

1. Simple and Compound Interest

Simple Interest

- In simple interest there is one deposit and one repayment.

$$C_1 = C_0(1+i) \quad C_0 = \frac{1}{1+i} C_1$$

- Over several time units, this becomes $C_n = C_0(1+ni)$.

- There are different conventions for the number of days in a year. ACT/365 assumes that there are 365 days.
- Calculations regarding simple interest must be done with respect to one fixed time point, called the focal point or valuation date.

Compound Interest

- Interest is given on top of existing interest. Assuming constant rate:

$$C_n = C_0(1+i)^n$$

- The discount factor is then $v = \frac{1}{1+i}$

- If an investment produces interest i_1 , then the interest i_a adjusted for inflation i_o is given by $i_a = \frac{i_1 - i_o}{1 + i_o}$

- By definition, the nominal rate of interest compounded monthly, $i^{(m)}$, leads to a growth of $(1 + \frac{i^{(m)}}{m})^m$, in one time period.

e.g. a $\underbrace{\$1000}_{C_0}$ loan for one year @ 15% p.a compounded monthly $\underbrace{i^{(m)}}_{\text{m=12}}$

$$\begin{aligned} C_n &= C_0 \left(1 + \frac{i^{(m)}}{m}\right)^{mn} \Rightarrow C_1 = C_0 \left(1 + \frac{i^{(m)}}{m}\right)^m \\ &= 1000 \left(1 + \frac{0.15}{12}\right)^{12} \end{aligned}$$

- The effective interest is the effective growth in one time period (annual by default).

$$i = \left(1 + \frac{i^{(m)}}{m} \right)^m - 1$$

- alternatively, if the period p is given ($p = \frac{1}{m}$), then

$i_{eq} = p i^{(m)} = \frac{i^{(m)}}{m}$ is called the effective interest rate over period p .

- effective rates for different time periods can be converted with:

$$(1 + i_{eq})^{1/p} = (1 + i)^{1/p}$$

- nominal rate (period p) = frequency \times effective rate (freq = $1/p$)

- Always assume given interest rates are compounded annually, unless the time period is less than one year, in which case you should assume simple interest.

- The accumulation factor is the accumulated value after one time period, i.e. $A(p) = (1+i)^p = [1 + p i]^p$

- In continuous compounding, we let $m \rightarrow \infty$:

$$i = \lim_{m \rightarrow \infty} \left(\left(1 + \frac{i^{(m)}}{m} \right)^m - 1 \right) = e^{\delta} - 1$$

- δ is the force of interest, i.e. $\lim_{m \rightarrow \infty} i^{(m)}$

$$c_n = c_0 e^{n\delta}$$

The Discount rate

- The interest rate on a deposit can also be called the discount rate on the repayment, i.e. $PV \times \text{interest rate} = FV \times \text{discount rate}$

- The discount rate is linked to the discount factor by $v = 1 - d$

- For a simple discount rate, $c_0 = (1 - nd)c_n$ by definition. Thus:

$$(1 + ni)(1 - nd) = 1$$

- Investment terminology:

- the **coupon** is an interest payment based on the face/nominal value of some instrument.
- the **yield** is the interest rate that can be earned, and is implied by the market price relative to the maturity price.
- a **discount instrument** does not pay a coupon: simple interest is used for calculations because there is only one repayment.

- As with interest, one can have nominal or effective discount rates

$$1-d = \left(1 - \frac{d^{(m)}}{m}\right)^m \Rightarrow \left(1 + \frac{i^{(m)}}{m}\right)\left(1 - \frac{d^{(m)}}{m}\right) = 1$$

- If we borrow \$1 at $t=0$ for repayment at $t=1$ with annual interest i :
 - repaying with equal instalments at the end of each m^{th} subinterval leads to a total interest payment of $i^{(m)}$
 - repaying with equal instalments at the start of each m^{th} subinterval leads to a total interest payment of $d^{(m)}$
- For continuous compounding, $d_{\text{cp}} = 1 - e^{-ps}$

Time dependent interest rates

- Suppose we have a case of continuous compounding where interest varies with time, i.e. $i_p(t)$ gives the nominal interest rate for period p starting at time t . Then: $A(t+p) = A(t)[1 + p i_p(t)]$

- In the limit as $p \rightarrow 0$ of $i_p(t)$, we find that:

$$\delta(t) = \lim_{p \rightarrow 0} i_p(t) = \lim_{p \rightarrow 0} \left[\frac{1}{A(t)} \frac{A(t+p) - A(t)}{p} \right] = \frac{A'(t)}{A(t)}.$$

Thus: $A(t_2) = A(t_1) \exp\left(\int_{t_1}^{t_2} \delta(s) ds\right)$

- The present value corresponding to the force of interest function is given by

$$V(t) = \frac{1}{A(t)} = \exp\left(-\int_0^t \delta(s) ds\right)$$

2. Cash flows, equations of value, and project appraisal

Cash flows

- Zero-coupon bonds provide a specified cash amount at a future time
 - Fixed interest security: coupon is paid at $t=1, 2, 3, \dots, n$, and a lump sum is paid at $t=n$
- | time | 0 | 1 | \dots | $n-1$ | n |
|-----------|--------|-----|---------|-------|-------|
| cash flow | $-c_0$ | c | \dots | c | $c+s$ |
- Indexed-link securities have coupons linked to some index
 - Annuity: investor pays a premium at time 0 and receives annual payments
- | time | 0 | 1 | \dots | $n-1$ | n |
|-----------|--------|-----|---------|-------|-----|
| cash flow | $-c_0$ | c | \dots | c | c |
- Equity: after purchase, you will receive a dividend at constant intervals
 - Term assurance: premium paid at time 0. If death occurs before $t=n$, the beneficiary will receive an amount c .
 - Loans and mortgages: borrower receives c_0 at $t=0$, then repays it (capital and interest) in a series of payments
 - Insurance: company receives a premium, with the amount and frequency of negative cash flows being uncertain.

Net present value and discounted cash flow

- The DCF formula states that the NPV of a series of cash flows is given by:

$$NPV(\underline{c}) = c_0 + \frac{c_1}{1+i} + \frac{c_2}{(1+i)^2} + \dots + \frac{c_n}{(1+i)^n} = \sum_{j=0}^n \frac{c_j}{(1+i)^j}$$

- Properties of NPV:

- decreasing in i , and decreases faster when later payments are larger
- if $a_j \geq b_j$, $j=0, \dots, n$, then $NPV(\underline{a}) \geq NPV(\underline{b})$ for $i \in [0, \infty)$

- If $v(t)$ is the present value of a unit amount at time t ,

$$NPV(\xi) = \sum_{j=1}^n c_j v(t_j)$$

- To find the NPV for continuous cash flows $\rho(t)$, we let $c_t = \rho(t)dt$ and take the limit of the above.

$$NPV(\rho) = \int_0^t \rho(u) v(u) du$$

- In the special case of constant interest i , we have:

$$NPV(\rho) = \int_0^t \frac{\rho(u)}{(1+i)^u} du$$

- For a general force of interest function:

$$NPV(\rho) = \int_0^t \rho(u) \exp\left(-\int_0^u \delta(s) ds\right) du$$

- The accumulated value is just $NPV \exp\left(\int_0^t \delta(s) ds\right)$, i.e.:

$$A(t) = \int_0^t \rho(u) \exp\left(\int_u^t \delta(s) ds\right) du$$

The equation of value and IRR: comparing projects

- The equation of value / yield equation of i is generally given by $f(i)=0$, where $f(i)=NPV(\xi, \rho)$
 - If $f(i)=0$ has exactly one root i_0 , with $i_0 > -1$, then the internal rate of return (IRR), or the money weighted rate of return, is defined to be i_0 . The IRR is the interest that zeroes the NPV .
 - If $i > IRR$, $NPV < 0$.
 - Numerical methods for finding the IRR:
 - expand $1/(1+r)^j$ as $1-jr$
 - multiply out $(1+r)^n$, then expand $(1+r)^j$ as $1+jr + \frac{j(j-1)}{2}r^2$
 - assume all future cash-flows equal the average cash flow
- | | | | | | | | |
|---|--------|-----------|-----------|-----|-----------|-----------|-----------------------------------|
| t | 0 | 1 | 2 | ... | $n-1$ | n | with $\bar{c} = \sum_{j=1}^n c_j$ |
| C | $-c_0$ | \bar{c} | \bar{c} | ... | \bar{c} | \bar{c} | → |

Then, we have $c_0 = \frac{x}{r} \left[1 - \left(\frac{1}{1+r} \right)^n \right] = x \alpha_{\bar{n}, r}$, where $\alpha_{\bar{n}, r} = \frac{1}{r} \left[1 - \left(\frac{1}{1+r} \right)^n \right]$.

The values for $\alpha_{\bar{n}, r}$ for different r can be found in tables.

- Distributing capital gains: if a cash flow consists of an interest payment x in years $0 \rightarrow n-1$ and a final payment of $c_n - x$, we can approximate $i = \frac{x + (c_n - x - c_0)}{c_0} / n$

- Disadvantages of IRR:

- there may be zero solutions or multiple solutions
- the IRR assumes that accumulated funds / loans will grow at the internal rate
- Projects ~~will~~ have different profitability depending on interest rates. It may be useful to graph $NPV_i(a)$ and $NPV_i(b)$ as functions of i . The rate that gives equal NPVs is called the cross-over rate.
- The discounted payback period (DPP) is the smallest time such that the accumulated value of the project becomes positive
 - especially useful if capital is scarce
 - does not quantify profit: projects with late but large cash inflows may have higher DPP but also higher NPV.
- The payback period is the time when net cash flow is positive - it is inferior because it doesn't discount future cash flows
- Overall, NPV is the best metric when i is known. Sensitivity analyses should be performed to observe how NPV varies with i .

Measuring investment performance

- Many projects, e.g. dividend-paying funds, will have cash flows of the form:

Time, t	t_0	t_1	\dots	t_n
Fund value at $t=0$	$f_{0=0}$	$f_{1=0}$	\dots	$f_{n=0}$
Net cash flow at t	c_0	c_1	\dots	c_n
Fund value at $t+0$	$f_0 = f_{0=0} + c_0$	$f_1 = f_{1=0} + c_1$	\dots	$f_n = f_{n=0} + c_n$

- The internal rate of return solves the equation of value, which equates all inflows with the eventual accumulation:

$$f_0(1+i)^t + c_1(1+i)^{t-1} + \dots + c_n(1+i)^0 = f_n$$

- It is also called the money-weighted rate of return (MWRR) because it is influenced more by the performance when the fund is larger.
- The MWRR has some drawbacks:

- sensitive to the amount and timing of cash flows, which are not controlled by fund managers
- may not have a unique solution

- Alternatively, the time-weighted rate of return (TWRR) ignores the size of the fund and instead 'averages' the growth between cashflows

$$(1+i)^t = \frac{f_{1=0}}{f_0} \cdot \frac{f_{2=0}}{f_1} \dots \frac{f_{n=0}}{f_{n-1}}$$

- However, the TWRR requires knowledge of all cash flows, and the value of the fund on the cashflow date.

- The linked internal rate of return (LIRR) can be used to approximate the TWRR: it calculates the IRR at specified sub-intervals then combines them with the TWRR formula:

$$(1+i)^{t_n} = (1+i_1)^{t_1} (1+i_2)^{t_2-t_1} (1+i_3)^{t_3-t_2} \dots (1+i_n)^{t_n-t_{n-1}}$$

- the LIRR can be used if the fund is not valued at every cashflow
- LIRR = TWRR if the interval lengths are the same

3. Perpetuities, Annuities and Loans

Perpetuities

- A perpetuity is a special case of an annuity that pays a fixed amount every year forever.

time	0	1	2	3	...
cash flow	0	1	1	1	...

- The present value at time zero is denoted as $a_{\bar{\infty}, i}$:

$$a_{\bar{\infty}, i} = r + r^2 + r^3 + \dots + r^n = \frac{r}{1-r} = \frac{1}{i}$$

- The present value at time 1 is $\ddot{a}_{\bar{\infty}}$. This quantity is equal to the value of the perpetuity with payments made in advance.

Clearly $\ddot{a}_{\bar{\infty}} = (1+i) a_{\bar{\infty}}$

Annuities

- The most common form has constant annual payments

time	0	1	2	...	$n-1$	n
cash flow	0	1	1	...	1	1

- $a_{\bar{n}}$ is the value at time zero, called the present value of the annuity-immediate a . This corresponds to payments in arrears.

$$a_{\bar{n}} = r + r^2 + \dots + r^n = \frac{r(1-r^n)}{1-r} = \frac{1-r^n}{i}$$

- $\ddot{a}_{\bar{n}}$ is the value at time 1, the present value of the annuity-due, which corresponds to payments in advance.

$$\ddot{a}_{\bar{n}} = (1+i) a_{\bar{n}} = \frac{1-r^n}{1-r}$$

- $s_{\bar{n}}$ is the accumulated value of the annuity-immediate

$$s_{\bar{n}} = (1+i)^n a_{\bar{n}}$$

- A deferred annuity has its first payment at time $k+1$. Its present value is the same as that of a standard annuity, just scaled down by r^k , i.e.: $k! \bar{a}_{\overline{n}} = r^k \bar{a}_{\overline{n}}$

Annuities payable monthly

time	0	$1/m$	$2/m$	\dots	$(nm-1)/m$	nm/m	$(nm+1)/m$
c	0	$1/m$	$1/m$	\dots	$1/m$	$1/m$	$1/m$
	↑	↑				↑	↑
	$a_{\overline{n}}^{(m)}$	$\ddot{a}_{\overline{n}}^{(m)}$				$s_{\overline{n}}^{(m)}$	$\ddot{s}_{\overline{n}}^{(m)}$

$a_{\overline{n}}^{(m)} = \frac{1}{m} \sum_{j=1}^{nm} r^{j/m} = \frac{1-r^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}}$
 and $\ddot{a}_{\overline{n}}^{(m)} = (1+i)^{1/m} a_{\overline{n}}^{(m)}$

Increasing annuities

time	0	1	2	\dots	n
c	0	1	2	\dots	n

$$(Ia)_{\overline{n}} \quad (I\ddot{a})_{\overline{n}}$$

$$(Ia)_{\overline{n}} = r + 2r^2 + \dots + nr^n = \frac{1}{i} \left[\frac{1-r^n}{1-r} - nr^n \right] = \frac{a_{\overline{n}} - nr^{n+1}}{1-r}$$

General results about increasing annuities can be found by considering equivalent cash flows

time	0	1	2	3	\dots	n
e.g. c	0	P	$P+Q$	$P+2Q$	\dots	$P+(n-1)Q$
= C ₁	0	$P-Q$	$P-Q$	$P-Q$	\dots	$P-Q$
+ C ₂		Q	$2Q$	$3Q$	\dots	nQ

$$\therefore NPV = (P-Q) a_{\overline{n}} + Q (Ia)_{\overline{n}}$$

Continuously payable annuities

$$\bar{a}_{\bar{n}} = \int_0^n p(t)v(t)dt = \int_0^n v^t dt = \frac{1-v^n}{\delta} = \frac{i}{\delta} a_{\bar{n}}$$

Many of the previous results apply:

- $\bar{s}_{\bar{n}} = (1+i)^n \bar{a}_{\bar{n}}$
- $k_1 \bar{a}_{\bar{n}} = v^k \bar{a}_{\bar{n}}$
- $\bar{a}_{\bar{m}} = \frac{i}{\delta} a_{\bar{m}}$

For a continuously payable annuity increasing in steps:

$$(I\bar{a})_{\bar{n}} = \sum_{j=1}^n \int_{j-1}^j j v^t dt = \frac{(1+i) \bar{a}_{\bar{n}} - n v^n}{\delta}$$

If payment increases continuously (i.e $p(t)=t$)

$$(\widehat{I\bar{a}})_{\bar{n}} = \int_0^n t v^t dt = \frac{\bar{a}_{\bar{n}} - n v^n}{8}$$

Loan schedules

Loan schedules are represented in tables:

year	loan outstanding before repayment	repayment	interest due	capital repaid	loan outstanding after repayment
1	l_0	x_1	$i l_0$	$x_1 - i l_0$	$l_1 = l_0 - (x_1 - i l_0)$
2	l_1	x_2	$i l_1$	$x_2 - i l_1$	$l_2 = l_1 - (x_2 - i l_1)$
...
k	l_{k-1}	x_k	$i l_{k-1}$	$x_k - i l_{k-1}$	$l_k = l_{k-1} - (x_k - i l_{k-1})$
...
n	l_{n-1}	x_n	$i l_{n-1}$	$x_n - i l_{n-1}$	$l_n = 0$

The equation of value for a loan is:

$$l_0 = x_1 v + x_2 v^2 + \dots + x_n v^n$$

There are two methods for finding l_k , the outstanding loan value immediately after the k th payment:

1. The retrospective loan calculation:

$$\text{value of loan} = \frac{\text{value at time } k}{\text{of original loan}} - \frac{\text{accumulated value at time } k \text{ of repayments}}{\text{of } k \text{ of repayments}}$$

$$\text{i.e. } l_k = l_0(1+i)^k - [x_1(1+i)^{k-1} + x_2(1+i)^{k-2} + \dots + x_k]$$

2. The prospective loan calculation (generally easier):

$$\text{value of loan} = \frac{\text{value at time } k}{\text{of all future repayments}}$$

$$\text{i.e. } l_k = vx_{n+1} + v^2x_{n+2} + \dots + v^{n-k}x_n$$

- In the special case of **level payments** (equal instalments), the equation of value is $l_0 = x a_{\overline{n}}$
 - the prospective calculation shows that $l_k = x a_{\overline{n-k}}$
 - the k th payment consists of interest $x(v^{n-k+1})$ and a capital repayment of xv^{n-k+1} .

- For the more general case of monthly payments of $1/m \times x$, we have $l_0 = m x a_{\overline{n}}^{(m)}$

Flat rate and APR

- Banks sometimes quote a **flat rate of interest**, which means that interest is charged on the full amount of the loan, regardless of capital repayments
 $\text{total interest paid} = \text{flat rate} \times \text{size of loan} \times \text{length of loan}$
- The effective rate tends to be just under double the flat rate.
- The **annual percentage rate (APR)** is the effective rate rounded to the lower 0.1%

4. Basic Financial Instruments

Markets, interest rates and instruments

- A primary market is where a security is originally issued.
- If the security is negotiable, it can subsequently be traded in a secondary market.
- The capital market refers to all long-term financial instruments
- A security is a bearer security if payment is made to whoever is holding it. Registered securities include a central record of ownership.
- The money market is the market in short-term (< 1 year) financial instruments which are based on an interest rate.
 - Treasury bill / T-bill : borrowing by the government, negotiable and bearer security. Carries a coupon. Called a Euro bill in Europe
 - Time deposit : a non-negotiable borrowing by a bank
 - Certificate of deposit (CD) : negotiable bearer borrowing by a bank, usually carrying a coupon
 - Commercial paper (CP) : negotiable bearer borrowing by company
 - Repurchase agreement (repo) : non-negotiable security to borrow against a long term instrument.
 - Futures contract : a deal to buy or sell an instrument at a future date.
- Functions of the money market:
 - banks can lend to other banks with money shortages
 - companies may have surplus/shortage of money
- The Bank of England can lend money to eligible commercial banks by buying their financial instruments, at the Official Dealing Rate.
- Banks in the UK will lend to each other at the LIBOR.

Fixed government borrowings

- A borrower who issues a bond agrees to pay interest at a specified rate (coupon payments) until the maturity/redemption date, at which time a fixed redemption value is paid.
- The coupon rate is applied to the face/par value of the bond, which is the value that should be paid at maturity.
 - if face value = redemption value, the bond is redeemable at par
 - if face value < redemption, the bond is redeemable at a premium
 - if face value > redemption, the bond is redeemable at a discount
- Some bonds have variable redemption dates - the borrower decides.
- Some government bonds have coupon and redemption linked to an inflation index.
- Investment banks can decompose a government bond into strips (separately traded and registered interest and principal security).
- A UK government bond is called a gilt.

Corporate / bank fixed borrowings

- Debentures (corporate bonds) are issued by companies and are usually secured against specific assets. Because they are more risky, investors will expect a higher yield.
- Foreign bonds are issued in the local currency but by a foreign borrower, e.g:
 - Samurai bonds in Japan (yen)
 - Yankee bonds in the US
 - Bulldog bonds in the UK (£)
- Eurobonds are issued in a different currency in a different country e.g if I need USD in the Swiss financial market.
- Eurobonds were developed to make eurocurrency (currency deposited outside of its home country) lending more marketable.

- Certificates of deposit (CDs) are issued by banks in return for a deposit. There is an active secondary market, but this flexibility reduces yield.
 - because they are short term, simple interest is typically used
 - the maturity proceeds are calculated by simple accumulation of the face value with coupon rate i_c over d_T days

$$M_p = f \left(1 + \frac{i_c d}{d_T} \right)$$

d_T is 360 for ACT/360,
365 for ACT/365

- if the CD is bought at time d_a for price p_a , and subsequently sold at time d_b for price p_b , the yield is

$$\left(\frac{p_b}{p_a} - 1 \right) \frac{d_T}{d_b - d_a}$$

can be rearranged in terms of the yields
at time d_a and d_b .

Investments with uncertain returns

- Ordinary shares entitle holders to a share in the net profits of a company, the dividend.
 - they are the last to be repaid during bankruptcy, so are the most risky
 - if a share is bought ex-dividend, the seller receives the next dividend payment.
- Preference shares normally pay a fixed dividend, and have a higher debt priority than ordinary shares
- Convertibles are loans that can be converted into ordinary shares at a fixed price and a fixed date
 - investors accept a lower interest payment because of possible capital gains from the rise in share prices
 - companies don't have to immediately dilute earnings and dividends as would be the case for issuing shares.
- Properties provide uncertain rental incomes and volatile capital gains.
- The running yield of an investment is the annual income divided by the current market price (ignoring capital gains). Thus:
 $\text{bonds} > \text{property} > \text{equity}$.

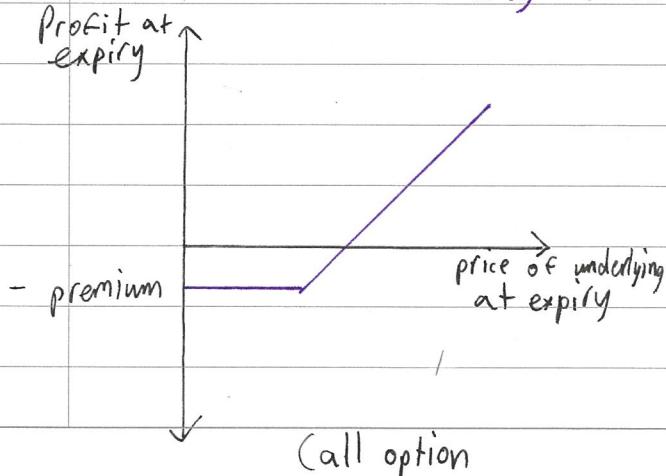
Derivatives

- Derivatives are instruments whose value depends on other financial assets. They can be used to hedge positions or for speculation
- Market risk is the risk that market conditions change adversely, while credit risk is the risk that the counterparty in an agreement will default on its payments.
- A forward contract is a legally binding contract to buy/sell an agreed quantity of an asset at a specified time in the future.
- A futures contract has the same definition; the difference is that forward contracts are over-the-counter (OTC), while futures are traded on an exchange and are thus more standardised.
- At maturity, the short side (seller) delivers the asset to the long side (long buyer) for the exchange delivery settlement price (EDSP).
- Often, a cash settlement is made instead of actually transferring assets.
- Each party in a futures contract must deposit some money, called the margin, to the clearing house to control credit risk.
- A financial future is a contract based on a financial asset.
- A bond future involves a notional bond - the exact list of eligible bonds must be agreed on. At maturity, the short side will deliver the cheapest bond.
- Stock index futures can be used to speculate on the movement of the market as a whole.
- Currency futures require the delivery of a specified amount of a given currency on a specified date.
- A forward rate agreement (FRA) is an interest rate hedge in which the buyer (normally a borrower) fixes the interest rate on his loan, while the seller receives the actual floating rate. Thus a buyer of an FRA profits if interest rates rise.

- An interest rate future is a futures-contract equivalent of an FRA. The price of an interest rate future rises as interest rates fall; they are usually priced at $100 - i$.
- A currency swap is an agreement to exchange interest payments and a capital sum of one currency for those in another currency. This is useful because some parties may be able to get favourable rates in their home country.
- In an interest rate swap, two parties swap a series of interest payments. Usually, one party will want a fixed instead of floating rate.
 - with a cap, the bank agrees to pay any interest above a certain rate
 - with a floor, the bank agrees to always pay you at least a certain amount of interest
 - a combination of cap and floor is called a collar.

Options

- A call option gives the holder the right to buy an asset for a specified price (the strike price) at a specified time in the future
- A put option lets you sell at the strike price.
- An American option can be exercised at any date before expiry, while a European option can only be exercised on the expiry date.
- An option is in-the-money if immediate execution would be profitable, and out-of-the-money otherwise.



- The maximum loss for the holder of an option is the premium paid.
- For a call, the holder breaks even if the price of the underlying at expiry equals the exercise price plus the premium paid.

5. Bonds, equities and inflation

Bonds

- A borrower who issues a bond pays a coupon on the face value of the bond at specified intervals until maturity, at which time they pay a fixed redemption value (almost always the same as the face value).

Fixed by the terms:

f = face / par value

r = coupon rate per year

C = redemption value

Vary throughout bond's lifetime:

P = current price

i = yield to maturity. (same as IRR).

n = time to redemption

- By writing down the cashflows of the bond, it can be seen that:

$$P = f r \frac{a_{n,i}^{(m)}}{1+i} + C v^n$$

- this is the equation of value

- price varies inversely as yield, given fixed n .

- if $P = f = C$, then $i^{(m)} = r$.

- If income (coupon payments) are taxed at rate t :

$$P = f r (1-t) a_{n,i}^{(m)} + C v^n$$

- Ratio of coupon to current price has many names: gross interest yield, flat yield, direct yield, current yield, gross running yield. These exclude tax. Net interest yield or net running yield includes tax.

- The yield to maturity, also called the gross/net yield to redemption, includes all subsequent coupons as well as the final redemption.

- To calculate yield to maturity, we can approximate it as being equal to (coupon per year + capital gains per year) / price, i.e.:

$$i \approx \frac{1}{P} (f r + \frac{100 - P}{n}) \quad \text{≈ overestimates } i.$$

- For simple cases, if $i > r$, then the bond is trading at a discount, and if $i < r$ then the bond is trading at a premium.
- To generalise slightly, let $g = fr/c$ (reduces back to the coupon rate if bond is redeemed at par). Then:
 - if $i^{(m)} \leq (1-t_1)g$, it means that $P > C$ (premium)
 - $i^{(m)} \geq (1-t_1)g \Rightarrow P < C$ (discount).
- In the latter case, capital gains tax (t_2) applies:

$$P = \frac{(1-t_1)fr a_{\frac{1}{n}}^{(m)} + (1-t_2)(v^n)}{1-t_2 v^n}$$

- Some bonds have their redemption date variable (at the bond issuer's option). The investor cannot know the exact yield, and must assume the worst case:
 - if $i^{(m)} < (1-t_1)g$, there is no capital gain and should thus expect earliest possible redemption
 - if $i^{(m)} > (1-t_1)g$, assume latest redemption as issuer would like to defer paying out.
- If a bond is sold between coupon dates, the seller feels entitled to some accrued interest. He thus sells the bond for the dirty price:

dirty price = NPV of future cash flows

clean price = dirty price - accrued interest

- Under ACT/365:

$$\text{accrued interest} = \text{annual coupon} \times \frac{\text{days since last coupon}}{365}$$

- Bonds quoted cum dividend mean that the next coupon will go to the buyer
- Ex dividend means that the buyer will not receive the coupon, so accrued interest is negative.

Equity calculations and interest with inflation

- Equities are characterised by a stream (often assumed to be infinite) of uncertain dividend payments.
- The money rate of interest of an investment ignores ^{inflation} interest; the effective rate of interest includes the inflation adjustment.

$$1 + i_R = \frac{1 + i_m}{1 + q} , \text{ where } q \text{ is the inflation rate.}$$

- For a general cash flow:

Time	0	1	2	...	n
cash flow	-P	C ₁	C ₂	...	C _n
inflation	Q(0)	Q(1)	Q(2)	...	Q(n)

The equation of value is given by:

$$P = \sum_{k=1}^n C_k \frac{Q(0)}{Q(k)} v^k , \text{ with } v = \frac{1}{1+i}$$

- For the special case of constant inflation

$$P = \sum_{k=1}^n C_k v_m^k = \sum_{k=1}^n \frac{C_n}{(1+q)^k} v_m^k , \text{ using } v_m = (1+q)v_m$$

- Some securities have their payments/coupons linked to an inflation index. In theory, this means that $C_k' = C_n Q(k)/Q(0)$, but in practice there may be a lag time.

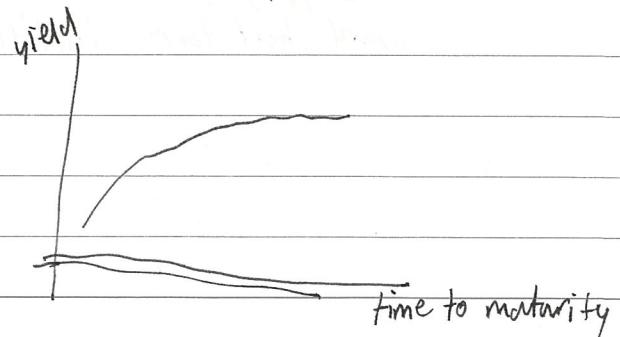
6. Interest Rate Problems

Spot rates, forward rates and the yield curve

- The term structure of interest rates is the relationship between interest rates and maturities.
 - Instead of a growth $(1+i)^t$, we have i as a function of t , so the accumulation will be $(1+y_t)^t$.
 - y_t is called the t -year spot rate of interest
 - Any fixed interest security can be analysed in terms of spot rates by looking at it as a sum of zero-coupon bonds.
 - For example, a bond with coupon x annually and redemption C obeys:
- $$P = x \left[\frac{1}{(1+y_1)} + \frac{1}{(1+y_2)^2} + \dots + \frac{1}{(1+y_n)^n} \right] + \frac{C}{(1+y_n)^n}$$
- y_1 can be determined from the yield of 1y T Bills, then y_2 from the price of 2y bonds, and so on:
- $$P = \frac{fr}{(1+y_1)} + \frac{fr+c}{(1+y_2)^2} \quad \text{if } y_1 \text{ known, so } y_2 \text{ can be found.}$$
- A zero coupon bond can also be constructed by subtracting bonds with different coupons.
 - The n -year par yield of a bond is the coupon rate that causes the bond price to be equal to its face value, assuming redemption at par. If r is the par yield, r satisfies:
- $$r \sum_{k=1}^n \frac{1}{(1+y_k)^k} + \frac{1}{(1+y_n)^n} = 1$$

- The coupon bias is the difference between the par yield and the spot rate.

- Plotting y_t against t gives the yield curve.



- The forward interest rate $f_{T,k}$ is the interest on money borrowed from time T to time $T+k$.

- like the spot rate, but doesn't necessarily start at $t=0$.

$$f_{0,k} = y_k$$

- sometimes $f_{T,k}$ means money borrowed from T to k .

- the forward rate is the geometric mean of $1+y$ can be calculated from spot rates

$$(1+y_2)^2 = (1+y_1)(1+f_{1,1})$$

$$(1+y_3)^3 = (1+y_2)^2(1+f_{2,1}) = (1+y_1)(1+f_{1,1})(1+f_{2,1})$$

$$(1+y_n)^n = (1+f_{0,1})(1+f_{1,1}) \dots (1+f_{k-1,1})$$

- i.e. the spot rate is the geometric mean of 1y forward rates.

- In the continuous case, we have the spot force of interest, and the forward force of interest (denoted by γ_t and F_t)

$$(1+y_t)^t = e^{t\gamma_t}$$

$$(1+f_{t,k})^k = e^{kF_{t,k}}$$

- The instantaneous forward rate can be found by taking the limit

$$\lim_{k \rightarrow 0} F_{t,k}, \text{ leading to } P_t = \exp(-\int_0^t F_t dt)$$

- Expectations theory explains the yield curve in terms of expected future movements in interest rates: interest rate expected to increase \Rightarrow long term bonds less attractive \Rightarrow yields increase to compensate
- Liquidity Preference theory: short term bonds are more flexible and inherently less risky as a result. So long term bonds must offer higher yield.
- Market segmentation theory suggests that there are different forces of supply and demand for bonds of different lengths: e.g. banks want short term, institutions want long term, etc.

Vulnerability to interest rate movements

- If P is the present value of a series of cashflows, then the Macaulay duration / duration / discounted mean term is given by:

$$d_m(i) = -\frac{1}{P} \frac{dP}{di} = \frac{1}{P} \sum_{k=1}^n \frac{t_k c_{t_k}}{(1+i)^{t_k}}$$

- the duration is the mean term of the cash flows weighted by their PVs. It has units of years
- the duration measures sensitivity to interest rates
- as the yield decreases, duration increases.

- The effective duration / modified duration / volatility of a cash flow is defined as:

$$d(i) = -\frac{1}{P} \frac{dP}{di} = \frac{1}{P} \sum_{k=1}^n \frac{t_k c_{t_k}}{(1+i)^{t_k+1}}$$

- it is related to the Macaulay duration by $d_m(i) = (1+i)d(i)$

- if the yield decreases by a factor of ϵ , price increases by a factor of $\epsilon d(i)$.

- The convexity of a cash flow is the (positive) second derivative of price w.r.t yield, per unit price.

$$c(i) = \frac{1}{P} \frac{d^2P}{di^2} = \frac{1}{P} \sum_{k=1}^n \frac{t_k(t_k+1)c_{t_k}}{(1+i)^{t_k+2}}$$

- higher positive convexity is good for the investor

- convexity is related to variance: a more spread out cash flow is usually more convex.

- A fund has Redington immunisation at i_0 subject to 3 conditions:

$$\textcircled{1} \quad NPV(\text{assets}) = NPV(\text{liabilities})$$

$$\textcircled{2} \quad d_m(\text{assets}) = d_m(\text{liabilities})$$

$$\textcircled{3} \quad c(\text{assets}) > c(\text{liabilities})$$

- Condition $\textcircled{3}$ is equivalent to:

$$\sum_{k=1}^n t_k^2 a_{t_k} v^{t_k} > \sum_{k=1}^n t_k^2 l_{t_k} v^{t_k}$$

Stochastic interest rate problems

- If i_t is the interest rate from time $t-1$ to t , the accumulated value of a unit investment at time n is:

$$S_n = (1+i_1)(1+i_2)\dots(1+i_n)$$

- If i_t is a random variable with mean μ and variance σ^2 , and i_1, i_2, \dots, i_n are independent:

$$E(S_n) = (1+\mu)^n$$

$$E(S_n^2) = [(1+\mu)^2 + \sigma^2]^n$$

$$\therefore \text{Var}(S_n) = [(1+\mu)^2 + \sigma^2]^n - (1+\mu)^{2n}$$

- If there are multiple investments, $E(A_n)$ and $E(A_n^2)$ can be calculated by noting that $A_n = (1+i_n)(1+A_{n-1})$.

- For a random variable Y , if $Y \sim N(\mu, \sigma^2)$, then:

$$(\ln Y \sim \text{Lognormal}(\mu, \sigma^2)) \quad \text{not the mean/var of the lognormal dist!}$$

- If $Y_i \sim \text{lognormal}(\mu, \sigma^2)$, $\prod_{i=1}^n Y_i \sim \text{lognormal}(n\mu, n\sigma^2)$, i.e. the product of lognormal distributions is a lognormal variable.
- Expectation of a lognormal dist = $e^{\mu + \frac{1}{2}\sigma^2}$
- Variance = $e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$
- Probability calculations will require the table of $\Phi(x)$, usually with interpolation.

7. Arbitrage

- If investment A costs s_A and B costs s_B , arbitrage exists in any of the following cases:
 - $NPV(A) = NPV(B)$ and $s_A \neq s_B$
 - $NPV(A) \neq NPV(B)$ and $s_A = s_B$
 - $s_A > s_B$ and $NPV(A) < NPV(B)$
- Realising the arbitrage involves buying some of A and shorting some B.

- A forward contract specifies that A will buy asset S from B for price K, at time T.
- In the simplest case of a constant force of interest with no income, the contract can be priced by considering the NPV

$$S_0 = K e^{-\delta T}$$

- If there are fixed coupons, we can amend the above to include $ce^{-\delta t_i}$.
- However, it is different when the dividend yield is paid on the (unknown) price of the asset. By the no-arbitrage assumption, at time T we require that $S = K$. Then we have:

$$S_0 e^{-rt} = ke^{-\delta T}$$

at an intermediate time

- The value of a long contract can be calculated by finding $K_t - K_0$, where t is the time of valuation, then discounting.

$$V_t = S_t - S_0 e^{\delta t}$$