University of Cambridge Mathematical Tripos Part II

Differential Geometry

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Course schedule

Smooth manifolds in $\mathbb{R}^n$, tangent spaces, smooth maps and the inverse function theorem. Examples, regular values, Sard's theorem (statement only). Transverse intersection of submanifolds. [4]

Manifolds with boundary, degree mod 2 of smooth maps, applications. [3]

Curves in 2-space and 3-space, arc-length, curvature, torsion. The isoperimetric inequality. [2]

Smooth surfaces in 3-space, first fundamental form, area. [1]

The Gauss map, second fundamental form, principal curvatures and Gaussian curvature. Theorema Egregium. [3]

Minimal surfaces. Normal variations and characterization of minimal surfaces as critical points of the area functional. Isothermal coordinates and relation with harmonic functions. The Weierstrass representation. Examples. [3]

Parallel transport and geodesics for surfaces in 3-space. Geodesic curvature. [2]

The exponential map and geodesic polar coordinates. The Gauss-Bonnet theorem (including the statement about classification of compact surfaces). [4]

Global theorems on curves: Fenchel's theorem (the total curvature of a simple closed curve is greater than or equal to 2π); the Fary-Milnor theorem (the total curvature of a simple knotted closed curve is greater than 4π). [2]

Recommended books

J. Milnor *Topology from the differentiable viewpoint.* Princeton University Press, 1997.
M. Do Carmo *Differential Geometry of Curves and Surfaces.* Pearson Higher Education, 1976

Contents

1 Differential topology

Definition Smooth map on an open subset
Let $U \subset \mathbb{R}^n$. We say that $f\colon U \to \mathbb{R}^m$ is smooth if all partial derivatives to all orders exist and are continuous.

Definition Smooth map
Let $X \subset \mathbb{R}^n$. We say that $f\colon X \to \mathbb{R}^m$ is smooth if, for each $x \in X$, there exists (i) an open neighbourhood $U \subset \mathbb{R}^n$ of x and (ii) a smooth map $\tilde{f}\colon U \to \mathbb{R}^m$ such that $\tilde{f}|_{X \cap U} = f|_{X \cap U}$.

Definition Diffeomorphism
Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$. We say that $f\colon X \to Y$ is a diffeomorphism if f is a smooth bijection with a smooth inverse. If such a map exists, we say that X and Y are diffeomorphic.

Exercise. Give an example of a smooth bijection that is not a diffeomorphism.

Definition k-dimensional manifold
We say that $X \subset \mathbb{R}^N$ is a k-dimensional manifold if, for each $x \in X$, there exists an open neighbourhood $V \subset X$ of x such that V is diffeomorphic to an open subset $U \subset \mathbb{R}^k$. A diffeomorphism $\varphi\colon U \to V$ is called a local parametrisation of V, whereas its inverse $\psi := \varphi^{-1}\colon V \to U$ is called a coordinate system or a chart on V.

Remarks

- By composing φ^{-1} with the projections $\pi_i\colon \mathbb{R}^k \to \mathbb{R}, (x_1, \cdots, x_k) \mapsto x_i$, we get smooth maps $x_i := \pi_i \circ \varphi^{-1}$ which we call *coordinate functions.*
- WLOG, we can replace 'diffeomorphic to an open subset $U \subset \mathbb{R}^k$" with "diffeomorphic to an open ball in R^k".
- It is easy to see that, if $X \subset \mathbb{R}^N$ is both a k-dimensional manifold and a $\tilde{k}$-dimensional manifold, then $k = \tilde{k}$.

Definition Dimension
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold. The dimension of X is k, and it is denoted by $\dim X$.

Example *Some trivial examples*

- $X = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W
- $X = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^N$

Example S^n
$S^n := \{x \in \mathbb{R}^{n+1}\colon ||x||_2 = 1\}$ is an n-dimensional manifold. To see this, consider the projection $\Pi_k\colon \mathbb{R}^{n+1} \to \mathbb{R}^n, (x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1})$. It is easy to verify that maps of the form $\psi_k^{\pm} = \Pi_k|_{S^n \cap \{\mathrm{sign}(x_k) = \pm 1\}}$ are diffeomorphisms $S^n \cap \{\mathrm{sign}(x_k) = \pm 1\} \to B_1(0)$.

Remark. It is easy to show that X is a 0-dimensional manifold iff X is a discrete subset of $\mathbb{R}^N$.

Exercise. Show that, if X and Y are manifolds, then $X \times Y$ is also a manifold, with $\dim X \times Y = \dim X + \dim Y$.

Definition Submanifold
Let $X, Y \subset \mathbb{R}^N$ be manifolds. If $Y \subset X$, then we say that Y is a submanifold of X. The codimension of Y in X is defined as

$$\operatorname*{codim}_X Y := \dim X - \dim Y$$

1.1 Tangent spaces

We first recall some basic facts from our youth. Let $U \subset \mathbb{R}^k$ be open. The *differential* of a smooth map $f\colon U \to \mathbb{R}^m$ at $x \in U$ is defined by

$$\begin{aligned} df_x\colon \mathbb{R}^k &\to \mathbb{R}^N \\ h &\mapsto \lim_{t\to 0} \frac{f(x+th)-f(x)}{t} \end{aligned}$$

This is a linear map, with matrix representation

$$df_x = \left(\frac{\partial f^i}{\partial x^j}\right)_{i,j}$$

Moreover, differentials satisfy the chain rule: given (i) two smooth maps $f\colon U \to \mathbb{R}^l$ and $g\colon V \to \mathbb{R}^m$ with $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$ open and (ii) a point $x \in U$ with $f(x) \in V$, we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Definition Tangent space
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold and $x \in X$. Choose a local parametrisation $\varphi\colon U \to V$ around x. We then define the tangent space T_xX of X at x to be

$$T_xX := \operatorname{im} d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k)$$

Of course, before we can safely proceed, we must show that T_xX is well-defined:

Lemma 1.1
Let X be as above. T_xX is independent of φ, and $\dim T_xX = k$.

Proof. Let $\tilde{\varphi}\colon \tilde{U} \to \tilde{V}$ be another local parametrisation near x. WLOG, by restricting if necessary, we may assume $\tilde{V} = V$. By the chain rule, we have

$$d\varphi_{\varphi^{-1}(x)} = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)} \circ d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}$$

Since $\tilde{\varphi}^{-1} \circ \varphi$ is a diffeomorphism of open subsets of $\mathbb{R}^n$, the corresponding differential $d(\tilde{\varphi}^{-1} \circ \varphi)$ is a linear isomorphism. Thus,

$$d\varphi_{\varphi^{-1}(x)}(\mathbb{R}^k) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(d(\tilde{\varphi}^{-1} \circ \varphi)_{\varphi^{-1}(x)}(\mathbb{R}^k)) = d\tilde{\varphi}_{\tilde{\varphi}^{-1}(x)}(\mathbb{R}^k)$$

as claimed.

Now, it remains to show that $\dim T_xX = k$. By definition, there exists an open set $\hat{V} \subset \mathbb{R}^N$ and a smooth map $\Psi\colon \hat{V} \to \mathbb{R}^k$ that extends the chart $\psi := \varphi^{-1}$. Note that $\Psi \circ \varphi = \mathrm{id}_U$, so by the chain rule,

$$d\Psi_x \circ d\varphi_{\varphi^{-1}(x)} = \mathrm{id}_{\mathbb{R}^k}$$

Then, $d\varphi_{\varphi^{-1}(x)}$ must be an isomorphism $\mathbb{R}^k \to T_xX$, and hence $\dim T_xX = k$. ■

Example *Tangent spaces for our trivial examples*
Returning to the trivial examples we previously gave, we now state the corresponding tangent space for an arbitrary point x on each manifold.

- $X = \mathbb{R}^N$: $T_xX = \mathbb{R}^N$
- $X = W \subset \mathbb{R}^N$ for any open subset W: $T_xX = \mathbb{R}^N$
- $X = \{(x_1, \cdots, x_k, 0, \cdots, 0)\} \subset \mathbb{R}^N$: $T_xX = X$

Example *Tangent spaces for S^n*
From any given chart, we can compute (φ and) $d\varphi$:

$$\frac{\partial \varphi}{\partial x^1} = (1, 0, \cdots, 0, -x_1/x_{n+1})$$

and similarly for $\partial\varphi/\partial x^i$. Manifestly, each partial derivative is perpendicular to x. Thus, $T_xX \subset x^\perp := \{v \in \mathbb{R}^{n+1} \colon \langle v, x\rangle = 0\}$. Since we know from the above lemma that $\dim T_xX = n$, we conclude that $T_xX = x^\perp$.

Definition Differential map for manifolds
Let $f \colon X \to Y$ be a smooth map between manifolds and $x \in X$. Choose a local parametrisation φ_1 around x and φ_2 around $f(x) \in Y$. We define the differential $df_x \colon T_xX \to T_{f(x)}Y$ of f at x by

$$df_x = d{\varphi_2}_{\varphi_2^{-1}(f(x))} \circ d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)} \circ (d{\varphi_1}_{\varphi_1^{-1}(x)})^{-1}$$

Lemma 1.2
df_x is independent of the choice of local parametrisations.

Proof. Trivial exercise. ■

Proposition 1.3 Chain rule for manifolds
Let $f \colon X \to Y$ and $g \colon Y \to Z$ be smooth maps between manifolds. For any $x \in X$,

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Proof. Trivial exercise. ■

Theorem 1.4 Inverse function theorem
Let $f \colon X \to Y$ be a smooth map between manifolds and $x \in X$. Suppose $df_x \colon T_xX \to T_{f(x)}Y$ is an isomorphism. Then f is a local diffeomorphism*, i.e., each $x \in X$ has an open neighbourhood $V \subset X$ such that $f|_V \colon V \to f(V)$ is a diffeomorphism.*

Proof. Since df_x is an isomorphism, it follows that $d(\varphi_2^{-1} \circ f \circ \varphi_1)_{\varphi_1^{-1}(x)}$ is also an isomorphism. We can then use the usual inverse function theorem to deduce the result. ■

1.2 Regular values and Sard's theorem

Definition Critical and regular points
Let $f \colon X \to Y$ be a smooth map between manifolds. We say that $x \in X$ is a critical point of f if $df_x \colon T_xX \to T_{f(x)}Y$ is not surjective. Otherwise, it is a regular point.

Notation. We denote by C the set of all critical points of f.

Remark. If $\dim Y > \dim X$, then $C = X$ and the pre-image of any regular value is $\varnothing$.

Definition Critical and regular values
Let $f \colon X \to Y$ be a smooth map between manifolds. We say that $y \in Y$ is a critical value of f if $y = f(x)$ for some $x \in C$. Otherwise, we say that y is a regular value of f.

Theorem 1.5 Pre-image theorem
Let $f \colon X \to Y$ be a smooth map between manifolds. Suppose $y \in Y$ is a regular value of f. If $f^{-1}(y) \neq \varnothing$, then $f^{-1}(y) \subset X$ is a submanifold of X with $\operatorname{codim}_X f^{-1}(y) = \dim Y$.

Proof. Fix $x \in f^{-1}(y)$. Since y is a regular value, we know that $df_x \colon T_xX \to T_yY$ is surjective. By the rank-nullity theorem, $\dim\ker df_x = \operatorname{codim}_X Y$. Suppose $X \subset \mathbb{R}^N$, and pick a linear map $T \colon \mathbb{R}^N \to \mathbb{R}^{\operatorname{codim}_X Y}$ such that $\ker T \cap \ker df_x = \{0\}$.[1]

Now, extend f to the map $F \colon X \to Y \times \mathbb{R}^{\operatorname{codim}_X Y}$ given by $z \mapsto (f(z), T(z))$. Note that the differential of F at x is given by

$$dF_x = (df_x, dT_x) = (df_x, T)$$

Since $\ker T \cap \ker df_x = \{0\}$, we have $\ker dF_x = \{0\}$, i.e., dF_x is injective. By the inverse function theorem for manifolds, there exists an open neighbourhood $U \subset X$ of x such that $F|_U \colon U \to V$ is a diffeomorphism to an open neighbourhood V of $(y, T(x))$. Hence, $F|_{f^{-1}(y)\cap U}$ is a local parametrisation of $(\{y\} \times \mathbb{R}^{\operatorname{codim}_X Y}) \cap V$, proving that $f^{-1}(y)$ is a manifold of dimension $\operatorname{codim}_X Y$. ∎

Exercise. Show that, under the conditions of the pre-image theorem, $T_x f^{-1}(y) = \ker df_x$.

Corollary 1.6
Let $f \colon X \to Y$ be a smooth map between manifolds of the same dimension, with X compact. If y is a regular value of f, then $f^{-1}(y)$ is finite.

Proof. By the pre-image theorem, $f^{-1}(y)$ is a 0-dimensional manifold, i.e., a collection of points. Since X is compact, such a collection must be finite. ∎

With just a bit more analysis and topology, we can actually say more than just finiteness:

Theorem 1.7 Stack of records theorem
Let $f \colon X \to Y$ be a smooth map between manifolds of the same dimension, with X compact. Let y be a regular value of f, and list the elements of $f^{-1}(y)$ as $x_1, \cdots, x_n$. There exists an open neighbourhood $V \subset Y$ of y and a collection of open neighbourhoods $W_i \subset X$ of each x_i such that

$$f^{-1}(V) = \bigsqcup_{i=1}^{n} W_i$$

and each $f|_{W_i} \colon W_i \to V$ is a diffeomorphism.

Proof. By the inverse function theorem for manifolds, we can pick open neighbourhoods W_i of x such that each $f|_{W_i}$ is a diffeomorphism to an open neighbourhood of y. By shrinking neighbourhoods if necessary, W_i can be taken WLOG to be pairwise disjoint. Now, set

$$V = \left[\bigcap_{i=1}^{n} f(W_i)\right] \setminus f\left(X \setminus \bigcup_{i=1}^{n} W_i\right)$$

Note that $f(X \setminus \bigcup_{i=1}^{n} W_i)$ is a compact set that does not contain y, so V is an open neighbourhood of y. Finally, note that $f^{-1}(V) = \bigsqcup_{i=1}^{n} W_i$ by construction. ∎

Now, the pre-image theorem can be a powerful tool for generating manifolds or showing that a certain set is one.

Application S^n *is a manifold*
Consider the map $f \colon \mathbb{R}^{n+1} \to \mathbb{R}, (x_1, \cdots, x_{n+1}) \mapsto x_1^2 + \cdots + x_{n+1}^2$. Note that $f^{-1}(1) = S^n$, so to show that S^n is a manifold, it suffices to show that 1 is a regular point. Indeed, note that $df_x = (2x_1, \cdots, 2x_{n+1})$, which is not surjective only if $x = 0 \notin f^{-1}(1)$.

[1] It is easy to constructively show using IB Linear Algebra that such a map exists. [*Exercise!*]

Application *Orthogonal group as a manifold*
Denote by $M(n)$ [resp. $S(n)$] the space of all [resp. symmetric] $n \times n$ matrices with entries in $\mathbb{R}$. Consider the orthogonal group $O(n) = \{A \in M(n)\colon AA^t = I\} \subset M(n) = \mathbb{R}^{n^2}$.

Let $f\colon M(n) \to O(n)$ be the map $A \mapsto AA^t$. This is smooth since multiplication and addition in $\mathbb{R}$ are smooth. Since $O(n) = f^{-1}(I)$, it suffices to show that I is a regular value of f. Note that

$$df_A(H) = \lim_{t \to 0} \frac{f(A+tH) - f(A)}{t} = AH^t + HA^t$$

Now, fix $A \in M(n)$. Given $B \in S(n)$, observe that

$$df_A\left(\frac{1}{2}CA\right) = \frac{1}{2}AA^tC^t + \frac{1}{2}CAA^t = \frac{1}{2}C + \frac{1}{2}C = C$$

completing the proof that I is a regular value of f.

Remark. Recall that, besides being a manifold as we've just shown, $O(n)$ is also a group. In fact, the group operations $(A, B) \mapsto AB$ and $A \mapsto A^{-1} = A^t$ are smooth. Hence, we see that $O(n)$ is a *Lie group.*

Now, the pre-image theorem raises the question: how easy is to find regular values? This leads us to Sard's theorem.

Definition Measure-zero subsets of $\mathbb{R}^N$
We say that $S \subset \mathbb{R}^N$ is of measure zero in $\mathbb{R}^N$ if, for each $\varepsilon > 0$, there exists a countable family $\{R_i\}$ of sets of the form $R_i = \prod_{j=1}^{N}[x_i^{(j)}, y_i^{(j)}]$ such that $S \subset \bigcup_i R_i$ and $\sum_i \operatorname{vol}(R_i) < \varepsilon$.

Definition Measure zero subsets of manifolds
Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold. We say that $A \subset X$ is of measure zero in X if, for all local parametrisations $\varphi\colon U \to V$ of X, $\varepsilon^{-1}(V \cap A) \subset \mathbb{R}^k$ has measure zero in $\mathbb{R}^k$.

Exercise. Let $U, \tilde{U} \subset \mathbb{R}^k$ be open and $\psi\colon U \to \tilde{U}$ a diffeomorphism. Show that, if $A \subset U$ is of measure zero in $\mathbb{R}^k$, then $\tilde{T} = \psi(T)$ is of measure zero in $\mathbb{R}^k$.

Remarks

- In view of the above exercise, $A \subset X$ is of measure zero in X iff $\varphi_i^{-1}(S \cap V_i)$ is of measure zero for all $\varphi_i\colon U \to V$ in an atlas of local parametrisations.
- If $\dim Y = 0$, then Y is of measure zero. If $\dim Y > 0$, then every non-empty open subset $V \subset Y$ is not of measure zero in Y.
- If $S \subset X$ is of measure zero in X, then any $\tilde{S} \subset S$ is also of measure zero in X.

Theorem 1.8 Sard's theorem
Let $f\colon X \to Y$ be a smooth map between manifolds. Then the set of critical values of f is of measure zero in Y.

Proof. Non-examinable — see Milnor's book if interested. ■

Corollary 1.9
The set of regular values of a smooth map $f\colon X \to Y$ between manifolds is dense in Y.

Proof. Any open set $V \subset Y$ cannot lie entirely in $f(C)$ since it has measure zero. ■

1.3 Transversality

Definition Transversal
Let $f\colon X \to Y$ be smooth and $Z \subset Y$ a submanifold of Y. We say that f is transversal to Z if, for each $x \in f^{-1}(Z)$,

$$T_{f(x)}Y = T_{f(x)}Z + \operatorname{im} df_x$$

We then write $f \pitchfork Z$.

Remarks

- If $f(X) \cap Z = \varnothing$, then f is transversal to Z.
- If $Z = \{y\}$, then f is transversal to Z iff y is a regular value of f. Thus, transversality is really a generalisation of the notion of regular values.

Exercise. Let X also be a submanifold of Y and $\iota\colon X \hookrightarrow Y$ the inclusion map. Show that $d\iota_x$ is just the inclusion map $T_xX \hookrightarrow T_xY$ of the tangent spaces. Thus, $\iota \pitchfork Z$ iff $T_xX + T_xZ = T_xY$ for all $x \in X \cap Z$.

Now, we state a generalisation of the pre-image theorem for transversal maps:

Theorem 1.10
Let $f\colon X \to Y$ be smooth map that is transversal to a submanifold $Z \subset Y$ of Y. If $f^{-1}(Z) \neq \varnothing$, then $f^{-1}(Z) \subset X$ is a submanifold of X, with $\operatorname{codim}_X f^{-1}(Z) = \operatorname{codim}_Y Z$.

Remark. If $Z = \{y\}$, then $\operatorname{codim}_Y Z = \dim Y$ as in the pre-image theorem.

Sketch of proof (non-examinable). Fix $z \in Z$ with $z = f(x)$ for some $x \in X$. Note that, for some open neighbourhood $V \subset Y$ of z, there exists a smooth map $h\colon V \to \mathbb{R}^{\operatorname{codim}_Y Z}$ such that $Z \cap V = h^{-1}(0)$ and dh_z is surjective. Locally around $x \in X$, $f^{-1}(Z) = (h \circ f)^{-1}(0)$. Thus, by the pre-image theorem, it suffices to show that 0 is a regular value of $h \circ f$.

Now, since $f \pitchfork Z$, we have $T_zY = T_zZ + \operatorname{im} df_x$. By the exercise after the pre-image theorem, we have $dh_z = T_zZ$. Moreover, $f \pitchfork Z$ gives us

$$T_zY = T_zZ + \operatorname{im} df_x = \ker dh_z + \operatorname{im} df_x$$

This then implies that $\operatorname{im} dh_z = \operatorname{im}(dh_z \circ df_z) = \operatorname{im} d(h \circ f)_x$. Since dh_z is surjective, $d(h \circ f)_z$ is also surjective and hence 0 is a regular value of $h \circ f$. ∎

Exercise. Construct the required map h.

Remark. Transversality is both a stable and generic property. It is stable in the sense that small perturbations of f remain transversal to a given submanifold. It is generic in the sense that any given smooth map may be deformed by arbitrarily small amounts into a map that is transversal to Z. See *Differential Topology* by Guillemin and Pollack for more details.

1.4 Manifolds with boundary

Consider the closed upper half plane

$$\mathbb{H}^k := \{(x_1, \cdots, x_k) \in \mathbb{R}^n \colon x_k \geq 0\}$$

We denote its boundary by $\partial\mathbb{H}^k = \{x_k = 0\}$.

Definition Manifold with boundary
We say that $X \subset \mathbb{R}^N$ is a (smooth) k-dimensional manifold with boundary if every $x \in X$ has an open neighbourhood $V \subset X$ that is diffeomorphic to an open subset $U \subset \mathbb{H}^k$.

Remark. Note that a diffeomorphism $\varphi \colon U \to V$ has a smooth extension defined on an open subset of $\mathbb{R}^k$. This allows us to deduce as before that, if X is both k- and $\tilde{k}$-dimensional, then $k = \tilde{k}$.

Definition Dimension
The dimension of a k-dimensional manifold with boundary X is k.

Definition Boundary of a manifold with boundary
Let X be a k-dimensional manifold with boundary. Its boundary is defined to be

$$\partial X := \{x \in X \colon \exists \text{ open nhood } V \subset X \text{ and diffeomorphism } \psi \colon V \to \psi(V) \text{ s.t. } x \in \psi^{-1}(\partial \mathbb{H}^k)\}$$

Remarks

- In fact, if $x \in \psi^{-1}(\partial \mathbb{H}^k)$ for some diffeomorphism $\psi \colon V \to \psi(V) \subset \mathbb{H}^k$ on an open nhood V of x, then it is true for *all* diffeomorphisms on an open nhood of x to its image in $\mathbb{H}^k$.
- In the definition of manifold with boundary, we may take $U = \mathbb{R}^k$ or $U = \mathbb{H}^k$ WLOG.

Exercise. Prove the first remark.

Definition Interior of manifold with boundary
Let X be a manifold with boundary. We define its interior to be

$$\operatorname{int} X := X \backslash \partial X$$

Proposition 1.11
Let X be a k-dimensional manifold with boundary. Then $\operatorname{int} X$ is a manifold of dimension k and ∂X is a manifold of dimension $k-1$.

Proof. $\operatorname{int} X$ is a manifold of dimension k because we can always restrict our diffeomorphisms $\varphi \colon V \to U$ such that $U \cap \partial \mathbb{H}^k = \varnothing$. See Example Sheet 1 for ∂X. ∎

Example

- Trivially, $\mathbb{H}^k$ is a k-dimensional manifold with boundary
- As we will prove later, $B^n := \{x \in \mathbb{R}^n \colon ||x||_2 \leq 1\}$ is an n-dimensional manifold with boundary. Note that $\partial B^n = S^{n-1}$ is a manifold of dimension $n-1$.
- $[0,1] \times [0,1]$ is not a manifold with boundary (see Example Sheet 1)
- If X is a manifold with boundary and Y is a manifold, then $X \times Y$ is a manifold with boundary, with $\partial(X \times Y) = (\partial X) \times Y$. (Of course, the previous example is a counterexample to the case that Y is also a manifold with boundary.)

Remark. Note that ∂X and $\operatorname{int} X$ are note the same as the topological notions of 'boundary' and 'interior' as subsets of $\mathbb{R}^N$. Indeed, if $\dim X < N$, the topological interior of X is empty, whereas $\operatorname{int} X$ is not.

Definition Tangent space
Let X be a k-dimensional manifold with boundary and $x \in X$. Let $\varphi \colon U \to V$ be a diffeomorphism from an open set $U \subset \mathbb{H}^k$ to an open neighbourhood $V \subset X$ of x. Since φ is smooth, there exists

a smooth extension $\tilde{\varphi}$ on an open subset of $\mathbb{R}^k$, with $d\tilde{\varphi}_{\varphi^{-1}(x)}$ well-defined. We then define the tangent space to be

$$T_xX := \operatorname{im} d\tilde{\varphi}_{\varphi^{-1}(x)}$$

Remark. As before, T_xX is well-defined.

Exercise. Show that, for every $x \in \partial X$, $T_x\partial X \subset T_xX$.

Lemma 1.12
Let X be a manifold of dimension k. Let $f\colon X \to \mathbb{R}$ be smooth, with 0 a regular value of f. Then $f^{-1}([0,\infty)) \subset X$ is a k-dimensional manifold with boundary $\partial(f^{-1}([0,\infty))) = f^{-1}(0)$.

Proof. The subset $f^{-1}((0,\infty)) \subset X$ is open in X and thus a submanifold of X. This means that we can restrict a local parametrisation to X such that its image lies in $f^{-1}((0,\infty))$ and get the diffeomorphism we need.

It remains to consider $x \in f^{-1}(0)$. Extend f to a map $\mathbb{R} \to \mathbb{R} \times \mathbb{R}^{k-1}$ as in the proof of the pre-image theorem (Theorem 1.5). We can then proceed as before using the inverse function theorem. ■

Corollary 1.13
B^n is an n-dimensional manifold with boundary.

Proof. This is immediate from the above lemma. ■

Theorem 1.14 Pre-image theorem for manifolds with boundary
Let X be a manifold with boundary and Y a manifold, with $\dim X > \dim Y$. Suppose $f\colon X \to Y$ is smooth and $y \in Y$ is a regular value of both f and $f|_{\partial X}$. Then $f^{-1}(y) \subset X$ is a manifold with boundary, with $\operatorname{codim}_X f^{-1}(y) = \dim Y$ and $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial X$.

Proof. WLOG, we may assume that $X = \mathbb{H}^m$ and $Y = \mathbb{R}^n$ since we are always working locally. The easy case $x \in f^{-1}(y) \cap \operatorname{int} \partial\mathbb{H}^m$ is left as an exercise. Now, suppose $x \in f^{-1}(y) \cap \partial\mathbb{H}^m$. Then there exists an open subset $U \subset \mathbb{R}^m$ such that $f|_{U\cap\mathbb{H}^m}$ extends to a smooth map $F\colon U \to \mathbb{R}^n$. Since y is a regular value of $f|_{U\cap\mathbb{H}^m}$, dF_x is surjective. Since the map $z \mapsto dF_z$ (defined on U) is smooth, we can shrink U such that dF_z is surjective for all $z \in U$.[2] Applying the pre-image theorem to F, we have that $F^{-1}(y)$ is a submanifold of U with $\operatorname{codim}_{\mathbb{R}^m} F^{-1}(y) = \dim Y$. Let $\pi\colon F^{-1}(y) \to \mathbb{R}, (x_1, \cdots, x_m) \mapsto x_m$. Note that

$$\left(f|_{U\cap\mathbb{H}^m}\right)^{-1}(y) = \pi^{-1}([0,\infty))$$

It then suffices to show that 0 is a regular value of π since, by the previous lemma, it would follow that $\pi^{-1}([0,\infty))$ is a submanifold of $F^{-1}(y)$ with boundary $\pi^{-1}(0) = F^{-1}(y) \cap \partial\mathbb{H}^m = f^{-1}(y) \cap U \cap \partial\mathbb{H}^m$.

Now, to show that, for any $z \in \pi^{-1}(0)$, the map $d\pi_z\colon T_zF^{-1}(y) \to \mathbb{R}$ is surjective, it suffices to show that $T_zF^{-1}(y) = \ker dF_z = \ker df_z \not\subset \ker d\pi_z = \mathbb{R}^{m-1} \times \{0\} = T_z\partial\mathbb{H}^m$. Indeed, note that

$$df_z|_{T_z\partial\mathbb{H}^m} = d\left(f|_{\partial\mathbb{H}^m}\right)_z$$

is surjective. If $\ker df_z \subset T_z\partial\mathbb{H}^m$, then $\ker(df_z|_{T_z\partial\mathbb{H}^m}) = \ker(df_z)$, but these have different dimensions by the rank-nullity theorem — a contradiction! ■

[2] Indeed, we know that some submatrix of dF_x has nonzero determinant. By continuity of det, there is some open neighbourhood $\tilde{U} \subset U$ of x on which the determinant of that submatrix remains nonzero and thus $\dim \operatorname{im} dF_z = n$ for all $z \in \tilde{U}$.

Theorem 1.15
Let X be a manifold with boundary and Y a manifold with $Z \subset Y$ a submanifold. Let $f\colon X \to Y$ be smooth such that $f \pitchfork Z$ and $f|_{\partial X} \pitchfork Z$. Then $f^{-1}(z) \subset X$ is a manifold with boundary, with $\operatorname{codim}_X f^{-1}(Z) = \operatorname{codim}_Y Z$ and $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$.

1.5 Degree modulo 2

Definition Smooth homotopy
Let $f, g\colon X \to Y$ be smooth maps between manifolds. A smooth homotopy between f and g is a smooth map $F\colon X \times [0,1] \to Y$ such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$. If such a map exists, we say that f and g are smoothly homotopic and write $f_0 \simeq f_1$.

Exercise. Show that $\simeq$ is an equivalence relation.

Definition Smooth isotopy
Let $f, g\colon X \to Y$ be diffeomorphisms. A smooth isotopy between f and g is a smooth homotopy $F\colon X \times [0,1] \to Y$ for which $F|_{X\times\{t\}}$ is a diffeomorphism for all $t \in [0,1]$. If such a map exists, we say that f and g are smoothly isotopic.

Lemma 1.16 Homotopy lemma
Suppose $f, g\colon X \to Y$ are smoothly homotopic, with X compact and $\dim X = \dim Y$. If y is a regular value of both f and g, then

$$\# f^{-1}(y) \equiv \# g^{-1}(y) \bmod 2$$

Proof. Let $F\colon X \times [0,1] \to Y$ be a smooth homotopy between f and g. We first suppose that y is also a regular value of F. By Theorem 1.14, $F^{-1}(y)$ is a 1-dimensional manifold with boundary $\partial F^{-1}(y) = F^{-1}(y) \cap (X \times \{0,1\})$. (To be continued) ■