ELEC 378 – Spring 2023
Homework 2

Due: Friday January 27, 5PM

1 The Harmonic Sinusoid Basis

Let $\mathbf{s}_k = \begin{bmatrix} s_k[0] & \dots & s_k[N-1] \end{bmatrix}^\mathsf{T}$ denote the k^{th} harmonic sinusoid in N dimensions, where $s_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N}n}$ for each $k = 0, \dots, N-1$. The inner product between two complex valued N-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{n=0}^{N-1} u[n] v[n]^\star,$$

where $\star$ denotes complex conjugation. Using this definition, show that the harmonic sinusoids $\{\mathbf{s}_k\}_{k=0}^{N-1}$ form a basis for $\mathbb{C}^N$, and provide a formula for the analysis weights α_k of a vector $\mathbf{x}$ having signal representation in this basis. Does it look familiar?

2 SVD via EVD

Cauchy proved that every real, symmetric matrix is diagonalizable, i.e., that if $\mathbf{B} = \mathbf{B}^\intercal$ then there exists a unitary matrix of orthonormal eigenvectors $\mathbf{W}$ and diagonal matrix of eigenvalues $\mathbf{\Lambda}$ such that $\mathbf{B} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{\mathrm{H}}$. This result can be used to justify the existence of the SVD for any matrix. That is, one can use the spectral theorem to construct matrices $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$ so that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}}$ for *any* real matrix $\mathbf{A}$.

(a) Let $\mathbf{B}_1 = \mathbf{A}\mathbf{A}^\intercal$. Show that $\mathbf{B}_1$ is symmetric and write its eigendecomposition.

(b) Suppose there exist matrices $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$ so that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}}$. Find an expression for $\mathbf{U}$ in terms of the eigenvectors of $\mathbf{B}_1$. (Write $\mathbf{B}_1$ in terms of the supposed SVD of $\mathbf{A}$.) Does $\mathbf{U}$ exist for every $\mathbf{A}$?

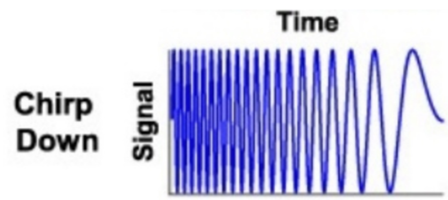
(c) Let $\mathbf{B}_2 = \mathbf{A}^\intercal\mathbf{A}$. Show that $\mathbf{B}_2$ is symmetric and write its eigendecomposition.

(d) Suppose there exist matrices $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$ so that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}}$. Find an expression for $\mathbf{V}$ in terms of the eigenvectors of $\mathbf{B}_2$. (Write $\mathbf{B}_2$ in terms of the supposed SVD of $\mathbf{A}$.) Does $\mathbf{V}$ exist for every $\mathbf{A}$?

(e) Find an expression for $\mathbf{\Sigma}$ in terms of the eigenvalues of $\mathbf{B}_1$ or $\mathbf{B}_2$. Does $\mathbf{\Sigma}$ exist for every $\mathbf{A}$?

3 Cauchy-Schwarz Decoding II

In Homework 1, you decoded a BFSK signal from noisy samples `y`. However, you solved a simplified problem in which exact transmitter-receiver synchronicity was assumed, i.e., the receiver started recording at the exact moment the transmitter started transmitting, and the receiver stopped recording at the exact moment the transmitter stopped transmitting. In practice this never happens – the receiver is "always listening", and the desired signal has to be found as a contiguous sub-sequence of the recorded signal. To facilitate this process, a known start/stop tone waveform is appended to the beginning and end of the transmitted signal. A common start/stop "tone" used in practice is the *chirp* signal:



The setup remains the same as in Homework 1, except now the BFSK signal starts and ends with a "chirp down" signal (as pictured above), and there are unknown delays between the start of recording and the start of the BFSK signal and between the end of the BFSK signal and the end of recording.

(a) Design a pre-processing system to extract the desired signal from the recorded signal so that the output of your pre-processing system can be properly decoded by the system you designed in Homework 1. Be sure to justify why the pre-processing system should work via the Cauchy-Schwarz inequality.

(b) Download `cauchy_schwarz_decoding_2.mat` from the Resources section of Piazza and modify your system from Homework 1 to decode the corrupt, delayed signal `y2`. You will need the chirp waveform `chrp`, also included, as well as the carrier signals `c0` and `c1` from Homework 1. Write the decoded bits as a `.jpg` file and include the resulting image in your write-up. Does your system decode the corrupt signal correctly?

Submission Instructions

Every student must submit their work in PDF format, providing intermediate and final results as well as any necessary code. Submit your homework on Gradescope.

Collaboration Policy

Collaboration both inside and outside class is encouraged. You may talk to other students for general ideas and concepts, but individual write-ups must be done independently.

Plagiarism

Plagiarism of any form will not be tolerated. You are expected to credit all sources explicitly. Homework, tests, and solutions from previous offerings of this course are off limits, under the honor code.