DO: Vector identities cheatsheet
DECIDE: Use bold or vector ntation?
REDO: General concepts and skills

1. numerical methods: Euler $y_{i+1} = y_i + f(t_i, y_i)\Delta t$

REDO: Data analysis
REDO: Mathematicasa

1. $\frac{d}{dt}\int_0^t f(x)dx = f(t) + \int_0^t \frac{t}{dt} f(x)dx$
2. Derivations:
   $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, $\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$
   $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{arccot}(x) = \frac{-1}{1+x^2}$
   $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{arcsinh}(x) = \frac{1}{\sqrt{x^2+1}}$, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{arccosh}(x) = \frac{1}{\sqrt{x^2-1}}$
   $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{arctanh}(x) = \frac{1}{1-x^2}$ and $1 > x^2$
   $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{arccoth}(x) = \frac{1}{1-x^2}$ and $1 < x^2$
3. Fourier:
   $f(x) = \sum c_n e^{inx}$ where $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$
4. Dot and Cross products are distributive.
5. $\nabla\cdot(\nabla\times\vec{A}) = 0$ and $\nabla\times(\nabla f) = 0$
6. Triple product $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$
7. BAC-CAB rule: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

I. KINEMATICS

1. $\vec{v} = \mathrm{d}\vec{x}/\mathrm{d}t$, $\vec{a} = \mathrm{d}\vec{v}/\mathrm{d}t$, $t = \int v_x^{-1}\mathrm{d}x = \int a_x^{-1}\mathrm{d}v_x$, $x = \int v_x/a_x \mathrm{d}v_x$, where integrals could be found from plot as an area under graph.
2. If $a = Const.$, then: $x = (v^2 - v_0^2)/2a$.
3. Do we want here eqs. for movement in g field?
4. Rotational motion: $\omega = d\phi/dt$, $\epsilon = d\omega/dt$
   $\vec{a} = \mathrm{d}v/\mathrm{d}t\hat{e}_\tau + v^2/R\hat{e}_n$
5. Motion of a rigid body. **a)** $v_a\cos\alpha = v_b\cos\beta$; $\vec{v}_a$, $\vec{v}_b$ — velocities of points A and B; **b)** The instanteous center of rotation can be found as the intersection pt. of perpediculars to $\vec{v}_a$ and $\vec{v}_b$ from the points A and B, or(if $\vec{v}_a, \vec{v}_b \perp AB$) as the ntersection pt. of AB whith the line connecting endpoints of v_a and v_b
6. Centrifugal force $\vec{F} = m\omega^2\vec{r}$, Coriolis force $\vec{F} = m\vec{v}\times\vec{\omega}$
7. Ballistic problem reachable region
   $y \leq v_0^2/(2g) - gx^2/(2v_0^2)$
   For and optimal balistic trajectory, initial and final velocities are perpendiculars.
8. Parabola is equidistant from focus and directrix (or use line parallel to direcrtix passing throught focus). Rays parallel to y-axis are reflected to focal point.
9. Elastic bounce of ball from moving wall:
   $\vec{v}_n{}' = -\vec{v}_n + 2\vec{u}$
10. For findings fastest paths, Snell's, Fermat's and Huygen's principles can be used.
11. To find a vector (velocity, acceleration) its enought to find its direction and projection to single axes.
12. Methods of solving prob.: Vector and Geometry, Differentiat. approach (calc. infinitezimal change of system), Equations of motion, Investigation in specific frame.

II. STATICS

1. For a 2D equilibrium of a rigid body: 2 eqns. for Forces, 1 eq. for torque. 1(2) eq. for force can be substituted whith 1(2) for torque. Torque is often better then boring forces.
2. Normal and frictional force can be combined into a sngle force, applied under $arctan\,\mu$ with respect to the normal.
3. $F_t \leq \mu mg$, there are several types: static, kinetic (sliding, dynamic).
4. Pulleys: a) Force in pulley axis = Σ forces from rope.
   b) Σ of center shifts of all the rope parts = 0
   c) If you draw a line through system of pulleys, then Σ of forces "acting" on the line is in the equilirium zero. d) Virtual work e) Noether theorem
5. Virtual work: in equilibrium
   $\delta W = \Delta\Pi - \Sigma F\Delta x = 0$
6. For studying stability (instability) use principle of minimum (maximum/saddle point) of potential energy or principle of virtual work (small displacement).

III. LAWS OF CONSERVATION

1. Momentum: no net external forces or conseved along axis $\perp$ to the net external force.
2. Energy: elastic bodies, no friction, conservative fields $\nabla\times\vec{F} = 0$
3. Angular momentum: no net external torque (arm = 0), can be written rel. to 2(3) pts. then substitutes conservation of lin. mom.
4. Number of nucleons and total charge.

IV. DYNAMICS

1. Newton's 1 law: There exists a frame of reference (called inertial), in which each isolated point mass moves uniformly in a straight line.
2. Newton's 2 law: For each point mass there exists a constant m and a vector function $\vec{F}$ such that its motion with respect to the inertial system is given by $m\ddot{\vec{x}} = \vec{F}$, $\left(\vec{F}_\Sigma = \mathrm{d}\vec{p}/\mathrm{d}t\right)$.
3. Mass accretion ($\mathrm{d}m > 0$) / ejection: ($\mathrm{d}m < 0$) $m\vec{a} = \vec{F}_{ext} + \vec{v}_{rel}\frac{dm}{dt}$.
4. Index C denotes quantities rel. to the centre of mass: $\vec{P}_C = \sum m_i\vec{v}_i$, $K = \sum m_i v_i^2/2$ $\vec{P}_i = \vec{P}_{Ci} - M_\Sigma\vec{v}_C$, $K_i = K_{Ci} + M_\Sigma v_C^2/2$
5. If forces are applied only to 2 points, the net force application lines coincide; for 3 points, the lines meet at a single point.
6. Tilted coordinates (motion on an inclined plane).
7. Generalized coordinate q: $\mathcal{M}\ddot{q} = -\mathrm{d}\Pi(q)/\mathrm{d}q$ and $T = \mathcal{M}\dot{q}^2/2$
8. Non-inertial frames of ref.: inertial force $-m\vec{a}$, centrifugal force $m\omega^2\vec{r}$, Coriolis force $2m\vec{v}\times\vec{\Omega}$ (better to avoid it, it does not create any work)
9. Stoke's drag: force of viscosity on a small sphere moving through a viscous fluid $F = 6\pi\eta R v$
10. Drag equation: $F = \rho C S v^2/2$, where ρ - density of the fluid, S - reference area, C - drag coeficient.
11. Collision of 2 bodies: conserved are
 a) net momentum b) net angular mom.
 c) total enery (for elastic collisions)
 d) if sliding stops during the impact, final velocities of the contact points will have equal projections to the contact plane;
 e) if sliding doesn't stop, the momentum delivered during the impact froms angle $arctan\,\mu$ with the normal.
12. Tension T in a string: horizontal component is conctant, vertical $\propto$ mass underneath. On pulley with friction tension change $e^{\mu\phi}$ times. Presure force N per unit lenght (resting on a smooth surface) is $\propto$ its radius R: $N = T/R$
13. Adiabatic invariant: if a relative change of an oscillating system is small during one period, the area I of the loop drawn in p-x coordinates is conversed with a very high accuracy.
14. Continuity condition: $\mathrm{d}m/\mathrm{d}t = \rho S v$

V. ROTATIONAL MOTION

1. $\vec{\tau} = I\vec{\epsilon} = \mathrm{d}\vec{L}/\mathrm{d}t$, where $\vec{L} = \sum\vec{r}_i\times\vec{P}_i$
2. $\vec{L} = I\vec{\omega}$, $\vec{L} = \sum m_i\vec{r}_i\times\vec{v}_i$, $\vec{L}_i = \vec{L}_{Ci} + M_\Sigma\vec{R}_C\times\vec{v}_C$
3. Point masses & fixed origin: $\vec{\tau} = \vec{r}\times\vec{F}^{ext}$
4. System of point masses & fixed origin: $\vec{\tau} = \sum\vec{r}_i\times\vec{F}_i^{ext}$
5. System of point masses & non-fixed origin: $\vec{\tau} = \sum(\vec{r}_i - \vec{r}_0)\times\vec{F}_i^{ext} - (\vec{r}_C - \vec{r}_0)\times(M_C\ddot{\vec{r}}_0)\hat{r}_0$ There are 3 cases when the second term is zero.
6. Rotation of body can be discribed by ortogonal ($A^T = A^{-1}$) matrix $A_{ij}(t)$ such, that $\hat{e}'_i = A_{ij}\hat{e}_j$, where $\hat{e}'_i$ is co-rotating and $\hat{e}_j$ inertial basis.
7. Tensor of angular veloc.: $\Omega = \frac{\mathrm{d}A}{\mathrm{d}t}A^T = -A^T\frac{\mathrm{d}A}{\mathrm{d}t}$.
8. $\Omega_{ij} = -\Omega_{ji}, \quad \omega_k = \frac{1}{2}\epsilon_{kij}\Omega_{ij}, \quad \Omega_{ij} = \epsilon_{ijk}\omega_k$
9. $\Omega^A = -\Omega^{-A}$ where A denotes rotational-defining matrix of angular vel. tensor.
10. $\Omega^C = \Omega^B + \Omega^A$, where $\hat{e}''_k = C_{ki}\hat{e}_i = B_{kj}A_{ji}\hat{e}_i$.
11. Ω has same components in the inertial and co-rotating frame (it points in the direction of invariant axis).
12. $\frac{\mathrm{d}\vec{w}}{\mathrm{d}t}\Big|_{prostor} = \frac{\mathrm{d}\vec{w}}{\mathrm{d}t}\Big|_{teleso} + \vec{\omega}\times\vec{w}$
13. Coeficients for I: cylinder 1/2; solid sphere 2/5; thin spherical shell 2/3; rod 1/12 (rel. to end-point 1/3); square 1/6;
14. $I = \sum r^2 m_i = \int r^2 dm$ where r is distance from the axis.
 Its possible to use scaling when finding I.
15. Steiner's (Parallel axis) theorem (a - distance of the mass center from rot. axis): $I = I_C + ma^2$ $I_{ij} = I^C_{ij} + (\delta_{ij}a_k a_k - a_i a_j)m$
16. Perpendicular axis theorem: if body lies in the xy plane, then $I_z = I_x + I_y$. The axis xyz must all intersect in a single point in the plane.
17. Mom. of inertia rel. to the z-axis through the mass center $I_{z0} = \sum m_i m_j r_{ij}^2/2M_\Sigma$
18. Mom. of inertia tensor - component def.:
 $$\tilde{I} = \int\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+y^2 & -yz \\ -zx & -zy & y^2+z^2\end{pmatrix}\mathrm{d}m$$
 $I_{ij} = \int\left(\delta_{ij}x_k x_k - x_i x_j\right)\rho\mathrm{d}V$
19. Mom. of inertia tensor - algebraric def.: $I(\vec{\xi},\vec{\omega}) = \sum_a m_a(\vec{\xi}\times\vec{r}^{\,a})\cdot(\vec{\omega}\times\vec{r}^{\,a}) = \vec{L}\cdot\vec{\xi}$
20. $I(\vec{\omega},\vec{\omega}) = 2T$, $I(\vec{\xi},\vec{\omega}) = I_{ij}\xi_i\omega_j$, $I_{ij} = I(\hat{e}_i,\hat{e}_j)$
21. Rot. energy $T = \frac{1}{2}I_{ij}\omega_i\omega_j$, ang. mom. $L_i = I_{ij}\omega_j$
22. Principal axis along ω_i with principal mom. I_i must satisfy $\tilde{I}\vec{\omega}_i = I_i\vec{\omega}_i$. Principal axis are $\perp$.

23. If two principal moments are equal ($I_1 = I_2 = I$), then any axis (through the chosen origin) in the plane of the corresponding principal axes is a principal axis (and its moment is also I). If all three principal moments are equal ($I_1 = I_2 = I_3 = I$), then any axis (through the chosen origin) in space is a principal axis (and its moment is also I).

24. $\tilde{I}_{\hat{u}}$ in the direction of unit vector $\hat{u}$: $\tilde{I}_{\hat{u}} = \hat{u}^T \tilde{I} \hat{u}$
$I_u = I_{ij} u_i u_j = \int (\vec{r} \cdot \vec{r} - (\vec{r} \cdot \vec{u})^2) \rho \mathrm{d}V$

25. Elipsoid of inertia $I_1 \xi_1^2 + I_2 \xi_2^2 + I_3 \xi_3^2 = 1$
an elipsoid whose semi-axis are equal to $1/\sqrt{\text{principal mom.}}$. Then $I_u = 1/|\vec{\xi}|^2$ where $\vec{u} = \vec{\xi}/|\vec{\xi}|$.

26. If a pancake object is symmetric under a rotation through an angle $\theta \neq \pi$ in the $x - y$ plane (for example a hexagon), then every axis in the $x - y$ plane (with the origin chosen to be the center of the symmetry rotation) is a principal axis.

27. Strike a rigi body with an impulse, what is the motion immediately after? Solution: Find the $\vec{L}$ rel. to the CM using the angular impulse, then calc. principal mom. and find $\vec{\omega}$ then add on the CM motion (lin. impulse);

28. Frequency due to torque? Solution: Calc. principal mom., find $\vec{L}$, find $\mathrm{d}\vec{L}/\mathrm{d}t$, calc. torque and equate it with $\mathrm{d}\vec{L}/\mathrm{d}t$;

29. $\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\big|_{prostor} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\big|_{teleso} + \vec{\omega} \times \vec{L}$,+

30. Euler's equations: in reference frame connected with principal axis of body: $\vec{\tau} = \tilde{I}\dot{\vec{\omega}} + \vec{\omega} \times \tilde{I}\vec{\omega}$ or ($1 \to 2 \to 3$): $\tau_1 = I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3$; usefull when $\tau = 0$, precession of free spining top.

31. Tenis racket theorem: rotation of an object ($I_1 > I_2 > I_3$) around its first and third principal axes is stable, while rotation around its second principal axis (or intermediate axis) is not. (prove by Euler's eqs.)

32. Acceleration in non-inertial rotating frame:
$\vec{a}_p = \vec{a}_t + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \vec{v}_t + \vec{\omega} \times (\vec{\omega} \times \vec{r})$.

33. Euler angles: φ - preces., ϑ - nutation, ψ - rotat.
For co-rotating frame x, y, z
$\omega_x = \dot{\varphi} \sin\vartheta \sin\psi + \dot{\vartheta} \cos\psi$,
$\omega_y = \dot{\varphi} \sin\vartheta \cos\psi - \dot{\vartheta} \sin\psi$,
$\omega_z = \dot{\varphi} \cos\vartheta + \dot{\psi}$.

34. For gyroscope precession assume no nutation.

35. Nutation: assume that precesion and nutation is small $\dot{\theta}$, $\dot{\Phi} << 1$.

VI. OSCILLATIONS AND WAVES

1. Damped oscillator: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$
General sol.: $x = e^{-\gamma t}(Ae^{i(\omega t + \varphi)} + Be^{-i(\omega t + \psi)})$
where $\omega = \sqrt{\omega_0^2 - \gamma^2}$
Underdapmed $\gamma < \omega_0$: $x = x_0 e^{-\gamma t} \cos(\omega t + \phi)$
Critically damped $\gamma = \omega_0$: $x = (A + Bt)e^{-\omega_0 t}$
Overdamped $\gamma > \omega_0$.

2. Energy of spring: $E = kx^2/2$.

3. Energy decay: $\mathrm{d}\langle E\rangle/\mathrm{d}t = -2\gamma\langle E\rangle$, what has a sol: $\langle E\rangle = \frac{1}{2}kA^2 e^{-2\gamma t} = E_0 e^{-\frac{\omega_0}{Q}t}$

4. Factor of quality $Q = 2\pi \frac{\text{Energy stored}}{\text{En. lost per cycle}} = \frac{\omega_0}{2\gamma}$

5. Driven oscillations: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F}{m}\cos\omega t$.
Particular solution is $x = Ae^{i(\omega t + \varphi)}$, solve for A and φ.

6. If a system described with a generalized coordinate ξ and $T = \mu\xi^2/2$ has an equilibrium state at $\xi = 0$ for small oscillations $\Pi \approx \kappa\xi^2/2$ then $\kappa = \Pi''(0)$ and $\omega_0^2 = \kappa/\mu$.

7. Eq. of motion for a system of coupled oscillators: $\ddot{x}_i = \sum_j K_{ij} x_j$, in tensor notat. $(\tilde{K} - m\omega_i^2 \tilde{I})\vec{A}_i = 0$, $m\omega_i^2$ are eigenvalues and A_i eigenvectors of $\tilde{K}$, thf. $\det(\tilde{K} - m\omega_i^2 \tilde{I}) = 0$ and sol. is $\vec{x} = \vec{A}_i \exp(-i\omega_i t)$.

8. A system of N coupled oscillators has N different eigenmodes when all the oscillators oscillate with the same frequecy ω_i, $\vec{x}_j = \vec{x}_{j0}\cos(\omega_i t + \varphi_j)$, and each eigenmode has its own eigenfrequency ω_i (which can be same for more eigenmodes). General sol. is superposition of all eigenmodes with $2N$ integration constants x_{j0}, φ_j.

9. Wave equation $\Delta u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$ has general solution $u(x,t) = f(x \pm vt)$, special case $u(x,t) = Ae^{\pm i(kx - \omega t)}$.

10. D'Alembert method: if $\xi = x - ct$, $\eta = x + ct$ then wave equation has a form $\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$.

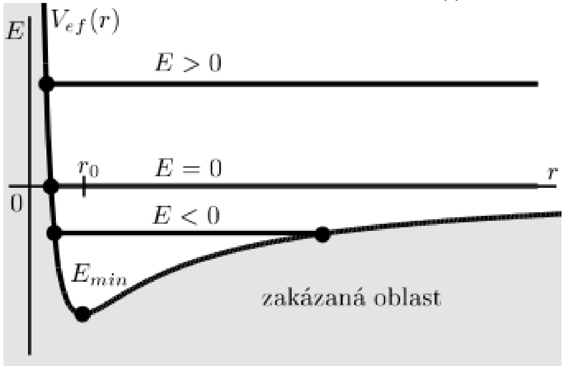
11. D'Alembert solution of oscillating infinite string with initial wave $u(x,0) = U_0(x)$ such, that $\int u_{,t}(x,0)\mathrm{d}x = V_0(x)$ is
$u(x,t) = \frac{1}{2}\left[U_0(\xi) + U_0(\eta) - \frac{1}{c}V_0(\xi) + \frac{1}{c}V_0(\eta)\right]$.

12. Bernoulli-Fourier sol. of oscillating infinite string is $u(x,t) = X(x)T(t)$ where $\omega_n = n\pi c/l$ and
$u(x,t) = \sum_n \sin(\omega_n x/c)\left[a_n \cos(\omega_n t) + b_n \cos(\omega_n t)\right]$,
and for initial wave $u(x,0) = U_0(x)$ such, that $\int u_{,t}(x,0)\mathrm{d}x = V_0(x)$ we have
$a_n = \frac{2}{l}\int_0^l U_0(x)\sin(\omega_n x)\mathrm{d}x$,
$b_n = \omega_n \frac{2}{l}\int_0^l V_0(x)\sin(\omega_n x)\mathrm{d}x$.

13. For zero initial speed Bernoulli-Fourier sol. gives all $b_n = 0$.
14. The phase of wave is $\varphi(x,t) = kx - \omega t + \varphi_0$. Phase velocity $v = \omega/k$ and group velocity $v_g = \mathrm{d}\omega/\mathrm{d}k$.
15. Standing wave is the sum of two identical counterpropagating waves: $e^{i(kx-\omega t)} + e^{i(-kx-\omega t)} = 2e^{-i\omega t}\cos(kx)$.
16. Speed of transverse waves: $v = \sqrt{T/\mu}$ where T is tension and μ is linear mass density.
17. Sp. of sound in gas: $v^2 = (\partial p/\partial \rho)_{\text{adb}} = \gamma p/\rho = \gamma RT/M$
18. Speed of sound in elastic materal: $v = \sqrt{Y/\rho}$.
19. Speed of shallow $h << \lambda$ water waves: $v = \sqrt{gh}$.

VII. CELESTIAN MECHANICS AND GRAVITY

1. $F = GMm/R^2$, $\Pi = -GMm/R$, $\alpha = GMm$
2. Kepler's I law: $F \propto R^{-2}$ interaction of 2 point masses: trajectory of each of them is a circle, eclipse, parabola or hyperbola, with a focus at the center of mass of the system.
3. Kepler's II law (conserv. of angular. mom.): for a point mass in a central force fiels, radius vector cover equal areas in equal times: $\mathrm{d}S/\mathrm{d}t = L/2m$
4. Kepler's III law: for two point masses at elliptic orbits in $F \propto r^{-2}$, periods and longer semi-axes obey: $T_1^2/T_2^2 = a_1^3/a_2^3$, generaly: $T^2/a^3 = 4\pi^2/G(M+m) \approx 4\pi^2/GM_\odot$
5. $L = mR^2\dot{\varphi}$, $m\left(\dot{R}^2 + \dot{\varphi}^2R^2\right)/2 + V(R) = E$, $m\dot{R}^2/2 + V_{eff}(R) = E$; where eff. potental is $V_{eff}(R) = L^2/2mR^2 + V(R)$
(For gravity $V(R) = -\alpha/R$).
6. Celestain motion is possible only if $V_{eff}(R) \leq E$.



7. Binet's equation: $\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = -\frac{m}{L^2}\frac{\mathrm{d}V}{\mathrm{d}u}$ where $V(u)$ is a central potential and u inverse distance. Solutions to $F(u) \sim u^3$ are Cotes spirals.
8. $r = p/(1+\epsilon\cos\varphi)$, where $p = L^2/\alpha m$ and $\epsilon^2 - 1 = 2L^2E/\alpha^2 m$.
9. Full energy $K + \Pi$ of a body in a gravity field:$E = -GMm/2a$
10. Vis-Viva equation: $v^2 = GM\big(2/r - 1/a\big)$.
11. For small eccentricities $\epsilon = f/a \ll 1$, trajectories can be considered as ahaving a circular shapes, with shifted foci.
12. Properties of an ellipse: $l_1 + l_2 = 2a$ (l - distances from the foci), light from one focus is reflected to the other (angles with normal are same), $S = \pi ab$
13. A circle and an ellipse with a focus at the circle's center can touch each other only at the longer axis.
14. Gauss's law for gravity field $\vec{g}$:
$\oint_{\partial V} \vec{g}\cdot\vec{dS} = -4\pi GM$ or $\nabla\cdot\vec{g} = -4\pi G\rho$
$\oint_{\partial S} \vec{g}\cdot\vec{dl} = 0$ or $\nabla\times\vec{g} = 0$ (conservative field)
15. Laplace-Runge-Lenz vector (LRL or the eccentricity vector), where $\alpha = GMm$ and $\epsilon = A/m\alpha$
$$\vec{\epsilon} = \frac{\vec{v}\times\vec{L}}{GMm} - \vec{e}_R \qquad \text{or} \qquad \vec{A} = \vec{p}\times\vec{L} - m\alpha\vec{e}_R$$
16. Reduced mass: 2 body interaction lagrangian in CM: $\mathcal{L} = \mu\dot{r}^2/2 + V(r)$, where $1/\mu = 1/m_1 + 1/m_2$ is the reduced mass and r is distance between them.
17. Bertrand's theorem: only central forces $F \propto R^{-2}$ and $F \propto R$ (Harmonic osc.) give rise to closed orbits independently of initial conditions.
18. Virial theorem for finite movement:
If $F \propto r^n$, then $n\langle K\rangle = \langle\Pi\rangle$ (time avarages)
19. Tsiolkovsky rocket equation:
$\Delta v = u\ln(M_{init}/M_{fin})$
20. Rutheford scattering $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Q_1Q_2e^2}{8\pi\epsilon_0 m v_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$
21. add Tidal forces

VIII. LAGRANGIAN FORMALISM

1. Generalized coordinate q: $\mathcal{M}\dot{q}^2/2 + \Pi(q) = E$
$\mathcal{M}\ddot{q} = -\mathrm{d}\Pi(q)/\mathrm{d}q$ and $T = \mathcal{M}\dot{q}^2/2$
2. Constraints are **a)** two-side $\phi = 0$ or one-side $\phi \geq 0$, **b)** Scleronomic $\phi(q_i) = 0$ or Reonomic $\phi(q_i,t) = 0$, **c)** Holonomic $\phi(q_i,t) = 0$ or non-holonomic (kinetic) $\phi(q_i,\dot{q}_i,t) = 0$.
3. Lagrange eqs. of 1st kind: $m\ddot{x}_i = F_i + T_i + \lambda\partial_i\phi$, where $\phi(x^j,t) = 0$, F_i is net ext. force and T_i is net friction force. In vector notation:
$m\ddot{\vec{x}} = \vec{F} + \vec{T} + \lambda\nabla\phi$.

4. Lagrange eqs. of 1st kind for N particles and V constraints: $m\ddot{x}_i = F_i + \sum_{v=1}^{V} \lambda^v \partial\phi_v / \partial x^i$ for $i \in \{1, \ldots, 3N\}$ and $\phi_v(x^1, \ldots, x^{3N}, t) = 0$ for $v \in \{1, \ldots, V\}$.
5. Stationary constrained particle under an action of force $\vec{F}$ obeys: $\vec{F} + \lambda\nabla\phi = 0$ and $\vec{F} \times \nabla\phi = 0$.
6. Virtual work: in equilibrium $\delta W = \delta\Pi - \sum_{i=1}^{3N} F_i \delta x^i = 0$
7. Differential principles: **a)** D'Alembert's principle: $\sum_{i=1}^{3N}(F_i - m\ddot{x}_i)\delta x^i = 0$, **b)** Jourdain principle: $\sum_{i=1}^{3N}(F_i - m\ddot{x}_i)\delta\dot{x}^i = 0$, **c)** Gauss's principle of least constraint: $\sum_{i=1}^{3N}(F_i - m\ddot{x}_i)\delta\ddot{x}^i = 0$. All virtual displacements have to obey constraints and be reversible (for every δx^i there exists $-\delta x^i$).
8. D'Alembert's principle in geometrical form: $(m\ddot{\vec{x}} - \vec{F}) \cdot \vec{t} = 0,\ \forall \vec{t} \in T_P Q$.
9. D'Alembert's principle for all virtual displacements: $\sum_{i=1}^{3N}(m\ddot{x}_i - F_i - \sum_{v=1}^{V} \lambda_v \partial\phi_v / \partial x^i)\delta x^i = 0$
10. $\partial q^i / \partial \dot{q}^j = 0 = \partial \dot{q}^i / \partial q^j$, $\partial q^i / \partial q^j = \delta^i_j = \partial \dot{q}^i / \partial \dot{q}^j$
11. Generalized force: $Q_j = \partial x^i / \partial q^j F_i$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}^j}\right) - \frac{\partial T}{\partial q^j} = Q_j = -\frac{\partial V}{\partial q^j}$
12. Generalized potential of force Q_j: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial \dot{q}^j}\right) - \frac{\partial V}{\partial q^j} = Q_j$
13. Lagrange's eqs. of 2nd kind: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = 0$ where $L = T - \Pi$.
14. L is a function on TQ.
15. Trajectory of system in $t \in [t_1, t_2]$ is such, that action $S = \int_{t_1}^{t_2} L(q^j(t), \dot{q}^j(t), t)\mathrm{d}t$ is stationary, $\delta S = 0$.
16. $f(q_j(t), \dot{q}_j(t))$ is an integral of motion if it is constant in time over the trajectory $q_j(t)$. $(\mathrm{d}f(t)/\mathrm{d}t = 0)$.
17. If q^i is a cyclic coordinate (L is not a funct. of q^i), then $\partial L / \partial \dot{q}^i$ is an integral of motion.
18. If L is not a funct. of t, then generalized energy $h(q^i, \dot{q}^i) = \frac{\partial L}{\partial \dot{q}^j}\dot{q}^j - L$ is an integral of motion.
19. If forces are conservative and constraints are both holonomic and ccleronomic, then $h = T + \Pi = const$.
20. Lagrangian for a system of coupled oscillators: $L = \frac{1}{2}\left(T_{ij}\dot{q}_i\dot{q}_j - \Pi_{ij}q_i q_j\right)$ which yields $\tilde{T}\ddot{\vec{q}} - \tilde{\Pi}\vec{q} = 0$ what can be solved using eigen-decomposition.
21. Euler-Ostrogradsky eq. for $L(y(x^i), y_{,i}(x^i), x^i)$: $\sum_i \frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial y_{,i}}\right) - \frac{\partial L}{\partial y} = 0$ where $y_{,i} = \partial y / \partial x^i$.
22. Euler-Poisson eq. for $L(q^j, \dot{q}^j, \ldots, q^{(k)j}, t)$: $-(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial q^{(k)j}}\right) + \cdots + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = 0$.
23. Euler-Lagrange constrained eq. $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(L+\lambda\phi)}{\partial \dot{q}^j}\right) - \frac{\partial(L+\lambda\phi)}{\partial q^j} = 0$
24. $S = \int_\Omega \mathcal{L}(\Phi, \Phi_{,\mu}, x^\mu)\mathrm{d}\Omega$ where $\mathrm{d}\Omega = \mathrm{d}V\mathrm{d}t$ and $\mathcal{L}$ is Lagrangian density.
25. $\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \Phi_{,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0$
26. Noether's theorem: if change of coord. $t' = t + \epsilon T(q^j, t)$, $q'^j = q^j + \epsilon Q^j(q^j, t)$ does not change L, $(L(q^j(t), \dot{q}^j(t), t) = L'(q'^j(t'), \dot{q}'^j(t'), t'))$ then the integral of motion is $\mathcal{Z} = \sum_j \frac{\partial L}{\partial \dot{q}^j}(Q^j - \dot{q}^j T) + LT$.
27. $L + \mathrm{d}F/\mathrm{d}t$ where F is any smooth function gives the same eqs. of motion as L.
28. For EM. field $L = mv^2/2 - e(\varphi - \vec{v} \cdot \vec{A})$.
29. Calibration of EM. fields: $\varphi' = \varphi - e^{-1}\partial F / \partial t$, $\vec{A}' = \vec{A} + e^{-1}\nabla F$.
30. Add my coordinate transform of E-L equation.

IX. HAMILTONIAN FORMALISM

1. Canonical (generalized) momentum: $p_j = \partial L / \partial \dot{q}^j$
2. $\partial p_i / \partial q^j = 0$, $\partial q^i / \partial q^j = \delta^i{}_j$, $\partial p_i / \partial p_j = \delta_i{}^j$
3. The physical state of a system is given by a point in the phase space.
4. Canonical variables q^i and p_i are implicit functs. of t: they depend on trajectory $q^i = q^i(\gamma(t))$, $p_i = p_i(\gamma(t))$. Hence $\partial q^i / \partial t = \partial p_i / \partial t = 0$.
5. Every singular point in the phase space is a stable equilibrium state of the system.
6. Hamiltonian is defined as $H(q^j, p_j, t) = \sum p_i \dot{q}^i - L(q^j, \dot{q}^i, t)$ where $\dot{q}^i = \dot{q}^i(q^j, p_j, t)$ is an inverse of $p_j = p_j(q^i, \dot{q}^i, t)$.
7. H is a function on T^*Q.
8. Lagrange's and Hamilton's formalisms are equivalent $\Leftrightarrow$ if L has no inflex points, $\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0$.
9. $L(q^j, \dot{q}^j, t) = \sum p_i \dot{q}^i - H$ where p_i is an inverse of $\dot{q}^i = \partial H / \partial p_i$.
10. Hamilton's canonical equations: $\mathrm{d}q^i / \mathrm{d}t = \partial H / \partial p_i$ and $\mathrm{d}p_i / \mathrm{d}t = -\partial H / \partial q^i$.
11. Legendre transformation: Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ a convex function; then its Legendre transform is the function $f^* : I^* \to \mathbb{R}$ defined by $f^*(x^*) = \sup_{x \in I}(x^* x - f(x))$ where $x^* \in I^*$.

12. Legedre transformation $TQ \leftrightarrow T^*Q$ is given by $L(q^j, \dot{q}^i, t) \leftrightarrow H(q^j, p_j, t)$, hence $H = p_i\dot{q}^i - L \leftrightarrow L = p_i\dot{q}^i - H$ and $p_i = \partial L/\partial\dot{q}^i$, $\dot{q}^i = \partial L/\partial p_i$.

13. Point mass in potential V:
a) $H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z)$
b) $H = \frac{1}{2m}(p_R^2 + \frac{p_\theta^2}{R^2} + p_z^2) + V(R, \theta, z)$
c) $H = \frac{1}{2m}(p_R^2 + \frac{p_\vartheta^2}{R^2} + \frac{p_\varphi^2}{R^2\sin^2\vartheta}) + V(R, \vartheta, \varphi)$

14. For EM. field $H = (\vec{p} - e\vec{A})^2/2m + e\varphi$.

15. Poisson brackets: $\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right)$

16. **a)** $\{f, g\} = -\{g, f\}$
b) $\{c_1 f_1 + c_2 f_2, g\} = c_1\{f_1, g\} + c_2\{f_2, g\}$
c) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$
d) $\{f, f\} = 0$
e) $\{fg, h\} = \{f, h\}g + f\{g, h\}$
f) $\frac{\partial}{\partial t}\{f, g\} = \{\frac{\partial f}{\partial t}, g\} + \{f, \frac{\partial g}{\partial t}\}$

17. Fundamental Poisson brackets:
$\{q^i, q^j\} = 0$, $\{p_i, p_j\} = 0$, $\{q^i, p_j\} = \delta^i{}_j$

18. $\{x_i, L_j\} = \epsilon_{ijk}x_k$, $\{p_i, L_j\} = \epsilon_{ijk}p_k$, $\{L_i, L_j\} = \epsilon_{ijk}L_k$ where $L_i = \epsilon_{ijk}x_j p_k$ is an angular mom.

19. $\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t}$

20. Function $f(q^i, p_i)$ is an integral of motion if $\{f, H\} = 0$.

21. If H is not a funct. of t, $H = H(q^i, p_i)$ is an integral of motion.

22. If f and g are integrals of motion, then also $\{f, g\}$ is an integral of motion.

23. $\mathrm{d}q^i/\mathrm{d}t = \{q^i, H\}$ and $\mathrm{d}p_i/\mathrm{d}t = \{p_i, H\}$

24. $H(q^i, p_i, t) \to H'(Q^j(q^i, p_i, t), P_j(q^i, p_i, t), t)$ is a canonical transformation if it preserves the form of Hamilton's canonical eqs.

25. Canonical transformations are generated by functions in 1st col. Requirements for canonicity are in 2nd col. and for integrability are in 3rd col.

$F_1(q^j, Q^j, t)$	$\frac{\partial F_1}{\partial q^i} = p_i$ $\frac{\partial F_1}{\partial Q^k} = -P_k$	$\frac{\partial p_i}{\partial Q^k} = -\frac{\partial P_k}{\partial q^i}$
$F_2(q^j, P_j, t)$	$\frac{\partial F_2}{\partial q^i} = p_i$ $\frac{\partial F_2}{\partial P_k} = Q^k$	$\frac{\partial p_i}{\partial P_k} = \frac{\partial Q^k}{\partial q^i}$
$F_3(p_j, Q^j, t)$	$\frac{\partial F_3}{\partial p_i} = -q^i$ $\frac{\partial F_3}{\partial Q^k} = -P_k$	$\frac{\partial q^i}{\partial Q^k} = \frac{\partial P_k}{\partial p_i}$
$F_4(p_j, P_j, t)$	$\frac{\partial F_4}{\partial p_i} = -q^i$ $\frac{\partial F_4}{\partial P_k} = Q^k$	$\frac{\partial q^i}{\partial P_k} = -\frac{\partial Q^k}{\partial p_i}$

First test if transformation is integrable (3rd col.), then find F_a (2nd col.), finally use gen. funct. as $H'(Q^j, P_j, t) = H(q^j, p_j, t) + \partial F_a/\partial t$.

26. Solving for generating function is analogic to solving for potential of conservative force.

27. Generating functions F_a are connected with Legendre dual transformation: $F_1(q, Q)$,
$F_2(q, P) = F_1 + P_i Q^i$, $F_3(p, Q) = F_1 - p_i q^i$,
$F_4(p, P) = F_1 + P_i Q^i - p_i q^i$.

28. Transformation is canonical $\Leftrightarrow$
$\{Q^i, P_j\} = \delta^I{}_j$ and $\{Q^i, Q^j\} = 0 = \{P_i, P_j\}$.

29. Set of all canonical transformations is a group, which group operation is composition of two transformations.

30. Poisson brackets are invariant under canonical transformations. $\{f, g\}_{q,p} = \{f, g\}_{Q,P}$

31. Volume of phase-space is invariant under canonical transformations.

32. Phase space distribution $\rho(p, q)$ determines the probability $\rho(p, q)\mathrm{d}^n q\mathrm{d}^n p$ that the system will be found in the infinitesimal phase space volume $\mathrm{d}^n q\mathrm{d}^n p$.

33. Liouville theorem: the phase-space distribution function is constant along the trajectories of the system, $\partial\rho/\partial t + \{\rho, H\} = 0$.

34. Hamilton-Jacobi theory

X. PROPERTIES OF MATERIALS

1. Hooke's law: $\sigma = Y\epsilon = F/S$; where σ is tensile stress, $\epsilon = \Delta L/L$ is extension, $Y = kL/S$ is Young's modulus;

2. Energy density of deformation: $w = E/V = Y\epsilon^2/2$

3. $l = l_0(1 + \alpha\Delta T)$, $V = V_0(1 + \beta\Delta T)$, $\rho = \rho_0(1 - \beta\Delta T)$;
for isotropic materials $\beta \approx 3\alpha$

4. Speed of sound in elastic material: $c_s = \sqrt{Y/\rho}$

XI. FLUID MECHANICS

1. Hydrostatic pressure: $p_h = h\rho g$; NB! atmospheric pressure $p = p_A + p_h$; Buoyancy: $F_B = V\rho g$, (V-sunked);

2. Continuity equation: $S\rho v = const.$; Special case - continuity condition: $S\rho v = dm/dt$

3. Bernoulli eq. - incompr. fluid: $p + \rho\varphi + \rho v^2/2 = const.$
In homog. fiels, in gravit. potential: $\varphi = gh$;

4. Torricelli's law: $v = \sqrt{2gh}$ if Energy is conserved or $v = \sqrt{gh}$ if momentum is conserved;

5. Liquid surface takes equipot. shape (neglecting σ); in incompr. liquid: $p = p_0 - w$, w is vol. dens. of pot. en.

6. Speed of shallow $(h \ll \lambda)$ water waves: $v = \sqrt{gh}$
derivation via Bernoulli and Continuity eqs;

7. Surface tension: $U = S\sigma$, $F = l\sigma$, $p = 2\sigma/R$;
Generally $p = \sigma \sum 1/R_i$, In case of 2 surfaces F, p times 2;

8. Young-Laplace eq: $\Delta p = \sigma \nabla \cdot \vec{n}$

9. Jurin's law: liquid height $h = 2\sigma \cos\theta / \rho g r_0$
r_0 - tube radius; θ - contact angle;

10. Contact angle of droplet with the underlay:
$\cos\theta = (\sigma_{SG} - \sigma_{SL})/\sigma_{GL}$