Scalable Semidefinite Programming

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Based on joint work with
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Quiz

1. What is the memory required to solve an underconstrained SDP with $n \times n$ decision variable $X \in \mathbf{S}_+^n$ and $O(n)$ linear constraints, assuming the rank of the solution $\operatorname{rank}(X^\star) \leq 5$?
   A. $O(\log n)$
   B. $O(n)$
   C. $O(n \log(n))$
   D. $O(n^2)$
   E. $O(n^6)$
2. There exist optimization algorithms that can be run without explicit access to the optimization variable. (e.g., with access only to a linear image of that variable.)
   A. true
   B. false

Outline

Motivation

Large scale SDP

Complementary slackness

SketchyCGM

SketchyCGAL

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Goal

Can we develop algorithms that provably solve a problem using *storage* controlled by the size of the *problem data* and the size of the *solution*?

Problem data: $\mathcal{O}(n)$

$\downarrow$

Working memory: $\mathcal{O}(???)$

$\downarrow$

Solution: $\mathcal{O}(n)$

Target problem: semidefinite program (SDP)

consider primal SDP with decision variable $X \in \mathbf{S}^n_+$:

$$\begin{array}{ll} \text{minimize} & \mathrm{tr}(CX) \\ \text{subject to} & \mathcal{A}X = b \\ & \mathrm{tr}\, X \le \alpha, \quad X \succeq 0 \end{array} \tag{$\mathcal{P}$}$$

problem data:

- cost matrix $C \in \mathbf{S}^n$
- linear map $\mathcal{A} : \mathbf{S}^n \to \mathbf{R}^m$
- righthand side $b \in \mathbf{R}^m$

Are desiderata achievable?

suppose $(\mathcal{P})$ has

- *compact specification*: problem data use $\mathcal{O}(n)$ storage
- *compact solution*: solution $X_\star$ has constant rank $r_\star$
  $\implies$ solution uses $\mathcal{O}(n)$ storage

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$(\mathcal{P})$, using any first order method:

$$\boxed{\text{Problem data: } \mathcal{O}(n)}$$
$$\downarrow$$
$$\boxed{\text{Working memory: } \mathcal{O}(n^2)}$$
$$\downarrow$$
$$\boxed{\text{Solution: } \mathcal{O}(n)}$$

Are desiderata achievable?

$(\mathcal{P})$, using methods from this talk:

Problem data: $\mathcal{O}(n)$

$\downarrow$

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Solution: $\mathcal{O}(n)$

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Max-Cut [Goemans and Williamson 1995]	Matrix Completion [Srebro and Shraibman 2005]
$\begin{array}{ll} \text{minimize} & \mathrm{tr}(-LX) \\ \text{subject to} & \mathbf{diag}(X) = \mathbf{1} \\ & X \succeq 0 \end{array}$	$\begin{array}{ll} \text{minimize} & \mathrm{tr}(W_1) + \mathrm{tr}(W_2) \\ \text{subject to} & X_{ij} = \bar{X}_{ij},\ (i,j) \in \Omega \\ & \begin{bmatrix} W_1 & X \\ X^* & W_2 \end{bmatrix} \succeq 0 \end{array}$

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  - 10^9 users, 10^9 products
  - $\implies$ SDP with 10^{18} variables

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- MaxCut:
  - 10^9 people in social network
  - $\implies$ SDP with 10^{18} variables
- Phase retrieval:
  - $10^3 \times 10^3$ discretization of sample
  - $\implies$ SDP with 10^{12} variables

Why compact?

why a $\mathcal{O}(n)$ specification?

- data is expensive
- collect constant data per column (=user or sample)
- if solution is compact, compact specification should suffice

why a $\mathcal{O}(n)$ solution?

- the world is simple and structured
- nice latent variable models are of log rank
- given d observations, there is a solution with rank $\mathcal{O}(\sqrt{d})$
  Barvinok 1995, Pataki 1998

Optimal Storage

What kind of storage bounds can we hope for?

- Assume black-box implementation of primitives

$$v \mapsto Cv, \qquad v \mapsto \mathcal{A}(vv^*), \text{and} \qquad (v, y) \mapsto (\mathcal{A}^* y)v,$$

where $v \in \mathbf{R}^n$, $y \in \mathbf{R}^m$.

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Definition. An algorithm to return a rank r (approximate) solution to $(\mathcal{P})$ has **optimal storage** if it uses working storage

$$\Theta(rn).$$

Related work: convex methods

- **Interior point methods**: storage $\Theta(n^2 + m^2)$
  - [Nesterov and Nemirovski 1989, Nesterov and Nemirovskii 1994, Alizadeh 1991; 1995]; ...

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  - [Rockafellar 1970, Auslender and Teboulle 2006, O'Donoghue et al. 2016, Yurtsever et al. 2018]; ...

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- **Storage-efficient first order methods**: storage $\mathcal{O}(nk)$ for k iters
  - multiplicative weights [Arora et al. 2005]; CGM [Hazan 2008, Jaggi 2013]; truncated CGM [Rao et al. 2013]; spectral bundle method [Helmberg and Rendl 2000]; spectral low rank optimization [Friedlander and Macedo 2016]; ...

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  - control storage or guarantee convergence
  - not both

Related work: non-convex methods

Burer-Monteiro approach: Solve

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(CFF^*) \\ \text{subject to} & \mathcal{A}(FF^*) = b \end{array} \tag{BM}$$

with variable $F \in \mathbf{R}^{n \times r}$

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- $(+)$: if
$$\frac{r(r+1)}{2} > m,$$
any second order stationary point is globally optimal [Burer and Monteiro 2003, Boumal et al. 2016]; can solve locally with Riemannian optimization when $\mathcal{A}(FF^*) = b$ is smooth manifold [Absil et al. 2009, Boumal et al. 2014]

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$\implies$ BM approach is not storage optimal

Storage optimal methods for SDP

two approaches (so far) to storage-optimal SDP:

- **Complementary slackness.** [Ding et al. 2019]
  1. Use any dual solver to solve dual problem
  2. Insight: optimality conditions identify range of primal
  3. Recover primal by solving smaller SDP
  4. Requires strict complementarity, exact low rank solution

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- **SketchySDP.** [Yurtsever et al. 2017; 2019b]; ...
  1. Use primal-dual solver to solve dual problem
  2. Insight: can sketch primal during iteration
  3. Recover from sketch
  4. Can find low rank approximation to high rank solution

Approximate eigenvectors

Definition

For a symmetric matrix $M \in \mathbf{S}^n_+$, we say a unit vector v is an ϵ-*approximate minimum eigenvector* if

$$v^* M v \le \lambda_{\min}(M) + \epsilon \|M\| .$$

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how to find approximate minimum eigenvector (efficiently)?

- (-) Krylov methods (e.g., ARPACK `eigs`)
  - unstable, hard to control precision
- (+) shifted power method
  - converges in $\mathcal{O}(\epsilon^{-1})$ iterations, needs $\mathcal{O}(n)$ storage
- (+) randomized Lanczos method
  - converges in $q = \mathcal{O}(\epsilon^{-1/2})$ iterations, needs $\mathcal{O}(nq)$ storage

Approximate eigenvectors: shifted power method

- use power iteration to find max eigenvalue
- min eigenvalue of M is max eigenvalue of $\|M\| I - M$

Algorithm ApproxMinEvec via randomized shifted power method

Input: $M \in \mathbf{S}_n$, and maxiters q

Output: Approximate minimum eigenpair $(\xi, v) \in \mathbb{R} \times \mathbb{R}^n$ of M

function $\textsc{ApproxMinEvec}(M; q)$

 $\sigma \leftarrow \|M\|$

 $v \leftarrow \texttt{randn}(n, 1)/\sqrt{n}$

 for $i \leftarrow 1, 2, 3, \ldots, q$ **do**

  $v \leftarrow \sigma v - Mv$

  $v \leftarrow v / \|v\|$

 return $(v^*(Mv), v)$

Power method: guarantees

Fact (Randomized shifted power method (Kuczyński and Woźniakowski 1992))

Let $M \in \mathbf{S}_n$. For $\epsilon \in (0,1]$ and $\delta \in (0,1]$, the shifted power method computes a unit vector $u \in \mathbb{R}^n$ that satisfies

$$u^* M u \le \lambda_{\min}(M) + \epsilon \|M\| \quad w/prob \ge 1 - \delta$$

after $q \ge \frac{1}{2} + \epsilon^{-1} \log(n/\delta^2)$ iterations.[1]

- arithmetic cost is $\mathcal{O}(q)$ matrix–vector multiplies with M and $\mathcal{O}(qn)$ extra operations
- working storage is about $2n$ numbers

[1] All logarithms are base-e.

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Dual problem

consider dual SDP with decision variable $y \in \mathbf{R}^m$:

$$\begin{array}{ll} \text{maximize} & b^* y \\ \text{subject to} & C - \mathcal{A}^* y \succeq 0 \end{array} \qquad (\mathcal{D})$$

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- any first order method for $(\mathcal{D})$ uses optimal storage
- (just compute minimal eigenvalue with iterative method)
- e.g., subgradient method, AdaGrad [Duchi et al. 2011], AdaNGD [Levy 2017], AccelGrad [Levy et al. 2018], ...

Assumptions

Assumption (Regular SDP)

- primal SDP has a unique solution $X_\star$
- dual SDP has a unique solution $y_\star$
- primal and dual SDP satisfy strong duality

$$0 = \mathrm{tr}(CX_\star) - b^* y_\star = \mathrm{tr}(X_\star(C - \mathcal{A}^* y_\star)) = X_\star(C - \mathcal{A}^* y_\star)$$

- (for storage optimality) strict complementary slackness

$$\mathrm{rank}(X_\star) + \mathrm{rank}(C - \mathcal{A}^* y_\star) = n$$

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Note: these conditions hold generically [Alizadeh et al. 1997, Ding and Udell 2020]

Primal recovery via exact complementarity

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- let $V_\star$ be a basis for nullspace of dual slack matrix $Z_\star$
- constrain $X = V_\star S V_\star^*$ in primal SDP for some $S \in \mathbf{S}_\star^{r_\star}$.
  $\Longrightarrow$ solution is preserved!

SDP via exact complementarity: algorithm

Algorithm SDP via exact complementarity

Given: problem data C, $\mathcal{A}$, b

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3. solve **reduced primal** $(\mathcal{P}_{V_\star})$ with variable $S \in \mathbf{S}_+^{r_\star}$

$$\begin{array}{ll} \text{minimize} & \mathrm{tr}(C V_\star S V_\star^*) \\ \text{subject to} & \mathcal{A}(V_\star S V_\star^*) = b \\ & S \succeq 0 \end{array} \qquad (\mathcal{P}_{V_\star})$$

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Reduced primal is easy: e.g., when $r_\star$=1, $(\mathcal{P}_{V_\star})$ solves a 1D problem over $S \in \mathbf{R}_+$

SDP via exact complementarity: storage complexity

SDP via exact complementarity uses optimal storage:

1. compute $y_\star$ with any subgradient method

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(in practice, use an IPM to solve $(\mathcal{P}_{V_\star})$; it's tiny!)

Primal recovery via exact complementarity: picture

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- $(X \approx X_\star)$? solve reduced SDP $(\mathcal{P}_V)$ with variable $S \in \mathbf{S}_+^r$

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(CVSV^*) \\ \text{subject to} & \mathcal{A}(VSV^*) = b \\ & S \succeq 0 \end{array} \qquad (\mathcal{P}_V)$$

to find primal solution $X = VSV^*$

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```
Status:  Infeasible
Optimal value (cvx_optval):  +Inf
```

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SDP via approximate complementarity

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Given: problem data C, $\mathcal{A}$, b; target rank $r \geq r_\star$

1. Compute approximate dual solution y.
2. Compute basis V for eigenspace of $C - \mathcal{A}^* y$ with r smallest eigenvalues.

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3. To find primal solution $X_\star = V_\star S V_\star^*$, $S \in \mathbf{S}_r^+$, minimize infeasibility

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\left\|\mathcal{A}(VSV^*) - b\right\|^2 \\ \text{subject to} & S \succeq 0 \end{array} \qquad \text{(MinFeasSDP)}$$

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or accept infeasibility δ and minimize loss

$$\begin{array}{ll} \text{minimize} & \mathrm{tr}(CVSV^*) \\ \text{subject to} & \|\mathcal{A}(VSV^*) - b\| \leq \delta \\ & S \succeq 0. \end{array} \qquad \text{(MinObjSDP)}$$

Exact vs approximate complementarity

Exact complementarity	**Approximate complementarity**
Compute dual solution $y_\star$	Compute approximate dual solution y
Compute basis $V_\star$ for $\textbf{null}(C - \mathcal{A}^* y_\star)$	Compute basis V for eigenspace of $C - \mathcal{A}^* y$ with r smallest eigenvalues
Solve $\mathcal{P}_{V_\star}$	Solve MinFeasSDP or MinObjSDP.

Theoretical guarantees: approximate complementarity

Theorem (Recovery guarantees)

Suppose $(\mathcal{P})$ and $(\mathcal{D})$ satisfy genericity assumptions.
If $r = r_\star$ (MinFeasSDP) or $r \geq r_\star$ (MinObjSDP),
primal recovery via approximate complementarity from an
ϵ-suboptimal dual solution y produces a $\sqrt{\epsilon}$-suboptimal primal solution X:

	MinFeasSDP	MinObjSDP
suboptimality $\mathrm{tr}(CX) - \mathrm{tr}(CX_\star)$	$\mathcal{O}(\kappa\sqrt{\epsilon})$	$\mathcal{O}(\sqrt{\epsilon})$
infeasibility $\|\mathcal{A}X - b\|_2$	$\mathcal{O}(\kappa\sqrt{\epsilon})$	$\mathcal{O}(\sqrt{\epsilon})$
distance to solution $\|X - X_\star\|_F$	$\mathcal{O}(\kappa\sqrt{\epsilon})$	$\mathcal{O}(\epsilon^{\frac{1}{4}})$

where the condition number $\kappa = \frac{\sigma_{\max}(\mathcal{A})}{\sigma_{\min}(\mathcal{A}|_{V_\star})}$.

Core Lemma

Lemma (Projected solution)

Suppose

- $(\mathcal{P})$ and $(\mathcal{D})$ admit solutions and satisfy strong duality;
- $y \in \mathbf{R}^m$ is feasible and ϵ-suboptimal for $(\mathcal{D})$;
- and the threshold $T := \lambda_{n-r}(C - \mathcal{A}^* y) > 0$.

Then for any solution $X_\star$ of the primal SDP $(\mathcal{P})$,

$$\left\| X_\star - V \underbrace{V^* X_\star V}_{\text{feasible } S \text{ for MinFeasSDP}} V^* \right\| \leq \frac{\epsilon}{T} + \sqrt{\frac{2r\epsilon}{T} \|X_\star\|_{\text{op}}}$$

where $\|\cdot\|$ is either the nuclear or Frobenius norm.

A practical algorithm for SDP

how to choose parameters?

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- infeasibility tolerance δ
  - solve MinFeasSDP first, then solve MinObjSDP

A practical algorithm for SDP

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- infeasibility tolerance δ
  - solve MinFeasSDP first, then solve MinObjSDP
- rank target r
  - bigger is better
  - use spectrum of slack matrix $C - \mathcal{A}^* y$:
    need $T = \lambda_{n-r}(C - \mathcal{A}^* y) > 0$

A practical algorithm for SDP

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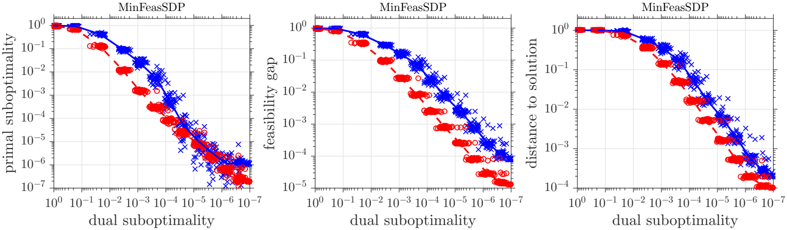
    need $T = \lambda_{n-r}(C - \mathcal{A}^* y) > 0$

Algorithm Primal recovery via approximate complementarity

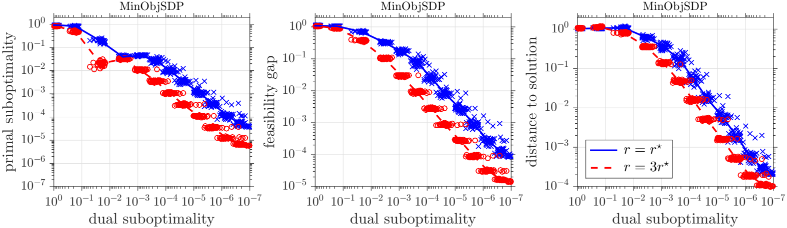
Given: approximate dual solution y, rank target r

- compute basis V for eigenspace of $C - \mathcal{A}^* y$ with r smallest eigenvalues
- solve (MinFeasSDP) to obtain a solution X_{infeas},
- then solve (MinObjSDP) with $\delta = 1.1 \, \|\mathcal{A} X_{\text{infeas}} - b\|_2$

Primal recovery works



Max-Cut



Matrix Completion

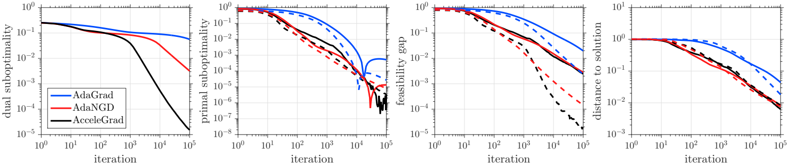
Don't try this at home

Algorithm for plot:

For $k = 1, 2, \ldots,$

- compute k-th dual iterate y_k using dual solver (here, AdaGrad [Duchi et al. 2011], AdaNGD [Levy 2017], and AccelGrad [Levy et al. 2018])
- recover a primal iterate X_k from y_k using RobustPrimalRecovery

Approximate complementarity solves primal SDP



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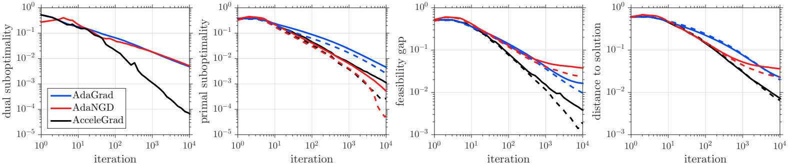


Matrix Completion

((dashed) $r = r_\star$ (solid) $r = 3r_\star$)

Outline

*

Model problem: low rank matrix optimization

consider a convex problem with decision variable $X \in \mathbf{S}^n_+$

compact matrix optimization problem:

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X) \\ \text{subject to} & X \in \alpha\Delta_n \end{array} \qquad \text{(CMOP)}$$

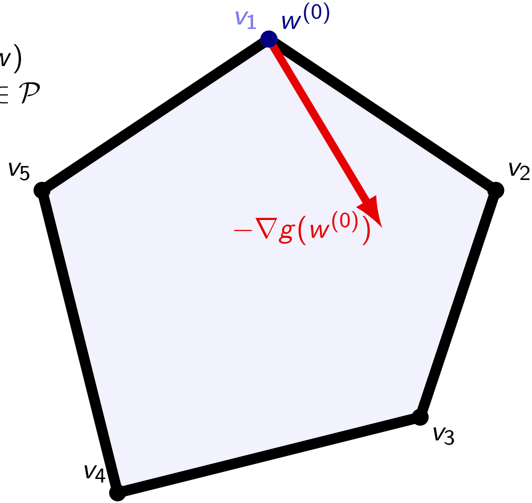
- $\mathcal{A} : \mathbf{S}^n \to \mathbb{R}^m$
- $f : \mathbb{R}^m \to \mathbb{R}$ convex and smooth
- $\Delta_n = \{X : \operatorname{tr} X \leq 1, X \succeq 0\}$

assume

- *compact specification*: problem data use $\mathcal{O}(n)$ storage
- *compact solution*: $\operatorname{rank} X_\star = r$ constant

Conditional gradient method (Frank-Wolfe)

$$\begin{array}{ll} \text{minimize} & g(w) \\ \text{subject to} & w \in \mathcal{P} \end{array}$$



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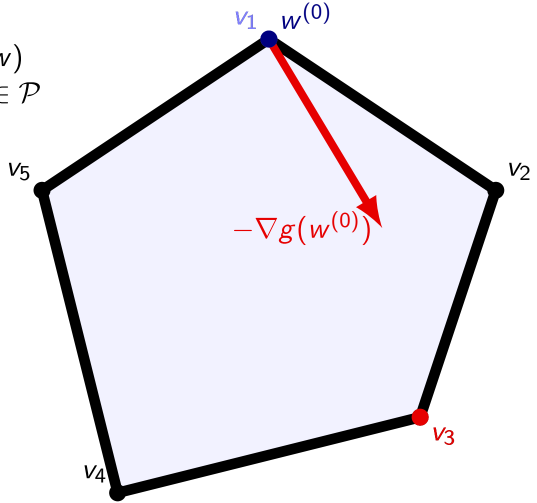

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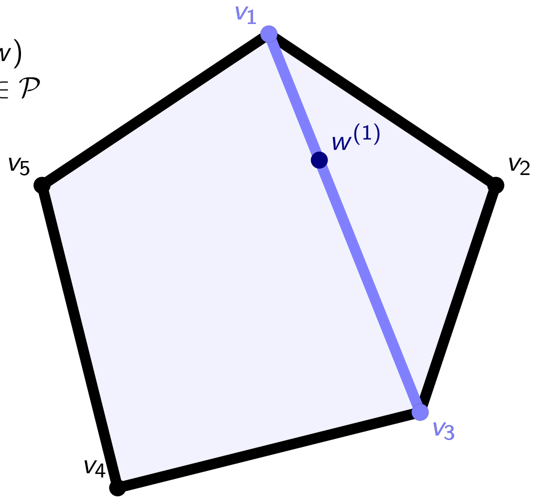

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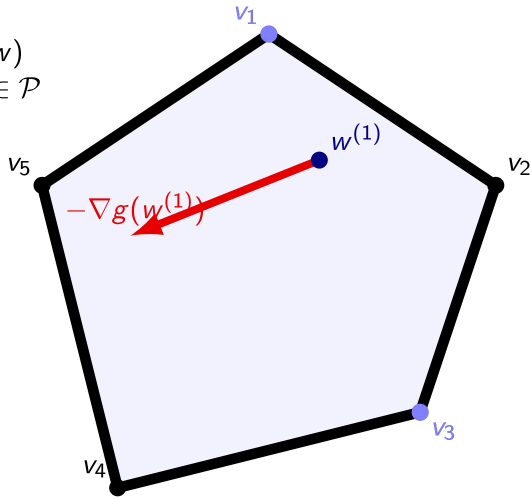

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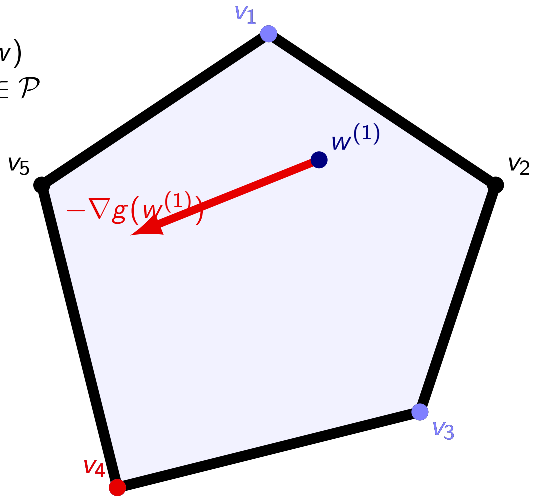

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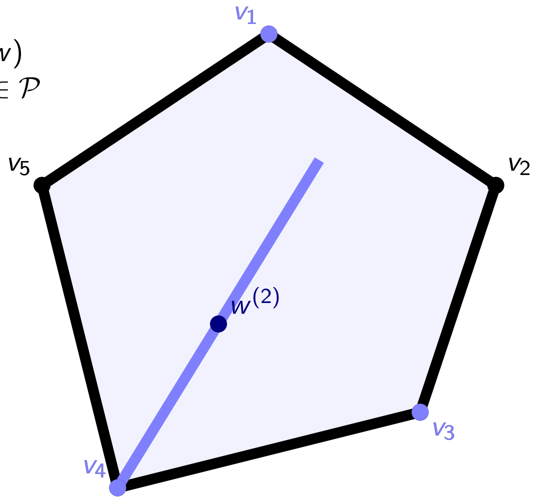

Linear optimization oracle for MOP

compute search direction

$$\underset{H \in \alpha\Delta_n}{\operatorname{argmin}} \langle H, G \rangle$$

- solution given by minimum eigenvector of G:

$$G = \sum_{i=1}^{n} \lambda_i v_i v_i^* \qquad \Longrightarrow \qquad H = \alpha v_1 v_1^*$$

- use shifted power method or randomized Lanczos method: apply $G = \mathcal{A}^*(\nabla f(\mathcal{A}X))$ and its conjugate to compute

$$\texttt{ApproxMinEvec}(G)$$

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$$\texttt{ApproxMinEvec}(G)$$

  (without forming G)

Conditional gradient descent

Algorithm CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ϵ
Output: Approximate solution X

```
function CGM
    X ← 0
    for t ← 0, 1, ... do
        (ξ, v) ← ApproxMinEvec(𝒜*(∇f(𝒜X)))
        H ← −α vv*
        if ⟨𝒜X − 𝒜H, ∇f(𝒜X)⟩ ≤ ε then break for
        η ← 2/(t + 2)
        X ← (1 − η)X + ηH
    return X
```

Two crucial ideas

To solve the problem using optimal storage:

- Use the low-dimensional "dual" variable

$$z_t = \mathcal{A}X_t \in \mathbb{R}^m$$

  to drive the iteration.
- Recover solution from small (randomized) sketch.

Never write down X until it has converged to low rank.

Conditional gradient descent

Algorithm CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ϵ
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function CGM
  $X \leftarrow 0$
  for $t \leftarrow 0, 1, \ldots$ **do**
    $(\xi, v) \leftarrow \texttt{ApproxMinEvec}(\mathcal{A}^*(\nabla f(\mathcal{A}X)))$
    $H \leftarrow -\alpha\, vv^*$
    if $\langle \mathcal{A}X - \mathcal{A}H, \nabla f(\mathcal{A}X) \rangle \leq \epsilon$ **then break for**
    $\eta \leftarrow 2/(t+2)$
    $X \leftarrow (1-\eta)X + \eta H$
  return X

Conditional gradient descent

Introduce "dual variable" $z = \mathcal{A}X \in \mathbb{R}^m$; eliminate X.

Algorithm Dual CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ϵ
Output: Approximate dual solution z^*

function $\textsc{dual}\mathrm{CGM}$
  $z \leftarrow 0$
  for $t \leftarrow 0, 1, \ldots$ **do**
    $(\xi, v) \leftarrow \texttt{ApproxMinEvec}(\mathcal{A}^*(\nabla f(z)))$
    $h \leftarrow \mathcal{A}(-\alpha v v^*)$
    if $\langle z - h, \nabla f(z) \rangle \leq \epsilon$ **then break for**
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---

we've solved the problem... but where's the solution?

Two crucial ideas

1. Use the low-dimensional “dual” variable

$$z_t = \mathcal{A}X_t \in \mathbb{R}^d$$

   to drive the iteration.
2. Recover solution from small (randomized) sketch.

How to catch a low rank matrix

if $\hat{X}$ has the same rank as $X^\star$,
and $\hat{X}$ acts like $X^\star$ (on its range and co-range),
then $\hat{X}$ is $X^\star$

use single-pass randomized sketch [Tropp et al. 2019; 2017a;b]

- see a series of additive updates
- remember how the matrix acts on random subspace
- reconstruct a low rank matrix that acts like $X^\star$
- storage cost for sketch and arithmetic cost of update are $\mathcal{O}(rn)$; reconstruction is $\mathcal{O}(r^2 n)$

Single-pass randomized sketch

- Draw and fix an standard normal matrix

$$\Omega \in \mathbb{R}^{n \times R}$$

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$$S = X\Omega \in \mathbb{R}^{n \times R} \quad \text{and} \quad \tau = \operatorname{tr} X \in \mathbb{R}$$

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- The sketch supports linear rank-one updates to X:
$$X' = (1-\eta)X + \eta vv^*$$
$$\Downarrow$$
$$S' = (1-\eta)S + \eta v(v^*\Omega)$$
$$\text{and} \quad \tau = (1-\eta)\tau + \eta \|v\|^2$$

Single-pass randomized sketch

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- Cost $\mathcal{O}(rn)$ to store and update sketch

Recovery from sketch

To recover approximation $\hat{X}$ from the sketch, compute

$$\hat{X} = S(\underbrace{\Omega^* S}_{R \times R})^\dagger S^* = X\Omega(\Omega^* X\Omega)^\dagger X\Omega^*$$

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Error estimate: use the trace τ to determine the error

$$\left\| X - \hat{X}_r \right\|_* = \tau - \operatorname{tr} \hat{X}_r$$

[Yurtsever et al. 2019b]

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[Yurtsever et al. 2019b]

Theorem (A priori error [Tropp et al. 2017a])

For each $r < R$, the approximation above yields a rank-r matrix $\hat{X}_r$ with

$$\mathbb{E}_\Omega \|X - \hat{X}_r\|_* \leq \left(1 + \frac{r}{R - r - 1}\right) \|X - [X]_r\|_* .$$

Similar bounds hold with high probability.

SketchyCGM

Algorithm SketchyCGM for the model problem (CMOP)

Input: Problem data; suboptimality ϵ; target rank r
Output: Rank-r approximate solution $\hat{X} = V \Lambda V^*$

function $\textsc{SketchyCGM}$
  $\textsc{Sketch.Init}(n, r)$
  $z \leftarrow 0$
  for $t \leftarrow 0, 1, \ldots$ **do**
    $(\xi, v) \leftarrow \texttt{ApproxMinEvec}(\mathcal{A}^*(\nabla f(z)))$
    $h \leftarrow \mathcal{A}(-\alpha v v^*)$
    if $\langle z - h, \nabla f(z) \rangle \leq \epsilon$ **then break for**
    $\eta \leftarrow 2/(t+2)$
    $z \leftarrow (1-\eta) z + \eta h$
    $\textsc{Sketch.CGMUpdate}(\alpha, v, \eta)$
  $(\Lambda, V) \leftarrow \textsc{Sketch.Reconstruct}(\,)$
  return (Λ, V)

Guarantees

Suppose

- $X_{\mathrm{cgm}}^{(t)}$ is tth CGM iterate
- $\lfloor X_{\mathrm{cgm}}^{(t)} \rfloor_r$ is best rank r approximation to CGM solution
- $\hat{X}^{(t)}$ is SketchyCGM reconstruction after t iterations

Theorem (Convergence to CGM solution)

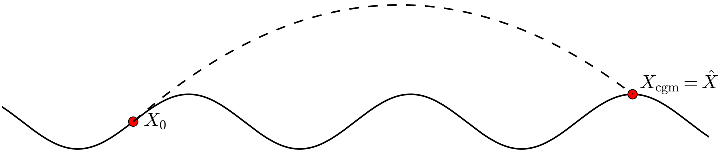
After t iterations, the SketchyCGM reconstruction satisfies

$$\mathbb{E} \, \| \hat{X}^{(t)} - X_{\mathrm{cgm}}^{(t)} \|_{\mathrm{F}} \leq 2 \, \| \lfloor X_{\mathrm{cgm}}^{(t)} \rfloor_r - X_{\mathrm{cgm}}^{(t)} \|_{\mathrm{F}} \, .$$

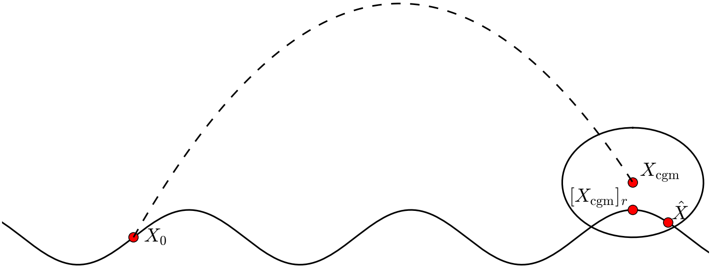
If in addition $X^\star = \lim_{t \to \infty} X_{\mathrm{cgm}}^{(t)}$ has rank r, then RHS $\to 0$!

[Yurtsever et al. 2017]

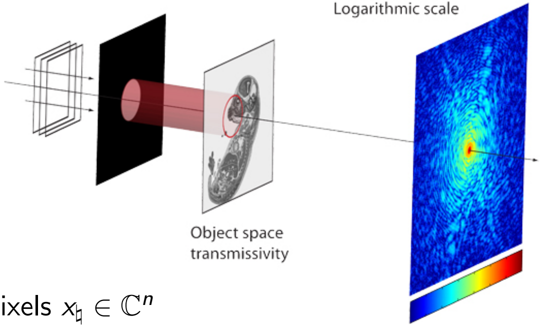
Convergence when rank($X_{\mathbf{cgm}}$) $\leq r$



Convergence when $\text{rank}(X_{\textbf{cgm}}) > r$



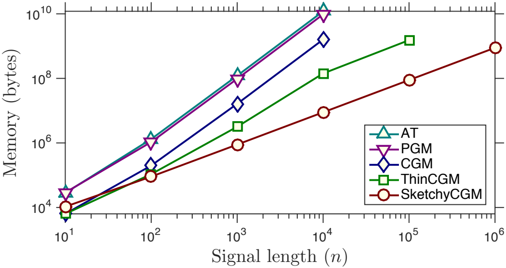
Application: Phase retrieval



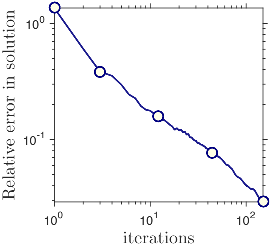
- image with n pixels $x_\natural \in \mathbb{C}^n$
- acquire noisy measurements $b_i = |\langle a_i, x_\natural \rangle|^2 + \omega_i$
- recover image by solving

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X; b) \\ \text{subject to} & \operatorname{tr} X \le \alpha \\ & X \succeq 0. \end{array}$$

SketchyCGM is scalable



(a) Memory usage for five algorithms



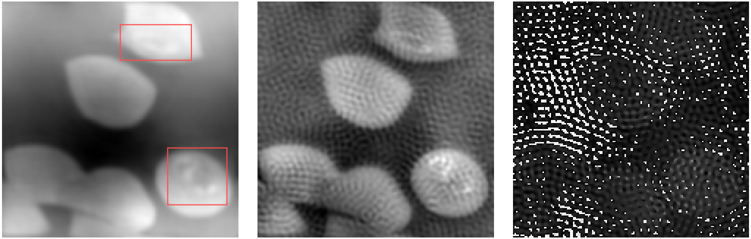
(b) Convergence for $n = 8 \cdot 10^6$.

PGM	=	proximal gradient (via TFOCS [Becker et al. 2011])
AT	=	accelerated PGM [Auslender and Teboulle 2006] (via TFOC
CGM	=	conditional gradient method [Jaggi 2013]
ThinCGM	=	CGM with thin SVD updates [Yurtsever et al. 2015]
SketchyCGM	=	ours, using $r = 1$

SketchyCGM is reliable

Fourier ptychography:

- imaging blood cells with $\mathcal{A}$ = subsampled FFT
- $n = 25,600$, $d = 185,600$
- $\text{rank}(X_\star) \approx 5$ (empirically)



(a) SketchyCGM (b) Burer–Monteiro (c) Wirtinger Flow

- brightness indicates phase of pixel (thickness of sample)
- red boxes mark malaria parasites in blood cells

Outline

Motivation

Large scale SDP

Complementary slackness

SketchyCGM

SketchyCGAL

*

Augmented Lagrangian

$$L_\beta(X; y) = \mathrm{tr}(CX) + \langle y, \mathcal{A}X - b\rangle + \frac{\beta}{2}\left\|\mathcal{A}X - b\right\|^2$$

- large β penalizes violations of constraint $\mathcal{A}X = b$
- dual variable y

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- dual variable y

saddle point of $L_\beta(X; y)$ gives solution to $(\mathcal{P})$:

- $\frac{\partial}{\partial y} L_\beta(X; y) = 0 \iff \mathcal{A}X = b$
- X is optimal for $(\mathcal{P})$ if in addition

$$X \in \operatorname*{argmin}_{X \in \alpha\Delta_n} L_\beta(X; y)$$

Conditional gradient augmented Lagrangian method

Augmented Lagrangian method: repeat

- $X \leftarrow \mathrm{argmin}_{X \in \alpha\Delta_n} L_\beta(X; y)$
- $y \leftarrow y + \eta(\mathcal{A}X - b)$

downside: minimization over X is too expensive

Conditional gradient augmented Lagrangian method

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- $X \leftarrow \operatorname{argmin}_{X \in \alpha\Delta_n} L_\beta(X; y)$
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downside: minimization over X is too expensive

Conditional gradient augmented Lagrangian method:

- update X with approx CGM step on (smooth) $L_\beta(X; y)$ over constraint set $\alpha\Delta_n$:

$$
\begin{aligned}
(\xi, v) &\leftarrow \mathrm{ApproxMinEvec}(C + \mathcal{A}^*(y + \beta(\mathcal{A}X - b)); q_t) \\
X &\leftarrow (1-\eta)X + \eta\alpha vv^*
\end{aligned}
$$

- $y \leftarrow y + \eta(\mathcal{A}X - b)$

Conditional gradient augmented Lagrangian method

Algorithm CGAL [Yurtsever et al. 2019a]

Input: Problem data $\mathcal{A}$, b, C; maxiters T

Output: Approximate solution X_T

function $\mathrm{CGAL}(T)$
  $X \leftarrow 0_{n \times n}$ and $y \leftarrow 0_m$
  for $t \leftarrow 1, 2, 3, \ldots, T$ **do**
    $\beta \leftarrow \sqrt{t+1}$ and $\eta \leftarrow 2/(t+1)$
    $(\xi, v) \leftarrow \text{ApproxMinEvec}(C + \mathcal{A}^*(y + \beta(\mathcal{A}X - b)); q_t)$
    $X \leftarrow (1-\eta)\, X + \eta\, (\alpha\, vv^*)$
    $y \leftarrow y + \gamma(\mathcal{A}X - b)$
  return X

Conditional gradient augmented Lagrangian method

Algorithm CGAL [Yurtsever et al. 2019a]

Input: Problem data $\mathcal{A}$, b, C; maxiters T

Output: Approximate solution X_T

function $\mathrm{CGAL}(T)$
- $X \leftarrow 0_{n\times n}$ and $y \leftarrow 0_m$
- **for** $t \leftarrow 1, 2, 3, \ldots, T$ **do**
  - $\beta \leftarrow \sqrt{t+1}$ and $\eta \leftarrow 2/(t+1)$
  - $(\xi, v) \leftarrow \text{ApproxMinEvec}(C + \mathcal{A}^*(y + \beta(\mathcal{A}X - b)); q_t)$
  - $X \leftarrow (1-\eta)\,X + \eta\,(\alpha\, vv^*)$
  - $y \leftarrow y + \gamma(\mathcal{A}X - b)$
- **return** X

- set $T \approx \epsilon^{-1}$ to achieve ϵ-optimal solution
- CGAL provably works with $\epsilon \sim 1/\sqrt{t}$-approx eigenvectors:
  - with shifted power method, set $q_t = 8t^{1/2}\log n$
  - with randomized Lanczos method, set $q_t = t^{1/4}\log n$

CGAL: convergence rate

Fact (CGAL: Convergence [Yurtsever et al. 2019b])

Assume the SDP satisfies strong duality. CGAL with approximate eigenvector computations yields a sequence $\{X_t : t = 1, 2, 3, \ldots\} \subset \alpha\Delta_n$ *that satisfies*

$$\|\mathcal{A}X_t - b\| \leq \frac{\mathrm{Const}}{\sqrt{t}} \quad \textit{and} \quad |\langle C, X_t\rangle - \langle C, X_\star\rangle| \leq \frac{\mathrm{Const}}{\sqrt{t}}.$$

- ϵ-optimal iterate X_T after $\mathcal{O}(\epsilon^{-2})$ iterations
- ...which requires $\mathcal{O}(\epsilon^{-3})$ calls to primitives (mat-vec)
- constant depends on problem data $(C, \mathcal{A}, b, \alpha)$ and norm of dual solution $\|y_\star\|$

Two crucial ideas

1. Use the low-dimensional "dual" variable

$$z_t = \mathcal{A}X_t \in \mathbb{R}^d$$

   to drive the iteration.

2. Recover solution from small (randomized) sketch.

How to catch a low rank matrix

if $\hat{X}$ has the same rank as $X^\star$,
and $\hat{X}$ acts like $X^\star$ (on its range and co-range),
then $\hat{X}$ is $X^\star$

use single-pass randomized sketch [Tropp et al. 2019; 2017a;b]

- see a series of additive updates
- remember how the matrix acts on random subspace
- reconstruct a low rank matrix that acts like $X^\star$
- storage cost for sketch and arithmetic cost of update are $\mathcal{O}(rn)$; reconstruction is $\mathcal{O}(r^2 n)$

Single-pass randomized sketch

- Draw and fix an standard normal matrix

$$\Omega \in \mathbb{R}^{n \times R}$$

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- The sketch supports linear rank-one updates to X:
$$X' = (1-\eta)X + \eta vv^*$$
$$\Downarrow$$
$$S' = (1-\eta)S + \eta v(v^*\Omega)$$
$$\text{and} \quad \tau = (1-\eta)\tau + \eta \|v\|^2$$

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- Cost $\mathcal{O}(rn)$ to store and update sketch

Recovery from sketch

To recover approximation $\hat{X}$ from the sketch, compute

$$\hat{X} = S(\underbrace{\Omega^* S}_{R \times R})^\dagger S^* = X\Omega(\Omega^* X\Omega)^\dagger X\Omega^*$$

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Error estimate: use the trace τ to determine the error

$$\left\| X - \hat{X}_r \right\|_* = \tau - \operatorname{tr} \hat{X}_r$$

[Yurtsever et al. 2019b]

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[Yurtsever et al. 2019b]

Theorem (A priori error [Tropp et al. 2017a])

For each $r < R$, the approximation above yields a rank-r matrix $\hat{X}_r$ with

$$\mathbb{E}_\Omega \|X - \hat{X}_r\|_* \leq \left(1 + \frac{r}{R - r - 1}\right) \|X - [X]_r\|_* .$$

Similar bounds hold with high probability.

Sketchy CGAL

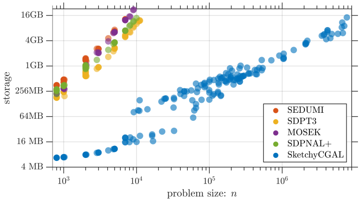
Algorithm SketchyCGAL Yurtsever et al. 2019a

Input: Problem data $\mathcal{A}$, b, C; maxiters T

Output: Approximate solution X_T

function $\mathrm{CGAL}(T)$
  $\textsc{Sketch.Init}(n, r)$
  $z \leftarrow 0_m$ and $y \leftarrow 0_m$
  for $t \leftarrow 1, 2, 3, \ldots, T$ **do**
    $\beta \leftarrow \sqrt{t+1}$ and $\eta \leftarrow 2/(t+1)$
    $(\xi, v) \leftarrow \mathrm{ApproxMinEvec}(C + \mathcal{A}^*(y + \beta(z - b)); q_t)$
    $\textsc{Sketch.CGMUpdate}(\alpha, v, \eta)$
    $z \leftarrow (1-\eta)z + \eta\alpha\mathcal{A}(vv^*)$
    $y \leftarrow y + \gamma(z - b)$
  $(\Lambda, V) \leftarrow \textsc{Sketch.Reconstruct}(\,)$
  return (Λ, V)

SketchyCGAL is scalable



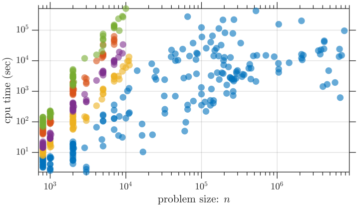


Figure: **MaxCut SDP: Scalability.** Storage cost [left] and runtime [right] of SketchyCGAL with sketch size $R = 10$ as compared with four standard SDP solvers. Each marker is one dataset.

SketchyCGAL for Quadratic Assignment Problem

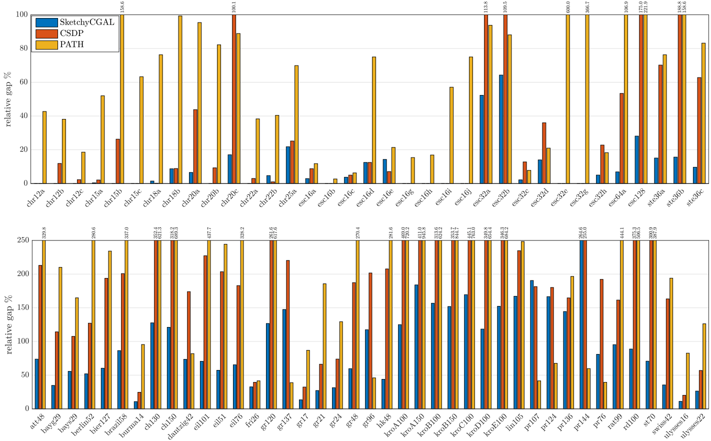


Figure: Using SketchyCGAL, CSDP [Bravo Ferreira et al. 2018], and PATH [Zaslavskiy et al. 2009], to solve SDP relaxations of QAPs from QAPLIB [top] and TSPLIB [bottom]. Bars compare the cost of the computed solution against the (known) optimal value; shorter is better.

Conclusion

Approximate complementarity approach provably solves (regular) SDP with optimal storage

- (+) uses $\mathcal{O}(nr)$ storage to find rank r solution
- (+) recovers approximate primal from approximate dual
- (+) parameters are easy to choose
- (+/-) relies on any subgradient solver for dual

Sketching methods provably solve large scale SDP efficiently

- (+) uses $\mathcal{O}(nr)$ storage to find rank r solution
- (+/-) requires compatible primal-dual solver (CGM, CGAL)
- (+) SketchyCGAL obtains ϵ-approximate solution after $\mathcal{O}(\epsilon^{-3})$ matrix-vector products

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Outline

Motivation

Large scale SDP

Complementary slackness

SketchyCGM

SketchyCGAL

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