Oet for single traces

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1 Basic relations of the Dirac operator

In twisted basis the Dirac operators obeys the relation:

$$M_u - M_d = 2\mu i\gamma_5. \tag{1}$$

Multiplying from the left by the inverse of the Dirac operator of the up quark, and from the right by the down one, one gets

$$M_d^{-1} - M_u^{-1} = 2\mu i M_u^{-1}\gamma_5 M_d^{-1}, \tag{2}$$

Now rotating to physical basis:

$$e^{i\pi\gamma_5/4}S_d^{-1}e^{i\pi\gamma_5/4} - e^{-i\pi\gamma_5/4}S_u^{-1}e^{-i\pi\gamma_5/4} = 2\mu i e^{-i\pi\gamma_5/4}S_u^{-1}\gamma_5 S_d^{-1}e^{i\pi\gamma_5/4}, \tag{3}$$

which means

$$S_d^{-1}e^{i\pi\gamma_5/2} - e^{-i\pi\gamma_5/2}S_u^{-1} = 2\mu i e^{-i\pi\gamma_5/2}S_u^{-1}\gamma_5 S_d^{-1}e^{i\pi\gamma_5/2}, \tag{4}$$

remembering that $e^{\pm i\pi\gamma_5/2} = \pm i\gamma_5$,

$$S_d^{-1}\gamma_5 + \gamma_5 S_u^{-1} = 2\mu\gamma_5 S_u^{-1}\gamma_5 S_d^{-1}\gamma_5, \tag{5}$$

multiplying by γ_5 from the right one gets:

$$S_d^{-1} + \gamma_5 S_u^{-1}\gamma_5 = 2\mu\gamma_5 S_u^{-1}\gamma_5 S_d^{-1}. \tag{6}$$

2 Application to the single trace

Taking the trace with a certain Γ,

$$\mathrm{Tr}\left[\Gamma S_d^{-1} \pm \Gamma S_u^{-1}\right] = 2\mu \mathrm{Tr}\left[\Gamma\gamma_5 S_u^{-1}\gamma_5 S_d^{-1}\right], \tag{7}$$

where +(-) must be taken if Γ commutes (anticommutes) with γ_5, which means:

$$\begin{array}{cccccccccccccccc} S & V_x & V_y & V_z & V_t & P & A_x & A_y & A_z & A_t & T_1 & T_2 & T_3 & B_1 & B_2 & B_3 \\ + & - & - & - & - & + & - & - & - & - & + & + & + & + & + & + \end{array} \tag{8}$$

So if we are interested in computing the sum, which is the physically-meaningful quantity, we can apply the method only to

$$S,\ P,\ T,\ B \tag{9}$$

3 What does it mean?

Remembering that:

$$\pm\mathrm{Tr}\left[\Gamma S_u^{-1}\right] = \mathrm{Tr}\left[\gamma_5\Gamma\gamma_5 S_u^{-1}\right] = \mathrm{Tr}\left[\Gamma\gamma_5 S_u^{-1}\gamma_5\right] = \mathrm{Tr}\left[\Gamma\left(S_d^{-1}\right)^\dagger\right] = \pm'\mathrm{Tr}\left[\Gamma^\dagger\left(S_d^{-1}\right)^\dagger\right] = \pm'\mathrm{Tr}\left[\Gamma S_d^{-1}\right]^* \tag{10}$$

one has

$$\mathrm{Tr}\left[\Gamma S_d^{-1} \pm \Gamma S_u^{-1}\right] = \mathrm{Tr}\left[\Gamma S_d^{-1}\right] \pm' \mathrm{Tr}\left[\Gamma S_d^{-1}\right]^* = 2\mathrm{Re/Im}\mathrm{Tr}\left[\Gamma S_d^{-1}\right] \tag{11}$$

where the imaginary part holds if Γ is antihermitean, as per the following table:

$$\begin{array}{cccccccccccccccc} S & V_x & V_y & V_z & V_t & P & A_x & A_y & A_z & A_t & T_1 & T_2 & T_3 & B_1 & B_2 & B_3 \\ Re & Re & Re & Re & Re & Re & Im & Im & Im & Im & Im & Im & Im & Im & Im & Im \end{array} \tag{12}$$

Hence, the OET is actually a trick to compute the real part, or the imaginary, but not both. This relation holds no matter the regularization! It is true also in Wilson regularization, where it simply reads:

$$\mathrm{Re/Im}\mathrm{Tr}\left[\Gamma S^{-1}\right] = \mu \mathrm{Tr}\left[\Gamma\gamma_5 S^{-1}\gamma_5 S^{-1}\right] \tag{13}$$

Physically, one has

$$\mathrm{Tr}\left[\Gamma S_r^{-1}(x,x)\right]^* = \mathrm{Tr}\left[\Gamma^\dagger S_r^{-1}(x,x)^\dagger\right] = \pm\pm' \,\mathrm{Tr}\left[\Gamma S_{-r}^{-1}(x,x)\right] \tag{14}$$

Now since the value of r is immaterial in the physical world, the trace is real or imaginary depending on its commutation with γ_5 and hermiticity, according to the following relation

$$\begin{array}{cccccccccccccccc} S & V_x & V_y & V_z & V_t & P & A_x & A_y & A_z & A_t & T_1 & T_2 & T_3 & B_1 & B_2 & B_3 \\ Re & Im & Im & Im & Im & Re & Re & Re & Re & Re & Im & Im & Im & Im & Im & Im \end{array} \tag{15}$$

Combining this with the previous table, one reads immediately that the method allows one to compute only the physical part of the following bubbles:

$$S,\ P,\ T,\ B \tag{16}$$

4 Why is this supposedly better?

Ok we have understood that writing

$$C_\Gamma(t) \equiv \mu \mathrm{Tr}\sum_{\vec{x},y}\left[\Gamma\gamma_5 S_u^{-1}(\vec{x},t;y)\,\gamma_5 S_d^{-1}(y;\vec{x},t)\right], \tag{17}$$

is equivalent to

$$D_\Gamma(t) \equiv \mathrm{Tr}\sum_{\vec{x}}\left[\Gamma S_{d/u}^{-1}(\vec{x},t;\vec{x},t)\right], \tag{18}$$

for $\Gamma \in \{S,\ P,\ T,\ B\}$. Since this is an analytical property, it is hard to understand why this should be better. It is actually possible to evaluate $C_\Gamma(t)$ and $D_\Gamma(t)$ in two different ways. One can either create a volume source $\eta(x)$ or a wall source $\eta_t(x) = \eta(x)\,\delta_{x_0,t}$. For the time being we focus on the volume source.

One defines the stochastic propagator:

$$\phi(x) = S_u^{-1}(x;y)\,\eta(y), \tag{19}$$

then one get the estimate of $D_\Gamma(t)$ as:

$$d_\Gamma(t) \equiv \sum_{\vec{x}}\eta^\dagger(\vec{x},t)\,\Gamma\phi(\vec{x},t), \tag{20}$$

in facts it is easy to see that the expectation value is

$$\langle d_\Gamma(t)\rangle = \sum_{\vec{x},y}\left\langle\eta^\dagger(\vec{x},t)\,\Gamma S_u^{-1}(x;y)\,\eta(y)\right\rangle = \sum_{\vec{x}}\mathrm{Tr}\left[\Gamma S_u^{-1}(\vec{x},t;\vec{x},t)\right]. \tag{21}$$

When it comes to $C_\Gamma(t)$ one can use the following estimator:

$$c_\Gamma(t) \equiv \mu\sum_{\vec{x}}\left[\phi^\dagger(\vec{x},t)\,\Gamma\phi(\vec{x},t)\right], \tag{22}$$

which has the correct expectation value since

$$\begin{aligned}\langle c_\Gamma(t)\rangle &= \mu\sum_{\vec{x},y,z}\left[\eta^\dagger(z)\,S_u^{-1}(\vec{x},t;z)^\dagger\,\Gamma S_u^{-1}(\vec{x},t;y)\,\eta(y)\right] =,\\ &= \mu\mathrm{Tr}\sum_{\vec{x},y}\left[\gamma_5 S_d^{-1}(y;\vec{x},t)\,\gamma_5\Gamma S_u^{-1}(\vec{x},t;y)\right],\end{aligned}$$

which is what needed. Since in both cases we have used 1 stochastic propagator, it is not clear what is the advantage. To understand it, we should estimate the variance. We should notice that introducing the stochastic source is equivalent to reintroduce the quark fields (with bosonic statistics). Let us start from the disconnected version:

$$\sigma^2 d_\Gamma(t) = \left\langle |d_\Gamma(t)|^2 \right\rangle - |\langle d_\Gamma(t)\rangle|^2 \tag{23}$$

$$\left\langle |d_\Gamma(t)|^2 \right\rangle = \langle d_\Gamma(t)\, d_\Gamma^*(t)\rangle = \sum_{\vec{x},\vec{y}} \left\langle \mathrm{Tr}\left[\eta^\dagger(\vec{x},t)\,\Gamma\phi(\vec{x},t)\right] \mathrm{Tr}\left[\phi^\dagger(\vec{y},t)\,\Gamma^\dagger\eta(\vec{y},t)\right]\right\rangle = \tag{24}$$

$$= \sum_{\vec{x},\vec{y},z,w} \left\langle \eta^\dagger(\vec{x},t)\,\Gamma S_u^{-1}(\vec{x},t;z)\,\eta(z)\,\eta^\dagger(w)\,S_u^{-1\dagger}(\vec{y},tw;)\,\Gamma^\dagger\eta(\vec{y},t)\right\rangle. \tag{25}$$

If $\left\langle \prod_{i=1}^N \eta(x_i)\right\rangle = 0$ for each value of N, as is the case of gaussian noise, the only contractions are:

$$\left\langle |d_\Gamma(t)|^2 \right\rangle = \left|\mathrm{Tr}\left[\Gamma S_u^{-1}(\vec{x},t;\vec{x},t)\right]\right|^2 + \\ + \mathrm{Tr}\sum_y \left[S_u^{-1}(\vec{x},t;y)\,\gamma_5 S_d^{-1}(y;\vec{x},t)\,\gamma_5\right]$$

The first line is the same of the modulus square of $\langle d_\Gamma(t)\rangle$, whereas using γ_5 hermiticity one can rewrite the second as:

$$\sigma^2 d_\Gamma(t) = \mathrm{Tr}\left[\sum_y \left|S_u^{-1}(\vec{x},t;y)\right|^2\right]. \tag{26}$$

Coming to the second estimator, one has

$$\sigma^2 c_\Gamma(t) = \left\langle |c_\Gamma(t)|^2 \right\rangle - |\langle c_\Gamma(t)\rangle|^2 \tag{27}$$

with

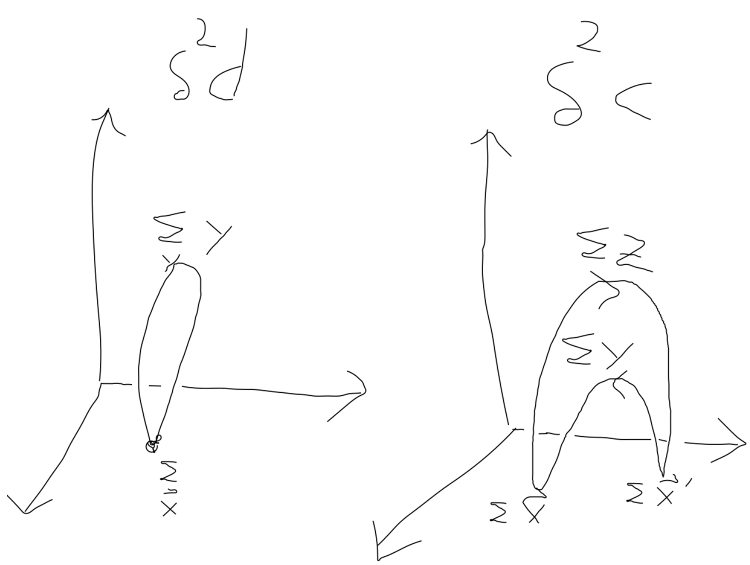
$$\left\langle |c_\Gamma(t)|^2 \right\rangle = \langle c_\Gamma(t)\, c_\Gamma^*(t)\rangle = \tag{28}$$

$$= \mu^2 \sum_{\vec{x},y,z} \left[\eta^\dagger(z)\,S_u^{-1}(\vec{x},t;z)^\dagger\,\Gamma S_u^{-1}(\vec{x},t;y)\,\eta(y)\right] \sum_{\vec{x}',y',z'} \left[\eta^\dagger(y')\,S_u^{-1\dagger}(\vec{x}',t;y')\,\Gamma^\dagger S_u^{-1}(\vec{x}',t;z')\,\eta(z')\right]. \tag{29}$$

Dropping from the beginning the disconnected contraction, one gets

$$\sigma^2 c_\Gamma(t) = \mu^2 \mathrm{Tr}\sum_{y,z}\left|\sum_{\vec{x}} \gamma_5 S_d^{-1}(z;\vec{x},t)\,\gamma_5\Gamma S_u^{-1}(\vec{x},t;y)\right|^2. \tag{30}$$

Is this better? If yes, why?



5 Now the other combination

In twisted basis the Dirac operators obeys the relation:

$$M_u + M_d = 2M_0. \tag{31}$$

Multiplying from the left by the inverse of the Dirac operator of the up quark, and from the right by the down one, one gets

$$M_d^{-1} + M_u^{-1} = 2M_u^{-1} M_0 M_d^{-1}, \tag{32}$$

Now rotating to physical basis:

$$\begin{aligned} e^{i\pi\gamma_5/4} S_d^{-1} e^{i\pi\gamma_5/4} + e^{-i\pi\gamma_5/4} S_u^{-1} e^{-i\pi\gamma_5/4} = \\ = 2e^{-i\pi\gamma_5/4} S_u^{-1} e^{-i\pi\gamma_5/4} M_0 e^{i\pi\gamma_5/4} S_d^{-1} e^{i\pi\gamma_5/4} = \\ = ie^{-i\pi\gamma_5/4} S_u^{-1} \left(\gamma_5 S_{d,0} + S_{u,0}\gamma_5\right) S_d^{-1} e^{i\pi\gamma_5/4} \end{aligned}$$

which means

$$e^{+i\pi\gamma_5/2} S_d^{-1} + S_u^{-1} e^{-i\pi\gamma_5/2} = 2ie^{-i\pi\gamma_5/2} S_u^{-1} S_{u,0}\gamma_5 S_d^{-1} e^{i\pi\gamma_5/2}, \tag{33}$$

$$\gamma_5 S_d^{-1} - S_u^{-1}\gamma_5 = 2\gamma_5 S_u^{-1} S_{u,0}\gamma_5 S_d^{-1}\gamma_5, \tag{34}$$

multiplying by γ_5 from the left one gets:

$$S_d^{-1} - \gamma_5 S_u^{-1}\gamma_5 = 2S_u^{-1} S_{u,0}\gamma_5 S_d^{-1}\gamma_5. \tag{35}$$

Taking the trace with a certain Γ,

$$\text{Tr}\left[\Gamma S_d^{-1} \mp \Gamma S_u^{-1}\right] = 2\text{Tr}\left[\Gamma\gamma_5 S_u^{-1} S_{u,0}\gamma_5 S_d^{-1}\right] . \tag{36}$$

One would be tempted to use the same approach, and insert the stochastic identity between S_d^{-1} and Γ, this way one gets

$$2\text{Tr}\left[\Gamma\gamma_5 S_u^{-1} S_{u,0}\gamma_5 S_d^{-1}\right] = \tag{37}$$

$$= 2\text{Tr}\left[\Gamma\gamma_5 S_u^{-1} S_u\gamma_5 S_d^{-1}\right] - 2\mu\text{Tr}\left[\Gamma\gamma_5 S_u^{-1}\gamma_5 S_d^{-1}\right] = \tag{38}$$

$$= 2\text{Tr}\left[\Gamma S_d^{-1}\right] - \text{Tr}\left[\Gamma S_d^{-1} \pm \Gamma S_u^{-1}\right] = \tag{39}$$

$$= \text{Tr}\left[\Gamma S_d^{-1} \pm \Gamma S_u^{-1}\right] \tag{40}$$