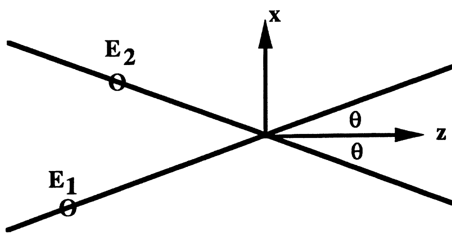
7. Interference

Consider two waves, polarised in the y direction, travelling as shown



$$E_1 = E_0\ exp - jkx\ sin\theta\ exp - jkz\ cos\theta \qquad (E_0\ real)$$

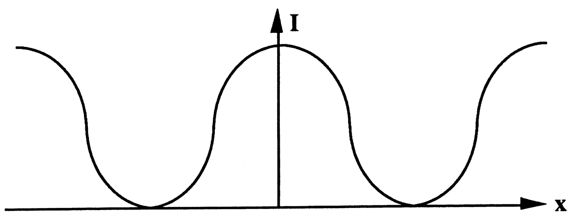
$$E_2 = E_0\ exp + jkx\ sin\theta\ exp - jkz\ cos\theta$$

Total field is the (vector) sum of these two fields

$$E = E_1 + E_2 = 2E_0\ cos\ (kx\ sin\theta)\ .\ exp - jkz\ cos\theta$$

This represents, as before, a standing wave in the x-direction and a travelling wave in z. At a particular value of z the intensity is given by

$$I \propto EE^* = 4E_0^2\ cos^2\ (kx\ sin\theta)$$



i.e. we would see bright and dark fringes.

We can confirm that nothing "funny" has happened to the intensity in the interference. We started off with intensity of each beam as E_0^2 i.e. $2E_0^2$. If we integrate the expression above over x we obtain

$$\langle I \rangle = 2E_0^2 = \langle I_1 \rangle + \langle I_2 \rangle$$

If we now consider interference at two arbitrary angles

$$E_1 = E_{10}\ exp(-jkx\ sin\theta_1)\ exp(-jkz\ cos\theta_1)$$

$$E_2 = E_{20}\ exp(jkx\ sin\theta_2)\ exp(-jkz\ cos\theta_2)$$

We keep E_{10} but permit E_{20} to be complex to account for phase. Again $E = E_1 + E_2$ and the intensity is given by $I \sim EE^*$.

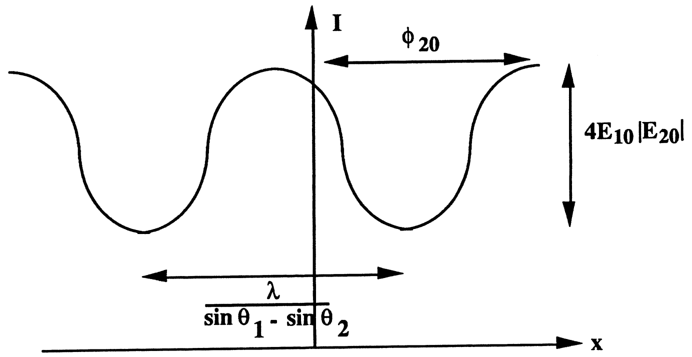
$$I = |E_{10}|^2 + |E_{20}|^2 + 2\,Re\left\{E_{10}\,E_{20}^*\,exp - jkx\left(sin\,\theta_1 - sin\,\theta_2\right).\;exp - jkz\left(cos\;\theta_1 - cos\;\theta_2\right)\right\}$$

If we choose to look at the fringes at a particular position, say $z = 0$, we find

$$I = E_{10}^2 + |E_{20}|^2 + 2\,Re\left\{E_{10}\,E_{20}^*\,exp - jkx\left(sin\,\theta_1 - sin\,\theta_2\right)\right\}$$

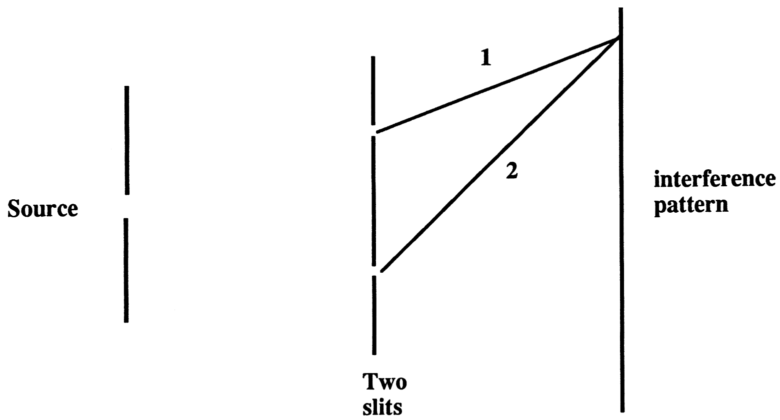
Let $E_{20} = |E_{20}|\;exp\;j\varphi_{20}$

$$I = E_{10}^2 + |E_{20}|^2 + 2\,E_{10}\,|E_{20}|cos\left(\varphi_{20} - kx\left(sin\,\theta_1 - sin\,\theta_2\right)\right)$$



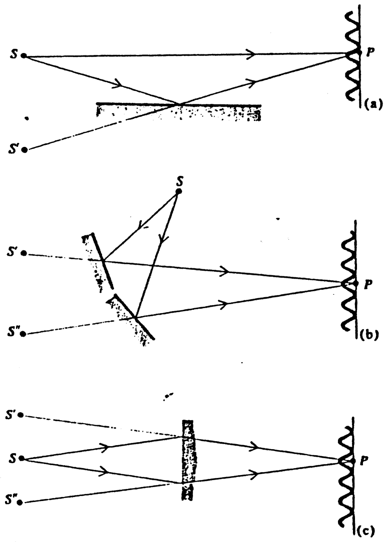
This is the basis of *many* measurement systems. As an example we can measure φ_{20} by fringe shift or λ *if* θ_1 *and* θ_2 are known.

Interference set-ups include the well-known Young's slits



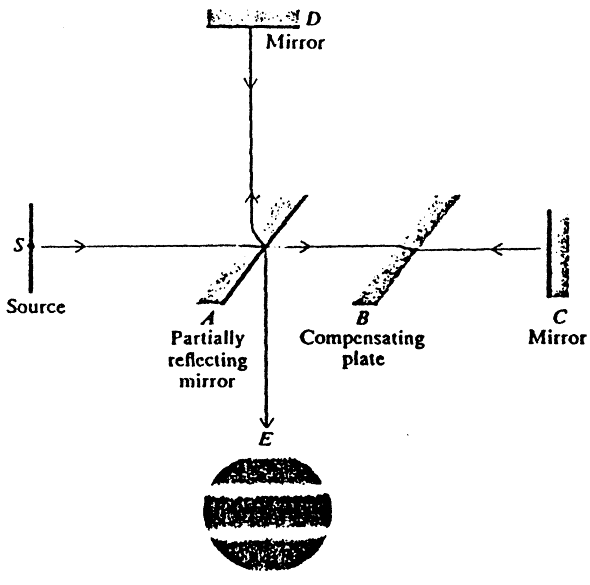
If the phase difference between beams 1 and 2 is φ then if the beams are of equal amplitude the total field in the detector plane is given by $E_{total} = E_0\left(1 + exp\;j\varphi\right)$ and hence the intensity fringes will vary as $cos^2\varphi/2$. Later, when we have considered *diffraction* we will return to this arrangement and consider the effects of finite slit widths.

Other forms of interferometer are shown on the next page. They are all of the “division of wavefront” type. They also all use point sources of light as such the beams add ***coherently***.



If the beams were mutually ***incoherent*** no interference would take place. Indeed, as we shall see later, the amount of interference that does take place can be used as a measure of the degree of temporal coherence of the source.

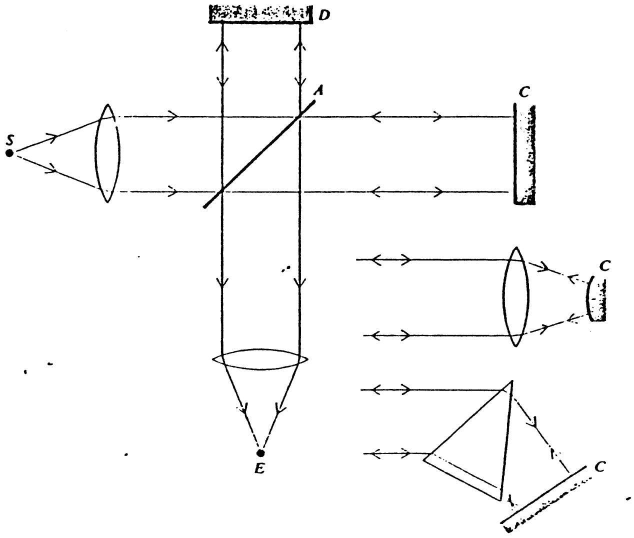
Other forms of interferometer are based on the "division of amplitude" principle. Examples include the Michelson interferometer.



Optical paths in the Michelson interferometer.

Usually a compensating plate B is inserted in one beam in order that the two optical paths include the same thickness of glass. One of the many uses of this interferometer is in the determination of the refractive index of gases. This is done by observing the change in the fringes as the gas is allowed to enter one of the arms of the interferometer.

Another modification of the Michelson scheme is due to Twyman and Green for testing optical components such as lenses and mirrors. Collimated light is used and imperfections are seen as distortions in the fringe pattern.



The Twyman-Green modification of the Michelson interferometer.

Fringe visibility

All we have said so far has concerned the observation of fringes at one wavelength by coherent superposition of the fields. Let us now ask what is the effect if using a broadband source of light. This means that the incoming wave will consist of a number of frequencies rather than just being monochromatic. If these frequencies are independent of each other then the interference pattern which we see will simply be the sum of the interference patterns due to the individual frequencies, i.e.

$$I_T = \sum I(\omega)$$

or, if the source has a *distribution* of frequencies rather than a discrete set the summation becomes an integral.

$$I_T = \int I(\omega)\, d\omega$$

It is customary to work in terms of wavelength rather than frequency.

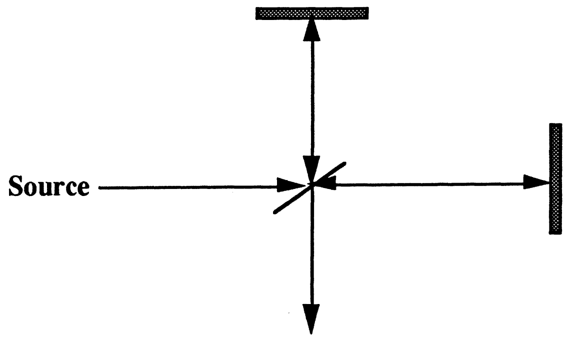
$$I_T \sim \int I\left(\frac{1}{\lambda}\right) d\left(\frac{1}{\lambda}\right)$$

which, essentially, might also be written as

$$I_T \sim \int I(k)\, dk$$

where $k = 2\pi/\lambda$ is the wavenumber.

In order to see the effects of the spectrum of the source let us consider the fringes in a simple Michelson interferometer.



If the path difference between the two arms is x then the interference pattern at one wavelength is given by

$$I \sim (1 + \cos k' x)$$

Suppose now that k_0 is the wavenumber at the centre of the spectrum produced by the source and that *k* represents the variation from this centre i.e.

$$k^1 = k_0 + k$$

If the source is now characterised by $S(k)$ we can write the final interference pattern as

$$I(x) = \int S(k) \{1 + \cos(k_0 + k)x\} \, dk$$

It is clear from this expression that in the limiting case when $S(k) = 1$ for all values of *k* the intensity $I(x)$ does not depend on *x* and so no interference (fringe pattern) will be seen. The other extreme of $S(k) = \delta(k)$, corresponding to a strictly monochromatic source, leads to $\cos^2$ fringes which are readily seen.

Using simple trigonometry we can expand the above equation as

$$I(x) = P + C \cos k_0 x - S \sin k_0 x$$

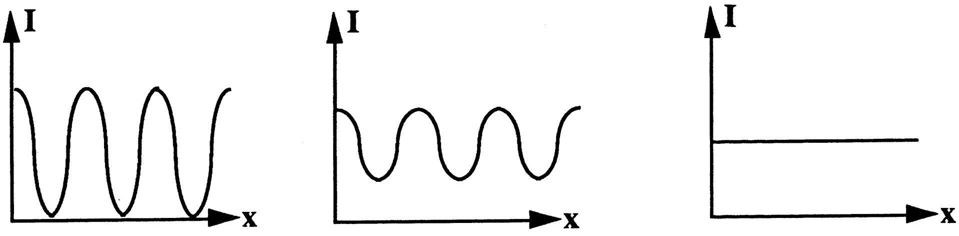
with
$$P = \int S(k) \, dk$$

$$C = \int S(k) \cos kx \, dk$$

$$S = \int S(k) \sin kx \, dk$$

In order to see how the bandwidth of the source affects the fringes define a ***fringe visibility***

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$



V=1.0
high contrast
fringes

V= 1/3
less sharp
fringes

V = 0
No fringes

If we now assume that the spectrum of the lamp is narrow i.e. $k << k_0$ then it is reasonable to assume that *C* and *S* are essentially constant. Simple differentiation with respect to *x* of the equation on the previous page leads to expressions for I_{max} and I_{min}, which then permit us to write

$$V = \frac{\sqrt{C^2 + S^2}}{P}$$

or if the lamp spectrum is symmetrical, $S(k) = S(-k)$, the expression simplifies to

$$V = C/P$$

Examples of light sources

Low pressure electric discharge lamps have spectral lines of a *Gaussian* shape.

$$S(k) \sim A\ exp - k^2/a^2$$

Typical of radiation gas whose natural shape is normally very narrow is broadened by thermal velocities of the atoms. This is called ***Doppler Broadening***. This visibility is now given by

$$V = exp - \left(\frac{ax}{2}\right)^2$$

and hence we can obtain reasonable fringes providing the product *ax* is not too large.

As an example let us suppose that the spectral half width is $2\pi\ x\ 1.6\ cms$ we can see that, if we assume fringes are seen until $V \sim e^{-1}$ that we would expect to see fringes up to a path difference *x* of

$$x = \frac{2}{a} = \frac{2}{2\pi\ x\ 1.6}\ cms \sim \underline{2\,mm}$$

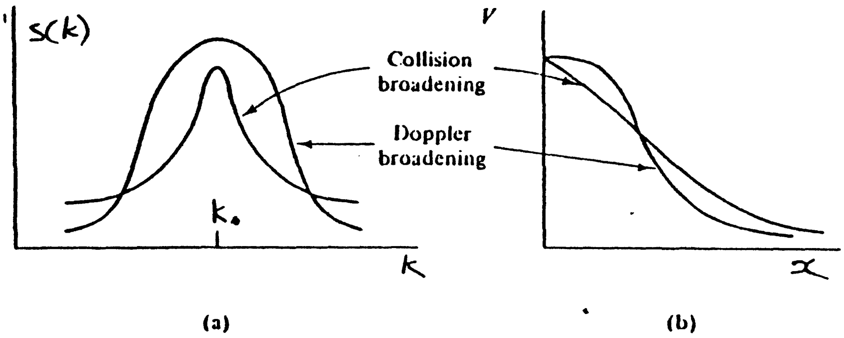
which is what is often seen in practice.

High pressure lamps have a ***Lorentzian profile*** due to the atoms undergoing frequent collisions. This is called ***collision broadening*** and the lamp spectrum is described by

$$S(k) = \frac{A}{k^2 + a^2}$$

which leads to

$$V \sim exp - |ka|$$

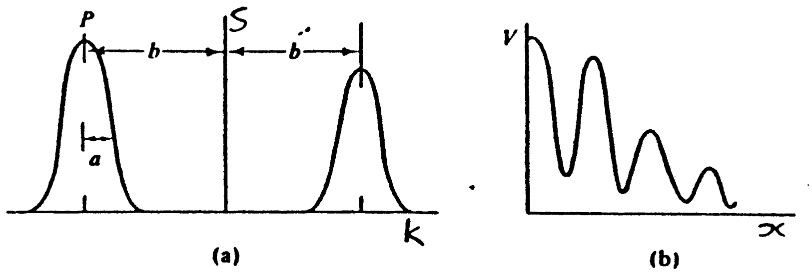


The shape of a Doppler-broadened and collision-broadened line. In (a), the integrated intensities of the lines are the same. In (b), the collision-broadened line has the straighter V curve.

Sodium vapour has a spectrum like

$$S(k)\ A\left\{exp - \left(\frac{k+b}{a}\right)^2 + 0.8\ exp - \left(\frac{k-b}{a}\right)^2\right\}$$

where one line is at 589.0nm and the other, at 589.5nm, is about 80% as intense.



The spectrum (a) and visibility curve (b) produced by light emitted by sodium vapor near 589 nm (5890 A).

In all the cases we have seen here the finite bandwidth of the source has caused a limit to be placed on the path difference over which interference fringes could be seen. This ties in with the idea of temporal coherence. If we think of the source as having a bandwidth $\Delta\upsilon\ H_z$ then we can introduce a "coherence" time

$$\tau_L = \frac{1}{\Delta\upsilon}$$

and hence a coherence length $l_c = c/\Delta\upsilon$ over which we might expect to be able to see interference effects.

We conclude this section with a discussion of the ***ideal laser***. Here

$$S(k) = \delta(k)$$

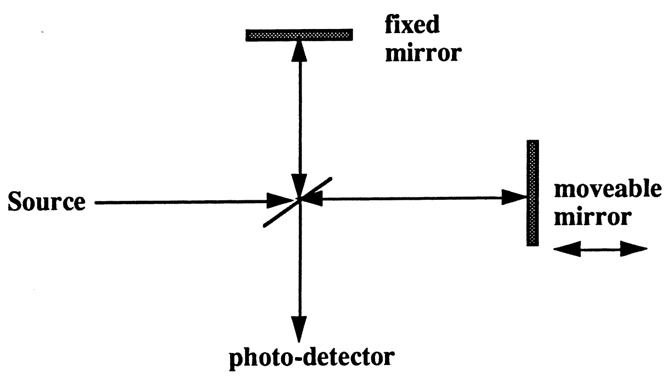
and we can easily see that this implies

$$V = 1$$

i.e. visible fringes irrespective of the path difference. Here $\Delta\upsilon = 0$ and so this result ties in with the idea of an infinite coherence length. In practice the linewidth is not identically zero, however, a good laser should give visible fringes over a path difference exceeding 1km. This is basically why lasers are very useful in applications such as holography where an interference pattern must be recorded over path lengths, often, of several meters.

Fourier Transform Spectroscopy

Having shown that the spectrum of the light source can make a difference to the fringe pattern in an interferometer we can now take advantage of this fact to use the interferometer as a ***spectrometer*** i.e. use the measured $I(x)$ to find $S(k)$. A Michelson type interferometer is often used.



We can scan the movable mirror to introduce a phase shift *kx* between the two beams. As before if the source spectrum is $S(k)$ then the detected signal will be given by

$$I(x) = \int S(k)\,(1 + \cos kx)\, dk$$
$$I(x) = \int S(k)\, dk + \int S(k) \cos kx\; dx$$

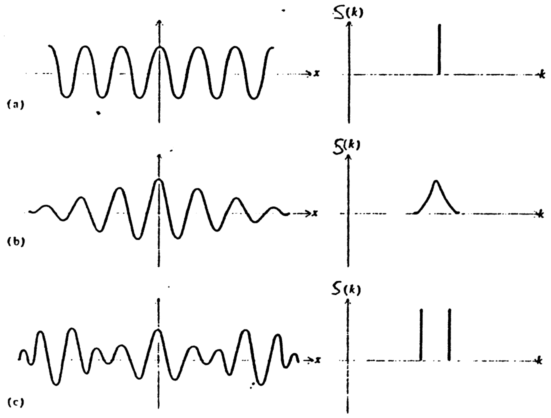
We recognise $\int S(k)\, dk$ as $1/2\; I(0)$ i.e. one half the signal at zero path difference.

$$I(x) - \frac{1}{2} I(0) = \int S(k) \cos kx\; dx$$

Thus we see that $I(x) - I(0)/2$ and $S(k)$ represents a ***Fourier transform*** pair. It is therefore easy, in the computer, to invert the transform pair to give

$$S(k) \sim \int \left[I(x) - \frac{1}{2} I(0) \right] cos\ kx\ dx$$

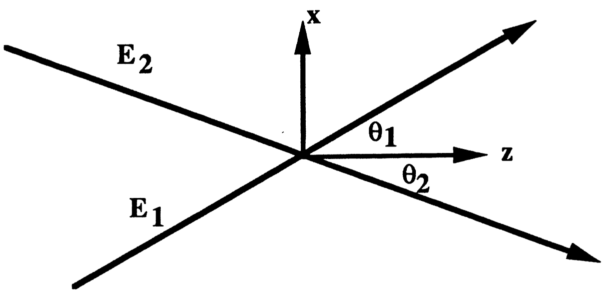
This is the basis of F.T. spectroscopy which is very useful for analysing complicated absorption processes in gases in I.R. and also weak sources as all the available light is used in the spectroscopy.



Intensity functions and their spectra. (a) A single monochromatic line; (b) a single broad line; (c) two narrow lines. (Spectra for positive values of k only are shown.)

Heterodyning

We now consider very briefly the effect of interfering two beams where one has a slightly different frequency from the other. This can be arranged very easily in practice. We revert to the geometry below where the two angles are different



$$E_1 = E_{10}\ exp - jkx\ sin\ \theta_1\ exp - jkz\ cos\ \theta_1$$

$$E_2 = |E_{20}| e^{j\varphi_{20}}\ .\ exp\ jkx\ sin\ \theta_2\ .\ exp - jkz\ cos\ \theta_2\ exp\ j\Delta\omega t$$

where the second beam is frequency sifted by $\Delta\omega$ with respect to the first. At the plane, $z = 0$, the intensity is given by

$$I = |E_{10}|^2 + |E_{20}|^2 + 2\ Re\left\{ E_{10}\ E_{20}^*\ exp - jkx\ (sin\ \theta_1 - sin\ \theta_2) e^{j\Delta\omega t} \right\}$$

i.e. the fringes move with time at the beat frequency, $\Delta\omega$. If we look at the component at frequency $\Delta\omega$ we have

$$\begin{aligned} I_{\Delta\omega} &= 2\,Re\left\{E_{10}\,E_{20}^{*}\,exp - jkx\left(sin\,\theta_1 - sin\,\theta_2\right) exp^{-j\Delta\omega t}\right\} \\ &= E_{10}\left|E_{20}\right| cos\left\{\varphi_{20} - kx\left(sin\,\theta_1 - \theta_2\right) - \Delta\omega t\right\} \\ &\qquad\qquad \uparrow \text{phase of } \varphi_{20} \end{aligned}$$

Thus we can get all the same information as in the earlier case with $\Delta\omega = 0$. However if we detect this signal electrically, as we would do in order to isolate this term, we must use a photodetector. This device gives an output given by

$$\int_{-\infty}^{\infty} I\,.\left(x\right).\,dx$$

Thus the output is determined by the integral of cosine, which is usually zero when averaged over many cycles *unless*

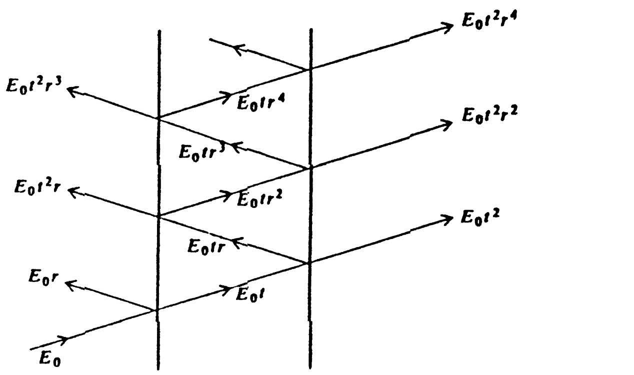
$$sin\ \theta_1 - sin\ \theta_2 = 0 \qquad \text{i.e.} \qquad \theta_1 = \theta_2$$

Application. To pick out (modulus and phase) an angular component of a general beam of an angle $\theta_2 = \theta_1$.

Multiple Beam Interferometry

Having considered interference caused by two beams we will now move on to consider the effects of many beams interfering and concentrate particularly on the Fabry-Perot interferometer. We will then go on to consider the use of many dielectric layers sandwiched together to make interference filters.

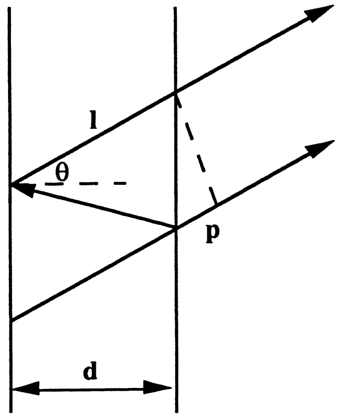
Consider two identical semi-reflecting interfaces



Paths of light rays in multiple reflection between two parallel mirrors. (For simplicity the mirrors are considered to be infinitely thin.)

E_0 is the primary beam, *r* and *t* are the reflection and transmission coefficients respectively. Thus the amplitude of successive transmitted waves are E_0t^2, $E_0t^2r^2$. $E_0t^2r^4 \cdots$. The phase difference between these waves must also be taken into account. The path difference Δ between two adjacent components is given by

$$\Delta = 21 - p = 2\text{d}\ cos\ \theta$$



and hence the phase difference δ is given by

$$\delta = \frac{2\pi}{\lambda} \cdot \Delta = \frac{4\pi}{\lambda} d \cos \theta$$

When the various beams overlap the total field is given by

$$\begin{aligned} E_T &= E_0 t^2 + E_0 t^2 r^2 e^{-j\delta} + E_0 t^2 r^4 e^{-2j\delta} + \cdots \\ &= E_0 t^2 \left(1 + r^2 e^{-j\delta} + r^4 e^{-2j\delta} + \cdots\right) \end{aligned}$$

i.e. a geometrical progression with common ratio $r^2 e^{-j\delta}$. Thus

$$E_T = \frac{t^2 E_0}{1 - r^2 e^{-j\delta}}$$

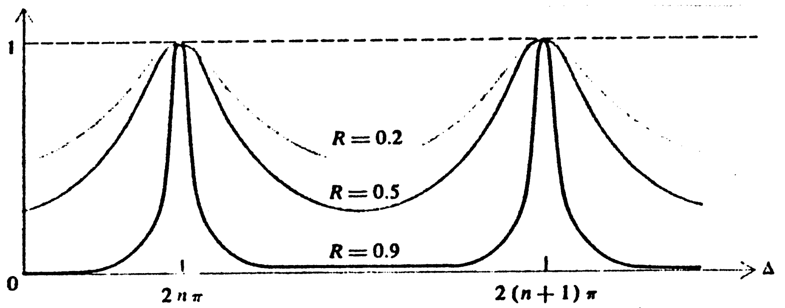
the intensity is given by $\sim |E_T|^2$. which may be written as

$$I_T = I_0 \cdot \frac{1}{1 + F \sin^2 \delta/2}$$

where $R = |r|^2,\ T = |t|^2$ and

$$F = \frac{4R}{(1-R)^2}$$

The function $1/\left(1 + F \sin^2 \delta/2\right)$ is called an airy function and takes the form shown below

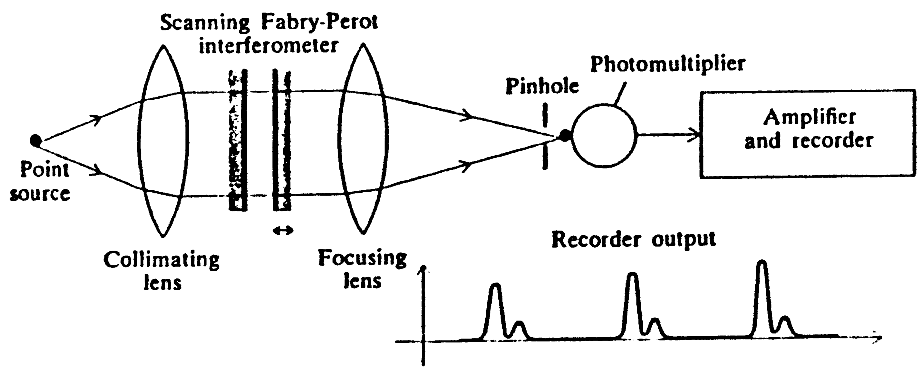


Graphs of the Airy function giving the intensity distribution of fringes in multiple-beam interference.

We see that when $\delta = 2N\pi$, $sub\ \delta/2 = 0$ and we find $I_T = 1$. N is known as the order of interference. This is true whatever the reflectivity of the mirrors – i.e. we can have two very highly reflecting mirrors and still obtain 100% transmission! The sharpness of the fringes decreases as the reflectivity decreases. In essence we only have interference between a small number of beams and the fringe pattern begins to look like the $\cos^2$ fringes in a two beam interference experiment.

The Fabry-Perot Interferometer

This is used to measure wavelengths with high precision and to study the fine structure of spectrum lines. It consists of two optically flat $(\lambda/20 \text{ to } \lambda/100)$ reflecting surfaces held accurately parallel. If the plate spacing can be varied the device is called an ***interferometer*** whereas if the spacing is held constant it is called an *etalon*.



There the output consists, essentially, of a sum of response functions $\sim\left[1 + F\ sin^2\ \delta/2\right]^{-1}$ for each wavelength component of the source.

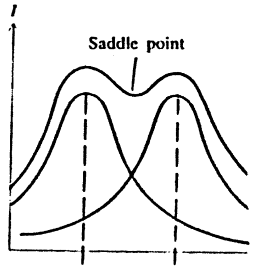
Resolution of Fabry-Perot Instruments

Suppose the spectrum consists of $\lambda\ and\ \lambda + \Delta\lambda$ where $\Delta\lambda$ is small. How small can $\Delta\lambda$ be to be resolved. The fringe pattern, or signal, is given by

$$I \sim \left(1 + F\ sin^2\ \frac{\delta_1}{2}\right)^{-1} + \left(1 + F\ sin^2\ \frac{\delta_2}{2}\right)^{-1}$$

where

$$\delta_1 = \frac{4\pi}{\lambda}\, d\,; \delta_2 = \frac{4\pi\ d}{\lambda + \Delta\lambda} \approx \frac{4\pi\ d}{\lambda} - \frac{4\pi\ d\ \Delta\lambda}{\lambda^2} \qquad \left(for\ \cos\theta \sim 1\right)$$



Resolution Criterion - we need to be able to see a dip in the total response. One criterion would be to say that the individual curves cross at half-intensity points, so that total intensity at the saddle point is equal to the maximum intensity of either line alone, i.e.

$$1 = 2\left(1 + F\ \sin^2\left(\frac{\delta_1 + \delta_2}{2}\right)\right)^{-1}$$

where

$$\frac{\delta_1 + \delta_2}{2} = \frac{2\pi\ d\ \Delta\lambda}{\lambda^2}$$

We assume this is sufficiently small to permit us to approximate

$$\sin\left(\frac{\delta_1 + \delta_2}{2}\right) \sim \frac{\delta_1 + \delta_2}{2}$$

Whence

$$\frac{2\pi\ d\ \Delta\lambda}{\lambda^2} = \frac{1}{\sqrt{F}}$$

or

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda}{2\pi \mathrm{d}} \cdot \frac{1}{\sqrt{F}}$$

If we now introduce the ***finesse***, $\Im$, of the interferometer as $\Im = \left(\pi/2\right)\sqrt{F}$ we can recast the chromatic resolving power, $\lambda/\Delta\lambda$, as

$$\frac{\lambda}{\Delta\lambda} = 4\left(\frac{d}{\lambda}\right)\Im$$

A good Fabry-Perot instrument can have a resolving power of 10^6 which is 10 to 100 times that of a prism or small grating spectroscope.

Our equations seem to imply that we can make $\frac{\lambda}{\Delta\lambda}$ take *any* value by a suitable choice of d and R. This is not, in practice, the case. The principal limitation is that the plates are *not* perfectly flat. We can model this *very crudely* by saying that the actual spacing varies by a fraction of a wavelength λ / M . This introduces a variation of δ of

$$\frac{4\pi}{\lambda} \cdot \frac{\lambda}{M} = 4\pi / M$$

and this must be less than the differences in the two peaks, i.e.

$$\frac{4\pi}{M} < \frac{1}{\sqrt{F}}$$

or

$$\frac{1}{M} < \frac{(1 - R)}{8\pi\sqrt{R}}$$

Typically mirrors have high reflectivities – high finesse, and M must be large. If R = 0.93 than M = 90 i.e. the mirrors need to be flat to $\lambda / 90$ and more. They also need to be *aligned parallel* to each other to the same extent!

If the wavelength separation of two components is sufficiently large, the displacement between the two patterns may be greater than the distance between adjacent maxima. If this happens the orders are said to overlap. The wavelength difference corresponding to a displacement of one order $|\Delta\delta| = 2\pi$, is called the *free spectral range* for the interferometer. For almost normal incidence this is given by

$$(\Delta\lambda)_{SR} = \frac{\lambda^2}{2\ d}$$

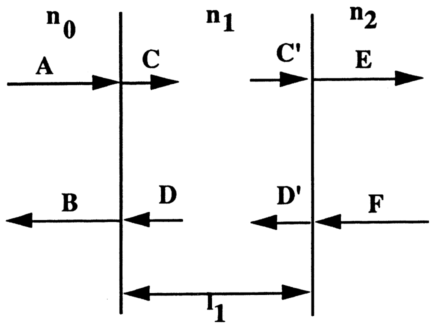
or $(\Delta\lambda)_{SR}$ is approximately equal to $\Im$ multiplied by the least resolvable wavelength difference. With typical values of $\Im$, 30-40, the spectral range is very small when the resolving power is high. As an example,
$\Im = 30$, $d = 4\,mm$, $\lambda = 0.5\mu\mathrm{m}$ *gives* $(\Delta\lambda)_{SR} = 0.3\mathrm{A}^{\circ}$ and $(\lambda / \Delta\lambda) \sim 10^5$.

Theory of Multi-layer Films

Optical surfaces having virtually any reflectance and transmittance may be made by means of thin film coatings. These are usually deposited on a suitable substrate such as glass or a metal by vacuum evaporation techniques. Examples are coatings on lenses, $\lambda / 4$ transformer, one way mirrors and optical filters.

We begin by considering the case of a single layer of dielectric and then develop the model to cope with multi-layers. For simplicity we also consider normal incidence – it is easy to generalise to the oblique case.

In the most general case a single layer may have waves incident on and reflected from both boundaries. We discuss this situation first since it will enable us to handle complicated multi-layers by straightforward matrix multiplication



The boundary conditions at boundary (1) lead to

$$A + B = C + D; \qquad n_0(A - B) = n_1(C - D)$$

or, in matrix notation,

$$\begin{pmatrix} 1 & 1 \\ n_0 & -n_0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ n_1 & -n_1 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$

which we will also write as

$$R_0\begin{pmatrix} A \\ B \end{pmatrix} = R_1\begin{pmatrix} C \\ D \end{pmatrix}$$

Transmission through a length, l_1, of material of refractive index n_1 may be described by

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} exp\ jk_1 l_1 & 0 \\ 0 & exp - jk_1 l_1 \end{pmatrix}\begin{pmatrix} C^1 \\ D^1 \end{pmatrix}$$

or

$$\begin{pmatrix} C \\ D \end{pmatrix} = T_1\begin{pmatrix} C^1 \\ D^1 \end{pmatrix}$$

and hence

$$R_0\begin{pmatrix} A \\ B \end{pmatrix} = R_1\, T_1\begin{pmatrix} C^1 \\ D^1 \end{pmatrix}$$

Applying similar techniques to interface (2) permits us to write

$$R_1\begin{pmatrix} C^1 \\ D^1 \end{pmatrix} = R_2\begin{pmatrix} E \\ F \end{pmatrix}$$

It should be clear that the *R* matrices are all of the same form with the appropriate refractive index used. Combining these equations yields

$$\begin{pmatrix} A \\ B \end{pmatrix} = R_0^{-1} \left[\left(R_1 \, T_1 \, R_1^{-1} \right) \right] R_2 \begin{pmatrix} E \\ F \end{pmatrix}$$

It is straightforward, although a little bit messy, to multiply out these matrices. In the end the result will look like

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix}$$

If our intention were to find the reflection coefficient from a single layer then it is clear that no wave can be incident from the right hand side and so we must set $F = 0$. The reflection coefficient is then given by $\Gamma = B/A$ which is easily seen to be equal to c/a - multiplying the matrices out gives

$$\Gamma = \frac{n_1 \left(n_0 - n_2 \right) \cos k_1 l_1 + j \left(n_0 \, n_2 - n_1^2 \right) \sin k_1 \, l_1}{n_1 \left(n_0 + n_2 \right) \cos k_1 \, l_1 + j \left(n_0 \, n_2 + n_1^2 \right) \sin k_1 \, l_1}$$

Suppose now that the layer is $\lambda/4$ thick such that $k_1 l_1 = \pi/2$ then

$$|\Gamma|^2 = \left(\frac{n_0 \, n_2 - n_1^2}{n_0 \, n_2 + n_1^2} \right)^2$$

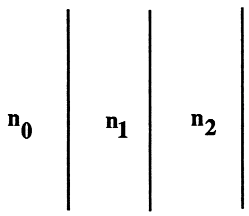
which is **zero** if

$$n_1 = \sqrt{n_0 \, n_2}$$

i.e. the geometric mean $n_0 n_2$ (c.f. quarter wave transformer).

This is an example of an ***anti-reflection*** coating. If the two media n_1 *and* n_2 are air and glass respectively then we need antireflection coating of index $\sqrt{1.5} = 1.225$. Magnesium fluoride has an index of 1.35 and is often used as a reasonable compromise.

Now let us move on to consider a system of two such layers of index n_1 *and* n_2



Application of the same techniques we have just used yields

$$\begin{pmatrix} A \\ B \end{pmatrix} = R_0^{-1} \left[\left(R_1 \, T_1 \, R_1^{-1} \right) \left(R_2 \, T_2 \, R_2^{-1} \right) \right] R_3 \begin{pmatrix} E \\ F \end{pmatrix}$$

or, if we had M layers

$$\begin{pmatrix} A \\ B \end{pmatrix} = R_0^{-1} \left[\prod_{1=1}^{1=M} \left(R_1 \, T_1 \, R_1^{-1} \right) \right] R_{M+1} \begin{pmatrix} E \\ F \end{pmatrix}$$

where R_0 refers to the superstrate, often air, and R_{M+1} to the substrate. Again it is clear that to obtain the reflection coefficient we set $F = 0$ and again find $\Gamma = c/a$.

If we specialise to layers which are $\lambda/4$ thick it is straightforward to calculate

$$\left(RTR^{-1} \right) = \frac{j}{n} \begin{pmatrix} 0 & 1 \\ n^2 & 0 \end{pmatrix}$$

Thus if we have a two layer system where the layers have refractive indices n_1 *and* n_2 it is easy to write

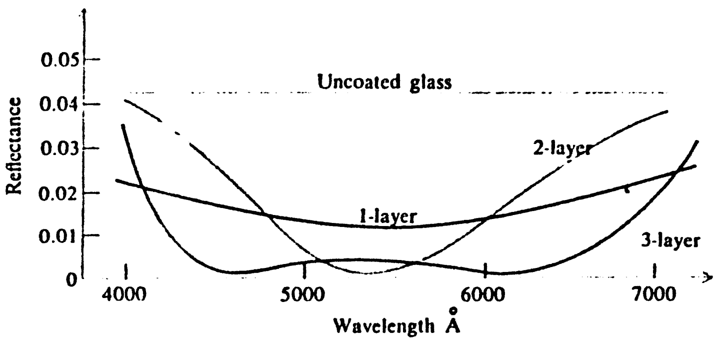
$$\Gamma = c/a = \frac{n_0 n_2^2 - n_3 n_1^2}{n_0 n_2^2 + n_3 n_1^2}$$

which can be made zero if

$$\left(\frac{n_2}{n_1} \right)^2 = \left(\frac{n_3}{n_0} \right)$$

Usually $n_3 > n_0$ and so we need to have $n_2 > n_1$. Thus the layers must be arranged as air-low *n* material –high *n* material – glass, say.

Other combinations can be used to achieve zero reflectivity at several wavelengths over the visible spectrum using readily available materials.



Curves of reflectance versus wavelength of antireflecting films.

High Reflectance Films

A high value of reflectance can also be obtained by stacking a large number of $\lambda/4$ thick layers of film of index n_1 *and* n_2 . The matrix describing these two layers is given by

$$\left(R_1\ T_1\ R_1^{-1}\right)\left(R_2\ T_2\ R_2^{-1}\right) = \begin{pmatrix} -\dfrac{n_2}{n_1} & 0 \\ 0 & \dfrac{n_1}{n_2} \end{pmatrix}$$

Which if we have $2N$ layers, i.e. N sets of double layers gives an overall matrix

$$\begin{pmatrix} \left(-\dfrac{n_2}{n_1}\right)^N & 0 \\ 0 & \left(-\dfrac{n_1}{n_2}\right)^N \end{pmatrix}$$

which leads to a reflection coefficient

$$|\Gamma|^2 = \left[\frac{n_0\left(\dfrac{n_2}{n_1}\right)^{2N} - n_s}{n_0\left(\dfrac{n_2}{n_1}\right)^{2N} + n_s}\right]^2$$

where n_s is the refractive index of the substrate.

This clearly tends to unity as N becomes large. Typical film layers are made of magnesium fluoride (n = 1.35), titanium dioxide (n = 2.6) and zinc sulphide (n = 2.3). In this way reflectivities greater than 0.900 can be achieved at one wavelength. This is better than can be obtained with pure silver. Such multilayer mirrors are often used in lasers.