Combinatorial Optimization and Modern Heuristics: Assignment 1

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Chapter 1 Problems

Problem 1(d)

Find a cylinder with a given surface area A that has the largest volume V.

Solution:

For the solution, it is sufficient to give only a feasible set F and cost function c.

$$F = \{r, h \in \mathbb{R} : 2\pi r(h + r) = A\} \tag{1}$$

$$c = \frac{1}{\pi r^2 h}, c : F \to \mathbb{R} \tag{2}$$

Problem 3

Show that the neighborhood defined in Example 1.5 for the MST is exact.

Example 1.5

In the MST, an important neighborhood is defined by

$N(f) = \{g : g \in F$ and g can be obtained from f as follows: add an edge e to the tree f, producing a cycle; then delete any edge on the cycle$\}$ $\square$

Solution:

Assume there is a spanning tree L which is locally optimal but not globally optimal. Also take the globally optimal spanning tree G which *has the most edges in common with L*. We pick an edge e_1 from G which is not in L and divide the vertices into sets G_1 and G_2. We define the set Q to be the set of edges in L such that each edge in Q both has one vertice in G_1 and one vertice in G_2 and belongs to the cycle created when e_1 is added to L. We select e_2 as an arbitrary edge in Q.

From the assumption that L is locally optimal, we know that replacing e_1 with e_2 in L would lead to an increase in (or no change to) the cost of L, thus $\text{len}(e_1) \geq \text{len}(e_2)$. However, the assumption that G is globally opimal similarly implies that $\text{len}(e_1) \leq \text{len}(e_2)$. Thus we know that $\text{len}(e_1) = \text{len}(e_2)$.

Since e_1 and e_2 have the same length, then e_1 in G could be replaced with e_2 to give the spanning tree G' which is also a globally minimal spanning tree. This contradicts the assumption that G is a globally minimal spanning tree which shares the most edges with L, as we have found globally optimal G' which shares one additional edge with L. Thus we conclude that our assumption that L could be locally optimal but not globally optimal was incorrect.

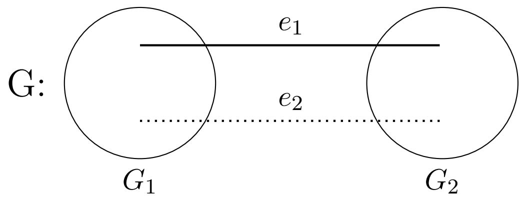

Figure 1: Globally optimal tree G with edge e_1 in the tree and e_2 not in the tree.

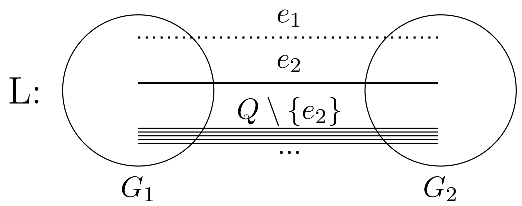

Figure 2: Locally optimal tree L with edge e_2 in the tree that would form a cycle is e_1 were added to the tree. Set of superfluous edges Q is also shown.

Problem 6

Suppose we are given a set S containing $2n$ integers, and we wish to partition it into two sets S_1 and S_2 so that $|S_1| = |S_2| = n$ and so that the sum of the numbers in S_1 is as close as possible to the sum of those in S_2. Let the neighborhood N be determined by all possible interchanges of two integers between S_1 and S_2. Is N exact?

Solution:

The given neighborhood is not exact. There exists two sets such that, in this neighborhood, they are locally optimal but they are not the global optimum. The sets are $S_1 = \{1, 2, 8, 9\}$ and $S_2 = \{3, 4, 5, 6\}$. Their sums are 20 and 18, respectively. By exchanging any two elements, the difference in their sums cannot be decreased. However, the sets $\{1, 3, 6, 9\}$ and $\{2, 4, 5, 8\}$ sum to 19 and 19 respectively: a difference of 0. Thus the neighborhood is not exact.

Problem 9

Let $f(x)$ be convex in R^n. Fix $x_2, ..., x_n$ and consider the function $g(x_1) = f(x_1, ..., x_n)$. Is g convex in R^1?

Solution:

Using the definition of convexity, we have

$$f(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \lambda f(\bar{x}) + (1-\lambda)f(\bar{y}), \lambda \in \mathbb{R} \text{ and } 0 \leq \lambda \leq 1 \tag{3}$$

We can rewrite $\bar{x}$ and $\bar{y}$ as vectors $\langle x_1, \dots, x_n\rangle$ and $\langle y_1, \dots, y_n\rangle$.

$$f(\lambda\langle x_1, \dots, x_n\rangle + (1-\lambda)\langle y_1, \dots, y_n\rangle) \leq \lambda f(\langle x_1, \dots, x_n\rangle) + (1-\lambda)f(\langle y_1, \dots, y_n\rangle), \lambda \in \mathbb{R} \tag{4}$$

Then do some linear algebra to arrange this so that the substitution for $g(x_1)$ becomes possible

$$f(\langle \lambda x_1 + (1-\lambda)y_1, \dots, \lambda x_n + (1-\lambda)y_n\rangle) \leq \lambda f(\langle x_1, \dots, x_n\rangle) + (1-\lambda)f(\langle y_1, \dots, y_n\rangle) \tag{5}$$

$$g(\lambda x_1 + (1-\lambda)y_1) \leq \lambda g(x_1) + (1-\lambda)g(x_1) \tag{6}$$

Which is precisely the statement that $g(x_1)$ is convex in $\mathbb{R}^1$.

Problem 10

Let $f(x_i)$ be a convex function of the single variable x_i. Then $g(x) = f(x_i)$ can also be considered a function of $x \in R^n$. Is $g(x)$ convex in R^n?

Solution:

Since g ignores all of the elements of $\bar{x}$, then it is constant along all dimensions except for the ith dimension. Thus $g(\bar{x})$ is convex. To prove this formally, first we state f's convexity formally.

$$f(\lambda x_i + (1-\lambda)y_i) \leq \lambda f(x_i) + (1-\lambda)f(y_i), \lambda \in \mathbb{R} \text{ and } 0 \leq \lambda \leq 1 \tag{7}$$

The right hand side can be substituted with g, creating vectors $\bar{x}$ and $\bar{y}$ which have x_i and y_i as their ith elements, respectively. The left hand side is more complicated. When the substitution is performed, a vector with $\lambda x_i + (1-\lambda)y_i$ as the ith element is made, with the remaining elements arbitrary. This vector is equal to $\lambda\bar{x} + (1-\lambda)\bar{y}$ at index i, and the remaining values in the vector are ignored. Thus we can finally state

$$g(\lambda\bar{x} + (1-\lambda)\bar{y}) \leq \lambda g(\bar{x}) + (1-\lambda)g(\bar{y}), \lambda \in \mathbb{R} \text{ and } 0 \leq \lambda \leq 1 \tag{8}$$

Chapter 2 Problems

Problem 8

Show that the set of optimal points of an instance of LP is a convex set.

Solution:

The set of feasible solutions $C : \mathbb{R}^n$ for an LP are defined as

$$C = \{\bar{x} \in \mathbb{R}^n | \boldsymbol{A}\bar{x} \geq \bar{b}\}. \tag{9}$$

With a linear cost function, $c : \mathbb{R}^n \to \mathbb{R}$, we can define the set of optimal points, S, as

$$S = \{\bar{x} \in \mathbb{R}^n | \boldsymbol{A}\bar{x} \geq \bar{b} \text{ and } c(\bar{x}) = v\} \tag{10}$$

where v is the value such that $\forall x \in C : c(x) \leq v$.

Using the definition of a convex set:

$$\forall \bar{y_1}, \bar{y_2} \in C : \lambda\bar{y_1} + (1-\lambda)\bar{y_2} \in C, 0 \leq \lambda \leq 1 \tag{11}$$

First we show that the set of feasible solutions is convex. Given $\bar{y}_1, \bar{y_2} \in C$, we show that their convex combination is also in C.

$$\lambda\boldsymbol{A}(\bar{y_1}) \geq \lambda b \tag{12}$$
$$(1-\lambda)\boldsymbol{A}(\bar{y_2}) \geq (1-\lambda)b \tag{13}$$
$$\boldsymbol{A}(\lambda\bar{y_1} + (1-\lambda)\bar{y_2}) = \lambda\boldsymbol{A}(\bar{y_1}) + (1-\lambda)\boldsymbol{A}(\bar{y_2}) \tag{14}$$
$$\lambda\boldsymbol{A}(\bar{y_1}) + (1-\lambda)\boldsymbol{A}(\bar{y_2}) \geq \lambda b + (1-\lambda)b = b \tag{15}$$

Next we need to show that the set of optimal points is convex. Given $\bar{y}_1, \bar{y_2} \in S$, we want to show that $\lambda\bar{y_1} + (1-\lambda)\bar{y_2} \in C$. Since we already know that C is a convex set, we only need to take care of the optimality constraint. Given $c(\bar{y_1}) = v, c(\bar{y_2}) = v$, we need to show that their convex combination also equals v. This is fairly easy thanks to c's linearity.

$$c(\lambda\bar{y_1} + (1-\lambda)\bar{y_2}) = \lambda c(\bar{y_1}) + (1-\lambda)c(\bar{y_2}) = \lambda v + (1-\lambda)v = v \tag{16}$$

Taken together, this means the set of optimal points (that is, points that are both feasible and have a cost equal to v) is convex.