Ecole Centrale de Nantes

Humanoid Robots Report Lab 1

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1 Introduction

The goal of this laboratory is to model a compass-walking robot walking down a slope. To run my lab and check that the calculations are correct, it is sufficient to run the "*Main.m*" file that comes along with the report. These files call the 3 functions developed in this lab and show the result. The geometric and mathematical modelling is provided in figure 1. The robot is considered in single support. The

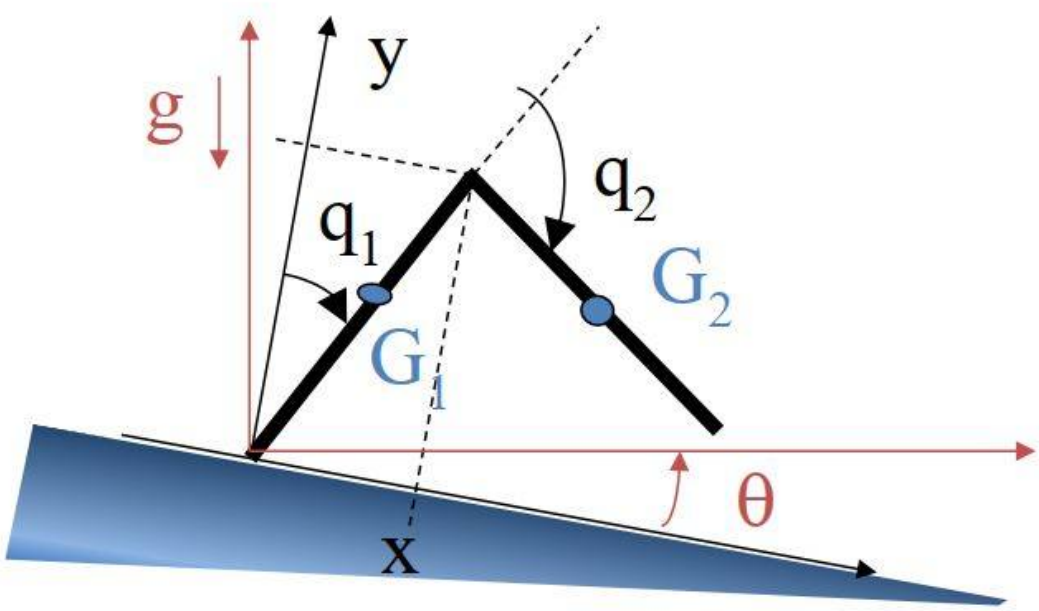


Figure 1: Compass Robot

known quantities are:

- Link Length $l = 0.8m$
- Link Mass $m = 2kg$
- Inertia $I = 0.1kgm^2$
- Center of Mass (G1, G2)
- distance from the hip $S = 0.5m$
- Gravity $g = 0.8m * s^{-1}$
- Slope Angle $\theta = rad$

In this report, we present the steps followed to obtain the required results and matrices and *MATLAB* files. We created a dynamic model of the robot in single support, a reaction force model an impact model. **NB**: in this report, we mainly present the results and the mathematical steps followed; to this report, it is also attached a *.PDF* file containing my computations made by hand. Any reference to the calculations in this report can be retrieved in the written notes.

2 Dynamic Model

First of all, we retrieve the dynamic model of the robot, found via the Lagrange Formalism approach (recalls from the **AMORO** exam are frequent in the following section) The model can be summarized in the following form:

$$A \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + H = B\tau \tag{1}$$

2.1 Center of Mass Position

Finding the centre of mass means determining the position of points **G1** and **G2**. This means finding the x and y position of the two vectors $\overrightarrow{OG_1}$ and $\overrightarrow{OG_2}$. By using simple geometry, it is possible to find the x and y position of these 2 vectors.

$$\overrightarrow{\mathbf{G1}} = (l - S) \begin{bmatrix} -sin(q_1) \\ cos(q_1) \end{bmatrix} \tag{2}$$

$$\overrightarrow{\mathbf{G2}} = l \begin{bmatrix} -sin(q_1) \\ cos(q_1) \end{bmatrix} + S \begin{bmatrix} -sin(q_1 + q_2) \\ cos(q_1 + q_2) \end{bmatrix} \tag{3}$$

2.2 Center of Mass Velocity

By deriving equations 2 and 3, it is possible to find their velocity:

$$\overrightarrow{\mathbf{VelocityG1}} = -(l - S) \begin{bmatrix} cos(q_1) \\ sin(q_1) \end{bmatrix} \dot{q}_1 \tag{4}$$

$$\overrightarrow{\mathbf{VelocityG2}} = -l \begin{bmatrix} cos(q_1) \\ sin(q_1) \end{bmatrix} \dot{q}_1 - S \begin{bmatrix} cos(q_1 + q_2) \\ sin(q_1 + q_2) \end{bmatrix} (\dot{q}_1 + \dot{q}_2) \tag{5}$$

2.3 Kinetic Energy

By using the well known Kinetic Energy Formula, it is possible to compute the total kinetic energy of the robot, $E_{tot} = E_1 + E_2$, with:

$$\mathbf{E1} = 0.5m(l - S)^2\dot{q}_1^2 + 0.5I\dot{q}_1^2 \tag{6}$$

$$\mathbf{E2} = 0.5ml^2\dot{q}_1^2 + 0.5mS^2(\dot{q}_1 + \dot{q}_2)^2 + mlS\dot{q}_1(\dot{q}_1 + \dot{q}_2)cos(q_2) + I(\dot{q}_1 + \dot{q}_2)^2 \tag{7}$$

2.4 Inertia Matrix

From the formula proposed in the lab assignment, and from the above-calculated equation 7, it is possible to deduce the matrix **A**, which ultimately contains the parameters of the kinetic energy.

$$\mathbf{A} = \begin{bmatrix} cm(l - S)^2 + mS^2 + ml^2 + 2I + 2mlScos(q2) & mS^2 + 2I + 2mlScos(q2) \\ mS^2 + I + 2mlScos(q2) & mS^2 + I \end{bmatrix} \tag{8}$$

2.5 Potential Energies

Finding the potential energies proved a bit trickier than expected as a change of frame is required. Since the gravity $\overrightarrow{g}$ is expressed in the direction Y_{plane}, we need to perform a rotation around Z of θ degrees, in order to get the gravity component in frame F_0. This way, we obtain the gravity vector in the frame we need:

$$\overrightarrow{\mathbf{g}} = \begin{bmatrix} -gsin(\theta) \\ gcos(\theta) \\ 0 \end{bmatrix} \tag{9}$$

This way, we can apply the potential energy formula and obtain for bot links (in the hand calculation we group the terms to the trigonometry formula of difference of angles):

$$\mathbf{G1} = -mg(l - S)cos(\theta - q_1) \tag{10}$$

$$\mathbf{G2} = -mg\left[lcos(\theta q_1) - Scos(\theta - q_1 - q_2\right] \tag{11}$$

Finally, we obtain the Potential Energy by summing up the terms: $U_{tot} = G_1 + G_2$

2.6 Gravity Effect

The gravity effect is found by summing 10 and 11, deriving them with respect to q_1 and q_2 and stacking them into a matrix $\mathbf{Q}$:

$$\mathbf{Q} = \begin{bmatrix} mg(l - S)sin(\theta - q_1) + mglsin(\theta - q_1)m + gSsin(\theta - q_1 - q_2) \\ mgSsin(\theta - q_1 - q_2) \end{bmatrix} \tag{12}$$

2.7 H Vector

Finally to find the vector $\overrightarrow{H}$, we use the formula:

$$\overrightarrow{\mathbf{H}} = B\dot{q}_1\dot{q}_2 + C \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} \tag{13}$$

Following the procedure detailed in the lab assignment, the computations for matrices $\mathbf{B}$ and $\mathbf{C}$ were done by hand, and it here are their final values:

$$\mathbf{B} = \begin{bmatrix} -2mlSsin(q2) \\ 0 \end{bmatrix} \tag{14}$$

$$\mathbf{C} = \begin{bmatrix} 0 & -mlSsin(q_2) \\ mglSsin(q_2) & 0 \end{bmatrix} \tag{15}$$

3 Reaction Force

To compute ground reaction force of the compass robot, we follow the model proposed in the laboratory assignment:

$$\overrightarrow{\mathbf{F}} = 2m\overrightarrow{\ddot{X}} - 2m\overrightarrow{g} \tag{16}$$

with $\overrightarrow{g}$ defined in 9. We can sum equations 2 and 3 and divide them by 2, to obtain the CoM of the robot, defined in:

$$\overrightarrow{\mathbf{Xg}} = \begin{bmatrix} -lsin(q_1) + 0.5Ssin(q_1) - 0.5Ssin(q_1 + q_2) \\ lcos(q_1) - 0.5Scos(q_1) + 0.5Scos(q_1 + q_2) \end{bmatrix} \tag{17}$$

Deriving once we obtain the velocity:

$$\overrightarrow{\mathbf{VelocityXg}} = \begin{bmatrix} -lcos(q_1)\dot{q_1} + 0.5Scos(q_1)\dot{q_1} - 0.5Scos(q_1 + q_2)(\dot{q_1} + \dot{q_2}) \\ -lsin(q_1)\dot{q_1} + 0.5Ssin(q_1)\dot{q_1} - 0.5Ssin(q_1 + q_2)(\dot{q_1} + \dot{q_2}) \end{bmatrix} \tag{18}$$

Lastly, by deriving the velocity, we can obtain the acceleration, which is what we need.

$$\overrightarrow{\mathbf{AccXg}} = \begin{bmatrix} lsin(q_1)\dot{q_1}^2 - lcos(q_1)\ddot{q_1} + 0.5Scos(q_1)\ddot{q_1} - 0.5Ssin(q_1)\dot{q_1}^2 + 0.5Ssin(q_1+q_2)(\dot{q_1}+\dot{q_2})^2 - 0.5Scos(q_1+q_2)(\ddot{q_1}+\ddot{q_2}) \\ -lcos(q_1)\dot{q_1}^2 - lsin(q_1)\ddot{q_1} + 0.5Ssin(q_1)\ddot{q_1} + 0.5Scos(q_1)\dot{q_1}^2 - 0.5Scos(q_1+q_2)(\dot{q_1}+\dot{q_2})^2 - 0.5Ssin(q_1+q_2)(\ddot{q_1}+\ddot{q_2}) \end{bmatrix} \tag{19}$$

Finally, by using 9 and 19 into 16, we can lastly obtain our reaction force vector.

4 Impact Model

Lastly, the impact model has been calculated. In this section we show the difference w.r.t. the first model, as now the second link is touching the ground while the first one takes off. The model is described by 20

$$\begin{bmatrix} A_1 & J_R^T \\ J_R^T & A_1 \end{bmatrix} \begin{bmatrix} \dot{x}^+ \\ \dot{y}^+ \\ \dot{q_1}^+ \\ \dot{q_2}^+ \\ I_{R_{2x}} \\ I_{R_{2y}} \end{bmatrix} = \begin{bmatrix} A_1 \\ 0_{2x4} \end{bmatrix} \begin{bmatrix} \dot{x}^- \\ \dot{y}^- \\ \dot{q_1}^- \\ \dot{q_2}^- \end{bmatrix} \tag{20}$$

We now also consider the parameters $\dot{x}\dot{y}$ for the computations of $\overrightarrow{OG_1}$, $\overrightarrow{OG_2}$, $\overrightarrow{VelocityOG_1}$ and $\overrightarrow{VelocityOG_2}$, leading to a different kinetic energy and matrix $\mathbf{A}$. The computations can be seen in the handwritten notes, leading to the final result:

$$\mathbf{A} = \begin{bmatrix} 2m & 0 & mScos(q1) - mScos(q_1+q_2) & -mScos(q_1+q_2) \\ 0 & 2m & mScos(q_1) - mSsin(q_1+q_2) & -mSsin(q_1+q_2) \\ mScos(q_1) - mScos(q_1+q_2) & mScos(q_1) - mSsin(q_1+q_2) & 2mS^2 + 2I & mS^2 + I; \\ -mScos(q_1+q_2) & -mSsin(q_1+q_2) & mS^2 + I & mS^2 + I \end{bmatrix} \tag{21}$$

Then, from the model $\dot{x} = J_R\dot{q}$ it is possible to extract the J_R matrix:

$$J_R = \begin{bmatrix} 1 & 0 & -lcos(q_1 + q_2) & -lcos(q_1 + q_2) \\ 0 & 1 & -lsin(q_1 + q_2) & -lsin(q_1 + q_2) \end{bmatrix} \tag{22}$$

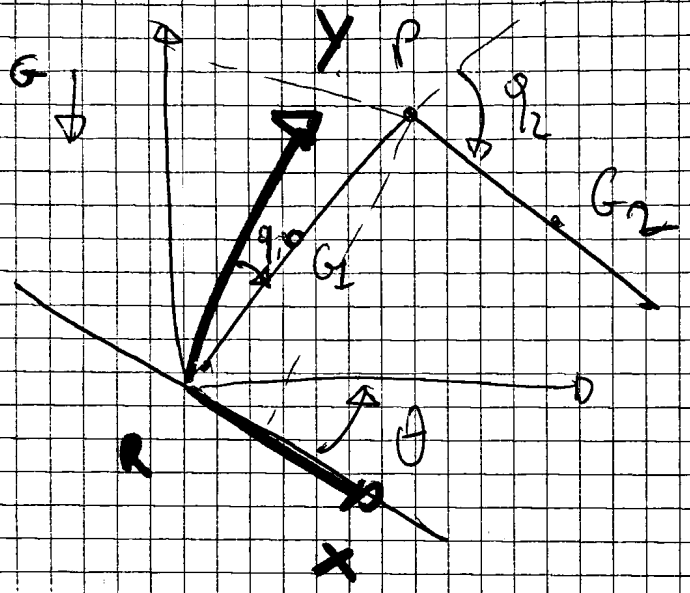
This way, according to the lab instruction, we have found the relation between the moments before and after impact.

5 Appendix: Dynamic Model

$$\boxed{\xi = \xi_a - B^T A^{-T} \xi_x}$$

HUMRO CAB

Show l m s I g

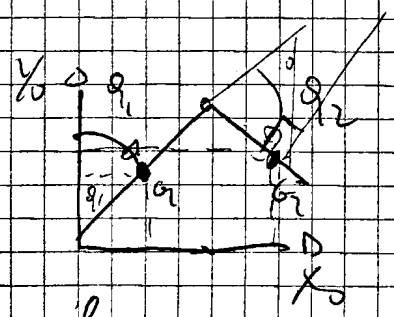


$[A, H] = \text{function_dyn}(q_1, q_2, \dot{q}_1, \dot{q}_2, \ldots)$

$$A \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + H = B\xi$$

PART 2 (1) → that q_1 is negative! – clockwise

* convention

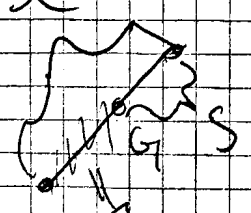


$$G_1 = \begin{bmatrix} X \\ Y \end{bmatrix} = l \cdot \begin{bmatrix} \sin q_1 \\ \cos q_1 \end{bmatrix} = (l-s) \begin{bmatrix} \sin(-q_1) \\ \cos(-q_1) \end{bmatrix}$$

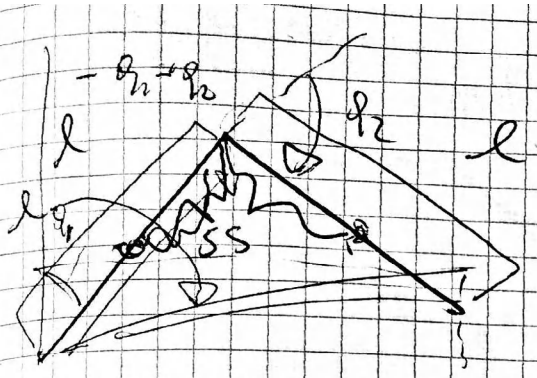
$$G_2 = \begin{bmatrix} X \\ Y \end{bmatrix} = l \begin{bmatrix} s q_1 \\ c q_1 \end{bmatrix}$$

by ANGOLA ASSOCIATI

$$= (l-s) \begin{bmatrix} -s q_1 \\ c q_1 \end{bmatrix}$$



$= l - s$



$$\overline{OP} = \begin{bmatrix} -l\, s\varphi_1 \\ l\, c\varphi_1 \end{bmatrix} + s \begin{bmatrix} s(\varphi_1+\varphi_2) \\ c(\varphi_1+\varphi_2) \end{bmatrix} \qquad \overline{PG_2}$$

by associated angles

$$\overline{OG_2} = l\begin{bmatrix} -s\varphi_1 \\ c\varphi_1 \end{bmatrix} + s\begin{bmatrix} -s(\varphi_1+\varphi_2) \\ c(\varphi_1+\varphi_2) \end{bmatrix}$$

2) Velocities, derivate w.r.t. tiro

$$\dot{\overline{OG_1}} = (l-s)\begin{bmatrix} -s\varphi_1\dot{\varphi}_1 \\ -s\varphi_1\dot{\varphi} \end{bmatrix} = (l-s)\dot{\varphi}_1\begin{bmatrix} c\varphi_1 \\ -s\varphi_1 \end{bmatrix} = (s-l)\dot{\varphi}_1\begin{bmatrix} c\varphi_1 \\ s\varphi_1 \end{bmatrix}$$

$$-(l-s)\dot{\varphi}_1\begin{bmatrix} c\varphi_1 \\ s\varphi_1 \end{bmatrix}$$

$$\dot{\overline{OG_2}} = -l\,\dot{\varphi}_1\begin{bmatrix} c\varphi_1 \\ s\varphi_1 \end{bmatrix} + s\begin{bmatrix} -c(\varphi_1+\varphi_2) \\ -s(\varphi_1+\varphi_2) \end{bmatrix}(\dot{\varphi}_1+\dot{\varphi}_2) =$$

$$= -\left[l\dot{\varphi}_1\begin{bmatrix} c\varphi_1 \\ s\varphi_1 \end{bmatrix} + s\begin{bmatrix} c(\varphi_1+\varphi_2) \\ s(\varphi_1+\varphi_2) \end{bmatrix}(\dot{\varphi}_1+\dot{\varphi}_2) \right]$$

3)

$$\omega_1 = \dot{\varphi}_1 \quad (\dot{\varphi}_1+\dot{\varphi}_2) = \omega_2$$

4)

$$T_1 = \frac{1}{2} m_1 \cdot (s-l)^2 \left[c\varphi_1 \; s\varphi_1\right]\begin{bmatrix} c\varphi_1 \\ s\varphi_1 \end{bmatrix}\dot{\varphi}_1^2 =$$

$$= \frac{1}{2} m_1 (s-l)^2 \cancel{\left[c^2\varphi_1 + s^2\varphi_1\right]}\dot{\varphi}_1^2 = \underbrace{\frac{1}{2} m_1 (s-l)^2}_{l-s}\dot{\varphi}_1^2$$

$$+ \frac{1}{2} I_{zz}\dot{\varphi}_1^2$$

$$\frac{1}{2} m_2 l^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 s^2 (\dot{\varphi}_1 + \dot{\varphi}_2)^2 + m_2 l s \cancel{\tfrac{1}{2}} \dot{\varphi}_1 (\dot{\varphi}_1 + \dot{\varphi}_2) c\varphi_2$$

$$+ 2 I_{zz} (\dot{\varphi}_1 + \dot{\varphi}_2)^2$$

$$\cancel{-l\dot{\varphi}_1 c\varphi_1 - s c(\varphi_1+\varphi_2)(\dot{\varphi}_1+\dot{\varphi}_2) - l\dot{\varphi}_1 s\varphi_1 - s s(\varphi_1+\varphi_2)(\dot{\varphi}_1+\dot{\varphi}_2)}$$

$$\begin{bmatrix} -l\dot{\varphi}_1 \\ -l\dot{\varphi}_1 c\varphi_1 \end{bmatrix}$$

(5) Inertia matrix

$$\sigma_{TOT} = \sigma_1 + \sigma_2 = \frac{1}{2} \begin{bmatrix} \dot{\varphi}_1 & \dot{\varphi}_2 \end{bmatrix} A \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}$$

don't forget $+ m l s c\varphi_2$

$$\frac{1}{2} \begin{bmatrix} \dot{\varphi}_1 & \dot{\varphi}_2 \end{bmatrix} A = \begin{bmatrix} m_1 (l-s)^2 + m_2 s^2 + 2 I_{zz} + m l s c\varphi_2 & m_1 s^2 + I \\ m s^2 + I + m l s c\varphi_2 & m_2 s^2 + I \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}$$

(6) Compute potential energies.

$$U = -m \left({}^0 g^T \right) \overline{OG_i}$$

$$U_{G_1} = -m \begin{bmatrix} 0 & -g & 0 & 0 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 & \\ s\theta & c\theta & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m \end{bmatrix}$$

or, much simpler, $-m\begin{bmatrix}0 & -g & 0\end{bmatrix}\begin{bmatrix}(l-s)(-sq_1)\\(l-s)\,cq_1\\0\end{bmatrix} =$

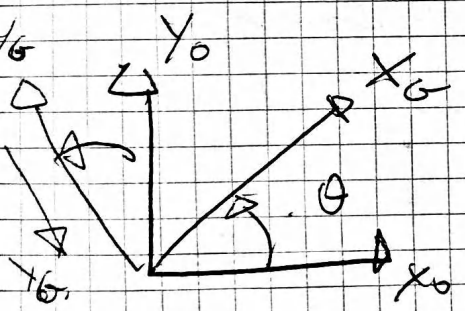
$+m(l-s)cq_1 = U_1$

$U_2 = -m\begin{bmatrix}0 & -g & 0\end{bmatrix}\begin{bmatrix}- - - -\\ l\,cq_1 + s\,c(q_1+q_2)\\0\end{bmatrix}$

$= +mg\left(lcq_1 + sc(q_1+q_2)\right) = U_2$

CHANGE OF FRAME

$\Rightarrow$ need ∇_G pick g in the proper frame



Y_G
Y_0
X_G
θ
Y_G
X_0

change of fram.

Need Y_G in X_0, Y_0

$$\boxed{Y_G = -s\theta\,\overline{X_0} + c\theta\,\overline{Y_0}}$$

$\hookrightarrow -g\overline{Y_G} = gs\theta\,\overline{X_0} - c\theta\,\overline{Y_0}$

in fram θ

$U_1 = -m\begin{bmatrix}+s\theta g & -c\theta g & 0\end{bmatrix}\begin{bmatrix}(l-s)(-sq_1)\\(l-s)\,cq_1\\0\end{bmatrix}$

$U_1 = +m\left[gs\theta\,(l-s)(-sq_1)\right.$

$\quad - g(l-s)\,c\theta\,cq_1$

$-s\alpha\,s\beta + c\alpha\,c\beta = -\left[s\alpha\,s\beta + c\alpha\,c\beta\right] = c(\theta -$

$$U_1 = -mg(l-s)\,c(\theta - q_1)$$

$U_2 = ?$

$$-m\begin{bmatrix} g s\theta & -g c\theta & 0 \end{bmatrix}\begin{bmatrix} -l s q_1 & -S s(q_1+q_2) \\ +l c q_1 & +S c(q_1+q_2) \end{bmatrix}$$

$$= -mg\left[-s\theta\left[l s q_1 + S s(q_1+q_2)\right] - c\theta\left[l c q_1 + S c(q_1+q_2)\right]\right]$$

$$= -mg\left[-l s\theta s q_1 - l c\theta c q_1 - S s\theta s(q_1+q_2) - S c\theta c(q_1+q_2)\right]$$

~~Recalgorsm~~ MINUS FOR TRIGONOM

by using some trigonometric function.

$$+mg\left[-l c(\theta - q_1) - S c(\theta - q_1 - q_2)\right] =$$

$$mgl\,c(\theta - q_1) + mgS\,c(\theta - q_1 - q_2) = U_2$$

$$U_{TOT} = U_1 + U_2$$

$$Q_i = \frac{\partial U}{\partial q_i} = \begin{bmatrix} \frac{\partial U}{\partial q_1} \\ \frac{\partial U}{\partial q_2} \end{bmatrix}$$

$$\frac{\partial U}{\partial q_1} = -mg(l-s)\,s(\theta - q_1) - mgl\,s(\theta - q_1)$$

$$- mg\, S\, s(\theta - q_1 - q_2)$$

$$\frac{\partial U}{\partial q_2} = -mg\, S\, s(\theta - q_1 - q_2)$$

8) $H = B\underbrace{\dot{q}\dot{q}}_{} + C\dot{q}$

$$\hookrightarrow \begin{bmatrix} \dot{q}_1 \dot{q}_2 & \dot{q}_1 \dot{q}_1 \\ \dot{q}_2 \dot{q}_1 & \end{bmatrix}^2 \qquad \dot{q}^2 = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix}^T$$

$$B = \frac{\partial A}{\partial q_k} + \frac{\partial A}{\partial q_j} - \frac{\partial A_{jk}}{\partial q_i}$$

$$C = \frac{\partial A}{\partial q_j} - \frac{1}{2}\frac{A}{\partial q_i}$$

Our case.

$$\dot{q}^2 = \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} \qquad B = [2\times 1] \qquad C = [2\times 2]$$

$$\dot{q}\dot{q} = \dot{q}_1 \dot{q}_2$$

$$\cancel{i=1 \quad j=1 \quad k=2 \quad = B_{1,2}}$$

$$\cancel{B = \left[\frac{\partial A_{12}}{\partial q_2} + \frac{\partial A_{12}}{\partial q_1} - \frac{\partial A_{12}}{\partial q_1} \right.}$$

APPLY B SIMILARLY

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$i=1 \quad J=1 \quad K=2$

$$B_{1,2} = \left[\underbrace{\frac{\partial A_{11}}{\partial q_2}}_{=0} + \underbrace{\frac{\partial A_{12}}{\partial q_1}}_{=0} - \underbrace{\frac{\partial A_{12}}{\partial q_1}}_{=0} \right] = 0$$

$i=2 \quad J=1 \quad K=2$

$$B_{2,2} = \left[\underbrace{\frac{\partial A_{21}}{\partial q_2}}_{=-s\,mlS_{sq_2}} + \underbrace{\frac{\partial A_{22}}{\partial q_1}}_{=0} - \underbrace{\frac{\partial A_{12}}{\partial q_2}}_{=+mlS_{sq_2}} \right] = -mlS_{sq_2}$$

$i=1 \quad J=1 \quad K=2$

first clerk of A

$$B_{1,2} = \left[\frac{\partial A_1}{\partial q_2} + \underbrace{\frac{\partial A_2}{\partial q_1}}_{=0} - \underbrace{\frac{\partial A_2}{\partial q_1}}_{=0} \right] = -2mlS_{cq_2}$$

$i=2 \quad J=1 \quad K=2$

$$B_{2,2} = \left[\frac{\partial A_2}{\partial q_2} + \frac{\partial A_4}{\partial q_1} - \frac{\partial A_2}{\partial q_2} \right] = 0$$

$$B = \begin{bmatrix} -2mlS_{cq_2} \\ 0 \end{bmatrix}$$

$i = 1 \quad j = 1$

$$C_{11} = \left[\frac{\partial A_{11}}{\partial q_1} - \frac{1}{2}\frac{A_{11}}{\partial q_1}\right] = 0$$

$i = 2 \quad j = 1$

$$C_{21} = \left[\underbrace{\frac{\partial A_{21}}{\partial q_1}}_{=0} - \frac{1}{2}\frac{\partial A_{11}}{\partial q_2}\right] = +mgSl\,sq_2$$

$i = 1 \quad j = 2$

$$C_{12} = \left[\underbrace{\frac{\partial A_{12}}{\partial q_2}}_{-mlS\,sq_2} - \frac{1}{2}\underbrace{\frac{\partial A_{22}}{\partial q_1}}_{=0}\right] = -mlS\,sq_2$$

$i = 2 \quad j = 2$

C_{22}

$$C_{22} = \left[\underbrace{\frac{\partial A_{22}}{\partial q_2}}_{=0} - \frac{1}{2}\underbrace{\frac{\partial A_{22}}{\partial q_2}}_{=0}\right] = 0$$

$$C = \begin{bmatrix} 0 & -mlS\,sq_2 \\ mgSl\,sq_2 & 0 \end{bmatrix}$$

$$M = (q, \dot{q})\; B\,\dot{q}_1\dot{q}_2 + C\begin{bmatrix}\dot{q}_1^2 \\ \dot{q}_2\end{bmatrix}$$

6 Appendix: Reaction Forces

Impact Model

$[F]$ = function - reaction force $(q_1, q_2, \dot{q}_1, \dot{q}_2, \ddot{q}_1, \ddot{q}_2)$

$\overrightarrow{G1}$, $\overrightarrow{G2}$

$$COM_x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{\cancel{m}(x_1 + x_2)}{2\cancel{m}}$$

$$COM_y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{\cancel{m}(y_1 + y_2)}{2\cancel{m}}$$

$$X_G = \frac{1}{2}\Big[\underbrace{(l-s)\, s(-q_1)}_{-(l-s)sq_1} - l s q_1 - s \sin(q_1 + q_2)\Big]$$

$$= -\frac{1}{2}\Big[(l-s)\sin(q_1) + l s \sin q_1 + s \sin(q_1 + q_2)\Big]$$

$$= +l s q_1 + \frac{1}{2} s s q_1 - \frac{1}{2} s \sin(q_1 + q_2)$$

$$Y_G = \frac{1}{2}\Big[(l-s)\cos q_1 + l \cos q_1 + s \cos(q_1 + q_2)\Big]$$

$$= l c q_1 - \frac{1}{2} s \cos q_1 + \frac{1}{2} s \cos(q_1 + q_2)$$

$\dot{X}_G$ and $\dot{Y}_G$

$$\dot{X}_G = -\frac{1}{2}\Big[(l-s)\cos(q_1)\dot{q}_1 + l \cos q_1 \dot{q}_1 + s \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2)\Big]$$

$$\dot{Y}_G = \frac{1}{2}\Big[-(l-s)\sin q_1 \dot{q}_1 - l \sin q_1 \dot{q}_1 - \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2)$$

let's draw some forces

$\cancel{\dot{X}_G = -l\cos q_1\,\dot{q}_1 + S\cos q_1\,\dot{q}_1 + S\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)}$

NODO

$$X_G = \begin{bmatrix} -l s q_1 + \frac{1}{2} S s q_1 - \frac{1}{2} S \sin(q_1+q_2) \\ l c q_1 - \frac{1}{2} S c q_1 \dot{q}_1 + \frac{1}{2} S \cos(q_1+q_2) \end{bmatrix}$$

$$\dot{X}_G = \begin{bmatrix} -l c q_1 \dot{q}_1 + \frac{1}{2} S c q_1 \dot{q}_1 - \frac{1}{2} S c(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ -l s q_1 \dot{q}_1 + \frac{1}{2} S s q_1 \dot{q}_1 - \frac{1}{2} S \sin(q_1+q_2)(\dot{q}_1+\dot{q}_2) \end{bmatrix}$$

$$\ddot{X}_G = \begin{bmatrix} +l s q_1 \dot{q}_1^2 - l c q_1 \ddot{q}_1 + \frac{1}{2} S c q_1 \ddot{q}_1 - \frac{1}{2} S s q_1 \dot{q}_1^2 + \frac{1}{2} S s(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2 - \frac{1}{2} S c(q_1+q_2)(\ddot{q}_1+\ddot{q}_2) \\ -l c q_1 \dot{q}_1^2 - l s q_1 \ddot{q}_1 + \frac{1}{2} S s q_1 \ddot{q}_1 + \frac{1}{2} S c q_1 \dot{q}_1^2 - \frac{1}{2} S s(q_1+q_2)(\ddot{q}_1+\ddot{q}_2) - \frac{1}{2} S c(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2 \end{bmatrix}$$

↓ point velocity & acceleration

The reaction force is $F = M\ddot{X}_G - M\bar{g}$ $\quad M = 2m$

$$F = 2m\ddot{X}_g - 2mg$$

$$\bar{g} = [g\sin\theta \quad -g\cos\theta]$$

FUNCTION IMPACT

$A_1 \quad J_R$ = function IMPACT.

$$\begin{bmatrix} A_1 & -J_{R2}^T \\ J_{R2} & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{x}^+ \\ \dot{y}^+ \\ \dot{q}_1^+ \\ \dot{q}_2^+ \\ \Delta_{2\times 1} \end{bmatrix} = \begin{bmatrix} A_1 \\ 0_{2\times 1} \end{bmatrix} \begin{bmatrix} \dot{x}^- \\ \dot{y}^- \\ \dot{q}_1^- \\ \dot{q}_2^- \end{bmatrix}$$

7 Appendix: Impact Model

$$\ddot{x}_G = \begin{bmatrix} [illegible] \\ -lc\varphi_1\dot{\varphi}_1^2 - ls\varphi_1\ddot{\varphi}_1 + \frac{1}{2}Ss\varphi_1\ddot{\varphi}_1 + \frac{1}{2}Sc\varphi_1\dot{\varphi}_1^2 - \frac{1}{2}Ss(\varphi_1+\varphi_2)(\ddot{\varphi}_1 \end{bmatrix}$$

↓ found velocity & acceleration.

The reaction force is $F = M\ddot{X}_G - M\bar{g}$

$$F = 2m\ddot{X}_g - 2m$$

$$\bar{g} = = [g\sin\theta \quad -g\cos\theta]$$

FUNCTION IMPACT

$A_1 \quad J_R$ = function IMPACT.

$$\begin{bmatrix} A_1 & -J_{R2}^T \\ J_{R2} & 0_{2\times2} \end{bmatrix} \begin{bmatrix} \dot{X}^+ \\ \dot{Y}^+ \\ \dot{\varphi}_1^+ \\ \dot{\varphi}_2^+ \\ I_{R2x} \\ I_{R2y} \end{bmatrix} = \begin{bmatrix} A_1 \\ 0_{2\times4} \end{bmatrix} \begin{bmatrix} \dot{X}^- \\ \dot{Y}^- \\ \dot{\varphi}_1^- \\ \dot{\varphi}_2^- \end{bmatrix}$$

A_1 already defined, (6)

$$T = T_{c1} + T_{c2} = \frac{1}{2}\begin{bmatrix}\dot{x} & \dot{y} & \dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix}A_1\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

Known the position (X, Y) $\qquad \dot{x}_j = J_R \dot{q}_e$

$$X = l \sin q_1$$
$$Y = l \cos q_1$$

now X and Y are known

$$G_1 = \begin{bmatrix} X - S \sin q_1 \\ Y - S \cos q_1 \end{bmatrix} \qquad \dot{G}_1 = \begin{bmatrix} \dot{X} - S \cos q_1 \, \dot{q}_1 \\ \dot{Y} + S \sin q_1 \, \dot{q}_1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} X - S \sin(q_1 + q_2) \\ Y + S \cos(q_1 + q_2) \end{bmatrix} \qquad \dot{G}_2 = \begin{bmatrix} \dot{X} - S \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \\ \dot{Y} + S \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \end{bmatrix}$$

$$\omega_1 = \dot{q}_1 \qquad \omega_2 = (\dot{q}_1 + \dot{q}_2)$$

$$T_1 = \frac{1}{2} m \underbrace{\begin{bmatrix} \dot{x} - S c q_1 \dot{q}_1 & \dot{y} + S \sin q_1 \dot{q}_1 \end{bmatrix}}_{1 \times 2} \underbrace{\begin{bmatrix} \dot{x} - S c q_1 \dot{q}_1 \\ \dot{y} + S \sin q_1 \dot{q}_1 \end{bmatrix}}_{2 \times 1} + \frac{1}{2} I \omega^2$$

$$\cancel{T_1 = \frac{1}{2} m [\dot{x}^2}$$

$$= \frac{1}{2} m \left[\dot{x}^2 - 2\dot{x} S c q_1 \dot{q}_1 + S c^2 q_1 \dot{q}_1 + \dot{y}^2 + 2\dot{y} S \sin q_1 \dot{q}_1 + S \sin^2 q_1 \dot{q}_1 \right] \leftarrow 1$$

$$= \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2 + 2 S \dot{q}_1 (-\dot{x} c q_1 + \dot{y} \sin q_1) + S \dot{q}_1 \right]$$

$$\mathcal{T}_2 = m\frac{1}{2}\begin{bmatrix}\dot{x} - s\,c(q_1+q_2)(\dot{q}_1+\dot{q}_2) & \dot{y} - s\,s(q_1+q_2)(\dot{q}_1+\dot{q}_2)\end{bmatrix}\begin{bmatrix}\dot{x} - s\,c(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ \dot{y} - s\,s(q_1+q_2)(\dot{q}_1+\dot{q}_2)\end{bmatrix}$$

$$= \frac{1}{2}m\Big[\dot{x}^2 - 2s\,c(q_1+q_2)(\dot{q}_1+\dot{q}_2)\dot{x} + s\,c^2(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2$$
$$+ \dot{y}^2 - 2s\,s(q_1+q_2)(\dot{q}_1+\dot{q}_2)\dot{y} + s\,s^2(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2\Big] + \frac{1}{2}I(\dot{q}_1+\dot{q}_2)^2$$

$$= \frac{1}{2}m\Big[\dot{x}^2+\dot{y}^2 - 2s(\dot{q}_1+\dot{q}_2)\big(\dot{x}\,c(q_1+q_2) + \dot{y}\,s(q_1+q_2)\big) + s(\dot{q}_1+\dot{q}_2)^2\Big] + \frac{1}{2}I(\dot{q}_1+\dot{q}_2)^2$$

$\Rightarrow$ from this, we can extrapolate mapping terms carefully the A matrix, $\boxed{\mathcal{T}_{tot} = \mathcal{T}_1 + \mathcal{T}_2}$

$$\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{q}_1 \\ \dot{q}_2 \end{array}\begin{bmatrix} 2m & 0 & m\,s\,c\,q_1 - m\,s\,c(q_1+q_2) & -m\,s\,c(q_1+q_2) \\ 0 & 2m & m\,s\,c\,q_1 - s\,s\,s\,w(q_1+q_2) & -m\,s\,s(q_1+q_2) \\ m\,s\,c(q_1) - m\,s\,c(q_1+q_2) & m\,s\,s(q_1) - m\,s\,s(q_1+q_2) & 2m\,s^2 + 2I & m\,s^2 + I \\ -m\,s\,c(q_1+q_2) & -m\,s\,s(q_1+q_2) & m\,s^2 + I & m\,s^2 + I \end{bmatrix}$$

2m of $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{x}$

A found.

It is also easily found

$$\dot{x} = J_R\,\dot{q}_e \qquad J_R = \begin{bmatrix} 1 & 0 & -l\,c\,q_1 & -l\,c(q_1+q_2) \\ 0 & 1 & -l\,s\,q_1 & -l\,s(q_1+q_2) \end{bmatrix}$$