Cheatsheet on linear algebra.
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I. Notes

1. **Def.** = definition, **St.** = statement, **Lemma.** = Lemma, **Th.** = theorem, **Note.** = note
2. $m, n \in \mathbb{N}, R^{m\times n}$ is always written only as $R^{m\times n}$.

II. Matrices

1. **St.** (Asociativity of matrix multiplication). Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{p\times q}$. Then $(AB)\,C = A\,(BC)$.
2. **St.** (Distributivity of matrix multiplication). Let $A, B \in \mathbb{R}^{m\times n}, C, D \in \mathbb{R}^{n\times p}$. Then $(A+B)\,C = AB + AC$, $A\,(C+D) = AC + AD$.
3. **St.** If $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times p}$, then $(AB)^T = B^T A^T$.
4. **St.** Let $A, B, C \in \mathbb{R}^{n\times n}$ so that $BA = AC = E$. Then $B = C$.
5. **St.** Let $A, B \in \mathbb{R}^{n\times n}$, then $(AB)^{-1} = B^{-1}A^{-1}$.
6. **St.** Let $A \in \mathbb{R}^{n\times n}$, then $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.
7. **Def.** Let $A \in \mathbb{C}^{m\times n}$, then $A^+ = \bar{A}^T$ is called the *hermitian conjugate* of A. If $A = A^+$ the matrix A is called a *hermitian matrix.*
8. **St.** Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, then $(AB)^+ = B^+A^+$.
9. **St.** Let $A \in \mathbb{C}^{n\times n}$, then $\left(A^+\right)^{-1} = \left(A^{-1}\right)^+$.
10. **St.** If $A \in \mathbb{R}^{m\times n}$, then $\mathrm{Ker}A \leq \mathbb{R}^n$.
11. **St.** The map $f_A : \mathbb{R}^n \to \mathbb{R}^m$ is bijective if and only if A is a square matrix and there exists a matrix X, such that $AX = E$.

III. Vector spaces

1. **Def.** Group: Let $\circ : G \times G \to G$ be a binary operation over the set G. Then $(G, \circ)$ is a group if:
   (a) $\forall a, b, c \in G : (a \circ b) \circ c = a \circ (b \circ c)$ (asociaticity)
   (b) $\exists e \in G : \forall a \in G : a \circ e = e \circ a = a$ (identity element)
   (c) $\forall a \in G : \exists a^{-1} \in G : a \circ a^{-1} = a^{-1} \circ a = e$ (inverse element).
2. **Def.** If $(G, \circ)$ is a group and $\forall a, b \in G : a \circ b = b \circ a$, then it is an abelian (commutative) group.
3. **Def.** Commutative field: Set $\mathbb{F}$ is a commutative field if there are two operations $+$ and $\cdot$ defined on that set, such that:
   (a) $(\mathbb{F}, +)$ is a commutative group.
   (b) If 0 is an identity element of $(\mathbb{F}, +)$, then $(\mathbb{F} \setminus \{0\}, \cdot)$ is a commutative group.
   (c) $\forall a.b.c \in \mathbb{F} : a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity).
4. **Def.** Vector space:
5. **Def.** Let V be a vector space over $\mathbb{F}$ and $W \subset V$, $W \neq \emptyset$, such that $\forall \boldsymbol{v}, \boldsymbol{w} \in W, \forall r, s \in \mathbb{F} : r\boldsymbol{v} + s\boldsymbol{w} \in W$. Then W is a subspace of V $(W \leq V)$.
6. Let V be a vector space over $\mathbb{F}$ and $M \subset V$. Then $\langle M \rangle \leq V$.
7. Let V be a vector space over $\mathbb{F}$. Then following holds:
   (a) $M \subset V$ having at least two elements is lineary dependent if and only if there exist $v \in M$, which can be written as a linear combination of M other elements.
   (b) Let M be generating V. Then M is lineary dependent, if and only if there exists $N \subset M$ that is generating V.

IV. Basis and dimension

1. **Def.** Basis: Let V be a vector space over $\mathbb{F}$. Set that is generating V and is lineary independent is called basis of V.
2. **St.** Let V be a vector space over $\mathbb{F}$. Set M is a basis of V if and only if every vector from V can be written as only one linear combination of M elements.
3. **Def.** Vector space V over $\mathbb{F}$ is called a vector space of a finite dimension if there esists a finite basis.
4. Let V be a vector space over $\mathbb{F}$. Then following statements are equivalent:
   (a) V is of finite dimension.
   (b) In V exists a finite set of generators.
   (c) From every set of V's generators is possible to take out a V's finite basis.
5. **Th.** Given a vector space V of finite dimension, then all the bases of the vector space have the same number of elements.
6. **Lemma.** Steinitz exchange lemma: Let V be a vector space over $\mathbb{F}$, M it's n-element set of generators and $N = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ a lineary independet set in V. Then $k \leq n$ and elemets of M is possible to reorder into $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ so that $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{u}_{k+1}, \dots, \boldsymbol{u}_n\}$ generates V.
7. **St.** Let V be a vector space over $\mathbb{F}$ of dimension n and $M \subset V$.
   (a) If M is lineary independent, then $|M| \leq n$.
   (b) If M generates V, then $|M| \geq n$.
   (c) If $|M| = n$, then M is a basis of V.
8. **St.** Let V be a vector space of dimension n and W its subspace. Then W is a space of finite dimension and $\dim W \leq n$.
9. **St.** Let V be a vector space of dimension n and W its subspace, N a basis of W. Then there exists a set $M \supset N$, which is basis of V.

V. Rank

1. **St.** Let $A = (\boldsymbol{a}_1 | \dots | \boldsymbol{a}_n) \in \mathbb{F}^{m\times n}$, where $\boldsymbol{a}_i \in \mathbb{F}^m$. Then $\mathrm{Ker}A = \{\boldsymbol{x} \in \mathbb{F}^n | A\boldsymbol{x} = \boldsymbol{o}\} \leq F^n$,
   $\mathrm{Im}A = \langle \boldsymbol{a}_1, \dots, \boldsymbol{a}_n \rangle \leq \mathbb{F}^m$.
2. **St.** Let $A = (\boldsymbol{a}_1 | \dots | \boldsymbol{a}_n) \in \mathbb{F}^{m\times n}$ and $R \in \mathbb{F}^{m\times m}$ be regular. Then (a) $\mathrm{Ker}\,(RA) = \mathrm{Ker}A$, (b) $\mathrm{Im}\,(RA)^T = \mathrm{Im}A^T$.
3. **St.** Let $A = (\boldsymbol{a}_1 | \dots | \boldsymbol{a}_n) \in \mathbb{F}^{m\times n}$ and $Q \in \mathbb{F}^{n\times n}$ be regular. Then (a) $\mathrm{Ker}\,(AQ)^T = \mathrm{Ker}A^T$, (b) $\mathrm{Im}\,(AQ) = \mathrm{Im}A$.
4. **Lemma.** Let $A = (\boldsymbol{a}_1 | \dots | \boldsymbol{a}_n) \in \mathbb{F}^{m\times n}$ and $R \in \mathbb{F}^{m\times m}$ be regular. Let $RA = A' = (\boldsymbol{a}'_1 | \dots | \boldsymbol{a}'_n)$. If $(s_1, \dots, s_n) \in \mathbb{F}^n$, then
   $\sum_{i=1}^n s_i \boldsymbol{a}_i = \boldsymbol{o} \iff \sum_{i=1}^n s_i \boldsymbol{a}'_i = \boldsymbol{o}$.
5. **Th.** If $A \in \mathbb{F}^{m\times n}$ then $\dim \mathrm{Im}A = \dim \mathrm{Im}A^T$.
6. **Def.** Let $A \in \mathbb{R}^{m\times n}$. Then rank of A is defined as $\mathrm{rank}A = \dim \mathrm{Im}A = \dim \mathrm{Im}A^T$.
7. **St.** Let $A \in \mathbb{R}^{m\times n}, R \in \mathbb{R}^{p\times m}, Q \in \mathbb{R}^{n\times q}$. Then
   (a) $\mathrm{rank}(RA) \leq \mathrm{rank}(A)$, (b) $\mathrm{rank}(AQ) \leq \mathrm{rank}(A)$,
   (c) $\mathrm{rank}(A) = \mathrm{rank}(A^T)$,
   (d) if $p = m$ and R is regular $\mathrm{rank}(RA) = \mathrm{rank}(A)$,
   (e) if $n = q$ and Q is regular $\mathrm{rank}(AQ) = \mathrm{rank}(A)$.
8. **Th.** Rank-nullity theorem: Let $A \in \mathbb{R}^{m\times n}$. Then $\dim \mathrm{Ker}A + \dim \mathrm{Im}A = n$.
9. **St.** Let $A \in \mathbb{R}^{m\times n}$. Then $\mathrm{Ker}A^TA = \mathrm{Ker}A$ and $\mathrm{rank}A^TA = \mathrm{rank}A$.
10. **Th.** Frobenius theorem: Let $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{F}^m$. System $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if $\mathrm{rank}A = \mathrm{rank}(A|b)$.

VI. Vector representation and linear maps

1. **St.** Let V be a vector space over $\mathbb{F}$ of finite dimension, B its basis, $\boldsymbol{u}, \boldsymbol{v} \in V, r, s \in \mathbb{F}$. Then $[r\boldsymbol{u} + s\boldsymbol{v}]^B = r\,[\boldsymbol{u}]^B + s\,[\boldsymbol{v}]^B$.

2. **Def.** Definition of linear map: Let V, W be two vector spaces over $\mathbb{F}$, and $f : V \to W$ such that $\forall r, s \in \mathbb{F}, \forall \boldsymbol{u}, \boldsymbol{v} \in V : f(r\boldsymbol{u} + s\boldsymbol{v}) = rf(\boldsymbol{u}) + sf(\boldsymbol{v})$. Then f is a linear map.
3. **Def.** Matrix of linear map: Let V, W be two vector spaces over $\mathbb{F}$, $f : V \to W$ is a linear map, $B = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ basis of V, $C = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m)$ basis of W. Then matrix $[f]_B^C = \left([f(\boldsymbol{v}_1)]^C | \ldots | [f(\boldsymbol{v}_n)]^C\right)$ is called matrix of linear map in respect to basis B and C.
4. **Def.** Let V, W be two vector spaces over $\mathbb{F}$, $f : V \to W$ is a linear map, B basis of V, C basis of W, $\boldsymbol{v} \in V$. Then $[f(\boldsymbol{v})]^C = [f]_B^C [\boldsymbol{v}]^B$.
5. **Def.** Let $f : V \to W$ be a linear map. Then $\mathrm{Ker} f = \{\boldsymbol{v} \in V | f(\boldsymbol{v}) = \boldsymbol{o}\}$ and $\mathrm{Im} f = \{\boldsymbol{w} \in W | \exists \boldsymbol{v} \in V : f(\boldsymbol{v}) = \boldsymbol{w}\}$.
6. **St.** Let V, W be two vector spaces over $\mathbb{F}$, $f : V \to W$ is a linear map, $B = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ basis of V, $C = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m)$ basis of W. Then $[\mathrm{Ker} f]^B = \mathrm{Ker}\,[f]_B^C$ and $[\mathrm{Im} f]^C = \mathrm{Im}\,[f]_B^C$.
7. **Def.** Let V be a vector space over $\mathbb{F}$ of dimension n, B, C its basis. Matrix $[\mathrm{Id}]_B^C \in \mathbb{F}^{n \times n}$ is called matrix of transform from basis C to basis B.
8. **St.** Let V, W be two vector spaces over $\mathbb{F}$, $f, g : V \to W$ are linear maps, B basis of V, C basis of W, $r, s \in \mathbb{F}$. Then $[rf + sg]_B^C = r[f]_B^C + s[g]_B^C$.
9. **St.** Let U, V, W be three vector spaces over $\mathbb{F}$, $f : U \to V$, $g : V \to W$ are linear maps, B basis of U, C basis of V and D basis of W. Then $[g \circ f]_B^D = [g]_C^D [f]_B^C$.
10. **St.** Let V, W be two vector spaces of finite dimension over $\mathbb{F}$, $f : V \to W$ is a linear map, B, B' bases of V, C, C' bases of W. Then $[f]_{B'}^{C'} = [\mathrm{Id}]_B^{C'} [f]_B^C [\mathrm{Id}]_{B'}^C$.
11. **Def.** Homomorphism = linear morphism, monomorphism = injective hom., epimorphism = surjective hom., isomorphism = bijective hom., endomorphism = hom. into itself, automorpism = injective monomorhism.
12. **St.** Let U, V, W be three vector spaces over $\mathbb{F}$, $f \in \mathrm{Hom}(V, W)$, $g \in \mathrm{Hom}(U, V)$. Then
    (a) f is a monomorphism $\iff \mathrm{Ker} f = \emptyset$,
    (2) f is an epimorphism $\iff \mathrm{Im} f = W$,
    (3) f, g are monomorphisms $\implies f \circ g$ is a monomorph.,
    (4) f, g are epimorphisms $\implies f \circ g$ is an epimorphism,
    (5) $f \circ g$ is monomorphism $\implies g$ is a monomorphism,
    (6) $f \circ g$ is an epiomorphism $\implies f$ is a epimorphism
13. **Th.** Two vector spaces V, W of finite dimension over $\mathbb{F}$ are isomorphic if and only if they have same dimension.
14. **Th.** Let V, W be two vector spaces. Then $\dim V = n \wedge \dim W = m \iff \dim \mathrm{Hom}(V, W) = mn$.
15. **Th.** Homomorphism given by values of basis: Let V, W be two vector spaces over $\mathbb{F}$, $f : V \to W$ is a linear map, $B = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ basis of V, $C = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m)$ basis of W. Then there exists just one $f \in \mathrm{Hom}(V, W)$ such that $f(\boldsymbol{v}_i) = w_i$. Furthemore, f is monomorphism if and only if C is lineary independent and f is epimorphism if and only if C generates W.
16. **Th.** Rank-nullity theorem: Let V, W be two vector spaces over $\mathbb{F}$, $\dim V = n$, $f \in \mathrm{Hom}(V, W)$. Then $\dim \mathrm{Ker} f + \dim \mathrm{Im} f = n$.
17. **St.** Let V be a vector space of finite dimension over $\mathbb{F}$, $f \in \mathrm{End}(V)$. If f is mono- or epimorphism, then it is also an automorphism.

VII. Permutations, determinant

1. **Def.** The symmetric group S_n is a group of all bijective functions from $M = \{1, 2, \ldots, n\}$ to M.
2. **Def.** Let $A \in \mathbb{F}^{n \times n}$, then a determinant of a matrix A is defined as $\det A = \sum_{\pi \in S_n} \mathrm{sgn} \pi a_{1\pi(1)} a_{2\pi(2)} \ldots a_{n\pi(n)}$.
3. **Th.** Let $A \in \mathbb{F}^{n \times n}$, then $\det A = \det A^T$ and $\det A^+ = \overline{\det A}$.
4. **St.** Let $i \in \{1, \ldots, n\}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n, \boldsymbol{a}'_i \in \mathbb{F}^n, r, r' \in \mathbb{F}, \rho \in S_n$. Then
   (a) $\det(\boldsymbol{a}_1 | \ldots | r\boldsymbol{a}_i + r'\boldsymbol{a}'_i | \ldots | \boldsymbol{a}_n) = r \det(\boldsymbol{a}_1 | \ldots | \boldsymbol{a}_i | \ldots | \boldsymbol{a}_n) + r' \det(\boldsymbol{a}_1 | \ldots | \boldsymbol{a}'_i | \ldots | \boldsymbol{a}_n)$,
   (b) $\det\left(\boldsymbol{a}_{\rho(1)} | \ldots | \boldsymbol{a}_{\rho(n)}\right) = \mathrm{sgn}(\rho) \det(\boldsymbol{a}_1 | \ldots | \boldsymbol{a}_n)$.
5. **St.** Let $A \in \mathbb{F}^{n \times n}$, then if A has two equal columns/rows or one zero column/row, then $\det A = 0$.
6. **St.** Let $A \in \mathbb{F}^{n \times n}$, then for elementary column/row operation of (a) column/row switching the sign of determinant is changed, (b) column/row multiplication by $r \in \mathbb{F}$ the determinant of A is multiplied by r, (c) addition of multiple of column/row to another column/row does not change determinant.
7. **Th.** Matrix $A \in \mathbb{F}^{n \times n}$ is regular if and only if $\det A \neq 0$.
8. **Def.** Let $A \in \mathbb{F}^{m \times n}$ and $k \in \mathbb{N} : 0 < k \leq m, k \leq n$. A minor of order k of A is the determinant of a $k \times k$ matrix obtained from A by deleting $m-k$ rows and $n-k$ columns.
9. **Th.** Let $A \in \mathbb{F}^{m \times n}$. Then $\mathrm{rank}(A) = k$ if and only if the highest order of nonzero minor of A is k.
10. **Def.** The i, j cofactor of the matrix $A \in \mathbb{F}^{n \times n}$ is the scalar C_{ij} defined by $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the i, j minor of A, that is, the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting the i-th row and the j-th column of A.
11. **Def.** Let $A \in \mathbb{F}^{m \times n}$. Then the matrix formed by all of the cofactors of A is called the cofactor matrix C.
12. **Th.** Laplace expansion: Suppose $A \in \mathbb{F}^{n \times n}$ and fix any $i, j \in \{1, 2, \ldots, n\}$. Then $\det A = \sum_{j'=1}^n a_{ij'} C_{ij'} = \sum_{i'=1}^n a_{i'j} C_{i'j}$.
13. **Th.** Cramer's rule: Let $A \in \mathbb{F}^{n \times n}$ be a regular matrix, $\boldsymbol{b} \in \mathbb{F}^n$. Then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution, and i-th value of $\boldsymbol{x}$ is given by $x_i = \det A_{i,\boldsymbol{b}} / \det A$, where $A_{i,\boldsymbol{b}}$ is the matrix formed by replacing the i-h column of A by the column vector $\boldsymbol{b}$.
14. **Def.** Let $A \in \mathbb{F}^{m \times n}$. Then the adjugate of A is the transpose of the cofactor matrix C of A, $\mathrm{adj}(A) = C^T$.
15. **Th.** Let $A \in \mathbb{F}^{m \times n}$ be regular. Then $A^{-1} = \mathrm{adj}(A) / \det(A)$.
16. **Th.** Let $A, B \in \mathbb{F}^{m \times n}$. Then $\det AB = \det A \det B$.
17. **St.** Let $A, R \in \mathbb{F}^{m \times n}$, where R is regular. Then $\det(R^{-1}) = 1/\det(R)$, and $\det(R^{-1}AR) = \det(A)$.
18. **St.** Let V be a vector space of finite dimension over $\mathbb{F}$, B it's basis and $f \in \mathrm{End}(V)$. Then $\det [f]_B^B$ is called *determinant of endomorphism* f, and is for every base B equal.
19. **Def.** Let $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ be a base of $\mathbb{R}^n$. The base B is positively oriented (right-handed) if $\det(\boldsymbol{b}_1 | \ldots | \boldsymbol{b}_n) > 0$, otherwise it's negatively oriented (left-handed).
20. **Def.** Let $A \in \mathbb{F}^{n \times n}$. Then $\mathrm{Tr} A = \sum_{i=1}^n a_{ii}$ is called a *trace* of A.
21. **St.** Let $A \in \mathbb{F}^{n \times p}$, $B \in \mathbb{F}^{p \times q}$, $C \in \mathbb{F}^{q \times n}$. Then $\mathrm{Tr} ABC = \mathrm{Tr} BCA$.
22. **St.** Let $A, R \in \mathbb{F}^{n \times n}$ and R be regular. Then $\mathrm{Tr} R^{-1}AR = \mathrm{Tr} A$.

VIII. Diagonalization

1. **Def.** Let V be a vector space over $\mathbb{F}$ and $f \in \mathrm{End}(V)$ and $\boldsymbol{v}$ be a nonzero vector in V, then v is an eigenvector of f if $f(\boldsymbol{v}) = \lambda \boldsymbol{v}$ where $\lambda \in \mathbb{F}$, known as the eigenvalue associated with $\boldsymbol{v}$.
2. **Def.** The *characteristic polynomial of matrix* $A \in \mathbb{F}^{n \times n}$

is the polynomial defined by $p_A(\lambda) = \det(A - \lambda E)$. The *characteristic polynomial of endomorphism* f is $\det(f - \lambda E)$, or the characteristic polynomial of its arbitrary matrix $[f]_B^B$.

3. **St.** Let $A \in \mathbb{F}^{n\times n}$. Then its characteristic polynomial can be written as $p_A(\lambda) = (-\lambda)^n + \mathrm{Tr}(A)(-\lambda)^{n-1} + \ldots + \det(A)$.
4. **Th.** Every non-constant polynomial with complex coeficients has at least one root in $\mathbb{C}$.
5. **St.** Let $p(x)$ be a polynome with complex coeficients of degree $n > 0$. Then it has together n roots with theirs multiplicities.
6. **Def.** Set of all eigenvalues of matrix A is called *spectrum* of A, denoted $\sigma(A)$. For $\lambda_i \in \sigma(A)$ is it's *algebraric multiplicity* defined as multiplicity of root λ_i of $p_A(\lambda)$, and it's *geometric multiplicity* defined as $\dim \mathrm{Ker}\,(A - \lambda_i E)$. These concepts are defined similary for endomorphisms.
7. **Lemma.** Let V be a vector space over $\mathbb{F}$ of dimension n, $f \in \mathrm{End}(V)$ and for all elements of set $M = \{\lambda_1, \ldots, \lambda_k\}$ let be chosen an arbitrary base B_i of eigenspace V_{λ_i}. Then the set $B = B_1 \cup \ldots \cup B_n$ is lineary independent.
8. **St.** Let $A \in \mathbb{C}^{n\times n}$ be a hermitian matrix. Then all of its eigenvalues are real.
9. **St.** Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, λ, μ two different eigenvalues of A, $\boldsymbol{v} \in V_\lambda$, $\boldsymbol{w} \in V_\mu$. Then $\boldsymbol{v} \perp \boldsymbol{w}$.
10. **St.** Let $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ be an ortonormal base of $\mathbb{R}^n$, $U = [\mathrm{Id}]_B^K$. Then $U^T = U^{-1}$.
11. **Def.** Let $U \in \mathbb{F}^{n\times n}$. If $U^T U = E$ then U is called an *ortonormal matrix.*

IX. Direct sum

1. **Def.** Let V be a vector space over $\mathbb{F}$ and $W_1, W_2 \leq V$. Then the *sum of subspaces* W_1, W_2, denoted as $W_1 + W_2$, is the set of all vectors that could be written as $\boldsymbol{w}_1 + \boldsymbol{w}_2$, where $\boldsymbol{w}_i \in W_i$. Moreover, if $W_1 \cap W_2 = 0$, $W_1 + W_2$ is called the *direct sum of subspaces*, denoted as $W_1 \oplus W_2$. If $W_1 \oplus W_2 = V$, then W_2 is a *complement* of W_1 into V.
2. **St.** Every element of $W_1 \oplus W_2$ is possible to write as combination of vector $\boldsymbol{w}_1 \in W_1$ and vector $\boldsymbol{w}_2 \in W_2$ with only one way.
3. **St.** If V is a vector space over $\mathbb{F}$ and $W_1, W_2 \leq V$, then $W_1 + W_2 = \langle W_1 + W_2 \rangle$, hence the sum of subspaces is also a subspace. If M_1 and M_2 are sets of generators of W_1 and W_2, then $M_1 \cup M_2$ is generating $W_1 + W_2$.
4. Let $\mathcal{W}$ be a set of subspaces in V. Then thiers *sum* is defined as $\sum_{W\in\mathcal{W}} W := \langle \cup_{W\in\mathcal{W}} W \rangle$.
5. **Th.** (About dimension of sum and intersection). Let $W_1, W_2 \leq V$ be both of finite dimension. Then $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.
6. **Def.** Let $W_1, \ldots, W_k \leq V$. Then sum $W = W_1 + \ldots + W_k$ is *direct*, if every vector $\boldsymbol{w} \in W$ is possible to write as $\boldsymbol{w_1} + \ldots + \boldsymbol{w_k}$, where $\boldsymbol{w_i} \in W_i$, only one way. We denote it then as $W = W_1 \oplus \ldots \oplus W_k = \bigoplus_{i=1}^k W_i$.
7. **Th.** Let $W_1, \ldots, W_k \leq V$ be subspaces of finite dimesion. Then $\dim(W_1 \oplus \ldots \oplus W_k) = \dim W_1 + \ldots + \dim W_k$.
8. **St.** Let $n = n_1 + n_2$, $m = m_1 + m_2$, $p = p_1 + p_2$, $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{n\times p}$ with blocks $A_{ij} \in \mathbb{F}^{m_i\times n_j}$, $A_{ij} \in \mathbb{F}^{n_i\times p_j}$. Then product $C := AB$ has a block notation $C_{ij} = A_{i1}B_{ij} + A_{i2}B_{2j}$.
9. **St.** Let $A = \mathrm{diag}(A_{11}, \ldots, A_{kk})$ be a block diagonal matrix. Then (a) $\forall p \in \mathbb{N} : A^p = \mathrm{diag}(A_{11}^p, \ldots, A_{kk}^p)$ and if are all A_{ii} regular then also $A^{-p} = \mathrm{diag}(A_{11}^{-p}, \ldots, A_{kk}^{-p})$,
(b) $\mathrm{rank}(A) = \mathrm{rank}(A_{11}) + \ldots + \mathrm{rank}(A_{kk})$,
(c) $\mathrm{Tr}(A) = \mathrm{Tr}(A_{11}) + \ldots + \mathrm{Tr}(A_{kk})$,
(d) $\det(A) = \det(A_{11}) \ldots \det(A_{kk})$,
(e) $\sigma(A) = \sigma(A_{11}) \cup \ldots \cup \sigma(A_{kk})$.
10. **St.** Let $A, B \in \mathbb{F}^{n\times n}$. Then $\sigma(AB) = \sigma(BA)$ including multiplicativity.

X. Scalar (dot) product

1. **Def.** Let V be a vector space over $\mathbb{F}$. A map $g : V \times V \to \mathbb{F}$ is called an *inner prroduct* on V, if for $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V, r, s \in \mathbb{F}$ it satisfies following:
(a) *conjugate symmetry* $g(\boldsymbol{u}, \boldsymbol{v}) = \overline{g(\boldsymbol{v}, \boldsymbol{u})}$,
(b) $g(r\boldsymbol{u} + s\boldsymbol{v}, \boldsymbol{w}) = \bar{r}g(\boldsymbol{u}, \boldsymbol{w}) + \bar{s}g(\boldsymbol{v}, \boldsymbol{w})$,
(c) $\forall \boldsymbol{u} \in V, \boldsymbol{u} \neq 0 : g(\boldsymbol{u}, \boldsymbol{u}) > 0$.
2. **Def.** Form $g : V \times V \to \mathbb{F}$ is called a *billinear form* if it is linear in both arguments. If it is linear in first argument and antilinear in second it is called an *sesquilinear form.* If $forall \boldsymbol{u} \in V, \boldsymbol{u} \neq 0 : g(\boldsymbol{u}, \boldsymbol{u}) > 0$ it is called a *possitively definite form.*
3. **Def.** *Standard* inner product on $\mathbb{C}^n$ is defined as $g(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{u}^+ \boldsymbol{v}$ and is denoted as $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$.
4. Let $(V, \langle,\rangle)$ be an inner product space of a finite dimension, $C = (\boldsymbol{w}_i)_1^n$ its ortonormal base, $\boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{x} = [\boldsymbol{u}]^C, \boldsymbol{y} = [\boldsymbol{v}]^C$. Then $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^n \bar{x}_i y_i$.
5. **Note.** If $B = (\boldsymbol{u}_i)_1^n$ is a basis of (V, g) then $[g]_B$ can be find using $[g_{ij}]_B = \left([\boldsymbol{u}_i]^B\right)^T [g]_B [\boldsymbol{u}_j]^B$.
6. **Def.** A base B in V we called a *polar base of form* g, if $[g]_B$ is diagonal.
7. **Th.** Every symetric billinear or hermitian sequilinear form on vector space V of finite dimension has a polar base.
8. **Def.** If B is a polar base of (V, g) then we call it an *ortogonal* base. Moreover if $[g]_B = E$, we say that B is an *ortonormal* base.
9. **St.** If g is hermitian sequilinear form on vector space V over $\mathbb{C}$, then $\forall \boldsymbol{u}, \boldsymbol{v} \in V$ holds following
(1) $\Re g(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}\left(g(\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v}) - g(\boldsymbol{u}, \boldsymbol{u}) - g(\boldsymbol{v}, \boldsymbol{v})\right)$
(2) $\Im g(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}\left(g(i\boldsymbol{u} + \boldsymbol{v}, i\boldsymbol{u} + \boldsymbol{v}) - g(\boldsymbol{u}, \boldsymbol{u}) - g(\boldsymbol{v}, \boldsymbol{v})\right)$.
10. **Def.** *Orthogonal projection* $P_{\boldsymbol{v}} : V \to V$ of $\boldsymbol{u} \in V$ onto nonzero vector $\boldsymbol{v} \in V$ is defined as $P_{\boldsymbol{v}}(\boldsymbol{u}) = \frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{||\boldsymbol{v}||^2}\boldsymbol{v}$.
11. **Def.** Orthogonal projec. onto *orthogonal complement* $\boldsymbol{v}^\perp$ is defined as $P_{\boldsymbol{v}^\perp} = \mathrm{Id} - P_{\boldsymbol{v}}$ or $P_{\boldsymbol{v}^\perp}(\boldsymbol{u}) = \boldsymbol{u} - \frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{||\boldsymbol{v}||^2}\boldsymbol{v}$.
12. **St.** Let $(V, \langle,\rangle)$ be an inner product space over $\mathbb{F}$, $\boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{v} \neq 0$. Then $P_{\boldsymbol{v}}, P_{\boldsymbol{v}^\perp}$are linear maps, $P_{\boldsymbol{v}}(\boldsymbol{v}) = \boldsymbol{v}$, $\mathrm{Ker}P_{\boldsymbol{v}} = \boldsymbol{v}^\perp$, $\mathrm{Im}P_{\boldsymbol{v}} = \langle \boldsymbol{v} \rangle$, $\mathrm{Ker}P_{\boldsymbol{v}^\perp} = \langle \boldsymbol{v} \rangle$, $\mathrm{Im}P_{\boldsymbol{v}^\perp} = \boldsymbol{v}^\perp$, $P_{\boldsymbol{v}} \circ P_{\boldsymbol{v}} = P_{\boldsymbol{v}}$ and $P_{\boldsymbol{v}^\perp} \circ P_{\boldsymbol{v}^\perp} = P_{\boldsymbol{v}^\perp}$.
13. **Def.** Let $(V, \langle,\rangle)$ be an inner product space, $(\boldsymbol{w}_i)_1^k$ an ortonormal sequence in V, W its span. Then a *projection onto a subspace* $P_W : V \to V$ is defined as $P_W(\boldsymbol{u}) = \sum_{i=1}^k P_{\boldsymbol{w}_i}(\boldsymbol{u})$.
14. **St.** Let $(V, \langle,\rangle)$ be an inner product space, $(\boldsymbol{w}_i)_1^k$ an ortonormal sequence in V, W its span. Then $\boldsymbol{u} \in W \iff P_W(\boldsymbol{u}) = \boldsymbol{u}$, $\mathrm{Im}P_W = W$, $P_W \circ P_W = P_W$, $\boldsymbol{u} - P_W(\boldsymbol{u}) \in W^\perp$, $\forall \boldsymbol{u} \in V : ||P_W(\boldsymbol{u})|| \leq ||\boldsymbol{u}||$ and $\forall \boldsymbol{u} \in V, \forall \boldsymbol{v} \in W : ||\boldsymbol{u} - P_W(\boldsymbol{u})|| \leq ||\boldsymbol{u} - \boldsymbol{v}||$.
15. **St.** Let $(V, \langle,\rangle)$ be an inner product space, $W \leq V$ a subspace of a finite dimension. Then $W \oplus W^\perp = V$, $\mathrm{Ker}P_W = W^\perp$, $\left(W^\perp\right)^\perp = W$ and if V is of a finite ddimension, then $\dim W = \dim V - \dim W^\perp$.
16. **Th.** (Pythagora's). If $\boldsymbol{u} \perp \boldsymbol{v}$, then $||\boldsymbol{u} + \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$.
17. **Th.** (Schwarz inequality). Let $(V, \langle,\rangle)$ be an inner product space, $\boldsymbol{u}, \boldsymbol{v} \in V$. Then $|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq ||\boldsymbol{u}||||\boldsymbol{v}||$, while equality holds if $\boldsymbol{u}$ and $\boldsymbol{v}$ are collinear.
18. **Th.** (Triangle inequality). Let $(V, \langle,\rangle)$ be an inner product space, $\boldsymbol{u}, \boldsymbol{v} \in V$. Then $||\boldsymbol{u} + \boldsymbol{v}|| \leq ||\boldsymbol{u}|| + ||\boldsymbol{v}||$.

XI. Orthogonalization, orthogonal diagonalization

1. **Th.** (Gram–Schmidt process). Let $(V, \langle,\rangle)$ be an inner product space and $B = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$ its base. Then there exists an ortonormal base $C = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$ of V, such that $\forall k \in \{1, \ldots, n\} : \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \rangle = \langle \boldsymbol{w}_1, \ldots, \boldsymbol{w}_k \rangle$.
2. **Note.** (Gram–Schmidt process). First we ortogonalize the base using $\boldsymbol{v}_k = \boldsymbol{u}_k - \sum_{i=1}^{k-1} P_{\boldsymbol{v}_i}(\boldsymbol{u}_k)$ and then we ortonormalize using $\boldsymbol{w}_i = \boldsymbol{v}_i/||\boldsymbol{v}_i||$.
3. **St.** (Fourier's coefficients). Let $C = (\boldsymbol{w}_i)_1^n$ be an ortonormal base of an inner product space $(V, \langle,\rangle)$. Then every vector $\in V$ can be represented in respect to C as $[\boldsymbol{u}]^C = (\langle \boldsymbol{w}_1, \boldsymbol{u} \rangle, \ldots, \langle \boldsymbol{w}_n, \boldsymbol{u} \rangle)^T \in \mathbb{C}^n$, where $\langle \boldsymbol{w}_i, \boldsymbol{u} \rangle$ are called a *Fourier's coefficient.*
4. **Th.** (Parselval's equality). Let $(V, \langle,\rangle)$ be an inner product space of a finite dimension, $C = (\boldsymbol{w}_i)_1^n$ its ortonormal base and $\boldsymbol{u} \in V$. Then $\sum_{i=1}^n |\langle \boldsymbol{w}_i, \boldsymbol{u} \rangle|^2 = ||\boldsymbol{u}||^2$.
5. **Th.** (QR decomposition). Let $A \in \mathbb{C}^{m\times n}$ is a matrix with lineary independent columns. Then there exists just one $Q \in \mathbb{C}^{m\times n}$ and one $R \in \mathbb{C}^{m\times n}$ such, that $A = QR$ where Q is an ortonormal set with respect to a standard scalar product on $\mathbb{C}^n$, and where R is an upper triangular matrix with possitive numbers on diagonal (*note that R is formed from Fourier's coeff.*).
6. **Def.** Square matrix $U \in \mathbb{C}^{n\times n}$, such that $U^+U = UU^+ = E$ is called an *unitary matrix.* If U is real then its called an *orthogonal matrix.*
7. **St.** If $U, V \in \mathbb{C}^{n\times n}$ are unitary matrices, then also $U^+ = U^{-1}$ and UV are unitary matrices.
8. **St.** Let $(V, \langle,\rangle)$ be an inner product space, $B = (\boldsymbol{v}_i)_1^n$, $C = (\boldsymbol{w}_i)_1^n$ its ortonormal basis. Then $U := [\mathrm{Id}]_B^C$ is an unitary matrix.
9. **Def.** *Linear operator* is linear map defined as $\mathbb{A} : V \to W$. An *adjoint operator* to A, denoted as $A^* : W \to V$, is an operator such that $\forall \boldsymbol{v} \in V, \boldsymbol{w} \in W : \langle \boldsymbol{w}, \mathbb{A}\boldsymbol{v} \rangle_W = \langle \mathbb{A}^*\boldsymbol{w}, \boldsymbol{v} \rangle_V$.
10. **St.** Let $B = (\boldsymbol{v}_i)_1^n$ be an ortonormal base of V and $C = (\boldsymbol{w}_i)_1^n$ of W, $A = [\mathbb{A}]_B^C \in \mathbb{C}^{m\times n}$. Then there exists $\mathbb{A}^*$ and $[\mathbb{A}^*]_C^B 2 = A^+ \in \mathbb{C}^{n\times m}$.
11. **St.** Let V, W, U be inner product spaces, $\mathbb{A}, \mathbb{B} : V \to W, \mathbb{G} : U \to V$ an operators, for which there exists an adjoint operators, $\alpha, \beta \in \mathbb{C}$. Then $(\alpha\mathbb{A} + \beta\mathbb{B}) = \bar{\alpha}\mathbb{A}^* + \bar{\beta}\mathbb{B}^*$, $(\mathbb{A}\mathbb{G})^* = \mathbb{G}^*\mathbb{A}^*$, $(\mathbb{A}^*)^* = \mathbb{A}$, $\mathrm{Ker}\mathbb{A} = (\mathrm{Im}\mathbb{A}^*)^\perp$, $\mathrm{Ker}\mathbb{A}^* = (\mathrm{Im}\mathbb{A})^\perp$, $\mathrm{Ker}\mathbb{A} = \mathrm{Ker}\mathbb{A}^*\mathbb{A}$. Moreover if $\dim V = n \in \mathbb{N}$, then $\mathrm{rank}(\mathbb{A}) = \mathrm{rank}(\mathbb{A}^*)$ and $\mathrm{rank}(\mathbb{A}) = \mathrm{rank}(\mathbb{A}^*\mathbb{A})$.
12. **Def.** An operator $\mathbb{A} : V \to V$ is called *self-adjoint* if $\mathbb{A} = \mathbb{A}^*$, *unitary* if $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A} = \mathrm{Id}$ and *normal* if $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A}$.
13. **Lemma.** Let V be an inner product space of dimension n and $(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k)$ an ortonormal sequence in V. Then this sequence can be completed into an ortonormal base $(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$ of V.
14. **Th.** (Schur decomposition/triangulation). Let V be an inner product space of a finite dimension, $\mathbb{A} : V \to V$ such that $A := [\mathbb{A}]_K^K \in \mathbb{C}^{n\times n}$. Then there exists an ortonormal basis B in V such, that $R := [\mathbb{A}]_B^B$ is an upper triangular matrix, $A = URU^+$, where $U = [\mathrm{Id}]_B^K$.
15. **Th.** (Ortonormal diagonalization of a normal operator). Let V be an inner product space of a finite dimension. Then for an operator $\mathbb{A} : V \to V$ there exists an ortonormal basis of V, with respect to it is matrix of $\mathbb{A}$ diagonal, if and only if $\mathbb{A}$ is normal.
16. **Th.** (Spetral decomposition). Let $\mathbb{A} : V \to V$ be a normal operator, $\mathbb{P}_\lambda : V \to V$ be an operator of orthogonal projection onto an eigenspace asiciated with eigenvalue λ. Then $\mathbb{A} = \sum_{\lambda\in\sigma(\mathbb{A})} \lambda\mathbb{P}_\lambda$.
17. **Th.** An operator $\mathbb{A} : V \to V$ is self-adjoint if and only if all its eigenvalues are real.
18. **St.** If $\mathbb{A} \in \mathbb{R}^{n\times n}$ is symmetric or $\mathbb{A} \in \mathbb{C}^{n\times n}$ is hermitian, then there exists an ortogonal matrix U, such that U^+AU is diagonal.
19. **St.** Let V over $\mathbb{C}$ be an inner product space of a dimension n and $\mathbb{U} : V \to V$ be an unitar operator. Then $\forall \boldsymbol{v} \in V : ||\mathbb{U}\boldsymbol{v}|| = ||\boldsymbol{v}||$, $\forall \boldsymbol{v}, \boldsymbol{u} \in V : \langle \mathbb{U}\boldsymbol{v}, \mathbb{U}\boldsymbol{u} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$. If B is an ortonormal basis of V then $U := [\mathbb{U}]_B^B$ is an unitar matrix. If $(\boldsymbol{w}_i)_1^n$ is an ortonormal basis of V then also $(\mathbb{U}\boldsymbol{w}_i)_1^n$ is. $\mathbb{U}$ is an normal operator whose eigenvalues lies on an unit circle.

XII. Singular value decomposition

1. **Def.** If $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ is a set of vectors in an inner product space $(V, \langle,\rangle)$ over $\mathbb{F}$ then a *Gram matrix* of this set is $G \in \mathbb{F}^{n\times n}$, the Hermitian matrix of inner products, whose entries are given by $G_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$.
2. **St.** Gram matrix of a set $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ is regular if and only if is this set lineary independent.
3. **Def.** Let $A \in \mathbb{C}^{m\times n}$, then a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ is called *Moore–Penrose pseudoinverse* of A, if $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$ and if $AA^\dagger$ and $A^\dagger A$ are hermitian.
4. **St.** Let $A \in \mathbb{C}^{m\times n}$ be a matrix with lineary independent columns, then $A^\dagger = (A^+A)^{-1}A^+$ and $A^\dagger$ constitutes a left inverse $A^\dagger A = E$. However, if $A \in \mathbb{C}^{m\times n}$ is a matrix with lineary independent rows, then $A^\dagger = A^+(AA^+)^{-1}$ and $A^\dagger$ constitutes a right inverse $AA^\dagger = E$.
5. **St.** Let $A \in \mathbb{C}^{m\times n}$ be a matrix with lineary independent columns and K a kanonical basis, then $[P_{\mathrm{Im}A}]_K^K = AA^\dagger$. However, if $A \in \mathbb{C}^{m\times n}$ is a matrix with lineary independent rows and K a kan. basis, then $[P_{\mathrm{Im}A^+}]_K^K = A^\dagger A$.
6. **St.** Let $A \in \mathbb{C}^{m\times n}$ and matrices $B, C \in \mathbb{C}^{n\times m}$ be Moore–Penrose pseudoinverses of A. Then $B = C$.
7. **Th.** Let $A \in \mathbb{C}^{m\times n}, \boldsymbol{b} \in \mathbb{C}^m$. Then vector $A^\dagger\boldsymbol{b}$ is an approximated solution of system $A\boldsymbol{x} = \boldsymbol{b}$ which has the smallest possible norm.
8. **Def.** An operator is called a *possitively semidefinite* operator if all its eigenvalues are not negative.
9. **St.** An operator $\mathbb{B}$ is possitively semidefinite if and only if $\langle \boldsymbol{v}, \mathbb{B}\boldsymbol{v} \rangle \geq 0$ for every $\boldsymbol{v} \in V$.
10. **Def.** Let $\mathbb{B} : V \to V$ be a possitively semidefinite operator, C an ortonormal basis of V and $D = U^+[\mathbb{B}]_C^C U = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_i \in \sigma([\mathbb{B}])$. Then a <u>possitively semidefinite square root of</u> $\mathbb{B}$ is defined as $[\sqrt{\mathbb{B}}]_C^C = U\mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})U^+$.
11. **St.** Let $\mathbb{A} : V \to V$ be an operator. Then $\mathbb{A}^*\mathbb{A}$ is a possitively semidefinite operator.
12. **Def.** A *modul of operator is defined as* $|\mathbb{A}| := \sqrt{\mathbb{A}^*\mathbb{A}}$.

XIII. Quadratic forms, quadrics

1. **Def.** Let g be a billinear form on V. The map $Q_g : V \to \mathbb{R}$, defined by $Q_g(\boldsymbol{u}) = g(\boldsymbol{u}, \boldsymbol{u})$, is called *quadratic form* associated with g.
2. **Def.** Let g be a billinear form on V. Its *symmetrization* and *antisymmetrization* are defined $\forall \boldsymbol{u}, \boldsymbol{v} \in V$ by $g_S(\boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}(g(\boldsymbol{u}, \boldsymbol{v}) + g(\boldsymbol{v}, \boldsymbol{u}))$ and $g_A(\boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}(g(\boldsymbol{u}, \boldsymbol{v}) - g(\boldsymbol{v}, \boldsymbol{u}))$. Then $g(\boldsymbol{u}, \boldsymbol{v}) = g_S(\boldsymbol{u}, \boldsymbol{v}) + g_A(\boldsymbol{u}, \boldsymbol{v})$.
3. **St.** The quadratic form Q_g is conclusively deifned by symmetric part g_S of g.
4. **Th.** (Sylvester's law of inertia). Let B, C be two polar bases of quadratic form Q_g on V. Then number of possitive, negative values and zeros are same on diagonals of $[g]_B$, $[g]_C$.
5. **Def.** (Signature). For quadratic form Q_g we define the *signature* as $\mathrm{sign}Q_g = (p, q, n)$, where p, q, n is a number of possitive, negative numbers and zeros on diagonal in any poalr basis of Q_g.
6. **Def.** Let $G \in \mathbb{R}^{n\times n}$. We denote $G_k \in \mathbb{R}^{k\times k}$ a matrix

created form G by deleting last $n-k$ rows and columns.

7. **Th.** (Jacobi-Sylvester theorem). Let g be a symmetric billinear form on V, $B = (\boldsymbol{u}_i)_1^m$ a basis ov V, such that $\forall k \in \{1, \ldots, m\}, G := [g]_B : \det G_k \neq 0$. Then g has a polar basis $C = (\boldsymbol{v}_i)_1^m$ such, that $\boldsymbol{v}_k \sum_{j=1}^k r_{jk} \boldsymbol{u}_j$ where $R \in \mathbb{R}^{m \times m}$ is an upper triangular matrix, and $[g]_C = \operatorname{diag}(\frac{1}{\det G_1}, \frac{\det G_1}{\det G_2}, \ldots, \frac{\det G_{m-1}}{\det G_m})$. Therefore, a signature of g is $(p, q, 0)$ where q is number of sign changes in $(1, \det G_1, \ldots, \det G_m)$.
8. **Note.** By knowing a signature of a symmetric form Q_g, we can determine it polar form as $[Q_g] = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1, 0, \ldots)$.
9. **Def.** *Null set* of quadratic form Q_g is a set defined by $N_g := \{\boldsymbol{v} \in V | Q_g(\boldsymbol{v}) = 0\}$.
10. **Def.**
11. **Def.**
12. **Def.**
13. **Def.**
14. **St.**
15. **Def.**
16. **Th.**

XIV. Jordan normal form

1.

XV. Tenzors

1.

XVI. Others

1.