Gaudin Model: Bethe Ansatz Solution

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1 Introduction

1.1 Gaudin Model

In this set of notes, we consider the Gaudin [1] Hamiltonian given by

$$H_i = \sum_{j \neq i} \frac{\vec{\sigma_i} \cdot \vec{\sigma_j}}{\epsilon_i - \epsilon_j}. \tag{1}$$

Let us fix $i = 0$ to be our central spin:

$$H_0 = \sum_{j \neq 0} \frac{\vec{\sigma_0} \cdot \vec{\sigma_j}}{\epsilon_0 - \epsilon_j}. \tag{2}$$

This Hamiltonian is equivalent to the central-spin model

$$H = g\mu_B \vec{S} \cdot \vec{B} + \vec{S} \cdot \sum_j A_j \vec{I}_j \tag{3}$$

where central spin $\vec{S}$ is coupled to environment spin $\vec{I}_j$ via hyperfine couplings A_j and to an external magnetic field $\vec{B}$ via g factor g and Bohr magneton μ_B [2]. For a system of N spins, we have $j = 1, \ldots, N-1$.

Hamiltonian (2) is an integrable model. It commutes with the $N-1$ operators (1):

$$[H_0, H_i] = 0. \tag{4}$$

These $N-1$ operators along with the total spin squared operator

$$\vec{\sigma} \cdot \vec{\sigma} = \left(\sum_i \vec{\sigma}_i \right)^2 \tag{5}$$

form a set of N commuting symmetry operators.

1.2 Bethe Ansatz

The model (1)/(2) is solvable via the Bethe Ansatz [1, 2]. In particular, the eigenstates are of the form

$$|w_1, \ldots, w_m\rangle = F(w_1) \cdots F(w_m) \left|\uparrow\uparrow \cdots \uparrow\right\rangle \tag{6}$$

$$F(w) = \sum_i \frac{\sigma_i^-}{w - \epsilon_i} \tag{7}$$

where $w_1, \ldots, w_m$ are solutions to the Bethe equations

$$\sum_i \frac{1}{w_k - \epsilon_i} + \sum_{l=1, l\neq k}^{m} \frac{2}{w_l - w_k} = 0. \tag{8}$$

Eq. (8) is a system of m equations, one for each parameter w_k. The eigenvalues corresponding to a particular solution of eq. (8) are then

$$E_i(w_1, \ldots, w_m) = \sum_{l=1}^{m} \frac{2}{w_m - \epsilon_i} + \sum_{j\neq i} \frac{1}{\epsilon_i - \epsilon_j}. \tag{9}$$

2 N = 2

First, we consider the simplest case of a system with two spins, one "central" spin and one "environment" spin. The Hamiltonian can be written as

$$\begin{aligned} H &= \frac{1}{\epsilon_0 - \epsilon_1} \vec{\sigma}_0 \cdot \vec{\sigma}_1 \\ &= \frac{1}{\epsilon_0 - \epsilon_1} (\sigma_0^x \sigma_1^x + \sigma_0^y \sigma_1^y + \sigma_0^z \sigma_1^z). \end{aligned} \tag{10}$$

The Hilbert space has dimension $2^2 = 4$, so there will be four eigenstates and associated energies.

2.1 State Construction with Young's Tableaux

The possible eigenstates that satisfy permutation symmetry can be constructed using Young's Tableaux [3]. For two spins, there are two possible configurations:

$$\begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} = \begin{array}{|c|c|}\hline & \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \tag{11}$$

The first is totally symmetric and the second is totally antisymmetric. We label the spin up state as 1 and the spin down state as 2. Then the rule for constructing Young's Tableaux is that the numbers may only increasing moving down and cannot decreasing moving to the right. Therefore, the possible configurations of the totally symmetric state are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \quad \text{and} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}. \tag{12}$$

Symmetry is implied by the horizontal nature of the tableaux, so the middle one must be a superposition of the two possible configurations. Thus, the three symmetric eigenstates are

$$\begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \,. \end{cases} \tag{13}$$

These states form a triplet (spin 1).

On the other hand, there is only one possible configuration of the totally antisymmetric state:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}. \tag{14}$$

Since antisymmetry is implied, it must also be a superposition. The one antisymmetric eigenstate is then

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{15}$$

which constitutes a singlet state (spin 0).

2.2 Exact Diagonalization

The Hamiltonian (10) can be solved using exact diagonalization. The computations in this section were performed using Mathematica. The Hamiltonian in matrix form is given by

$$H = \begin{pmatrix} \frac{1}{\epsilon_0-\epsilon_1} & 0 & 0 & 0 \\ 0 & -\frac{1}{\epsilon_0-\epsilon_1} & \frac{2}{\epsilon_0-\epsilon_1} & 0 \\ 0 & \frac{2}{\epsilon_0-\epsilon_1} & -\frac{1}{\epsilon_0-\epsilon_1} & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon_0-\epsilon_1} \end{pmatrix}. \tag{16}$$

Then diagonalizing gives the following eigenvalues and eigenvectors, where we write each eigenvector also in its normalized form in ket notation:

$$\begin{aligned}
\vec{v}_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |\uparrow\uparrow\rangle\,,\; E_1 = \frac{1}{\epsilon_0-\epsilon_1} \\
\vec{v}_2 &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),\; E_2 = \frac{1}{\epsilon_0-\epsilon_1} \\
\vec{v}_3 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |\downarrow\downarrow\rangle\,,\; E_3 = \frac{1}{\epsilon_0-\epsilon_1} \\
\vec{v}_4 &= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),\; E_4 = -\frac{3}{\epsilon_0-\epsilon_1}.
\end{aligned} \tag{17}$$

We see that we reproduce the three triplet states, which are degenerate in energy, as well as the singlet state.

2.3 Bethe Ansatz Solution

The Hamiltonian (10) can also be solved using the Bethe Ansatz described in Sec. 1.2. We must find solutions to the Bethe equations (8) and their corresponding states (6) and energies (9).

For two spins, the Bethe equations become

$$\frac{1}{w_k-\epsilon_0} + \frac{1}{w_k-\epsilon_1} + \sum_{l=1,l\neq k}^{m} \frac{2}{w_l-w_k} = 0. \tag{18}$$

We first consider the case of $m = 1$, i.e. we only have a single parameter w:

$$\frac{1}{w-\epsilon_0} + \frac{1}{w-\epsilon_1} = 0. \tag{19}$$

This is actually enough to reproduce all of our states. One solution is obtained by letting $\omega \to \infty$. In this case, $F(\omega) \to 0$, and we are left with the state

$$|w\rangle = |\uparrow\uparrow\rangle\,. \tag{20}$$

The associated energy (9) is then

$$E = \frac{1}{\epsilon_0-\epsilon_1}. \tag{21}$$

We have reproduced our first state of the triplet with the correct energy. This is the state of highest weight within the triplet (meaning maximum value of the total z-spin operator σ_z). To obtain the other state of the triplet, we may successively apply the total spin lowering operator and normalize:

$$\sigma^- \left|\uparrow\uparrow\right\rangle = (\sigma_0^- \otimes 1 + 1 \otimes \sigma_1^-) \left|\uparrow\uparrow\right\rangle = \frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle) \tag{22}$$

$$\sigma^- (\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle) = \left|\downarrow\downarrow\right\rangle \tag{23}$$

$$\sigma^- \left|\downarrow\downarrow\right\rangle = 0. \tag{24}$$

We have thus obtained our three states of the triplet which have the same energy.

Next, we consider finite solutions to eq. (19). There is only one, and it is

$$w = \frac{\epsilon_0 + \epsilon_1}{2}. \tag{25}$$

We can compute the state associated to this w:

$$\begin{aligned} \left|w\right\rangle &= \sum_i \frac{\sigma_i^-}{w - \epsilon_i} \left|\uparrow\uparrow\right\rangle \\ &= \left(\frac{2\sigma_0^-}{\epsilon_1 - \epsilon_0} + \frac{2\sigma_1^-}{\epsilon_0 - \epsilon_1}\right) \left|\uparrow\uparrow\right\rangle \\ &= \frac{2}{\epsilon_0 - \epsilon_1}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle) \\ &\to \frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle) \end{aligned} \tag{26}$$

where in the last line we normalized the state. We have reproduced our singlet state. Its energy is

$$\begin{aligned} E &= \frac{2}{w - \epsilon_0} + \frac{1}{\epsilon_0 - \epsilon_1} \\ &= \frac{4}{\epsilon_1 - \epsilon_0} + \frac{1}{\epsilon_1 - \epsilon_0} \\ &= -\frac{3}{\epsilon_0 - \epsilon_1} \end{aligned} \tag{27}$$

as expected.

3 N = 3

References

[1] M. Gaudin, "Diagonalisation d'une classe d'hamiltoniens de spin", Journal de Physique **37**, 1087–1098 (1976).

[2] J. Schliemann, A. Khaetskii, and D. Loss, "Electron spin dynamics in quantum dots and related nanostructures due to hyperfine interaction with nuclei", Journal of Physics: Condensed Matter **15**, R1809–R1833 (2003).

[3] J. J. Sakurai and S. F. Tuan, *Modern quantum mechanics*, eng, rev. ed (Addison-Wesley, Reading (Mass.), 1994).