4. Optical Waves

Although we can learn a great deal from the geometrical optics analysis of systems we can't learn everything. In order to understand phenomena such as interference or diffraction we have to appeal to the wave nature of light. As almost everything else in this course, as well as many of the other optics courses, will be concerned with waves, it is appropriate to review, briefly, how they can be described mathematically. We begin, inevitably, with Maxwell's equations, which we repeat here in their most useful 'optical form'

$\nabla \cdot \hat{\boldsymbol{D}} = div\, \hat{\boldsymbol{D}} = \rho$ Gauss Law

$\nabla \cdot \hat{\boldsymbol{B}} = div\, \hat{\boldsymbol{B}} = 0$ No Free magnet poles

$\nabla \times \hat{\boldsymbol{E}} = curl\, \hat{\boldsymbol{E}} = -\dfrac{\partial \hat{\boldsymbol{B}}}{\partial t}$ Faraday's Law

$\nabla \times \hat{\boldsymbol{H}} = curl\, \hat{H} = \dfrac{\partial \hat{\boldsymbol{D}}}{\partial t} + \hat{\boldsymbol{J}}$ Ampere's Law & Maxwell's displacement current

Note that ^ denotes time varying (instantaneous) values.

We also have the **Material** or **Constitutive Equations**

$\hat{\boldsymbol{D}} = \varepsilon \hat{\boldsymbol{E}}$ $\varepsilon =$ permittivity – see later, particularly under crystal optics

$\hat{\boldsymbol{B}} = \mu \hat{\boldsymbol{H}}$ $\mu =$ permeability

$\hat{\boldsymbol{J}} = \sigma \hat{\boldsymbol{E}}$ Ohm's Law (usually σ is small in optical case)

If we now consider the usual case of a source free $(\rho = o)$ medium we can easily obtain the wave equation as

$$\nabla \times \left(\nabla \times \hat{\boldsymbol{E}}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \hat{\boldsymbol{B}}\right)$$

$$= -\mu \frac{\partial}{\partial t}\left(\nabla \times \hat{\boldsymbol{H}}\right)$$

and since μ is independent of time

$$= -\mu\, \sigma \frac{\partial \hat{\boldsymbol{E}}}{\partial t} - \mu\, \varepsilon \frac{\partial^2 \hat{\boldsymbol{E}}}{\partial t^2}$$

Now we can expand $\nabla \times \left(\nabla \times \hat{\boldsymbol{E}}\right)$ by the vector identity

$$\nabla \times (\nabla \times \hat{\boldsymbol{E}}) = \nabla (\nabla \cdot \hat{\boldsymbol{E}}) - \nabla^2 \hat{\boldsymbol{E}}$$

Thus the wave equation becomes

$$\nabla^2 \hat{\boldsymbol{E}} - \nabla (\nabla \cdot \hat{\boldsymbol{E}}) - \mu\, \sigma \frac{\partial \hat{\boldsymbol{E}}}{\partial t} - \mu\, \varepsilon \frac{\partial^2 \hat{\boldsymbol{E}}}{\partial t^2} = 0$$

Now, ***if*** ε is constant in space, we can substitute $\hat{\boldsymbol{E}} = 1/\varepsilon\ \hat{\boldsymbol{D}}$ and write

$$\nabla (\nabla \cdot \hat{\boldsymbol{E}}) = \frac{1}{\varepsilon} \nabla (\nabla \cdot \hat{\boldsymbol{D}}) = \frac{1}{\varepsilon} \nabla (\nabla \cdot \boldsymbol{0}) = 0$$

Note that ε=constant is a sufficient condition for $\nabla (\nabla \cdot \hat{\boldsymbol{E}}) = 0$, but it is not necessary. (Can you think of other conditions that would lead to this with a specific variation of ε?)

Thus the wave equation becomes, in the presence of loss,

$$\nabla^2 \hat{\boldsymbol{E}} - \mu\, \sigma \frac{\partial \hat{\boldsymbol{E}}}{\partial t} - \mu\, \varepsilon \frac{\partial^2 \hat{\boldsymbol{E}}}{\partial t^2} = 0$$

If we can ignore loss, as is often reasonable, we obtain the familiar form of

$$\nabla^2 \hat{\boldsymbol{E}} = \mu\, \varepsilon \frac{\partial^2 \hat{\boldsymbol{E}}}{\partial t^2}$$

An equivalent series of steps for the $\hat{\boldsymbol{H}}$ vector will give

$$\nabla^2 \hat{\boldsymbol{H}} = \mu\, \varepsilon \frac{\partial^2 \hat{\boldsymbol{H}}}{\partial t^2}$$

Harmonic Solutions

In this case we take the time dependence as $\hat{\boldsymbol{E}} = \boldsymbol{E}\ exp\ j\omega t$ and so the wave equation becomes

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0$$

where

$$k^2 = \omega^2\ \mu\varepsilon$$

This relationship between k and ω is the ***dispersion*** relation which we shall return to later. In essence it tells us how waves of different frequencies propagate.

Plane Waves

Let us further suppose that the electric field E is polarised in only one direction, say,

$$\boldsymbol{E} = E\,\boldsymbol{i} \qquad \boldsymbol{i} = \text{suitable unit vector}$$

Then the vector wave equation becomes a scalar wave equation

$$\nabla^2 E + k^2 E = 0$$

Which, if we take $E = E(z)$ only, that is to say we replace $\nabla^2 E$ *by* $d^2 E/dz^2$, has a solution

$$E = E_0 \, exp \pm jkz$$

and hence the full solution is

$$\hat{\boldsymbol{E}} = E_0 \, \boldsymbol{i} \; exp \; j\omega t \; exp \pm jkz$$

which represents a plane polarised wave travelling in the + or –z direction. Specifically

$$exp \; j(\omega t - kz)$$

represents a plane wave travelling in the +z direction. It has a phase velocity given by

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

and a group velocity given by

$$v_g = \frac{\partial \omega}{\partial k}$$

It is clear that in vacuum

$$v_p = v_g = \frac{1}{\sqrt{\mu\varepsilon}} = 3\text{x}\,10^8 \; ms^{-1} = \text{ speed of light}$$

Our harmonic solution here is obtained at a particular frequency ω and it may well be that the permittivity ε, in particular, is a function of frequency. Usually μ is fairly constant in non-magnetic material and we will approximate it to μ_0 in the following. Thus we can write the dispersion relation as

$$k^2 = \omega^2 \; \mu_0 \; \varepsilon(\omega)$$

In optics it is usual to reference the behaviour in a medium to that in free space via the ***refractive index***. One definition would be the ratio of the phase velocities in vacuum and a medium: i.e.,

$$n = \frac{v_p(vacuum)}{v_p(medium)} = \frac{\sqrt{\mu_0\,\varepsilon}}{\sqrt{\mu_0\,\varepsilon_0}} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$$

hence

$$n(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

and the dispersion relation now becomes

$$k^2 = \omega^2\,\mu_0\,\varepsilon_0\,n^2(\omega)$$

or
$$\frac{k}{k_0} = n(\omega)$$

Let us now remind ourselves what k is physically. If the wave $exp\;j(\omega t - kz)$ is frozen in time we can see that the constant phase is given by

$$kz = constant + m\;2\pi$$

where m in an integer. If we move a distance in z such that the change in phase 2π then we have defined the wavelength, λ, of the wave. It is clear that we must have

$$k\lambda = 2\pi$$

or
$$k = \frac{2\pi}{\lambda}\quad \textbf{\textit{always}}.$$

Our previous solution or the plane wave was for the electric vector. Assuming $\hat{\boldsymbol{E}} = E_0\,\boldsymbol{i}_x\;exp\;j\omega t\;exp \pm jkz$, the $\hat{\boldsymbol{H}}$ solution is readily obtained as

$$\hat{\boldsymbol{H}} = \pm\frac{E_0}{\eta}\,\boldsymbol{i}_y\;exp\;j\omega t\;.\;exp \pm jkz$$

with $\eta = \sqrt{\frac{\mu}{\varepsilon}} =$ impedance of medium (120π in free space)

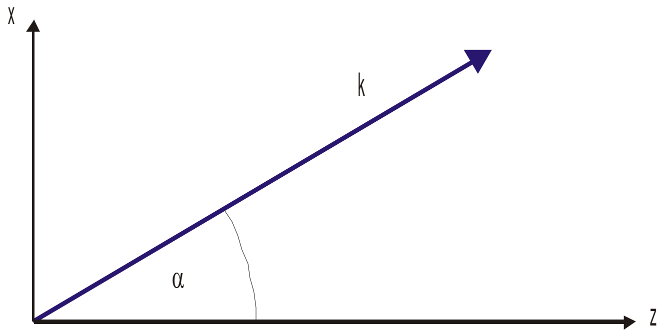
It is clear that $\hat{\boldsymbol{H}}$, $\hat{\boldsymbol{E}}$ and **k** form an orthogonal set. $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{H}}$ are also in phase (for the $\sigma = 0$ case we have considered).

Other solutions to the wave equation

We have considered a specific case of a plane wave travelling in the z-direction as a solution of the wave equation. In essence we took $\nabla^2 E$ to be $\frac{\partial^2 E}{\partial z^2}$. If we had, instead, considered a general plane wave we would have found a solution of the form

$$exp\ j(\omega t - \boldsymbol{k} \cdot \boldsymbol{r})$$

where $|\boldsymbol{k}| = k = 2\pi/\lambda$. **k** is called the 'k' vector and the plane wave propagates in the r direction. As a specific example we will consider a plane wave propagating at an angle α



The wave may be described as

$$exp\ j\omega t\ .\ exp - j\boldsymbol{k} \cdot \boldsymbol{r}$$

with $\boldsymbol{k} \cdot \boldsymbol{r} = k_x \cdot x + k_z \cdot z = k\ sin\,\alpha\, x + k\ cos\,\alpha\, z$, where $k = \frac{2\pi}{\lambda}$,

and so the full expression for the plane wave travelling at the angle α to the z-axis is

$$exp\ j\omega t\ exp - jk\ (x\ sin\,\alpha + z\ cos\,\alpha)$$

We have implicitly obtained plane wave solutions by writing the Laplacian in Cartesian form. If we had used spherical co-ordinates we would have obtained spherical wave solutions of the form

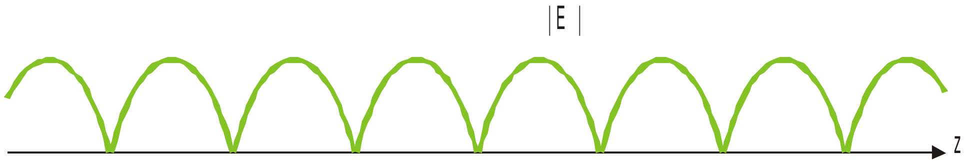
$$E_0 \frac{exp\ j\ (\omega t - kr)}{r}$$

In fact, because we have assumed the materials equations to be linear (i.e. we assumed that **D** was proportional to **E** and this may not always be the case – see later in the discussion of how waves propagate in crystals) and because Maxwell's equations are linear the resulting wave equation is also linear. This means that the solutions may be added together to produce new solutions – ***superposition***. For example, the spherical wave solution could have been obtained from an (infinite) sum of plane wave solutions. This is the basis of an important technique which we will discuss later whereby any realisable wave can be decomposed into an ***angular spectrum of plane wave solutions***.

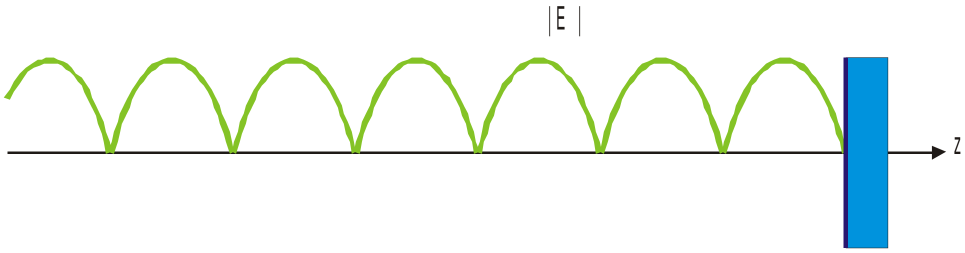
We now return to consider simple plane waves and consider that we have two, of equal strength, travelling in opposite directions along the z axis.

$$\begin{aligned} \boldsymbol{E} &= \left[E_0 \boldsymbol{i}\, exp(-jkz) + E_0 \boldsymbol{i}\, exp(jkz) \right] exp(j\omega t) \\ &= 2E_0 \boldsymbol{i} \cos kz \, exp(j\omega t) \end{aligned}$$

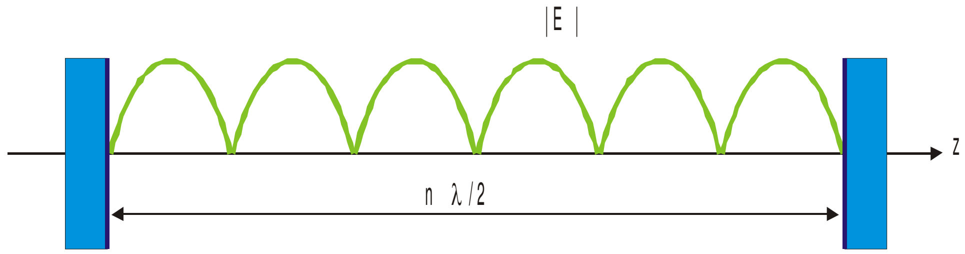
i.e. standing wave set up. The field would look something like the following



This can be produced easily, in practice, by reflecting a wave from a perfect conductor. At perfect conductor the transverse **E** field vanishes.

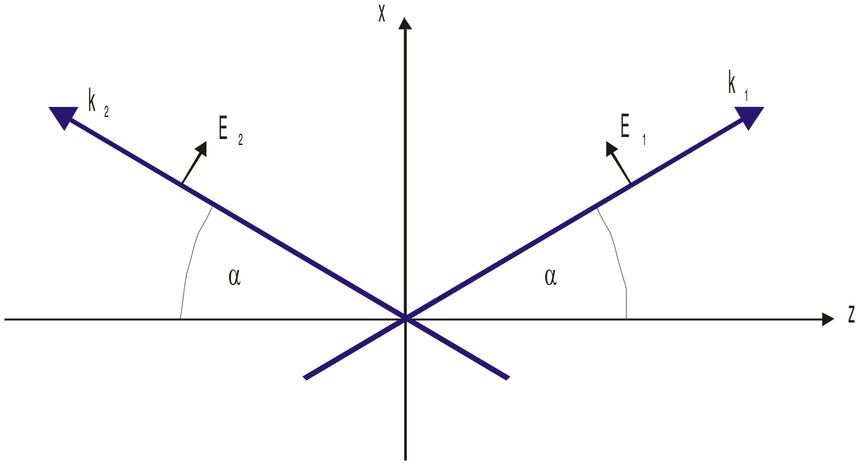


Similarly if we have two mirrors placed at any of the nodes the light bounces back and forth and we have built a ***resonator***.

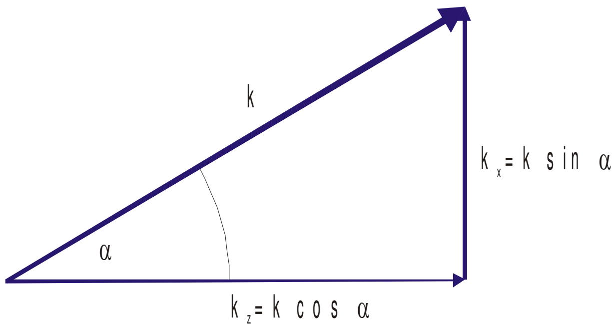


The spacing is an integer number of half-wavelengths. This is the basis of the Fabry-Perot resonator. Resonators will be discussed more fully in a later course. They are very important in many fields of optics, not least, in lasers where together with a suitable gain medium they provide the necessary interaction length to achieve laser action.

Now suppose that the two waves are counter-propagating at an angle α to the z-axis



The wave vector of wave (1) may be decomposed as



The wave vector of wave (2) may be similarly decomposed such that we can write the two plane waves as

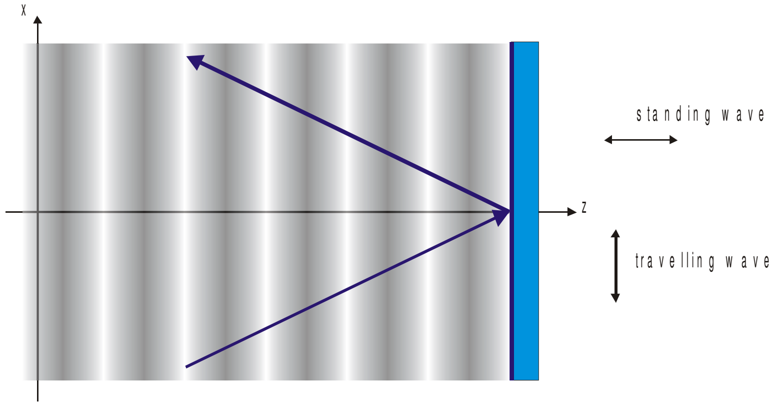
$$\boldsymbol{E}_1 = E_0 \left(\cos\,\alpha\,\boldsymbol{i}_x - \sin\,\alpha\,\boldsymbol{i}_z\right) \exp\left\{-\,jk\left(z\,\cos\,\alpha + x\,\sin\,\alpha\right)\right\}$$
$$\boldsymbol{E}_2 = E_0 \left(\cos\,\alpha\,\boldsymbol{i}_x + \sin\,\alpha\,\boldsymbol{i}_z\right) \exp\left\{jk\left(z\,\cos\,\alpha - x\,\sin\,\alpha\right)\right\}$$

The total field is now

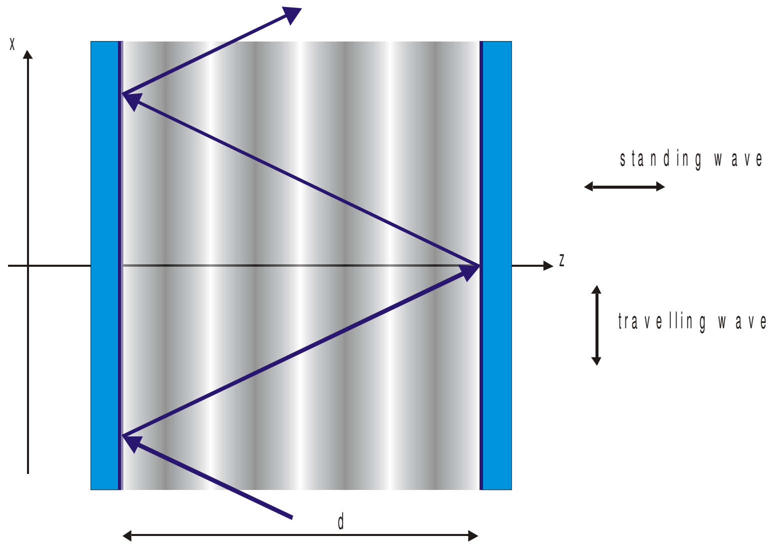
$$\boldsymbol{E}_1 + \boldsymbol{E}_2 = 2\,E_0\,\exp\left\{-\,j\,k\,x\,\sin\,\alpha\right\}\left[\cos\,\alpha\,\cos\,\left(k\,z\,\cos\,\alpha\right)\boldsymbol{i}_x + j\,\sin\,\alpha\,\sin\left(k\,z\,\cos\,x\right)\boldsymbol{i}_z\right]$$

This is a very important expression. The **first term**, $\exp - j\,k\,x\,\sin\,\alpha$ represents a **travelling wave** in the x-direction. The **term in square brackets** represents a **standing wave** in the z-direction. Two such counter-propagating waves can be achieved by reflecting a wave from a perfect mirror at an appropriate angle.

Graphically, the result can be represented as



Conductor (mirror) coincides with the nodes and so satisfies the boundary conditions. If, instead, we use two reflectors we can build a parallel plate **waveguide**



The transverse E field must vanish on the conductors and so, from the last expression we must have

$$kd \ \cos \ \alpha \leq n \ \pi \quad n \ \text{integer}$$

which for a given λd means only a limited number of values of n are possible.

Here we have considered reflections from perfect conductors, but is important to realise that optical waveguides are built of dielectric materials and so it is important to know how optical waves are reflected and transmitted when they are incident on a boundary between two dielectrics. However before we consider this it will be useful to remind ourselves of what we mean by the state of ***polarisation*** of a wave.