Modelling of physiological and pathological processes

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Lecture 4

Recall our simple reaction-diffusion system:

$$\frac{\partial u}{\partial t} = D\nabla^2 u + F(u)$$

Slightly different, simplified form:

•Diffusion coefficient is there, D

•Only one equation/concentration

Can we enrich and generalise this equation to account for *“more”* ?

The general transport equation:

$$\underbrace{\frac{\partial(\rho\Phi)}{\partial t}}_{\text{Unsteady}} + \underbrace{\nabla\bullet(\rho\vec{V}\Phi)}_{\text{Convection}} = \underbrace{\nabla\bullet(D\nabla\Phi)}_{\text{Diffusion}} + \underbrace{S_{M_{x_i}}}_{\text{Everything else}}$$

Equations for general fluid flow:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0$$

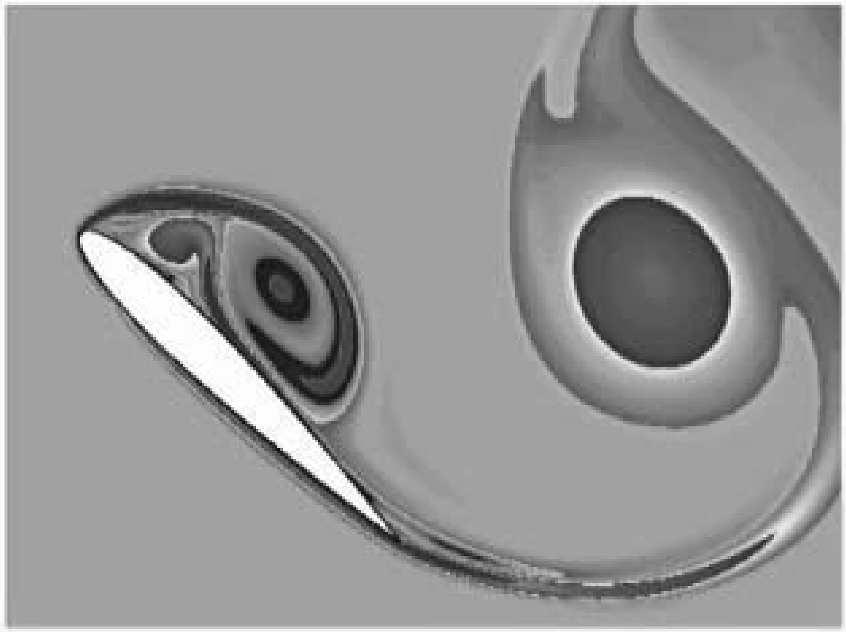
$$\frac{\partial(\rho u_i)}{\partial t} + \nabla \bullet (\rho \vec{V} u_i) = \nabla \bullet (\mu \nabla u_i) + (S_{M_{x_i}} - \frac{\partial p}{\partial x_j})$$

Equations for incompressible fluid flow:

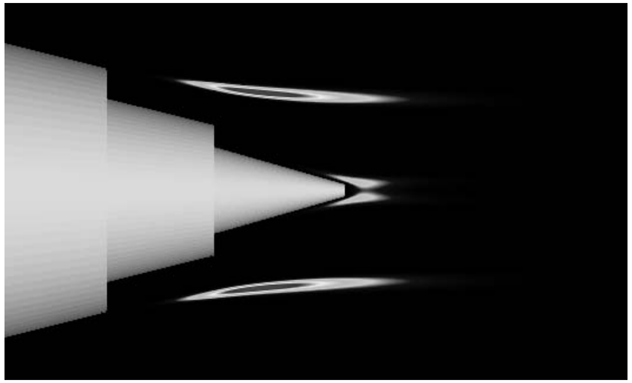
$$\nabla \bullet \vec{V} = 0$$

$$\frac{\partial(u_i)}{\partial t} + \nabla \bullet (\vec{V} u_i) = \nabla \bullet (\mu \nabla u_i) + (S_{M_{x_i}} - \frac{\partial p}{\partial x_j})$$

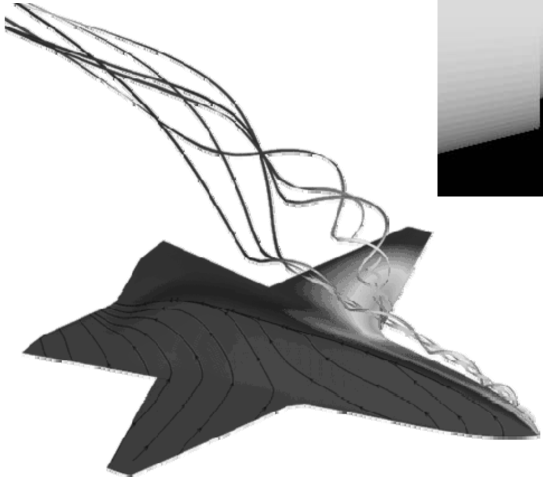
Aerodynamics



Dr. Craig Hunter
NASA Langley



Noise sources in a jet plume. The data set has about 35 variables at 17 million cells.



Pressure coefficient contours and volume streamribbons (highlighting the vortex flow field) are shown on the right side of the advanced tailless fighter model. Surface streamtraces are shown on the left side.

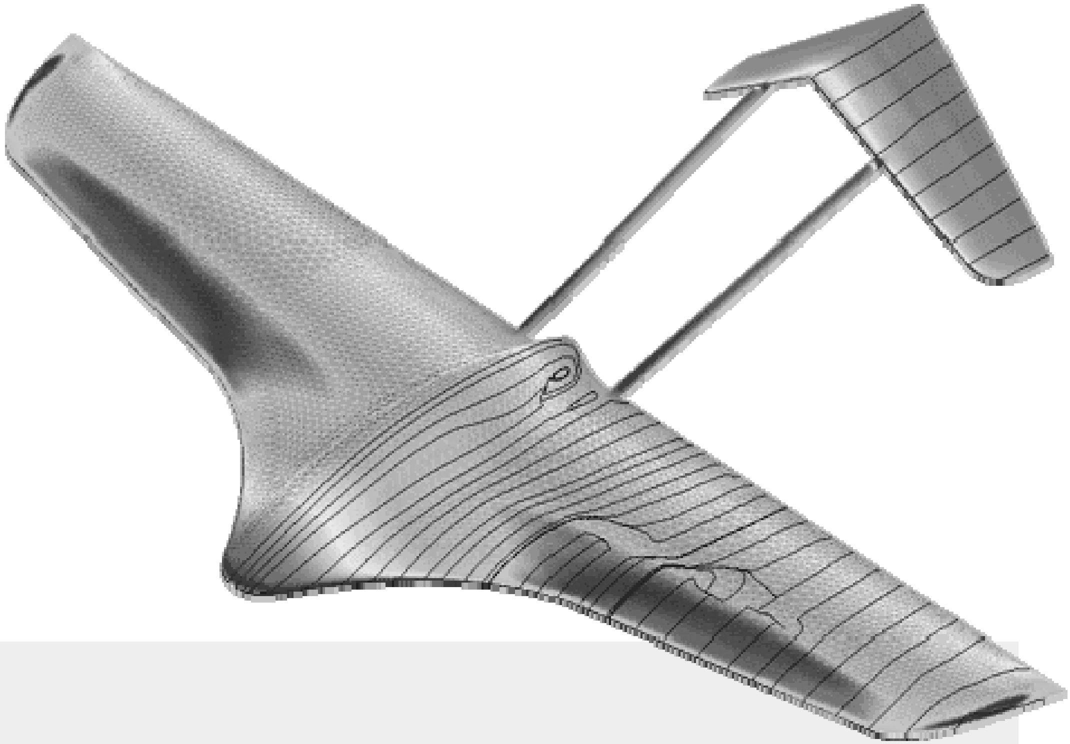
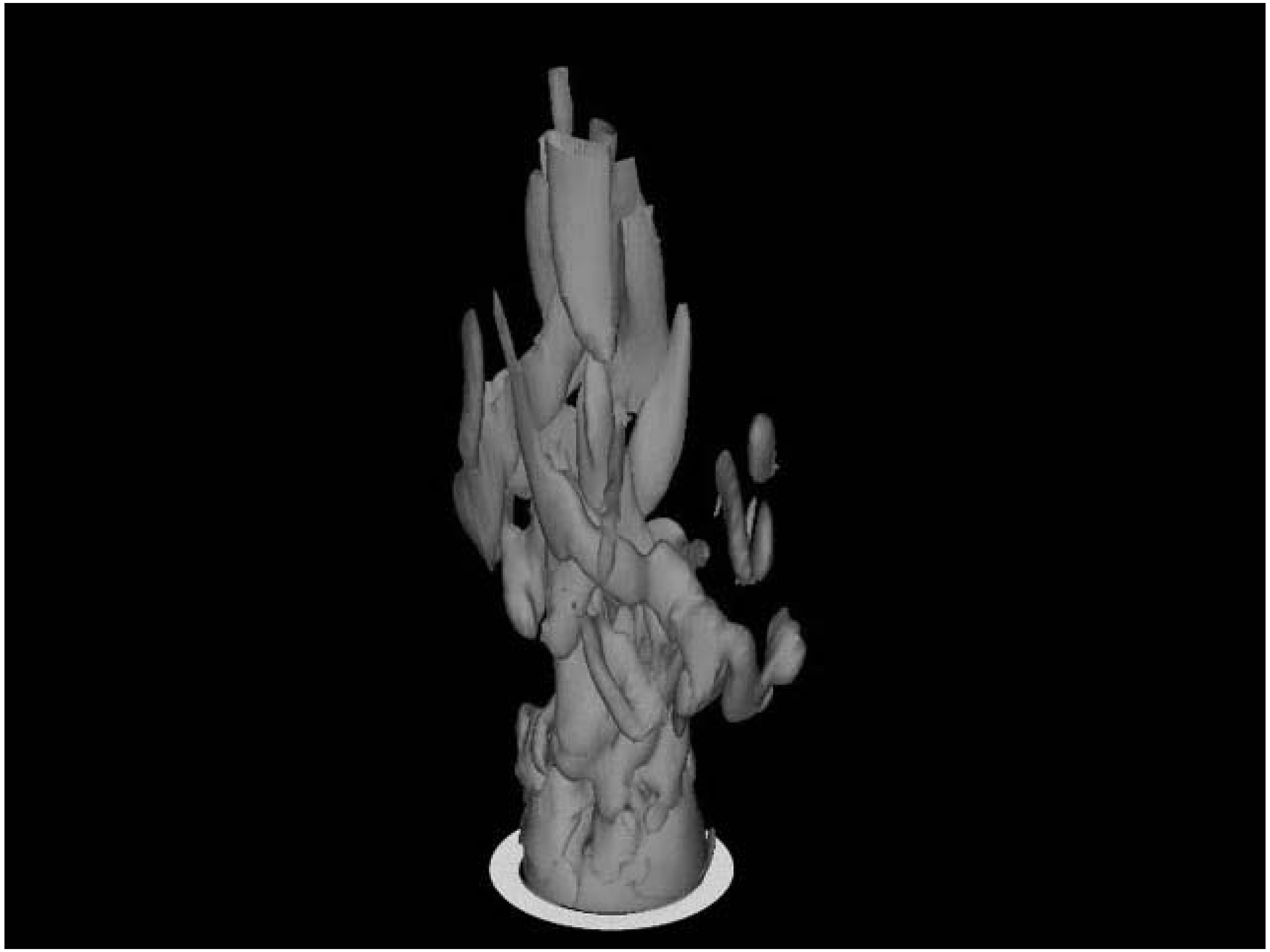
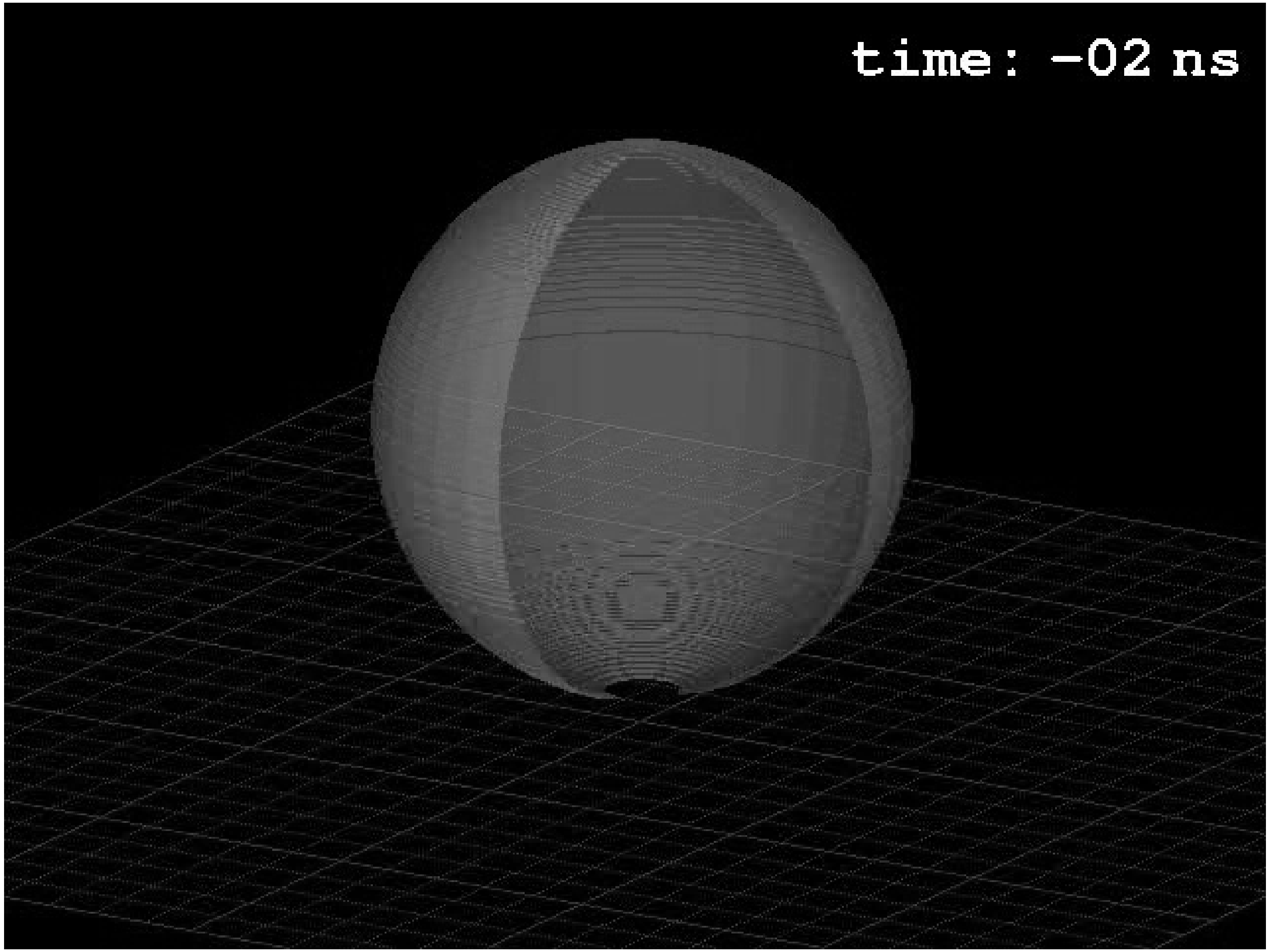


Figure 3: The Eagle, a high altitude concept demonstrator of the Mars Scout ARES aircraft.

Jets and Combustion



Micro-nano flows, droplets, sprays etc.



Why is any of this of any interest to us?

"In the Western world, **cardiovascular disease** (for example myocardial infarction, stroke, etc.) are the most common diseases causing death, and the formation of a blood clot in an artery is directly responsible in about 90% of all patients for such diseases. Together they account for **more than 40% of all deaths**, whereas cancer, the second leading cause of mortality, causes about 25%"

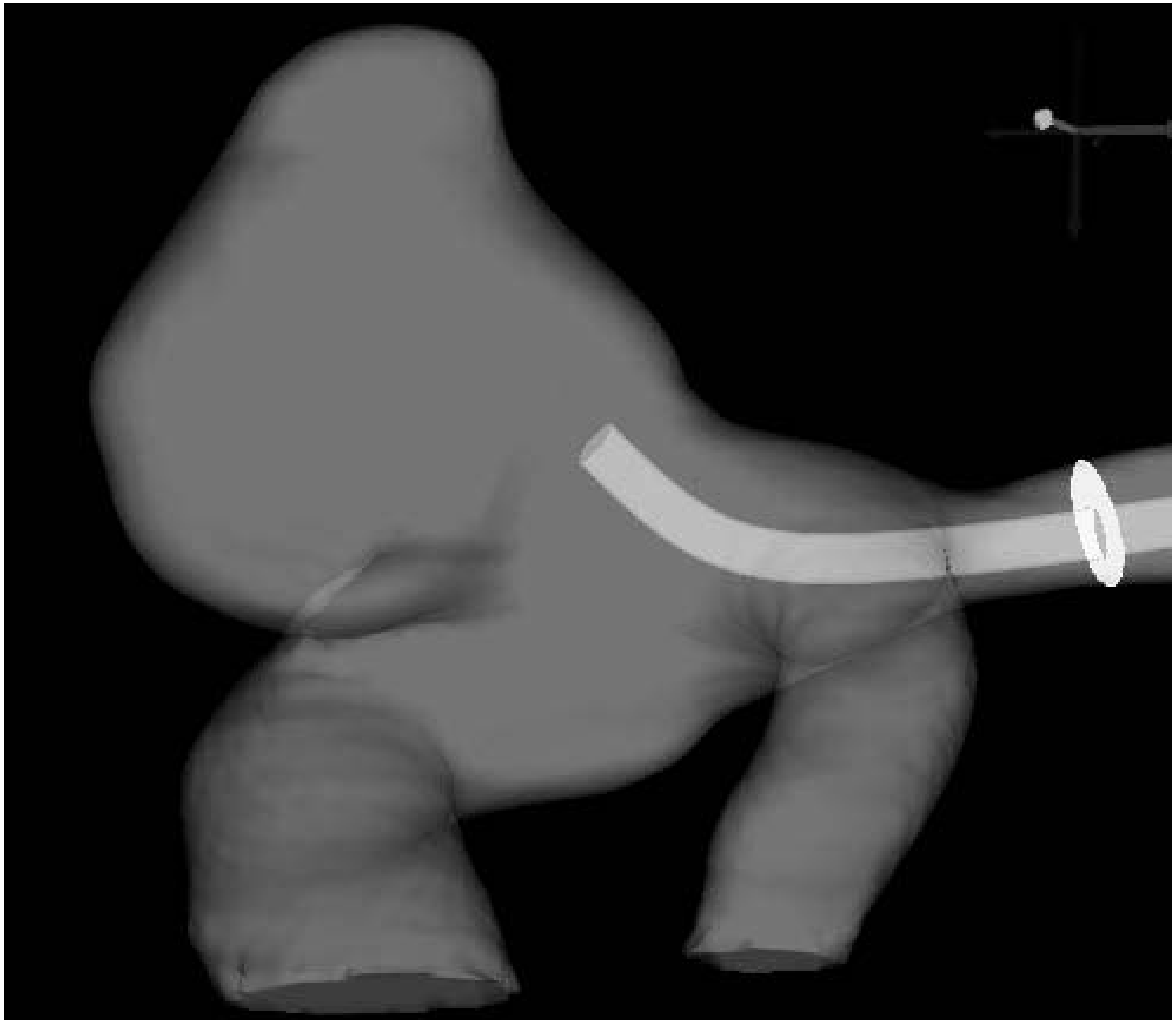
Mortality is the single issue with the highest societal impact, of similar importance is another trend though: between 1990 and 2000 the number of people age 60 and over remained approximately constant, **whereas those age 75 to 84 increased by 21%;** those **age 85 and older increased by 38%.**

Alzheimer's disease, (8th cause of death altogether), inflicts serious problems to the quality of life for a large percentage of this aging population: For the USA, **an estimated 5 million people suffer from the disease**; the number doubled since 1980. By 2050 the number of individuals with Alzheimer's could range from **11.3 million to 16 million**. Discovery of a treatment that could delay the onset of Alzheimer's disease by a **mere five years** could reduce the number of individuals suffering by nearly 50% after 50 years.

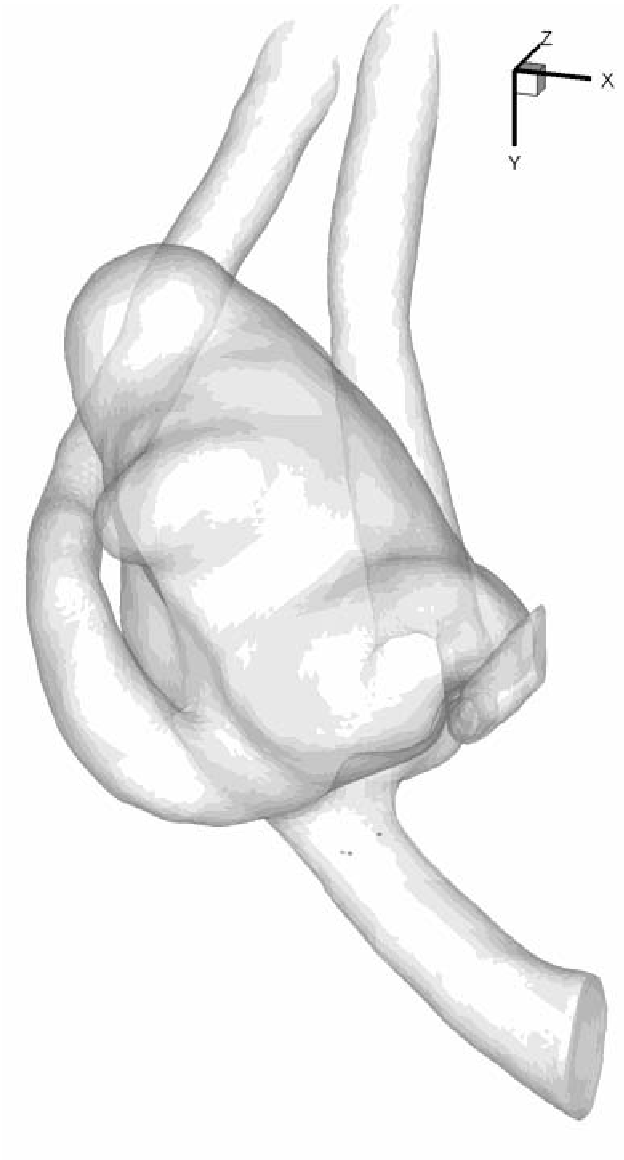
Asthma cases, especially among **children under 15 years old**, have **quadrupled in the last 20 years**.

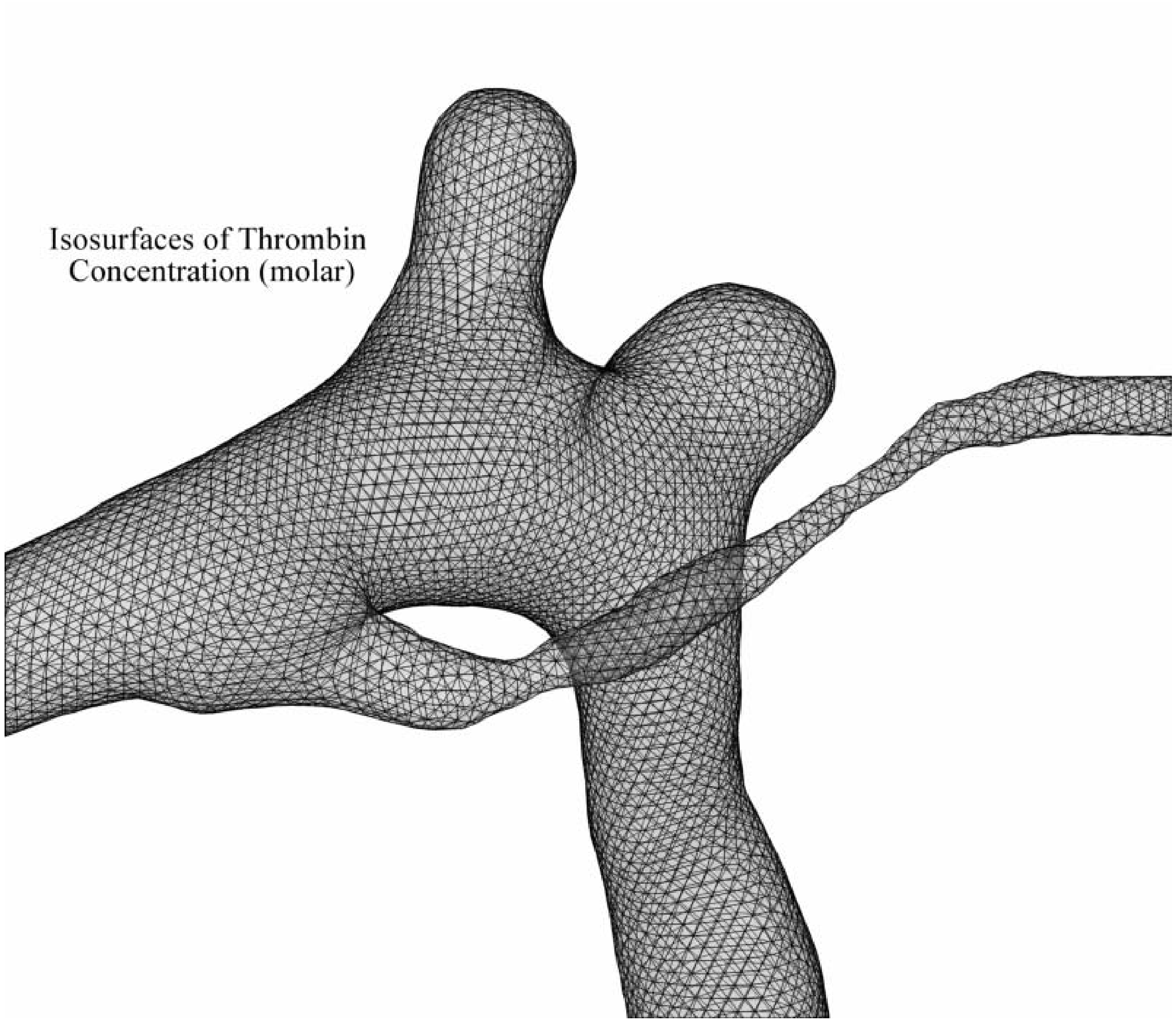
The underlying theme? Transport

Modelling of biological-biomedical-biomechanics has a lot to offer…



**…on cases that can be flow only,
or flow & biochemistry or anything you might imagine…**





For example, for flow and biochemical reactions:

$$\nabla \bullet \vec{V} = 0$$

$$\frac{\partial(u_i)}{\partial t} + \nabla \bullet (\vec{V} u_i) = \nabla \bullet (\mu \nabla u_i) + (S_{M_{x_i}} - \frac{\partial p}{\partial x_j})$$

+

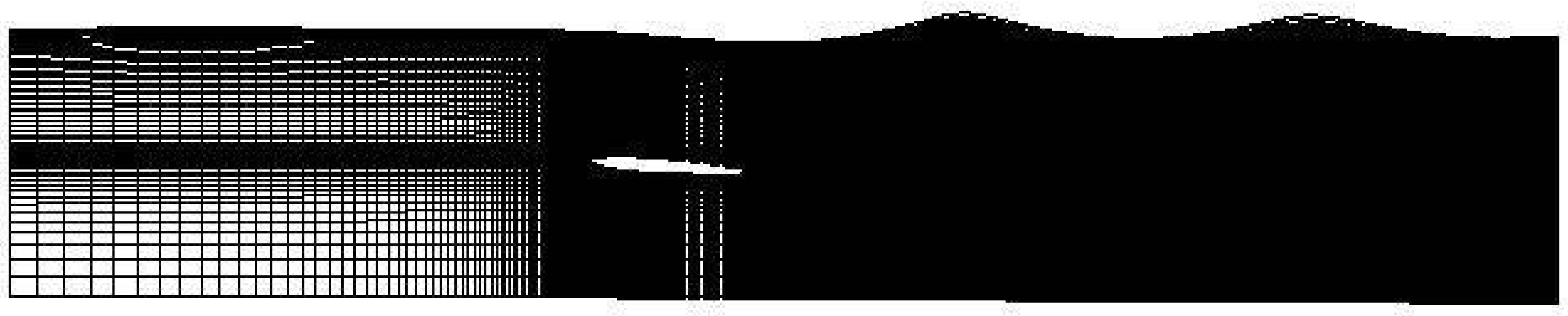
$$\frac{\partial(\rho C_1)}{\partial t} + \nabla \bullet (\rho \vec{V} C_1) = \nabla \bullet (D_1 \nabla C_1) + S_{M_1}$$

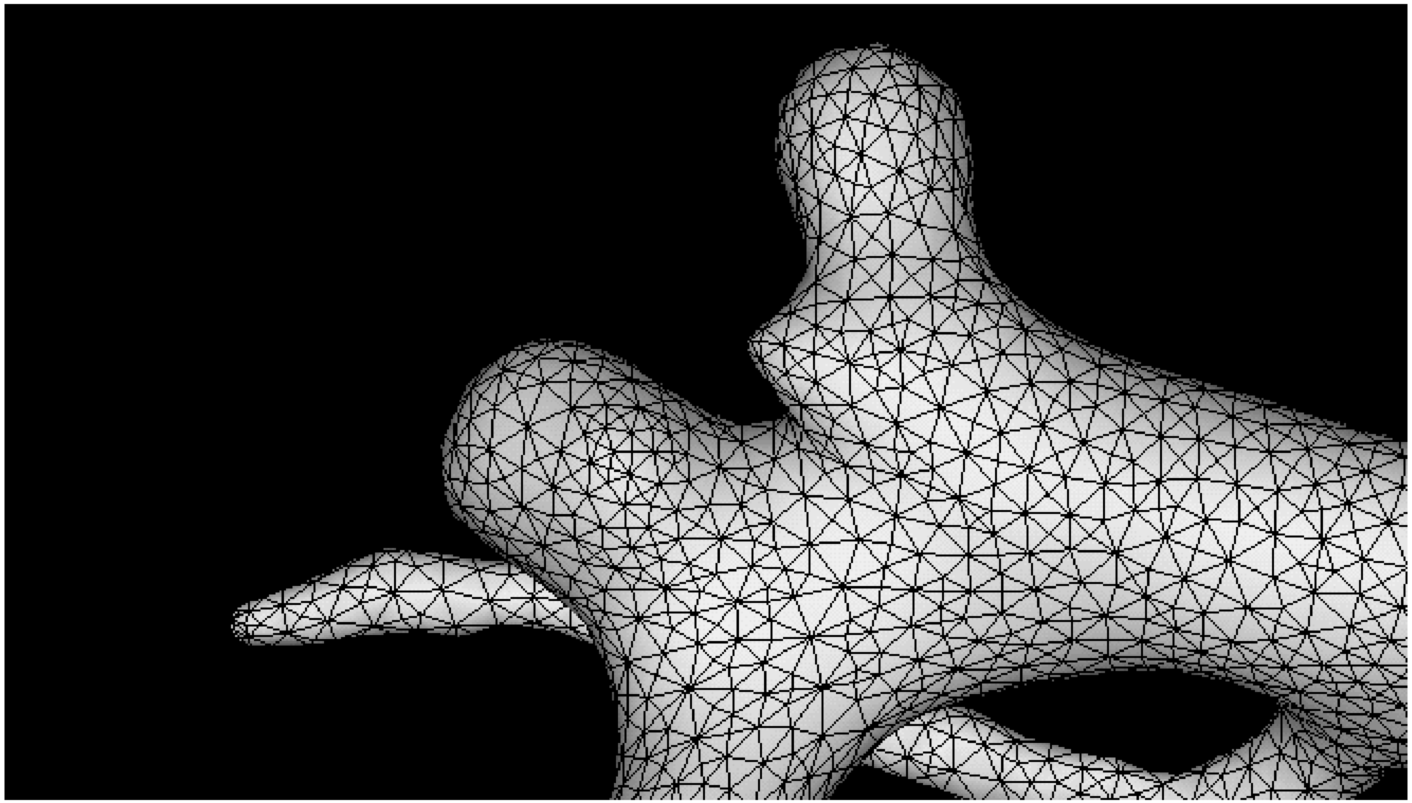
$$\frac{\partial(\rho C_2)}{\partial t} + \nabla \bullet (\rho \vec{V} C_2) = \nabla \bullet (D_2 \nabla C_2) + S_{M_2}$$

...

$$\frac{\partial(\rho C_n)}{\partial t} + \nabla \bullet (\rho \vec{V} C_n) = \nabla \bullet (D_n \nabla C_n) + S_{M_n}$$

In all cases you need a grid (mesh)…





… a proper casting of the governing equations …

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial(\rho u)}{\partial t}+\nabla\bullet(\rho\vec{V}u)=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+S_{M_x}$$

$$\frac{\partial(\rho v)}{\partial t}+\nabla\bullet(\rho\vec{V}v)=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial(-p+\tau_{yy})}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+S_{M_y}$$

$$\frac{\partial(\rho w)}{\partial t}+\nabla\bullet(\rho\vec{V}w)=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial(-p+\tau_{zz})}{\partial z}+S_{M_z}$$

In this case τ_{xx} etc. is the stress tensor

… and a good numerical technique.

- Finite elements
- Finite differences
- Finite volumes
- Particle methods
- Variants & combinations of the above

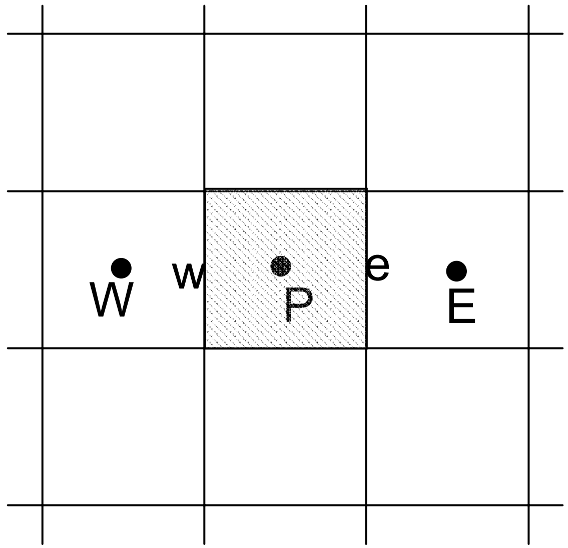
Books: too many for our own good, but…

- S. V. Patankar, “Numerical Heat Transfer and Fluid Flow”, Hemisphere Pub. Corp.

- J.H. Ferziger, M. Periç, “Computational Methods for Fluid Dynamics”, Springer

Finite volumes: Let's build the case…

$$\frac{\partial}{\partial x}\left(D\frac{\partial \Phi}{\partial x}\right) + S = 0$$



If D is constant:

$$D\frac{\partial^2 \Phi}{\partial x^2} + S = 0$$

Integrate over the "control volume":

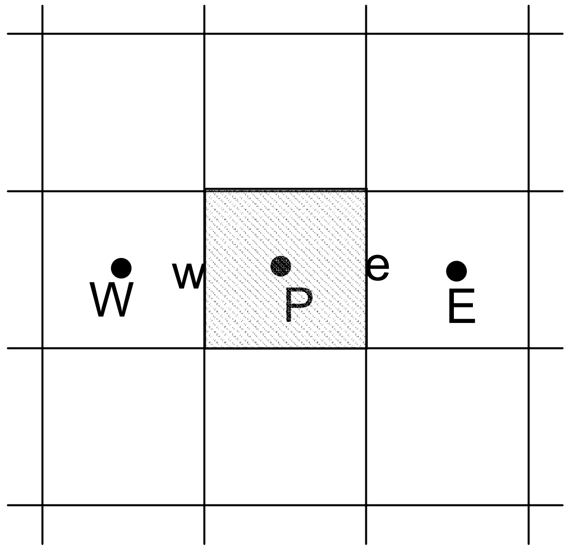
$$\int_{CV}\left[\frac{\partial}{\partial x}\left(D\frac{\partial \Phi}{\partial x}\right)\right]dx + \int_{CV} S dx = 0$$

Finite volumes: Let's build the case…

$$\cancel{\int_{CV}\left[\frac{\partial}{\partial x}\right.}\left.(D\frac{\partial\Phi}{\partial x})\right]dx+\int_{CV}Sdx=0$$

$$\left[D\frac{\partial\Phi}{\partial x}\right]_w^e+\int_w^e Sdx=0$$

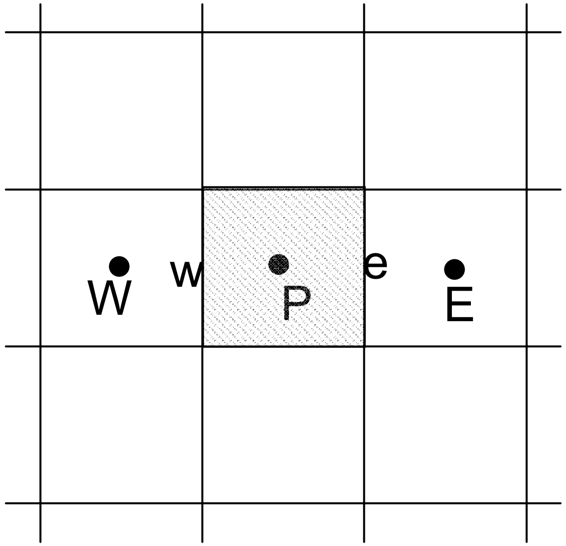
$$\left[D\frac{\partial\Phi}{\partial x}\right]_e-\left[D\frac{\partial\Phi}{\partial x}\right]_w+\int_w^e Sdx=0$$



I still have a gradient. What do I do?

Finite volumes: Let's build the case…

$$\left[D\frac{\partial\Phi}{\partial x}\right]_e - \left[D\frac{\partial\Phi}{\partial x}\right]_w + \int_w^e S\,dx = 0$$



I assume a piece-wise linear profile for $\frac{\partial\Phi}{\partial x}$ (good old finite differences, central scheme!)

Believe it or not, I am done:

$$\frac{D_e(\Phi_E - \Phi_P)}{(\delta x)_e} - \frac{D_w(\Phi_P - \Phi_W)}{(\delta x)_w} + \bar{S}\Delta x = 0$$

Beautify:

Finite volumes: Let's build the case…

$$a_P \Phi_P = a_E \Phi_E + a_w \Phi_w + b$$

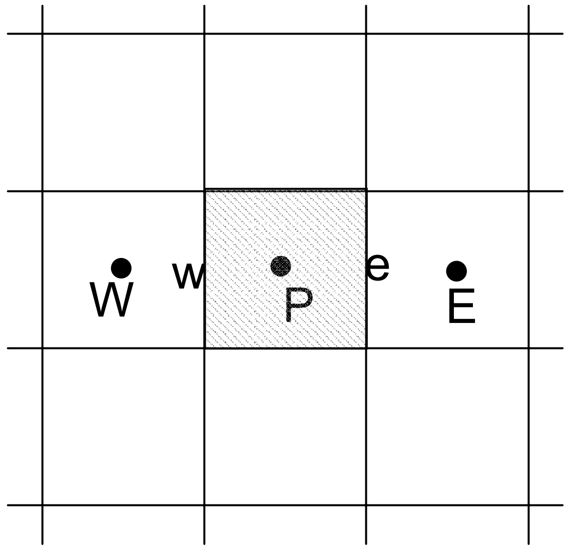
where:

$$a_E = \frac{D_E}{(\delta x)_e}$$

$$a_W = \frac{D_W}{(\delta x)_w}$$

$$a_P = a_E + a_W$$

$$b = \bar{S} \Delta x$$



Finite volumes: Let's build the case…

System is m x n

$$\begin{bmatrix} & & 0 \\ & 0 & \\ 0 & & \\ a_{w,i} & a_{P,i} & a_{E,i} \\ 0 & & \end{bmatrix} \begin{bmatrix} \bullet \\ \bullet \\ \Phi_i \\ \bullet \\ \bullet \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ b_i \\ \bullet \\ \bullet \end{bmatrix}$$

If the system is linear, solve once and that's it.
If non-linear, iterations are required.

There are 2 important complications, especially when dealing with fluid systems:

$$p?+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial(\rho u)}{\partial t}+\nabla\bullet(\rho\vec{V}u)=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+S_{M_x}$$

The (very) non-linear nature of the convection term of the momentum equations $\nabla \bullet (\rho \vec{V} u)$

If you repeat the derivation we performed in the case with diffusion only, for the case with convection and diffusion (this will be in your problem sheet, with instructions), you will get:

$$a_E = \frac{D_e}{(\delta x)_e} - \frac{(\rho u)_e}{2}$$

$$a_W = \frac{D_W}{(\delta x)_w} + \frac{(\rho u)_w}{2}$$

$$a_P = a_E + a_W + (\rho u)_e - (\rho u)_w$$

Why is this a problem?

It is a problem, because it is not “bounded”:

$$\frac{D_e}{(\delta x)_e} = \frac{D_W}{(\delta x)_w} = 1$$

$$(\rho u)_e = (\rho u)_w = 4$$

$$\text{then, if}$$

$$\Phi_E = 100 \,\&\, \Phi_w = 100 \rightarrow \Phi_P = 50$$

$$\Phi_E = 100 \,\&\, \Phi_w = 200 \rightarrow \Phi_P = 250$$

The answer is, either finer grids, or “smarter” numerical schemes.

The stupidest “smarter” scheme is upwind:

$$\Phi_e = \Phi_P \ldots \text{if} \ldots (\rho u)_e > 0$$
$$\Phi_e = \Phi_E \ldots \text{if} \ldots (\rho u)_e < 0$$

This is how the interpolation must be done.

“whichever is bigger” operator

$$a_E = \frac{D_E}{(\delta x)_e} + \left[\!\left[\frac{(\rho u)_e}{2}, 0 \right]\!\right]$$
$$a_W = \frac{D_W}{(\delta x)_w} + \left[\!\left[\frac{(\rho u)_w}{2}, 0 \right]\!\right]$$

Unfortunately, upwinding is only first-order accurate. Many variants of this, (hybrid etc.) as well as more sophisticated, higher order schemes exist.

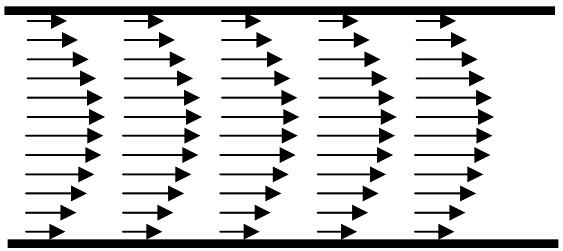
Boundary Conditions:

Dirichlet → $\Phi = \text{const.}$

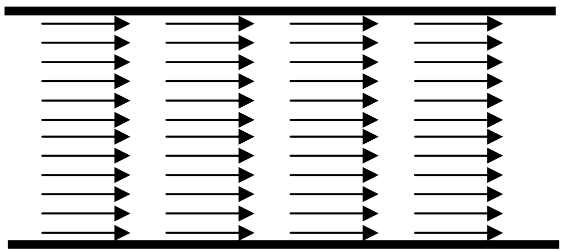
Neumann → $\dfrac{\partial \Phi}{\partial \text{n}} = \text{const.}$

Initial Conditions:

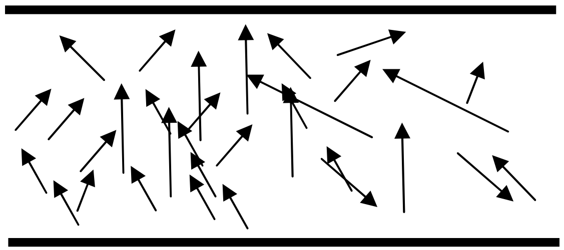
No magic recipe. You should start with a “guess” that is as close to your solution as possible.



If



then



but not!

The issue of pressure

$$\cancel{p?} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \nabla \bullet (\rho \vec{V} u) = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{M_x}$$

Many ways to approach this: an interesting and very popular one is to integrate the continuity equation over the control volume and to utilise its “deficit” to drive a pressure correction equation (note that pressure does not appear anywhere in the equations, only pressure gradient!).

This approach (pressure correction methodology) has led to many algorithms with very cool names: SIMPLE[1], SIMPLER, SIMPLEC, PISO etc…

Semi-**I**mplicit **M**ethod for **P**ressure-**L**inked **E**quations

Things we have not covered at all:

- Turbulence
- Particles and Molecular Dynamics
- Cell-level simulations (FE, motility, etc.)
- Large scale integrative models (autoregulation etc.)
- Functional aspects and interactions
- …
- …
- and...

Validation: Experiments, self-consistency