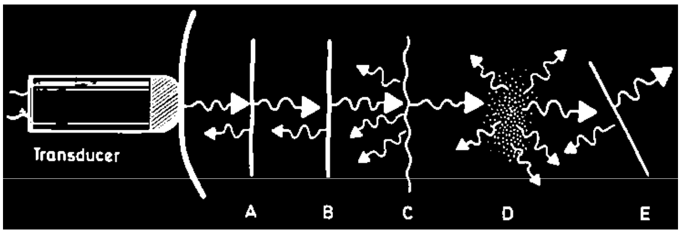
BME2 – Biomedical Ultrasonics

Lecture 2: Transmission, Reflection & Refraction of Plane Waves



Dr. Constantin-C. Coussios

Biomedical Ultrasonics & Biotherapy Laboratory

Office: 43 Banbury Road

Tel: 01865-(2)74750

Email: constantin.coussios@eng.ox.ac.uk

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Department of Engineering Science

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2.1 The 1-D Wave Equation: Harmonic Plane Waves

- If pressure only varies along a single direction *x,* the 3-D wave equation [1.11] written out in Cartesian coordinates can be simplified to:

$$\boxed{\frac{\partial^2 p(x,t)}{\partial t^2} = c^2\left(\frac{\partial^2 p(x,t)}{\partial x^2}\right)} \qquad [2.1]$$

which is known as the 1-D wave equation.

- Integrating both sides, this suggests solutions that satisfy:

$$\frac{\partial p(x,t)}{\partial t} = \pm c\left(\frac{\partial p(x,t)}{\partial x}\right)$$

- This implies that the general solution to the 1-D wave equation must be of the form:

$$\boxed{p(x,t) = f\underbrace{\left(t - \frac{x}{c}\right)}_{\text{“Plane wave” traveling in the positive x direction}} + g\underbrace{\left(t + \frac{x}{c}\right)}_{\text{“Plane wave” traveling in the negative x direction}}} \qquad [2.2]$$

- Furthermore, if we assume that the plane wave is harmonic at frequency *ω,* it can be written in the form

$$\boxed{p(x,t) = A\cos\left[\omega\left(t \pm \frac{x}{c}\right)\right] = A\cos(\omega t \pm kx) = \mathrm{Re}\left[\tilde{A}e^{\omega t \pm kx}\right]} \qquad [2.3]$$

- Note that harmonic plane waves are periodic both in space and time.
- Note that quantities such as pressure, density and velocity must always be REAL. However, it is convenient to work with the complex exponential form, and complex quantities can be defined such that:

$$\boxed{p(x,t) = \mathrm{Re}[\tilde{p}(x,t)]}$$
$$\boxed{\rho(x,t) = \mathrm{Re}[\tilde{\rho}(x,t)]} \qquad [2.4]$$
$$\boxed{\mathbf{u}(x,t) = \mathrm{Re}[\tilde{\mathbf{u}}(x,t)]}$$

In many textbooks, the ‘Real part of’ is often implied and therefore omitted altogether. However, it is important to remember that all physical quantities must ultimately be REAL.

2.2 Acoustic and Characteristic Impedance

- Many complex systems are characterized by their impedance
- Impedance is the ratio of a stimulus to a response
- Impedance is generally complex, as it generally affects both the amplitude and phase of the response.
  - Electrical $\tilde{Z} = \tilde{V}/I = \text{Voltage/Current}$
  - Mechanical $\tilde{Z} = \text{Force/Velocity}$
  - Acoustical $\tilde{Z} = \text{Acoustic Pressure / Particle Velocity}$
- Acoustic impedance is a measure of how much a material “complies” to a dynamic pressure disturbance.
- Acoustic impedance determines the relative magnitude and phase of pressure and velocity at a point, and this play an important role in establishing boundary conditions for acoustic propagation.

- In general, acoustic impedance is a COMPLEX quantity that varies with position:

$$\tilde{Z} = \frac{\tilde{p}}{\tilde{v}} \qquad [2.5]$$

- However, for the special case of a harmonic plane wave, the 1-D form of the momentum equation [1.7] takes the form:

$$\rho_0 \frac{\partial \tilde{\mathbf{v}}(x,t)}{\partial t} = -\left(\frac{\partial \tilde{p}(x,t)}{\partial x} \right) \mathbf{i}$$

- Substituting the positive-going harmonic plane wave solution for (complex) pressure and integrating with respect to time gives:

$$\tilde{\mathbf{v}}(x,t) = \pm \frac{\tilde{A}}{\rho_0 c} e^{-i(\omega t \pm kx)} = \pm \frac{\tilde{p}}{\rho_0 c} \qquad [2.6]$$

- This means that, *for harmonic plane waves*, the acoustic impedance is given by

$$\tilde{Z} = \frac{\tilde{p}}{\tilde{v}} = \pm \rho_0 c = \pm r \qquad [2.7]$$

where the quantity *r* is called the *characteristic impedance* of a medium and is a *material property.* It has units of Pa.s / m

- Therefore, for the special case of a *harmonic plane wave*, the impedance is REAL, of magnitude equal to the *characteristic impedance,* with a sign that depends on the direction of wave propagation.

Material	ρ (kg/m ^{3})	C (m/sec)	*r* (Pa•sec/m)
Air	1.21	343	415
Water	998	1481	1.48 x 10 ^{6}
Steel	7700	6100	47.0 x 10 ^{6}

Stiffer, denser materials have greater characteristic impedances

2.3 Energetics of Plane Waves

Acoustic Intensity:

$$\vec{I}_i = p\vec{v} = \pm \frac{p^2}{\rho_o c}\hat{n} \quad ; \quad I = \pm P_{rms} V_{rms} = \pm \frac{P_{rms}}{\rho_o c}$$

Acoustic Energy Density:

$$e_i = \tfrac{1}{2}\rho_o\left(v^2 + \frac{p^2}{\rho_o^2 c^2}\right) = \frac{pv}{c} = \rho_o v^2$$

$$e = \frac{P_{rms} V_{rms}}{c}$$

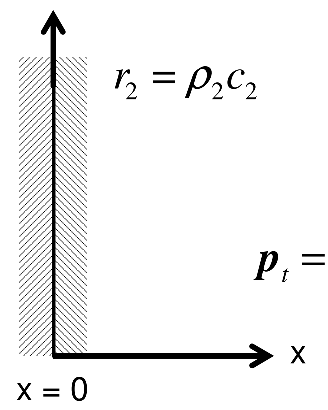
2.2 Reflection and Transmission of Plane Waves at Normal Incidence

$$r_1 = \rho_1 c_1 \qquad r_2 = \rho_2 c_2$$

$$\boldsymbol{p}_i = \mathbf{P}_i e^{i(\omega t - k_1 x)}$$

$$\boldsymbol{p}_r = \mathbf{P}_r e^{i(\omega t + k_1 x)}$$

$$\boldsymbol{p}_t = \mathbf{P}_t e^{i(\omega t - k_2 x)}$$



$$\boldsymbol{p}_1 = \boldsymbol{p}_i + \boldsymbol{p}_r \qquad \boldsymbol{p}_2 = \boldsymbol{p}_t$$

Superposition

Match the Boundary Conditions

$$\boldsymbol{p}_i + \boldsymbol{p}_r = \boldsymbol{p}_t$$

$$\boldsymbol{v}_i + \boldsymbol{v}_r = \boldsymbol{v}_t$$

Continuity of pressure at x = 0

Continuity of normal velocity at x = 0

From superposition:

$$\left.\frac{\boldsymbol{p}_i + \boldsymbol{p}_r}{\boldsymbol{v}_i + \boldsymbol{v}_r}\right|_{x=0} = \left.\frac{\boldsymbol{p}_t}{\boldsymbol{v}_t}\right|_{x=0} \Rightarrow \left.\boldsymbol{z}_1\right|_{x=0} = \left.\boldsymbol{z}_2\right|_{x=0}$$

- The boundary condition calls for the continuity of specific acoustic impedance. Note that this *does not* imply that the characteristic impedance is continuous.
- What are the implications for sound transmission and reflection?

- Pressure Transmission Coeff: $\mathbf{T} \equiv \frac{\mathbf{P}_t}{\mathbf{P}_i}$ $\quad T_I \equiv \frac{I_t}{I_i} = \frac{r_1}{r_2}|\mathbf{T}|^2$
- Pressure Reflection Coeff: $\mathbf{R} \equiv \frac{\mathbf{P}_r}{\mathbf{P}_i}$ $\quad R_I \equiv \frac{I_r}{I_i} = |\mathbf{R}|^2$
- Substituting and rearranging yields

$$\mathbf{R} = \frac{r_2 - r_1}{r_2 + r_1} \qquad \mathbf{T} = \frac{2r_2}{r_2 + r_1} \qquad R_I = \left(\frac{r_2 - r_1}{r_2 + r_1}\right)^2 \qquad T_I = 4\frac{r_1/r_2}{(1 + r_1/r_2)^2}$$

- Note that: $\mathbf{T} = 1 + \mathbf{R}$

Limiting Behavior

- For $r_1 = r_2$: R = 0 & T = 1
  - complete transmission (impedance matched)
- For $r_1 << r_2$: R = +1 & T = 2
  - complete in-phase reflection (rigid boundary)
  - pressure at y = 0 is doubled
  - velocity at y = 0 is zero
- For $r_1 >> r_2$: R = -1 & T = 0
  - Complete anti-phase reflection (pressure release)
  - pressure at y = 0 is zero
  - velocity at y = 0 is doubled
- Mismatched configurations result in scattering and, in some cases, standing waves.
  - If you want to deliver sound to the system, the impedances must be matched at all points in the path.

Reflection From a “Hard” (rigid) Boundary

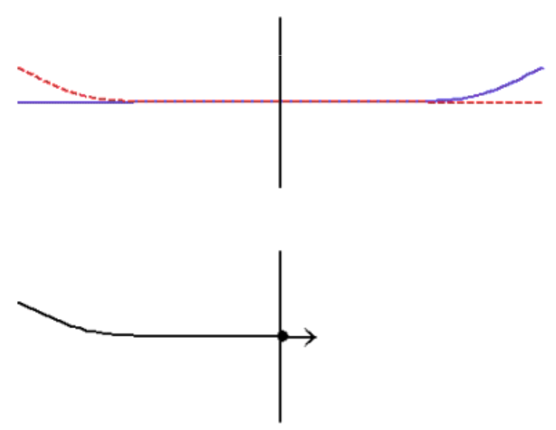
$r_1 << r_2 = \infty$ R = +1 & T = 2

- You have in-phase reflection of the pressure wave, hence R = +1. The contribution of the boundary is equivalent to having an in-phase wave coming in from the right. In essence, you replace the reflecting boundary with a plane wave source on the right hand side

Reflection From a “Hard” Boundary

$r_1 << r_2 = \infty$ R = +1 & T = 2

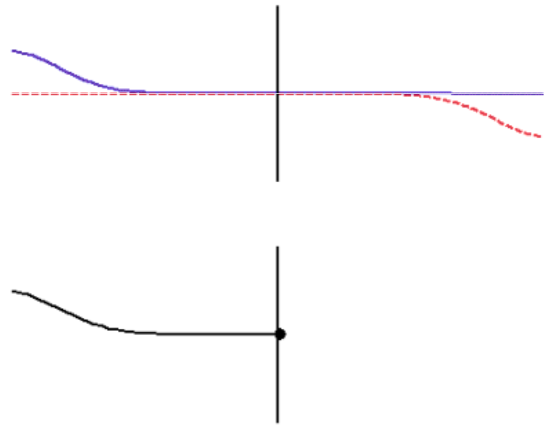
- The total pressure at the boundary is doubled. This is the pressure of the transmitted wave, hence, T = 2.



Reflection From a “Hard” Boundary

$r_1 << r_2 = \infty$ R = +1 & T = 2

- The reflected velocity wave is out of phase. The total particle velocity at the boundary equals 0. This is a *rigid* boundary.



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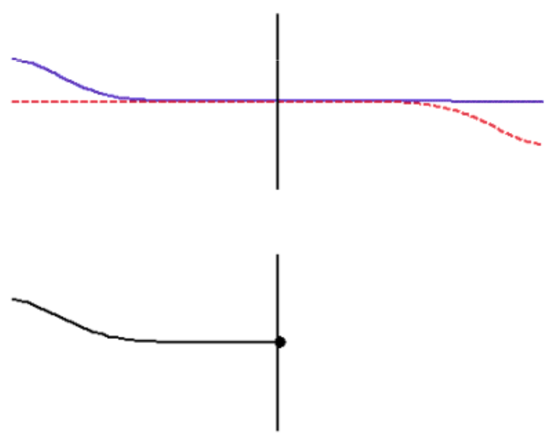
Reflection From a “Soft” Boundary

$r_1 >> r_2 = 0$ R = -1 & T = 0

- The reflected pressure wave is out of phase. The total pressure at the boundary equals 0. This is a *pressure release* boundary.



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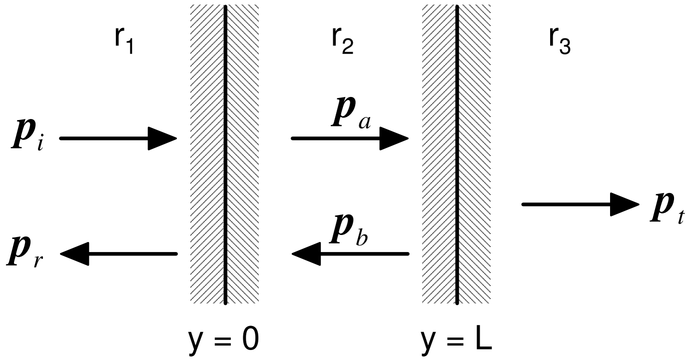
Transmission Through a Layer

- In the case of layers, the problem is solved in the same manner, except you have two boundaries to deal with.



Transmission Through a Layer

- Applying the boundary conditions and working though the algebra:

$$\mathbf{R} = \frac{\left(1 - \frac{r_1}{r_3}\right)\cos k_2 L + i\left(\frac{r_2}{r_3} - \frac{r_1}{r_2}\right)\sin k_2 L}{\left(1 + \frac{r_1}{r_3}\right)\cos k_2 L + i\left(\frac{r_2}{r_3} + \frac{r_1}{r_2}\right)\sin k_2 L}$$

$$T_I = \frac{4}{2 + \left(\frac{r_3}{r_1} + \frac{r_1}{r_3}\right)\cos^2 k_2 L + \left(\frac{r_2^2}{r_1 r_3} + \frac{r_1 r_3}{r_2^2}\right)\sin^2 k_2 L}$$

Engineering Application: Acoustic Windows

- Same fluid at inlet and outlet: $r_1 = r_3$ $\quad T_I = \left[1 + \frac{1}{4}\left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right)^2 \sin^2 k_2 L \right]^{-1}$

- For $L \ll \lambda$, you get perfect transmission
- For $k_2 L = n\pi$, you get perfect transmission: $\quad f_n = n\frac{c_2}{2L}$
- For $k_2 L = (2n-1)\pi/2$, you have: $\quad L = \frac{(2n-1)}{4}\lambda$

$$T_I = 4 r_1 r_3 \Big/ \left(r_2 + \frac{r_1 r_3}{r_2} \right)^2 \quad ; \quad T_I \approx 1 \text{ for } r_2 = \sqrt{r_1 r_3}$$