, this contradicts the fact that degree mod 2 is a homotopy invariant. ■

Corollary 1.20
There does not exist a smooth retraction of B^{n+1} onto its boundary, i.e. there is no smooth map $f: B^{n+1} \to \partial B^{n+1}$ such that $f|_{\partial B^{n+1}} = \mathrm{id}$.

Proof. Suppose, on the contrary, that there exists such a map f. Define

$$\begin{aligned} F: S^n \times [0,1] &\to S^n \\ (x,t) &\mapsto f(xt) \end{aligned}$$

This is smooth since f is smooth. Moreover, we have $F|_{S^n \times \{0\}}$ is the constant map $S^n \to S^n, x \mapsto f(0)$, whereas $F|_{S^n \times \{1\}} = \mathrm{id}_{S^n}$. However, this contradicts Lemma 1.19. ■

Theorem 1.21 (Smooth) Brouwer's fixed point theorem
Every smooth map $f: B^n \to B^n$ has a fixed point.

Proof. Suppose, on the contrary, that $f: B^n \to B^n$ is a smooth map without a fixed point. We will obtain a contradiction by constructing a smooth retraction $g: B^n \to \partial B^n$.

For each $x \in B^n$, $f(x) \neq x$ so there exists a line joining $f(x)$ to x in B^n. Extending this line to ∂B^n, we define $g(x)$ to be the unique intersection point. It is easy to show that g as we've constructed it is well-defined and smooth. But this contradicts Corollary 1.20! ■

Corollary 1.22 (Topological) Brouwer fixed point theorem
Every continuous map $f: B^n \to B^n$ has a fixed point.

Proof. Suppose not. Then there exists $\varepsilon > 0$ such that $|f(x) - x| \geq \varepsilon$ for all $x \in B^n$. By convolutions (cf. II Analysis of Functions), we can show that $C^\infty(B^n, B^n)$ is dense in $C(B^n, B^n)$. We can then obtain a contradiction to the smooth Brouwer fixed point theorem. ■

Now, we consider a generalisation of degree mod 2 for regular values to submanifolds via transversality. Let $f: X \to Y$ be a smooth map, with X compact. Suppose $Z \subset Y$ is a closed submanifold of Y, with $f \pitchfork Z$ and $\dim X + \dim Z = \dim Y$. Note that $f^{-1}(Z)$ is a closed 0-dimensional submanifold of X.

Definition Intersection number modulo 2
We define the intersection number mod 2 $I_2(f, Z)$ of f with Z as

$$I_2(f, Z) := \# f^{-1}(Z) \bmod 2$$

Proposition 1.23
If g is another such map $X \to Y$ and is smoothly homotopic to f, then $I_2(f, Z) = I_2(g, Z)$.

Remarks

- Recall that we noted the genericity of transversality. This property allows us to define intersection number mod 2 even for maps f that are not transversal to Z. In that case, we define it to be $I_2(\tilde{f}, Z)$ where $\tilde{f}$ is a small perturbation of f with $\tilde{f} \pitchfork Z$.

- For $X \subset Y$, we can define the intersection number mod 2 of X and Z in Y via $I_2(X, Z) = I_2(\iota, Z)$ where $\iota \colon X \to Y$ is the inclusion map.
- If $X \subset Y$ and $\dim X = \frac{1}{2} \dim Y$, we can define the self-intersection number mod 2 of X.

1.6 Abstract manifolds

Now, we will briefly discuss the more abstract notion of manifolds instead of the manifolds embedded in $\mathbb{R}^N$ that we've been dealing with.

Definition Abstract smooth k-dimensional manifold
An abstract smooth k-dimensional manifold is a second-countable, Hausdorff topological space X together with a collection $\{\varphi_\alpha\}_{\alpha \in I}$ of maps $\varphi_\alpha \colon U_\alpha \to X$ (where $U_\alpha \subset \mathbb{R}^k$ is open) such that

(i) *φ_α is a homeomorphism onto its image*

(ii) *$X = \bigcup_\alpha V_\alpha$ where $V_\alpha := \varphi_\alpha(U_\alpha)$*

(iii) *$\varphi_\beta^{-1} \circ \varphi_\alpha \Big|_{U_\alpha \cap \varphi_\alpha^{-1}(V_\alpha \cap V_\beta)}$ is a diffeomorphism $U_\alpha \cap \varphi_\alpha^{-1}(V_\alpha \cap V_\beta) \to U_\beta \cap \varphi_\beta^{-1}(V_\alpha \cap V_\beta)$*

(iv) *$\{\varphi_\alpha\}_{\alpha \in I}$ is maximal, i.e. if $\tilde{\varphi} \colon \tilde{U} \to X$ is such that $\{\varphi_\alpha\}_{\alpha \in I} \cup \{\tilde{\varphi}\}$ (i) to (iii), then $\tilde{\varphi} = \varphi_\alpha$ for some $\alpha \in I$*

The collection $\{\psi_\alpha = \varphi_\alpha^{-1} \colon V_\alpha \to U_\alpha\}_{\alpha \in I}$ is called a smooth atlas.

Remark. By changing the regularity conditions on the transition maps, we can define abstract C^k manifolds and C^ω manifolds.

Theorem 1.24 Whitney's embedding theorem
Let X be an abstract smooth k-dimensional manifold. Then there exists a smooth embedding (diffeomorphism to its image) $X \to \mathbb{R}^{2k+1}$.

2 Curves and surfaces

2.1 Curves in 3-space

Definition Curve
Let $X \subset \mathbb{R}^N$ be a manifold. Let $I \subset \mathbb{R}$ an interval. A curve in X is a smooth map $\alpha\colon I \to X$. We say that α is regular if α is an immersion, i.e. $d\alpha_t(1) \neq 0$ for all $t \in I$.

Notation. We will sometimes write $\dot{\alpha}(t)$ to mean $d\alpha_t(1)$.

In what follows, we will for the most part be discussing curves in $\mathbb{R}^3$.

Definition Arc length
The arc length of a curve $\alpha\colon I \to \mathbb{R}^3$ from a point $t_0 \in I$ to any $t \in I$ is

$$s(t) := \int_{t_0}^{t} |\dot{\alpha}(\tau)| \, d\tau$$

where $|\cdot|$ denotes the Euclidean norm. If the interval I has endpoints a, b with $a < b$, then the length of α is

$$\operatorname{length}(\alpha) := \int_{a}^{b} |\dot{\alpha}(\tau)| \, d\tau$$

Remarks

- Arc length is a *Euclidean invariant.* Indeed, consider a general Euclidean transformation T of $\mathbb{R}^n$ of the form $v \mapsto Rv + b$, where $R \in O(n)$ and $b \in \mathbb{R}^n$. Applying T to the curve α, we get the new curve $\tilde{\alpha}(s) = (R \circ \alpha)(s) + b$. Then the arc length parameter s for α is also an arc length parameter for $\tilde{\alpha}$ since $|\dot{\tilde{\alpha}}(s)| = |R\dot{\alpha}(s)| = |\dot{\alpha}(s)| = 1$.
- Note that we can think of s as a smooth map $I \to \tilde{I} \subset \mathbb{R}$. For a regular curve, $\tau \mapsto s(\tau)$ is a strictly increasing function and thus has a smooth inverse $\tau = \tau(s)$. We can then reparametrise α by s: $s \mapsto \alpha(\tau(s))$. Thus, we may always assume WLOG that regular curves are parametrised by arc length.

In what follows, $\alpha\colon I \to \mathbb{R}^3$ is a regular curve in 3-space that is parametrised by arc length.

Definition Tangent vector
The tangent vector to the curve $\alpha\colon I \to \mathbb{R}^3$ at $s \in I$ is $t(s) := \dot{\alpha}(s)$.

Remark. We can think of t as a map $I \to \mathbb{S}^2$.

Definition Curvature and normal vector
The curvature of α at $s \in I$ is

$$\kappa(s) := |\ddot{\alpha}(s)|$$

If $\kappa(s) \neq 0$, we define the normal vector at s to be

$$n(s) := \frac{1}{\kappa(s)} \ddot{\alpha}(s)$$

Definition Binormal vector
The binormal vector to the curve $\alpha\colon I \to \mathbb{R}^3$ at $s \in I$ is

$$b(s) := t(s) \times n(s)$$

Remark. For any $s \in I$, $\{t(s), n(s), b(s)\}$ forms an orthonormal basis of $\mathbb{R}^3$, which is called the *Frenet frame.*

Now, we compute

$$\dot{b}(s) = \dot{t}(s) \times n(s) + t(s) \times \dot{n}(s) = t(s) \times \dot{n}(s)$$

Differentiating the relation $b(s) \cdot b(s) = 1$, we get $\dot{b}(s) \cdot b(s) = 0$. Putting everything together, we have that $\dot{b}$ is orthogonal to both $t(s)$ and $b(s)$, so it must be parallel to $n(s)$. We may thus write

$$\dot{b}(s) = \tau(s)n(s)$$

Definition Torsion
The quantity $\tau(s)$ defined above is called the torsion of α at $s \in I$.

Remark. Curvature and torsion are proper Euclidean invariants. Fix $R \in SO(3)$ and $b \in \mathbb{R}^3$. Let $\alpha, \tilde{\alpha}\colon I \to \mathbb{R}^3$ be regular curves parametrised by arc length s, related by $\tilde{\alpha} = R \circ \alpha + b$. Observe that $\tilde{\kappa}(s) = |\ddot{\tilde{\alpha}}(s)| = |R\ddot{\alpha}(s)| = |\ddot{\alpha}(s)| = \kappa(s)$, so curvature is indeed a proper Euclidean invariant. Note also that $\tilde{t}(s) = Rt(s)$ and

$$\tilde{n}(s) = \frac{\ddot{\tilde{\alpha}}(s)}{\tilde{\kappa}(s)} = \frac{R\ddot{\alpha}(s)}{\kappa(s)} = Rn(s)$$

Since $R \in SO(3)$, we have that

$$\tilde{b}(s) = \tilde{t}(s) \times \tilde{n}(s) = (Rt(s)) \times (Rn(s)) = R(t(s) \times n(s)) = Rb(s)$$

so $\tilde{\tau}(s)(Rn(s)) = \tilde{\tau}(s)\tilde{n}(s) = R(\tau(s)n(s)) = \tau(s)(Rn(s))$ and hence $\tilde{\tau} \equiv \tau$.

Proposition 2.1 Frenet equations
The vectors t, n, b satisfy the system of ODEs

$$\begin{bmatrix} \dot{t} \\ \dot{n} \\ \dot{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

Remark. Remark about antisymmetry of matrix to follow

Theorem 2.2 Fundamental theorem of curves in $\mathbb{R}^3$
Let $I \subset \mathbb{R}$ be an interval. If $\kappa\colon I \to \mathbb{R}_{>0}$ and $\tau\colon I \to \mathbb{R}$ are smooth, then there exists a regular curve $\alpha\colon I \to \mathbb{R}^3$ parametrised by arc length such that $\kappa(s)$ and $\tau(s)$ are the curvature and torsion at s, respectively. Moreover, if $\tilde{\alpha}\colon I \to \mathbb{R}^3$ is another regular curve parametrised by arc length such that $\tilde{\kappa}(s) = \kappa(s)$ and $\tilde{\tau}(s) = \tau(s)$ for every $s \in I$, then $\tilde{\alpha} = R \circ \alpha + a$ for some $R \in SO(3)$ and $a \in \mathbb{R}^3$.

Proof. Pick $t_0, n_0 \in \mathbb{R}^3$ orthonormal vectors. Define $b_0 = t_0 \times n_0$. Fix $s_0 \in I$. By standard ODE theory, there exists a unique solution $t, n, b\colon I \to \mathbb{R}^3$ to the Frenet equations (Proposition 2.1). We then define α by

$$\alpha(s) = \int_{s_0}^{s} t(\tilde{s})\, d\tilde{s}$$

Now, let $\tilde{\alpha}$ be another such curve as in the theorem statement. Since $\{t, n, b\}$ and $\{\tilde{t}, \tilde{n}, \tilde{b}\}$ are both orthonormal bases for $\mathbb{R}^3$, we can pick $R \in SO(3)$ that sends $(t, n, b) \mapsto (\tilde{t}, \tilde{n}, \tilde{b})$. Define $a := \tilde{\alpha}(s_0) - \alpha(s_0)$. Let $\tilde{\tilde{\alpha}}(s) := (R \circ \alpha)(s) + a$, with $(\tilde{\tilde{t}}, \tilde{\tilde{n}}, \tilde{\tilde{b}})(s_0) = (\tilde{t}, \tilde{n}, \tilde{b})(s_0)$. By uniqueness of solution to the Frenet equations, $(\tilde{\tilde{t}}, \tilde{\tilde{n}}, \tilde{\tilde{b}}) = (\tilde{t}, \tilde{n}, \tilde{b})$ on I. We then have that

$$\tilde{\alpha}(s) = \tilde{\alpha}(s_0) + \int_{s_0}^{s} \tilde{t}(\hat{s})\, d\hat{s} = \tilde{\tilde{\alpha}}(s_0) + \int_{s_0}^{s} \tilde{\tilde{t}}(\hat{s})\, d\hat{s} = \tilde{\tilde{\alpha}}(s)$$

as required. ■

Lemma 2.3 Plane curves
A regular curve $\alpha\colon I \to \mathbb{R}^3$ with $\kappa(s) \neq 0$ for all $s \in I$. Then α lies in a plane iff $\tau(s) = 0$ for all $s \in I$.

Proof. ($\Longleftarrow$) If $\tau \equiv 0$, then $\dot{b} = 0$, so $(t \times n)(s) = (t \times n)(s_0)$ for all $s \in I$. Let Π be the plane spanned by $\{t(s), n(s)\}$. Then we have that

$$\alpha(s) = \alpha(s_0) + \int_{s_0}^{s} t(\hat{s})\, d\hat{s} \in \alpha(s_0) + \Pi$$

($\Longrightarrow$) WLOG set $\alpha(s_0) = 0$, so $\alpha(s) \in \Pi$ for some plane Π through the origin. By differentiation, we see that $\dot{\alpha}, \ddot{\alpha} \in \Pi$. Then $t \times n$ is the unique normal to Π, so $\dot{b} \equiv 0$. Hence, $\tau \equiv 0$. ■

Remark. For plane curves, we can assign a sign to the curvature κ. In particular, we can define the (signed) curvature $\kappa(s)$ via

$$\dot{t}(s) = \kappa(s) n(s)$$

Note that $|\kappa(s)|$ coincides with our previous definition of curvature. The sign essentially arises from demanding that the basis $\{t(s), n(s)\}$ has the same orientation as $\{e_1, e_2\}$.

2.2 Isoperimetric inequality

In this subsection, let $I = [a, b]$ be a closed interval.

Definition Simple, closed, regular curve
We say that a regular curve $\alpha\colon I \to \mathbb{R}^3$ is simple, closed if $\alpha^{(n)}(a) = \alpha^{(n)}(b)$ for all $n \in \mathbb{N}_0$.

Remark. Equivalently, we can view α as an injective map $\mathbb{S}^1 \to \mathbb{R}^3$.

Lemma 2.4 Jordan curve theorem
Let $\alpha\colon I \to \mathbb{R}^2$ be a simple, closed, regular curve in $\mathbb{R}^2$. Then there exists an open subset $U \subset \mathbb{R}^2$ such that $U \cup \alpha(I)$ is a compact manifold with boundary.

Lemma 2.5 Wirtinger's inequality
Let $f\colon \mathbb{R} \to \mathbb{C}$ be a smooth periodic function with period L and $\int_0^L f(x)\, dx = 0$. Then

$$\int_0^L |f(x)|^2\, dx \leq \frac{L^2}{4\pi^2} \int_0^L |f'(x)|^2\, dx$$

with equality iff $f = ae^{2\pi ix/L} + be^{-2\pi ix/L}$ for some $a, b \in \mathbb{C}$.

Proof. Since f is a smooth periodic function, we can write it as

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi inx/L}$$

Since $\int_0^L f(x)\, dx = 0$, then we must have $a_0 = 0$. Since f is smooth, we can differentiate term by term to get

$$f'(x) = \sum_{n \in \mathbb{Z}} \frac{2\pi in}{L} a_n e^{2\pi inx/L}$$

By Plancherel,

$$\frac{1}{L}\int_0^L |f(x)|^2\,dx = \sum_{n\in\mathbb{Z}} |a_n|^2, \qquad \frac{1}{L}\int_0^L |f'(x)|^2\,dx = \sum_{n\in\mathbb{Z}} \frac{4\pi^2 n^2}{L^2}|a_n|^2$$

It then follows that

$$\int_0^L |f'(x)|^2\,dx = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{4\pi^2 n^2}{L^2}|a_n|^2 \geq \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{4\pi^2}{L^2}|a_n|^2 = \frac{4\pi^2}{L^2}\int_0^L |f(x)|^2\,dx$$

as required. ■

Remark. The regularity conditions in the above statement are overkill. In fact, the inequality even holds in Sobolev spaces.

Theorem 2.6 Isoperimetric inequality
Let $\alpha\colon I \to \mathbb{R}^2$ be a simple, closed, regular curve in $\mathbb{R}^2$, with Ω the domain bounded by α given by the Jordan curve theorem. Then

$$\operatorname{length}(\alpha)^2 \geq 4\pi \operatorname{area}(\Omega)$$

with equality iff Ω is a disc.

Proof. WLOG, we may assume that α is parametrised by arc length, $\alpha\colon [0,\ell] \to \mathbb{R}^2$, where $\ell = \operatorname{length}(\alpha)$. Write $\alpha(s) = (x(s), y(s))$. Define the vector field $V\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x,y)$. By the divergence theorem, we have

$$\int_0^\ell \langle V, n\rangle\,ds = \int_\Omega \operatorname{div} V\,dxdy = 2\operatorname{area}(\Omega)$$

It then follows that

$$2\operatorname{area}(\Omega) = \left|\int_0^\ell \langle V,n\rangle\,ds\right| \leq \int_0^\ell |\langle V,n\rangle|\,ds \leq \int_0^\ell |V|\,ds = \int_0^\ell \sqrt{x(s)^2 + y(s)^2}\,ds$$

By translation, we may assume WLOG that

$$\int_0^\ell x(s)\,ds = \int_0^\ell y(s)\,ds = 0$$

This is possible by the intermediate value theorem. By the Cauchy-Schwarz inequality and Wirtinger's inequality (Lemma 2.5), we have

$$\begin{aligned}
2\operatorname{area}(\Omega) &\leq \int_0^\ell \sqrt{x(s)^2 + y(s)^2}\,ds \\
&\leq \left(\int_0^\ell x(s)^2 + y(s)^2\,ds\right)^{1/2} \left(\int_0^\ell 1\,ds\right)^{1/2} \\
&\leq \sqrt{\ell}\left(\int_0^\ell x(s)^2\,ds + \int_0^\ell y(s)^2\,ds\right)^{1/2} \\
&\leq \sqrt{\ell}\left(\frac{\ell^2}{4\pi^2}\int_0^\ell \dot{x}(s)^2 + \dot{y}(s)^2\,ds\right)^{1/2} \\
&= \frac{\ell^2}{2\pi}
\end{aligned}$$

Upon rearranging, we get $\operatorname{length}(\alpha)^2 \geq 4\pi\operatorname{area}(\Omega)$. Finally, note that equality in Cauchy-Schwarz is achieved above iff $x(s)^2 + y(s)^2 = \text{const}$, i.e. Ω is a disc. ■

2.3 First fundamental form

Definition Surface
A surface is a 2-dimensional manifold $X \subset \mathbb{R}^N$.

In what follows, we will for the most part be discussing surfaces in $\mathbb{R}^3$.

Let $S \subset \mathbb{R}^3$ be a surface and $I = [a, b]$ with $a < b$. Let $\alpha\colon I \to \mathbb{R}^3$ be a regular curve with $\alpha(I) \subset S$. As before, the length of α is

$$\text{length}(\alpha) = \int_a^b \langle \dot\alpha(\lambda), \dot\alpha(\lambda)\rangle^{1/2}\, d\lambda$$

where $\dot\alpha(\lambda) = d\alpha_\lambda(1) \in T_{\alpha(\lambda)}S \subset \mathbb{R}^3$ and $\langle \cdot, \cdot\rangle$ is the inner product on $\mathbb{R}^3$. Of course, for the surface S, we do not really need the full inner product on $\mathbb{R}^3$, leading us to the following definition.

Definition First fundamental form
Let $S \subset \mathbb{R}^3$ be a surface. The first fundamental form of S at $p \in S$ is the map $I_p\colon T_pS \times T_pS \to \mathbb{R}$ given by the restriction $\mathrm{I}_p = \langle \cdot, \cdot\rangle|_{T_pS \times T_pS}$.

Remark. For any Euclidean transformation Φ that sends $x \mapsto Rx + b$ for some $R \in O(3)$ and $b \in \mathbb{R}^3$, we can define $\phi = \Phi|_S\colon S \to \tilde S := \Phi(S)$. Let I_p be the FFF of S at $p \in S$ and $\tilde{\mathrm{I}}_{\phi(p)}$ the FFF of $\tilde S$ at $\phi(p)$. Observe that $\mathrm{I}_p(v, w) = \tilde{\mathrm{I}}_{\phi(p)}(d\phi_p(v), d\phi_p(w))$ for all $v, w \in T_pS$: indeed, this follows from $d\phi_p = d\Phi_p|_{T_pS} = R|_{T_pS}$ and $\langle Rv, Rw\rangle = \langle v, w\rangle$.

Definition Isometry
Let $S, \tilde S \subset \mathbb{R}^3$ be surfaces. We say that a diffeomorphism $\phi\colon S \to \tilde S$ of surfaces is an isometry if, for every $p \in S$ and $v, w \in T_pS$,

$$\mathrm{I}_p(v, w) = \tilde{\mathrm{I}}_{\phi(p)}(d\phi_p(v), d\phi_p(w))$$

Remark. By the previous remark, the restriction of a Euclidean transformation is an isometry.

Lemma 2.7
If $\phi\colon S \to \tilde S$ is a diffeomorphism of surfaces, then

$$\text{length}_{\tilde S}(\phi \circ \alpha) = \text{length}_S(\alpha)$$

for any curve $\alpha\colon [a, b] \to \mathbb{R}^3$ with $\alpha([a, b]) \subset S$.

Proof. Trivial computation. ■

Exercise. Show that the converse holds as well.

> Let $X \subset \mathbb{R}^N$ be a manifold. A *(smooth) Riemannian metric* on X is a collection $(g_p)_{p\in X}$ of maps $g_p\colon T_pX \times T_pX \to \mathbb{R}$ such that
> (i) for every $v, w \in T_pX$, $g_p(v, w) = g_p(w, v)$
> (ii) for every $v_1, v_2, w \in T_pX$ and $\lambda, \mu \in \mathbb{R}$, $g_p(\lambda v_1 + \mu v_2, w) = \lambda g_p(v_1, w) + \mu g_p(v_2, w)$
> (iii) for every $v \in T_pX$, $g_p(v, v) \geq 0$, with equality iff $v = 0$
> and $p \mapsto g_p$ is smooth.

> This allows us to define length of a curve $\alpha\colon [a,b] \to (X,g)$ by
> $$\text{length}_g(\alpha) = \int_a^b \sqrt{g_{\alpha(\lambda)}(\dot\alpha(\lambda), \dot\alpha(\lambda))}\, d\lambda$$
> We then define an isometry to be a diffeomorphism $\phi\colon X \to \tilde X$ that satisfies
> $$g_p(v,w) = \tilde g_{\phi(p)}(d\phi_p(v), d\phi_p(w))$$
> for all $v, w \in T_pX$. Again, a diffeomorphism is an isometry iff it preserves the lengths of all curves.

Now, let $S \subset \mathbb{R}^3$ be a surface, with $\varphi\colon U \to W$ a local parametrisation from $U \subset \mathbb{R}^2_{(u,v)}$ to an open set $W \subset S$. Note that $\mathrm{I}_{\varphi(u,v)}$ is completely determined by

$$\begin{aligned} E(u,v) &:= \mathrm{I}_{\varphi(u,v)}(\varphi_u, \varphi_u) = \langle \varphi_u, \varphi_u \rangle \\ F(u,v) &:= \mathrm{I}_{\varphi(u,v)}(\varphi_u, \varphi_v) = \langle \varphi_u, \varphi_v \rangle \\ G(u,v) &:= \mathrm{I}_{\varphi(u,v)}(\varphi_v, \varphi_v) = \langle \varphi_v, \varphi_v \rangle \end{aligned}$$

In terms of E, F, G, we can write the length of α (viewed as $\alpha(\lambda) = \varphi(u(\lambda), v(\lambda))$) as

$$\text{length}(\alpha) = \int_a^b \sqrt{E\dot u^2 + 2F\dot u \dot v + G\dot v^2}\, d\lambda$$

Remarks

- If $\phi\colon S \to \tilde S$ is an isometry and φ is a local parametrisation of S, then $\phi \circ \varphi$ is a local parametrisation of $\tilde S$. Writing $E = \langle \varphi_u, \varphi_u \rangle$ and $\tilde E = \langle (\phi\circ\varphi)_u, (\phi\circ\varphi)_u \rangle$ (and similarly for F and G), it can be shown that
$$E = \tilde E, \qquad F = \tilde F, \qquad G = \tilde G$$
- Under a different local parametrisation $\tilde\varphi$, we get different functions $\tilde E(\tilde u, \tilde v), \tilde F(\tilde u, \tilde v), \tilde G(\tilde u, \tilde v)$. However, it is easy to verify that the expressions for length are equivalent under the two parametrisations.

Definition Area
Let $S \subset \mathbb{R}^3$ be a surface and $\varphi\colon U \to W$ be a local parametrisation of S. Let $\Omega \subset S$ be an open subset such that $\Omega \subset W$. The area of Ω is defined to be

$$\text{area}(\Omega) := \int_{\varphi^{-1}(\Omega)} |\varphi_u \times \varphi_v|\, du\, dv$$

Exercise. Show that the area of Ω is well-defined.

Lemma 2.8
The area of open subset $\Omega \subset S$ can be expressed as

$$\text{area}(\Omega) = \int_{\varphi^{-1}(\Omega)} \sqrt{EG - F^2}\, du\, dv$$

Exercise. Prove the above lemma.

Example *Torus*
Consider

$$\begin{aligned} \varphi\colon [0,2\pi] \times [0,2\pi] &\to \mathbb{R}^3 \\ (u,v) &\mapsto ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u) \end{aligned}$$

Note that $\operatorname{im}\varphi$ defines a torus T^2, with $\varphi|_{(0,2\pi)^2}$ a local parametrisation away from a negligible subset. It is easy to compute $E = r^2$, $F = 0$, $G = (r\cos u + a)^2$ and thus $\operatorname{area}(T^2) = 4\pi^2 ra$.

2.4 Second fundamental form

Let $S \subset \mathbb{R}^3$ be a surface and $\varphi\colon U \to W$ a local parametrisation of S. Note that

$$\frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}$$

defines a unit vector that is normal to $T_{\varphi(u,v)}S$.

Definition Gauss map
A Gauss map of S is a smooth map $N\colon S \to S^2$ such that $N(p) \perp T_pS$.

Remark. If N is a Gauss map of S and $\varphi\colon U \to W$ a local parametrisation of S, with U connected, then

$$N|_W(\varphi(u,v)) = \pm\frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}(u,v)$$

Definition Orientable and oriented surface
If there exists a Gauss map on S, then we say that S is orientable. In that case, if we equip S with a choice of Gauss map N, then we say that (S, N) is oriented.

Example

- S^2 is an orientable surface since $N = \pm\,\mathrm{id}$ both define Gauss maps.
- The Möbius strip is not orientable.

Now, let (S, N) be an oriented surface. Note that the differential of N at $p \in S$ is a map $T_pS \to T_{N(p)}S^2$. But $T_{N(p)}S^2 = T_pS$ since both are planes with normal $N(p)$. Thus, dN_p is a linear map $T_pS \to T_pS$.

Proposition 2.9
dN_p is self-adjoint with respect to I_p, i.e. for every $v, w \in T_pS$,

$$\mathrm{I}_p(dN_p(w_1), w_2) = \mathrm{I}_p(w_1, dN_p(w_2))$$

Proof. By linearity, it suffices to show that it holds for $w_1, w_2 \in \{\varphi_u, \varphi_v\}$. The result is trivial if $w_1 = w_2$, so in fact we just have to show that $\mathrm{I}_p(dN_p(\varphi_u), \varphi_v) = \mathrm{I}_p(\varphi_u, dN_p(\varphi_v))$. To see this, observe that $\langle N, \varphi_u\rangle = 0$ implies that

$$\langle N_v, \varphi_u\rangle + \langle N, \varphi_{uv}\rangle = 0$$
$$\langle N_u, \varphi_v\rangle + \langle N, \varphi_{uv}\rangle = 0$$

Thus, $\mathrm{I}_p(dN_p(\varphi_u), \varphi_v) = \langle N_u, \varphi_v\rangle = \langle N_v, \varphi_u\rangle = \mathrm{I}_p(\varphi_u, dN_p(\varphi_v))$. ■

Definition Second fundamental form
Let (S, N) be an oriented surface. The second fundamental form of (S, N) at $p \in S$ is the map $\mathrm{II}_p\colon T_pS \times T_pS \to \mathbb{R}$ defined by $(v, w) \mapsto -\mathrm{I}_p(dN_p(v), w)$.

Remarks

- By self-adjointness of dN_p (Proposition 2.9), II_p is symmetric. Moreover, it is bilinear. The associated quadratic form is written as $\mathrm{II}_p(v) := \mathrm{II}_p(v, v)$.

- II_p is a proper Euclidean invariant: for any proper Euclidean transformation $x \mapsto Rx + b$ for $R \in SO(3)$ and $b \in \mathbb{R}^3$, we can define the set $\tilde{S} := R(S) + b$. It is easy to see that $\tilde{S}$ is a surface with orientation $\tilde{N}$ defined by $\tilde{N} \circ \psi = R \circ N$. Let $\phi\colon S \to \tilde{S}$ be the restriction of the proper Euclidean transformation to S. Then

$$\begin{aligned}\tilde{\mathrm{II}}_{\phi(p)}(d\phi_p(v), d\phi_p(w)) &= -\tilde{\mathrm{I}}_{\phi(p)}(d\tilde{N}_{\phi(p)}(d\phi_p(v)), d\phi_p(w)) \\ &= -\tilde{\mathrm{I}}_{\phi(p)}(d\phi_p(dN_p(v)), d\phi_p(w)) \\ &= -\mathrm{I}_p(dN_p(v), w) \\ &= \mathrm{II}_p(v, w)\end{aligned}$$

where we have used the Euclidean invariance of I_p in going to the penultimate line.

As before, let (S, N) be an oriented surface and $p \in S$. We define

$$\kappa_1(p) := \max_{\substack{v \in T_pS \\ \mathrm{I}_p(v,v)=1}} \mathrm{II}_p(v, v), \qquad \kappa_2(p) := \min_{\substack{v \in T_pS \\ \mathrm{I}_p(v,v)=1}} \mathrm{II}_p(v, v)$$

which exist by compactness of the set $\{v \in T_pS\colon \mathrm{I}_p(v, v) = 1\}$.

Proposition 2.10
There exists an orthonormal basis $\{e_1, e_2\}$ of T_pS such that, for $i \in \{1, 2\}$,

$$dN_p(e_i) = -\kappa_i e_i$$

Exercise. Prove the above proposition.

Definition Principal curvatures and directions
We call $\kappa_1(p)$ and $\kappa_2(p)$ the principal curvatures of S at p. Moreover, e_1 and e_2 are called the principal directions at p.

Remark. The principal curvatures are proper Euclidean invariants.

Definition Normal curvature
Let (S, N) be an oriented surface and $\alpha\colon I \to S$ be a regular curve parametrised by arc length. The normal curvature of α at $s \in I$ is

$$\kappa_n(s) := \langle \ddot{\alpha}(s), (N \circ \alpha)(s) \rangle = \langle \kappa(s)n(s), (N \circ \alpha)(s) \rangle$$

Remark. It is immediate from the definition that $|\kappa_n(s)| \leq \kappa(s)$.

Proposition 2.11 Meusmer's theorem
Let (S, N) and $\alpha\colon I \to S$ be as above. Then the normal curvature at $s \in I$ is given by

$$\kappa_n(s) = \mathrm{II}_p(\dot{\alpha}(s))$$

Definition Mean and Gauss curvatures
Let (S, N) be an oriented surface. The mean curvature of S at $p \in S$ is

$$H(p) := \frac{1}{2}\operatorname{tr}(-dN_p)$$

The Gauss curvature of S at $p \in S$ is

$$K(p) := \det(-dN_p)$$

Lemma 2.12
The mean and Gauss curvatures are also given by

$$H(p) = \frac{1}{2}(\kappa_1(p) + \kappa_2(p)), \qquad K(p) = \kappa_1(p)\kappa_2(p)$$

Proof. This follows immediately from Proposition 2.10. ∎

Remark. Note that changing $N \mapsto -N$ will result κ_1, κ_2 changing sign. Thus, H changes sign but K does not.

Example

- For a sphere S^2, we have $\kappa_1 = \kappa_2 = 1$ so $H = K = 1$.
- For a cylinder, $\kappa_2 = 0$ so $K = 0$. However, $\kappa_1 > 0$ so $H > 0$.
- Consider a torus T^2. By considering various circles, one can see that K can have any sign depending on where one is on T^2.

Definition Umbilic point
We say that a point $p \in S$ is umbilic if $\kappa_1(p) = \kappa_2(p)$.

Remark. If p is an umbilic point, then dN_p can be diagonalised to $-\kappa_1 \,\mathrm{id}$ and takes form $\mathrm{diag}(-\kappa_1, -\kappa_1)$ in any orthonormal basis.

Now, let $S \subset \mathbb{R}^3$ be a surface, with $\varphi\colon U \to W$ a local parametrisation from $U \subset \mathbb{R}^2_{(u,v)}$ to an open set $W \subset S$. Note that $\mathrm{II}_{\varphi(u,v)}$ is completely determined by

$$\begin{aligned}
e(u,v) &:= \mathrm{II}_{\varphi(u,v)}(\varphi_u, \varphi_u) = -\langle N_u, \varphi_u \rangle = \langle N, \varphi_{uu} \rangle \\
f(u,v) &:= \mathrm{II}_{\varphi(u,v)}(\varphi_u, \varphi_v) = -\langle N_u, \varphi_v \rangle = \langle N, \varphi_{uv} \rangle \\
g(u,v) &:= \mathrm{II}_{\varphi(u,v)}(\varphi_v, \varphi_v) = -\langle N_v, \varphi_v \rangle = \langle N, \varphi_{vv} \rangle
\end{aligned}$$

where we have used the fact that $\langle N, \varphi_u \rangle = \langle N, \varphi_v \rangle = 0$.

Proposition 2.13 Curvatures in local coordinates
In local coordinates, the mean and Gauss curvatures are given by

$$\begin{aligned}
K &= \frac{eg - f^2}{EG - F^2} \\
H &= \frac{eG - 2fF + gE}{2(EG - F^2)}
\end{aligned}$$

Proof. Trivial computations. ∎

Now, one might ask: are there more fundamental forms? It turns out that the first and second fundamental forms are already sufficient to characterise a surface up to rigid motion. This result is known as the fundamental theorem of surface theory (cf. IB Geometry Example Sheet 2 Question E3).

2.5 Theorema egregium

Theorem 2.14 Theorema egregium
Let $S, \tilde{S} \subset \mathbb{R}^3$ be surfaces. If $\phi\colon S \to \tilde{S}$ is an isometry, then $K = \phi_* \tilde{K}$.

Remark. In other words, the Gauss curvature only depends on the first fundamental form.

Proof. We have previously remarked that isometries preserve the first fundamental form, so it suffices to show that K can be expressed in terms of only E, F, G.

We express the derivatives of φ_u, φ_v in terms of the basis $\{\varphi_u, \varphi_v, N\}$:

$$\varphi_{uu} = \Gamma^u_{uu} \varphi_u + \Gamma^v_{uu} \varphi_v + eN$$

$$\varphi_{uv} = \Gamma^u_{uv}\varphi_u + \Gamma^v_{uv}\varphi_v + fN$$
$$\varphi_{vu} = \Gamma^u_{vu}\varphi_u + \Gamma^v_{vu}\varphi_v + fN$$
$$\varphi_{vv} = \Gamma^u_{vv}\varphi_u + \Gamma^v_{vv}\varphi_v + gN$$

We call the coefficients Γ^k_{ij} the *Christoffel symbols.* Since $\varphi_{uv} = \varphi_{vu}$, we have $\Gamma^u_{uv} = \Gamma^u_{vu}$ and $\Gamma^v_{uv} = \Gamma^v_{vu}$. To find the Christoffel symbols, we simply have to take inner products:

$$\Gamma^u_{uu}E + \Gamma^v_{uu}F = \langle \varphi_{uu}, \varphi_u \rangle = \frac{1}{2}E_u$$
$$\Gamma^u_{uu}F + \Gamma^v_{uu}G = \langle \varphi_{uu}, \varphi_v \rangle = F_u - \frac{1}{2}E_v$$

Since $EG - F^2 \neq 0$, the above linear system can be solved for Γ^u_{uu} and Γ^v_{uu}. In a similar way, we obtain expressions for other Christoffel symbols.

Now, by symmetry of mixed partials, we have $\varphi_{uuv} = \varphi_{uvu}$ and after some lengthy algebraic manipulations, we can compare coefficients of ϕ_v to obtain

$$(\Gamma^v_{uv})_u - (\Gamma^v_{uu})_v + \Gamma^u_{uv}\Gamma^v_{uu} + \Gamma^v_{uv}\Gamma^v_{uv} - \Gamma^v_{uu}\Gamma^v_{vv} - \Gamma^u_{uu}\Gamma^v_{uv} = -EK$$

which gives us the *Gauss formula* for K in terms of only E, F, G and derivatives thereof. ■