

# Implicit Differentiation

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```

1  caseInsensitive = function(a,b) {
2      return a.toLowerCase() == b.toLowerCase();
3  };

```

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**Dylan:** Woah! What's up with this?

**Julia:** I didn't know functions were explicit!

**Dylan:** The  $x$  and  $y$  are on the same side of the equation! I can't deal with this.

**James:** Functions can be explicit or implicit! And it not the way you're thinking Julia...

## Introduction

So far we have dealt only with explicitly defined functions, where  $y = f(x)$ . Here  $y$  is dependent variable and it is given in terms of the independent variable  $x$ . Functions given in terms of both independent and dependent variables are called *implicit* functions.

## Guided Example

**Question 1** Which of the following equations defined  $y$  as a function of  $x$  implicitly?

**Select All Correct Answers:**

(a)

$$y = x^2 + 5x - 7$$

(b)

$$y = \sin(x)$$

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Learning outcomes:  
Author(s):

(c)  $x^2 + y^2 = 1$

✓

(d)  $y = \sqrt{(x - 3)}$

(e)  $x^2y^3 + y = 5x + 8y$  ✓

Graph the curve defined by this equation:

$$x^2 + y^2 = 1$$

Graph of

Now, in the following Sage cell, solve for  $y$ . For help using the solve command refer to the [documentation](#) here.

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SAGE

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```

1 x,y = var("x, y")
2 #eqn = x^2+y^2==1, this sets eqn to the unit circle
3 #use the solve command to solve eqn for y

```

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Graph the two explicit equations on the same axis below.

Graph of

**Question 2** Which of the following are true?

Select All Correct Answers:

(a)  $x^2 + y^2 = 1$  is a function of  $x$ .

(b)  $y = -\sqrt{1 - x^2}$  is a function of  $x$ . ✓

(c)  $y = \sqrt{1 - x^2}$  is a function of  $x$ . ✓

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SAGE

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1 **Question 3**

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Using the functions you found, differentiate to find the slope of the tangent lines at the point  $\left(\left(\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}\right)\right)$ . You may do this in the above Sage cell or by hand.

-1

Unfortunately not all implicit equations can be easily solved for  $y$ , which is why we use implicit differentiation!

**Explanation.** Starting with

$$x^2 + y^2 = 1$$

we first differentiate each term using  $\frac{d}{dx}$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$$

You can already fill in 2 of the terms

$$\boxed{2x} + \frac{d}{dx}y^2 = \boxed{0}$$

For the term  $\frac{d}{dx}y^2$  you can imagine  $y = f(x)$ , and hence by the chain rule

$$\begin{aligned}\frac{d}{dx}y^2 &= \frac{d}{dx}(f(x))^2 \\ &= 2 \cdot f(x) \cdot f'(x) \\ &= 2y \frac{dy}{dx}\end{aligned}$$

Thus we have the following:

$$2x + 2y \frac{dy}{dx} = 0$$

Solving for  $\frac{dy}{dx}$  we get

$$\boxed{\frac{-x}{y}}$$

**Question 4** Use the equation obtained from the above explanation to find  $\frac{dy}{dx}$  at  $\left(\left(\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}\right)\right)$  -1

**Feedback (attempt):** Using both methods you can obtain the same answer, but for many equations the first method is much more work!

**Question 5** We can fairly easily use Sage to do this process for us, to illustrate the process evaluate the following Sage cell.

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SAGE

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```

1 var('x,y')
2 y(x)=function('y')(x)
3 eq=x^2+y^2==1
4 eq.substitute(y =y(x))
5 diff(eq,x)
6 solve(diff(eq,x),diff(y(x)))

```

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What did you get as output from your Sage cell? (copy just the answer portion after the “==” and before the “]”)  $-x/y(x)$

**Feedback (attempt):** Notice that Sage uses  $y(x)$  for  $y$  in the output.

## On Your Own

In each of the following problems, evaluate the derivative by hand, and use the Sage cells as a check.

Consider the equation  $y^4 + xy = x^3 - x + 2$ . Using the following Sage cell implicitly differentiate to find  $\frac{dy}{dx}$  using the same commands as shown in the previous question.

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SAGE

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```

1 var('x,y')

```

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**Question 6** What did you get as output from your Sage cell? (copy just the answer portion after the “==” and before the “]”)

$(3 * x^2 - y(x) - 1)/(4 * y(x)^3 + x)$

**Question 7** Using your result in the previous section, evaluate  $\frac{dy}{dx}$  at the point  $(1, 1)$ . 1/5

**Question 8** Now use Sage Math again to find  $\frac{dy}{dx}$  for  $\sin(x^2) = \cos(xy^2)$ , copy your answer in the same way as indicated in the previous section.

\_\_\_\_\_ **SAGE** \_\_\_\_\_

`var('x,y')`

$-1/2 * (\sin(x * y(x)^2) * y(x)^2 + 2 * x * \cos(x^2)) / (x * \sin(x * y(x)^2) * y(x))$

## Perpendicular at a Point

**Julia:** Wow, implicit differentiation is rough.

**Dylan:** You're telling me... I've been doing this for hours! I wish we could at least do a little more with it if I have to learn it.

**James:** Did I hear that you guys want to know more about using implicit differentiation?

**Julia and Dylan:** James! Tell us more!

**James:** Alright guys, you can use implicit differentiation with implicit functions to tell if two functions are perpendicular at a point!

**Julia:** But how?

**Dylan:** Yeah, I don't see how that helps.

**James:** It's easy - all we have to do is see if the tangent lines are perpendicular at that point, and if they are, then so are the curves!

**Question 9** Graph  $3x - 2y + x^3 - x^2y = 0$  and  $x^2 - 2x + y^2 - 3y = 0$  on the same set of axes.

Graph of

Do they look perpendicular anywhere?

**Multiple Choice:**

(a) Yes ✓

(b) No

**Hint:** To show two lines are perpendicular you must show that the slope of one is the opposite inverse of the other

Show the two curves are (or are not) perpendicular at the origin. You can do this in the Sage cell provided or by hand.

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SAGE

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Slope of  $3x - 2y + x^3 - x^2y = 0$  at the origin:

3/2

Slope of  $x^2 - 2x + y^2 - 3y = 0$  at the origin:

-2/3

Are the lines perpendicular at the origin?

## In Summary

There are two main methods to differentiate implicit equations

- (a) Solve for  $y$  and then differentiate.
- (b) Treat  $y$  as  $y(x)$  and differentiate with respect to the variable  $x$ , eventually solving for  $\frac{dy}{dx}$  to give the value of the derivative at any point  $(x, y)$ .