Idzorek's method minimisation step: 23/4/20

Substitute $s_K \equiv \frac{1}{\omega_K}$ (scalar)

$y \equiv w_{K\%}$ is the tilted weight vector.

Solve: $\min_{s_K} \| y - w_K \|^2, \quad s_K > 0.$

$$\text{where } w_K = (\delta \Sigma)^{-1} \underbrace{\left[(\tau\Sigma)^{-1} + s_K p_K^T p_K \right]^{-1}}_{[A \,+\, B]^{-1}} \left[(\tau\Sigma)^{-1} \Pi + s_K p_K^T Q_K \right]$$

If B has rank 1 (it does, since p_K is a $1 \times N$ row vector):

$$(A+B)^{-1} = A^{-1} - \frac{1}{1+g} A^{-1} B A^{-1} \quad \text{where} \quad g = tr\left(BA^{-1}\right)$$

$$\therefore \left[(\tau\Sigma)^{-1} + s_K p_K^T p_K \right]^{-1} = \tau\Sigma - \frac{1}{1+g} (\tau\Sigma) s_K p_K^T p_K (\tau\Sigma)$$

$$= \tau\Sigma - s_K \frac{\tau^2}{1+g} \Sigma p_K^T p_K \Sigma$$

$$\Rightarrow w_K = (\delta\Sigma)^{-1} \Big\{ \Pi + s_K \Big[\tau\Sigma p_K^T Q_K - \frac{\tau^2}{1+g} \Sigma p_K^T p_K \Sigma (\tau\Sigma)^{-1} \Pi \Big] - s_K^2 \frac{\tau^2}{1+g} \Sigma p_K^T p_K \Sigma p_K^T Q_K \Big\}$$

$$= (\delta\Sigma)^{-1} \Big\{ \Pi + s_K \tau \Big[\Sigma p_K^T Q_K - \frac{1}{1+g} \Sigma p_K^T p_K \Pi \Big] - s_K^2 \frac{\tau^2}{1+g} \Sigma p_K^T p_K \Sigma p_K^T Q_K \Big\}$$

$=$ $n\times n$ $n\times 1$ $(n\times n)(n\times 1)(1\times 1)$ $(n\times n)(n\times 1)(1\times n)(n\times 1)$ $(n\times n)(n\times 1)(1\times n)(n\times n)(n\times 1)(1\times 1)$

Dimensions ✓

$$w_K = (\delta\Sigma)^{-1} \Pi + \frac{s_K \tau}{\delta} \Big[p_K^T Q_K - \frac{1}{1+g} p_K^T p_K \Pi \Big] - \frac{s_K^2 \tau^2}{\delta(1+g)} p_K^T p_K \Sigma p_K^T Q_K$$

$$\therefore \quad w_K = a + b s_K + c s_K^2 \quad \text{where } a, b, c \in \mathbb{R}^n.$$

Let $J(s_k) \equiv \| y - w_k \|^2$ be our cost function.

We want the s_k that minimises J.

$$J(s_k) = (y - w_k)^T (y - w_k) = y^T y - y^T w_k - w_k^T y + w_k^T w_k.$$

$$= y^T y - y^T (a + b s_k + c s_k^2) - (a^T + b^T s_k + c^T s_k^2) y + w_k^T w.$$

$$\frac{\partial J}{\partial s_k} = -y^T b - 2y^T c\, s_k - b^T y - 2c^T y s_k + 2b^T b\, s_k + 4c^T c\, s_k^3$$
$$+ 2a^T b + 4a^T c\, s_k + 6b^T c\, s_k^2$$

$$= 2a^T b - y^T b - b^T y + (4a^T c + 2b^T b - 2y^T c - 2c^T y) s_k + 6b^T c\, s_k^2 + 4c^T c\, s_k^3 = 0.$$