Exam sheet

Federico Trionfetti 1991289

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1 Question 1 (Convergence in distribution)

Prove that, if X_n is uniform on $[0, 1/n]$, then the sequence $(X_n)_{n>0}$ converges in distribution (to what random variable)?

1. Direct proof

Here's a direct proof that

$$X_n \sim \text{Unif}[0, 1/n]$$

converges in distribution to the constant random variable $X \equiv 0$.

1. Identify the candidate limit

Let X be the degenerate (constant) random variable

$$X(\omega) = 0 \quad \text{for all } \omega.$$

Its distribution function is

$$F_X(x) = P(X \leq x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

2. Write down the CDF of X_n

For each n, the CDF of X_n is

$$F_{X_n}(x) = P\big(X_n \leq x\big) = \begin{cases} 0, & x < 0, \\ nx, & 0 \leq x \leq \frac{1}{n}, \\ 1, & x > \frac{1}{n}. \end{cases}$$

3. Take the pointwise limit

Fix any real x at which F_X is continuous (i.e. any $x \neq 0$). Then:

If $x < 0$, then for all n, $F_{X_n}(x) = 0$, so

$$\lim_{n\to\infty} F_{X_n}(x) = 0 = F_X(x).$$

If $x > 0$, then for sufficiently large n, $\frac{1}{n} < x$, so $F_{X_n}(x) = 1$. Hence

$$\lim_{n\to\infty} F_{X_n}(x) = 1 = F_X(x).$$

By the definition of convergence in distribution, this exactly means

$$X_n \xrightarrow{d} X \equiv 0.$$

Recall that convergence in distribution means

$$\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$$

at every continuity point of F_X .

2. Alternate viewpoint via convergence in probability

We recall from **Theorem 1.1** the relationships between various modes of convergence.

In particular, by part (2) of Theorem 1.1, convergence in probability implies convergence in distribution.

One also checks easily that for every $\varepsilon > 0$,

$$\mathbb{P}\big(|X_n - 0| \geq \varepsilon\big) = \mathbb{P}(X_n \geq \varepsilon) = \begin{cases} 1 - n\varepsilon, & \varepsilon \leq \frac{1}{n}, \\ 0, & \varepsilon > \frac{1}{n}, \end{cases}$$

which tends to 0 as $n \to \infty$. Thus, $X_n \to 0$ in probability, and by Theorem 1.1(2), this implies that $X_n \to 0$ in distribution.

3. Graphical intuition

Below are two plots showing, for $n = 1, 2, 5, 10$, the behavior of the uniform distribution on $[0, 1/n]$:

- **Density** $f_n(x)$ (see Figure 1 Panel A).
- **CDF** $F_n(x)$ (see Figure 1 Panel B).

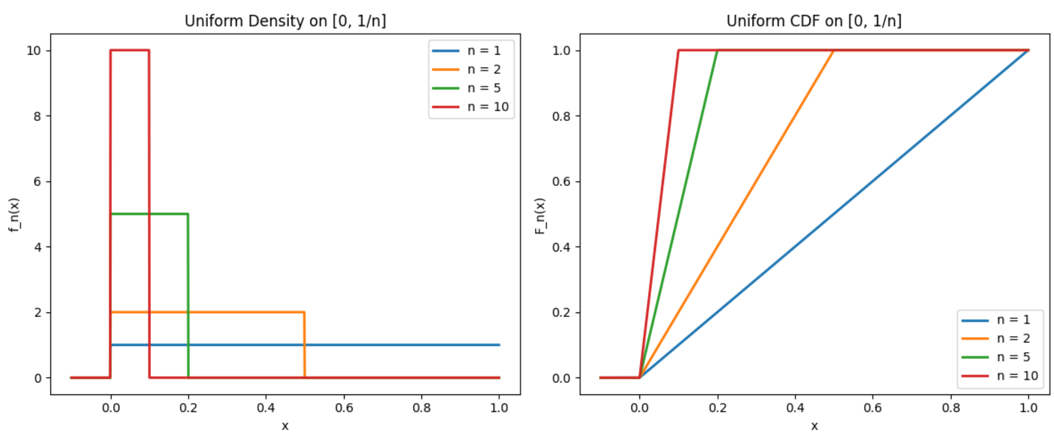

Figure 1: Panel A: Uniform density $f_n(x)$ on $[0, 1/n]$ for $n = 1, 2, 5, 10$. Panel B: Uniform CDF $F_n(x)$ on $[0, 1/n]$ for $n = 1, 2, 5, 10$.

As n increases, the density "spikes" near zero over a shrinking interval, and the CDF "jumps" to 1 ever more quickly past 0, illustrating convergence of $F_n(x)$ to the step function at 0.

Question 2 (Markov chains)

Write a 4 state transition probability matrix with a transient state and three recurrent states. What can you say of the time to absorption if you start from the transient state?

1. Example Transition Matrix

A canonical example of a Markov chain with both transient and recurrent states involves one or more states that lead to, but cannot be reached from, closed sets of states. An absorbing state is the simplest form of a closed, recurrent class.

Consider the state space $E = \{1, 2, 3, 4\}$ and the transition probability matrix P:

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.3 & 0.1 & 0.1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This matrix has one transient state (State 1) and three recurrent, absorbing states (States 2, 3, and 4).

Figure 2 shows the corresponding transition diagram.

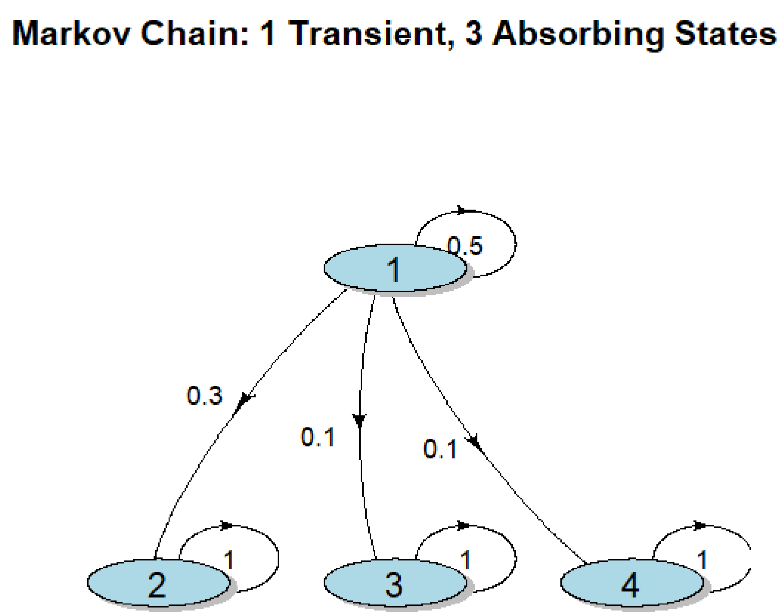

Figure 2: Transition diagram of the four-state Markov chain

2. Classification of States

We can formally classify each state by analyzing its long-term behavior.

States 2, 3, and 4: Recurrent

States 2, 3, and 4 are **absorbing states**, as $p_{ii} = 1$ for $i \in \{2, 3, 4\}$. An absorbing state is a special case of a recurrent state.

We can prove their recurrence using two key theorems.

1. Each state forms its own communicating class: $\{2\}$, $\{3\}$, and $\{4\}$.
2. Each of these classes is **closed**, as it is impossible to leave the class once entered (the probability of leaving is 0).
3. According to Theorem 1.2, every finite closed communicating class is recurrent.

Since each of the classes $\{2\}$, $\{3\}$, and $\{4\}$ is finite and closed, they are all recurrent classes.

State 1: Transient

State 1 is **transient**. We can demonstrate this in several ways.

1. **Using Return Probability (f_i):** According to Theorem 1.3, a state i is transient if and only if $f_i < 1$, where f_i is the probability of ever returning to state i after starting there. From state 1, the chain can transition to states 2, 3, or 4 with a combined probability of $0.3 + 0.1 + 0.1 = 0.5$. Since these three states are absorbing, if the chain enters any of them, it will never return to state 1. Thus, there is a non-zero probability of never returning to state 1, which means $f_1 < 1$.
2. **Using Sum of Probabilities:** According to Theorem 1.4, a state i is transient if and only if the expected number of visits is finite, which is equivalent to $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$. For state 1, the probability of being in state 1 after n steps is $p_{11}^{(n)} = (0.5)^n$. The sum is a convergent geometric series:
$$\sum_{n=1}^{\infty} p_{11}^{(n)} \;=\; \sum_{n=1}^{\infty} (0.5)^n \;=\; \frac{0.5}{1-0.5} \;=\; 1$$
Since the sum is finite ($1 < \infty$), state 1 is transient.
3. **Using Class Properties:** For a finite Markov chain, a communicating class is transient if and only if it is not closed Theorem 1.5. The class $\{1\}$ is not closed because it is possible to leave it (e.g., via the transition $1 \to 2$). Therefore, the class $\{1\}$ is transient.

3. Analysis of Time to Absorption

When a chain starts in a transient state and has accessible recurrent states, it is guaranteed to be absorbed into one of the recurrent states.

Distribution of Absorption Time

Let τ be the random variable representing the number of steps until the chain is absorbed (i.e., leaves state 1 for the first time). At each step, as long as the chain is in state 1, it either stays in state 1 with probability $p_{11} = 0.5$ or it gets absorbed into states $\{2, 3, 4\}$ with probability $1 - p_{11} = 0.5$.

This scenario describes a sequence of Bernoulli trials, meaning the time to the first "success" (absorption) follows a **Geometric distribution**. The probability that absorption occurs at exactly step k (for $k = 1, 2, \dots$) is:

$$\Pr(\tau = k) \;=\; (p_{11})^{k-1}(1 - p_{11}) \;=\; (0.5)^{k-1}(0.5) \;=\; (0.5)^k$$

Expected Time to Absorption

To find the mean time to absorption, we partition the matrix P into transient (T) and recurrent (R) states. Let the set of transient states be $T = \{1\}$ and recurrent states be $R = \{2, 3, 4\}$.

$$P = \begin{pmatrix} Q & R \\ \mathbf{0} & I \end{pmatrix}$$

Here, Q is the submatrix of transitions between transient states, and R is the submatrix of transitions from transient to recurrent states.

$$Q = [0.5] \qquad R_{\text{matrix}} = \begin{bmatrix} 0.3 & 0.1 & 0.1 \end{bmatrix}$$

The fundamental matrix, $N = (I - Q)^{-1}$, gives the expected number of times the process is in each transient state.

$$N \ = \ (I - Q)^{-1} \ = \ (1 - 0.5)^{-1} \ = \ (0.5)^{-1} \ = \ [2]$$

The expected time to absorption, $\mathbf{t}$, starting from transient state i is the sum of the entries in row i of N. Here, we have only one transient state:

$$\mathbf{t} \ = \ N\mathbf{1} \ = \ 2 \times 1 \ = \ 2 \text{ steps}$$

The expected time to absorption is 2 steps.
This result is consistent with the mean of the Geometric distribution we identified:
$\mathbb{E}[\tau] = \frac{1}{1-p_{11}} = \frac{1}{0.5} = 2.$

4. Absorption Probabilities

We can also determine the probability of being absorbed into each *specific* recurrent state. This is given by the matrix $B = NR_{\text{matrix}}$.

$$B \ = \ NR_{\text{matrix}} \ = \ [2] \begin{bmatrix} 0.3 & 0.1 & 0.1 \end{bmatrix} \ = \ \begin{bmatrix} 0.6 & 0.2 & 0.2 \end{bmatrix}$$

The entries of B give the probabilities of absorption into states 2, 3, and 4, respectively, starting from state 1.

- Probability of being absorbed into State 2 is **0.6**.
- Probability of being absorbed into State 3 is **0.2**.
- Probability of being absorbed into State 4 is **0.2**.

The sum of these probabilities is $0.6 + 0.2 + 0.2 = 1.0$, confirming that absorption into one of these states is certain.

Monte Carlo Simulation Verification in R

To corroborate our analytical findings, we conducted a Monte Carlo simulation in R. Using a fixed seed for reproducibility, we implemented a function `simulate_chain` that initializes each trial in state 1 and, at each discrete time step, draws the next state according to the transition probabilities in row 1 of P. The simulation halts as soon as the chain enters one of the absorbing states 2, 3, or 4.

We performed $n = 10\,000$ independent replications, recording for each run both (i) the number of steps elapsed until absorption and (ii) the absorbing state reached. The resulting empirical mean absorption time was

$$\widehat{t}_{\text{abs}} \ = \ 2.002 \text{ steps},$$

in excellent agreement with the theoretical expectation $t_{\text{abs}} = 2$. Likewise, the observed proportions of absorption into states 2, 3, and 4 were

$$\widehat{p}_2 = 0.6066, \quad \widehat{p}_3 = 0.1975, \quad \widehat{p}_4 = 0.1959,$$

which closely mirror the predicted probabilities $(0.6, 0.2, 0.2)$. For the complete R code, see Listing 2 in the Appendix.

Question 3 (Martingales)

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with 0 expected value and variance $\mathrm{Var}(X_i) = \sigma_i^2 < \infty$. Define

$$S_n = \sum_{i=1}^{n} X_i$$

and

$$T_n^2 = \sum_{i=1}^{n} \sigma_i^2.$$

Is

$$(S_n^2 - T_n^2)_{n>0}$$

a martingale?

Answer

Yes, the process $(M_n)_{n>0}$ is a martingale with respect to the natural filtration.

Let the natural filtration be defined as $F_n = \sigma(X_1, \ldots, X_n)$. The process

$$M_n = S_n^2 - T_n^2$$

is a martingale with respect to $(F_n)_{n\geq 1}$.

Proof. To prove this, we will verify the three conditions for a process to be a martingale as specified in Definition 1.6. A process $(M_n)_{n\geq 1}$ is a martingale if:

1. $(M_n)_{n\geq 1}$ is $(F_n)_{n\geq 1}$-adapted.
2. $E[|M_n|] < \infty$ for all $n \geq 1$ (Integrability).
3. $E[M_n|F_{n-1}] = M_{n-1}$ for each $n > 1$ (The Martingale Property).

1. Adaptedness

The process M_n must be F_n-measurable for every $n \geq 1$.

- The filtration $F_n = \sigma(X_1, \ldots, X_n)$ is the information generated by the first n random variables.
- $S_n = \sum_{i=1}^{n} X_i$ is a sum of F_n-measurable random variables, so S_n is also F_n-measurable.
- A Borel measurable function of an F_n-measurable variable is also F_n-measurable. Since $f(x) = x^2$ is a continuous (and thus Borel-measurable) function, S_n^2 is F_n-measurable.
- $T_n^2 = \sum_{i=1}^{n} \sigma_i^2$ is a deterministic constant for each n. A constant is measurable with respect to any σ-algebra.
- The difference of two F_n-measurable variables, $M_n = S_n^2 - T_n^2$, is therefore also F_n-measurable.

Thus, the process $(M_n)_{n\geq 1}$ is adapted to the filtration $(F_n)_{n\geq 1}$.

2. Integrability

We must show that $E[|M_n|] < \infty$ for all $n \geq 1$. By the triangle inequality, it suffices to show that $E[|S_n^2|] < \infty$ and $E[|T_n^2|] < \infty$.

- T_n^2 is a deterministic, finite value for each n, so $E[|T_n^2|] = T_n^2 < \infty$.
- Since $S_n^2 \geq 0$, we have $|S_n^2| = S_n^2$. We check its expectation. From the definition of variance, $Var(Y) = E[Y^2] - (E[Y])^2$.
- First, we find the mean of S_n:

$$E[S_n] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 0 = 0$$

- Next, we find the variance of S_n. Since the $\{X_i\}_{i=1}^{\infty}$ are independent, their covariances are zero. Thus, the variance of the sum is the sum of the variances:

$$Var(S_n) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \sigma_i^2 = T_n^2$$

- Therefore, the expectation of S_n^2 is:

$$E[S_n^2] = Var(S_n) + (E[S_n])^2 = T_n^2 + 0^2 = T_n^2$$

  Since each $\sigma_i^2 < \infty$, the finite sum T_n^2 is finite.

Both components are integrable, which means M_n is integrable.

3. The Martingale Property

We must show that $E[M_n|F_{n-1}] = M_{n-1}$ for $n > 1$. We begin by rewriting M_n in terms of quantities from time $n-1$:

$$\begin{aligned}
M_n &= S_n^2 - T_n^2 \\
&= (S_{n-1} + X_n)^2 - (T_{n-1}^2 + \sigma_n^2) \\
&= S_{n-1}^2 + 2S_{n-1}X_n + X_n^2 - T_{n-1}^2 - \sigma_n^2 \\
&= (S_{n-1}^2 - T_{n-1}^2) + 2S_{n-1}X_n + (X_n^2 - \sigma_n^2) \\
&= M_{n-1} + 2S_{n-1}X_n + X_n^2 - \sigma_n^2
\end{aligned}$$

Now, we take the conditional expectation with respect to F_{n-1}, applying the properties listed in Theorem 1.7.

$$E[M_n|F_{n-1}] = E[M_{n-1} + 2S_{n-1}X_n + X_n^2 - \sigma_n^2|F_{n-1}]$$

Using linearity of conditional expectation (Property 6 in Theorem 1.7), we get:

$$E[M_n|F_{n-1}] = E[M_{n-1}|F_{n-1}] + E[2S_{n-1}X_n|F_{n-1}] + E[X_n^2 - \sigma_n^2|F_{n-1}]$$

Let's evaluate each term:

- Because M_{n-1} is F_{n-1}-measurable, $E[M_{n-1}|F_{n-1}] = M_{n-1}$ (Property 2 in Theorem 1.7).
- For the second term, S_{n-1} is F_{n-1}-measurable, so we can take it out of the expectation (Property 3 in Theorem 1.7). The variable X_n is independent of F_{n-1}, so its conditional expectation is its unconditional expectation (Property 4 in Theorem 1.7).

$$E[2S_{n-1}X_n|F_{n-1}] = 2S_{n-1}E[X_n|F_{n-1}] = 2S_{n-1}E[X_n] = 2S_{n-1} \cdot 0 = 0$$

- For the third term, X_n^2 is independent of F_{n-1}, and σ_n^2 is a constant.

$$E[X_n^2 - \sigma_n^2 | F_{n-1}] = E[X_n^2 | F_{n-1}] - \sigma_n^2 = E[X_n^2] - \sigma_n^2$$

  We know $E[X_n^2] = Var(X_n) = \sigma_n^2$. Thus:

$$E[X_n^2 - \sigma_n^2 | F_{n-1}] = \sigma_n^2 - \sigma_n^2 = 0$$

Combining these results, we find:

$$E[M_n | F_{n-1}] = M_{n-1} + 0 + 0 = M_{n-1}$$

The martingale property holds. Since all three conditions are satisfied, $(M_n)_{n>0}$ is a martingale. □

Graphical Simulation

To illustrate the martingale property of

$$M_n = S_n^2 - T_n^2,$$

we simulated ten independent sample paths of length $n = 500$, with each increment $X_i \sim N(0, 1)$. Figure 3 shows the resulting trajectories of M_n over time.

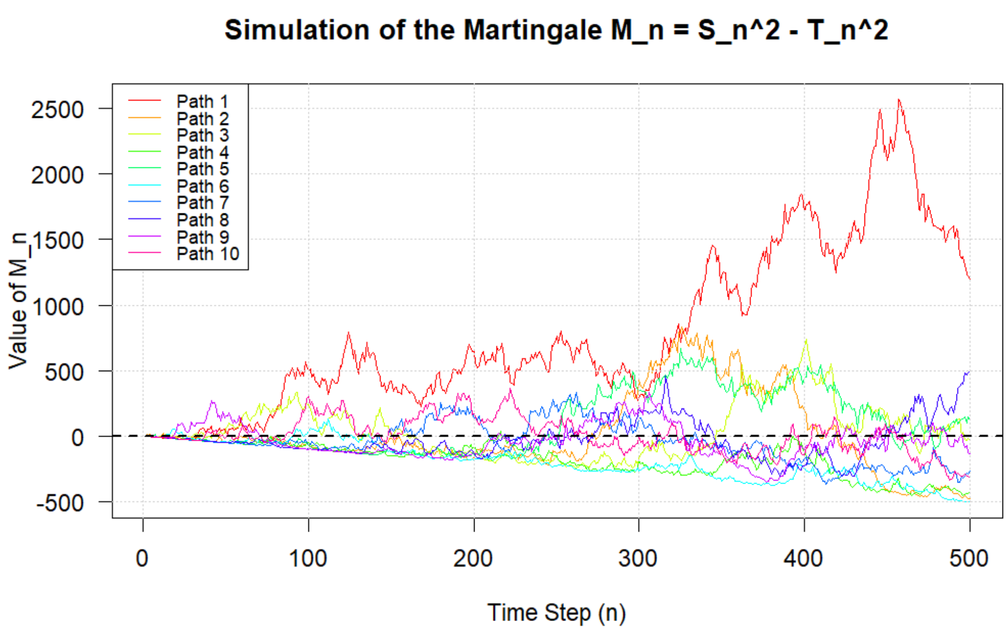

Figure 3: The output is a plot visualizing the martingale paths.

As expected, although individual paths wander above and below zero, there is no systematic drift: the ensemble average remains at 0 for all n, confirming that $(M_n)_{n\geq 1}$ is indeed a martingale.

The horizontal dashed line at 0 highlights that, despite the increasing variance of the underlying sum S_n, the process $S_n^2 - T_n^2$ fluctuates symmetrically around its constant mean. (see Listing 3 in the Appendix).

Appendix:

Supporting Materials Answer 1

1.1 Theorem

Theorem 1.1 (Theorem 1.2, Lecture Notes on Gaussian Processes, Lőrinczi, p. 4)**. *Theorem 1.2*** *Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables and X another real-valued random variable, all on $(\Omega, \mathcal{F}, \mathbb{P})$. The following implications hold:*

1. *If $X_n \xrightarrow{a.s.} X$ or $X_n \xrightarrow{L^p} X$ for some $p \geq 1$, then $X_n \xrightarrow{P} X$ as $n \to \infty$.*
2. *If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{d} X$ as $n \to \infty$.*
3. *If $X_n \xrightarrow{P} X$, then there exists a subsequence $(X_{n_k})_{k\geq 1}$ such that $X_{n_k} \xrightarrow{a.s.} X$ as $k \to \infty$.*
4. *If $X_n \xrightarrow{P} X$ and $(X_n)_{n\geq 1}$ is uniformly integrable, i.e.,*

$$\lim_{N\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}\left[|X_n| \cdot \mathbf{1}_{\{|X_n|\geq N\}}\right] = 0,$$

then $X_n \xrightarrow{L^1} X$ as $n \to \infty$.

1.2 Python Code for Simulation

```
import numpy as np
import matplotlib.pyplot as plt

# Define the values of n to compare
ns = [1, 2, 5, 10]

# Create a sequence of x values
x = np.linspace(-0.1, 1, 1000)

# FIGURE 1: Density functions f_n(x)
plt.figure()
for n in ns:
    f_n = np.where((x >= 0) & (x <= 1/n), n, 0)
    plt.plot(x, f_n, label=f"n = {n}", linewidth=2)
plt.title("Uniform Density on [0, 1/n]")
plt.xlabel("x")
plt.ylabel("f_n(x)")
plt.legend()
plt.show()

# FIGURE 2: CDF functions F_n(x)
plt.figure()
for n in ns:
    F_n = np.where(x < 0, 0, np.where(x <= 1/n, n * x, 1))
    plt.plot(x, F_n, label=f"n = {n}", linewidth=2)
plt.title("Uniform CDF on [0, 1/n]")
plt.xlabel("x")
plt.ylabel("F_n(x)")
plt.legend()
plt.show()
```

Supporting Materials Answer 2

1.3 Theorems

Theorem 1.2 (Theorem 2.7.10, Skript 2B, p. 3). *Every finite closed communicating class is recurrent.*

Theorem 1.3 (Theorem 2.7.3, Skript 2B, p. 1). *Let $i \in E$ be a state. Denote by f_i the probability that a Markov chain which starts at i returns to i at least once, then (1) The state i is recurrent if and only if $f_i = 1$. (2) The state i is transient if and only if $f_i < 1$.*

Theorem 1.4 (Theorem 2.7.5., Skript 2B, p. 1). *Let $i \in E$ be a state, then*

(1) The state i is recurrent if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$

(2) The state i is transient if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.

Theorem 1.5 (Finite State Chain Equivalency, Skript 2B, p. 3). *In a Markov chain with finitely many states we have the following equivalencies*

(1) A communicating class is recurrent if and only if it is closed.

(2) A communicating class is transient if and only if it is not closed.

R Code for Transition Matrix and State Diagram

Listing 1: R code to define the transition matrix and plot the Markov chain diagram.

```
# Define transition matrix
P <- matrix(c(
  0.5, 0.3, 0.1, 0.1,
  0.0, 1.0, 0.0, 0.0,
  0.0, 0.0, 1.0, 0.0,
  0.0, 0.0, 0.0, 1.0
), byrow = TRUE, nrow = 4)

states <- c("1", "2", "3", "4")

# Plot diagram (requires the 'diagram' package)
library(diagram)
plotmat(t(P),
        pos = c(1, 3),
        name = states,
        lwd = 1,
        box.size = 0.1,
        box.type = "circle",
        box.prop = 0.5,
        box.col = "lightblue",
        self.cex = 0.7,
        arr.length = 0.3,
        arr.width = 0.1,
        cex.txt = 0.8,
        main = "Markov Chain: 1 Transient, 3 Absorbing States")
```

1.4 R Code for Simulation Answer 2

Listing 2: R code to simulate absorption times and empirical absorption probabilities.

```
# 1. Define transition matrix P
```

```r
P <- matrix(c(
  0.5, 0.3, 0.1, 0.1,
  0.0, 1.0, 0.0, 0.0,
  0.0, 0.0, 1.0, 0.0,
  0.0, 0.0, 0.0, 1.0
), nrow = 4, byrow = TRUE)
rownames(P) <- colnames(P) <- 1:4

# 2. Helper function to simulate one chain until absorption
simulate_chain <- function(P, start = 1, max_steps = 100) {
  state <- start
  for (t in 1:max_steps) {
    state <- sample(1:4, 1, prob = P[state, ])
    if (state %in% 2:4) return(list(absorbed_in = state, steps = t))
  }
  return(list(absorbed_in = NA, steps = NA))
}

# 3. Monte Carlo simulation parameters
set.seed(12345)
n_sims <- 10000

# 4. Run simulations
res <- replicate(n_sims, simulate_chain(P), simplify = FALSE)
steps    <- sapply(res, `[[`, "steps")
absorbed <- factor(sapply(res, `[[`, "absorbed_in"), levels = 2:4)

# 5. Empirical results
cat("Empirical mean time to absorption:", round(mean(steps), 3), "steps\n")
cat("Empirical absorption probabilities:\n")
print(prop.table(table(absorbed)))
```

Supporting Materials Answer 3

1.5 Theorems and Definitions

Definition 1.6 (Definition 1.8, Lecture Notes On Gaussian Processes with Examples, Lőrinczi, p. 7)**.** *Let $(\Omega, \mathcal{F}, P)$ be a filtered space with given filtration $(\mathcal{F}_t)_{t\geq 0}$. The random process $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale whenever*

1. *$(X_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted*
2. *$E_P[|X_t|] < \infty$ for all $t \geq 0$*
3. *$E_P[X_t|\mathcal{F}_s] = X_s$ for each $s \leq t$.*

A martingale describes the model of a fair game, implying that the best prediction of the future net winnings/losses per unit stake over the interval $[s,t]$ is zero. Indeed,

$$E[X_t - X_s|\mathcal{F}_s] = E[X_t|\mathcal{F}_s] - E[X_s|\mathcal{F}_s] = E[X_t|\mathcal{F}_s] - X_s = 0.$$

The following result offers a simple method to disprove that a random process is a martingale.

Theorem 1.7 (Theorem 1.6,SKRIPT, Lőrinczi, p. 7)**.** *Let X be a random variable on (Ω, F, P), and $\mathcal{G} \subset F$ be a sub-σ-field. The following properties of conditional expectation hold:*

1. $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.

2. *If X is $\mathcal{G}$-measurable, then* $\mathbb{E}[X|\mathcal{G}] = X$ *a.s.*

3. *If Y is $\mathcal{G}$-measurable and bounded, then* $\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$ *a.s.*

4. *If X is independent of $\mathcal{G}$, then* $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$ *a.s.*

5. ***Tower property:*** *If $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, then* $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$ *a.s.*

6. ***Linearity:*** $\mathbb{E}[\alpha X + \beta Y|\mathcal{G}] = \alpha\mathbb{E}[X|\mathcal{G}] + \beta\mathbb{E}[Y|\mathcal{G}]$ *a.s., for all random variables X, Y and $\alpha, \beta \in \mathbb{R}$.*

7. ***Monotonicity:*** *If $X \leq Y$ a.s., then* $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$ *a.s.*

8. ***Conditional Jensen's Inequality:*** *If φ is convex and $\mathbb{E}[|X|], \mathbb{E}[|\varphi(X)|] < \infty$, then*

$$\varphi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\varphi(X)|\mathcal{G}] \quad a.s.$$

9. ***Fatou's Lemma:*** *Let $(X_n)_{n\geq 1}$ be a sequence of non-negative random variables. Then*

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n \,\middle|\, \mathcal{G}\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}] \quad a.s.$$

1.6 R Code for Simulation Answer 3

Listing 3: R code to simulate and plot the martingale paths.

```
# Simulation Parameters
set.seed(12345)
num_steps <- 500        # Number of steps in each path (n)
num_paths <- 10         # Number of martingale paths to simulate

# Initialize a matrix to store the results
# Each column will be a separate path
martingale_paths <- matrix(0, nrow = num_steps, ncol = num_paths)

# --- Simulation Loop ---
for (i in 1:num_paths) {
  # 1. Generate independent random variables
  # For simplicity, we use standard normal variables, X_i ~ N(0, 1).
  # So, E[X_i] = 0 and Var(X_i) = sigma_i^2 = 1.
  X <- rnorm(num_steps, mean = 0, sd = 1)

  # 2. Calculate the partial sums S_n
  S <- cumsum(X)

  # 3. Calculate the sum of variances T_n^2
  # Since sigma_i^2 = 1 for all i, T_n^2 is just n.
  T2 <- 1:num_steps

  # 4. Calculate the martingale M_n = S_n^2 - T_n^2
  M <- S^2 - T2

  # 5. Store the path
  martingale_paths[, i] <- M
}
```

```
# —— Plotting the Results ——
path_colors <- rainbow(num_paths)

matplot(1:num_steps, martingale_paths, type = 'l', lty = 1,
        col = path_colors,
        main = "Simulation of the Martingale M_n = S_n^2 - T_n^2",
        xlab = "Time Step (n)",
        ylab = "Value of M_n",
        las = 1)
grid()

# Add a horizontal line at y=0, which is the expected value of the martingale
abline(h = 0, col = "black", lwd = 2, lty = 2)

legend("topleft", legend = paste("Path", 1:num_paths),
       col = path_colors, lty = 1, cex = 0.8, bg="white")
```