5. Polarisation of plane waves

Consider a plane wave propagating in the z-direction

$$\boldsymbol{E}\,(z\,,t) = \boldsymbol{A}\ .\ exp\ \ j\ (\omega t - k\ \ z)$$

by which we mean, for example,

$$\boldsymbol{E}\,(z\,,t) = Re\,\{\boldsymbol{A}\ .\ exp\ \ j\,(\omega t - k\ \ z)\}$$

A is a complex vector in the (x, y) plane and so can be written as $\left(A_x\ exp\ \ j\delta_x\,,\ A_y\ exp\ \ j\delta_y\right)$.

Thus, fully,

$$E_x = A_x\,cos\,(\omega t - kz + \delta_x)$$

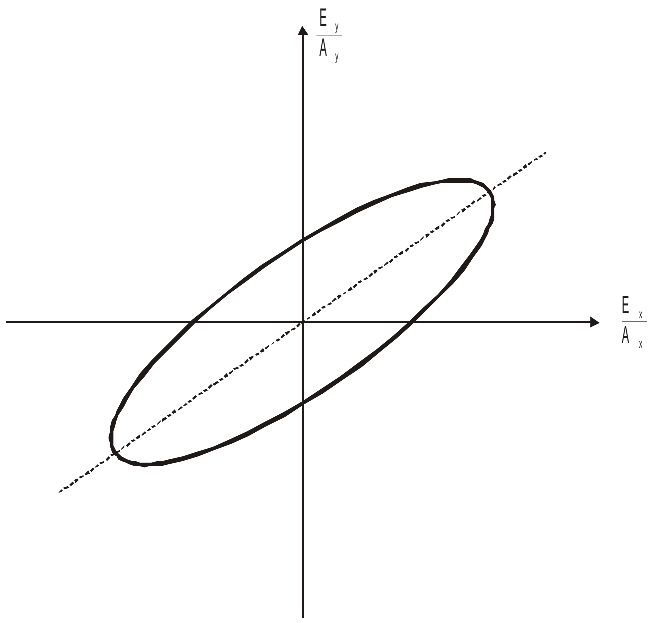
$$E_y = A_y\,cos\,(\omega t - kz + \delta_y)$$

Eliminate $(\omega t - kz)$ to obtain

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\ cos\ \delta\ .\ \left(\frac{E_x}{A_x}\right)\left(\frac{E_y}{A_y}\right) = sin^2\ \delta$$

where $\delta = \delta_x - \delta_y$.

This is an ellipse – the ***polarisation*** ellipse.



$$tan\ 2\ \varphi = \frac{2A_x\ A_y}{A_x^2 + A_y^2}\ cos\ \ \delta$$

Thus light is said to be, in general, ***elliptically*** polarised if the end of the electric vector describes an ellipse.

Linearly polarised if end of electric vector describes a straight line.

Special, important, cases

$$\delta = \delta_x - \delta_y$$

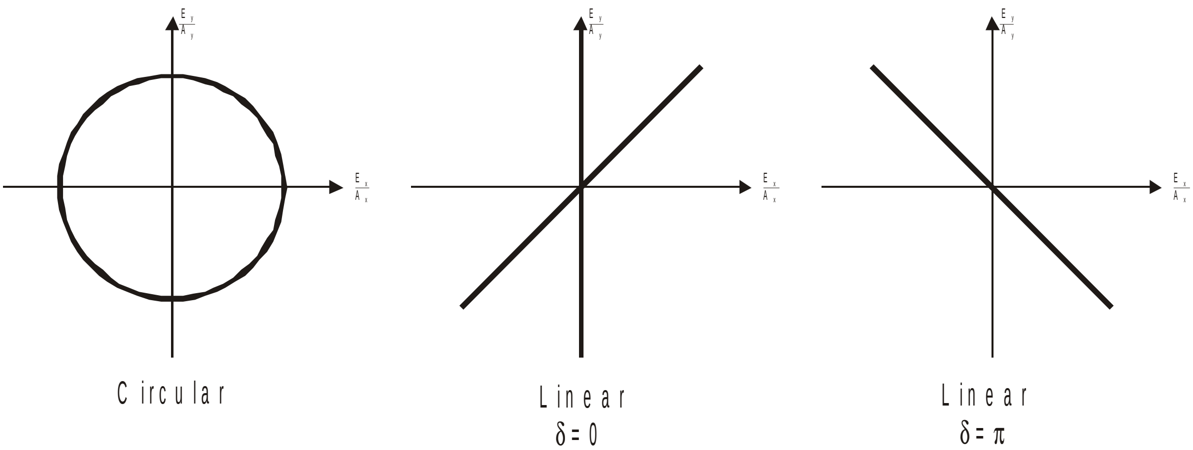
Linear $\delta = m\pi \quad m = \ldots -1,\, 0,\, 1 \ldots$

$$\frac{E_y}{E_x} = (-1)^m \frac{A_y}{A_x}$$

Circular $\delta = \pm\,\pi/2$ and $A_x = A_y = A$

$\delta = +\pi/2$ corresponds to right hand circular polarisation, whereas $\delta = -\pi/2$ corresponds to left hand circular polarisation. The polarisation ellipse is now a circle.

$$E_x^2 + E_y^2 = A^2$$



Mathematical description of polarisation state

So far our analysis of the different polarisation states has been based on the analysis of the amplitudes and phases of the electric field components of the wave. In particular we have considered the locus that the endpoint of the electric field vector defines as it changes in time during a full period of the wave: in the most general case this is an ellipse, which for special values of the amplitudes and/or phases degenerates into a circle or line. If the geometry of the polarisation ellipse is fully defined, then the polarisation state of the wave is fully defined. There are several ways in which an ellipse can be specified. For our purposes, as will be shown later, the most convenient is to use the column vector

$$\boldsymbol{J} = \begin{pmatrix} A_x \, exp \;\; j\delta x \\ A_y \, exp \;\; j\delta y \end{pmatrix},$$

which corresponds to the wave

$$\boldsymbol{E}\left(z,t\right)=\boldsymbol{A}\ .\ exp\ \ j\left(\omega t-k\ z\right)$$

with

$$E_x = A_x\cos\left(\omega t - kz + \delta_x\right)$$
$$E_y = A_y\cos\left(\omega t - kz + \delta_y\right)\cdot$$

The use of vector ***J*** was originally suggested by R.C. Jones in 1941, and is called the ***Jones vector***. Jones vectors are easy to use and make the mathematical treatment of polarisation effects very concise. A few examples of Jones vectors corresponding to different polarisation states:

Polarisation state	*Jones vector*
Horizontal linear	$\begin{pmatrix}1\\0\end{pmatrix}$
Vertical linear	$\begin{pmatrix}0\\1\end{pmatrix}$
Linear at 45°	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$
Linear at -45°	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\j\end{pmatrix}$
Right circular	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-j\end{pmatrix}$
Left circular	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\j\end{pmatrix}$

One drawback, however, of Jones vectors is that they are only applicable to fully polarised light, therefore they cannot be used to treat natural or partially polarised light. In 1825 G.G. Stokes developed a vector treatment that can be used generally; he suggested the use of a set of four parameters, the ***Stokes parameters***, which are normally written as a four-component column vector, the ***Stokes vector***. In contrast to the components of the Jones vector, the Stokes parameters are based on values that can be directly measured. Let us perform our measurements using a detector that is combined with one of four filters: first we use an isotropic 50% neutral density filter to obtain I_0. For filters two and three let us use linear polarisers oriented horizontally and at 45°, and record I_1 and I_2 respectively. Finally record I_3 using a circular polariser that will block all light with left circular polarisation. Having obtained I_0, I_1, I_2 and I_3 we can define the four Stokes parameters as:

$$S_0 = 2I_0$$
$$S_1 = 2I_1 - 2I_0$$
$$S_2 = 2I_2 - 2I_0$$
$$S_3 = 2I_3 - 2I_0$$

With these definitions the Stokes parameter S_0 will simply be proportional to the incident irradiance. S_1 will indicate whether the polarisation state of our light beam is closer to horizontal linear ($S_1>0$) or vertical linear ($S_1<0$). Similarly, S_2 will indicate whether the polarisation state is closer to 45° linear ($S_2>0$) or -45° linear ($S_2<0$). S_3 will give an indication whether the polarisation state is close to linear ($S_3\approx 0$), right circular ($S_3>0$), or left circular ($S_3<0$). A few examples of Stokes vectors corresponding to different polarisation states:

Polarisation state	*Stokes vector*
Horizontal linear	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$
Vertical linear	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$
Linear at 45°	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$
Linear at -45°	$\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$
Right circular	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$
Left circular	$\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$

For completely polarised light, a useful visual representation of the Stokes parameters, and the polarisation state they correspond to, can be built as follows. Let us take the general expressions for the x and y components of a completely polarised monochromatic wave:

$$E_x = A_x\, cos\,(\omega t - kz + \delta_x)$$
$$E_y = A_y\, cos\,(\omega t - kz + \delta_y)$$

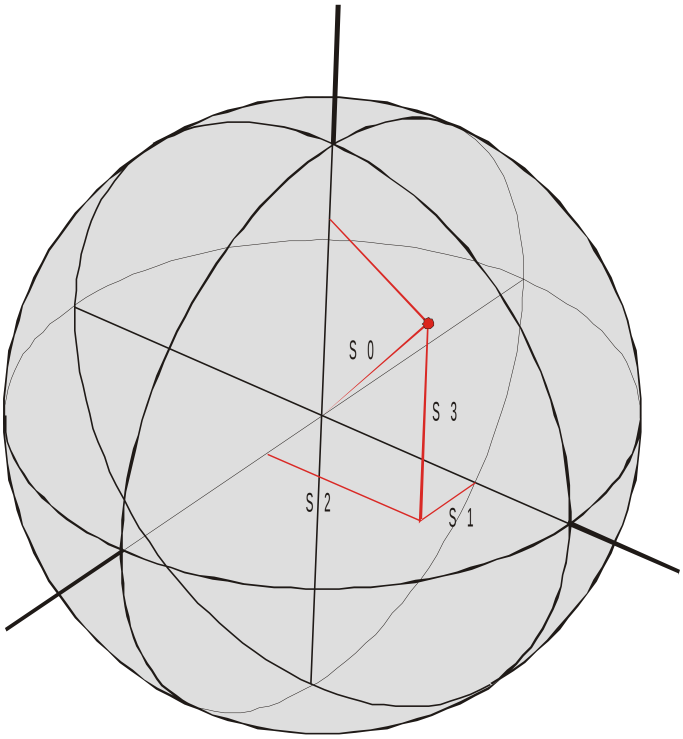
By making use of $I = \langle E_x^2\rangle + \langle E_y^2\rangle$, assuming $\delta = \delta_x - \delta_y$ and dropping a pre-multiplying constant, the four Stokes parameters can be written as

$$S_0 \propto A_x^2 + A_y^2$$
$$S_1 \propto A_x^2 - A_y^2$$
$$S_2 \propto 2A_x A_y cos\,\delta$$
$$S_3 \propto 2A_x A_y sin\,\delta$$

The above four Stokes parameters are clearly not independent, but are related through

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

This indicates that for a beam with constant total intensity, all the possible polarisation states can be represented by points on the surface of a sphere, where the x, y and z point co-ordinates are set to the S_1, S_2 and S_3 Stokes parameters respectively, and the radius of the sphere equals S_0. This visual representation of polarisation states is called the ***Poincare sphere***.



On the Poincare sphere different linear polarisation states correspond to points along the equator (z=0), right circular polarisation to the north pole and left circular polarisation to the south pole.

Waves in crystals

Although this will be discussed in greater detail later it is probably worth pausing to note that $\underline{D}$ and $\underline{E}$ are not usually related in crystals in the simple way we have assumed. Let us consider the following case

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \varepsilon_1 E_x \\ \varepsilon_2 E_y \\ \varepsilon_3 E_z \end{pmatrix}$$

that is to say that the permittivity depends on the polarisation.

Let us propagate a wave in the z-direction, $\boldsymbol{E}\ exp - \ jkz$, and find the value of k. In order to do this we must, as before, solve Maxwell's equations

$$\nabla \times \boldsymbol{H} = j\omega\, \boldsymbol{D}$$

$$\nabla \times \boldsymbol{E} = -\ j\,\omega\mu\, \boldsymbol{H}$$

substitution yields

$$\begin{pmatrix} j\ k\ H_y \\ -j\ k\ H_x \end{pmatrix} = j\omega \begin{pmatrix} \varepsilon_1\ E_x \\ \varepsilon_2\ E_y \end{pmatrix}$$

$$\begin{pmatrix} j\ k\ E_y \\ -\ j\ k\ E_x \end{pmatrix} = j\omega\mu \begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

From this it is clear that we can group the $\left(E_x , H_y\right)$ and $\left(E_y , H_x\right)$ components together. The $\left(E_x , H_y\right)$ equations are, specifically,

$$jkH_y = j\,\omega\varepsilon_1\, E_x$$

$$-\ jkE_x = j\,\omega\mu\ H_y$$

which leads to: $k^2 = \omega^2\ \mu\varepsilon_1$

and similarly for the $\left(E_y , H_x\right)$ combination

$$k^2 = \omega^2\ \mu\varepsilon_2$$

This implies that the propagation of the wave (i.e. its speed) depends on it's polarisation. If it is polarised in the x-direction it will "see" a permittivity ε_1 , whereas y-polarised light will "see" a permittivity ε_2 .

This permits us to tune the state of polarisation of an initially plane polarised beam. Suppose a beam is initially polarised such that $E_x = E_y = E_0$. After travelling a distance, 1, the phase of E_x becomes $exp -\ jk_1 l$ and that if $E_y\ exp -\ jk_2 l$. The phase difference being $exp -\ j\left(k_1 - k_2\right) l = exp -\ jk_0\,\Lambda$ where $\Lambda = \left(n_1 - n_2\right) l$ is the optical path difference. If we make $\Lambda = \lambda / 4$ then the phase difference is $\pi / 2$ and we have converted plane polarised light into circular polarised light. Such a device is called a ***quarter wave plate***.

Jones and Mueller matrices

For an optical component the polarisation state of a fully polarised input beam can be described by its Jones vector $\mathbf{J}_{in}$. Similarly the polarisation state of the output can be described by $\mathbf{J}_{out}$. It then follows that effect of the component can be described by a 2x2 matrix, the ***Jones matrix***, such that

$$\boldsymbol{J}_{out} = \underline{\underline{\boldsymbol{J}}}\ \boldsymbol{J}_{in}$$

Similarly the input and output Stokes vectors can be related via a 4x4 matix, the ***Mueller matrix***:

$$\boldsymbol{S}_{out} = \underline{\underline{\boldsymbol{M}}}\ \boldsymbol{S}_{in}$$

The table below lists the Jones and Mueller matrices for a few simple optical components:

Horizontal linear polariser	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Vertical linear polariser	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
45° linear polariser	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
-45° linear polariser	$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
λ/4 plate vertical fast axis	$e^{j\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & -j \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
λ/2 plate vertical fast axis	$\begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

The real utility of these matrices is best illustrated by giving a practical example. Let us consider a ferro-electric liquid crystal (FLC) cell in transmission. These cells are effectively half-wave plates, but with the special property that the orientation of their fast axis can be switched between two predefined directions by applying an external drive voltage. The switching angle depends on the LC material used. For this example let us assume that the fast axis orientation is horizontal in the 'off' state and is at 45° in the 'on' state. If the input beam has horizontal linear polarisation, what are the parameters of the output beam in the 'on' and 'off' states?
The Jones vector for the input beam is

$$\boldsymbol{J}_{in} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Rotations in the system can be represented by the rotation matrices

$$\underline{\underline{\boldsymbol{R}}}_{90}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad \underline{\underline{\boldsymbol{R}}}_{-90}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad \underline{\underline{\boldsymbol{R}}}_{45}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \qquad \underline{\underline{\boldsymbol{R}}}_{-45}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

The Jones matrix corresponding to a half-wave plate with vertical fast axis is

$$\underline{\underline{\boldsymbol{H}}}=\begin{pmatrix}j & 0\\ 0 & -j\end{pmatrix}$$

A vertical polariser is represented by

$$\underline{\underline{\boldsymbol{P}}}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

We can then write the Jones matrices corresponding to half-wave plates with fast axis set horizontally (0°) and at 45° as

$$\underline{\underline{\boldsymbol{H}}}_0=\underline{\underline{\boldsymbol{R}}}_{90}\ \underline{\underline{\boldsymbol{H}}}\ \underline{\underline{\boldsymbol{R}}}_{-90}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}j & 0\\ 0 & -j\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\underline{\underline{\boldsymbol{H}}}_{45}=\underline{\underline{\boldsymbol{R}}}_{45}\ \underline{\underline{\boldsymbol{H}}}\ \underline{\underline{\boldsymbol{R}}}_{-45}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}j & 0\\ 0 & -j\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

The output states can then be calculated directly as

$$\underline{\boldsymbol{J}}_{out,0}=\underline{\underline{\boldsymbol{P}}}\ \underline{\underline{\boldsymbol{R}}}_{90}\ \underline{\underline{\boldsymbol{H}}}\ \underline{\underline{\boldsymbol{R}}}_{-90}\ \underline{\boldsymbol{J}}_{in}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}j & 0\\ 0 & -j\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

indicating that for the ‘off’ state no light is getting through, which gives a dark pixel, and

$$\underline{\boldsymbol{J}}_{out,45}=\underline{\underline{\boldsymbol{P}}}\ \underline{\underline{\boldsymbol{R}}}_{45}\ \underline{\underline{\boldsymbol{H}}}\ \underline{\underline{\boldsymbol{R}}}_{-45}\ \underline{\boldsymbol{J}}_{in}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}j & 0\\ 0 & -j\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}0\\ -j\end{pmatrix}$$

indicating that for the ‘on’ state all the light gets through (there is also a phase shift), which gives a bright pixel.