SUPERCONGRUENCES VIA MODULAR FORMS

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Abstract. We prove two supercongruences for the coefficients of power series expansions in t of modular forms where t is a modular function. As a result, we settle two recent conjectures of Chan, Cooper and Sica. Additionally, we provide a table of supercongruences for numbers which appear in similar power series expansions and in the study of integral solutions of Apéry-like differential equations.

1. Introduction

In [7], the authors investigate sequences of integers that satisfy congruence properties similar to those of the Apéry numbers associated with the irrationality of $\zeta(3)$. They also conjecture seven congruences and supercongruences for coefficients of power series expansions in t of modular forms where t is a modular function. The term supercongruences appeared in [3] and was the subject of the Ph.D. thesis of Coster [10]. It originally referred to families of congruences that are stronger than ones suggested by formal group theory, but now includes individual congruences (see [1]). Let

$$f(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{m^2+mn+6n^2}, \quad t_1 = t_1(z) = \frac{\eta(z)\eta(23z)}{f}$$

and

$$F(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{2m^2+mn+3n^2}, \quad t_2 = t_2(z) = \frac{\eta(z)\eta(23z)}{F}.$$

where $\eta(z)$ is the Dedekind eta-function, $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. Write

$$f = f(z) = \sum_{n=0}^{\infty} f_n t_1^n \quad \text{and} \quad F = F(z) = \sum_{n=0}^{\infty} F_n t_2^n.$$

In [7], the authors make the following

Conjecture 1.1. *If p is a prime with $\left(\frac{p}{23}\right) = 1$ and $n \geq 1$, then*

$$f_{np} \equiv f_n \pmod{p}$$

and

$$F_{np} \equiv F_n \pmod{p}.$$

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The first few terms in the sequence $\{f_n\}_{n\geq 1}$ are

$$1,\ 2,\ 110,\ 1210,\ 14652,\ 188364,\ 2523444,\ 34842060,\ 492322160,\ \dots$$

while for $\{F_n\}_{n\geq 1}$, we have

$$1,\ 2,\ 26,\ 142,\ 876,\ 5790,\ 40020,\ 285582,\ 2087612,\ \dots.$$

Closed forms for f_n and F_n were not given in [7] and thus a combinatorial approach to Conjecture 1.1 is not yet available. The purpose of this note is to prove this conjecture via modular forms. We have the following.

Theorem 1.2. *If p is a prime with $\left(\frac{p}{23}\right) = 1$ and n, $r \geq 1$ are integers, then*

$$f_{np^r} \equiv f_{np^{r-1}} \pmod{p^r} \tag{1}$$

and

$$F_{np^r} \equiv F_{np^{r-1}} \pmod{p^r}. \tag{2}$$

In Section 2, we recall some preliminaries on power series expansions and Eisenstein series with characters and then prove Theorem 1.2. In Section 3, we provide a table of supercongruences for numbers appearing in other power series expansions found in [7] and in the study of integral solutions of Apéry-like differential equations (see [2], [5], [17]). Finally, we note that two other conjectural congruences from [7] which involve $f_{2,n}$ and $f_{3,n}$ (see Section 3) have recently been proven in [8].

2. Proof of Theorem 1.2

We first recall a recent result of Jarvis and Verrill (see Proposition 4.2 in [12] or Proposition 3 in [4]). This result is quite useful as it allows one to deduce congruence properties of coefficients in a power series expansion from those of another expansion.

Proposition 2.1. *Let t be a power series*

$$t = \frac{1}{m}\sum_{n=1}^{\infty} a_n u^{n/v},$$

convergent in a neighborhood of $u = 0$, with m, v positive integers, $a_n \in \mathbb{Z}$ and $a_1 = 1$. Suppose that in some neighborhood of $u = 0$ we have an equality of convergent power series given by

$$\sum_{n=1}^{\infty} b_n t^{n-1}\, dt = \sum_{n=1}^{\infty} c_n u^{n-1}\, du,$$

for some integers b_n and c_n, $n \geq 1$. Assume p is a prime not dividing m or v. If

$$c_{np^r} \equiv c_{np^{r-1}} \pmod{p^r},$$

then

$$b_{np^r} \equiv b_{np^{r-1}} \pmod{p^r}.$$

We now discuss the notion of Eisenstein series with characters. For further details, see Chapter 7 of [13]. Let $M_k(\Gamma_0(N),\epsilon)$ be the space of modular forms of weight k on $\Gamma_0(N)$ with character ϵ. Suppose χ and ψ are primitive Dirichlet characters modulo L and R, respectively. Let

$$E_{k,\chi,\psi}(q) := c_0 + \sum_{n=1}^{\infty}\Big(\sum_{d|n}\psi(d)\chi(n/d)d^{k-1}\Big)q^n \tag{3}$$

where

$$c_0 = \begin{cases} -\dfrac{B_{k,\chi}}{2k} & \text{if} \quad L=1, \\ \quad 0 & \text{otherwise.} \end{cases}$$

If t is a positive integer and $k \geq 3$ is an integer such that $\chi(-1)\psi(-1) = (-1)^k$, then $E_{k,\chi,\psi}(q^t)$ is in $M_k(\Gamma_0(RLt),\chi\psi)$. Moreover, given N and ϵ, the series $E_{k,\chi,\psi}(q^t)$ such that $RLt \mid N$ and $\chi\psi = \epsilon$ form a basis for the Eisenstein subspace $E_k(\Gamma_0(N),\epsilon)$ of $M_k(\Gamma_0(N),\epsilon)$.

Proof of Theorem 1.2. Let χ be the character $\left(\frac{\cdot}{23}\right)$ and ψ be the trivial character 1. We first note that

$$E_{3,\chi,1}(q) =: \sum_{n=1}^{\infty} a_n q^n$$

and

$$E_{3,1,\chi}(q) =: \sum_{n=1}^{\infty} e_n q^n$$

form a basis for the space $E_3\Big(\Gamma_0(23),\left(\frac{\cdot}{23}\right)\Big)$. By Lemma 0.3 in [16] and a finite computation, we have

$$f\frac{q\frac{dt_1}{dq}}{t_1} = F\frac{q\frac{dt_2}{dq}}{t_2} = -\frac{1}{24}E_{3,1,\chi}(q) - \frac{23}{24}E_{3,\chi,1}(q)$$

and so

$$f\frac{dt_1}{t_1} = F\frac{dt_2}{t_2} = \left[-\frac{1}{24}E_{3,1,\chi}(q) - \frac{23}{24}E_{3,\chi,1}(q)\right]\frac{dq}{q}. \tag{4}$$

By (3), we have

$$a_{np^r} - \left(\frac{p}{23}\right)a_{np^{r-1}} = \sum_{d'|n}\left(\frac{n/d'}{23}\right)(d'p^r)^2, \tag{5}$$

$$e_{np^r} - e_{np^{r-1}} = \sum_{d'|n}\left(\frac{d'p^r}{23}\right)(d'p^r)^2, \tag{6}$$

$e_n = a_n$ if $\left(\frac{n}{23}\right) = 1$ and $e_n = -a_n$ if $\left(\frac{n}{23}\right) = -1$. Letting $u = q$ in (4) implies that

$$\sum_{n=1}^{\infty} f_n t_1^{n-1}\, dt = \sum_{n=1}^{\infty} F_n t_2^{n-1}\, dt = \left[-\frac{1}{24}E_{3,\chi,1}(u) - \frac{23}{24}E_{3,1,\chi}(u)\right]\frac{du}{u}. \tag{7}$$

If we take $v = 1$, $m = 1$, $b_n = f_n$, F_n, respectively, and $c_n = -\frac{1}{24}a_n - \frac{23}{24}e_n$, then by (5), (6) and (7), we have

$$c_{np^r} \equiv c_{np^{r-1}} \pmod{p^r} \tag{8}$$

for primes $p \geq 3$ such that $\left(\frac{p}{23}\right) = 1$ and $r \geq 1$. An application of Proposition 2.1 then implies (1) and (2). The same argument holds for $p = 2$ and $r \geq 3$. The result then follows upon a routine check that (8) holds in the six cases: (i) $p = 2$, $r = 1$, $\left(\frac{n}{23}\right) = 1$, (ii) $p = 2$, $r = 1$, $\left(\frac{n}{23}\right) = -1$, (iii) $p = 2$, $r = 1$, $\left(\frac{n}{23}\right) = 0$, (iv) $p = 2$, $r = 2$, $\left(\frac{n}{23}\right) = 1$, (v) $p = 2$, $r = 2$, $\left(\frac{n}{23}\right) = -1$ and (vi) $p = 2$, $r = 2$, $\left(\frac{n}{23}\right) = 0$.

□

3. Tables

Using the methods in Section 2, we have proven supercongruences of the form

$$A(np^r) \equiv A(np^{r-1}) \pmod{p^r} \tag{9}$$

for all of the numbers $A(n)$ which appear in Tables 1, 2 and 3. For brevity, we only give the relevant modular function t and modular forms $f(t)$ and $M := f(t)\frac{q\frac{dt}{dq}}{t}$. The coefficients of M in Tables 1, 2 and 3 can be computed using (3), Chapter 4, Section 32 in [11], [13] or [14]. Given positive integers s_1, s_2, …, s_k and integers r_1, r_2, …, r_k, we write $s_1^{r_1}s_2^{r_2}\ldots s_k^{t_k}$ for the eta-quotient $\eta(s_1z)^{r_1}\eta(s_2z)^{r_2}\cdots\eta(s_kz)^{r_k}$. Table 1 consists of numbers $f_{i,n}$, $i = 2$, 3, 5, 7 and 11 which are coefficients in the power series expansion in t of the modular forms (see [7])

$$f_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{m^2+n^2}, \quad f_3 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{m^2+mn+n^2}, \quad f_5 = \frac{1^5}{5^1}, \quad f_7 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{m^2+mn+2n^2}$$

and

$$f_{11} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{m^2+mn+3n^2}.$$

We write $\chi_s := \left(\frac{\cdot}{s}\right)$ for $s = 3$, 5, 7, 11 and $\chi_{-4} := \left(\frac{-4}{\cdot}\right)$. The analogue of Theorem 1.2 is true for primes p satisfying $\chi_{-4}(p) = 1$ in (i), $\chi_3(p) = 1$ in (ii), $\chi_7(p) = 1$ in (iv), $\chi_{11}(p) = 1$ in (v). It is true for all primes in (iii). The only known closed forms are $f_{2,n} = \left(\frac{8^n\left(\frac{1}{4}\right)_n}{n!}\right)^2$ and $f_{3,n} = \frac{108^n\left(\frac{1}{6}\right)_n\left(\frac{1}{3}\right)_n}{(n!)^2}$. Table 2 lists numbers which arise in Beukers' [5] and Zagier's [17] study of integral solutions of second order Apéry-like differential equations. The choices of t and the parameterizations of f can be found in [16] and [17]. In case (ix), congruence (9) with $A(n)$ replaced by $2^n A(n)$ has been proven in [12]. Table 3 contains numbers listed in [2] as part of a discussion on third order Apéry-like differential equations. Here

$$L_1(z) := \frac{1}{240}E_4(z) + \frac{3}{560}E_4(2z) - \frac{1}{420}E_4(3z) - \frac{3}{20}E_4(6z),$$

$$L_2(z) := \frac{1}{60}E_4(z) - \frac{3}{20}E_4(2z) - \frac{1}{15}E_4(3z) + \frac{12}{5}E_4(6z),$$

and

$$L_3(z) := \frac{1}{240}E_4(z) - \frac{27}{80}E_4(2z) - \frac{1}{60}E_4(3z) + \frac{27}{20}E_4(6z)$$

where $E_4(z)$ is the usual weight 4 Eisenstein series on $SL_2(\mathbb{Z})$. The choices of t and the parameterizations of f can be found in [6], [9] and [15].

Finally, we have numerically observed extensions of (9) modulo p^{2r} (subject to the above conditions for p odd) in (i), (ii), (iv), (v), (vii), (viii), (ix) and (x) and modulo p^{3r} for (iii), (vi), (xi), (xii) and (xiii). Here $p \geq 5$ for (xi) and (xii). It might of interest to see if combinatorial techniques can be applied to some of these extensions.

Table 1

$A(n)$	t	M
(i) $f_{2,n}$	$\frac{2^{12}}{f_2^6}$	$-4E_{3,\chi_{-4},1}(q) - 16E_{3,1,\chi_{-4}}(q)$
(ii) $f_{3,n}$	$\frac{1^6 3^6}{f_3^6}$	$-9E_{3,\chi_3,1}(q) - 27E_{3,1,\chi_3}(q)$
(iii) $f_{5,n}$	$\frac{5^6}{1^6}$	$E_{4,\chi_5,1}(q)$
(iv) $f_{7,n}$	$\frac{1^3 7^3}{f_7^3}$	$-\frac{7}{8}E_{3,\chi_7,1}(q) - \frac{49}{8}E_{3,1,\chi_7}(q)$
(v) $f_{11,n}$	$\frac{1^2 11^2}{f_{11}^2}$	$-\frac{1}{3}E_{3,\chi_{11},1}(q) - \frac{11}{3}E_{3,1,\chi_{11}}(q)$

TABLE 2

$A(n)$	t	$f(t)$	M
(vi) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^3$	$\frac{1^3 6^9}{2^3 3^9}$	$\frac{2^1 3^6}{1^2 6^3}$	$-\frac{1^1 2^4 3^5}{6^4}$
(vii) $\displaystyle\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor}(-1)^k 3^{n-3k}\binom{n}{3k}\binom{3k}{k}\binom{2k}{k}$	$\frac{9^3}{1^3}$	$\frac{1^3}{3^1}$	$\frac{3^9}{9^3}$
(viii) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$	$\frac{1^4 6^8}{2^8 3^4}$	$\frac{2^6 3^1}{1^3 6^2}$	$-\frac{1^1 2^4 3^5}{6^4}$
(ix) $\displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 4^{n-2k}\binom{n}{2k}\binom{2k}{k}^2$	$\frac{1^4 4^2 8^4}{2^{10}}$	$\frac{2^{10}}{1^4 4^4}$	$\frac{2^4 4^6}{8^4}$
(x) $\displaystyle\sum_{k=0}^{n}\sum_{l=0}^{k}(-1)^k 8^{n-k}\binom{n}{k}\binom{k}{l}^3$	$\frac{1^5 3^1 4^5 6^2 12^1}{2^{14}}$	$\frac{2^{15} 3^2 12^2}{1^6 4^6 6^5}$	$\frac{2^7 6^{11}}{1^1 3^5 4^1 12^5}$

TABLE 3

$A(n)$	t	f	M
(xi) $\displaystyle\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2$	$\frac{1^{12} 6^{12}}{2^{12} 3^{12}}$	$\frac{2^7 3^7}{1^5 6^5}$	$-7L_1(z)$
(xii) $\displaystyle(-1)^n\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\frac{2^6 6^6}{1^6 3^6}$	$\frac{1^4 3^4}{2^2 6^2}$	$\frac{1}{2}L_2(z)$
(xiii) $\displaystyle(-1)^n\sum_{k=0}^{[n/3]}(-1)^k\frac{3^{n-3k}(3k)!}{(k!)^3}\binom{n}{3k}\binom{n+k}{k}$	$\frac{3^4 6^4}{1^4 2^4}$	$\frac{1^3 2^3}{3^1 6^1}$	$L_3(z)$

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