Vectors

- Combining a vector with its additive inverse gives the zero vector, with length 0 and undefined direction.
- A scalar product projects one vector onto another.
- We can resolve $\underset{\sim}{a}$ into // and $\perp$ vectors w.r.t some $\hat{\underset{\sim}{n}}$

$$\underset{\sim}{a} = \underbrace{\underset{\sim}{a} - (\underset{\sim}{a}\cdot\hat{\underset{\sim}{n}})\hat{\underset{\sim}{n}}}_{\underset{\sim}{a}_\perp} + \underbrace{(\underset{\sim}{a}\cdot\hat{\underset{\sim}{n}})\hat{\underset{\sim}{n}}}_{\underset{\sim}{a}_{//}}$$

- Distributive property of dot product can be proved diagrammatically.
- Derive cosine rule with $|\underset{\sim}{c}| = |\underset{\sim}{a}+\underset{\sim}{b}| \Rightarrow |\underset{\sim}{c}|^2 = (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b})$.

Vector product

- $\underset{\sim}{a}\times\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta\,\hat{\underset{\sim}{n}}$ ← only unique in 3D.
- $\underset{\sim}{a}\times\underset{\sim}{b} = -\underset{\sim}{b}\times\underset{\sim}{a}$ (anticommutative)
- $\underset{\sim}{a}\times\underset{\sim}{b} = 0 \Rightarrow \underset{\sim}{a} // \underset{\sim}{b}$ OR $\underset{\sim}{a}$ or $\underset{\sim}{b} = \underset{\sim}{0}$.
- $|\underset{\sim}{a}\times\underset{\sim}{b}|$ is the area of a parallelogram.
- Non-associative, i.e $\underset{\sim}{a}\times(\underset{\sim}{b}\times\underset{\sim}{c}) \neq (\underset{\sim}{a}\times\underset{\sim}{b})\times\underset{\sim}{c}$.

Vector area

- Vector area $\underset{\sim}{S}$ of a finite plane surface is defined such that $|\underset{\sim}{S}|$ = area, with $\underset{\sim}{S}$ pointing normal to surface.
- The area of a projection (e.g onto xy plane) is $\underset{\sim}{S}\cdot\hat{\underset{\sim}{z}}$
- We can define a total vector area for a composite surface as the sum of vector area elements, $\underset{\sim}{S} = \int d\underset{\sim}{S}$
  ↳ $\Sigma\underset{\sim}{S}$ for a closed surface $= 0$.

Triple products

- Scalar triple product: $[\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c}] \equiv \underset{\sim}{a}\cdot(\underset{\sim}{b}\times\underset{\sim}{c})$
  ↳ invariant under cyclic permutation, i.e $\underset{\sim}{a}\cdot(\underset{\sim}{b}\times\underset{\sim}{c}) = \underset{\sim}{c}\cdot(\underset{\sim}{a}\times\underset{\sim}{b}) = \underset{\sim}{b}\cdot(\underset{\sim}{c}\times\underset{\sim}{a})$
  ↳ gives the volume of a parallelepiped

- If scalar triple product is zero, vectors are coplanar.

- The vector triple product is $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c})$, which can be evaluated with the BAC-CAB rule:

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = \underset{\sim}{b}(\underset{\sim}{a} \cdot \underset{\sim}{c}) - \underset{\sim}{c}(\underset{\sim}{a} \cdot \underset{\sim}{b}).$$

  ↳ $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c})$ lies in the plane of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

Lines and planes

- A line is parameterised by λ: $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \hat{\underset{\sim}{l}}$
- Because $(\underset{\sim}{r} - \underset{\sim}{a}) \parallel \hat{\underset{\sim}{l}}$, we can also write: $\underset{\sim}{r} \times \hat{\underset{\sim}{l}} = \underset{\sim}{a} \times \hat{\underset{\sim}{l}}$

- For a plane: $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{p} + \mu \underset{\sim}{q}$

$$\Rightarrow \underset{\sim}{r} \cdot \hat{\underset{\sim}{n}} = \underset{\sim}{a} \cdot \hat{\underset{\sim}{n}} = d$$

  ↳ ~~pe~~shortest distance to the origin is $|d|$.

Orthogonal bases

- In 3D, any 3 non-coplanar vectors constitute a basis
  - basis spans the space, i.e $\underset{\sim}{r} = \lambda \underset{\sim}{a} + \mu \underset{\sim}{b} + \nu \underset{\sim}{c}$ where the components $\{\lambda, \mu, \nu\}$ are unique.
  - basis vectors will have linear independence.
- Components can be extracted using the reciprocal basis

cyclic order preserved. $\left\{ \underset{\sim}{A} = \dfrac{\underset{\sim}{b} \times \underset{\sim}{c}}{[\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}]} \quad \underset{\sim}{B} = \dfrac{\underset{\sim}{c} \times \underset{\sim}{a}}{[\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}]}, \quad \underset{\sim}{C} = \dfrac{\underset{\sim}{a} \times \underset{\sim}{b}}{[\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}]} \right.$

  ↳ the component is just dot product of $\underset{\sim}{r}$ with the appropriate reciprocal basis vector:

$$\lambda = \underset{\sim}{A} \cdot \underset{\sim}{r} \quad \mu = \underset{\sim}{B} \cdot \underset{\sim}{r} \quad \nu = \underset{\sim}{C} \cdot \underset{\sim}{r}$$

- A basis is orthonormal if all basis vectors are $\perp$ and have unit length
- Right-handed if $[\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}] > 0$
- Direction cosines are cosines of angles between $\underset{\sim}{a}$ and coordinate axes, i.e $\underset{\sim}{a} = |\underset{\sim}{a}|(\cos\theta_x, \cos\theta_y, \cos\theta_z)$ in Cartesian.

· In Cartesian, $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is invariant under rotation.

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \Leftarrow$$ i.e. transformed volume of a unit cube.

Polar coordinates

· Point specified by (r, ϕ)

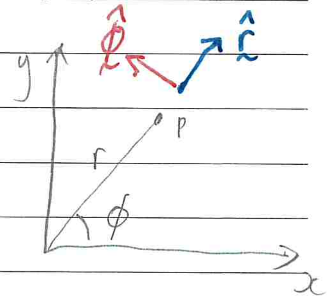
$$x = r\cos\phi \qquad y = r\sin\phi$$
$$r = \sqrt{x^2+y^2} \qquad \phi = \tan^{-1}(y/x).$$

· Circle described by $r = a$

· Straight line at angle α to y-axis with shortest dist $|d|$:

$$r\cos(\phi - \alpha) = d.$$

· We can use the following orthonormal basis:



$$\hat{\underset{\sim}{r}} = \cos\phi\,\hat{\underset{\sim}{i}} + \sin\phi\,\hat{\underset{\sim}{j}}$$
$$\hat{\underset{\sim}{\phi}} = -\sin\phi\,\hat{\underset{\sim}{i}} + \cos\phi\,\hat{\underset{\sim}{j}}$$

· We can evaluate $\dot{\underset{\sim}{r}}$:

$$\dot{\underset{\sim}{r}} = \dot{r}\hat{\underset{\sim}{r}} + r\dot{\phi}\hat{\underset{\sim}{\phi}}$$

· The area element will be $r\,dr\,d\phi$.

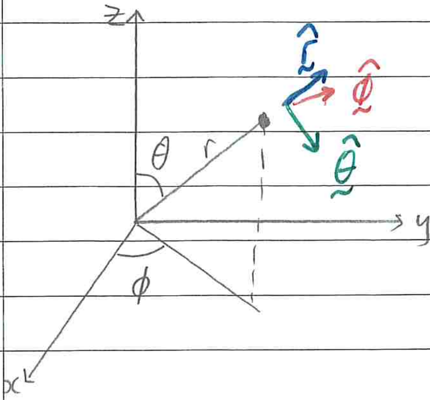
Cylindrical coordinates

· Extension of plane polar coordinates to include z.

$$x = r\cos\phi \qquad y = r\sin\phi \qquad z = z$$

· Volume element is: $dV = r\,dr\,d\phi\,dz$.

Spherical coordinates



- Points described by radius, polar angle, azimuthal angle (i-e r, θ, ϕ).

$x = r\sin\theta\cos\phi$ $\quad y = r\sin\theta\sin\phi$ $\quad z = r\cos\theta$

$r = \sqrt{x^2+y^2+z^2}$ $\quad \theta = \cos^{-1}\left(\frac{z}{r}\right)$ $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

- We can find the orthonormal basis vectors using:

$$\hat{r} = \frac{\partial \underset{\sim}{r}}{\partial r} \Big/ \left|\frac{\partial \underset{\sim}{r}}{\partial r}\right| \qquad \hat{\theta} = \frac{\partial \underset{\sim}{r}}{\partial \theta} \Big/ \left|\frac{\partial \underset{\sim}{r}}{\partial \theta}\right| \qquad \hat{\phi} = \frac{\partial \underset{\sim}{r}}{\partial \phi} \Big/ \left|\frac{\partial \underset{\sim}{r}}{\partial \phi}\right|$$

$$\Rightarrow \quad \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\underset{\sim}{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\underset{\sim}{k}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

- $dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi.$

Complex numbers

- Complex numbers are a closed field → all operations return $\mathbb{Z}$.
- Complex conjugate $z^* \equiv a - ib$ for $z = a + ib$
  - ↳ $zz^* = a^2 + b^2 > 0$
  - ↳ $z + z^* = 2\,\mathrm{Re}(z)$
  - ↳ $z - z^* = 2i\,\mathrm{Im}(z)$.
  - ↳ $\frac{1}{z} = \frac{z^*}{|z|^2}$
- Mutiplying corresponds to scaling and rotation.
- De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
  - ↳ can be used to derive trig identities

$$\text{e.g. } \cos 5\theta = \mathrm{Re}(\cos 5\theta + i\sin 5\theta) = \mathrm{Re}\left((\cos\theta + i\sin\theta)^5\right).$$

$$\text{e.g. } \cos\theta = \tfrac{1}{2}(z + z^{-1}) \Rightarrow \cos^5\theta = \tfrac{1}{2^5}(z + z^{-1})^5 \text{ etc.}$$

- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

- The nth roots of unity are the solutions to $z^n = 1$ for positive n.

$$e^{in\theta} = 1 \Rightarrow \theta = \frac{2\pi k}{n}, \quad k = 0, 1, 2, \dots n-1$$

$$\therefore \text{ roots are } 1, \omega, \omega^2, \dots \omega^{n-1} \text{ with } \omega \equiv e^{2\pi i/n}.$$

- We define the complex logarithm as:

$$\ln z = \ln(re^{i\theta}) = \ln r + i(\theta + 2\pi n) \quad n = 0, \pm1, \pm2, \dots$$

  - ↳ the principal value is $\ln r + i\theta$ for $\theta \in [0, 2\pi)$.
- Likewise, general powers will be multi-valued

$$z_1^{z_2} = e^{z_2 \ln z_1}$$

- The fundamental theorem of algebra states:

$$a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

has n complex roots for all possible complex coefficients.

Hyperbolic functions

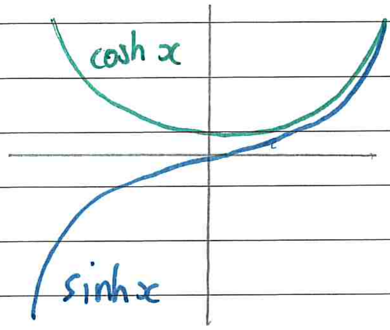
- Define: $\cos z \equiv \frac{1}{2}(e^{iz}+e^{-iz})$ and $\sin z \equiv \frac{1}{2i}(e^{iz}-e^{-iz})$.
- The hyperbolic functions are these functions evaluated on the imaginary axis:

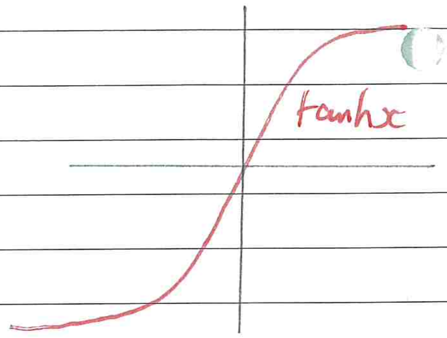
$$\cosh y \equiv \cos(iy) = \frac{1}{2}(e^{y}+e^{-y})$$

$$\cancel{\sinh y \equiv \frac{1}{i}\sin(iy) = \frac{1}{2i}(e^{-y}-e}$$

$$\sinh y = \frac{1}{i}\sin(iy) = \frac{1}{2}(e^{y}-e^{-y}).$$

We can then define tanh, sech, cosech etc.





- We can generate identities by substituting iy in and using $\cos iy = \cosh y$, $\sin(iy) = i\sinh y$.
  - ↳ $\cosh^2 y - \sinh^2 y = 1$
  - ↳ $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$
- Inverse hyperbolic functions can be expressed as elementary functions.

Calculus and Analysis

Limits

- Intuitively, $\lim_{x \to x_0} f(x) = k$ means $f(x)$ can be made arbitrarily close to k by making x close enough to x_0.
- The ε-δ definition: For real $f(x)$ defined on some open interval containing x_0 (but not necessarily at x_0), $\lim_{x \to x_0} f(x) = k$ means
  for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that:
  $$|f(x) - k| < \varepsilon \quad \text{for all} \quad 0 < |x - x_0| < \delta$$
  ↳ i.e if you give me an ε, I can find δ to stay within ε of k.
  ↳ in practice, we guess the limit then prove with ε-δ.
- Limits at infinity: $|f(x) - k| < \varepsilon$ for all $x > X$.
- Limits can be manipulated by addition and multiplication.

- If a quotient is indeterminate (top and bottom both 0 or $\pm\infty$), we can use L'Hôpital's rule:
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Continuity and differentiability

- A real function $f(x)$ is continuous at $x = a$ iff:
  i) $f(a)$ exists
  ii) $\lim_{x \to a} f(x)$ exists and equals $f(a)$.
- A function $f(x)$ is differentiable at $x = a$ iff:
  i) it is continuous at $x = a$
  ii) $f'(a)$ exists i.e $\lim_{\delta x \to 0} \frac{f(a + \delta x) - f(a)}{\delta x}$ exists.

Leibniz formula

- Used to find nth derivative of a product of functions (just like Binomial theorem):

$$\frac{d^n(fg)}{dx^n} = \sum_{m=0}^{n} \binom{n}{m} f^{(n-m)} g^{(m)}$$

$$= f^{(n)}g + nf^{(n-1)}g' + \frac{n(n-1)}{2} f^{(n-2)}g'' + \dots + fg^{(n)}$$

- Can be proved by induction.

Infinite Series

- Given a sequence of terms $u_0, u_1, u_2, \dots$ the nth partial sum is $S_n \equiv \sum_{k=0}^{n} u_k$
- If the partial sums have a finite limit as $n \to \infty$, the infinite series is convergent.
  - ↳ if it doesn't converge, it either diverges or oscillates.
- If $\sum_{k=0}^{\infty} |u_k|$ converges, the series is absolutely convergent (which also implies $\sum_{k=0}^{\infty} u_k$ converges)
  - ↳ otherwise if $\sum_{k=0}^{\infty} u_k$ converges but $|u_k|$ doesn't, series is conditionally convergent.
  - ↳ For absolutely convergent series we can rearrange terms.

Geometric progressions

$$S_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$ because $rS_n = r + r^2 + \dots + r^{n+1} = r^{n+1} + S_n - 1.$

- Series is absolutely convergent for $|r| < 1$

$$\therefore \sum_{k=0}^{\infty} r^n = \frac{1}{1-r}$$

- If $|r| \geq 1$, series cannot converge.

Convergence tests

1. $u_k \to 0$ as $k \to \infty$ is a necessary condition for convergence (but insufficient, e.g harmonic series).

2. Comparison test:
   - Compare with a series of known convergence, v_k
   - If all terms $\leq$ ~~terms~~ v_k for all $k \geq K$, S_n converges
   - If all terms $\geq v_k$ for divergent v, S_n diverges.
   - Try to compare with geometric series or Harmonic series
   - p-series test: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ → converges for $p>1$ by comparison with geometric series
     ↳ diverges for $p \leq 1$ by comp. with Harmonic

3. Ratio test

   If $\lim_{k\to\infty} \frac{u_{k+1}}{u_k} < 1$, S_n converges

   If $\lim_{k\to\infty} \frac{u_{k+1}}{u_k} > 1$, S diverges

   If ratio $=1$, test indeterminate.

4. Alternating series:
   - Use the Leibniz criterion:

     $\sum_{k=0}^{\infty} (-1)^{k+1} a_k$ with $a_k > 0$ converges if a_k is monotonic decreasing for large enough k and $\lim_{k\to\infty} a_k = 0$.

5. Integral test:
   - If $f(n)$ is continuous, positive, and decreasing on $[1, \infty)$:

     $\sum_{n=1}^{\infty} f(n)$ converges / diverges as $\int_1^{\infty} f(x)dx$.

Power series

- Series of the form $f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \dots$
- Either:
  - converges for $x = 0$ only
  - converges for all finite x
  - converges for $|x| < R$, diverges for $|x| > R$.
- Using ratio test and $L \equiv \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$:

  convergent for $|x| < 1/L$, divergent for $|x| > 1/L$.
- For a complex power series, this will define a circle of convergence

Taylor series

- $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$

- Or Maclaurin series when $a = 0$: $f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \dots$
- We can truncate the Taylor series and add a remainder term:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \dots + \frac{x^k}{k!} f^{(k)}(0) + R_n$$

with $R_n = \frac{1}{k!} \int_0^x (x-t)^k f^{(k+1)}(t)\, dt$.

↳ derived by $f(x) = f(0) + \int_0^x f'(t)\, dt$ (FTC) then IBP.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

Newton-Raphson

- Helps us find x^* such that $f(x^*)=0$
- If we have an initial guess x_0, we need h such that $f(x_0+h)=0$.

$$0 = f(x_0+h) \approx f(x_0) + hf'(x_0).$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Then we can iterate this to converge on x^*.
- If ε_i is the error in x_i:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \Rightarrow \varepsilon_{i+1} = \varepsilon_i - \frac{f(x^*+\varepsilon_i)}{f'(x^*+\varepsilon_i)}.$$

↳ approximating the last term with a Taylor expansion:

$$\varepsilon_{i+1} \approx \varepsilon_i^2 \frac{f''(x^*)}{2f'(x^*)} \quad \text{i.e rapid quadratic convergence}$$

- If there is a turning point between the root and x_i, it may not converge.

Integration

- Formally: $\int_a^b f(x)dx = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i)(x_{i+1}-x_i)$. ← area under curve.

- Hyperbolic substitutions:
  $\sqrt{x^2+a^2}$ ← $x = a\sinh y$
  $\sqrt{x^2-a^2}$ ← $x = a\cosh y$
  a^2-x^2 ← $x = a\tanh y$
- Integrate using complex numbers, e.g. $\int \cos x e^{ax} dx = Re\left(\int e^{(a+i)x} dx\right)$

- If $I(\alpha) \equiv \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)dx$

$$I'(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha} dx + \frac{db}{d\alpha} f(b;\alpha) - \frac{da}{d\alpha} f(a;\alpha)$$

Stirling's approximation

$\ln n! = \sum_{k=1}^{n} \ln n$. But $\int_1^n \ln x dx \le \sum_{k=1}^{n} \ln n \le \int_1^{n+1} \ln x dx$

$\therefore \ln n! \sim n\ln n - n$ for large n.

Cauchy-Schwarz inequality

$|\langle \underset{\sim}{a}, \underset{\sim}{b} \rangle|^2 \le ||\underset{\sim}{a}|| \, ||\underset{\sim}{b}||$ where $\langle \cdot , \cdot \rangle$ is the inner product.

- For ~~an~~ N-dimensional vectors,

$$\left(\sum_{i=1}^{N} a_i b_i\right)^2 \le \left(\sum_i^N a_i^2\right)\left(\sum_i^N b_i^2\right)$$

- Taking $N\to\infty$, we get Schwarz's inequality.

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx.$$

Multiple integrals

- $$\iiint f(\underline{x})\,dV = \lim_{\delta V \to 0} \sum f(r)\,\delta V.$$

- Cartesian: $dV = dx\,dy\,dz$
- Cylindrical: $dV = r\,dr\,d\phi\,dz$
- Spherical: $dV = r^2 \sin\theta\,dr\,d\theta\,d\phi$
- We can do the integrals in any order.
- If limits are independent, we can factor the integral out.

Gaussian ~~distribution~~ Integrals

- $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ is a common improper integral.
- Evaluate with polar coordinates:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dx\right) = \iint e^{-(x^2+y^2)}\,dx\,dy.$$

- Technically should use a in limits then $\lim_{a \to \infty}$

Probability

- Outcomes ω_i are mutually exclusive
- The sample space is the set of all possible outcomes: $\Omega = \{w_i\}$
- An event is ~~th~~ a subset of Ω

- $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \implies P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

(Bayes' Theorem)

- Law of total probability: $P(A) = \sum_i P(A|B_i) P(B_i)$

Random variables

- Map sample states to an allowed value of the random variable, such that the subsets partition the space.
- Assign a prob. distribution $P(x)$.

- Poisson distribution : $P(X = x) = \dfrac{e^{-\lambda} \lambda^x}{x!}$

↳ can be shown that it is the limit of a binomial dist as $n \to \infty$, with $np = \lambda$.

- For continuous random variables, the prob. density function is

$$f(x)\,dx \equiv P(x - \tfrac{dx}{2} \le X \le x + dx/2).$$

$$P(a \le X \le b) = \int_a^b f(x)\,dx \quad \text{with } \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$F(a) \equiv \int_{-\infty}^{a} f(x)\,dx.$$

- Median is a such that $F(a) = 1/2$
- Variance of a distribution is the same even when conditioned

Ordinary Differential Equations

- A first-order ODE has the form $F(y', y, x) = 0$.
- An nth-order ODE : $F(y^{(n)}, y^{(n-1)}, \dots, y', y, x) = 0$.
- A separable 1st order ODE :

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \Rightarrow \int g(y)\,dy = \int f(x)\,dx.$$

- The general solution (including a constant) can be fixed by an initial/boundary condition.

- A linear 1st order ODE : $\frac{dy}{dx} + p(x)y = f(x)$ — if $f(x) = 0$, it is homogeneous, and separable.
  - ↳ y and $\frac{dy}{dx}$ appear linearly
  - ↳ can be solved with an integrating factor, $\mu(x)$, such that $\mu(x) \cdot \text{LHS}$ is the derivative of something w.r.t x.

$$\Rightarrow \mu(x) = e^{\int p(x)dx}$$

  - ↳ $\therefore \frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$ which is easy to solve.

- Substitutions may be required to make an ODE linear/separable.
- Homogeneous ODE : $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ ← i.e f invariant when x and y scaled.
  - ↳ solve by sub $u = y/x$

$$\Rightarrow y = u(x)x \Rightarrow x\frac{du}{dx} + u = F(u) \leftarrow \text{separable.}$$

- Bernoulli ODE : $\frac{dy}{dx} + p(x)y = q(x)y^n$
  - ↳ sub $z = y^{1-n}$ $\Rightarrow \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

$$\therefore \frac{dz}{dx} = (1-n)[-p(x)z + q(x)] \leftarrow \text{linear.}$$

Second-order equations

- A linear 2^{nd} order ODE: $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)$.

↳ with the linear differential operator $\mathcal{L}$, we can rewrite

$$\mathcal{L} = \frac{d}{dx^2} + p(x)\frac{d}{dx} + q(x) \Rightarrow \mathcal{L}y = f(x)$$

↳ $\mathcal{L}(\alpha u) = \alpha \mathcal{L}(u)$ if α constant } because linear.
↳ $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$

- For a homogeneous 2^{nd} order ODE ($\mathcal{L}y = 0$), any linear combination of solutions is a solution ← principle of superposition.
- For inhomogeneous case, i.e. $\mathcal{L}y = f(x)$:
  - a particular integral is any solution of $\mathcal{L}y = f(x)$
  - the complementary function y_c is the general solution of $\mathcal{L}y = 0$
  - the general solution is the sum: $y(x) = y_c(x) + y_p(x)$.

- 2^{nd} order ODEs are generally hard to solve unless constant coefficients.
- Consider homogeneous 2^{nd} order linear ODE:

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0.$$

↳ sub $y = e^{\lambda x}$ as a trial gives the auxiliary equation $\lambda^2 + 2a\lambda + b = 0$
↳ if roots are ~~negative~~ complex, we have oscillatory behavior.
↳ if $\lambda_1 = \lambda_2$, we have critical damping: $y = (C_1 + C_2 x)e^{-\alpha x}$.

- For linear 2^{nd} order inhomogeneous ODEs with constant coefficients:
  - y_c can be found as above.
  - y_p can be found with trial solutions
  - if $f(x)$ is a polynomial, try y_p = polynomial of same degree
  - if $f(x) = ce^{kx}$, try $y_p = de^{kx}$
  - if $f(x) = c_1 \cos kx + c_2 \sin kx$, try $y_p = d_1 \cos kx + d_2 \sin kx$.
  - but if scalar multiples of these trial solutions are already solutions of the homogeneous eq, we may need to multiply by x or x^2 and try again

- Alternatively, since it is linear and differential operators commute, we can factorise : $\left(\frac{d}{dx}-\lambda_1\right)\left(\frac{d}{dx}-\lambda_2\right)=f(x)$
  ↳ let $z(x)=\left(\frac{d}{dx}-\lambda_2\right)y \Rightarrow \left(\frac{d}{dx}-\lambda_1\right)z=f(x)$.
  ↳ solve for z then for y.
  ↳ this gives us a particular integral

Multivariable calculus

- Mixed partial derivatives are always equal, and partial derivatives commute. $\therefore f_{xy} = f_{yx}$
- Integrating w.r.t one variable, we can treat others as constant but then we will need to add an arbitrary function.
- For $f(x,y)$, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.
- Taylor series becomes:

$$f(x+h, y+k) = f(x,y) + f_x(x,y)h + f_y(x,y)k + \tfrac{1}{2}f_{xx}h^2 + f_{xy}hk + \tfrac{1}{2}f_{yy}k^2 + \dots$$

- Suppose $f(x,y)$ where $x = x(u,v)$ $y = y(u,v)$. By an abuse of notation, we write $f(x,y) = f(u,v)$ even though they are different functions:

$$\left.\begin{aligned}\left(\frac{\partial f}{\partial u}\right)_v &= \left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial u}\right)_v \\ \left(\frac{\partial f}{\partial v}\right)_u &= \left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u\end{aligned}\right\} \text{multivariable chain rule}$$

e.g. $f(x,y) \to f(r,\phi)$: $x = r\cos\phi$, $y = r\sin\phi$

$\therefore \left(\frac{\partial f}{\partial r}\right)_\phi = \cos\phi\left(\frac{\partial f}{\partial x}\right)_y + \sin\phi\left(\frac{\partial f}{\partial y}\right)_x$ etc.

- If both x and y are functions of t:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)_y\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_x\frac{dy}{dt}$$

- If we have $F(x,y,z) = 0$, then the partial derivatives have reciprocity and are cyclic.

i.e $\left(\frac{\partial x}{\partial y}\right)_z = 1\Big/\left(\frac{\partial y}{\partial x}\right)_z$ and $\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_z = -1$

Exact differentials

- $\omega = P(x,y)dx + Q(x,y)dy$ is a differential form in x and y.
- ω is an exact differential if $\exists f(x,y)$ such that $df = Pdx + Qdy$.
  ↳ equivalently, exact iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
  ↳ if $Pdx + Qdy$ is exact, $f(x,y) = c$.
- We can make an inexact differential form exact with an integrating factor: $\mu(x,y)[Pdx + Qdy]$
  ↳ this is very difficult to solve for μ, so we instead try to find $\mu(x)$ or $\mu(y)$ only.
  e.g. $\mu(x)$: $\mu \frac{\partial P}{\partial y} = Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x}$ if exact

$$\Rightarrow \frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

  ↳ likewise for $\mu(y)$: $\frac{1}{\mu}\frac{d\mu}{dy} = -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$.

* ## Maxwell's relations

- Any two of (p, V, T, S) can describe the state of a gas.
- ~~Given a thermodynamic relation,~~ we
- The fundamental thermodynamic relation is
  $$dU = TdS - pdV$$
  ↳ if we treat U as a function of (S, V):
  $$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV \Rightarrow \left(\frac{\partial U}{\partial S}\right)_V = T \text{ and } \left(\frac{\partial U}{\partial V}\right)_S = -p$$
  $\therefore \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$ by mixed partials. This is one of Maxwell's relations.
- We can derive the others using Legendre transformations
  ↳ $F = U - TS \Rightarrow dF = -SdT - pdV$
  ↳ $H = U + PV \Rightarrow dH = TdS + VdP$
  ↳ $G = H - TS \Rightarrow dG = -SdT + VdP$.

- We can also derive a different type of relation:

$dU = TdS - pdV$ but let $U = U(T, S)$

~~$\therefore dU = TdS - p\left[\left(\frac{\partial V}{\partial T}\right)_S dT + \left(\frac{\partial V}{\partial S}\right)_T dS\right]$~~

then take partial derivatives and equate.

Stationary points

- Because $f(\underline{x}) = f(\underline{x_0}) + \nabla f(\underline{x_0}) \cdot \delta \underline{x}$, a point is stationary if $\nabla f(\underline{x_0}) = \underline{0}$.
- To find the character of the stationary points, we use the determinant of the Hessian: $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$.

$\det H > 0$ and $f_{xx} > 0 \Rightarrow$ minimum

$\det H > 0$ and $f_{xx} < 0 \Rightarrow$ maximum

$\det H < 0 \Rightarrow$ saddle

$\det H = 0$ inconclusive.

- For more variables:
  - ↳ if all eigenvalues > 0, min
  - ↳ if all eigenvalues < 0, max
  - ↳ else saddle.

Conditional stationary values.

- To optimise $f(x,y)$ subject to $g(x,y) = c$, solve

$\nabla f = \lambda \nabla g$, where λ is a Lagrange Multiplier

  - ↳ consider some displacement $d\underline{x}$
  - ↳ $d\underline{x}$ must be tangent to $g(x,y) = 0$ $\therefore (\nabla g) \cdot d\underline{x} = 0$
  - ↳ Likewise, $df = (\nabla f) \cdot d\underline{x} = 0$ by definition of a stationary point
  - ↳ $\therefore \nabla f \parallel \nabla g$.
- For more constraints: $\nabla f = \lambda \nabla g + \mu \nabla h$

Boltzmann distribution

- Consider a system ~~where~~ which has n possible discrete states, ~~with a~~ which holds N_i particles whose energy is E_i.
  - ↳ total number of particles is $N = \sum_{i=1}^{n} N_i$
  - ↳ total energy is $E = \sum_{i=1}^{n} N_i E_i$
- A given distribution of particles can be achieved in W ways:

$$W = \frac{N!}{N_1! N_2! \dots N_n!}$$

- The most likely state maximises W, or $\ln W$ equivalently

$$\ln W = \ln(N!) - \sum_{i=1}^{n} \ln(N_i!)$$

↳ in an isolated system, $N = \hat{N}$ and $E = \hat{E}$

$$\therefore \quad L = \ln(N!) - \sum_{i=1}^{n} \ln(N_i!) - \alpha\left(\sum_{i=1}^{n} N_i - \hat{N}\right) - \beta\left(\sum_{i=1}^{n} N_i E_i - \hat{E}\right).$$

↳ N_i are the variables, $\therefore$ need $\partial L / \partial N_i$.

$$\frac{\partial L}{\partial N_i} = \ln N - \ln N_i - \alpha - \beta E_i \quad \text{because} \quad \frac{\partial \ln N!}{\partial N_i} = \frac{\partial \ln N!}{\partial N}\frac{\partial N}{\partial N_i} = \ln N.$$

↳ Then set $\frac{\partial L}{\partial N_i} = 0$ and solve for N_i $\therefore N_i = N e^{-\alpha} e^{-\beta E_i}$

- This gives the Boltzmann dist.
- Different assumptions about particle states leads to different W.

Vector calculus

- Let $\phi(x,y,z)$ be a scalar field.

$$\text{grad } \phi \equiv \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

- The rate of change of ϕ in direction $\underset{\sim}{t}$ is the directional derivative

$$\frac{d\phi}{ds} = \underset{\sim}{t}\cdot\nabla\phi$$

- this implies that $\nabla\phi$ is the direction of most rapid increase.
- Given a surface $f(x,y,z)=c$, ∇f must be normal to the surface because f is constant along the surface

$$\Rightarrow \underset{\sim}{\hat{n}} = \frac{\nabla f}{|\nabla f|}.$$

Line integrals

- Consider a curve parameterised by t: $\underset{\sim}{r} = (x(t), y(t), z(t))$
- $d\underset{\sim}{r} = \frac{d\underset{\sim}{r}}{dt}\,dt$

- For a scalar field parameterised by an arc length s:

$$\int_C \phi\, ds = \int_{s_1}^{s_2} \phi(\underset{\sim}{r}(s))\, ds$$

- For a more general parameter t:

$$\int_C \phi\, ds = \int_{t_1}^{t_2} \phi(\underset{\sim}{r}(t)) \left|\frac{d\underset{\sim}{r}}{dt}\right| dt$$

- For a vector field $\underset{\sim}{F}(\underset{\sim}{r})$

$$\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_{t_1}^{t_2} \underset{\sim}{F}(\underset{\sim}{r}(t))\, \frac{d\underset{\sim}{r}}{dt}\, dt.$$

- The Gradient theorem:

$$\int_C (\nabla\phi)\cdot d\underset{\sim}{r} = \int_C d\phi = \phi(\underset{\sim}{r_2}) - \phi(\underset{\sim}{r_1})$$

Conservative fields

- Line integral independent of the path.
- $\underset{\sim}{F} = -\nabla\phi$ for some $\phi(\underset{\sim}{r})$
- $\underset{\sim}{F}\cdot d\underset{\sim}{r}$ is exact
- $\oint_c \underset{\sim}{F}\cdot d\underset{\sim}{r} = 0$ for all closed curves.
- $\nabla\times\underset{\sim}{F} = 0$

each implies the other.

Surface integrals

- For a general curved surfaces S in space, the vector area element is defined by $d\underset{\sim}{S} = \hat{\underset{\sim}{n}}\,dS$. The total vector area is $\int_S \hat{\underset{\sim}{n}}\,dS$
- The flux of $\underset{\sim}{F}$ through S is defined by:

$$\int_S \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_S \underset{\sim}{F}\cdot\hat{\underset{\sim}{n}}\,dS$$

Divergence

- $\text{div}\,F \equiv \nabla\cdot F \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ ← SCALAR
- The divergence theorem:

$$\iiint_V (\nabla\cdot\underset{\sim}{F})\,dV = \int_S \underset{\sim}{F}\cdot d\underset{\sim}{S}$$

- Can be used to define divergence:

$$\nabla\cdot\underset{\sim}{F} = \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta S}\underset{\sim}{F}\cdot d\underset{\sim}{S}$$

- If a surface is not closed, we can first construct a closed one then apply the divergence theorem.
- The Laplacian is the divergence of a gradient

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

↳ it is also a scalar

Curl

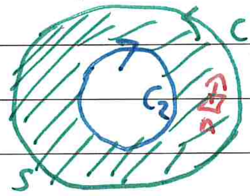
- $\nabla \times \underset{\sim}{F} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$ (Vector)

- Stokes theorem: $\int_S \nabla \times \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

  where C bounds S.

  ↳ use RH grip rule for direction.
- This leads to a geometric definition: $\hat{\underset{\sim}{n}} \cdot (\nabla \times \underset{\sim}{F}) = \lim_{\delta S \to 0} \frac{1}{\delta S} \int_{\delta C} \underset{\sim}{F} \cdot d\underset{\sim}{r}$
- For any ~~vector~~ conservative field $\underset{\sim}{F} = -\nabla\phi$, $\nabla \times \underset{\sim}{F} = 0$.
- Many different surfaces can be bounded by a closed curve, but only one volume is bounded by a closed surface
- A multiply connected surface may have multiple bounding curves

e.g annulus



$$\int_S (\nabla \times \underset{\sim}{F}) d\underset{\sim}{S} = \int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} + \int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

- For a planar surface, we can use Green's theorem, a special case.

Fourier Series

- Functions are orthogonal on an interval if their inner product is zero:

$$\int_a^b f(x)g(x)\,dx = 0$$

- On the interval $[-\pi, \pi]$, all $\cos nx$ and $\sin mx$, ~~fo~~ $n, m \in \mathbb{Z}$ are mutually orthogonal (but not normalised):

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 2\pi, & m=n=0 \\ \pi, & m=n\neq 0 \\ 0 & m\neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} \pi, & m=n\neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx = 0$$

- We can change bounds to $\pm L$ provided we scale $mx \to \frac{m\pi x}{L}$
  - ↳ then this will work for any $[a,b]$ such that $2L = b-a$.
- $\sin mx$ and $\cos nx$ thus form a basis, such that almost any $f(x)$ can be represented with a Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$
  - ↳ Fourier coefficients can be found by integrating ~~w/~~ after mult. with $\cos\left(\frac{m\pi x}{L}\right)$ or $\sin\left(\frac{m\pi x}{L}\right)$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx}$$

  - ↳ for even functions, all $b_n = 0$, so it is a cosine series
  - ↳ for odd functions, all $a_n = 0$, so it is a sine series

~~• Fourier coefficients decline like $1/n$, so we can approx functions.~~

- We can observe how fast the coefficients decline to understand convergence.
- Around a discontinuity, the Fourier series will always overshoot, even in the limit, though the width of the overshoot ↓. Gibbs phenomenon.
- Differentiating always reduces smoothness:
  - ↳ Fourier coefficients drop less rapidly.

- The mean-square value of a periodic function can be evaluated using Parseval's theorem: $\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{r=1}^{\infty}(a_n^2+b_n^2)$
- The set of values, for different n, is the power spectrum and describes how power is distributed amongst the harmonics.

Complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx$$

- $e^{in\pi x/L}$ is used as a basis
- For complex functions, $f(x)$ and $g(x)$ are orthogonal if:

$$\int_a^b [f(x)]^* g(x)\, dx = 0.$$