Crank-Nicolson scheme for 1D diffusive transport

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Assuming zero-flux boundary conditions, variable diffusivity and variable area over depth, we start with the one-dimensional diffusion equation:

$$\frac{\partial T}{\partial t} = \frac{1}{A}\frac{\partial}{\partial z}(K(z)A(z)\frac{\partial T}{\partial z}) \tag{1}$$

Expanding this we get:

$$\frac{\partial T}{\partial t} = \frac{1}{A}(\frac{\partial K}{\partial z}A(z)\frac{\partial T}{\partial z} + K(z)\frac{\partial A}{\partial z}\frac{\partial T}{\partial z} + K(z)A(z)\frac{\partial T}{\partial z}) \tag{2}$$

We discretise the expanded form using central second-order finite differences for both the first and second derivaties, as well as implicit midpoint method in time to derive:

$$\begin{aligned}\frac{T_n^{i+1} - T_n^i}{\Delta t} = \frac{1}{2}\frac{1}{A}(&\frac{K_{n+1} - K_{n-1}}{2\Delta z}A_n\frac{T_{n+1}^{i+1} - T_{n-1}^{i+1}}{2\Delta z} + K_n\frac{A_{n+1} - A_{n-1}}{2\Delta z}\frac{T_{i+1}^{i+1} - T_{n-1}^{n+1}}{2\Delta z} + \\ &K_nA_n\frac{T_{n+1}^{i+1} - 2T_n^{i+1} + T_{n-1}^{i+1}}{\Delta z^2} + \frac{K_{n+1} - K_{n-1}}{2\Delta z}A_n\frac{T_{n+1}^i - T_{n-1}^i}{2\Delta z} + K_n\frac{A_{n+1} - A_{n-1}}{2\Delta z}\frac{T_{n+1}^i - T_{n-1}^i}{2\Delta z} + \\ &K_nA_n\frac{T_{n+1}^i - 2T_n^i + T_{n-1}^i}{\Delta z^2})\end{aligned} \tag{3}$$

We can set $\alpha = \frac{\Delta t}{2A_n\Delta z^2}$, $K_n' = K_{n+1} - K_{n-1}$, and $A_n' = A_{n+1} - A_{n-1}$ to get a tridiagonal system of equations:

$$\begin{aligned}&T_{n-1}^{i+1}(\frac{\alpha K_n'A_n}{4} + \frac{\alpha K_nA_n'}{4} - \alpha K_nA_n) + T_n^{i+1}(1 + 2\alpha K_nA_n) + T_{n+1}^{i+1}(-\frac{\alpha K_n'A_n}{4} - \frac{\alpha K_nA_n'}{4} - \alpha K_nA_n) = \\ &T_{n-1}^i(-\frac{\alpha K_n'A_n}{4} - \frac{\alpha K_nA_n'}{4} + \alpha K_nA_n) + T_n^i(1 - 2\alpha K_nA_n) + T_{n+1}^i(\frac{\alpha K_n'A_n}{4} + \frac{\alpha K_nA_n'}{4} + \alpha K_nA_n)\end{aligned} \tag{4}$$

We can set up the tridiagonal matrix using $a = \frac{\alpha K_n'A_n}{4} + \frac{\alpha K_nA_n'}{4} - \alpha K_nA_n$, $b = 1 + 2\alpha K_nA_n$, and $c = -\frac{\alpha K_n'A_n}{4} - \frac{\alpha K_nA_n'}{4} - \alpha K_nA_n$ (here exemplary shown for a 4x4 problem):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_3 & b_3 & c_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1^{i+1} \\ T_2^{i+1} \\ T_3^{i+1} \\ T_4^{i+1} \end{pmatrix} = \begin{pmatrix} T_1^i \\ -a_2T_1^i + T_2^i(1 - 2\alpha_2K_2A_2) - cT_3^i \\ -a_2T_2^i + T_3^i(1 - 2\alpha_2K_2A_2) - cT_4^i \\ T_4^i \end{pmatrix} \tag{5}$$

where the first and last equations are just boundary conditions that are equal to the simulated temperatures at both ends.

We also need to treat the boundaries for A_n' as well as K_n' at 1 and N (surface and bottom, respectively) as

$$A_1' = 2(A_2 - A_1) \tag{6}$$

and

$$A_N' = 2(A_N - A_{N-1}) \tag{7}$$

(and similarly for K').