

2/17/2023

Problem Set #5

$$1. a. f(x) = x^2 \rightarrow \nabla f(x) = \left[\frac{d}{dx} x^2 \right] = [2x]$$

$$x[0] = -10$$

See code below and graph.

- b. The number of iterations to reach the minimum at $x=0$ does not depend on the initial guess in this case. See code below and graphs for $x[0] = 0.5$ and $x[0] = -1 \times 10^6$; both cases approach the minimum within 50 iterations.

For this function $f(x) = x^2$, the gradient ("slope") becomes larger as you move away from the minimum at $x=0$ $\rightarrow \nabla f(x) = [2x]$ which increases or decreases linearly with x . Thus, at extremely large (positive or negative) initial values, the steep gradient (large $\nabla f(x) = [2x]$) causes a faster descent towards the minimum.

- c. The learning rate must be < 1 to find the minimizer $x=0$. See code below and graphs for step sizes (learning rates) of 0.9, 1, and 1.1. At steps < 1 , the descent "jumps" across the minimum but still descends towards it. At a step $= 1$, the descent never reaches the minimizer:

$$x_1 = x_0 + \mu \nabla f(x_0)$$

$$\nabla f(x) = 2x$$

$$x_1 = (-10) + (1)(2(-10)) = +10$$

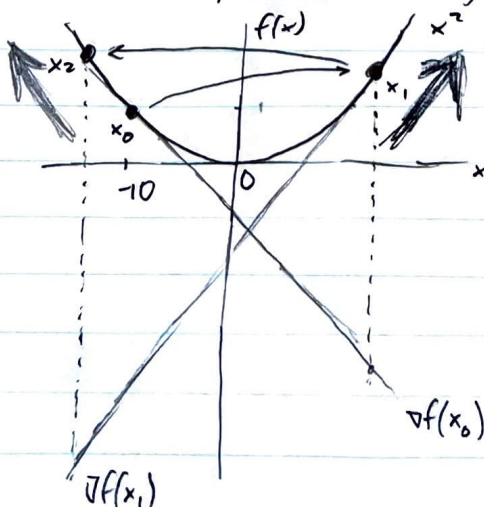
$$x_2 = (+10) - (1)(2(+10)) = -10$$

$$x_3 = (-10) - (1)(2(-10)) = +10$$

At $\mu=1$, the descent jumps between ± 10 each iteration since the step is too large

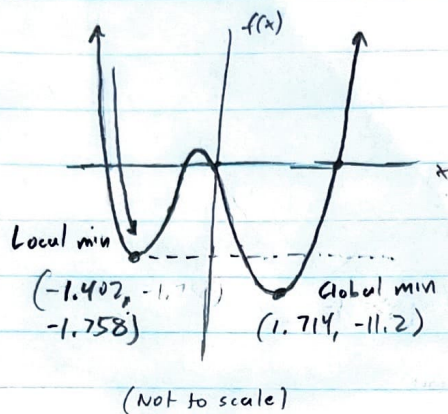
...

At steps > 1 , the curve diverges as it moves away from the minimizer:



The large learning rate causes the descent to overshoot the minimum and instead diverge away from it (shown as \Rightarrow arrows).

(1).d. The function $f(x) = x^4 - 5x^2 - 3x$ has 2 minima:



$$\nabla f(x) = \left[\frac{d}{dx} (x^4 - 5x^2 - 3x) \right] = [4x^3 - 10x - 3]$$
$$x[0] = -10$$

The gradient descent for $f(x)$ here only finds the local minimum, not the global minimum with a starting guess of $x[0] = -10$, which differs from part A since $f(x) = x^2$ only has 1 minimum.

This descent also requires a much smaller step size - see code below and graph. The step of 0.0005 is required because of the larger magnitude (steepness) of the gradient $4x^3 - 10x - 3$ (3rd power gradient compared to 1st power gradient in part A). The smaller step size is necessary to prevent the descent from overshooting the minimum.

To find the global minimum, a positive initial guess $x[0]$ is required to descend its "well", looking at the graph above.

PROBLEM 2 OUTPUT

RAW CLUSTER S	SIMPLIFIED CLUSTER ANNOTATIONS
[33 35]	#0 cluster: 5,5
[6 26]	#1 cluster: 6,6
[11 21]	#2 cluster: 1,1
[19 31]	#3 cluster: 9,9
[20 36]	#4 cluster: 0,0
[0 30]	#5 cluster: 0,0
[10 44]	#6 supercluster: 0 with cluster [0, 0]
[29 43]	#7 supercluster: 9 with cluster [9, 9]
[16 34]	#8 cluster: 6,6
[8 28]	#9 cluster: 8,8
[12 22]	#10 cluster: 2,2
[18 38]	#11 cluster: 8,8
[4 24]	#12 cluster: 4,4
[13 23]	#13 cluster: 3,3
[15 32]	#14 cluster: 5,5
[5 9]	#15 cluster: 5,9
[3 39]	#16 cluster: 3,9
[14 52]	#17 supercluster: 4 with cluster [4, 4]
[45 46]	#18 supercluster: cluster[0, 0] with cluster [0, [0, 0]]
[40 54]	#19 supercluster: cluster[5, 5] with cluster [5, 5]
[37 47]	#20 supercluster: 9 with cluster [9, [9, 9]]
[17 27]	#21 cluster: 7,7
[1 42]	#22 supercluster: 1 with cluster [1, 1]
[25 51]	#23 supercluster: 5 with cluster [8, 8]
[41 48]	#24 supercluster: cluster[6, 6] with cluster [6, 6]
[56 60]	#25 supercluster: cluster[3, 9] with cluster [9, [9, [9, 9]]]
[2 49]	#26 supercluster: 2 with cluster [8, 8]

[7 61]	#27 supercluster: 7 with cluster [7, 7]
[55 65]	#28 supercluster: cluster[5, 9] with cluster [[3, 9], [9, [9, [9, 9]]]]
[53 67]	#29 supercluster: cluster[3, 3] with cluster [7, [7, 7]]
[59 63]	#30 supercluster: cluster[[5, 5], [5, 5]] with cluster [5, [8, 8]]
[66 69]	#31 supercluster: cluster[2, [8, 8]] with cluster [[3, 3], [7, [7, 7]]]
[57 62]	#32 supercluster: cluster[4, [4, 4]] with cluster [1, [1, 1]]
[50 68]	#33 supercluster: cluster[2, 2] with cluster [[5, 9], [[3, 9], [9, [9, [9, 9]]]]]
[64 72]	#34 supercluster: cluster[[6, 6], [6, 6]] with cluster [[4, [4, 4]], [1, [1, 1]]]
[70 71]	#35 supercluster: cluster[[[5, 5], [5, 5]], [5, [8, 8]]] with cluster [[2, [8, 8]], [[3, 3], [7, [7, 7]]]]
[58 75]	#36 supercluster: cluster[[0, 0], [0, [0, 0]]] with cluster [[[5, 5], [5, 5]], [5, [8, 8]]], [[2, [8, 8]], [[3, 3], [7, [7, 7]]]]]
[73 76]	#37 supercluster: cluster[[2, 2], [[5, 9], [[3, 9], [9, [9, [9, 9]]]]]] with cluster [[[0, 0], [0, [0, 0]]], [[[5, 5], [5, 5]], [5, [8, 8]], [[2, [8, 8]], [[3, 3], [7, [7, 7]]]]]]]
[74 77]	#38 supercluster: cluster[[[6, 6], [6, 6]], [[4, [4, 4]], [1, [1, 1]]]] with cluster [[[2, 2], [[5, 9], [[3, 9], [9, [9, [9, 9]]]]]], [[[0, 0], [0, [0, 0]]], [[[5, 5], [5, 5]], [5, [8, 8]], [[2, [8, 8]], [[3, 3], [7, [7, 7]]]]]]]]]

DETAILED CLUSTER ANNOTATIONS

```
#0 cluster[33 35]: 5,5
#1 cluster[ 6 26]: 6,6
#2 cluster[11 21]: 1,1
#3 cluster[19 31]: 9,9
#4 cluster[20 36]: 0,0
#5 cluster[ 0 30]: 0,0
#6 supercluster[10 44]: 0 with cluster[4] (cluster[20 36]: 0,0)
#7 supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9)
#8 cluster[16 34]: 6,6
#9 cluster[ 8 28]: 8,8
#10 cluster[12 22]: 2,2
```

#11 cluster[18 38]: 8,8

#12 cluster[4 24]: 4,4

#13 cluster[13 23]: 3,3

#14 cluster[15 32]: 5,5

#15 cluster[5 9]: 5,9

#16 cluster[3 39]: 3,9

#17 supercluster[14 52]: 4 with cluster[12] (cluster[4 24]: 4,4)

#18 supercluster[45 46]: cluster[5] (cluster[0 30]: 0,0) with cluster[6] (supercluster[10 44]: 0 with cluster[4] (cluster[20 36]: 0,0))

#19 supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)

#20 supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9))

#21 cluster[17 27]: 7,7

#22 supercluster[1 42]: 1 with cluster[2] (cluster[11 21]: 1,1)

#23 supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8)

#24 supercluster[41 48]: cluster[1] (cluster[6 26]: 6,6) with cluster[8] (cluster[16 34]: 6,6)

#25 supercluster[56 60]: cluster[16] (cluster[3 39]: 3,9) with cluster[20] (supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9)))

#26 supercluster[2 49]: 2 with cluster[9] (cluster[8 28]: 8,8)

#27 supercluster[7 61]: 7 with cluster[21] (cluster[17 27]: 7,7)

#28 supercluster[55 65]: cluster[15] (cluster[5 9]: 5,9) with cluster[25] (supercluster[56 60]: cluster[16] (cluster[3 39]: 3,9) with cluster[20] (supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9))))

#29 supercluster[53 67]: cluster[13] (cluster[13 23]: 3,3) with cluster[27] (supercluster[7 61]: 7 with cluster[21] (cluster[17 27]: 7,7))

#30 supercluster[59 63]: cluster[19] (supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)) with cluster[23] (supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8))

#31 supercluster[66 69]: cluster[26] (supercluster[2 49]: 2 with cluster[9] (cluster[8 28]: 8,8)) with cluster[29] (supercluster[53 67]: cluster[13] (cluster[13 23]: 3,3) with cluster[27] (supercluster[7 61]: 7 with cluster[21] (cluster[17 27]: 7,7)))

#32 supercluster[57 62]: cluster[17] (supercluster[14 52]: 4 with cluster[12] (cluster[4 24]: 4,4)) with cluster[22] (supercluster[1 42]: 1 with cluster[2] (cluster[11 21]: 1,1))

#33 supercluster[50 68]: cluster[10] (cluster[12 22]: 2,2) with cluster[28] (supercluster[55 65]: cluster[15] (cluster[5 9]: 5,9) with cluster[25] (supercluster[56 60]: cluster[16] (cluster[3 39]: 3,9) with cluster[20] (supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9))))))

#34 supercluster[64 72]: cluster[24] (supercluster[41 48]: cluster[1] (cluster[6 26]: 6,6) with cluster[8] (cluster[16 34]: 6,6)) with cluster[32] (supercluster[57 62]: cluster[17] (supercluster[14 52]: 4 with cluster[12] (cluster[4 24]: 4,4)) with cluster[22] (supercluster[1 42]: 1 with cluster[2] (cluster[11 21]: 1,1)))

#35 supercluster[70 71]: cluster[30] (supercluster[59 63]: cluster[19] (supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)) with cluster[23] (supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8))) with cluster[31] (supercluster[66 69]: cluster[26] (supercluster[2 49]: 2 with cluster[9] (cluster[8 28]: 8,8)) with cluster[29] (supercluster[53 67]: cluster[13] (cluster[13 23]: 3,3) with cluster[27] (supercluster[7 61]: 7 with cluster[21] (cluster[17 27]: 7,7))))

#36 supercluster[58 75]: cluster[18] (supercluster[45 46]: cluster[5] (cluster[0 30]: 0,0) with cluster[6] (supercluster[10 44]: 0 with cluster[4] (cluster[20 36]: 0,0))) with cluster[35] (supercluster[70 71]: cluster[30] (supercluster[59 63]: cluster[19] (supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)) with cluster[23] (supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8))) with cluster[31] (supercluster[66 69]: cluster[26] (supercluster[2 49]: 2 with cluster[9] (cluster[8 28]: 8,8)) with cluster[29] (supercluster[53 67]: cluster[13] (cluster[13 23]: 3,3) with cluster[27] (supercluster[7 61]: 7 with cluster[21] (cluster[17 27]: 7,7))))

#37 supercluster[73 76]: cluster[33] (supercluster[50 68]: cluster[10] (cluster[12 22]: 2,2) with cluster[28] (supercluster[55 65]: cluster[15] (cluster[5 9]: 5,9) with cluster[25] (supercluster[56 60]: cluster[16] (cluster[3 39]: 3,9) with cluster[20] (supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9)))))) with cluster[36] (supercluster[58 75]: cluster[18] (supercluster[45 46]: cluster[5] (cluster[0 30]: 0,0) with cluster[6] (supercluster[10 44]: 0 with cluster[4] (cluster[20 36]: 0,0))) with cluster[35] (supercluster[70 71]: cluster[30] (supercluster[59 63]: cluster[19] (supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)) with cluster[23] (supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8))) with cluster[31] (supercluster[66 69]: cluster[26] (supercluster[2 49]: 2 with cluster[9] (cluster[8 28]: 8,8)) with cluster[29] (supercluster[53 67]: cluster[13] (cluster[13 23]: 3,3) with cluster[27] (supercluster[7 61]: 7 with cluster[21] (cluster[17 27]: 7,7))))

#38 supercluster[74 77]: cluster[34] (supercluster[64 72]: cluster[24] (supercluster[41 48]: cluster[1] (cluster[6 26]: 6,6) with cluster[8] (cluster[16 34]: 6,6)) with cluster[32] (supercluster[57 62]: cluster[17] (supercluster[14 52]: 4 with cluster[12] (cluster[4 24]: 4,4)) with cluster[22] (supercluster[1 42]: 1 with cluster[2] (cluster[11 21]: 1,1))) with cluster[37] (supercluster[73 76]: cluster[33] (supercluster[50 68]: cluster[10] (cluster[12 22]: 2,2) with cluster[28] (supercluster[55 65]: cluster[15] (cluster[5 9]: 5,9) with cluster[25] (supercluster[56 60]: cluster[16] (cluster[3 39]: 3,9) with cluster[20] (supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9)))))) with cluster[36] (supercluster[58 75]: cluster[18] (supercluster[45 46]: cluster[5] (cluster[0 30]: 0,0) with cluster[6] (supercluster[10 44]: 0 with cluster[4] (cluster[20 36]: 0,0))) with cluster[35] (supercluster[70 71]: cluster[30] (supercluster[59 63]: cluster[19] (supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)) with cluster[23] (supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8))) with cluster[31] (supercluster[66 69]: cluster[26] (supercluster[2 49]: 2 with cluster[9] (cluster[8 28]: 8,8)) with cluster[29] (supercluster[53 67]: cluster[13]

```
(cluster[13 23]: 3,3) with cluster[27] (supercluster[ 7 61]: 7 with cluster[21] (cluster[17  
27]: 7,7))))))
```


ELEC378-HW5

February 17, 2023

```
[1]: # ROBERT HEETER
      # ELEC 378 Machine Learning
      # 17 February 2023

      # PROBLEM SET 5
```

```
[2]: # PROBLEM 1

import numpy as np
import matplotlib.pyplot as plt

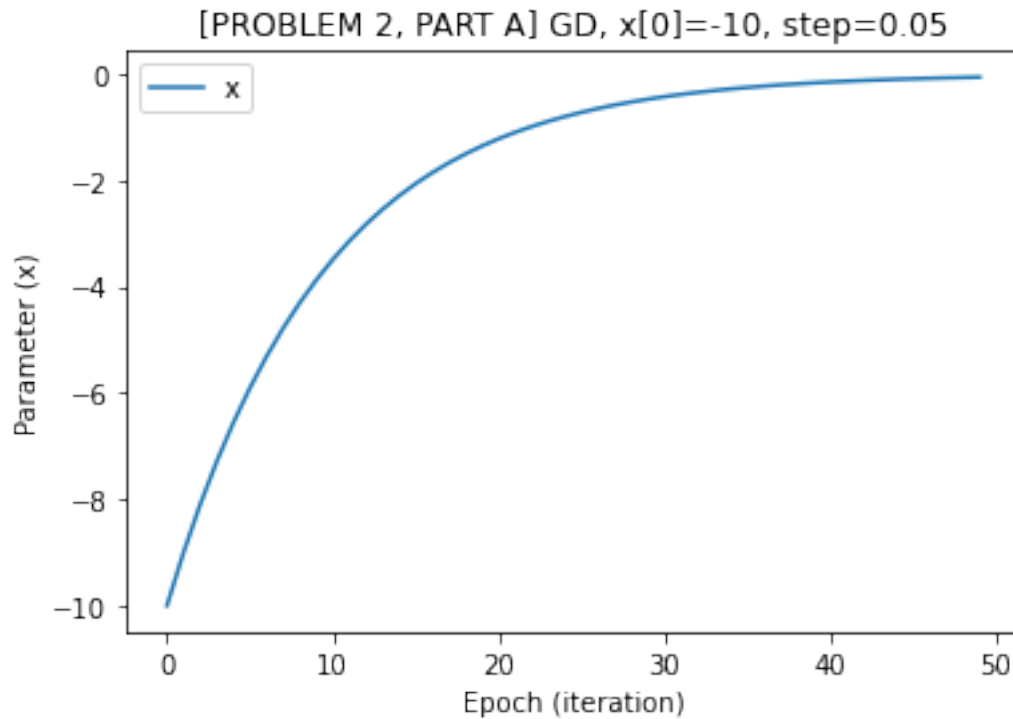
# PART A

T = 50 # number of epochs
x = np.empty((T, 1))
x[0] = -10 # initial guess

grad_L = lambda x: np.array([2*x[0]])
mu = 0.05

for t in range(1,T):
    x[t] = x[t-1] - mu*grad_L(x[t-1])

fig,ax = plt.subplots(1,1)
ax.plot(x[:,0])
ax.set_xlabel('Epoch (iteration)')
ax.set_ylabel('Parameter (x)')
ax.legend(['x'])
plt.title('[PROBLEM 2, PART A] GD, x[0]=-10, step=0.05')
plt.show()
```

```
[3]: # PART B

x = []
T = 50 # number of epochs
x = np.empty((T, 1))
x[0] = 0.5 # initial guess

grad_L = lambda x: np.array([2*x[0]])
mu = 0.05

for t in range(1,T):
    x[t] = x[t-1] - mu*grad_L(x[t-1])

fig,ax = plt.subplots(1,1)
ax.plot(x[:,0])
ax.set_xlabel('Epoch (iteration)')
ax.set_ylabel('Parameter (x)')
ax.legend(('x'))
plt.title('[PROBLEM 2, PART B] GD, x[0]=+0.5, step=0.05')
plt.show()

x = []
T = 50 # number of epochs
```

```

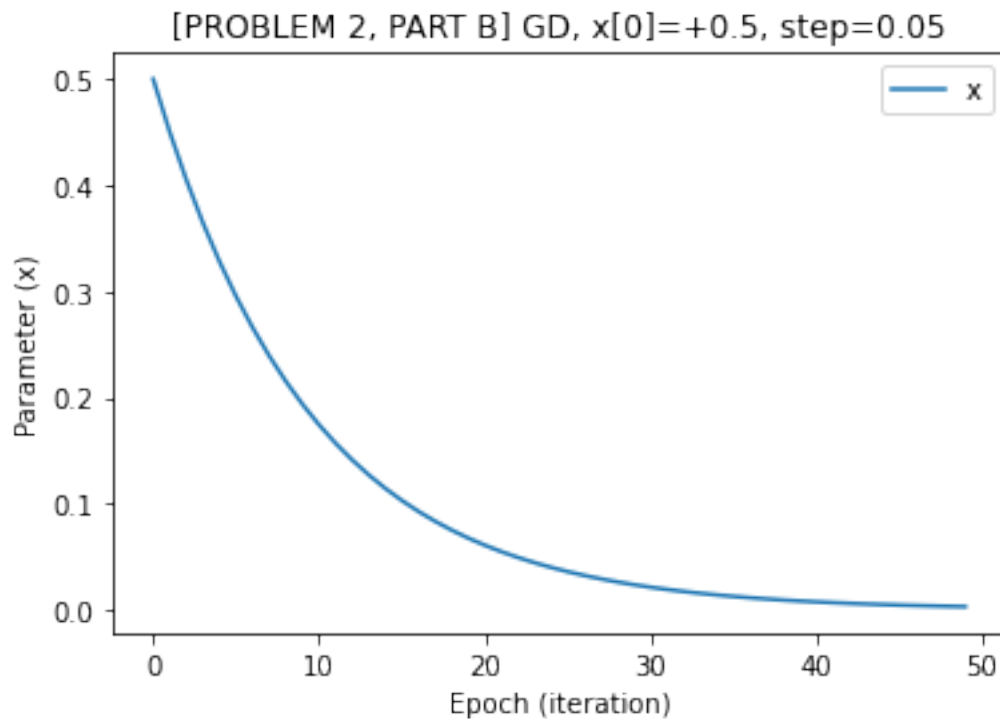
x = np.empty((T, 1))
x[0] = -1*(10**6) # initial guess

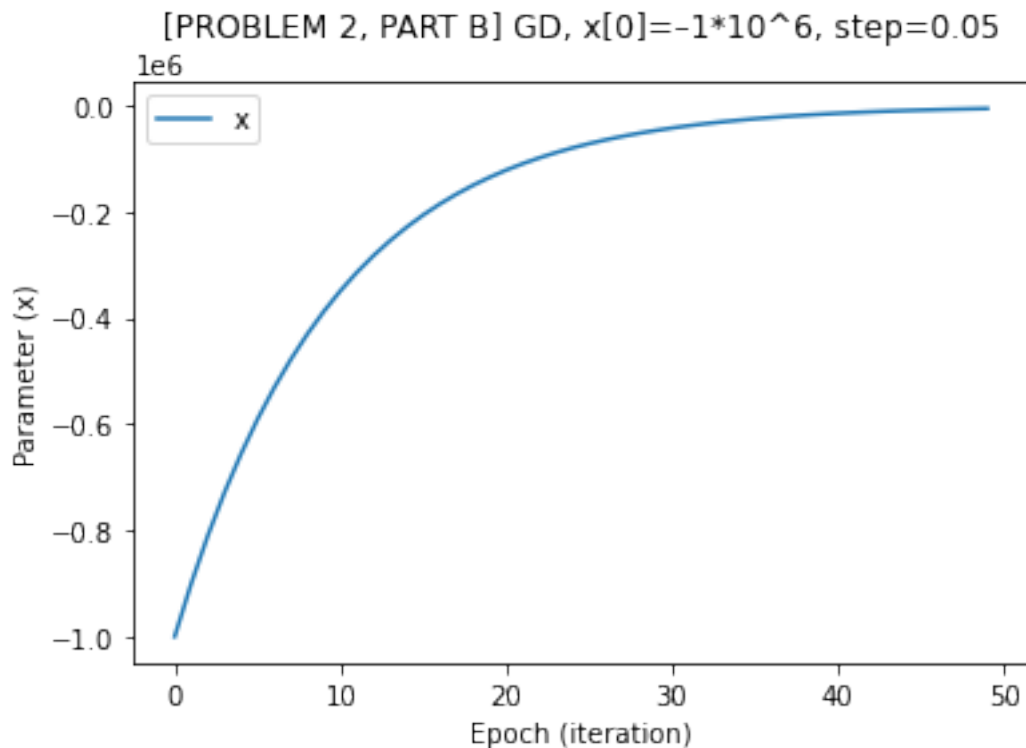
grad_L = lambda x: np.array([2*x[0]])
mu = 0.05

for t in range(1,T):
    x[t] = x[t-1] - mu*grad_L(x[t-1])

fig,ax = plt.subplots(1,1)
ax.plot(x[:,0])
ax.set_xlabel('Epoch (iteration)')
ax.set_ylabel('Parameter (x)')
ax.legend(['x'])
plt.title('[PROBLEM 2, PART B] GD, x[0]=-1*10^6, step=0.05')
plt.show()

```





```
[4]: # PART C

x = []
T = 50 # number of epochs
x = np.empty((T, 1))
x[0] = -10 # initial guess

grad_L = lambda x: np.array([2*x[0]])
mu = 0.9

for t in range(1,T):
    x[t] = x[t-1] - mu*grad_L(x[t-1])

fig,ax = plt.subplots(1,1)
ax.plot(x[:,0])
ax.set_xlabel('Epoch (iteration)')
ax.set_ylabel('Parameter (x)')
ax.legend(['x'])
plt.title('[PROBLEM 2, PART C] GD, x[0]=-10, step=0.9')
plt.show()

x = []
T = 50 # number of epochs
```

```

x = np.empty((T, 1))
x[0] = -10 # initial guess

grad_L = lambda x: np.array([2*x[0]])
mu = 1.1

for t in range(1,T):
    x[t] = x[t-1] - mu*grad_L(x[t-1])

fig,ax = plt.subplots(1,1)
ax.plot(x[:,0])
ax.set_xlabel('Epoch (iteration)')
ax.set_ylabel('Parameter (x)')
ax.legend(('x'))
plt.title('[PROBLEM 2, PART C] GD, x[0]=-10, step=1.1')
plt.show()

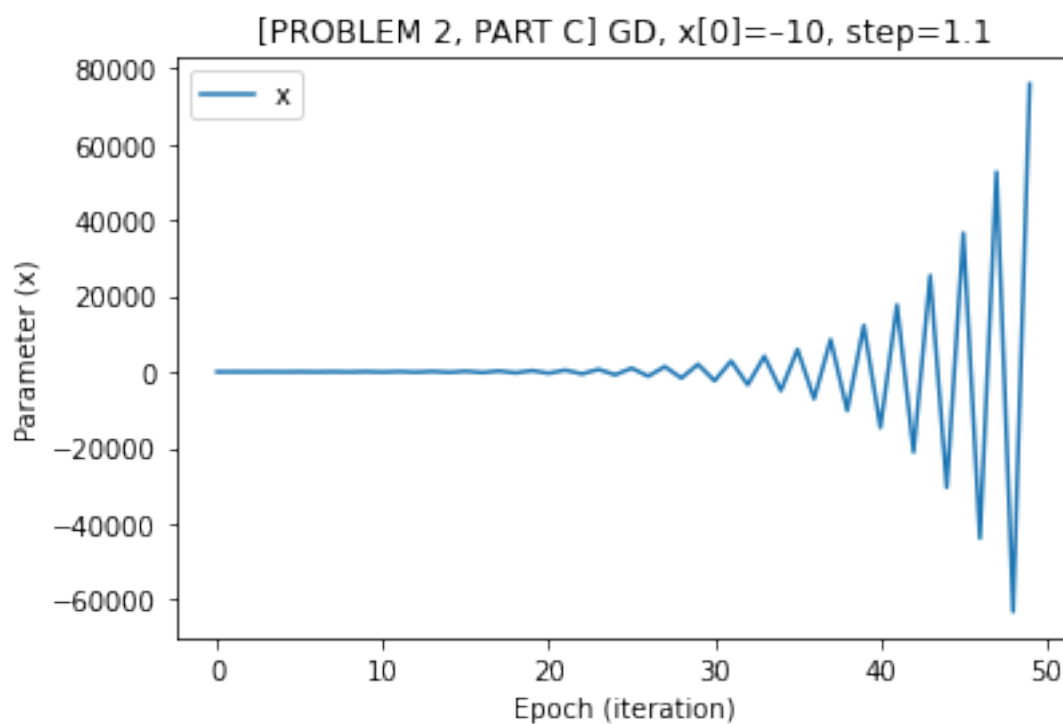
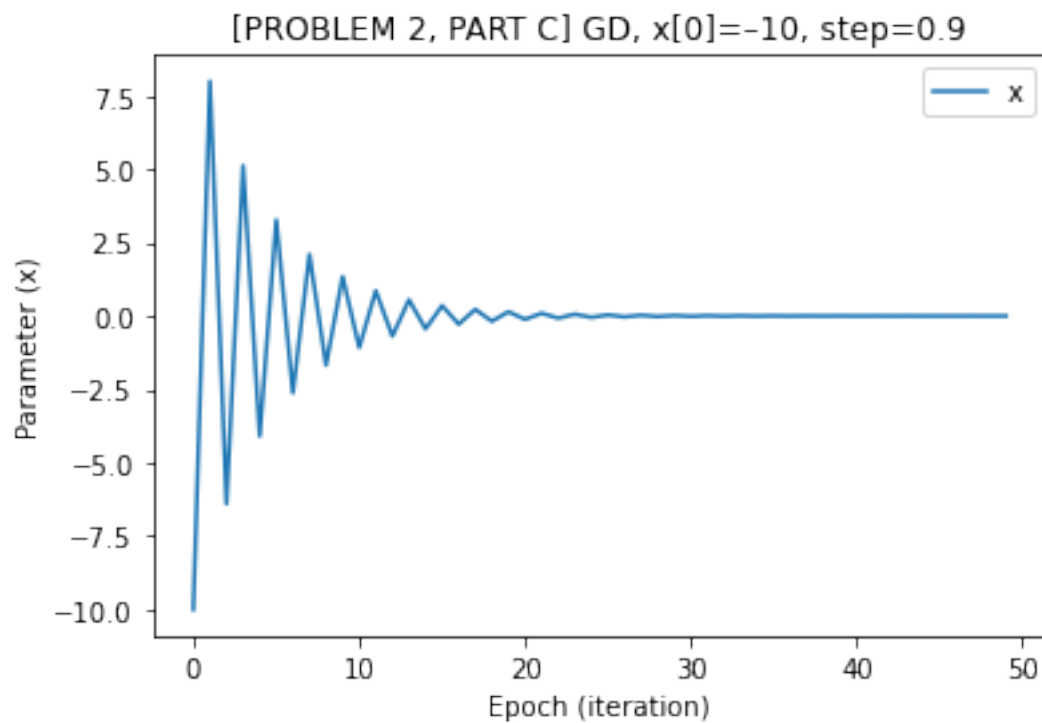
x = []
T = 50 # number of epochs
x = np.empty((T, 1))
x[0] = -10 # initial guess

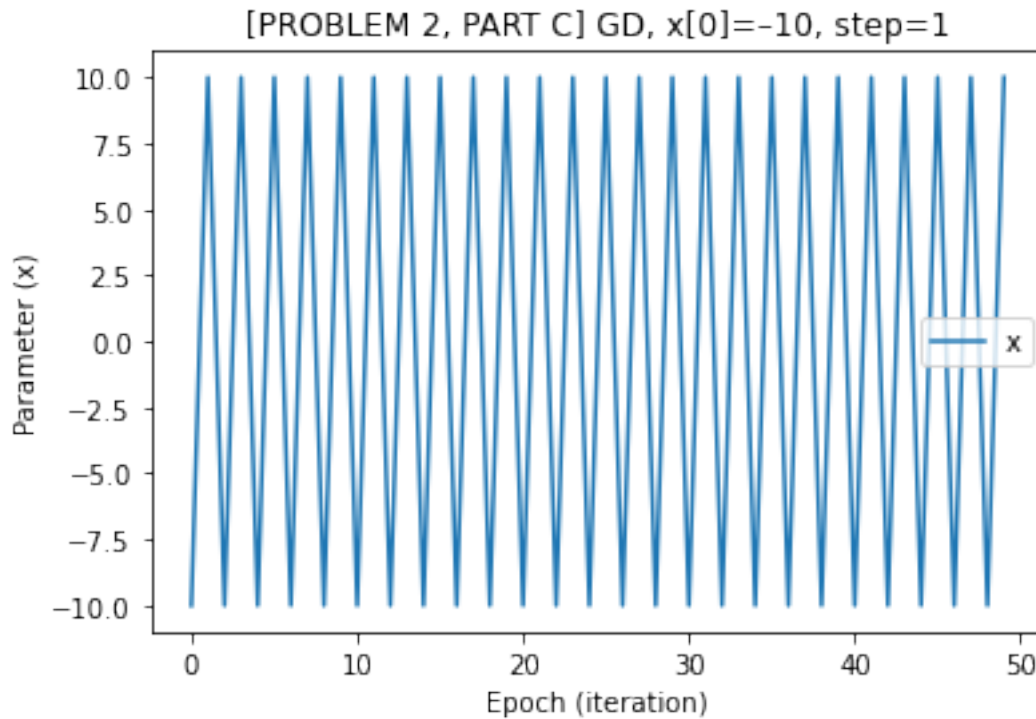
grad_L = lambda x: np.array([2*x[0]])
mu = 1

for t in range(1,T):
    x[t] = x[t-1] - mu*grad_L(x[t-1])

fig,ax = plt.subplots(1,1)
ax.plot(x[:,0])
ax.set_xlabel('Epoch (iteration)')
ax.set_ylabel('Parameter (x)')
ax.legend(('x'))
plt.title('[PROBLEM 2, PART C] GD, x[0]=-10, step=1')
plt.show()

```





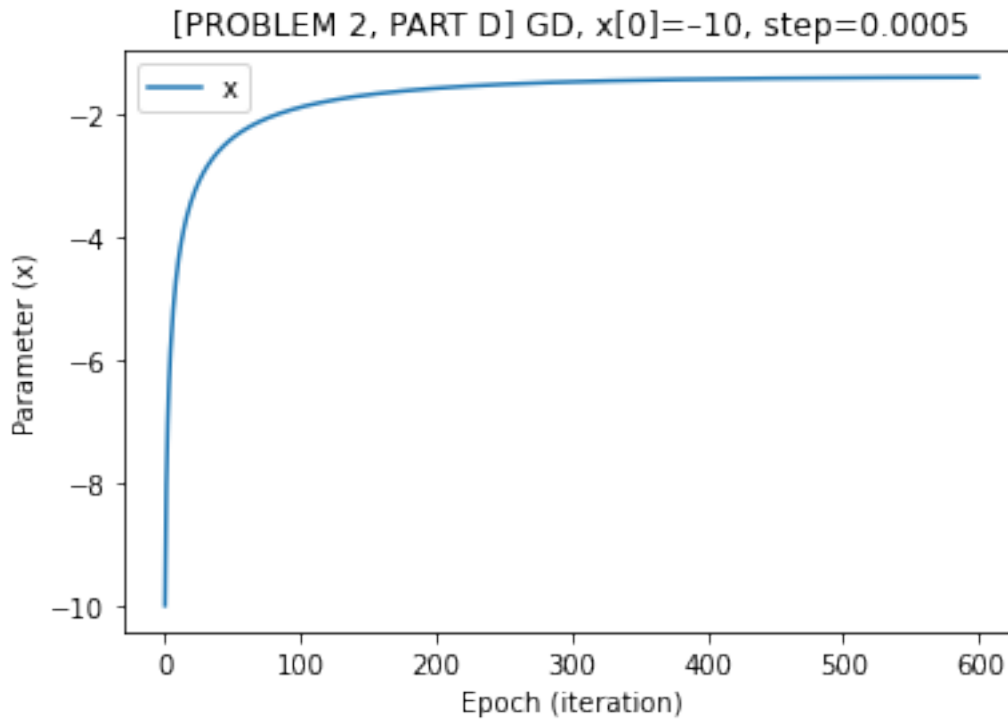
```
[5]: # PART D

x = []
T = 600 # number of epochs
x = np.empty((T, 1))
x[0] = -10 # initial guess

grad_L = lambda x: np.array([4*(x[0]**3) - 10*x[0] - 3])
mu = 0.0005

for t in range(1,T):
    x[t] = x[t-1] - mu*grad_L(x[t-1])

fig,ax = plt.subplots(1,1)
ax.plot(x[:,0])
ax.set_xlabel('Epoch (iteration)')
ax.set_ylabel('Parameter (x)')
ax.legend(('x'))
plt.title('[PROBLEM 2, PART D] GD,  $x[0]=-10$ , step=0.0005')
plt.show()
```

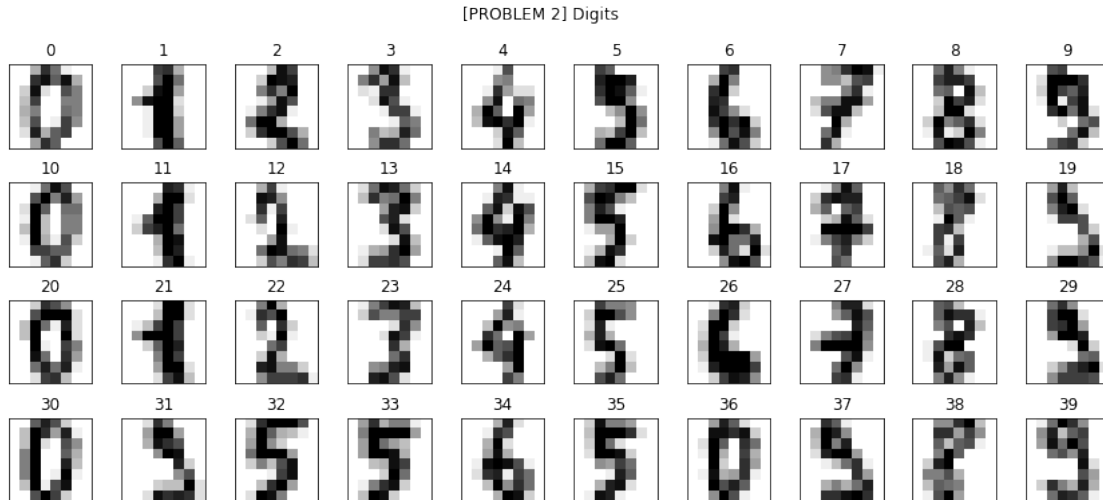


```
[6]: # PROBLEM 2

import numpy as np
import matplotlib.pyplot as plt

# load the data matrix containing as each row an unwrapped
# array representation of an image of a handwritten digit
digits = np.load('digits.npy')

# show digits
fig, axes = plt.subplots(4, 10, figsize=(15, 6), subplot_kw={'xticks': [],
    ↳ 'yticks': []}, gridspec_kw=dict(hspace=0.4, wspace=0.1))
fig.suptitle('[PROBLEM 2] Digits')
for i, ax in enumerate(axes.flat):
    ax.set_title(i)
    ax.imshow(digits[i].reshape(8, 8), cmap='binary',
    ↳ interpolation='nearest', clim=(0, 16))
plt.show()
```

```
[7]: from sklearn.cluster import AgglomerativeClustering
```

```
X = digits
```

```
clst = AgglomerativeClustering(n_clusters=None, distance_threshold=0).fit(X)
```

```
[8]: d = _
→ [0,1,2,3,4,5,6,7,8,9,0,1,2,3,4,5,6,7,8,9,0,1,2,3,4,5,6,7,8,9,0,9,5,5,6,5,0,9,8,9] _
→ # digit values

cls = [] # string clusters
cdig = [] # clustered digits
for c in clst.children_:
    if c[0]<len(d) and c[1]<len(d):
        cdig.append([d[c[0]],d[c[1]]])
        cls.append('cluster: ' + str(d[c[0]]) + ',' + str(d[c[1]]))
    if c[0]<len(d) and c[1]>=len(d):
        sc = c[1]-len(d) # subcluster index
        cdig.append([d[c[0]],cdig[sc]])
        cls.append('supercluster: ' + str(d[c[0]]) + ' with cluster ' + _
→str(cdig[sc]))
    if c[0]>=len(d) and c[1]<len(d):
        sc = c[0]-len(d) # subcluster index
        cdig.append([cdig[sc],d[c[1]]])
        cls.append('supercluster: cluster ' + str(cdig[sc]) + ' with ' + _
→str(d[c[1]]))
    if c[0]>=len(d) and c[1]>=len(d):
        sc0 = c[0]-len(d) # subcluster index
        sc1 = c[1]-len(d) # subcluster index
        cdig.append([cdig[sc0],cdig[sc1]])
```

```

        cls.append('supercluster: cluster' + str(cdig[sc0]) + ' with cluster ' +
↪+ str(cdig[sc1]))

print('SIMPLIFIED CLUSTER ANNOTATIONS')
for i in range(0,len(cls)):
    print('#' + str(i) + ' ' + cls[i] + '\n')

```

SIMPLIFIED CLUSTER ANNOTATIONS

#0 cluster: 5,5

#1 cluster: 6,6

#2 cluster: 1,1

#3 cluster: 9,9

#4 cluster: 0,0

#5 cluster: 0,0

#6 supercluster: 0 with cluster [0, 0]

#7 supercluster: 9 with cluster [9, 9]

#8 cluster: 6,6

#9 cluster: 8,8

#10 cluster: 2,2

#11 cluster: 8,8

#12 cluster: 4,4

#13 cluster: 3,3

#14 cluster: 5,5

#15 cluster: 5,9

#16 cluster: 3,9

#17 supercluster: 4 with cluster [4, 4]

#18 supercluster: cluster[0, 0] with cluster [0, [0, 0]]

```

#19 supercluster: cluster[5, 5] with cluster [5, 5]

#20 supercluster: 9 with cluster [9, [9, 9]]

#21 cluster: 7,7

#22 supercluster: 1 with cluster [1, 1]

#23 supercluster: 5 with cluster [8, 8]

#24 supercluster: cluster[6, 6] with cluster [6, 6]

#25 supercluster: cluster[3, 9] with cluster [9, [9, [9, 9]]]

#26 supercluster: 2 with cluster [8, 8]

#27 supercluster: 7 with cluster [7, 7]

#28 supercluster: cluster[5, 9] with cluster [[3, 9], [9, [9, [9, 9]]]]

#29 supercluster: cluster[3, 3] with cluster [7, [7, 7]]

#30 supercluster: cluster[[5, 5], [5, 5]] with cluster [5, [8, 8]]

#31 supercluster: cluster[2, [8, 8]] with cluster [[3, 3], [7, [7, 7]]]

#32 supercluster: cluster[4, [4, 4]] with cluster [1, [1, 1]]

#33 supercluster: cluster[2, 2] with cluster [[5, 9], [[3, 9], [9, [9, [9, 9]]]]]

#34 supercluster: cluster[[6, 6], [6, 6]] with cluster [[4, [4, 4]], [1, [1, 1]]]

#35 supercluster: cluster[[[5, 5], [5, 5]], [5, [8, 8]]] with cluster [[2, [8, 8]], [[3, 3], [7, [7, 7]]]]

#36 supercluster: cluster[[0, 0], [0, [0, 0]]] with cluster [[[[5, 5], [5, 5]], [5, [8, 8]]], [[2, [8, 8]], [[3, 3], [7, [7, 7]]]]]

#37 supercluster: cluster[[2, 2], [[5, 9], [[3, 9], [9, [9, [9, 9]]]]]] with cluster [[[0, 0], [0, [0, 0]]], [[[[5, 5], [5, 5]], [5, [8, 8]]], [[2, [8, 8]], [[3, 3], [7, [7, 7]]]]]]]

#38 supercluster: cluster[[[6, 6], [6, 6]], [[4, [4, 4]], [1, [1, 1]]]] with cluster [[[2, 2], [[5, 9], [[3, 9], [9, [9, [9, 9]]]]]], [[[[0, 0], [0, [0, 0]]], [[[[5, 5], [5, 5]], [5, [8, 8]]], [[2, [8, 8]], [[3, 3], [7, [7, 7]]]]]]]]]

```

```

[9]: cls = []
for c in clst.children_:
    if c[0]<len(d) and c[1]<len(d):
        cls.append('cluster' + str(c) + ': ' + str(d[c[0]]) + ',' +
→str(d[c[1]]))
    if c[0]<len(d) and c[1]>=len(d):
        sc = c[1]-len(d) # subcluster index
        cls.append('supercluster' + str(c) + ': ' + str(d[c[0]]) + ' with
→cluster[' + str(sc) + '] (' + str(cls[sc]) + ')')
    if c[0]>=len(d) and c[1]<len(d):
        sc = c[0]-len(d) # subcluster index
        cls.append('supercluster' + str(c) + ': cluster[' + str(sc) + '] (' +
→str(cls[sc]) + ') with ' + str(d[c[1]]))
    if c[0]>=len(d) and c[1]>=len(d):
        sc0 = c[0]-len(d) # subcluster index
        sc1 = c[1]-len(d) # subcluster index
        cls.append('supercluster' + str(c) + ': cluster[' + str(sc0) + '] (' +
→str(cls[sc0]) + ') with cluster[' + str(sc1) + '] (' + str(cls[sc1]) + ')')

print('DETAILED CLUSTER ANNOTATIONS')
for i in range(0,len(cls)):
    print('#' + str(i) + ' ' + cls[i] + '\n')

```

DETAILED CLUSTER ANNOTATIONS

#0 cluster[33 35]: 5,5

#1 cluster[6 26]: 6,6

#2 cluster[11 21]: 1,1

#3 cluster[19 31]: 9,9

#4 cluster[20 36]: 0,0

#5 cluster[0 30]: 0,0

#6 supercluster[10 44]: 0 with cluster[4] (cluster[20 36]: 0,0)

#7 supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9)

#8 cluster[16 34]: 6,6

#9 cluster[8 28]: 8,8

#10 cluster[12 22]: 2,2

```

#11 cluster[18 38]: 8,8

#12 cluster[ 4 24]: 4,4

#13 cluster[13 23]: 3,3

#14 cluster[15 32]: 5,5

#15 cluster[5 9]: 5,9

#16 cluster[ 3 39]: 3,9

#17 supercluster[14 52]: 4 with cluster[12] (cluster[ 4 24]: 4,4)

#18 supercluster[45 46]: cluster[5] (cluster[ 0 30]: 0,0) with cluster[6]
(supercluster[10 44]: 0 with cluster[4] (cluster[20 36]: 0,0))

#19 supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with cluster[14]
(cluster[15 32]: 5,5)

#20 supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with
cluster[3] (cluster[19 31]: 9,9))

#21 cluster[17 27]: 7,7

#22 supercluster[ 1 42]: 1 with cluster[2] (cluster[11 21]: 1,1)

#23 supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8)

#24 supercluster[41 48]: cluster[1] (cluster[ 6 26]: 6,6) with cluster[8]
(cluster[16 34]: 6,6)

#25 supercluster[56 60]: cluster[16] (cluster[ 3 39]: 3,9) with cluster[20]
(supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3]
(cluster[19 31]: 9,9)))

#26 supercluster[ 2 49]: 2 with cluster[9] (cluster[ 8 28]: 8,8)

#27 supercluster[ 7 61]: 7 with cluster[21] (cluster[17 27]: 7,7)

#28 supercluster[55 65]: cluster[15] (cluster[5 9]: 5,9) with cluster[25]
(supercluster[56 60]: cluster[16] (cluster[ 3 39]: 3,9) with cluster[20]
(supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3]
(cluster[19 31]: 9,9))))

#29 supercluster[53 67]: cluster[13] (cluster[13 23]: 3,3) with cluster[27]
(supercluster[ 7 61]: 7 with cluster[21] (cluster[17 27]: 7,7))

```

```

#30 supercluster[59 63]: cluster[19] (supercluster[40 54]: cluster[0]
(cluster[33 35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)) with cluster[23]
(supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8))

#31 supercluster[66 69]: cluster[26] (supercluster[ 2 49]: 2 with cluster[9]
(cluster[ 8 28]: 8,8)) with cluster[29] (supercluster[53 67]: cluster[13]
(cluster[13 23]: 3,3) with cluster[27] (supercluster[ 7 61]: 7 with cluster[21]
(cluster[17 27]: 7,7)))

#32 supercluster[57 62]: cluster[17] (supercluster[14 52]: 4 with cluster[12]
(cluster[ 4 24]: 4,4)) with cluster[22] (supercluster[ 1 42]: 1 with cluster[2]
(cluster[11 21]: 1,1))

#33 supercluster[50 68]: cluster[10] (cluster[12 22]: 2,2) with cluster[28]
(supercluster[55 65]: cluster[15] (cluster[5 9]: 5,9) with cluster[25]
(supercluster[56 60]: cluster[16] (cluster[ 3 39]: 3,9) with cluster[20]
(supercluster[37 47]: 9 with cluster[7] (supercluster[29 43]: 9 with cluster[3]
(cluster[19 31]: 9,9))))))

#34 supercluster[64 72]: cluster[24] (supercluster[41 48]: cluster[1] (cluster[
6 26]: 6,6) with cluster[8] (cluster[16 34]: 6,6)) with cluster[32]
(supercluster[57 62]: cluster[17] (supercluster[14 52]: 4 with cluster[12]
(cluster[ 4 24]: 4,4)) with cluster[22] (supercluster[ 1 42]: 1 with cluster[2]
(cluster[11 21]: 1,1)))

#35 supercluster[70 71]: cluster[30] (supercluster[59 63]: cluster[19]
(supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with cluster[14]
(cluster[15 32]: 5,5)) with cluster[23] (supercluster[25 51]: 5 with cluster[11]
(cluster[18 38]: 8,8))) with cluster[31] (supercluster[66 69]: cluster[26]
(supercluster[ 2 49]: 2 with cluster[9] (cluster[ 8 28]: 8,8)) with cluster[29]
(supercluster[53 67]: cluster[13] (cluster[13 23]: 3,3) with cluster[27]
(supercluster[ 7 61]: 7 with cluster[21] (cluster[17 27]: 7,7))))

#36 supercluster[58 75]: cluster[18] (supercluster[45 46]: cluster[5] (cluster[
0 30]: 0,0) with cluster[6] (supercluster[10 44]: 0 with cluster[4] (cluster[20
36]: 0,0))) with cluster[35] (supercluster[70 71]: cluster[30] (supercluster[59
63]: cluster[19] (supercluster[40 54]: cluster[0] (cluster[33 35]: 5,5) with
cluster[14] (cluster[15 32]: 5,5)) with cluster[23] (supercluster[25 51]: 5 with
cluster[11] (cluster[18 38]: 8,8))) with cluster[31] (supercluster[66 69]:
cluster[26] (supercluster[ 2 49]: 2 with cluster[9] (cluster[ 8 28]: 8,8)) with
cluster[29] (supercluster[53 67]: cluster[13] (cluster[13 23]: 3,3) with
cluster[27] (supercluster[ 7 61]: 7 with cluster[21] (cluster[17 27]: 7,7))))))

#37 supercluster[73 76]: cluster[33] (supercluster[50 68]: cluster[10]
(cluster[12 22]: 2,2) with cluster[28] (supercluster[55 65]: cluster[15]
(cluster[5 9]: 5,9) with cluster[25] (supercluster[56 60]: cluster[16] (cluster[
3 39]: 3,9) with cluster[20] (supercluster[37 47]: 9 with cluster[7]
(supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9)))))) with

```

```

cluster[36] (supercluster[58 75]: cluster[18] (supercluster[45 46]: cluster[5]
(cluster[ 0 30]: 0,0) with cluster[6] (supercluster[10 44]: 0 with cluster[4]
(cluster[20 36]: 0,0))) with cluster[35] (supercluster[70 71]: cluster[30]
(supercluster[59 63]: cluster[19] (supercluster[40 54]: cluster[0] (cluster[33
35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)) with cluster[23]
(supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8))) with
cluster[31] (supercluster[66 69]: cluster[26] (supercluster[ 2 49]: 2 with
cluster[9] (cluster[ 8 28]: 8,8)) with cluster[29] (supercluster[53 67]:
cluster[13] (cluster[13 23]: 3,3) with cluster[27] (supercluster[ 7 61]: 7 with
cluster[21] (cluster[17 27]: 7,7))))))

```

```

#38 supercluster[74 77]: cluster[34] (supercluster[64 72]: cluster[24]
(supercluster[41 48]: cluster[1] (cluster[ 6 26]: 6,6) with cluster[8]
(cluster[16 34]: 6,6)) with cluster[32] (supercluster[57 62]: cluster[17]
(supercluster[14 52]: 4 with cluster[12] (cluster[ 4 24]: 4,4)) with cluster[22]
(supercluster[ 1 42]: 1 with cluster[2] (cluster[11 21]: 1,1)))) with
cluster[37] (supercluster[73 76]: cluster[33] (supercluster[50 68]: cluster[10]
(cluster[12 22]: 2,2) with cluster[28] (supercluster[55 65]: cluster[15]
(cluster[5 9]: 5,9) with cluster[25] (supercluster[56 60]: cluster[16] (cluster[
3 39]: 3,9) with cluster[20] (supercluster[37 47]: 9 with cluster[7]
(supercluster[29 43]: 9 with cluster[3] (cluster[19 31]: 9,9)))))) with
cluster[36] (supercluster[58 75]: cluster[18] (supercluster[45 46]: cluster[5]
(cluster[ 0 30]: 0,0) with cluster[6] (supercluster[10 44]: 0 with cluster[4]
(cluster[20 36]: 0,0))) with cluster[35] (supercluster[70 71]: cluster[30]
(supercluster[59 63]: cluster[19] (supercluster[40 54]: cluster[0] (cluster[33
35]: 5,5) with cluster[14] (cluster[15 32]: 5,5)) with cluster[23]
(supercluster[25 51]: 5 with cluster[11] (cluster[18 38]: 8,8))) with
cluster[31] (supercluster[66 69]: cluster[26] (supercluster[ 2 49]: 2 with
cluster[9] (cluster[ 8 28]: 8,8)) with cluster[29] (supercluster[53 67]:
cluster[13] (cluster[13 23]: 3,3) with cluster[27] (supercluster[ 7 61]: 7 with
cluster[21] (cluster[17 27]: 7,7))))))

```

```

[10]: print('RAW CLUSTERS')
      for c in clst.children_:
          print(str(c) + '\n')

```

RAW CLUSTERS

[33 35]

[6 26]

[11 21]

[19 31]

[20 36]

[0 30]

[10 44]

[29 43]

[16 34]

[8 28]

[12 22]

[18 38]

[4 24]

[13 23]

[15 32]

[5 9]

[3 39]

[14 52]

[45 46]

[40 54]

[37 47]

[17 27]

[1 42]

[25 51]

[41 48]

[56 60]

[2 49]

[7 61]

[55 65]

[53 67]

[59 63]

[66 69]

[57 62]

[50 68]

[64 72]

[70 71]

[58 75]

[73 76]

[74 77]

[]: