Introduction

Julia: There are so many tests to keep track of for series!

Dylan: Did you say test?! When??? WHY HAS NO ONE TOLD ME?!

James: Calm down Dylan! Julia was talking about *series* tests, not an exam.

Dylan: Oh, good, I was worried there for a second. Sucks how many series tests we have to keep track of, huh?

Julia: Do you think you could help us out James? Maybe a trick to remember?

James: Well, there's no better way to remember something than repetition!

Strategies for Applying Series Tests

In Section 11.5, the text gives a very detailed description of when to try and apply each of our series tests. Here is a brief outline to follow when looking at a series $\sum a_n$:

- First thing to try: **the Divergence Test**. If $\lim_{n\to\infty} a_n \neq 0$, then we know that the series $\sum a_n$ diverges and we are done right away! Otherwise, $\lim_{n\to\infty} a_n = 0$ and we have to try an actual test.
- Next, try to see if the series is of a certain class that we know: a **Geometric Series**, a ***p*-series**, the Harmonic Series, etc.. Look for variations as well - if the series looks like a sum or difference of two geometric series, or a sum of a p-series and a geometric series, etc.. Remember, the sum or difference of two converging series converges (this is Theorem 1 in Sec. 11.2). (What happens if we add a diverging series with a converging series?)
- Identify if the series has all positive terms. If it does not, determine if it is an **Alternating Series** and try the *Alternating Series Test*.
- If the series has negative terms and is not alternating or does not apply, you can try to determine if the series is *absolutely convergent*, since absolute convergence implies convergence. Remember, this means determining whether $\sum |a_n|$ converges. You can either use the tests below for positive series, or use the **Ratio** or **Root Test**, as these are tests for absolute convergence. In particular, if there are any factorials involved, you likely want to use the *ratio test*.

- Lastly for a non-positive series, make sure you have answered the question! Does it ask for *absolute/conditional convergence*, or simply asks whether the series converges or diverges? If the former, make sure you fully investigate the series by checking for absolute convergence, especially if it is an alternating series.

- Now, if your series is positive, or you are examining $\sum |a_n|$, then we can apply the first three tests discussed in Sec. 11.3 - the **Direct & Limit Comparison Tests** and the **Integral Test**. It's a good idea to try *Direct comparison* first. If that fails due to the inequality going the wrong way, then use the *Limit comparison test*.

- If all of the above has failed you, then we have the *Integral Test* as our backup test.

To help summarize the strategies you know, we will provide you this handy table.

Series or Test	**Conclusions**	**Comments**
Divergence Test: For a series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. Otherwise, the test is inconclusive.	You cannot show that a series converges using this test - only divergence.
Geometric Series: $\sum_{n=0}^{\infty} cr^n$	If $\lvert r\rvert < 1$, then the series converges to $\frac{c}{1-r}$. Otherwise, if $\lvert r\rvert \geq 1$, the series diverges.	If the series starts at $n = M$, then if $\lvert r\rvert < 1$ we have $\sum_{n=M}^{\infty} cr^n = \frac{cr^M}{1-r}$
p-**Series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, then the series converges. If $p \leq 1$, then the series diverges.	Note that if $p = 1$, we have the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$.
Direct Comparison Test: Compare a positive series $\sum_{n=1}^{\infty} a_n$ with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n > N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $a_n \geq b_n$ for all $n > N$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.	Starting with a series, remove terms in the numerator or denominator and note how that changes the general term (bigger or smaller) until you obtain the general term for a series you know. Often, the known series will be a geometric or p-series.
Limit Comparison Test: For a positive series $\sum_{n=1}^{\infty} a_n$, compare with a known series $\sum_{n=1}^{\infty} b_n$.	Compute $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. If $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.	Note that this means both series either both converge or both diverge. Also, there are conclusions when $L = 0$ or $L = \infty$ - see the textbook Sec. 11.3.

Series or Test	**Conclusions**	**Comments**
Integral Test: Need a positive, continuous, decreasing function $f(x)$ such that $f(n) = a_n$ for all $n > N$. Evaluate $\int_N^\infty f(x)\,dx$.	If $\int_N^\infty f(x)\ dx$ converges, then so does $\sum_{n=1}^\infty a_n$. If $\int_N^\infty f(x)\ dx$ diverges, then so does $\sum_{n=1}^\infty a_n$.	Despite learning this first, we often use this as more of a last resort, since often comparison tests are easier. If using this test, make sure the function is something you can actually integrate! Integrands with natural logs are typical for this test.
Alternating Series: Series of the form $\sum_{n=1}^\infty (-1)^{n-1} b_n$ or $\sum_{n=1}^\infty (-1)^n b_n$.	If $\{b_n\}$ is decreasing, meaning $b_{n+1} \leq b_n$ for all n, and $\lim_{n\to\infty} b_n = 0$, then the series converges.	As the name suggests, this is only applicable to alternating series. If the decreasing condition does not hold, then you likely have some strange difference of series, potentially ones you know. If the limit is not zero, then you have yourself a diverging series by the divergence test.
Ratio Test: For a series $\sum_{n=1}^\infty a_n$, compute $L = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$.	If $L < 1$, then series converges absolutely. If $L > 1$, then diverges. If $L = 1$, the test is inconclusive.	Most commonly used when there are factorials or exponentials involved in the series. Also used extensively for *Power Series* and *Taylor Series*.
Root Test: For a series $\sum_{n=1}^\infty a_n$, compute $L = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$.	If $L < 1$, then series converges absolutely. If $L > 1$, then diverges. If $L = 1$, the test is inconclusive.	Most commonly used when $\lvert a_n \rvert = b_n^n$. Also used sometimes for *Power Series* and *Taylor Series*.

Practice Problems

Try to utilize the tests above to determine if the given series is converging or diverging. If the series has negative terms, determine if the series is absolutely convergent, conditionally convergent, or divergent.

Problem 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 2n}{n^3 + 3n^2 + 1}$

Multiple Choice:

(a) *Diverges* ✓

(b) *Conditionally Convergent*

(c) *Absolutely Convergent*

Problem 2 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent*

(c) *Absolutely Convergent* ✓

Problem 3 $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^4}$

Multiple Choice:

(a) *Diverges* ✓

(b) *Conditionally Convergent*

(c) *Absolutely Convergent*

Problem 4 $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent*

(c) *Absolutely Convergent* ✓

Problem 5 $\sum_{n=1}^{\infty} \frac{2^n - 5^n}{7^n}$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent*

(c) *Absolutely Convergent* ✓

Problem 6 $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent*

(c) *Absolutely Convergent* ✓

Problem 7 $\sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n}$

Multiple Choice:

(a) *Diverges* ✓

(b) *Conditionally Convergent*

(c) *Absolutely Convergent*

Problem 8 $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Multiple Choice:

(a) *Diverges* ✓

(b) *Conditionally Convergent*

(c) *Absolutely Convergent*

Problem 9 $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} - (\ln(n))^4}$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent*

(c) *Absolutely Convergent* ✓

Problem 10 $\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$

Multiple Choice:

(a) *Diverges* ✓

(b) *Conditionally Convergent*

(c) *Absolutely Convergent*

Problem 11 $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}}$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent* ✓

(c) *Absolutely Convergent*

Problem 12 $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent*

(c) *Absolutely Convergent* ✓

Problem 13 $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}(\ln(n))^2}$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent* ✓

(c) *Absolutely Convergent*

Problem 14 $\frac{1}{2} - \frac{1}{5} + \frac{1}{4} - \frac{1}{25} + \frac{1}{8} - \frac{1}{125} + \dots$

Multiple Choice:

(a) *Diverges*

(b) *Conditionally Convergent* ✓

(c) *Absolutely Convergent*
