The universal property of the Lebesgue integration

Food for thought | Based on [Lei20]

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- approximation of a general function by simple functions.
- **Lebesgue integration is a limit.**
- $L^1 := \mathcal{L}^1 / \sim$.

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Instructor (functional analyst):

- Riemann integration defines a norm $\|\bullet\|$ on C
- Complete C with respect to $\|\bullet\|$ to get L^1.
- **Extend Riemann integration as an operator acting on L^1.**

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Instructor (category theorist): Here is a statement that you might understand:

Theorem (A)

*L^1 is an **initial object** in the category of Banach spaces (with a little more data), so it is **characterized by its properties.***

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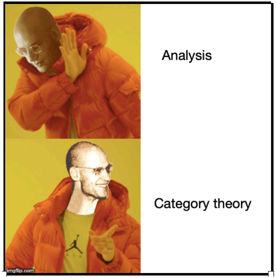

Student: Could you review category theory quickly and clarify what a little more data means?

Instructor: Sure! Here is an overview of what's coming next.

1. Categories and some analysis after all
2. Precise statements: Characterization of $L^1[0,1]$
3. Proof: Uniqueness of the map out of $L^1[0,1]$
4. Proof: Existence of the map out of $L^1[0,1]$

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Student: Can you give some examples?
Instructor: Set, **Ring**, **Ban**.

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- A **map** ϕ in **Ban** satisfies $\|\phi(v)\| \leq \|v\|$.
- Extra: $V \oplus W$ has the norm

$$\|v, w\| := \frac{1}{2}(\|v\| + \|w\|).$$

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and maps contract V while preserving the structure δ.

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$$(\gamma(f,g))(x) := \begin{cases} f(2x) & \text{if } x < 1/2, \\ g(2x-1) & \text{if } x > 1/2. \end{cases}$$

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2. Suppose $\mathbb{F}(\mathbb{C}$ or $\mathbb{R})$ is a field and $m : \mathbb{F} \to \mathbb{F}$ is the arithmetic mean. Then $(\mathbb{F}, 1, m)$ is an object of $\mathcal{A}$.

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Student: What's so interesting about an initial object?
Instructor: It is **unique** at two levels.

Student: Can you state the theorem again?

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Instructor:

Theorem (Main)

$(L^1[0,1], I, \gamma)$ *is the initial object of* $\mathcal{A}$.

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Instructor:

Corollary (Integration is averaging operator)

$\int_0^1$ *is the unique bounded linear functional on* $L^1[0,1]$ *such that* $\int_0^1 I = 1$ *and*

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left(\int_0^1 f\left(\frac{x}{2}\right)dx + \int_0^1 f\left(\frac{x+1}{2}\right)\right).$$

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- Use induction and at the fact that L^1 is (the closure of E_n's) the colimit of $E_0 \hookrightarrow E_1 \dashrightarrow$.

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- Θ_{n+1} extends Θ_n.
- $L^1[0,1]$ is the closure (colimit) of E_n, so there is $\Theta : L^1[0,1] \to V$ such that $\Theta|_{E_n} = \Theta_n$.

Student: Why is $\Theta : (L^1[0, 1], I, \gamma) \to (V, v, \delta)$ a map in $\mathcal{A}$?

Instructor:

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- Use density (colimit) again.

Student: Could you summarize what we just did?

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Instructor:

- Use E_n to construct a unique Θ for each E_n and use density (colimit) argument.

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- Consider $\mathbf{Ban}_*$ to be the category of pointed Banach spaces.
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- $\mathcal{A}$ is the category of T-algebras.
- Then we can construct the initial object using a theorem of Adámek [Adá74]: it is the co-limit of

$$\mathbb{R} \to T(\mathbb{R}) \to T^2\mathbb{R} \dashrightarrow$$

which is $L^1[0,1]$.

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- The universality of L^p comes from the self-similarity of $[0,1]$. One can use the same proof for $L^p(X)$ where X is self-similar.
- With a little more work, one can prove the universality of L^p for general measure spaces.

Instructor: I have some questions for you to think about.

Question

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1. *In what sense is the process of turning measurable spaces into Banach spaces universal?*
2. *Can you get hard analytical estimates just using category theory?*

Thank you!

References

[Adá74] Jiří Adámek, *Free algebras and automata realizations in the language of categories*, Commentationes Mathematicae Universitatis Carolinae **15** (1974), no. 4, 589–602.

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