One-End-Trick for single traces

June 16, 2024

1 Basic relations of the Standard One-end-trick

In physical basis the Dirac operators obeys the relation:

$$\gamma_5 S_u + S_d \gamma_5 = 2\mu\gamma_5. \tag{1}$$

Multiplying left by S_d^{-1} and right by $S_u^{-1}\gamma_5$ one gets:

$$S_d^{-1} + \gamma_5 S_u^{-1}\gamma_5 = 2\mu S_d^{-1}\gamma_5 S_u^{-1}\gamma_5. \tag{2}$$

2 Standard One-End-Trick at work

Taking the trace with a certain Γ, we can compute

$$\mathrm{Tr}\left[\Gamma S_d^{-1} + \gamma_5\Gamma\gamma_5 S_u^{-1}\right] = \mathrm{Tr}\left[\Gamma S_d^{-1} \pm \Gamma S_u^{-1}\right] = 2\mu\mathrm{Tr}\left[\Gamma\gamma_5 S_d^{-1}\gamma_5 S_u^{-1}\right], \tag{3}$$

where $\gamma_5\Gamma\gamma_5 = \pm\Gamma$ according to:

$$\begin{array}{cccccccccccccccc} S & V_x & V_y & V_z & V_t & P & A_x & A_y & A_z & A_t & T_1 & T_2 & T_3 & B_1 & B_2 & B_3 \\ + & - & - & - & - & + & - & - & - & - & + & + & + & + & + & + \end{array} \tag{4}$$

Remembering that:

$$\mathrm{Tr}\left[\gamma_5\Gamma\gamma_5 S_u^{-1}\right] = \mathrm{Tr}\left[\Gamma\gamma_5 S_u^{-1}\gamma_5\right] = \mathrm{Tr}\left[\Gamma\left(S_d^{-1}\right)^\dagger\right] = \pm\mathrm{Tr}\left[\Gamma\left(S_d^{-1}\right)\right]^* \tag{5}$$

one has

$$\mathrm{Tr}\left[\Gamma S_d^{-1} + \gamma_5\Gamma\gamma_5 S_u^{-1}\right] = \mathrm{Tr}\left[\Gamma S_d^{-1}\right] \pm \mathrm{Tr}\left[\Gamma S_d^{-1}\right]^* = 2\mathrm{Re}/i\mathrm{Im}\mathrm{Tr}\left[\Gamma S_d^{-1}\right] \tag{6}$$

where the real/imaginary part holds if Γ is hermitean/antihermitean, as per the following table:

$$\begin{array}{cccccccccccccccc} S & V_x & V_y & V_z & V_t & P & A_x & A_y & A_z & A_t & T_1 & T_2 & T_3 & B_1 & B_2 & B_3 \\ Re & Re & Re & Re & Re & Re & Im & Im & Im & Im & Im & Im & Im & Im & Im & Im \end{array} \tag{7}$$

Hence, the OET is actually a trick to compute the real part, or the imaginary, but not both. This relation holds no matter the regularization! It is true also in Wilson regularization, where it simply reads:

$$\mathrm{Re}/\mathrm{Im}\mathrm{Tr}\left[\Gamma S^{-1}\right] = \mu\mathrm{Tr}\left[\Gamma\gamma_5 S^{-1}\gamma_5 S^{-1}\right] \tag{8}$$

Since only the sum of the bubble has a physical meaning, one reads immediately that the method allows one to compute only the physical part of the following bubbles:

$$S,\ P,\ T,\ B \tag{9}$$

3 Stochastic estimators

How to apply it? We can use One End Trick and rewrite it as the scalar product of a suitable pair of fields, either putting a stochastic identity near or far of the Γ:

$$2\mu \mathrm{Tr}\left[\Gamma\gamma_5 S_d^{-1}\gamma_5 S_u^{-1}\right] = 2\mu\left(\phi,\phi\Gamma\right) = 2\mu\left(\phi,\Gamma\phi\right) \tag{10}$$

where

$$\phi_{\alpha\beta}\left(x\right) = S_{u;\alpha\beta}^{-1}\left(x,y\right)\eta\left(y\right)\,. \tag{11}$$

In the second case there are a number of advantages: the source can be undiluted, $\phi_\alpha\left(x\right) = S_{u;\alpha\beta}\left(x,y\right)\phi_\beta\left(y\right)$, and taking the sum over $\vec{x}$ at fixed time one has automatically the trace diluted over time. Most important, the stochastic source is closed over identity, and this has notoriously a number of advantages.

4 Equivalence of the first estimator to the "trivial" calculation

It must be noted that the first approach can be rewritten as

$$2\mu\left(\phi,\phi\Gamma\right) = 2\mu\phi_{\alpha\beta}^{\dagger}\left(x\right)\phi_{\beta\sigma}\left(x\right)\Gamma_{\sigma\alpha} = \eta^{\dagger}\left(y\right)\left(\gamma_5 S_d^{-1}\left(y,x\right)2\mu\gamma_5\right)_{\alpha\beta} S_{u;\beta\sigma}^{-1}\left(x,z\right)\Gamma_{\sigma\alpha}\eta\left(z\right), \tag{12}$$

such that using relation of eq.1, one obtains immediately:

$$2\mu\left(\phi,\phi\Gamma\right) = \eta^{\dagger}\left(x\right)\left(\gamma_5 S_u^{-1}\gamma_5 + S_d^{-1}\right)\left(x,z\right)_{\alpha\beta}\Gamma_{\beta\alpha}\eta\left(z\right) = 2\mathrm{Re}/i\mathrm{Im}\left(\eta,\phi\Gamma\right), \tag{13}$$

where real and imaginary are to be taken according to the relation of eq.4. So using the first approach we have not doing anything different from the "trivial" calculation of the trace using a volume source, which for the real and imaginary part reads

$$d_\Gamma \equiv \eta^{\dagger}\left(x\right)\Gamma\phi\left(x\right). \tag{14}$$

But the second estimator, cannot be brought back to the original form.

It is useful to note that both estimators are purely real or imaginary:

$$\left(\phi,\phi\Gamma\right)^{*} = \left(\phi\Gamma,\phi\right) = \left(\phi,\phi\Gamma^{\dagger}\right) = \pm\left(\phi,\phi\Gamma\right). \tag{15}$$

$$\left(\phi,\Gamma\phi\right)^{*} = \left(\Gamma\phi,\phi\right) = \left(\phi,\Gamma^{\dagger}\phi\right) = \pm\left(\phi,\Gamma\phi\right). \tag{16}$$

5 Why is this supposedly better?

Let us consider the time-diluted trace trivial estimator:

$$d_\Gamma\left(t\right) \equiv \sum_{\vec{x}}\eta^{\dagger}\left(\vec{x},t\right)\Gamma\phi\left(\vec{x},t\right), \tag{17}$$

and the second one-end-trick estimator (we drop the 2 for consistency of normalization with d which is the single quark estimator):

$$c_\Gamma\left(t\right) \equiv \mu\sum_{\vec{x}}\left[\phi^{\dagger}\left(\vec{x},t\right)\Gamma\phi\left(\vec{x},t\right)\right], \tag{18}$$

we should estimate the variance. We should notice that introducing the stochastic source is equivalent to reintroduce the quark fields (with bosonic statistics). Let us start from the disconnected version:

$$\sigma^2 d_\Gamma\left(t\right) = \left\langle\left|d_\Gamma\left(t\right)\right|^2\right\rangle - \left|\left\langle d_\Gamma\left(t\right)\right\rangle\right|^2 \tag{19}$$

$$\left\langle\left|d_\Gamma\left(t\right)\right|^2\right\rangle = \left\langle d_\Gamma\left(t\right)d_\Gamma^{*}\left(t\right)\right\rangle = \sum_{\vec{x},\vec{y}}\left\langle\mathrm{Tr}\left[\eta^{\dagger}\left(\vec{x},t\right)\Gamma\phi\left(\vec{x},t\right)\right]\mathrm{Tr}\left[\phi^{\dagger}\left(\vec{y},t\right)\Gamma^{\dagger}\eta\left(\vec{y},t\right)\right]\right\rangle = \tag{20}$$

$$= \sum_{\vec{x},\vec{y},z,w}\left\langle\eta^{\dagger}\left(\vec{x},t\right)\Gamma S_u^{-1}\left(\vec{x},t;z\right)\eta\left(z\right)\eta^{\dagger}\left(w\right)S_u^{-1\dagger}\left(\vec{y},tw;\right)\Gamma^{\dagger}\eta\left(\vec{y},t\right)\right\rangle. \tag{21}$$

If $\left\langle\prod_{i=1}^{N}\eta\left(x_i\right)\right\rangle=0$ for each value of N, as is the case of gaussian noise, the only contractions are:

$$\left\langle\left|d_\Gamma\left(t\right)\right|^2\right\rangle=\left|\mathrm{Tr}\left[\Gamma S_u^{-1}\left(\vec{x},t;\vec{x},t\right)\right]\right|^2+ \\ +\mathrm{Tr}\sum_y\left[S_u^{-1}\left(\vec{x},t;y\right)\gamma_5 S_d^{-1}\left(y;\vec{x},t\right)\gamma_5\right]$$

The first line is the same of the modulus square of $\left\langle d_\Gamma\left(t\right)\right\rangle$, whereas using γ_5 hermiticity one can rewrite the second as:

$$\sigma^2 d_\Gamma\left(t\right)=\mathrm{Tr}\left[\sum_y\left|S_u^{-1}\left(\vec{x},t;y\right)\right|^2\right]. \tag{22}$$

Coming to the second estimator, one has

$$\sigma^2 c_\Gamma\left(t\right)=\left\langle\left|c_\Gamma\left(t\right)\right|^2\right\rangle-\left|\left\langle c_\Gamma\left(t\right)\right\rangle\right|^2 \tag{23}$$

with

$$\left\langle\left|c_\Gamma\left(t\right)\right|^2\right\rangle=\left\langle c_\Gamma\left(t\right)c_\Gamma^*\left(t\right)\right\rangle= \tag{24}$$

$$=\mu^2\sum_{\vec{x},y,z}\left[\eta^\dagger\left(z\right)S_u^{-1}\left(\vec{x},t;z\right)^\dagger\Gamma S_u^{-1}\left(\vec{x},t;y\right)\eta\left(y\right)\right]\sum_{\vec{x}',y',z'}\left[\eta^\dagger\left(y'\right)S_u^{-1\dagger}\left(\vec{x}',t;y'\right)\Gamma^\dagger S_u^{-1}\left(\vec{x}',t;z'\right)\eta\left(z'\right)\right]. \tag{25}$$

Dropping from the beginning the disconnected contraction, one gets

$$\sigma^2 c_\Gamma\left(t\right)=\mu^2\mathrm{Tr}\sum_{y,z}\left|\sum_{\vec{x}}\gamma_5 S_d^{-1}\left(z;\vec{x},t\right)\gamma_5\Gamma S_u^{-1}\left(\vec{x},t;y\right)\right|^2. \tag{26}$$

So it is clear that they are genuinely different. Why the second is better, has to be better understood.

6 Now the other combination

Let us take the difference now,

$$\gamma_5 S_u-S_d\gamma_5=\gamma_5 S_u\left(\mu=0\right)-S_d\left(\mu=0\right)\gamma_5=2\gamma_5 S_u\left(\mu=0\right). \tag{27}$$

Multiplying left by S_d^{-1} and right by $S_u^{-1}\gamma_5$ one gets:

$$S_d^{-1}-\gamma_5 S_u^{-1}\gamma_5=2S_d^{-1}\gamma_5 S_u\left(\mu=0\right)S_u^{-1}\gamma_5. \tag{28}$$

Taking the trace with a certain Γ,

$$\mathrm{Tr}\left[\Gamma S_d^{-1}-\gamma_5\Gamma\gamma_5 S_u^{-1}\right]=2\mathrm{Tr}\left[\Gamma\gamma_5 S_d^{-1}\gamma_5 S_u\left(\mu=0\right)S_u^{-1}\right]. \tag{29}$$

We can play two different estimators:

$$2\left(\phi\Gamma,S_u\left(\mu=0\right)\phi\right)=2\Gamma\eta^\dagger\gamma_5 S_d^{-1}\gamma_5 S_u\left(\mu=0\right)S_u^{-1}\eta=\eta^\dagger\Gamma\left(S_d^{-1}-S_u^{-1}\right)\eta. \tag{30}$$

This estimator is, analytically equivalent to computing from the very beginning the naive estimator.

$$2\left(\phi,\Gamma S_u\left(\mu=0\right)\phi\right)=2\eta^\dagger\gamma_5 S_d^{-1}\gamma_5\Gamma S_u\left(\mu=0\right)S_u^{-1}\eta=2\eta^\dagger\Gamma S_d^{-1}\eta-2\eta^\dagger\gamma_5 S_d^{-1}\mu\gamma_5\Gamma S_u^{-1}\eta= \tag{31}$$

$$=2\eta^\dagger\Gamma S_d^{-1}\eta-2\mu\left(\phi,\Gamma\phi\right). \tag{32}$$

The second estimator is actually different, and in facts is not exactly equivalent. In particular, it is not exactly real or imaginary at fixed η. In practice the difference is real or imaginary when the estimator is imaginary or real, so nothing changes on the physical part: this estimator too is equivalent to the naive trace, for the real/imaginary part concerned. It is different from zero on the other part, but such difference averages to zero in the infinite η limit.