Name: Robert Heeter

EXAM 1 - BIOE 391
Take Home – 2022

This portion of the exam is **open book/open notes**. Any other resources used must be acknowledged. Please **READ ALL INSTRUCTIONS**, manage your time effectively and answer the questions concisely but completely. Submit a zip file with all your documents. The recommended time investment in this take-home exam should be of no more than **5 hours**, although you are allowed to use more.

On my honor, I have neither given nor received any unauthorized aid on this exam.

Signature: Robert Heeter

Read Carefully!

Please comment your code as much as possible. This will help us to grade and give YOU partial credit.

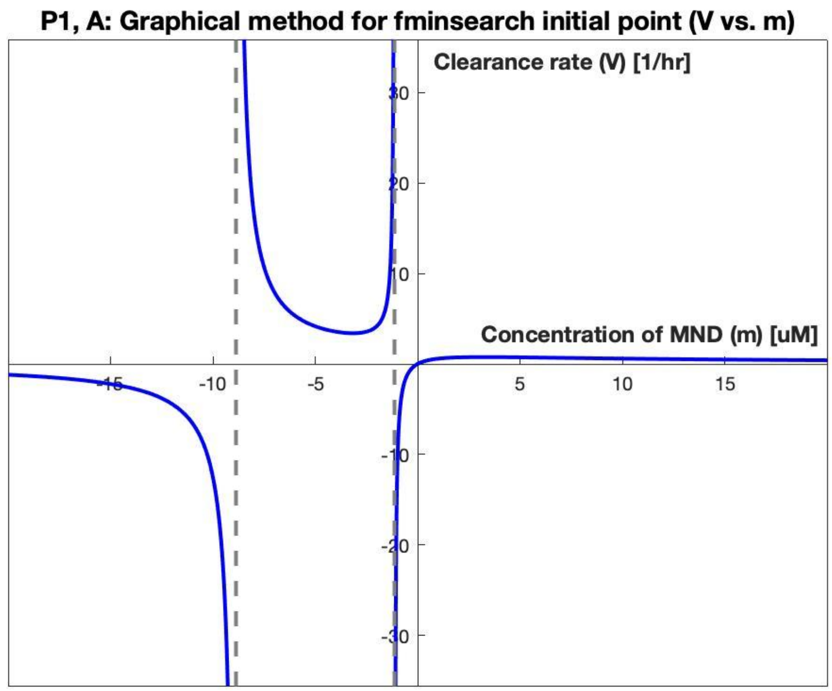
Robert Heeter

BIOE 391 Numerical Methods – Due 25 February 2022

Exam 1 Solutions

1.

a. Figure (for finding initial point)

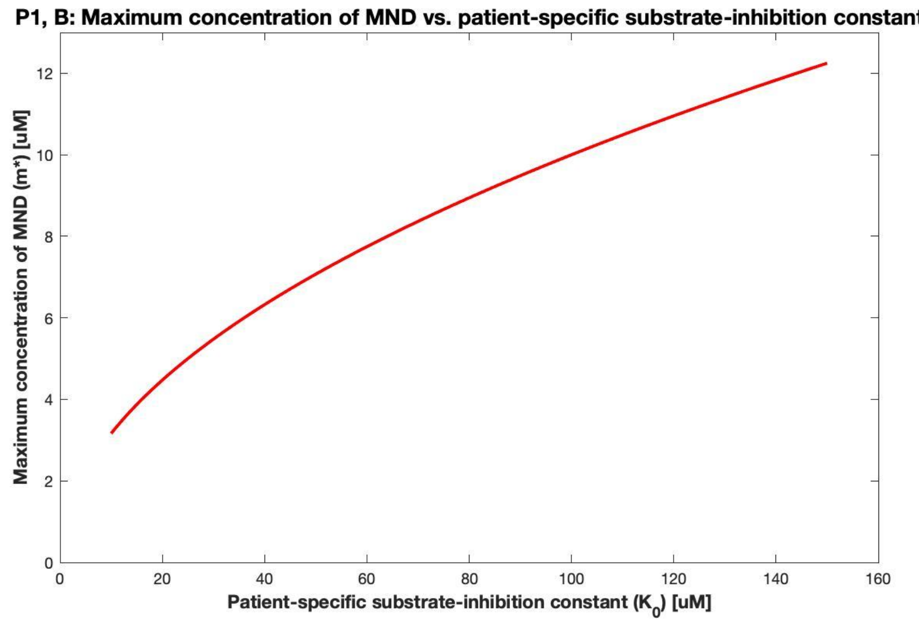


Output

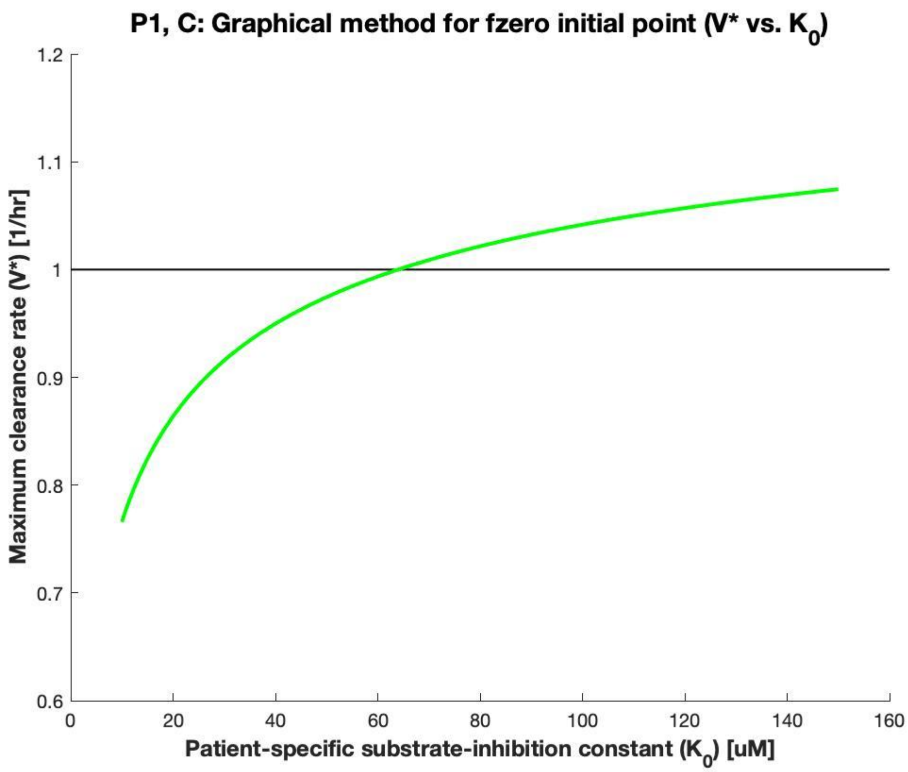
P1,A: Fminsearch method example output:
K_0 [uM] = 10
maximum (m_star, V_star) [uM, 1/hr] = (3.162250, 0.765718)

b. Figure

Robert Heeter
BIOE 391 Numerical Methods – Due 25 February 2022



c. **Figure** (for finding initial point)



Output
P1,C: Fzero method output:
K_0 [uM] = 64.000000
V_star [1/hr] = 1.000000

2.

Robert Heeter
BIOE 391 Numerical Methods – Due 25 February 2022

a. Analytical derivative & comparison

The analytical derivative can be calculated using the product rule:

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

This gives:

$$u'(x) = \frac{du}{dx} = (2x)\left(e^{-0.1x}\right) - 0.1\left(x^2\right)\left(e^{-0.1x}\right)$$

The CDD approximation (given approximate relative error threshold of 0.0001% and iteration limit of 50) for the derivative is very good and matches the analytical results with high accuracy—the true relative errors for the sample set of integer points from *x =1* to *x = 15* are all less than 1e-07. While the true relative error is small for all tested points, it still fluctuates by about one order of magnitude, shown in one of the figures below.

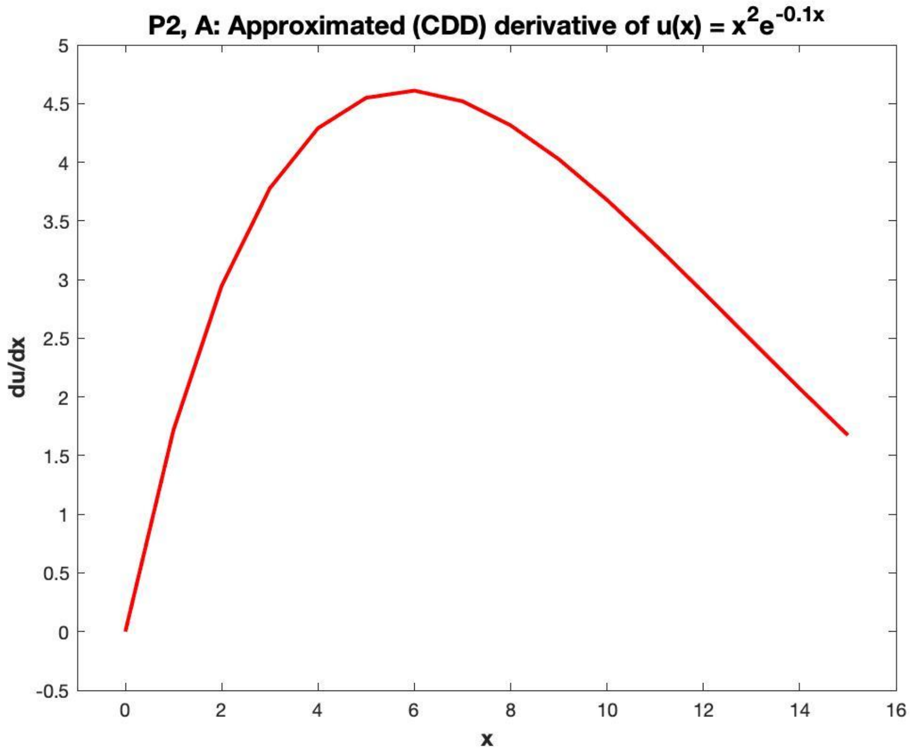
Output

P2,A: Comparing CDD approximation and analytical derivative:

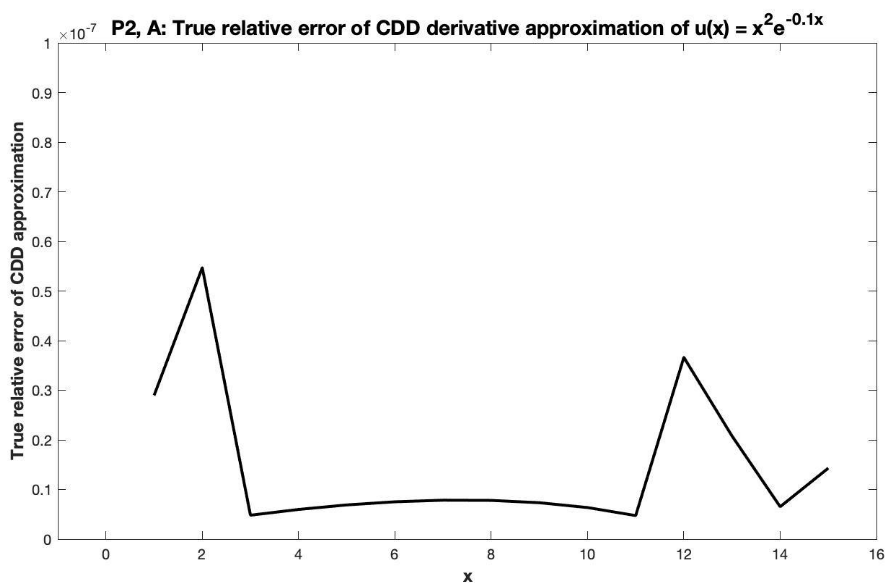
x:	du/dx (approx):	du/dx (exact):	True rel. error:
0.0	0.0000	0.0000	NaN
1.0	1.7192	1.7192	2.8965e-08
2.0	2.9474	2.9474	5.47057e-08
3.0	3.7782	3.7782	4.8133e-09
4.0	4.2900	4.2900	5.97614e-09
5.0	4.5490	4.5490	6.88775e-09
6.0	4.6100	4.6100	7.52041e-09
7.0	4.5189	4.5189	7.83993e-09
8.0	4.3136	4.3136	7.79906e-09
9.0	4.0250	4.0250	7.33533e-09
10.0	3.6788	3.6788	6.35801e-09
11.0	3.2954	3.2954	4.73955e-09
12.0	2.8915	2.8915	3.66213e-08
13.0	2.4800	2.4800	2.0781e-08
14.0	2.0714	2.0714	6.5271e-09
15.0	1.6735	1.6735	1.43052e-08

Figure

Robert Heeter
BIOE 391 Numerical Methods – Due 25 February 2022

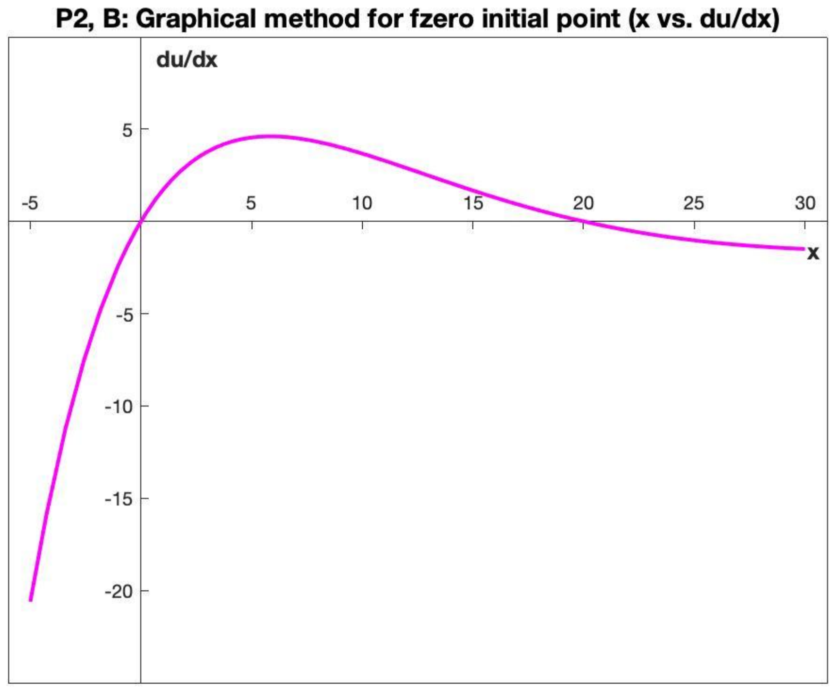


Figure



b. **Figure** (for finding initial point)

Robert Heeter
BIOE 391 Numerical Methods – Due 25 February 2022



Output
P2,B: Fzero method output:
root of dudx = (20.000000, 0.000000)

c. **Output**
P2,C: Fminbnd method output:
max of u(x) = (20.000014, 54.134113)

Comparison
The maximum using the *fminbnd* function occurs at the same *x*-value as the zero of the derivative *du/dx* (using *fzero*). The points with a zero derivative for a continuous function indicate maxima and minima extrema, so this makes sense. A derivative (*du/dx*) of zero also occurs at $x = 0$, seen in the graph of *du/dx* above, though this case was not considered for part C as it corresponds to a minimum of *u(x)*. The point $x = 20$ is known to be a maximum as *du/dx* switches from positive to negative at $x = 20$, which is partly shown in part A ($du/dx > 0$ for $x < 20$) and also in the graph for part B (sign change at $x = 20$).

Robert Heeter
BIOE 391 Numerical Methods – Due 25 February 2022

Complete MATLAB Code

```
% Robert Heeter
% BIOE 391 Numerical Methods
% EXAM 1 MATLAB SCRIPT

clc, clf, clear, close all

%% PROBLEM 1, PART A
disp('PROBLEM 1');

% Constants and equation
V_max = 1.25; % maximum degradation rate (1/hr)
K = 1; % Michaelis-Menten constant characterizing saturation of liver enzymes (uM)
Vm = @(m,K_0) (V_max.*m)./(K+m+(m.^2./K_0)); % kinetic equation as function of m (concentration of MND)
and K_0 (patient-specific substrate-inhibition constant)

% Graphical method to find initial point for fminsearch
figure
K_0_temp = 10;
fplot(@(m) Vm(m,K_0_temp),[-20, 20],'-b','LineWidth',2);
xlabel('Concentration of MND (m) [uM]','FontSize',12,'FontWeight','bold');
ylabel('Clearance rate (V) [1/hr]','FontSize',12,'FontWeight','bold');
title('P1, A: Graphical method for fminsearch initial point (V vs.
m)','FontSize',14,'FontWeight','bold');
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';

guess1 = 0; % graphical method shows approximate maximum near m ~ 0

% Example of fminsearch function, using initial point m ~ 0
K_0 = 10; % sample K_0 value to test vm_fminsearch function
[V_star,m_star] = vm_fminsearch(Vm,guess1,K_0); % vm_fminsearch function written and end of document

% Display results
disp('P1,A: Fminsearch method example output:');
fprintf('K_0 [uM] = %d\nmaximum (m_star, V_star) [uM, 1/hr] = (%f, %f)\n\n',K_0,m_star,V_star);


%% PROBLEM 1, PART B

% Determine points
guess1 = 0; % graphical method in part A shows approximate maximum near m ~ 0
K_0_int = (10:0.1:150)'; % interval of K_0 values
m_star_int = zeros(size(K_0_int)); % preallocate
for i = 1:length(K_0_int)
    [~,m_star_int(i)] = vm_fminsearch(Vm,guess1,K_0_int(i)); % find value of m_star at each K_0 value
end

% Make figure
figure
plot(K_0_int,m_star_int,'-r','LineWidth',2);
xlabel('Patient-specific substrate-inhibition constant (K_0) [uM]','FontSize',12,'FontWeight','bold');
ylabel('Maximum concentration of MND (m*) [uM]','FontSize',12,'FontWeight','bold');
title('P1, B: Maximum concentration of MND vs. patient-specific substrate-inhibition
constant','FontSize',14,'FontWeight','bold');
axis([0 160 0 13]);


%% PROBLEM 1, PART C

% Graphical method to find initial point for fzero
V_star_int = Vm(m_star_int,K_0_int); % use interval of m_star values from part B
```

Robert Heeter

BIOE 391 Numerical Methods – Due 25 February 2022

```
figure
hold on
fplot(1,'-k','LineWidth',1);
plot(K_0_int,V_star_int,'-g','LineWidth',2);
xlabel('Patient-specific substrate-inhibition constant (K_0) [uM]','FontSize',12,'FontWeight','bold');
ylabel('Maximum clearance rate (V*) [1/hr]','FontSize',12,'FontWeight','bold');
title('P1, C: Graphical method for fzero initial point (V* vs.
K_0)','FontSize',14,'FontWeight','bold');
axis([0 160 0.6 1.2]);
hold off

guess2 = 65; % graphical method shows V* ~ 1 near K_0 = 65

% Use fzero to find K_0 that gives V* = 1.0/hr
[K_0_crit,V_star_rel] = fzero(@(K_0) 1-(vm_fminsearch(Vm,guess1,K_0)),guess2); % minimize (1-maximum of
Vm) depending on K_0

% Display results
disp('P1,C: Fzero method output:')
fprintf('K_0 [uM] = %f\nV_star [1/hr] = %f\n\n',K_0_crit,V_star_rel+1);


%% PROBLEM 2, PART A
disp('PROBLEM 2');

% Example shear strain equation and analytical derivative
ux = @(x) (x.^2).*exp(-0.1.*x);
dudx = @(x) (2.*x.*exp(-0.1.*x)) - (0.1.*(x.^2).*exp(-0.1.*x));

% Determine points
x_int = (0:1:15)';
dudx_cdd = zeros(size(x_int));
for i = 1:length(x_int)
    dudx_cdd(i) = shearstrain(ux,x_int(i));
end
dudx_exact = dudx(x_int);
er = abs(dudx_exact - dudx_cdd)./dudx_exact;

% Display results
disp('P2,A: Comparing CDD approximation and analytical derivative:') % display results
fprintf(' x:        du/dx (approx):   du/dx (exact):   True rel. error: \n');
fprintf(' %4.1f     %7.4f           %7.4f           %g\n',[x_int,dudx_cdd,dudx_exact,er]');
disp(' ');

% Plot results
figure
plot(x_int,dudx_cdd,'-r','LineWidth',2);
xlabel('x','FontSize',12,'FontWeight','bold');
ylabel('du/dx','FontSize',12,'FontWeight','bold');
title('P2, A: Approximated (CDD) derivative of u(x) = x^2e^{-0.1x}','FontSize',14,'FontWeight','bold');
axis([-1 16 -0.5 5]);

figure
plot(x_int,er,'-k','LineWidth',2);
xlabel('x','FontSize',12,'FontWeight','bold');
ylabel('True relative error of CDD approximation','FontSize',12,'FontWeight','bold');
title('P2, A: True relative error of CDD derivative approximation of u(x) =
x^2e^{-0.1x}','FontSize',14,'FontWeight','bold');
axis([-1 16 0 1e-7]);


%% PROBLEM 2, PART B

% Graphical method to find initial point for fzero
figure
fplot(dudx,[-5,30],'-m','LineWidth',2);
```

Robert Heeter


```
xlabel('x','FontSize',12,'FontWeight','bold');
ylabel('du/dx','FontSize',12,'FontWeight','bold');
title('P2, B: Graphical method for fzero initial point (x vs.
du/dx)','FontSize',14,'FontWeight','bold');
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
axis([-6 31 -25 10]);

guess1 = 20; % graphical method shows du/dx ~ 0 near x = 20

% Use fzero to find du/dx = 0
[x_root,dudx_root] = fzero((@(x) shearstrain(ux,x)),guess1); % minimize shearstrain function (CDD
derivative approximation)

% Display results
disp('P2,B: Fzero method output:');
fprintf('root of dudx = (%f, %f)\n\n',x_root,dudx_root);


%% PROBLEM 2, PART C

% Use fminbnd to find maximum of u(x)
guess_lower = 10; % upper and lower guesses given maximum occurs around root (x = 20);
guess_upper = 30;

[x_max,ux_max] = fminbnd(@(x) -1*ux(x),guess_lower,guess_upper); % use fminbnd for negative u(x) for
maximum
ux_max = -1*ux_max; % ux_max is -1*fminbnd output

% Display results
disp('P2,C: Fminbnd method output:');
fprintf('max of u(x) = (%f, %f)\n\n',x_max,ux_max);


%% Additional Functions

function [V_star, m_star] = vm_fminsearch(Vm,guess,K_0)
% ABOUT: Implements fminsearch to find maximum of function Vm near an
% initial value.
% INPUTS: Vm = function; guess = initial point; K_0 = K_0 parameter for Vm
% OUTPUTS: V_star = Vm-value at maximum; m_star = m-value at maximum

[m_star,V_temp] = fminsearch(@(m) -1*Vm(m,K_0),guess); % use fminsearch for negative function for
maximum
V_star = -1*V_temp; % V_star is -1*fminbnd output

end

function [dudx] = shearstrain(ux,x0,h,ea)
% ABOUT: Implements centered divided-difference approximation for
% derivative of u(x) at x0 for incrementally smaller step sizes (h)
% INPUTS: ux = function; x0 = point for derivative; h = step size; ea =
% approximate relative error threshold
% OUTPUTS: dudx = derivative approximation at x0

% Check inputs
if nargin < 2 || isempty(ux) || isempty(x0)
    error('At least 2 input arguments required.')
end

if nargin < 3 || isempty(h)
    if x0 ~= 0
        h = abs(x0)/10;
    else
        h = 1;
```

```
    end
end

if nargin < 4 || isempty(ea)
    ea = 0.0001;
end

% Iterate CDD until iteration or error thresholds are reached
dudx_old = (ux(x0+h) - ux(x0-h))/(2*h); % centered divided difference formula
maxit = 50; % maximum number of iterations
iter = 0;
er = 100;

while (1)
    h = h/2;
    iter = iter+1;
    dudx = (ux(x0+h) - ux(x0-h))/(2*h); % centered divided difference formula
    if dudx ~= 0
        er = abs((dudx-dudx_old)/dudx)*100;
    end
    if er < ea || iter >= maxit
        break
    end
    dudx_old = dudx;
    h = h/2;
end

end
```