Cheatsheet on mathematical analysis.
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I. Notes

1. **Def.** = definition, **Lemma.** = Lemma, **Th.** = theorem, **St.** = statement, **Note.** = note

II. Introduction

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II. Cardinality

1. **Def.** (Cardinality of sets). Let A, B be two sets.
   (1) A and B have the *same cardinality*, $m(A) = m(B)$, if there exists a bijection from A to B,
   (2) A has *cardinality less than or equal to* the cardinality of B, $m(A) \leq m(b)$, if there exists an injective function from A into B,
   (3) A has cardinality *strictly less* than the cardinality of B, $m(A) < m(B)$, if $m(A) = m(B)$ and $m(A) \leq m(B)$ do not hold.
2. **Th.** (Cantor-Bernsstein theorem). Let A, B be two sets. If $m(A) \leq m(B)$ and $m(B) \leq m(A)$ then $m(A) = m(B)$.
3. **St.** Every infinite subset of $\mathbb{N}$ has same cardinality as $\mathbb{N}$, $\aleph_0 := m(\mathbb{N})$.
4. **St.** There can't be an infinite set that has smaller cardinality then $\aleph_0$.
5. **Def.** (Finite, countable and uncountable sets).
   (1) Set with cardinality less than $\aleph_0$ is a *finite set*,
   (2) set that has cardinality $\aleph_0$, is a *countably infinite*,
   (3) set with cardinality greater than $\aleph_0$, is *uncountable*.r
6. **Note.** (Cantor diagonalization). Consists of ordering a set A in a way $(\forall a_{ij\dots} \in A \mid i + j + \dots = n)_n^\infty$. From there we can see that rational numbers, $\mathbb{Q} \in \mathbb{N}^2$, are countable.
7. **St.** Set $\mathbb{R}$ is uncountable, $m(\mathbb{R}) = 2^{\aleph_0}$ and is called the cardinality of continuum.
8. **St.** Irrational numbers are uncountable.

III. Limits I, continuity, derivatives

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IV. Limits II

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V. Primitive functions

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VI. Continuous and differentiable functions

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VII. Taylor's polynomial

1. **Def.** Let $f : \mathbb{R} \to \mathbb{R}, x_0 \in \mathbb{R}, n \in \mathbb{N}_0$ and $f^{(n)}(x_0) \in \mathbb{R}$. Then a polynomial $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x - x_0)^k$ is called a *Taylor's polynomial* of degree n asociated with the function f at x_0.
2. **Th.** (Peano theorem). Let $f : \mathbb{R} \to \mathbb{R}, x_0 \in \mathbb{R}, n \in \mathbb{N}$ and $f^{(n)}(x_0) \in \mathbb{R}$. Then there exists just one polynomial Q_n of the maximum n-th degree, such that $f(x) - Q_n(x) = o((x - x_0)^n)$. Moreover, Q_n is a Taylor's polynomial.
3. **Th.** Let $f : \mathbb{R} \to \mathbb{R}, x > x_0, n \in \mathbb{N}_0$ and f has a finite n–-th derivative at $[x_0, x]$ and $n+1$-th derivative at (x_0, x). Let $\Phi : \mathbb{R} \to \mathbb{R}$ to has a finite non-zero derivative at (x_0, x) and to be continuous at $[x_0, x]$. Then there esists $\xi \in (x_0, x)$ such, that $R_{n+1} = \frac{(x-\xi)^n}{n!} \frac{\Phi(x) - \Phi(x_0)}{\Phi'(\xi)} f^{(n+1)}(\xi)$ is called a *remainder* of Taylor's polynomial.
4. **St.** (Lagrange's remainder). Let $\Phi(t) = (x - t)^{n+1}$ from remainder theorem. Then $R_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$.
5. **St.** (Cauchy's remainder). Let $\Phi(t) = t$ from remainder theorem. Then $R_{n+1} = \frac{f^{(n+1)}(x_0 + \Theta(x - x_0))}{n!}(1 - \Theta)^n (x - x_0)^{n+1}$ where $\Theta := \frac{\xi - x_0}{x - x_0} \in (0, 1)$.
6. **St.** Maclaurin series for basics functions:
   $e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n) \qquad \forall x \in \mathbb{R}$
   $\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) \qquad \forall x \in \mathbb{R}$
   $\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) \qquad \forall x \in \mathbb{R}$
   $\ln(1 + x) = \sum_{k=0}^{n} (-1)^{k-1} \frac{x^k}{k} + o(x^n) \qquad \forall x \in (-1, 1]$
   $(1 + x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + o(x^n) \qquad \forall x \in (-1, 1)$
   where $n \in \mathbb{N}, \alpha \in \mathbb{R}$ and $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$.

VIII. Newton's and Riemann's integral

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IX. Ordinary differerntial equations

1. **Def.** (ODR). Let $n \in \mathbb{N}$ and $f : \mathbb{R}^{n+2} \to \mathbb{R}$. Then $f(x, y, y', \dots, y^{(n)}) = 0$ is called *scalar ordinary differential equation* of n-th order. Function $y : (a, b) \to \mathbb{R}$ is called a *solution of ODR* $f(x, y, y', \dots, y^{(n)}) = 0$, if y has own derivatives of n-th order at (a, b) and $\forall x \in (a, b) : f(x, y, y', \dots, y^{(n)}) = 0$.
2. **Def.** (System of ODR of 1st order). Let $\boldsymbol{F} : \mathbb{R}^{m+1} \to \mathbb{R}^m$. Then $\boldsymbol{y}' = \boldsymbol{F}(x, \boldsymbol{y})$ is called a *system of ODR of 1st order* with solution $\boldsymbol{y} : (a, b) \to \mathbb{R}^m$ that has own derivatives of n-th order at (a, b) and $\forall x \in (a, b) : \boldsymbol{y}' = \boldsymbol{F}(x, \boldsymbol{y})$.
3. **Def.** (Cauchy problem). Let $\boldsymbol{F} : \mathbb{R}^{m+1} \to \mathbb{R}^m$. A *Cauchy problem* for an equation $\boldsymbol{y}' = \boldsymbol{F}(x, \boldsymbol{y})$ at (a, b) asks for its solution $\boldsymbol{y} : \mathbb{R}^m \to \mathbb{R}$ that obeys $\forall x \in (a, b) : \boldsymbol{y}' = \boldsymbol{F}(x, \boldsymbol{y})$ and $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$, where $x_0 \in (a, b)$ and $\boldsymbol{y}_0 \in \mathcal{D}_{\boldsymbol{F}} \subset \mathbb{R}^m$ are given values.

X. Series

1. **Def.** Let $\{a_k\} \subset \mathbb{R}$ be a sequence. Then $\sum_{k=1}^{\infty} a_k$ is denoting a (number) *series*. Number $s_n = \sum_{k=1}^{n} a_k$ is called the n-th *partial sums*.
2. **Def.** The series is *convergent* if $\lim_{n\to\infty} s_n \in \mathbb{R}$ or $\mathbb{C}$, is *divergent* if $\lim_{n\to\infty} |s_n| \to \infty \in \mathbb{R}^*$ or $\mathbb{C}^*$, or otherwise is *oscillating*.
3. **Note.** If series has only non-negative terms, it is convergent or divergent.
4. **Note.** A change in finite number of finite terms does not change a convergence of series, so we can cut off the beggining of series to have a series which can be tested by convergence tests.
5. **Th.** (The necessary condition of convergence). Let $\sum_{k=1}^{\infty} a_k$ be convergent. Then $lim_{k\to\infty} a_k = 0$.
6. **Th.** (B-C condition for series). Series $\sum_{k=1}^{\infty} a_k$ is convergent if it obeys B-C condition: $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} \cap [n_o, \infty), \forall p \in \mathbb{N} : |\sum_{k=n+1}^{n+p} a_k| < \epsilon$.
7. **Th.** (Arithemtic of series). Let $\sum_{k=1}^{\infty} a_k = A \in \mathbb{R}^*, \sum_{k=1}^{\infty} b_k = B \in \mathbb{R}^*, \alpha, \beta \in \mathbb{R}$. Then $\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) = \alpha A + \beta B$, when right side is meaningfull.
8. **Def.** A series $\sum_{k=1}^{\infty} a_k$ is convergent *absolutely* if $\sum_{k=1}^{\infty} |a_k|$ is convergent, and is convergent *not-absolutely* if if $\sum_{k=1}^{\infty} a_k$ is convergent $\sum_{k=1}^{\infty} |a_k|$ is not.
9. **Th.** If $\sum_{k=1}^{\infty} a_k$ is convergent absolutely, then is also clasically.
10. **Def.** Let $\{a_n\} \subset [0, \infty)$ be a sequence. Then $\sum_{k=1}^{\infty} (-1)^k a_k$ is called and *alternating* series.
11. **Th.** (Leibniz criterion). Let $\{a_n\}$ be non-negative non-increasing sequence. Then $\sum_{k=1}^{\infty} (-1)^k a_k$ is convergent $\iff \lim_{k\to\infty} a_k = 0$.
12. **Th.** (Comparison criterion). Let $a_k, b_k \subset \mathbb{R}$ such, that

$\exists k_0 \in \mathbb{N}, \exists C \in \mathbb{R}, \forall k \geq k_0 : |a_k| \leq C|b_k|$.
Then, if $\sum_{k=1}^{\infty} b_k$ is absolutely convergent $\Rightarrow \sum_{k=1}^{\infty} a_k$ is convergent (also absolutely).
And, if $\sum_{k=1}^{\infty} a_k$ is divergent $\Rightarrow \sum_{k=1}^{\infty} b_k$ is divergent.

13. **Th.** (Ratio comparison criterion). Let $a_k, b_k \subset (0, \infty)$ such, that $\exists k_0 \in \mathbb{N} \forall k \geq k_0 : a_{k+1}/a_k \leq b_{k+1}/b_k$.
Then, if $\sum_{k=1}^{\infty} b_k$ is convergent $\Rightarrow \sum_{k=1}^{\infty} a_k$ is convergent.
And, if $\sum_{k=1}^{\infty} a_k$ is divergent $\Rightarrow \sum_{k=1}^{\infty} b_k$ is divergent.
14. **Th.** (Limiting comparison criterion). Let $a_k, b_k \subset (0, \infty), k_0 \in \mathbb{N}$ and $lim_{k\to\infty} a_k/b_k \in (0, \infty)$. Then $\sum_{k=1}^{\infty} b_k$ is convergent $\iff \sum_{k=1}^{\infty} a_k$ is convergent.
Moreover, if $a_k, b_k \subset (0, \infty), k_0 \in \mathbb{N}$ and $lim_{k\to\infty} a_k/b_k \in [0, \infty)$. Then $\sum_{k=1}^{\infty} b_k$ is convergent $\Longrightarrow \sum_{k=1}^{\infty} a_k$ is convergent.
15. **Th.** (Integral criterion). Let $a \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous, possitive and non-increasing on $[a, \infty]$, Then $\sum_{k=a}^{\infty} f(k)$ is convergent $\iff (\mathcal{N}) \int_a^{\infty} f \, \mathrm{d}x \in \mathbb{R}$.
16. **Th.** (Chauchy's root crit.) Let $\{a_k\} \subset [0, \infty), k_0 \in \mathbb{N}$.
(1) $\exists q \in [0, 1), \forall k > k_0 : \sqrt[k]{a_k} \leq q \Rightarrow \sum_{k=1}^{\infty} a_k$ converges,
(2) $\forall k > k_0 : \sqrt[k]{a_k} \geq 1 \Longrightarrow \sum_{k=1}^{\infty} a_k$ diverges.
17. **Th.** (Limiting Cauchy's root crit.) Let $\{a_k\} \subset [0, \infty)$.
If $\lim_{k\to\infty} \sqrt[k]{a_k} < 1$, the series converges.
If $\lim_{k\to\infty} \sqrt[k]{a_k} > 1$, the series diverges.
18. **Th.**(d'Alembert's ratio crt.) Let $\{a_k\} \subset (0, \infty), k_0 \in \mathbb{N}$.
(1) $\exists q \in [0, 1), \forall k > k_0 : \frac{a_{k+1}}{a_k} \leq q \Rightarrow \sum_{k=1}^{\infty} a_k$ converges.
(2) $\forall k > k_0 : \frac{a_{k+1}}{a_k} \geq 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ diverges.
19. **Th.** (Limiting ratio criterion). Let $\{a_k\} \subset (0, \infty)$.
If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, the series converges,
If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$, the series diverges.
20. **Th.** (Raabe's criterion) Let $\{a_k\} \subset (0, \infty)$ and $k, k_0 \in \mathbb{N}$.
(1) $\exists q > 1, \forall k > k_0 : k(\frac{a_{k+1}}{a_k} - 1) \geq q \Rightarrow \sum_{k=1}^{\infty} a_k$ converges. If $\lim_{k\to\infty} k(\frac{a_{k+1}}{a_k} - 1) > 1$, the series converges.
(2) $\forall k > k_0 : k(\frac{a_{k+1}}{a_k} - 1) \leq 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ diverges. Moreover, $\lim_{k\to\infty} k(\frac{a_{k+1}}{a_k} - 1) < 1$, the series diverges.
21. **Th.** (Gauss's criterion) Let $\{a_k\} \subset (0, \infty)$. $\exists p, q \in \mathbb{R}$ and $\epsilon, C > 0$ such, that $\frac{a_k}{a_{k+1}} = p + \frac{q}{k} + \frac{t_k}{k^{1+\epsilon}}$, where $|t_k| \leq C$.
(1) If $p > 1$, $\sum_{k=1}^{\infty} a_k$ converges and if $p < 1$, diverges.
(2) If $p = 1$ and $q > 1$, the series converges.
(3) If $p = 1$ and $q \leq 1$, the series diverges.
22. **Th.** (Abel and Dirichlet criteria). Let $\{a_k\}, \{b_k\} \subset \mathbb{R}$, $\{a_k\}$ be monotonic. Dirichlet: If $\{a_k\} \to 0$ and $\{b_k\}$ has bounded partial sums, then $\sum_{k=1}^{\infty} a_k b_k$ converges. Abel: If $\{a_k\}$ is bounded and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k b_k$ converges.
23. **St.** Complex variant of Abel and Dirichlet criteria have $\{a_k\} \subset \mathbb{R}, \{b_k\} \subset \mathbb{C}$ because of monotony of $\{a_k\}$.
24. **St.** Let $a \in \mathbb{R}$. Then $\{\sin(ak)\}$ has bounded partial sums, and if a is not a multiple of 2π then $\{\cos(ak)\}$ has bounded partial sums.
25. **Def.** (Rearrangement of series). Let $\{a_k\} \subset \mathbb{R}$ and $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijetion. Then a series $\sum_{k=1}^{\infty} a_{\varphi(k)}$ is called the *rearrangement* of $\sum_{k=1}^{\infty} a_k$ with respect to bijection φ.
26. **Def.** Let $x \in \mathbb{R}$. We define a *possitive part* of x as $x^+ := \max\{x, 0\}$ and *negative part* as $x^- := \max\{-x, 0\}$.
27. **Th.** (Characterization of absolute/not-absolute convergence). Let $\{a_k\} \subset \mathbb{R}$. Then
(1) $\sum_{k=1}^{\infty} a_k$ converges absolutely $\Leftrightarrow \sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ converges.
(2) $\sum_{k=1}^{\infty} a_k$ converges not-absolutely $\Rightarrow \sum_{k=1}^{\infty} a_k^+ = \infty$ and $\sum_{k=1}^{\infty} a_k^- = -\infty$.
28. **Th.** (Rearrangement of abs. convergent series). Let $\{a_k\} \subset \mathbb{R}$ and series $\sum_{k=1}^{\infty} a_k$ converges absolutely. Then every rearrangement of it converges absolutely and has same sum.
29. **Th.** (Riemann rearrangement theorem). Let $\{a_k\} \subset \mathbb{R}$ and series $\sum_{k=1}^{\infty} a_k$ converges not-absolutely. Then for every $S \in \mathbb{R}^*$ there exists rearrangement of $\sum_{k=1}^{\infty} a_k$ with sum S.
30. **Def.** Let M be a countable set. A *generalized series* $\sum_{m\in M} a_m$ converges, if there exists a bijection $\varphi : M \to \mathbb{N}$ a bijection, such that $\sum_{k=1}^{\infty} a_{\varphi(k)}$ is absolutely convergent. Then we define $\sum_{m\in M} a_m := \sum_{k=1}^{\infty} a_{\varphi(k)}$.
31. **Th.** (Cauchy's series product theorem). Let $\{a_k\}, \{b_k\} \subset \mathbb{R}$ and let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be convergent absolutely. Then $\sum_{i,j=1}^{\infty} a_i b_j$ converges absolutely and $\sum_{i,j=1}^{\infty} a_i b_j = (\sum_{k=1}^{\infty} a_k)(\sum_{k=1}^{\infty} b_k)$.
32. **St.** (Cauchy's equation). Let $\{a_k\}, \{b_k\} \subset \mathbb{R}$, then $\sum_{i,j=1}^{\infty} a_i b_j = \sum_{n=1}^{\infty} (\sum_{i+j=n+1} a_i b_j)$.
33. **Lemma.** (Convergence of arhitmetic means). Let $\{a_k\} \subset \mathbb{R}$ such that $\lim_{k\to\infty} a_k = A \in \mathbb{R}^*$. We define $\{b_k\}$ so that $b_j = \frac{1}{j} \sum_{k=1}^{j} a_k$. Then $\lim_{k\to\infty} b_k = A$.
34. **Def.** (Cesàro summation). Let $\{a_k\} \subset \mathbb{R}$. $\forall n \in \mathbb{N}$: $s_n = \sum_{k=1}^{n} a_k, \sigma_n = \frac{1}{n} \sum_{k=1}^{n} s_k$. We say that $\sum_{k=1}^{\infty} a_k$ is *Cesàro summable*, if $\lim_{n\to\infty} \sigma_n = A \in \mathbb{R}$. Number A is then called *Cesàro sum* of $\sum_{k=1}^{\infty} a_k$, denoted as $(C, 1) \sum_{k=1}^{\infty} a_k = A$.
35. **Th.** (Cauchy condensation test). Let $\{a_k\} \in [0, \infty)$ be non-increasing sequence. Then $\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow$ $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.
36. **Th.** (The necessary condition of convergence). Let $\{p_k\} \subset (0, \infty)$ and let $\Pi_{k=1}^{\infty} p_k$ be convergent. Then $\lim_{k\to\infty} p_k = 1$.
37. **Th.** Let $\{p_k\} \subset (0, \infty)$ or $\{p_k\} \subset (-1, 0)$. Then $\Pi_{k=1}^{\infty}(1 + p_k)$ is convergent $\iff \sum_{k=1}^{\infty} p_k$ is convergent.

XI. Power series

1. **Def.** (Power series). Let $\{a_n\} \subset \mathbb{C}$ and $z_0 \subset \mathbb{C}$. Then a series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ is called a *power series* centered around z_0.
2. **Def.** (Radius of convergence). Let $\{a_n\} \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$. Then R is the *radius of convergence*, if $R \in \mathbb{R}^*, R > 0$ and the series converges if $|z - a| < R$ and diverges if $|z - a| > R$.
3. **Th.** (Convergence of power series). Let $\{a_n\} \in \mathbb{C}$, and let $R := (\limsup_{k\to\infty} \sqrt[k]{|a_k|})^{-1}$, usign $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. Then (1) the series $\sum_{k=0}^{\infty} a_k z^k$ has a radius of absolute convergence R,
(2) if $\lim_{k\to\infty} |\frac{a_{k+1}}{a_k}|$ exists, then it is equal to R,
(3) if $\lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists, then it is equal to $\frac{1}{R}$.
4. **Lemma.** Let $\{a_n\} \subset \mathbb{C}$. Then power series $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=1}^{\infty} k a_k z^{k-1}$ have a same radius of convergence.
5. **Th.** (Derivative of power series). Let $\{a_n\} \subset \mathbb{C}$. Then for $x \in (-R, R)$, where $R \geq 0$ is a radius of convergence of associated power series, holds $(\sum_{k=0}^{\infty} a_k z^k)' = \sum_{k=1}^{\infty} k a_k z^{k-1}$.
6. **St.** Every power series at its circle of convergence defines infinitely continuously differentiable function.
7. **Th.** (Integral of power series). Let $\{a_n\} \subset \mathbb{C}$.
(1) For $x \in \mathbb{R}$ inside circle of convergence we have $\int (\sum_{k=0}^{\infty} a_k z^k) dz = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1} + C$.
(2) If $a, b \in (-R, R)$, where R is radius of convergencce of $\sum_{k=0}^{\infty} a_k z^k$, then
$(\mathcal{R}) \int_a^b (\sum_{k=0}^{\infty} a_k z^k) dz = (\mathcal{N}) \int_a^b (\sum_{k=0}^{\infty} a_k z^k) dz = \sum_{k=0}^{\infty} (\mathcal{R}) \int_a^b a_k z^k dz = \sum_{k=0}^{\infty} (\mathcal{N}) \int_a^b a_k z^k dz$.
8. **Def.** (Taylor series). Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely

differentiable at $x_0 \in \mathbb{R}$. Then $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ we call a Taylor series of f at x_0.

9. **Th.** (Borel's lemma). Let $\{a_k\} \subset \mathbb{R}, x_0 \in \mathbb{R}$ and $\exists \delta > 0 : \sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges at $(x_0-\delta, x_0+\delta)$. Then $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is a Taylor series of its sum at x_0.
10. **Def.**
11. **Th.**
12. **Th.**
13. **Th.**
14. **Def.**
15. **Th.**

XII. Metric spaces

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XII. Functions of several variables

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XIII. Classical calculus of variations

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XIV. Series of functions

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XV. Lebesgue integral

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XVI. Lebesgue spaces

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XVII. Line and surface integrals

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XVIII. Differentail forms

1.

XIX. Fourier series

1.

XX. Fourier transform

1.

XXI. Complex analysis

1.

XXII. Partial differential equations

1.

XXIII. Functional analysis

1.

XXIV. Green's functions

1.