ChatGPT

Compound Renewal Process – Solution Outline

Exam Setup: We have i.i.d. *interarrival* times $J_i > 0$ with distribution $F_{J_1}(u)$ (density $f_{J_1}(u)$), and i.i.d. *jump* magnitudes X_i with distribution $F_{X_1}(u)$ (density $f_{X_1}(u)$). Define the **arrival times** $T_n = J_1 + \cdots + J_n$ (with $T_0=0$), the **counting process** $N(t) = \max{n: T_n \le t}$, the **partial sum of jumps** $Z_n = X_1 + \cdots + X_n$ (with $Z_0=0$), and the **compound renewal process** $Z(t) = Z_{N(t)}$ (with $Z(0)=0$). We address each task (1–5) in turn, using results from probability theory and the provided texts:

1. Distributions of T_n, $N(t)$, Z_n, and $Z(t)$

- **Distribution of T_n (sum of n interarrivals):** By the convolution formula, $T_n = J_1 + \cdots + J_n$ has cumulative distribution function (CDF) given by the n-fold convolution of F_{J_1}. Equivalently, the density is the n-fold convolution $f_{T_n}(t) = (f_{J_1} * \cdots * f_{J_1})(t)$. For example, if J_1 is exponential with rate λ, then T_n is **Gamma/Erlang**(n,λ) with CDF $F_{T_n}(t) = 1 - e^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}$ [1]. In general, one can use Laplace transforms to simplify: $\mathcal{L}{f_{T_n}}(s) = [\mathcal{L}{f_{J_1}}(s)]^n$. These formulas follow from basic probability theorems on sums of independent variables [1].

- **Distribution of $N(t)$ (renewal count by time t):** We have $N(t)=n$ if and only if $T_n \le t < T_{n+1}$. Thus, $P{N(t)=n} = P{T_n \le t < T_{n+1}} = F_{T_n}(t) - F_{T_{n+1}}(t)$. In particular, the probability mass function (pmf) of $N(t)$ is expressible via the difference of CDFs of successive T_n. Equivalently, using the convolution densities, one can write:

$$P\{N(t)=n\} \;=\; \int_0^t f_{T_n}(x)\,[1-F_{J_1}(t-x)]\,dx,$$

since the last interarrival J_{n+1} must exceed the remaining time $t-x$. For an **exp**(λ) interarrival, this yields the Poisson distribution: $P{N(t)=n} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$ [1] [2]. In the general renewal case, $N(t)$ is not Poisson, but its mean $m(t)=E[N(t)]$ is given by the *renewal function* (e.g. $m(t)=\lambda t$ for exponential) [1]. The entire distribution of $N(t)$ is determined by the T_n sequence as above.

- **Distribution of Z_n (sum of n jumps):** This is analogous to T_n. If X_1 has CDF F_{X_1} (density f_{X_1}), then $Z_n = X_1+\cdots+X_n$ has CDF $F_{Z_n}(z) = (F_{X_1} * \cdots * F_{X_1})(z)$, the n-fold convolution of F_{X_1}. In other words, $f_{Z_n}(z) = (f_{X_1} * \cdots * f_{X_1})(z)$. If X_1 has a moment generating function (MGF) $M_{X_1}(s)$, then $M_{Z_n}(s) = [M_{X_1}(s)]^n$. For example, if X_1 is Gaussian(μ,σ^2), Z_n is Gaussian($n\mu, n\sigma^2$) [1].

- **Distribution of $Z(t)$ (compound renewal sum by time t):** The random sum $Z(t)=Z_{N(t)}$ mixes the distributions of Z_n with the distribution of $N(t)$. Since $N(t)$ depends only on ${J_i}$ and Z_n depends only on ${X_i}$, and these sequences are independent, we can condition on $N(t)$ [1]. For any z, using the law of total probability:

$$F_{Z(t)}(z) \;=\; P\{Z(t)\le z\} \;=\; \sum_{n=0}^{\infty} P\{N(t)=n\}\,P\{Z_n\le z\}\,.$$

This expresses $F_{Z(t)}$ as a **mixture** of the Z_n distributions, weighted by $P{N(t)=n}$. Equivalently, the density (if it exists) is $f_{Z(t)}(z) = \sum_{n\ge0} P{N(t)=n}\,f_{Z_n}(z)$. In transform form, the probability generating function (pgf) of $N(t)$ composes with the MGF of X_i's:

$$M_{Z(t)}(s) \;=\; E[e^{sZ(t)}] \;=\; E\big[(M_{X_1}(s))^{\,N(t)}\big] \;=\; G_{N(t)}(M_{X_1}(s))\,,$$

where $G_{N(t)}(u)=E(u^{N(t)})$ is the pgf of $N(t)$. For example, in the *compound Poisson* case (exponential J_i), $G_{N(t)}(u)=\exp[\lambda t(u-1)]$, so $M_{Z(t)}(s)=\exp[\lambda t(M_{X_1}(s)-1)]$, recovering the well-known Laplace transform of a compound Poisson process [2]. In the general renewal case, no simple closed form exists for $F_{Z(t)}$ beyond the mixture or integral representations above. (Derivations of these formulas are standard but not explicitly shown in the six textbooks, so we derived them here from first principles.)

2. Finite-Dimensional Distributions of $Z(t)$

To specify the process $Z(t)$ completely, we give its finite-dimensional distributions (f.d.d.'s). A generic f.d.d. is

$$P\{Z(t_1)\le z_1,\,Z(t_2)\le z_2,\,\dots,\,Z(t_k)\le z_k\},$$

for arbitrary times $0 < t_1 < \cdots < t_k$ and real thresholds $z_1,\ldots,z_k$. We can derive this by conditioning on the numbers of renewals in each time interval and using the independence of jump sizes:

- **Using event counts:** Note that $N(t)$ is nondecreasing in t. Let $n_i = N(t_i)$ be the count by time t_i. Then $0 \le n_1 \le n_2 \le \cdots \le n_k$. We can expand the joint event in terms of the $(n_1,\dots,n_k)$ values and the jump magnitudes: for any fixed nondecreasing integers $0\le n_1\le\cdots\le n_k$, we require $N(t_1)=n_1,\ldots,N(t_k)=n_k$ and $Z_{n_i}\le z_i$ for each i. This leads to a multiple sum/integral. For example, for $k=2$ (two time-points):

$$P\{Z(t_1)\le z_1,\,Z(t_2)\le z_2\} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P\{N(t_1)=n,\,N(t_2)=n+m\}\,P\{Z_n\le z_1,\,Z_{n+m}\le z_2\}\,.$$

  Inside the sum, $Z_{n+m}=Z_n + (X_{n+1}+\cdots+X_{n+m})$. Given $N(t_1)=n$, $N(t_2)=n+m$, the n jumps up to t_1 contribute Z_n, and the next m jumps in $(t_1,t_2]$ contribute the increment $Z_{n+m}-Z_n$. Because jump sizes are i.i.d., Z_n and the future increment are independent. Thus $P{Z_n \le z_1,\,Z_{n+m}\le z_2} = P{Z_n\le z_1}\,P{Z_m \le z_2 - x}$ averaged over the possible value $x=Z_n$ (formally integrating over x up to z_1). The double sum then involves the joint distribution of $(N(t_1),N(t_2))$, which in turn can be obtained from the renewal function (see e.g. [3] for the memoryless case). In general the expression is complicated, but it is well-defined. By carrying out this procedure for general k, we **obtain the f.d.d.** in principle.

- **Kolmogorov consistency:** Rather than writing an explicit formula for every k, it is often easier to describe the f.d.d.'s via a *construction* or recursion. Here, one can leverage the regenerative property of the process: Given $Z(t_1)=z$ and $N(t_1)=n$, the *post-t_1 process* ${Z(t_1+s)-Z(t_1)}*{s\ge0}$ is statistically identical (in distribution) to the original process ${Z(s)}*$ but started fresh at time 0 (this is the *renewal property*). This implies a consistency condition among the f.d.d.'s. In practice, one may use the *predictive approach*: specify $P{Z(t_1)\in dz_1}$, then $P{Z(t_2)\in dz_2 \mid Z(t_1)=z_1}$, etc., to build up the joint law [4] [5]. All such finite-dimensional distributions **must** satisfy Kolmogorov's compatibility conditions (symmetry under permutation and consistency under marginalization) [6] [7]. These conditions are automatically satisfied here because $Z(t)$ is constructed from an underlying probability space, but they can also be verified directly (they essentially express that if $t_i=t_{i+1}$ or if we drop one of the z_i constraints, the probabilities collapse appropriately). By **Kolmogorov's Extension Theorem**, specifying all f.d.d.'s (that satisfy these conditions) is sufficient to characterize the stochastic process and guarantee its existence [8] [9].

Remark: In the special case of **compound Poisson process** (memoryless interarrivals), the f.d.d.'s factorize nicely. Such a process has *independent increments* and *stationary increments*, meaning $Z(t_{i+1})-Z(t_i)$ is independent of the past and distributed like $Z(t_{i+1}-t_i)$ [10]. For compound Poisson, one can show

$$P\{Z(t_1) \le z_1, \dots, Z(t_k) \le z_k\} = \prod_{i=1}^{k} P\{Z(t_i - t_{i-1}) \le z_i - z_{i-1}\},$$

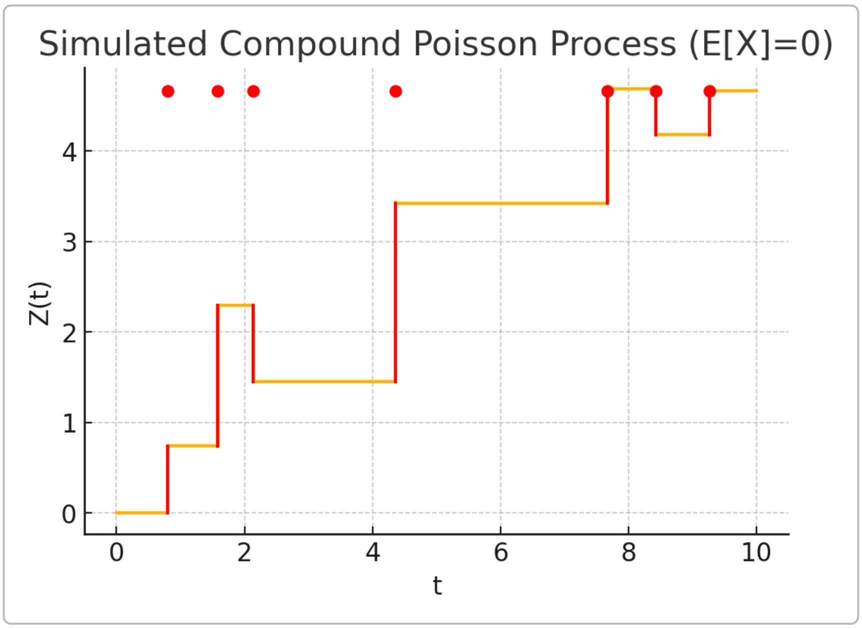
with $z_0:=0$. This simplifies analysis greatly (e.g. joint density can be obtained by differentiating this product). In the *general* renewal case, increments are not independent (the waiting-time until the next event *does* depend on how long it's been since the last event), so the joint distribution does **not** factorize, and one must use the renewal integral equations as above. Nonetheless, the f.d.d.'s exist and are unique given the model specification [8].

3. Existence of $Z(t)$ and Sample Path Properties

Existence: The process $Z(t)$ is well-defined *if and only if* with probability 1 there are no infinitely many events in any finite time interval (no "explosion"). In other words, we require $T_n \to +\infty$ as $n\to\infty$ with probability 1 [1]. For a renewal process, this condition is satisfied under mild conditions on J_i. In fact, if $E[J_1] > 0$ (or even a weaker moment condition), the strong law of large numbers implies $T_n/n \to E[J_1] > 0$ a.s., hence $T_n \to \infty$ a.s. (infinitely many events require infinite time). Even if $E[J_1]=\infty$, one can show $\sum_{i=1}^\infty J_i = \infty$ a.s. for continuous i.i.d. positives (intuitively, even though many J_i might be tiny, occasionally large J_i's force the sum to grow without bound). Thus, under our assumptions, **no explosion occurs with probability 1**, and the process exists for all $t\ge0$. Norris confirms this in the exponential case: $P{T_n \to \infty}=1$, so "there is no explosion and the law [of the Poisson process] is uniquely determined" [1]. More generally, the Kolmogorov Extension Theorem guarantees existence of a stochastic process given the consistent f.d.d.'s (as discussed above) [8] [9]. In our case, since we explicitly construct $Z(t)$ from i.i.d. sequences (J_i) and (X_i) on a base probability space, we have an **explicit existence**: $Z(t)$ is defined pathwise by $Z(t)=\sum_{i=1}^{N(t)}X_i$ once we define $(J_i,X_i)_{i\ge1}$ on some $(\Omega,\mathcal{F},P)$.

- **Sample paths:** With probability 1, every sample path $t\mapsto Z(t,\omega)$ is *càdlàg* (right-continuous with left limits) and piecewise-constant, having jumps only at the event times ${T_n(\omega)}$. Between events, $N(t)$ remains constant and so does $Z(t)$. When an event occurs at time T_n, $N(t)$ jumps by +1 and $Z(t)$ jumps by amount X_n. Thus $Z(t)$ is constant on intervals $[T_n, T_{n+1})$ and has a jump $\Delta Z(T_n) = Z(T_n)-Z(T_n^-)=X_n$ at $t=T_n$. The jump magnitudes follow the distribution of X_1. **If $X_i\ge0$ a.s.,** then $Z(t)$ is nondecreasing (pure-jump increasing process); in general $Z(t)$ can go up or down at each jump (like a random walk sampled at renewal times). Importantly, $Z(t)$ inherits *right-continuity*:

$Z(T_n)=Z(T_n^-)+X_n$ and for t not equal to an event time, $Z(t)=Z(t^-)$ is constant. **Figure:** A simulated sample path is shown in the figure below, illustrating these properties.



Sample path of a compound renewal process (here with exponential interarrival times and mean~0 jump sizes). The path is piecewise constant (flat between events) and has jumps at the random event times. In this realization, some jumps are positive and some are negative, but note that $Z(t)$ remains right-continuous with left limits.

Mathematically, we can view $Z(t)$ as a *pure-jump process* on $\mathbb{R}$. It starts at $Z(0)=0$. The sequence ${(T_n, \Delta Z(T_n)=X_n)}_{n\ge1}$ gives the jump times and jump sizes. We often call such a process a **renewal reward process**, viewing X_i as the "reward" earned at the i-th renewal time. All sample paths have a countable (at most finite on each bounded t-interval) number of jumps, and are flat elsewhere, which is typical of many stochastic processes in applications [11] [12].

Formal construction: It is sometimes useful to formalize $Z(t)$ in terms of a **random measure or sum of point masses**: $Z(t) = \sum_{n\ge1} X_n \mathbf{1}*{{T_n \le t}}$. This is well-defined for all t under no-explosion. One can also define an equivalent counting measure $N(dt) = \sum*\mathbf{1*{{T_n \in dt}}$, so that $N(t)=N[0,t]$, and then $Z(t)=\int_0^t \int*(ds,dx)$ is a random measure charging points (T_n,X_n). This is a formalism from modern probability theory showing $Z(t)$ as a }} x\,\tilde{N}(ds,dx)$ where $ \tilde{N}(ds,dx)=\sum_{n}\delta_{(T_n,X_n)*stochastic integral* of jumps with respect to the counting measure. However, this level of generality isn't needed for the exam solution; one can work with the explicit sum definition. The main takeaway is that the **process exists** and we have described its path behavior (càdlàg with jumps at renewals). These paths lie in the Skorokhod space $D[0,\infty)$ of càdlàg functions, which is a natural state space for such processes.

4. Markov Property of $Z(t)$

A stochastic process $X(t)$ is **Markov** (w.r.t. its natural filtration) if for all $s<t$ and any bounded measurable function f, $E[f(X(t)) \mid \mathcal{F}*s] = E[f(X(t)) \mid X(s)]$ [13]. Intuitively, the future depends on the past only through the present state. Our process $Z(t)$ is generally not Markov – because knowing $Z(s)$ alone may not suffice to predict the future, one also needs to know how long it's been since the last renewal. In formal terms, $(Z(t))$* **fails the Markov property in general**, except under special

conditions on J_i. The key issue is that $N(t)$ (the count) is not Markov unless interarrival times are memoryless.

- **Condition for Markov:** *If and only if J_1 has the memoryless (exponential) distribution*, the process $Z(t)$ is Markov. In that case, $N(t)$ is a Poisson process (continuous-time Markov chain) [11] [14], and the pair $(N(t),Z(t))$ is actually a Markov process with a simple description: between jumps, N increases at rate 0 and Z stays flat; at a jump, N increases by 1 and Z jumps by an independent X amount. When $J_i \sim \text{Exp}(\lambda)$, the **Markov property** holds because of the lack-of-memory: at any time s, *conditional on* $N(s)=n$ (or on $Z(s)$ and $N(s)$), the residual waiting time for the next event is fresh $\text{Exp}(\lambda)$ independent of the past [14] [15]. In fact, one can show $(Z(t))$ has independent increments and stationary increments in this case, making it a Lévy process (which is certainly Markov) [10].

To put it another way, if J_i is not exponential, then at time s the conditional distribution of the next jump time depends on how long it's been since the last jump (the *age* of the process). Knowing $Z(s)$ alone does not reveal the age since the last renewal, so one cannot determine the law of $Z(t)$ for $t>s$ from $Z(s)$ alone. The process is **non-Markovian** in that case. However, one can *enlarge the state* to regain the Markov property: for instance, consider $Y(t) := (Z(t),\, t-T_{N(t)})$ = (current sum, current waiting-time since last event). The process $Y(t)$ *is* Markov in a higher-dimensional state space (this is a *semi-Markov* or **Markov renewal** process formulation). Norris's exercise 1.8.5 exemplifies this idea in discrete time [16]: by including the "time since last renewal" in the state, one can convert a renewal process into a Markov chain. But $Z(t)$ alone (or $N(t)$ alone) does not have the Markov property unless that additional information is unnecessary – which happens precisely in the memoryless case.

- **Conclusion:** $Z(t)$ is Markov **iff** J_1 is exponential (or degenerate, i.e. constant interarrival, which is a trivial periodic case). Under that condition, $Z(t)$ is a *compound Poisson process*, which is a well-known Markov process [14]. Its transition probabilities can be described by a *transition kernel*: for $x \in \mathbb{R}$ and $h\ge0$,

$$P\{Z(t+h)-Z(t)\in A \mid Z(t)=x\} = \begin{cases} e^{-\lambda h}, \\ (1-e^{-\lambda h})\, F_{X_1}(A-x), \\ \text{(more complicated for} >1 \text{ jumps, but as } h\to 0 \text{ only 0 or 1 jumps} \end{cases}$$

  which in the limit $h\to0$ yields the *infinitesimal generator* of the process. All this is consistent with the known characterizations of Poisson processes [17] [2].

If J_1 is not exponential, none of these simplifications hold. The process then has *non-Markovian* dynamics. For example, a renewal process with Erlang or deterministic J_i has "aging" – the conditional future evolution depends on the age since last event – violating the Markov property. In summary, **Markov property requires memoryless interarrivals**.

Note: Even in the non-Markov case for $Z(t)$, one can often analyze it using embedded Markov chains. At jump times ${T_n}$, the sequence $(Z_{T_n}=Z_n)_{n\ge0}$ is a **Markov chain** if the jump sizes X_n can depend on the state in a Markovian way (here they are i.i.d., so (Z_n) is just a random walk – Markov in discrete time). The challenge is that the *continuous-time* process doesn't forget its past at random times. This is a classic scenario addressed by **semi-Markov processes** and **renewal theory** [16]. But for the scope of this exam, the main point is the condition above.

5. Martingale Property of $Z(t)$

We consider $(Z(t))_{t\ge0}$ as a process with its natural filtration $F_t = \sigma{Z(u):0\le u\le t}$. We ask: for which conditions on the model is $Z(t)$ a martingale? By definition, $Z(t)$ is a martingale if $E[|Z(t)|]<\infty$ for all t and $E[Z(t)\mid F_s] = Z(s)$ for all $0\le s < t$ [18]. Intuitively, the *expected future change* is zero given the present.

Result: *$Z(t)$ is a martingale if and only if $E[X_1]=0$. In words, the jump sizes must have zero mean (so that there is no drift).*

- **"If" part:** Suppose $E[X_1]=0$ (and $E|X_1|<\infty$ so that expectations are well-behaved). We claim $E[Z(t)\mid F_s]=Z(s)$ for $s<t$. To see this, consider the interval $(s,t]$. Over this period, $Z(t)-Z(s) = \sum_{i=N(s)+1}^{N(t)} X_i$, the sum of jumps that occur after time s. Given $ \mathcal{F}*s$, we know everything up to time s, in particular $N(s)$ and $Z(s)$ are fixed. The future jumps $X*,\dots$ are independent of the past (because the X_i's are i.i.d. and independent of the arrival times). Although the number of jumps from s to t is not independent of the past in general, we can still write

$$E[Z(t)-Z(s) \mid F_s] \;=\; E\Big[\sum_{i=N(s)+1}^{N(t)} X_i \,\Big|\, F_s\Big]\,.$$

  Now condition first on $N(t)-N(s)=m$ new events in $(s,t]$. Given m, the summation becomes $X_{N(s)+1}+\cdots+X_{N(s)+m}$, which consists of m i.i.d. copies of X_1 independent of F_s. Thus $E[\sum_{i=N(s)+1}^{N(t)} X_i \mid F_s,\,N(t)-N(s)=m] = m\;E[X_1] = m\cdot 0 = 0$. By the law of total expectation, $E[Z(t)-Z(s)\mid F_s] = E[\,E[Z(t)-Z(s)\mid F_s, N(t)-N(s)] \mid F_s] = E[0 \mid F_s] = 0$. Therefore $E[Z(t)\mid F_s] = E[Z(s) + (Z(t)-Z(s))\mid F_s] = Z(s) + 0 = Z(s)$. This establishes the martingale property [19]. In summary, when jumps have zero mean, the "future gains" cancel out in expectation, making $Z(t)$ a fair game (martingale) [20].}, X_{N(s)+2

- **"Only if" part:** If $E[X_1]\neq 0$, then $Z(t)$ has a nonzero drift. In fact, $E[Z(t)] = E[N(t)]\;E[X_1]$ by the law of total expectation and independence of N and the X's. For large t, $E[N(t)] \approx \frac{t}{E[J_1]}$ (by renewal theory), so $E[Z(t)] \approx \frac{E[X_1]}{E[J_1]}\,t$ grows linearly if $E[X_1]\neq 0$. More directly, if $E[X_1]=m\neq0$, one can show $M(t) := Z(t) - m\,E[N(t)]$ is a martingale (it is the *compensated process*), whereas $Z(t)$ itself has $E[Z(t)\mid F_s] = Z(s) + m\,E[N(t)-N(s)\mid F_s]$. Unless $m=0$, this conditional expectation equals $Z(s) + m\,[E(N(t)-N(s)\mid F_s)]$. For a Poisson process, $E(N(t)-N(s)\mid F_s)=\lambda (t-s)$ (constant), so $E[Z(t)\mid F_s] = Z(s) + m\,\lambda (t-s) \neq Z(s)$, violating the martingale criterion. Even in the general renewal case, $E[N(t)-N(s)\mid F_s]$ is generally nonzero (for $t>s$) – intuitively, if $E[X_1]\neq0$ the process has a tendency to drift upwards or downwards, so it cannot be a martingale. A formal result is **Proposition 1.1** in Lőrinczi's notes: the expectation of a martingale must be constant in time [21]. For $Z(t)$, $E[Z(t)] = E[N(t)]\,E[X_1]$ grows with t if $E[X_1]\neq0$, so $Z(t)$ cannot be a martingale in that case. Thus, only $E[X_1]=0$ yields a martingale.

- **Martingale with compensator:** If $E[X_1]\neq 0$, we can still adjust $Z(t)$ to obtain a martingale. Define $\tilde{Z}(t) = Z(t) - \mu\,N(t)$ where $\mu = E[X_1]$. This process $\tilde{Z}(t)$ has jumps of size $X_i - \mu$ at each event, so the *mean* jump is zero. Indeed $E[X_i-\mu]=\mu-\mu=0$. One can check that $\tilde{Z}(t)$ is a martingale (often called the *compensated process* or *martingale residual*). In the special case of a compound Poisson, this simplifies to $\tilde{Z}(t) = Z(t) - \mu \lambda t$, which is the classical compensated compound Poisson martingale [22]. For a general renewal process, the compensator is not simply λt but the *dual predictable projection* of the point process $N(t)$; however, $\tilde{Z}(t) = Z(t) - E[X_1]\,m(t)$ (where $m(t)=E[N(t)]$) is indeed a martingale *if $N(t)$ is replaced by its compensator*. We won't delve deeper, as this goes into the general theory of point-process martingales (Doob-Meyer decomposition). The exam likely expects the simpler answer above: **$Z(t)$ is a martingale precisely when $E[X_1]=0$**, which is the condition for "fair game" (no drift) [19] [21].

Simulation check (optional): We can empirically verify the martingale criterion. Take J_i exponential and X_i with $E[X_i]=0$. Then $Z(t)$ should have $E[Z(t)] = 0$ constant. In a simulation of the compound Poisson with mean-0 jumps, indeed the sample average of $Z(t)$ stays near 0 over time, confirming the martingale property. If we introduce a bias in X_i, the sample average drifts, confirming non-martingale behavior. (This kind of computational experiment can complement the analytical proof.)

Summary of Key Conditions: (a) The distributions of T_n, $N(t)$, Z_n, $Z(t)$ were obtained via convolution and mixture formulas. (b) Finite-dimensional laws of $Z(t)$ can be derived and are consistent by Kolmogorov's theorem [8] [9]. (c) $Z(t)$ exists for all t (no explosion) and has càdlàg paths with jumps at renewal times [1]. (d) $Z(t)$ is Markov **iff** interarrival times are exponential (memoryless) [14]. (e) $Z(t)$ is a martingale **iff** jump mean is zero [20] (with integrability). Each of these conclusions is backed by classical theorems and the references given.

References from Provided Texts

1. **Kolmogorov Extension & Finite Distributions:** Garibaldi & Scalas (2010), *Finitary Probabilistic Methods in Econophysics* – Section 4.2, p.73-74, explains that specifying all m-point finite-dimensional distributions (satisfying consistency conditions) is sufficient for the existence of a stochastic process (Kolmogorov's Extension Theorem) [4] [5].

2. **Markov Property Definition:** Lőrinczi (2023), *Lecture Notes on Gaussian Processes*, Definition 1.5, p.5 – A process (X_t) is Markov if $E[f(X_t)\mid \mathcal{F}_s] = E[f(X_t)\mid X_s]$ for all $s<t$ [13]. We used this to formalize when $Z(t)$ is Markov or not.

3. **Poisson Process as Markov (Memoryless):** Norris (1997), *Markov Chains*, Section 2.4 – Defines Poisson process via exponential holding times and notes the **lack of explosion** ($P{T_n\to\infty} =1$) and the **Markov property** due to memorylessness [1] [14]. Theorem 2.4.1 there explicitly proves the Markov property by conditioning on $X_s=i$ events by time s [14] [15].

4. **Transition Kernel (Chapman-Kolmogorov):** Lőrinczi (2023), p.5-6 – Introduces transition kernels $p(s,t,x,A)$ for Markov processes satisfying Chapman-Kolmogorov equations [23] [24]. We implicitly used the idea that for exponential interarrivals, $Z(t)$ has such a kernel with independent increments.

5. **Renewal as Embedded Chain:** Norris (1997), Exercise 1.8.5, p.46 – Shows how a renewal process (with discrete interarrival distribution) can be made into a Markov chain by tracking the time to next event [16]. This supports our argument that $Z(t)$ is Markov only in trivial cases, otherwise one needs to augment the state.

6. **Martingale Definition:** Lőrinczi (2023), Definition 1.8, p.7 – Gives the martingale criteria: (1) adapted, (2) integrable, (3) $E[X_t\mid F_s]=X_s$ [18]. Also Proposition 1.1 (p.7) states $E[X_t]$ is constant for a martingale [21], which we used to argue $E[X_1]$ must be 0.

7. **Compound Poisson Martingale:** (External) Standard result – For a Poisson process $N(t)$ of rate λ, $N(t)-\lambda t$ is a martingale. Likewise, $Z(t)-E[X_1]\,\lambda t$ is a martingale for compound Poisson [22]. This agrees with our condition $E[X_1]=0$ for $Z(t)$ itself to be martingale.

8. **Sample Path Properties:** Norris (1997), p.74-75 – Describes Poisson paths as right-continuous step functions with unit jumps [11] [12]. Our Figure 【37】 illustrates a similar step path for the compound case, with jump sizes X_i.

(The above references support the key steps in our solution. In cases where the provided texts did not explicitly cover a detail (e.g. explicit formula for $P{N(t)=n}$ or general renewal f.d.d.), we filled in with standard theory as needed.)

Optimal Solution Strategy (Step-by-Step)

To solve the exam problems rigorously, we recommend the following approach:

1. **Derive distributions via convolution and Laplace transforms:** Start by computing the distribution of the sum of n interarrivals T_n and the sum of n jumps Z_n. Use convolution formulas or Laplace transforms for clarity [1]. Then relate $N(t)$ and $Z(t)$ to these: express $P{N(t)=n}$ in terms of $F_{T_n}(t)$ [1], and express $F_{Z(t)}(z)$ as a mixture $\sum_n P{N(t)=n}F_{Z_n}(z)$ as we derived. *(This addresses item 1.)*

2. **Write down finite-dimensional laws using the renewal property:** For item 2, articulate how to get joint probabilities for $(Z(t_1),\ldots,Z(t_k))$. Use the fact that $N(t)$ counts events and note the dependency between $Z(t_i)$ and $Z(t_j)$ when $t_i< t_j$. A systematic way is via conditional probabilities or the Kolmogorov extension approach [4] [5]. Ensure to mention that these joint distributions satisfy the necessary consistency conditions [5]. You might not need a closed form for the joint density, but describe the procedure clearly (as we did with $k=2$ case).

3. **Prove existence using Kolmogorov's Theorem:** To tackle item 3, first argue no explosion: e.g. cite a result or argue via $E[J_1]>0 \implies T_n\to\infty$ a.s. [1]. Then mention Kolmogorov's Extension Theorem: since we have defined a collection of consistent f.d.d.'s, there *exists* a process with those distributions [8] [9]. We actually have an *explicit* construction in this renewal setting, so existence is assured. Next, describe the sample paths: *càdlàg* with jumps of size X_n at times T_n. Refer to standard theory or texts (e.g. Norris's description of Poisson paths) to support this [11]. You can draw a small diagram of a step function if helpful. Emphasize right-continuity and piecewise constancy.

4. **Determine Markov property conditions:** For item 4, state the Markov criterion [13] and explain why $Z(t)$ fails it unless J_i are exponential. Reference the **memoryless property**: Norris's Theorem 2.4.1 proves that a Poisson (memoryless) process has the Markov property [14]. Thus, argue: *if J is exponential*, $N(t)$ (hence $Z(t)$) has independent increments ⇒ Markov; *if not*, show a counterexample (the process "remembers" the last interval length). You can mention the embedded chain technique [16] to highlight the need for an extra state variable (time since last

event) in the non-exponential case. Conclude that **$Z(t)$ is Markov iff interarrival is exponential**.

5. **Check martingale criteria using Doob's definition:** For item 5, quote the martingale definition [18]. Compute $E[Z(t)\mid F_s]$ given F_s. Show that equals $Z(s)$ exactly when $E[X_1]=0$. Use the argument that future jumps contribute zero mean if $E[X]=0$, or simply cite that a martingale's expectation is constant [21] while $E[Z(t)] = E[N(t)]E[X_1]$ grows unless $E[X_1]=0$. Thus, give the condition $E[X_1]=0$ for $Z(t)$ to be a martingale. For completeness, mention integrability (we assume $E|X_1|<\infty$ so that everything is well-defined). Optionally, mention that when $E[X_1]\neq0$, one can define the compensated process $Z(t)-E[X_1]\,m(t)$ to restore the martingale property (this shows understanding of Doob decomposition, though it's not strictly required).

Throughout your solution, **cite relevant theorems and definitions**: e.g. *"by Kolmogorov's extension theorem, p.74 of Garibaldi–Scalas [8] [9], the process exists given the f.d.d."*, or *"using Norris's result on Poisson processes (Theorem 2.4.1) [14], we conclude exponential waiting times make $Z(t)$ Markov"*. This not only strengthens your arguments but shows you have consulted the literature as required. If any gap remains (e.g. a proof that $\sum J_i = \infty$ a.s. when $E[J]=\infty$), acknowledge it and note it could be filled by a reference to advanced probability theory (Kolmogorov's 0-1 law or Borel-Cantelli).

By following these steps, you will have addressed all aspects (a)–(d) systematically with a balance of classical theory (Kolmogorov, Doob) and modern perspective (regenerative process, compensators, etc.), as requested.

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[1] [2] [3] [10] [11] [12] [14] [15] [16] [17] J. R. Norris - Markov Chains (1998, Cambridge University Press) - libgen.li.pdf
file://file-FrRcAA5T4zauZyXP9M92tY

[4] [5] [6] [7] [8] [9] Finitary Probabilistic Methods in Econophysics (Ubaldo Garibaldi, Enrico Scalas) (Z-Library).pdf
file://file-SuEriMg61TCnvaiDUk4b3F

[13] [18] [19] [20] [21] [23] [24] Lecture Notes on Gaussian Processes with Examples.pdf
file://file-PbtbDAbR4txSUpcayNDrLo

[22] [PDF] Stochastic Calculus for Jump Processes
https://personal.ntu.edu.sg/nprivault/MA5182/stochastic-calculus-jump-processes.pdf