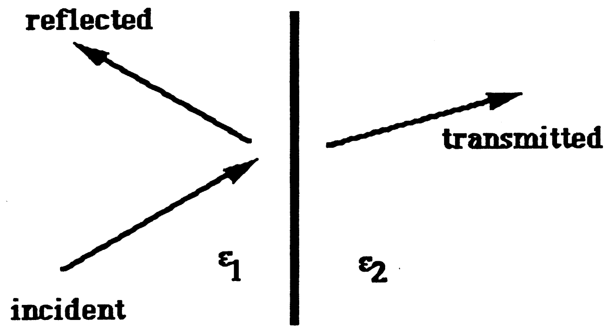
6. Reflection and Refraction at planar boundaries



We would like to know:

(i) Relationship between incident, reflected and transmitted angles i.e., law of reflection and refraction (Snell).

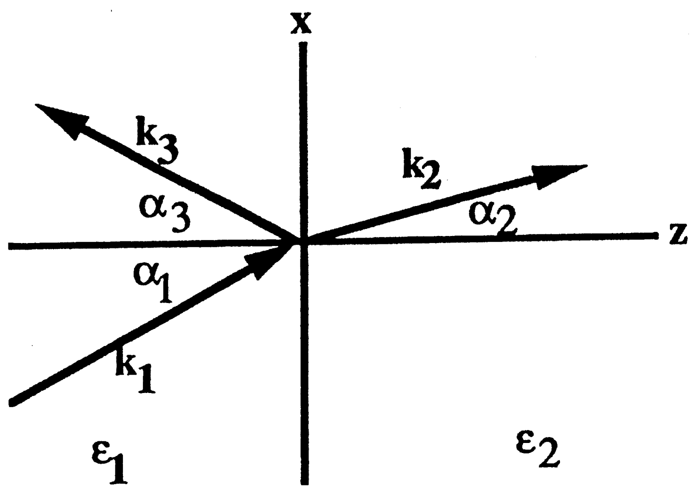
(ii) Transmitted and Reflected Intensities.

We would also like to know this for all possible polarisations of the input light.

The appropriate boundary conditions for dielectrics are

E and **H** ***parallel*** to the boundary are continuous.
B and **D** ***normal*** to the boundary are continuous.

The **very existence of boundary conditions** allows us to derive the law of reflection and Snell's law very easily.



Suppose we have an arbitrary incident wave $\boldsymbol{E}_i \, exp - j\boldsymbol{k}_1 \cdot \boldsymbol{r}$ and then we write the reflected wave as $\boldsymbol{E}_r \, exp - j\boldsymbol{k}_3 \cdot \boldsymbol{r}$ and the transmitted wave as $\boldsymbol{E}_t \, exp - j\boldsymbol{k}_2 \cdot \boldsymbol{r}$.

We would like to know a relationship between the angles α_1, α_2, *and* α_3 which we assume to be general, unknown, angles.

The boundary conditions must be met at the boundary $z = 0$ and ***everywhere*** on that boundary. This means that, ***on the boundary***, all three waves must have the same spatial variation in x. Thus the arguments of the exponential factors, $exp - j\boldsymbol{k}\cdot\boldsymbol{r}$, must be equal on the boundary, i.e.

$$\boldsymbol{k}_1\cdot\boldsymbol{r} = \boldsymbol{k}_2\cdot\boldsymbol{r} = \boldsymbol{k}_3\cdot\boldsymbol{r} \quad \text{on the boundary}$$

or
$$k_1 \sin\alpha_1 = k_2 \sin\alpha_2 = k_3 \sin\alpha_3$$

Now $k_1 = k_3$ because the two waves are travelling in the same material

$$\therefore \; k_1 \sin\alpha_1 = k_1 \sin\alpha_3$$

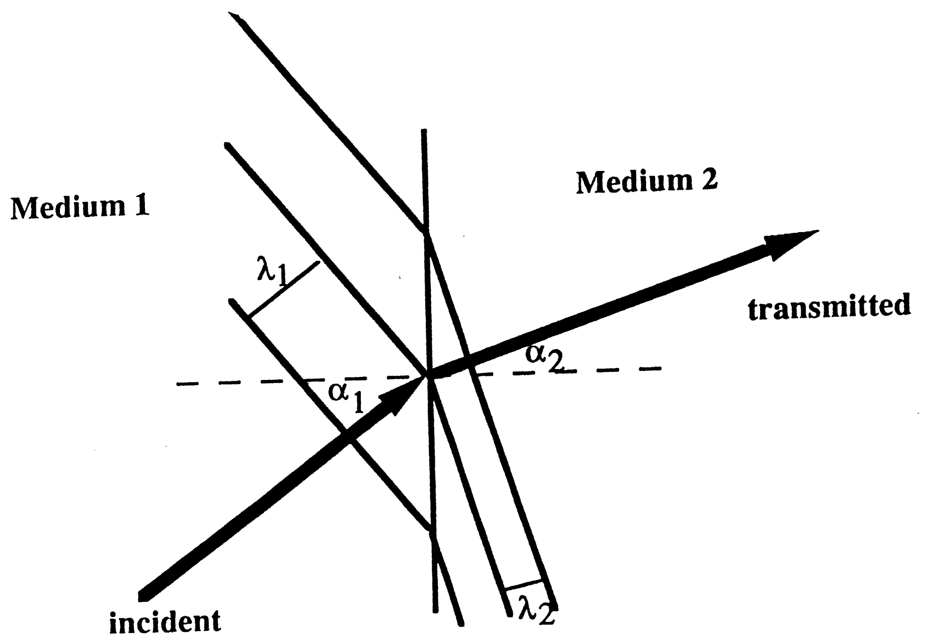
or
$$\alpha_1 = \alpha_3$$
law of reflection

Also
$$k_1 \sin\alpha_1 = k_2 \sin\alpha_2$$

or
$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2$$
Snell's law

These results have come simply from the existence of boundary conditions.

Physical Explanation of Snell's Law



At the boundary the period must be the same for the waves in medium 1 and medium 2. Simple geometry then gives

$$\frac{\lambda_1}{\sin\,\alpha_1} = \frac{\lambda_2}{\sin\,\alpha_2}$$

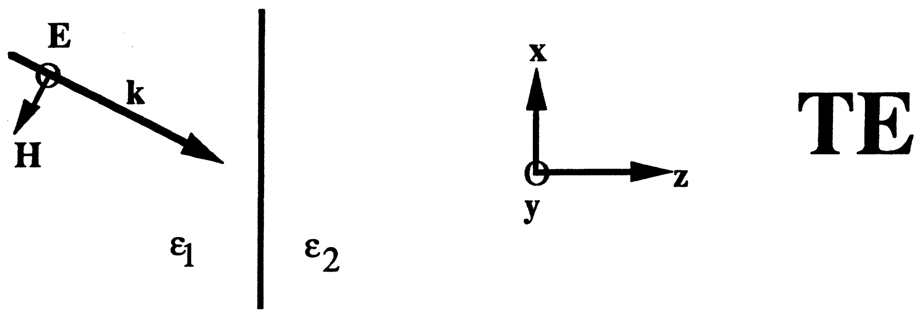
or $$n_1 \sin\alpha_1 = n_2 \sin\alpha_2$$ Snell, again.

The physical explanation we have presented here is based on matching the phase shifts at the boundary. It is also equivalent to matching the components of the k-vectors along the boundary. It is an example of ***phase matching***.

Fresnel Equations

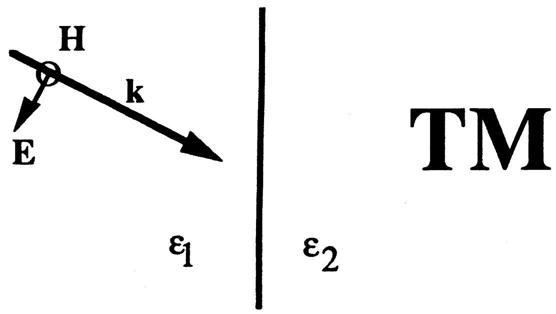
Our previous discussion has revealed the relations between the angles of incidence, reflection and refraction but has told us nothing about the amplitude reflection and transmission coefficients. In order to discuss this we need to write the appropriate boundary conditions down in full. We also need to find expressions that will apply to any state of polarisation on input beam.

In order to solve this problem fully we need only consider *two* polarisations as the general case is simply a linear combination of these two.



Here the H vector lies in the plane of the paper and the E vector only has a component E_y ***transverse*** to the plane. This is called the **TE (*transverse electric*)** case.

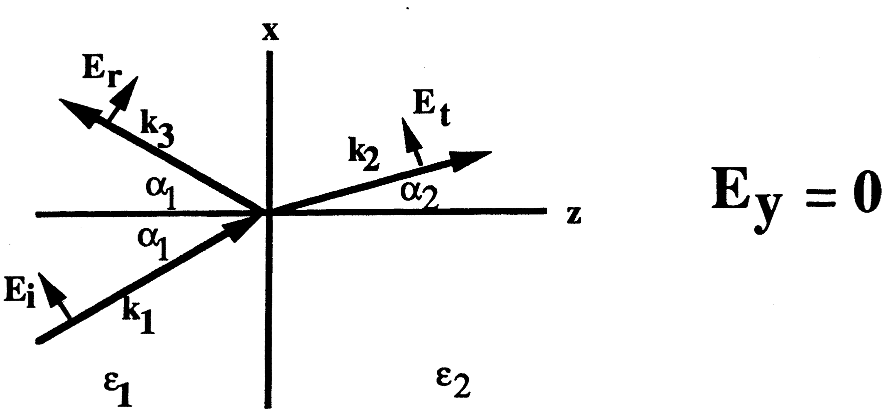
The other, complementary case, is



which is, naturally called the **TM (*transverse magnetic*)** case.

TM Case

We now derive the amplitude reflection and transmission coefficients for the TM case. Expressions for the TE case can be derived in a similar manner.



$$\boldsymbol{E}_i = E_0 \left(\cos\ \alpha_1\ \boldsymbol{i}_x - \sin\ \alpha_1\ \boldsymbol{i}_z \right) \exp -\ j\boldsymbol{k}_1 \cdot \boldsymbol{r}$$

$$\boldsymbol{E}_r = \Gamma E_0 \left(\cos\ \alpha_1\ \boldsymbol{i}_x + \sin\ \alpha_1\ \boldsymbol{i}_z \right) \exp -\ j\boldsymbol{k}_3 \cdot \boldsymbol{r}$$

$$\boldsymbol{E}_t = TE_0 \left(\cos\ \alpha_2\ \boldsymbol{i}_x - \sin\ \alpha_2\ \boldsymbol{i}_z \right) \exp -\ j\boldsymbol{k}_2 \cdot \boldsymbol{r}$$

where Γ and T are the *amplitude* reflection and transmission coefficients respectively.

The easiest boundary conditions here are probably the continuity of **E** tangential and **D** normal, i.e. $E_{x,1} = E_{x,2}$ and $D_{z,1} = D_{z,2}$ (or $\varepsilon_1 E_{z,1} = \varepsilon_2 E_{z,2}$).

These give, omitting the space varying factors, which must cancel on the boundary, as we have discussed before,

$$\cos\ \alpha_1 \left(1 + \Gamma \right) = T\ \cos\ \alpha_2$$

and

$$\varepsilon_1\ \sin\ \alpha_1 \left(1 - \Gamma \right) = T\ \varepsilon_2\ \sin\ \alpha_2\ ,$$

which may be recast using Snell's Law as

$$n_1 \left(1 - \Gamma \right) = n_2\, T\ .$$

Solving these equations gives

$$\Gamma_{TM} = \frac{n_1\ \cos\ \alpha_2 - n_2\ \cos\ \alpha_1}{n_1\ \cos\ \alpha_2 + n_2\ \cos\ \alpha_1}$$

and

$$T_{TM} = \frac{2\mathrm{n}_1\ \cos\ \alpha_1}{n_1\ \cos\ \alpha_2 + n_2\ \cos\ \alpha_1}\ .$$

Similar results may be derived for the TE case:

$$\Gamma_{TE} = \frac{n_1 \cos \alpha_1 - n_2 \cos \alpha_2}{n_1 \cos \alpha_1 + n_2 \cos \alpha_2}$$

and

$$T_{TE} = \frac{2 n_1 \cos \alpha_1}{n_1 \cos \alpha_1 + n_2 \cos \alpha_2} .$$

These equations may also be written in terms of components of the k-vectors of the waves. In the TM case the equations become

$$\Gamma_{TM} = \frac{n_1^2 k_{2,z} - n_2^2 k_{1,z}}{n_1^2 k_{2,z} + n_2^2 k_{1,z}} \text{ and } T_{TM} = \frac{2 n_1 n_2 k_{1,z}}{n_1^2 k_{2,z} + n_2^2 k_{1,z}} ,$$

and for the TE case

$$\Gamma_{TE} = \frac{k_{1,z} - k_{2,z}}{k_{1,z} + k_{2,z}} \text{ and } T_{TE} = \frac{2 k_{1,z}}{k_{1,z} + k_{2,z}} .$$

We will now concentrate on the reflection coefficients which may be written, using Snell's law to eliminate α_2 , as

$$\Gamma_{TM} = \frac{n_1 \sqrt{1 - \left(\frac{n_1 \sin \alpha_1}{n_2} \right)^2} - n_2 \cos \alpha_1}{n_1 \sqrt{1 - \left(\frac{n_1 \sin \alpha_1}{n_2} \right)^2} + n_2 \cos \alpha_1} .$$

$$\Gamma_{TE} = \frac{n_1 \cos \alpha_1 - n_2 \sqrt{1 - \left(\frac{n_1 \sin \alpha_1}{n_2} \right)^2}}{n_1 \cos \alpha_1 + n_2 \sqrt{1 - \left(\frac{n_1 \sin \alpha_1}{n_2} \right)^2}}$$

Special Cases

(i) α_1 small – normal incidence

$$\Gamma_{TE} = \Gamma_{TM} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)$$

i.e. both reflectivities are equal which our previous experience agrees with.

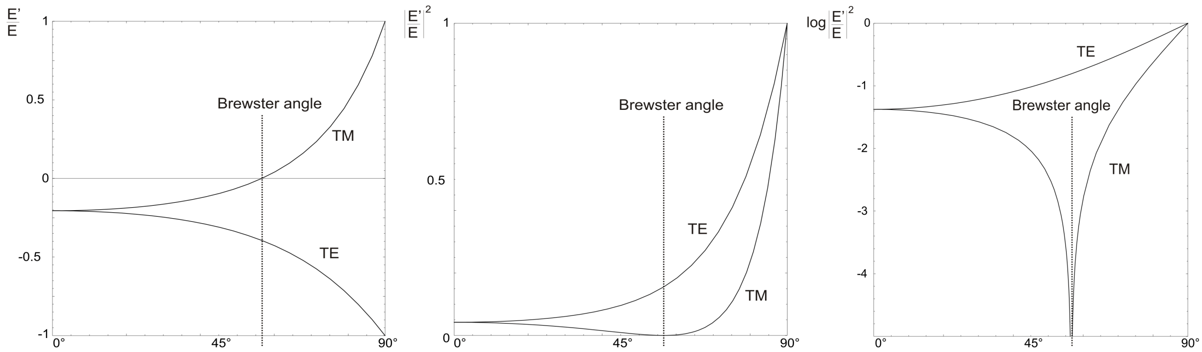
(ii) α_1 large – grazing incidence

$$\Gamma_{TM} \sim 1$$
$$\Gamma_{TE} \sim -1$$
$$|\Gamma_{TM}|^2 = |\Gamma_{TW}|^2 = 1$$

and this result is ***independent*** of the values of n_1 and n_2. This is one way to make very efficient reflectors.

External Reflection ($n_1 < n_2$)

Here $n_1 < n_2$, for example at an air glass interface



(i) $\alpha_1 = 0$

In this case $\Gamma_{TW} = \Gamma_{TM} = \dfrac{n_1 - n_2}{n_1 + n_2}$ is negative because $n_1 < n_2$.

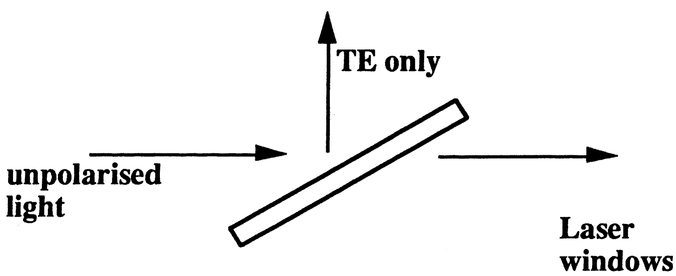
(ii) In TM case there is a value of α_1 at which there is ***no*** reflection $\left(\Gamma_{TM} = 0\right)$. This is called the ***Brewster angle***, simple algebra says that

$$\alpha_1 = \alpha_B = tan^{-1}\left(\frac{n_2}{n_1}\right)$$

Example

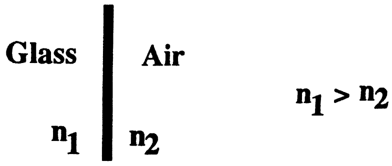
Air/Glass $\alpha_B = tan^{-1}\left(\dfrac{1.5}{1.0}\right) = 56°$

Many uses e.g.



Internal Reflection ($n_1 > n_2$)

For example at a glass air interface



Referring to the expressions for the reflection coefficients above we see that as the angle of incidence, α_1 , increases it eventually reaches a value when

$$1 - \left(\frac{n_1}{n_2} \sin \alpha_1 \right)^2 = 0$$

and $$|\Gamma_{TM}| = |\Gamma_{TE}| = 1$$

This is called the ***critical angle***

$$\alpha_1 = \alpha_{CRIT} = \sin^{-1}\left(\frac{n_2}{n_1} \right)$$

for glass/air interface $\alpha_{CRIT} = \sin^{-1}\left(\frac{1}{1.5} \right) \sim 42°$ as α_1 is further increased the square root becomes imaginary and the expressions are more conveniently written as

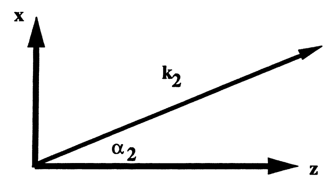
$$\Gamma_{TM} = \frac{jn_1 \sqrt{\left(\frac{n_1 \sin \alpha_1}{n_2} \right)^2 - 1} - n_2 \cos \alpha_1}{jn_1 \sqrt{\left(\frac{n_1 \sin \alpha_1}{n_2} \right)^2 - 1} + n_2 \cos \alpha_1}$$

$$\Gamma_{TE} = \frac{n_1 \cos \alpha_1 - jn_2 \sqrt{\left(\frac{n_1 \sin \alpha_1}{n_2} \right)^2 - 1}}{n_1 \cos \alpha_1 + jn_2 \sqrt{\left(\frac{n_1 \sin \alpha_1}{n_2} \right)^2 - 1}}$$

and clearly, for all α_1 , now $|\Gamma_{TM}|^2 = \left(\Gamma_{TE}\right)^2 = 1$. This is the case of ***total internal reflection***.

It is clear that the phase of Γ_{TE} and Γ_{TM} does vary with angle during total internal reflection and this can be taken advantage of to tune the state of polarisation of a beam. It is possible, for example, to change linear to circular polarisation by introducing a $\pi/2$ phase shift.

Let us now consider the form of the transmitted wave



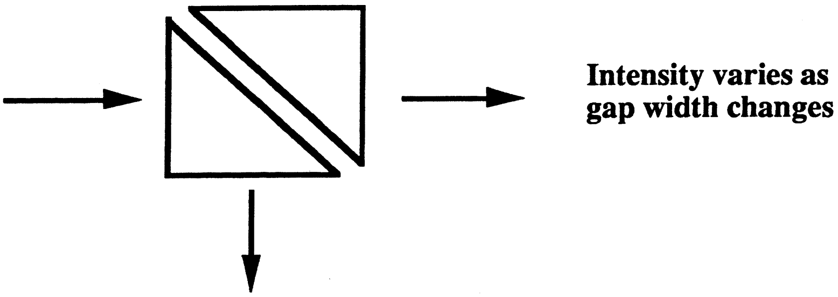
$exp - j\boldsymbol{k}_2 \cdot \boldsymbol{r} = exp - jk_2 \left(cos\ \alpha_2\ z + sin\ \alpha_2\ x \right)$ which, using Snell's Law, we can re-write in terms of α_1 , as

$$exp - jk_2 \left(\sqrt{1 - \left(\frac{n_1\ sin\ \alpha_1}{n_2} \right)^2}\ z + \frac{n_1}{n_2}\ sin\ \alpha_1\ x \right)$$

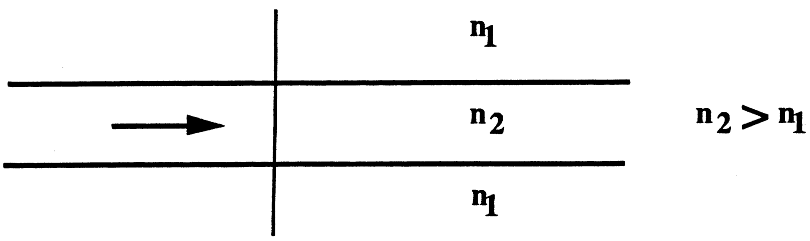
In the case of total internal reflection the square now becomes imaginary and so the field may be written (taking the physically sensible "j" square root).

$$exp - k_2 \sqrt{\left(\frac{n_1\ sin\ \alpha_1}{n_2} \right)^2 - 1}\ z\ .\ exp - jk_2\ \frac{n_1}{n_2}\ sin\ \alpha_1\ x$$

which is, of course, an ***evanescent wave***. The interesting point however, is that the field does penetrate a very small distance into the n_2 material. This does happen. For example, consider two glass prisms



Optical waveguides guide by total internal reflection.



Electric field penetrates cladding. This is the basis, for example, of waveguide couplers.

Power Conservation

We will conclude this discussion of reflection and transmission coefficients by considering whether the results we have derived "conserve power". This means, does the reflected power plus the transmitted power add up to the incident power? In order to simplify matters we will only consider normal incidence for which case the reflection coefficients are given by

$$\Gamma = \frac{n_1 - n_2}{n_1 + n_2} \text{ and } T = \frac{2n_1}{n_1 + n_2}$$

It is tempting to assume that power is proportional by $|\Gamma|^2$ for the reflected wave and to $|T|^2$ for the transmitted wave. Unfortunately $|T|^2 + |\Gamma|^2 \neq 1$, the reason being that the constants of proportionality are different in the two cases. To work things out properly we need to calculate the Poynting vector and integrate it across the interface

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}^* \text{ and } P = \iint \boldsymbol{S}\ dx\ dy$$

At normal incidence we can write the electric field as

$$\boldsymbol{E} = E_0\, \boldsymbol{i}_x\, exp - jkx$$

and hence the magnetic field as

$$\boldsymbol{H} = \frac{k}{\omega\mu} E_0\, \boldsymbol{i}_y\, exp - jkx$$

and hence the Poynting vector has only a component in the z-direction given by $\boldsymbol{S} = E_x\ H_y^*\, \boldsymbol{i}_z$. We can therefore write:

$$(\text{Incident power})_z \sim \frac{k_1}{\omega\mu} E_0^2$$

$$(\text{Reflected power})_z \sim -\frac{k_1}{\omega\mu} |\Gamma|^2\ E_0^2$$

$$(\text{Transmitted power})_z \sim \frac{k_2}{\omega\mu} |T|^2\ E_0^2 = \frac{\frac{n_2}{n_1} k_1}{\omega\mu} |\Gamma|^2\ E_0^2$$

Thus using the expressions for Γ and T above we can see that

$$|\Gamma|^2 + \frac{n_2}{n_1} |T|^2 = 1$$

and hence power is conserved.

Reflections from Good Conductors

If one medium is a good conductor, such as a metal, and the other, from which the light is incident is a dielectric, then the reflectivity behaviour is complicated by the complex value of ε for the metal. The theoretical analysis follows the same lines of our previous work. The results below are for an air/silver interface at $\lambda = 589.3\ nm$ (Sodium D line). Naturally we find high, but not constant reflectivity at all angles of incidence.