Combinatorial Optimization and Modern Heuristics: Simplex Mini-project

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1 Problem Statement

1.1 Formal Statement

A linear programming problem is defined as a set of n inequalities using m variables and an objective function to be minimized or maximized that is a linear combination of the m variables. Formally, we state the problem as

$$\begin{aligned} \text{min/maximize} \quad & \sum_{j=1}^{m} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} A_{j,i} \geq b_j, \qquad i = 1, ..., m \\ & x_j \geq 0, \quad j = 1, ..., n \end{aligned}$$

where c_n is the nth coefficient of the objective functions, b_n is the nth right hand side of each inequality, and $A_{m,n}$ are the coefficients of mth term of the nth inequality.

1.2 The Algorithm

To use the simplex algorithm, we need to rearrange the previous problem statement's inequalities into equalities by introducing m slack variables.

$$\begin{aligned} \text{min/maximize} \quad & \sum_{j=1}^{m} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} A_{j,i} + s_i = b_i, \qquad i = 1, ..., m \\ & x_j \geq 0, \quad j = 1, ..., n \end{aligned}$$

where s_m is the mth slack variable.

High-level pseudocode for the simplex algorithm to solve this problem is here:

1.3 Implementation

I used the linprog solver from Scipy[1], which includes an option to use the simplex algorithm. Their simplex implementation can be found on Github [2]. I created a Python script to generate linear programming problems, solve them using this solver, and record how long it takes to solve problems of various sizes. The results are recorded in Table 1, showing the average runtime over 10 runs for each (m,n) pair.

[1] `https://www.scipy.org/`

[2] `https://github.com/scipy/scipy/blob/master/scipy/optimize/_linprog_simplex.py`

Algorithm 1 Simplex Algorithm

Require: $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$
  complete ← false
  while not complete **do**
    pivot-col-found, pivot-col ← find-pivot-col(A)
    if not pivot-col-found **then**
      complete ← true
      RETURN A[:,-1]
    end if
    pivot-row-found, pivot-col ← find-pivot-row(A, b)
    if not pivot-row-found **then**
      ERROR solution not found
    end if
    apply-pivot(A, b, pivot-row, pivot-col)
  end while

Generating solvable LP problems from a given m and n was difficult, and I settled on filling A with random number from -5 to 5, with additional code to make about half of the values default to 1. Elements of b and c are all set to 1. This LP problem generation strategy is very inefficient when m is greater than n, as most of the generated LP problems are not solvable. Addressing this issue directly was outside of the scope of the project, and instead a trial and error method was used. If the solver terminated with failure, a new problem was generated until the solver succeeds. This may invalidate the timing results because it heavily biases the kinds of problems that the solver solves. The number of attempts to generate 10 valid LP problems is shown in Table 2.

2 Experimental Results

The following linear programming problem was given:

$$\begin{array}{lccccccccc} \min & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & - & x_4 & = 1 & \\ & -x_1 & - & 5x_2 & + & 2x_3 & + & 3x_4 & = 1 & \\ & & & & & x_i & \geq & 0 \ \forall i & & \end{array} \tag{1}$$

Which was converted to the following arrays, and found the solution, x. Solving it took 1.925 ms.

$$A = \begin{bmatrix} 1 & 2 & -1 & -1 \\ -1 & -5 & 2 & 3 \end{bmatrix} \tag{2}$$

$$b^T = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{3}$$

$$c = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \tag{4}$$

$$x^T = \begin{bmatrix} 2.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{5}$$

The randomly generated problem statements, solutions, and individual run times are included in the addendum "Linear Programming Problem Instances and Solutions".

3 Analysis

Increasing n and m increases runtime, but not in a consistent way. I suspect this has to do with the inefficiency of the implementation and the randomness of the generated data. Increasing

n \ m	2	6	10	14
4	1.547	2.198	1.829	2.256
10	1.477	4.884	5.726	6.863
20	1.631	5.602	8.634	15.813
30	2.500	7.932	9.726	14.775
40	2.464	7.472	11.317	16.587
50	2.286	7.680	13.277	16.673

Table 1: Average time in milliseconds to solve an LP problems for each m,n pair.

n \ m	2	6	10	14
4	11	98	820	16208
10	10	11	334	8706
20	10	10	13	96
30	10	10	10	13
40	10	10	10	10
50	10	10	10	10

Table 2: Number of attempts to generate 10 solveable LP problems for each m,n pair.

the number of constraints to be greater than or equal to the number of variables makes it very difficult to find solvable LP instances.

4 Conclusion

Although it has been superseded by the Interior Points algorithm, the Simplex algorithm is a simple local search algorithm which works well for smallish matrices and emerges naturally from the geometric interpretation of linear programming problems.

5 Code Listing

solver.py

```
from scipy.optimize import linprog
from random import randint as dice
import time
from collections import defaultdict

def randy():
    '''Generates random numbers for filling A, customized to make them more likely to be solvable'''
    if dice(0,1) == 0:
        return 1
    return dice(-5, 5)

def generate_data(m, n):
    ''' creates problems of the same kind as the example problem, with b and c habing only 1's'''
    A = [[randy() for _ in range(n)] for _ in range(m)]
    b = [[1] for _ in range(m)]
    c = [1 for _ in range(n)]
    return A, b, c

# Result tables
m_n_to_time = defaultdict(list)
m_n_to_tries = defaultdict(int)
m_n_to_matrices = dict()

# parameter space
ns = [4, 10, 20, 30, 40, 50]
ms = [2, 6, 10, 14]

# for each pair of parameters, get 10 successful calculations
for n in ns:
    for m in ms:
        successes = 0
        while successes < 1:
            m_n_to_tries[(m,n)] += 1
            A, b, c = generate_data(m, n)
            tic = time.perf_counter()
            result = linprog(c=c, A_eq=A, b_eq=b, method='simplex')
            toc = time.perf_counter()
            if result['success']:
                m_n_to_time[(m,n)].append(1000*(toc-tic))
```

6 Linear Programming Problem Instances and Solutions

```
Problem instance for m=2, n=4
A =
[[ 1 -2  1  1]
 [ 1  1  1 -4]]
b^T =
[[1 1]]
c =
[1 1 1 1]
x^T =
[1. 0. 0. 0.]
runtime = 1.84 ms

Problem instance for m=6, n=4
A =
[[ 1 -3  5  5]
 [ 1  1 -2  1]
 [ 1  1 -1 -2]
 [ 1  1  1  1]
 [ 1  5  3 -2]
 [ 1  1  1  1]]
b^T =
[[1 1 1 1 1 1]]
c =
[1 1 1 1]
x^T =
[1. 0. 0. 0.]
runtime = 2.93 ms

Problem instance for m=10, n=4
A =
[[ 1  1  1  0]
 [ 2  1  1  1]
 [ 1  1 -5  1]
 [-4  1 -1  1]
 [ 1  1  1  4]
 [-1  1  1  1]
 [ 4  1  1 -5]
 [-4  1  2  1]
 [-5  1  1  0]
 [-2  1 -2  1]]
b^T =
[[1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1]
x^T =
[0. 1. 0. 0.]
runtime = 1.49 ms

Problem instance for m=14, n=4
A =
[[ 3 -5  1 -4]
 [ 1  2  1 -1]
 [ 1  1  1  1]
 [ 1  1  1  1]
 [-1 -4  1  1]
 [-1  1  1  1]
 [-3  1  1  1]
 [ 1  2  1  1]
 [-1 -3  1  2]
 [ 1  1  1  1]
 [ 1  2  1 -2]
 [ 1  2  1  1]
 [-2  1  1  1]
 [-3  1  1  1]]
b^T =
[[1 1 1 1 1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1]
x^T =
[0. 0. 1. 0.]
runtime = 3.01 ms

Problem instance for m=2, n=10
A =
[[-5  4 -4 -1  1 -5  1 -4  1  3]
 [ 1  1  4  0 -1  5 -3  1  1  1]]
b^T =
[[1 1]]
c =
[1 1 1 1 1 1 1 1 1 1]
x^T =
[0.   0.4  0.   0.   0.   0.12 0.   0.   0.   0.  ]
runtime = 1.79 ms

Problem instance for m=6, n=10
A =
[[ 1 -2  1  1 -2  1  1  1  1  1]
 [ 1  1  1  1  1  1  0  4 -5  1]
 [ 1 -2 -4  1 -5  0  1 -3  1  1]
 [ 1  1 -2  1  1  1  1  1  0 -2]
 [-3 -4 -2  1  1  1  1  1  1 -2]
 [-3  1  0  1  1  1  1  1  1  1]]
b^T =
[[1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1]
x^T =
```

```
[0. 0. 0. 1. 0. 0. 0. 0. 0. 0.]
runtime = 5.62 ms

Problem instance for m=10, n=10
A =
[[ 1  1  1 -5 -4  1 -1  1  2  1]
 [ 4 -4 -2 -5 -3  1  1  1  1  1]
 [-5  1  0 -3  1 -5  1  1  1 -4]
 [ 1  1  1  0  1 -3  1  1  1  1]
 [ 1 -1  1  1 -3  4  1  1  1  3]
 [ 4  0  2  1 -4  1  1  1  1 -5]
 [-1  1  1 -4  1  3 -4  1  1  3]
 [ 1  1 -3  1  1  1  0  1 -4 -2]
 [ 1  1  1  1  1 -3  1  1  0 -3]
 [ 5  1  1 -1  3 -1 -4  1  1  1]]
b^T =
[[1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1]
x^T =
[0. 0. 0. 0. 0. 0. 0. 1. 0. 0.]
runtime = 4.62 ms

Problem instance for m=14, n=10
A =
[[ 1  1  1 -4 -1 -5  1  2  1  1]
 [ 1  1  4  1  4  1  1  1  0  1]
 [ 4  1  1  1 -1 -2 -4  1  3  1]
 [ 1  1  1  5  1  5  1 -1 -1  1]
 [ 1  1 -4  1  1  1  1  4 -3  1]
 [ 4  1  1  0  1  1  1  1  1  1]
 [-1  1 -5  1  1  1  3  5  1 -1]
 [ 1  1  4  3  1  1 -3 -2  1  1]
 [-5  1  1  1 -4  1  4  5  1  1]
 [-2  1  1  1  4  5  1 -2  1  1]
 [ 1  1  1  1 -1  1  1  0  1  1]
 [ 1  1  1  3  1  1  1 -4  0  5]
 [ 1  1 -5 -5 -3 -2 -2  4 -1  1]
 [ 1  1  1  1  1  5 -2  3 -4  0]]
b^T =
[[1 1 1 1 1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1]
x^T =
[0. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
runtime = 9.2 ms

Problem instance for m=2, n=20
A =
[[-1  3  1  5  1  1 -4  5  1 -3 -4 -2 -1  1  2  4  1  1 -1  1]
 [ 2 -2  3  1  2  1 -2  1  5  1  1  1 -5  0 -1 -5  0  2 -1  4]]
b^T =
[[1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.         0.         0.16666667 0.         0.         0.         0.         0.16666667 0.
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.        ]
runtime = 1.81 ms

Problem instance for m=6, n=20
A =
[[ 1  1  5 -1  1  1  1  1 -4 -2 -1  3  1  1  3  0  1  5  1]
 [ 1  1 -5  2  1 -3 -5  4  0 -2  2 -1  5  1  1  1  1  2  4  0]
 [-3  1  1  1  1  1 -5  4 -2  3  1 -5  1  1 -5  1 -4  1 -1  4]
 [ 1 -3 -1 -4  1  1 -2  5 -2  5  1  2  1  1  1  1 -1  1  1 -2]
 [ 1 -4 -3  2  5  1  2  1  1 -2  1  1  4  1 -5  1  1 -3  1  1]
 [ 0  1  1  1  1 -3  1  4 -4  1  1  1  2  1  1  1 -5  1  1  5]]
b^T =
[[1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.         0.06102476 0.         0.08865861 0.03511802 0.         0.16350029 0.         0.
 0.         0.         0.13356362 0.         0.         0.         0.         0.         0.         0.00690846]
runtime = 3.92 ms

Problem instance for m=10, n=20
A =
[[ 5  1  1  1  1  0 -3  1 -3  1  1  1  1 -3  5 -2  1 -1  5  1]
 [ 1  1  1 -4  1  1  1 -4 -4  1  1  1 -1 -5  1  1 -3  1 -2  1]
 [ 0  4  1  0 -3  1  1  1  3  1  1  4  5 -5  1  4  1  1  1  1]
 [ 2  3 -5  1  1  5 -3  1  1  1  1  1  1  5  1  1  1  3  1  1]
 [-3 -3  1  1  1  1  1 -4  4  1  1  3 -5 -2 -3  1  1  4  1  4]
 [ 1  1  1  4  1  1 -1  1 -4  1  0 -2  3  2  2  1  1  3 -1  4]
 [ 2  1  1 -1  1  0  1  0  1  4  1  1 -2 -4 -4 -3  1  2  0  1]
 [-5 -4  1 -2  0  4  1  1  1  1 -1  1  1 -5  1 -5  3  1  1  1]
 [ 1  1  4  1  1  1  1  1  1  1  2  2  1 -5  0 -2  1  1 -4  0 -1]
 [ 1 -1  1  1 -5  5  3  1 -5 -2  1  1  1 -4  1  1 -4  1 -5  1]]
b^T =
[[1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.07116803 0.         0.19575473 0.         0.07301701 0.22028366 0.         0.         0.         0.14767754
 0.         0.09427438 0.         0.         0.05956253 0.0268043  0.         0.11111283 0.         0.00034499]
runtime = 9.71 ms

Problem instance for m=14, n=20
A =
```

```
[[-2  1  1  3  4 -5 -3  1  0  1  1  1 -2  1 -5  1 -1  1 -2  1]
 [ 3  1  1  1 -4  1 -5  1  1  1  0  1  1 -5  1  1 -3  1 -3  3]
 [-2  1  1  1  1  1  2  1  1  0  0  1 -2  0  5  1  1  1  1 -2]
 [ 0  1  1  1 -5  1  1  3  1  2  2  1  0  1  1  1 -2  1  1 -3]
 [ 1  1  1 -3  0  5  1  1  1  1 -5  1  4  3  4 -4 -3  1  1 -5]
 [ 1  1  0  1  1  1  1  1  1  1 -2 -5 -4  1  3 -4  3  2  1  1]
 [ 2  4  1  1 -1  1  3  1  2 -1 -2 -4  1  1  2  1  2  1  1 -3]
 [-5  5  1  1  1  1  1 -3  1  1  1  1  1  1  0  1  1  1 -3  1]
 [ 1  1  4  1  1  4  1  1  1  5 -5  5 -1  1 -5  1  4  3  1 -4]
 [ 1  1  1 -2 -5  5 -3  1  1  3 -3 -3 -1  1  3  2  0 -2 -1  4]
 [ 1  1  1  2  4  1  2  1  4  0  1  1 -5  1  1  1  3  0  1 -5]
 [-5  5  3  4  0  1 -1 -5  1  3  4  0 -5 -1  0  1  1  3  1  2]
 [-1  1 -5  5 -5 -5  1 -5  1 -5  4  1  1  1  5  1 -2  1  1  1]
 [ 1  1  0  1  1  4 -3  1  5  1  1  0 -5  1  2  1  2  5  1 -5]]
b^T =
[[1 1 1 1 1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.13397386 0.         0.17540032 0.36008069 0.11950414 0.         0.         0.1246343  0.27151498 0.19324494
 0.06361242 0.07368348 0.23216934 0.26253284 0.20222192 0.         0.         0.06013901 0.         0.23189006]
runtime = 11.2 ms

Problem instance for m=2, n=30
A =
[[ 0 -3 -5  1 -5  2  5  3 -4  1  1  1  1  1 -5  1  1  1  1  1 -1 -5  5  0  0  1  1  1  1  1]
 [ 1  5 -4  5  2  1  1 -2  1 -2  1  1  1  1  1  1  1  1  1  1 -4 -2 -5  1  1 -1  2  1  1  1]]
b^T =
[[1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.         0.         0.16666667 0.         0.         0.16666667 0.         0.         0.
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.        ]
runtime = 2.44 ms

Problem instance for m=6, n=30
A =
[[ 1 -5 -2  1  1  1  1  4  1 -2  3 -1  1 -2 -5  1 -4  1  0 -5  1  1  1  1  1  1 -1  2 -3  1]
 [ 2  1  1  1 -1  1  1 -4 -5  5  1  1  1  1 -5  3  4  5  1  1  1 -5  1 -5  0 -5 -5  1 -5  1]
 [ 1  1  5  5  2  4  1  1 -2  1  2  0  1  1  3  1  1  0  3  2  1 -3  1 -4  3 -3 -4  1  2 -3]
 [ 1  2  0 -1  0  1  1  5  3  1  1  0 -3  1  4  1  2 -1  3  1 -5  1 -5  1  1 -1  5  1  1  1]
 [ 4  1  1  1  1  5 -4 -5  1  1 -3  3  1 -5  1 -2  1  1  1  1  5  0  4  2  1 -2  3  4  1  1]
 [ 1 -2  1 -1  1  1  1  1 -5  5  1  1  3  1  1  0 -2  1  1  1  1  1 -4 -3  1  4  1  4 -5 -2]]
b^T =
[[1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.22857143 0.         0.         0.         0.         0.         0.         0.04489796 0.         0.
 0.10408163 0.         0.         0.         0.         0.         0.         0.0755102  0.13877551 0.
 0.         0.         0.         0.         0.         0.         0.         0.10204082 0.         0.        ]
runtime = 6.65 ms

Problem instance for m=10, n=30
A =
[[ 1  1 -4  4  1  0  3  1  1  0  1  1 -2  3 -3  4  1  1  0 -1  0  4  1  1  1  2 -3  0 -4 -1]
 [-3  0  1  3  1  1 -5 -1  0  1  4  4  5  1  1  5 -3  1  1  1  1  1  1  1  3 -5  0  1  3  1]
 [-3 -2  1  1  1  0  1  1  1 -1 -2  1  1  1  4  1  1  4  1  1  1  4  1  5  0  3 -5  1  1  1]
 [-3  1  1  2  1  4  1  1 -5  1 -3  5  0  1 -1  5  1  4  5  1 -1 -5 -1  0  1  2  1  3 -3  2]
 [ 1  1  1  0  1  1  1 -2  1  5  0  1 -2 -5  4  1  1  1 -2  1  5  4  4 -2  1  1  4  1 -1 -1]
 [ 1  1  1 -2  1  0  1  1  1  1  1  4  5 -4  5  1  1  1  1  1  4  1  1  1 -5  1  1  5 -5  3]
 [ 1  1 -4  1 -5 -1 -4  5  1  1  3  1  1  2  1 -3  1  5  1  5 -5  1  1  5  3  3 -1  2  5  2]
 [ 4 -5  2  1 -2  2  1 -1  1  3  4  1  1  1 -1 -1 -2 -2 -5  1  1  0 -3  1  5  1  1  1 -5  5]
 [-1  3 -1 -3  1  1 -3 -3 -2 -1  5  5  1  1  5  1  1 -3 -1 -2  0 -3  1  0  1  0  2  1  1  1]
 [ 0  1  1  1  4  1  4  1  3  1  0  4  1  1 -4  1 -3  1  1  1  1  1  2 -5 -1  4 -5  1 -3  1]]
b^T =
[[1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.         0.         0.         0.         0.         0.         0.         0.         0.07074801
 0.04773323 0.18564942 0.         0.         0.         0.04488969 0.         0.         0.         0.
 0.02675908 0.06407899 0.         0.06586953 0.05050301 0.10752455 0.         0.         0.         0.00067902]
runtime = 8.97 ms

Problem instance for m=14, n=30
A =
[[ 4  2  1  1  1  1  3  1  1  1 -5  1  1  1  1 -2  4  1  1  1  1 -4  1 -3 -1  1 -1 -3  1  3]
 [ 2  1  1  4  1  1  1  4  3  1  1  1  1  1 -2  1  1  0  0  2  1  1  4 -4 -1  3  1  3  1  1]
 [-1  5  2  1  1  1 -4  1  1  1  0  1  1  1 -2  4  0  1 -2  2  1  1  1  1  1 -3 -2 -3  1  1]
 [ 3  1  1  0  1 -5  1  1 -4  1  1  1  1  1 -4  1  1  1 -5  1  1  1 -1  1  1  0  1  1  1  1]
 [ 3  4 -2  1  2  1  4  4  3  1  1 -4  1  1  1  1 -4  3  1  1  1  1 -1 -3  2  1  1 -1  5  1]
 [ 1  1 -1 -5  1 -1  1  1  1  1  1 -4  0  1  1  1 -2  1  1  1 -4  1  1  2  1  1 -2 -2  3  4]
 [ 1  1 -1 -4  1 -3  0  1  1  1  3  1 -5 -1  1  1 -1  1 -2  1  5  4 -4  1  1  1 -1 -5  2  1]
 [-5  1 -2  1  1  5  1  1  2  1  1  1  0  1 -5  1  1  1  0 -4  1  5  1  0 -3 -1  5  3  1  1]
 [ 1  1  1  1  1  3  1  1  1  1 -2  3 -4  1  1  5  1  4  1  4  2  1  1  1 -1 -3  1  1  1  4]
 [-2 -1  1  1  2  1 -3  1  1  1  1  0  1  1  1  1  1  1 -1 -5  1  1  3  1  1  1  1  1  1 -2]
 [ 1 -4  1  2  0 -5  1 -2  1 -5  1  1  1  1 -4  1  5  1  1  1  1  1  0  4  1  1  1 -1  1  2]
 [-3  3 -1 -1  1 -1  1 -4  0  1  1  1  1  1  1  1 -5  1  1 -3  1  2 -3  1  2  5  1  1  4 -5]
 [ 1  0  1  1  1 -2  1 -1  1 -3 -2  1  1 -1 -4 -2  1  4  1  1 -3 -2  3  1  1  1  1  1  1 -4]
 [ 1  1  3  1 -5  1 -2  1  1  1  1  1  1 -2 -4  1  5  1  1  1  3 -3 -3  1  3  0  1  1  3  0]]
b^T =
[[1 1 1 1 1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.00725891 0.16797687 0.         0.         0.08147869 0.         0.         0.05073843 0.         0.05636609
 0.         0.         0.         0.         0.         0.         0.18966513 0.14781667 0.         0.00586606
```

```
 0.         0.08387283 0.03974659 0.14806094 0.         0.17952198 0.         0.         0.08194737 0.00418079]
runtime = 13.4 ms

Problem instance for m=2, n=40
A =
[[-4  1  1  1  1  1  1 -2  1  1  3 -4  1  1  2  1  1  1 -4  1 -5  1  2  0 -1  1  1  1 -1  1 -1  1 -4  1  1  3  2  1  0
   1]
 [ 4  3 -1 -5  3 -4  1  3  1  0  1  1  0  1  4  1  2  1  1  1  1  1  1 -3  3  3 -4  4  1 -2  1 -4 -1  4 -3  1  0 -1  1
  -2]]
b^T =
[[1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.2 0.  0.  0.  0.2 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.
 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. ]
runtime = 4.86 ms

Problem instance for m=6, n=40
A =
[[ 3  1  1  1  1 -3  0 -1 -4  2 -3  1 -2 -1  1  5 -3 -4 -4  3  1  1  1  1 -5  1 -1 -1  0  2  1  1 -2 -4  1 -5  1 -4 -4
  -1]
 [ 1  1  1  1  1  1  1  0  4  5 -1  1  4 -1  1  3  1  4  5  3  2  1  1  1  1  1  1  4  1 -4 -4  1  1  1 -4  1  3  1 -5
   2]
 [ 1  3  1  1  2  1  1  1  1  5  1 -5  2 -3 -1  1  0  4 -4 -4 -3 -4  1  3  1  1  0  4 -4  1 -4  1  1  1  1  1  1 -4  1
  -2]
 [ 3  2  1  1  5  0  0  1  1  1  1  1  1  1  4  1 -1  1  1  1  1  1 -1  1 -5  1  0  1 -3  1  4  1 -5  1 -5  1 -4  1  1
   1]
 [ 1 -1  1 -2  1  1  1  1  4  1  3 -4 -2 -3 -5 -1  4  1  1  1  1 -5  3  1 -1  0  1  1  1  1  1  0 -4 -1  1  1 -4 -5  1
   1]
 [ 1  0  5  1  1  1 -1  1  4  1 -1 -3  1 -4 -1 -5  5  4 -3  1  4  1  0  1 -5  1  1 -2  4  2  1  1 -5  2  1  4  1  1  1
   5]]
b^T =
[[1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.32659438 0.         0.         0.         0.00441095 0.         0.         0.         0.         0.12093365
 0.         0.         0.         0.         0.         0.         0.10181952 0.         0.         0.
 0.         0.         0.04043374 0.         0.         0.         0.         0.         0.         0.01948171
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.        ]
runtime = 7.03 ms

Problem instance for m=10, n=40
A =
[[ 3  1  5  0 -5  4  0 -3  1  1  1  1  0  3 -3  2 -2  1  1  1  1  4 -5  1  1 -2  0  2  1  1  1  1  1 -5  1 -4 -5 -4  1
   1]
 [ 1  1 -1  1  1  1  1  1  1  1  1 -2 -3 -1  1  1 -3  1  1  1  1  1 -4  5  5  0 -4  1  4  1  4  4  2  5 -3  5 -5  1  2
   1]
 [ 1  1  1  1  1  3 -2  1  1  1 -3 -3  0 -2 -4  1  1  1 -5  1  1  1  1 -3  0  1  1  0  3  1  1  1  2  1  1  1  1  4 -4
   1]
 [-3  1 -5  1  5  1  1  1  1  1  1  1  1  1 -5  1 -5  1  1  2  4  1 -4  1  1 -1 -1  1  4 -4  1  1 -2 -1 -3  1  1  4  4
  -4]
 [-5 -5  1 -1  1  1  1  1  1  5  2  5  1 -3  1  1  1  1 -3  3  1  1  1  2  1  1 -4  1  2  1  1  1 -4  0 -4  1  3  1  1
   1]
 [ 0  1  1  1  0  1 -4  1  1  1  1  1 -3  2  1  1  1  0 -4  1  1  1  3  1  1  1  1 -1  5  2 -2  1  1 -3  3 -1  2 -1  0
   1]
 [ 1  3  0  0 -3  1  2  2  1  3  1  3  5  1  1  1  1  1  1 -5 -5  4  5 -5 -5  1  1  0 -3  5  1  1 -4  1  1  1  1 -4 -4
  -3]
 [ 2 -4 -5  3  1  5  2  0  1  0  1 -5  1 -1 -5  1  4  4  1  1  1  1  3  1 -5 -2  4  2  2  4  1  1  3  1  1  1  1  4  1
  -4]
 [ 1  1  5  1  1  4  1  1  3  1  1  1  1  4 -3  1  1 -5  1  1  1  1  1  1  1 -5 -1  5 -1 -3  1  1  1  1  1  3 -4  1  1
  -1]
 [ 5 -5  1  1  1  4  1 -5  1  0  1 -1  1  0  1 -5  1  1  1 -4 -4  1 -3  3 -4 -4  1  1  1 -4 -5 -1 -3 -5  0  1  1  1  1
   2]]
b^T =
[[1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.0039747  0.01382197 0.00613866 0.         0.         0.15745986 0.         0.         0.         0.
 0.         0.05591306 0.         0.01071794 0.         0.         0.         0.         0.         0.
 0.         0.19894956 0.         0.         0.         0.         0.         0.         0.1118444  0.
 0.         0.         0.         0.         0.         0.10989859 0.04847879 0.         0.         0.        ]
runtime = 11.0 ms

Problem instance for m=14, n=40
A =
[[ 1  4  1  5 -4  1  1  1  1  1 -4  1  0  1  1  1  1  3 -1  1  1  4 -4  1 -4  3  1  0  1  1 -1 -3 -2  1 -2 -3  1  1  3
   1]
 [ 2  2  3  0  1 -1 -4  1  3 -3  1  4 -3  1  0  1  1  1  5  0  1  1  4  3  1  1  1 -1  1  4  1  1  1 -5  1  1  1 -5  1
   0]
 [-5 -1  0  1  1 -1  1  1  1 -2  1 -3  2  3  3 -2  1  1  1  1 -2  1  1  1  1  3  3  1 -2 -4  1 -1  1 -3  1  1  1  1  1
   5]
 [ 2  1 -3  1 -2  3  1 -1 -5  0 -4  1 -5  1  1  1  1  1 -4  4  2 -2 -2  1 -4  1  1 -5  1  1  1  1  4 -1  1  1  1  3 -5
   5]
 [ 1  1  1  3  2 -2  1  1  1  1 -4  1  1 -4 -5 -3  1  1  2  1  5  2  1 -2  1  1  4  1 -4 -5  4  0  1  1  1  1  3  1  1
  -3]
 [-2  1  1  2  1  0  1 -4  1 -5  1  1  1 -3  1  1  1 -3  1  1  1  1  1  3  1  1  4  1  3  1 -5  3  1  1  1  0  1 -1 -2
   3]
 [ 1  1  1  1 -1  5  1 -3  1  5  1  0  1 -5  1  3 -3  1  1  1 -3  1  2 -1  1  1 -2  4  1  1  4  1  2  1  0 -2  4  1  1
   0]
 [ 1 -3  4  1  3  1  1  1  1  1  1  1  5 -1  4 -2  1 -2 -3  1  2  1 -2  1  1  1  1  1  4  1  1 -4  1 -4  3  1  1  0  1
   3]
 [ 1 -3  3  1  0  1  1  1 -3  1 -4  1 -1  1  1  1  1  1  1  4 -1 -4  1 -3  2  1 -3  1  1  1 -4 -1 -5  1  1 -4 -1  2  4
   0]
 [-3 -4 -5  1  3  1  1  3  1  1 -4  1  2  1 -2 -4  0  1  0  0  1 -3  3  1  1 -2 -4  1  1 -2  4 -1  1  3  1  1  1 -4 -1
   4]
 [-4  5 -1 -4  1  1  5 -5  1 -1 -3  1  2  1 -4  3 -4  1  1 -1  1  1 -2  1 -2 -3 -3  4  4  1 -2  0  0 -4 -4 -1  1  1 -2
  -3]
```

```
 [-1  1  3  4  1  1  1  1  1 -4  1 -2  1  1  1  1  5  1 -3  1  1  1 -2  1  1  1  1  0  1  3  1  0  1  1  1 -3  1  3  1
   0]
 [-1 -2 -2  1  1 -1  3  1  1  0  1 -3 -2  1 -1  1  1 -3 -3  1  1  1  5 -4  1  1 -3  1  5  1  5  5  1 -5  1 -1  0  0  1
   1]
 [ 1 -5 -3  3  1  1  1  1  1  1 -4  1  1  2 -5 -1 -2  1  1  1 -2  2 -3 -3  1  2  1  5  5  2 -3  1 -2  1 -5  5  0  1  0
   1]]
b^T =
[[1 1 1 1 1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.         0.04252207 0.07533956 0.0436461  0.         0.12468513 0.         0.         0.
 0.         0.         0.         0.10217773 0.         0.         0.         0.05927084 0.09077776 0.16483207
 0.06145056 0.         0.05077311 0.02937761 0.         0.         0.         0.         0.10958987 0.
 0.         0.         0.         0.         0.         0.         0.24992513 0.         0.         0.02938408]
runtime = 18.2 ms

Problem instance for m=2, n=50
A =
[[-1  2  1 -5 -5  1  1  1  0  5  1  1  3  1  4  1 -1  1  1  4 -3  5  1  5 -1  4  0  1  1  1 -2  1 -1 -3  1  5  1  1 -2
  -3 -5  1  1  1  1  1 -5 -1  2 -1]
 [ 1  0  1  3 -3  5 -4 -4  3  1  1  3 -1  1  1  0  4 -3  1  1 -1 -2  1 -3 -4  2  1  1  1  1 -3  1  1  3  3  1 -4  1  1
  -4  1  4  2  1  1 -1  5  1  0 -4]]
b^T =
[[1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.         0.         0.         0.         0.16666667 0.         0.         0.         0.16666667
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.        ]
runtime = 3.39 ms

Problem instance for m=6, n=50
A =
[[ 1  1  1  1  1  0  3  1  3  1 -3  0 -2  1  1 -3  1  1  1  4  1  1 -4  1  1  1 -2  4  1  4 -5 -3 -2  1  1  1  1  1  1
   1  1 -4  1  1 -1  1  1 -5  1 -3]
 [ 1 -4  1  1 -1  1  2  5 -1 -2 -4  0  2 -2  3  1  0  1  4  1  1 -1  1  1  2  1  1 -5  1  2  1 -2  2  1  1  3 -2  1  1
  -3  1 -3  1  1 -2  1  1  4  1  1]
 [-5  5  3  1 -1  1  1  1  1 -3 -5 -2  1  1  1  5 -2  0 -3  1  1 -5  3  1  1  1 -2  1  5  1 -5  1 -3 -1  1  5 -4  1 -2
   2  1  1  1  1  1  1  1  4 -3  5]
 [ 1 -3  2  1  1  1  1  1  1  1  1  1  1  1  4  4  0  1  1  1  1  1  2  5  5  1 -2  1  1 -5  3  1  2 -4  1  1 -1 -2  1
   1  1  1  1  1 -1  1  1  1 -4 -5]
 [ 3  1 -5  1  2  1  4 -3  1  1  1  1  1  1  0  1  1  1  0 -5  1  1  1  1  1 -4  5 -3 -2  2 -4  1 -1 -3  1 -4  5  1  1
   1  1  1  1  1  1 -1  1  1  0  1]
 [-3  1  1  1  1 -5  1  1  1  1  1 -5 -1  5  2  3  1  0  1  1 -1  1  1  1  1 -5 -4 -2  1 -2 -5  0 -5 -3 -5  1  2  1  0
  -4  1  3 -2 -5  5  1  1 -4  1  1]]
b^T =
[[1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.         0.         0.         0.         0.         0.28978224 0.         0.         0.
 0.         0.         0.         0.06281407 0.10720268 0.03685092 0.         0.         0.         0.
 0.         0.         0.         0.         0.         0.         0.         0.         0.01298157 0.
 0.         0.         0.         0.         0.         0.05820771 0.         0.         0.         0.
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.        ]
runtime = 9.88 ms

Problem instance for m=10, n=50
A =
[[ 1  5 -4  1 -5  1  5  0 -5  3  0  1  1 -5  1 -4  1  3 -1  1  0  4  1  2 -1 -4  2  1  1  1  1 -2 -4  0  1  4  2 -4  0
   1  2  2  1  1  1 -1  2  1  1  1]
 [ 1  1  1  1  5  1  1 -5  5  1  1 -4  1  1  1  1  1 -2  1  5 -1  1  1  1  4  5 -3  1  1  1 -5  1  3  1  1  3 -1  2 -4
   1  1  1 -4  1 -2  2  1  1  1  1]
 [ 1  1  1  1 -2  5  0  3  1  1 -5  1  2 -1  1  1  1 -2  1  5  1  1  3  1  1  1 -4  1  1  1  0  4  4  1  1  1  4  1 -1
   4  1  1 -4  3  1 -5  1  1  1  1]
 [ 4  5  1  1  1  1 -4 -5  1 -5  1  2  1  5  1  1  1 -3  1 -3  1 -2  0  1  1  5 -1  1 -2 -1  5  4  1  1  1  2 -1 -1  3
   3  1 -1 -2  3  1 -1  1 -2 -5  1]
 [ 2  4  2  1 -4 -2  1  4  1  0 -3 -1  1 -5  1  3  1  1 -3  1  4  2  1  1  1  1  1  1  1  1 -1  1  1  1  1 -4  1 -5  1
   1  1  1 -4  4 -1  0 -5  4  1  0]
 [ 1  1  1  3  1  1  1 -2 -2  0  1  0  3  4 -3  1  1  1  1  2  1  5  1  1  1  1  1  1  1  1  1  1  2  1 -3  1  0 -4  4
   1  1  1  0  1  1  0  1 -5  0  1]
 [ 1  3  1  1  1  1  4  1  1  3  1  5  1  4  4  3 -3  1  1  0  1  1 -5 -5  1  1 -3  5  2  1 -5 -2  3 -4  0  1 -5  1  1
  -3 -4 -1  1  3  1  1 -1  1  4  1]
 [ 1  1  5 -5  1 -2 -2  3  1 -5 -5  1 -3  1  1  1 -3  2  1 -1 -4 -3  1  1  3  1  4  1  1  1  1  3  1  1  1  1  5 -3 -1
  -4  0  1 -1 -3  4  1 -5 -5  2  1]
 [ 2 -2  3  1  1 -2  1  1  2  5  2  4  0  1  1  1 -5  1  5 -4  2  1  0  1  4 -5 -1 -4 -2  0  1  1  1  1  1  4  1  1  3
  -4  1  1  1  1  1  1  1  1  4  0]
 [ 1  2  1  1  5  1 -3  4 -4  1 -3  1  1  1  1  1  1  1  1 -5 -2 -4 -4  1 -5  1  4  0 -2  1  1  1  1 -2  1  1  1  1  1
   1  1  1  1  2  1  1 -3 -4 -5  1]]
b^T =
[[1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.16289006 0.13643994 0.         0.         0.         0.         0.         0.         0.03889791
 0.         0.         0.         0.         0.         0.         0.         0.         0.         0.
 0.         0.0365179  0.         0.         0.         0.         0.05223223 0.         0.         0.
 0.         0.         0.09830594 0.         0.         0.11758789 0.01737909 0.         0.         0.
 0.         0.10054519 0.         0.05110331 0.         0.         0.         0.         0.         0.        ]
runtime = 12.4 ms

Problem instance for m=14, n=50
A =
[[ 4  1  1  3  1  5  1  1  1 -2  1  3  1  1  1  1 -3  1  1  2 -3  1  1 -1  1  1  1 -2 -5  1  1 -3  1  1  0  1  2  2  0
  -1 -3  3  1  0 -3  1  1  1  3  1]
 [ 2 -1 -2  4  0 -3  1  1  3  1  5  3 -2  1  1  1  1  1  1  0  1  2 -2  0  1  1  1  4  1  4  1 -4  1  2  1 -3 -5  1 -3
```

```
   5 -4  1  1  1  3  1 -2  1  1  1]
 [ 3  4  1  1  2  0  1  1  1  1  1  1  1  1  1 -1  2  1 -3  1  1  4 -1  3  1 -4  4  1  1  1  1  1  1  1  1 -3  1  5  1
   1  4 -3  1 -1 -5  0  1  1  0  1]
 [ 1  0  1  1  1  1  1  3  0  4  1  1  1  1  4  1  1  1  1 -4  1 -1 -4  1  1  2  2  1 -5  2  1  1 -5  1  1  1 -1  1 -3
   4 -3 -2  1 -4  1  1 -3  1 -1 -4]
 [ 1  1  1  1  1  1 -4  2 -5  1  1  1  1  5  1  1  1  1  2  1  4  1  1  5  1  1  1  1  1  0  1  1  1  1  4 -2  2  1  1
   1  1  5  1  1  1  1 -4 -5  1  3]
 [ 1  1 -3  1  1  1 -3  1  1  1 -4 -3  5  1 -3 -2  1 -5  1  1 -1 -3  1 -1  1  1  1  1  1  1 -2  1 -4  1  1  1  1  1  1
   1  1 -5 -5  1  1 -2 -1  4  2  1]
 [-5 -2  1  1  0 -1  1  1  1 -1  0  1  1  1  1  4  1 -3 -5 -2  1  2  1 -5 -4  5  0  4  1 -3  2 -1  1 -2 -2  0  3  5  1
   1  5 -5  1  1  1 -4 -4  1 -3  2]
 [-3 -2  1  1  4  1  1 -2  2 -2  1 -5 -1  1  2  1  1  5 -4  4  3  1  3  0  1 -5  1  1  4  4  1 -5  2 -5  0  1  1  2  1
   1  1 -5  1  1  1  1  1  5  0 -4]
 [-4  1  1  1  1  1  5  5  1 -2 -1  1  1 -3  2  1  1  1 -5  1  1  1  3  1  5  1  0  1  1 -3 -4  1  1  2  1  1 -3 -5 -4
   1 -2  1  1  1  1  2  1 -1  2  1]
 [ 1 -5  1  1  5  3  1  1  1 -5 -5  1  5 -4  1 -1  2  1  1  1  1 -4 -5  1 -5  1  4  1  2  3  1  1  1  1  1  1 -4 -3  1
   3 -1  1  5 -1  1  1  1  1  1  0]
 [ 1  1  1  3  4  1 -1 -1  5  1  1  1 -4  1  1  1 -4  1  1  1  1  2  1  1  2  1  1  1 -4  1  0  5  1  1  1  1  1  0 -5
   1 -1  1  1  0  0  1 -4  1 -1  1]
 [ 0 -4  1  1  3 -4 -3  1  1  1  0 -4  1  1  4  1  1  1 -2 -3 -5  1 -5  1  1  1  1  1 -1  2 -1  1  1  5  2 -1  4  1  1
   4 -1  1  3  1  1  1  2  1  1  1]
 [ 3  1  1  1  1  1  1  1  1  1  4 -3  1 -5  1  2  0 -3  1  1  1 -5 -3 -3 -4  1  1  1  1  1  3  1  3  1 -2 -1  1  1 -3
   1  1  5  1  1  1 -2  1 -3  1  3]
 [ 2 -2  1  1 -2  1  1  1  1  1  1  5  1  4  1  1 -4  1  4  1 -2  3 -4  1  1  3 -2  2  1  1 -4 -1  1  1  1  1 -2  1 -5
   1  1 -3 -5  5 -2  3  2  5 -4 -3]]
b^T =
[[1 1 1 1 1 1 1 1 1 1 1 1 1 1]]
c =
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
x^T =
[0.         0.         0.         0.07414208 0.15646937 0.10114168 0.         0.1657347  0.         0.04168796
 0.04712164 0.         0.03274348 0.         0.         0.         0.         0.         0.         0.01569608
 0.         0.         0.         0.         0.         0.         0.         0.10645827 0.         0.
 0.         0.         0.         0.         0.         0.         0.04591572 0.03148155 0.         0.0552408
 0.         0.         0.         0.09022998 0.         0.         0.         0.03514275 0.         0.        ]
runtime = 24.7 ms
```