- Same fluid at inlet and outlet: $r_1 = r_3$ $\quad T_I = \left[1 + \frac{1}{4}\left(\frac{r_2}{r_1} - \frac{r_1}{r_2} \right)^2 \sin^2 k_2 L \right]^{-1}$

- For $L \ll \lambda$, you get perfect transmission

- For $k_2 L = n\pi$, you get perfect transmission: $\quad f_n = n\frac{c_2}{2L}$

- For $k_2 L = (2n-1)\pi/2$, you have: $\quad L = \frac{(2n-1)}{4}\lambda$

$$T_I = 4r_1 r_3 \Big/ \left(r_2 + \frac{r_1 r_3}{r_2} \right)^2 \quad ; \quad T_I \approx 1 \text{ for } r_2 = \sqrt{r_1 r_3}$$

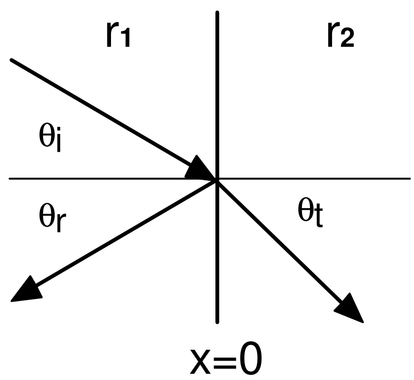
2.5 Reflection, Transmission & Refraction of Plane Waves at Oblique Incidence

- For oblique plane wave incidence at a fluid-fluid interface, we once more solve the problem by matching boundary conditions in pressure and velocity at the interface.

- However, this time, we only match the *normal* component of velocity at the interface, i.e.:

$$\boldsymbol{p}_i + \boldsymbol{p}_r = \boldsymbol{p}_t$$

$$\vec{\boldsymbol{v}}_i \cdot \hat{n}_o + \vec{\boldsymbol{v}}_r \cdot \hat{n}_o = \vec{\boldsymbol{v}}_t \cdot \hat{n}_o$$



- The pressure boundary condition yields Snell's Law:

$$\sin\theta_i = \sin\theta_r \quad ; \quad \frac{\sin\theta_i}{c_1} = \frac{\sin\theta_t}{c_2} \qquad [2.11]$$

$$\cos\theta_t = \sqrt{1-\left(\frac{c_2}{c_1}\right)^2 \sin^2\theta_i}$$

- The velocity condition yields the Rayleigh reflection coefficient:

$$R = \frac{r_2/r_1 - (\cos\theta_t/\cos\theta_i)}{r_2/r_1 + (\cos\theta_t/\cos\theta_i)} \qquad [2.12]$$

- Angle of incidence = angle of reflection

- For $c_1>c_2$: *θ_t is less than θ_i -- the wave bends towards the normal*

- For $c_1<c_2$: *θ_t is greater than θ_i -- the wave bends away from the normal but only up to a point*

- For $c_1<c_2$, there exists a critical angle θ_c such that, when $\theta_i> \theta_c$, $\cos\theta_t$ is imaginary. This critical angle is given by:

$$\boxed{\sin\theta_C \equiv \frac{c_1}{c_2}} \quad ; \quad \theta_C \text{ is the "critical angle"} \qquad [2.13]$$

- For $\theta_i> \theta_c$, the transmitted wave does not propagate in the y-direction... It is termed an evanescent wave

$$\mathbf{p_t} = \mathbf{P_t} e^{-i(\omega t - k_1(x\sin\theta_i - y\cos\theta_t))} = \mathbf{P_t} e^{\gamma y} e^{-i(\omega t - k_1 x \sin\theta_i)} \qquad [2.14]$$

$$\gamma = k_2\sqrt{(c_2/c_1)^2 \sin^2\theta_i - 1}$$

- This is a condition of *total internal reflection*.

- Now suppose that $r_2/r_1 = (\cos\theta_t/\cos\theta_i)$. By substitution in [2.12] we find that

$$R = \frac{r_2/r_1 - (\cos\theta_t/\cos\theta_i)}{r_2/r_1 + (\cos\theta_t/\cos\theta_i)} = 0$$

- The angle of incidence therefore results in perfect transmission and is called the *angle of intromission:*

$$\sin\theta_i = \sqrt{\frac{1-(r_1/r_2)^2}{1-(\rho_1/\rho_2)^2}} \qquad [2.15]$$

- This angle exists only if *r_1 and ρ_1 are both greater than or less than both r_2 and ρ_2.*

2.5 Engineering Applications

- Absorbers:
  - Sound absorbing surfaces must be impedance matched to the environment to minimize reflections.
  - "Slow" surfaces are better for they do not exhibit a critical angle.
  - Avoid grazing angle incidence -- promote normal incidence.
  - Facets promote multiple reflections at the interface.

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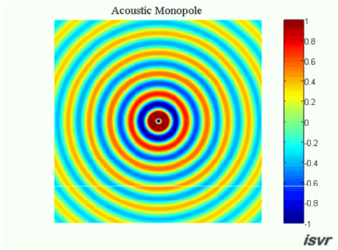
- Diffusers:
  - Multi-faceted surfaces reflect "wavelets" according to Snell's Law, where multiple facets launch reflections in multiple directions.
  - The end result is the plane wave coherent wavefront is broken up and the sound field becomes more diffuse upon reflection.
  - Useful for avoiding coherent modes in concern & lecture halls, and for avoiding dead zones in ultrasonic cleaning baths

BME2 – Biomedical Ultrasonics

Lecture 3: Sources of Sound – the Pulsating Sphere, the Point Source and the Discrete Line Array



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Department of Engineering Science

Contents

3.1 The 1-D Wave Equation: Harmonic Plane Waves

- The 3-D wave equation [1.11] can be written in spherical coordinates by using

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta^2}$$

- However, if spherical symmetry can be assumed, *p = p (r)* only in space and the wave equation reduces to:

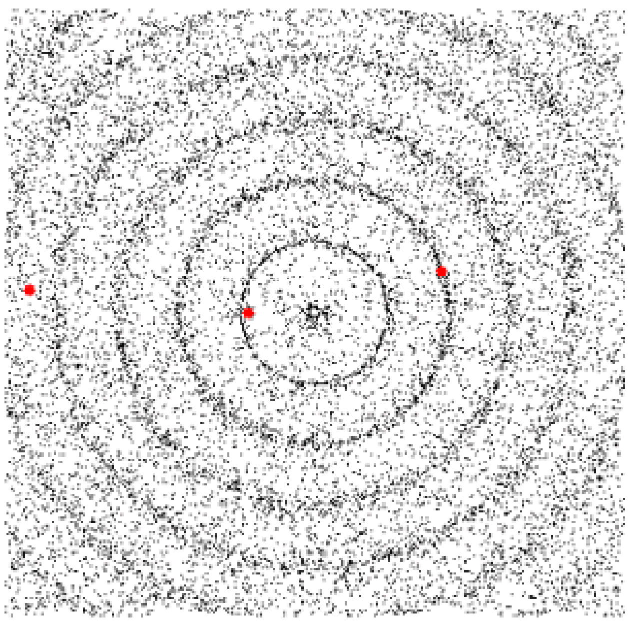
$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0} \qquad [3.1]$$

- The general solution to this 1-D wave equation must be of the form:

$$p(r,t) = \underbrace{\frac{f(r-ct)}{r}}_{\text{Outgoing spherical wave}} + \underbrace{\frac{g(r+ct)}{r}}_{\text{Incoming spherical wave}} \qquad [3.2]$$

- This describes the propagation of spherical waves:

Acoustic Monopole



- In general, a harmonic spherical wave of frequency *ω* and initial phase *φ* can be written in the form

$$\boxed{\begin{aligned} p(r,t) &= \frac{A}{r}\cos\left[\omega\left(t \pm \frac{r}{c}\right) - \varphi\right] = \frac{A}{r}\cos(\omega t \pm kr - \varphi) \\ &= \frac{1}{r}\mathrm{Re}\left[Ae^{i\phi}e^{-i(\omega t \pm kr)}\right] = \frac{1}{r}\mathrm{Re}\left[\tilde{A}e^{-i(\omega t \pm kr)}\right] \end{aligned}} \qquad [3.3]$$

- To obtain the velocity, take the gradient of the pressure (in spherical coordinates), substitute into the momentum equation [1.7] and integrate with respect to time:

$$\boxed{\mathbf{u}(r,t) = \frac{1}{\rho_0 c}\mathrm{Re}\left[\left(1 + \frac{i}{kr}\right)\tilde{p}(r,t)\right]\hat{\mathbf{r}}} \qquad [3.4]$$

- Note that, in general, velocity is NOT in phase with pressure for a spherical wave.
- The quantity *kr* is very important, as it gives a metric for determining acoustically small and acoustically large distances.

3.2 Acoustic Impedance

- Recall that acoustic impedance is a COMPLEX quantity that varies with position:

$$\tilde{Z} = \frac{\tilde{p}}{\tilde{v}}$$

- For a harmonic spherical wave, impedance can be obtained by dividing [3.3] and [3.4], giving

$$\mathbf{Z} = \frac{\mathbf{p}}{\mathbf{u}} = \frac{\rho_o c}{\left(1 + \frac{i}{kr}\right)} = \rho_o c \cos\theta\, e^{-i\theta} \quad \text{where} \quad \cos\theta = \frac{kr}{\sqrt{1+(kr)^2}} \quad \text{and} \quad \cot\theta = kr$$

$$\text{For } kr \text{ small, } p \text{ and } v \text{ are 90Þ out of phase} \qquad [3.5]$$

$$\text{For } kr \text{ large, } p \text{ and } v \text{ are in phase}$$

- This tells us that at acoustically large distances from the origin, *all spherical waves resemble a plane wave (since pressure in phase with velocity)*

- The nondimensional quantity “kr” holds special significance:

$$kr = 2\pi\left(\frac{r}{\lambda}\right)$$

- The distance from the origin relative to the wavelength determines the phase of Z:

  - For acoustically large ranges: $kr >> 1 \quad \text{and} \quad Z = \rho_o c \quad \text{(plane waves)}$
  - For acoustically small ranges: $kr << 1 \quad \text{and} \quad \mathbf{Z}(r) = \rho_o c(-ikr)$

- Consider a sound source consisting of a pulsating sphere of radius a<<λ:

  - Z(a) is purely reactive. The source cannot effectively transmit power into the medium because power transfer is related to the real part of any impedance.

  - Z(a) is small. Thus, for finite surface velocity, the pressure is small. *Acoustically small sources cannot generate intense pressure waves.*

3.3 Energetics of Spherical Waves

$$p(r,t) = P\cos(\omega t - kr) \quad ; \quad \vec{u}(r,t) = U\cos(\omega t - kr + \theta)\hat{r}$$

$$\vec{I} = \langle pu \rangle \hat{r} = \frac{1}{T}\int_0^T P\cos(\omega t - kr)\cos(\omega t - kr + \theta)dt$$

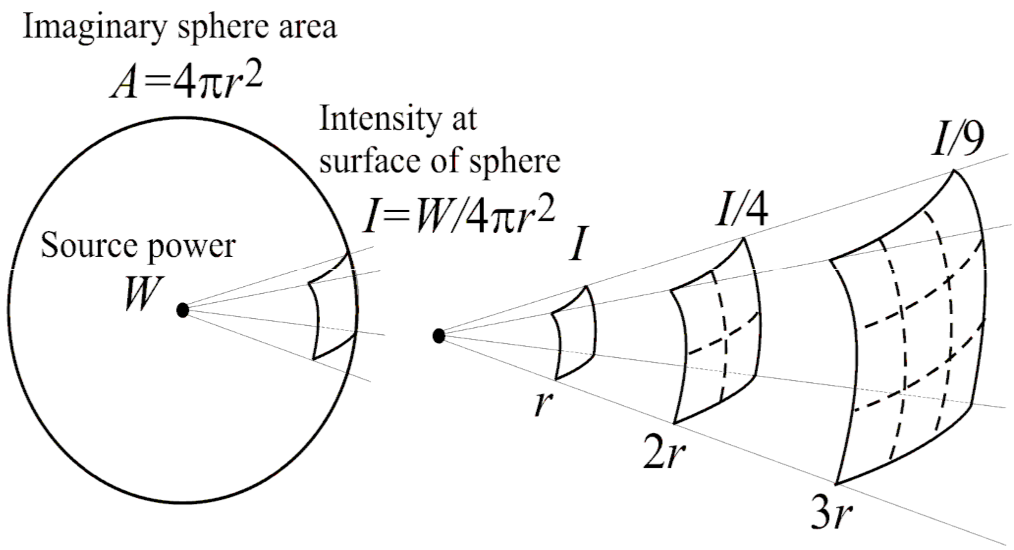
$$\vec{I} = \tfrac{1}{2} PU\cos\theta\,\hat{r} = P_{rms}U_{rms}\cos\theta\,\hat{r} \qquad [3.6]$$

The *cosθ* term is analogous to the power factor in electrical circuit theory:

$$U = \frac{P}{\rho_o c\cos\theta} \quad \text{and} \quad P = \frac{A}{r}$$

$$\vec{I} = \frac{1}{2}\frac{P^2}{\rho_o c}\hat{r} = \frac{1}{r^2}\frac{A^2}{2\rho_o c}\hat{r} \qquad [3.7]$$

- Also, note the important effects of *spherical spreading*



3.4 The pulsating sphere

- Consider a sphere with a radius that varies sinusoidally with time:

$$\mathbf{p}(r,t) = \frac{A}{r} e^{-i(\omega t - kr)}$$

- On the surface of the sphere of average radius a:

$$\mathbf{u}(a,t) = U_o e^{-i(\omega t)}$$

- To get the pressure on the boundary, consider the acoustic impedance $\mathbf{Z}(a) = \rho_o c \cos\theta_a e^{-i\theta_a} \quad ; \quad \cot\theta_a = ka$

- Substituting

$$\mathbf{p}(a,t) = \mathbf{Z}(a)\mathbf{u}(a) = \rho_o c U_o \cos\theta_a e^{-i(\omega t - kr + \theta_a)}$$

$$\boxed{p(r,t) = \rho_o c U_o \frac{a}{r} \cos\theta_a e^{-i(\omega t - kr + \theta_a)}} \qquad [3.8]$$

3.5 The point source

- For ka small, the acoustic impedance at the surface of the sphere is:

$$Z(a) \approx \rho_o c(-ika)$$

- Now, define the source strength as a volume velocity:

$$Q \equiv 4\pi a^2 U_o$$

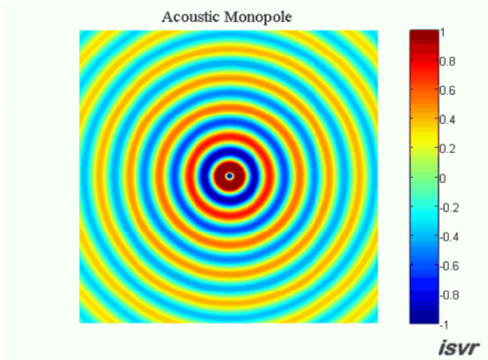
- In this limit of a "point source", the pressure becomes

$$\boxed{\mathbf{p}(r,t) = -i\rho_o c \frac{Qk}{4\pi r} e^{-i(\omega t - kr)}} \qquad [3.9]$$

- If you position the simple source very close to an an acoustically large rigid plane boundary called a baffle, the reflected pressure is in phase with the source pressure and the source pressure is doubled:

$$\boxed{\mathbf{p}(r,t) = -i\rho_o c \frac{Qk}{2\pi r} e^{-i(\omega t - kr)}} \qquad [3.10]$$

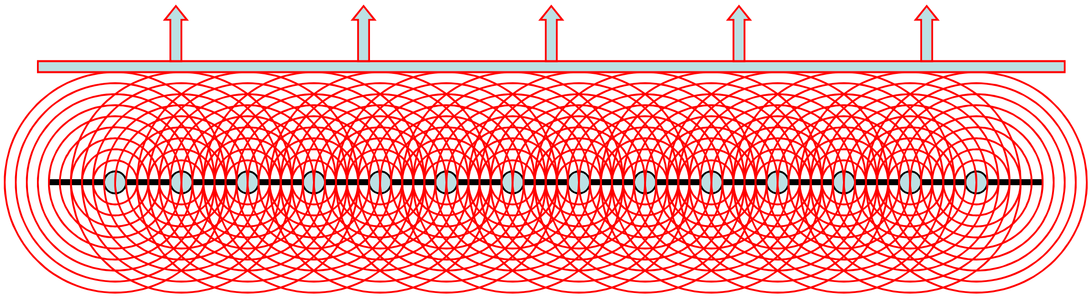
So, Why Do We Care About Simple Sources???



- Every radiating surface can be modeled as an array of simple sources pulsating independently
  - Analogous to the Huygens Wavelets from Optics
- By invoking **superposition**, the total radiated field generated by an array of simple sources is simply the sum of the individual fields
- Classic problems in the acoustics of sound sources:
  - The monopole ← Been There… Done That
  - The dipole
  - The continuous line source ← Engineering Technology Element
  - Discrete array sources ← Engineering Technology Element
  - The baffled piston source ←

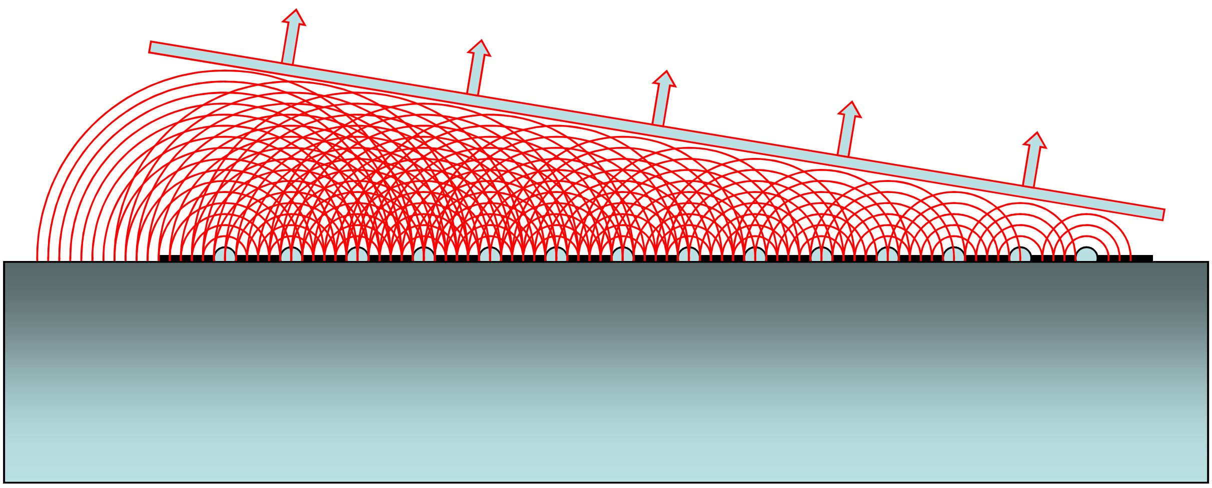
3.6 The discrete line array:

- Motivation: beamsteering



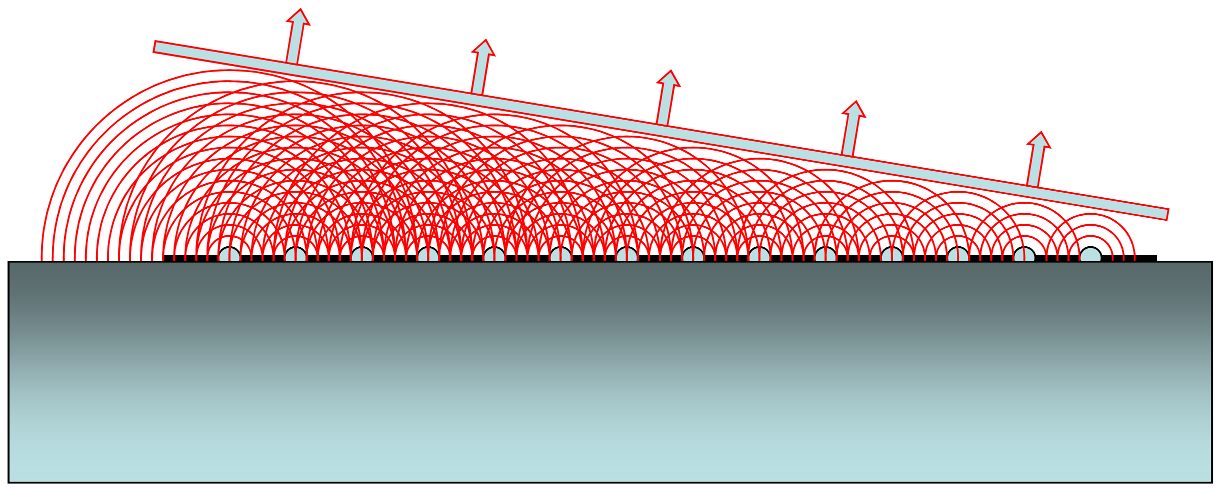
The Discrete Line Array

Steering the Beam By Delaying Elements



The Discrete Line Array

Steering the Beam Using a Soundspeed Gradient (refraction)



January 2008 - February 2008

January 2008

Mo	Tu	We	Th	Fr	Sa	Su
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

February 2008

Mo	Tu	We	Th	Fr	Sa	Su
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29		

	Monday	Tuesday	Wednesday	Thursday	Friday
14 - 18 Jan	**14 Jan**	**15**	**16**	**17**	**18**
	1st WEEK				
	15:00 16:00 BME Ultrasonics; Thom LR2			12:00 13:00 BME Ultrasonics; Thom LR2	
21 - 25 Jan	**21**	**22**	**23**	**24**	**25**
	2nd WEEK				
	15:00 16:00 BME Ultrasonics; Thom LR2			12:00 13:00 BME Ultrasonics; Thom LR2	
28 Jan - 1 Feb	**28**	**29**	**30**	**31**	**1 Feb**
	3rd WEEK				
	15:00 16:00 BME Ultrasonics; Thom LR2			12:00 13:00 BME Ultrasonics; Thom LR2	11:00 13:00 BME2 MSc Class; Thom LR5 14:00 15:00 BME2 C Class; Thom LR5 15:00 16:00 BME2 C Class; Thom LR5 16:00 17:00 BME2 C Class; Thom LR5
4 - 8 Feb	**4**	**5**	**6**	**7**	**8**
	4th WEEK				
	15:00 16:00 BME Ultrasonics; Thom LR2			12:00 13:00 BME Ultrasonics; Thom LR2	
11 - 15 Feb	**11**	**12**	**13**	**14**	**15**
	10:10 13:10 BME 6 Lab; 43 BR	14:00 17:00 BME 6 Lab; 43 BR		14:00 17:00 BME 6 Lab; 43 BR	11:00 13:00 BME2 MSc Class; Thom LR5 14:00 17:00 BME 6 Lab; 43 BR 14:00 15:00 BME2 C Class; Thom LR5 15:00 16:00 BME2 C Class; Thom LR5 16:00 17:00 BME2 C Class; Thom LR5