Constructing illoyal algebra-valued models of set theory

SYSMICS 2019

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joint work with Benedikt Löwe and Sourav Tarafder

January 22, 2019

ILLC, Universiteit van Amsterdam

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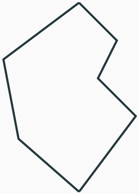

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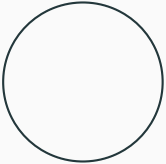
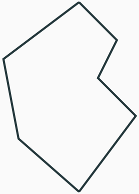


recursively defined set of names



A

$\{0, 1, \wedge, \vee, \rightarrow, \neg\}$-algebra

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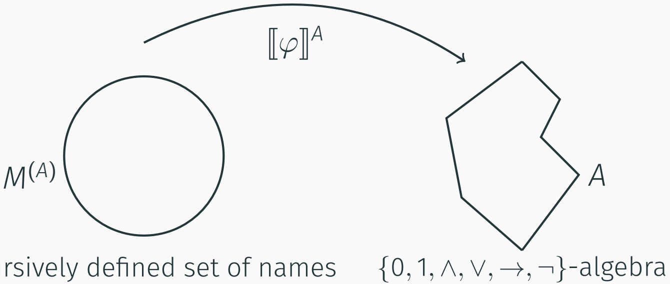

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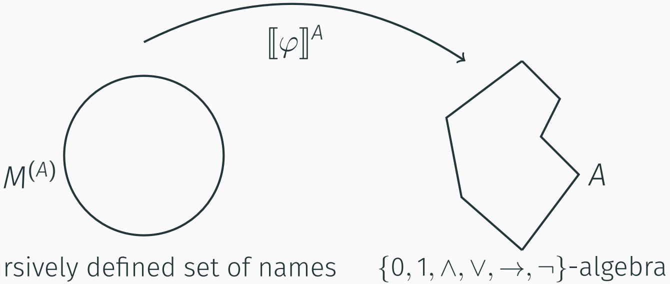

The logic $\mathbf{L}(M^{(A)})$ of $M^{(A)}$ is the *propositional logic* arising from set-theoretical sentences evaluated in the model (i.e., without parameters!).

Propositional Logic of $M^{(A)}$

The *propositional logic* $\mathsf{L}(M^{(A)})$ is the following set of propositional formulas:

$$\{\varphi \mid \text{ for all translations } \tau : \mathrm{Prop} \to \mathrm{Sent}_{\in} \text{ have } [\![\varphi^{\tau}]\!]^{A} = 1\}.$$

The general question

How much (logical) structure of the algebra is preserved when building a model of set theory on top of it?

There are two different degrees:

faithful preserving the structure,

loyal preserving the logic.

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Loyalty and Faithfulness

We call an algebra-valued model $M^{(A)}$:

faithful if for every $a \in A$, there is a set-theoretical sentence φ such that $[\![\varphi]\!]^A = a$, and,

loyal if the propositional logic of the algebra-valued model is the same as the propositional logic of the algebra, i.e., $\mathbf{L}(M^{(A)}) = \mathbf{L}(A)$.

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In this work, we are interested in *illoyal* algebra-valued models, i.e., models that are not loyal.

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Our general procedure:

1. Start with a complete atomic Boolean algebra B (i.e., power sets),
2. stretch or twist it to obtain an algebra A, and,
3. observe that $M^{(A)}$ is not loyal, i.e., $\mathsf{L}(A) \subsetneq \mathsf{L}(M^{(A)})$.

Illoyal models: A case study

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Illoyal models: Stretching and twisting

We'll consider three constructions:

1. tail stretches,
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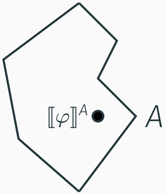

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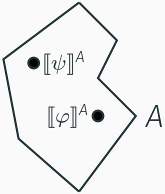

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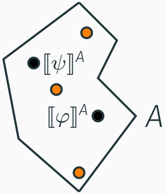

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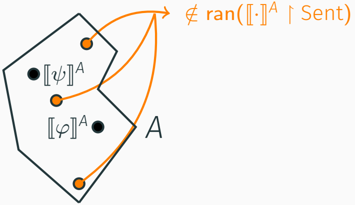

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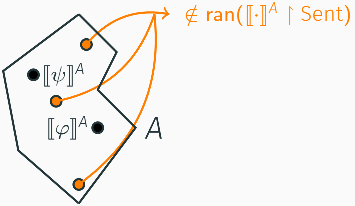

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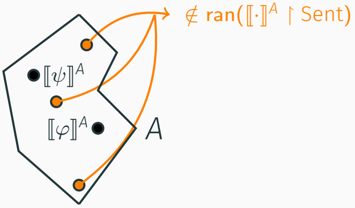

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If $f : A \to A$ is an automorphism such that $f(a) \neq a$, then $a \notin \mathsf{ran}(\llbracket\cdot\rrbracket^A \restriction \mathsf{Sent})$.

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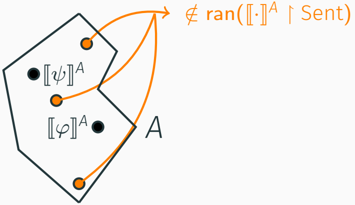

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By induction on formulas. □

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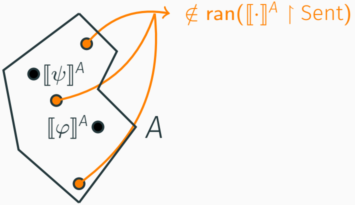

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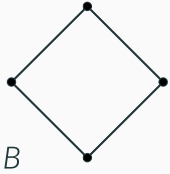
Proof.
By induction on formulas. $\square$

Hence, excluding values from $\mathsf{ran}(\llbracket\cdot\rrbracket^A \restriction \mathsf{Sent})$ allows us to learn about $\mathsf{L}(M^A)$.

Illoyal models: Tail stretch

Start from an atomic Boolean algebra.



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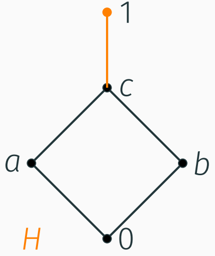

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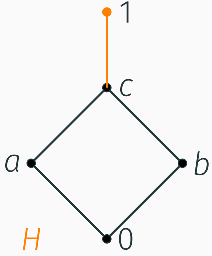

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$M^{(H)}$ is a model of IZF.

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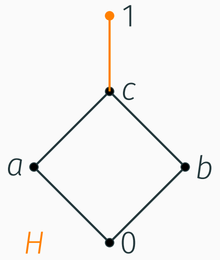

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$\varphi \to \psi \vee \psi \to \varphi \notin \mathsf{L}(H)$

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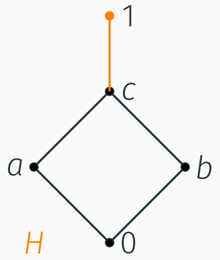

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Proof.
$a \to b \vee b \to a = b \vee a = c < 1$ $\square$

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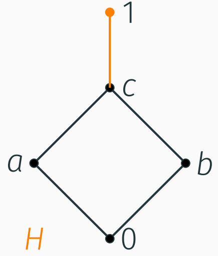

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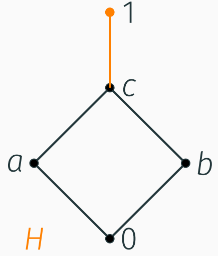

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Proof.
There is an automorphism of H swapping a and b. Therefore, by our main tool, a and b cannot be in the range of $[\![\cdot]\!]^A \restriction \mathrm{Sent}$. Hence, $[\![\cdot]\!]^A \restriction \mathrm{Sent}$ is a linear algebra. $\square$

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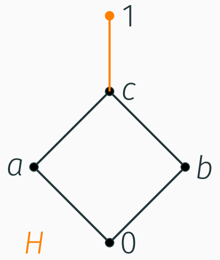

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The model $M^{(H)}$ is not loyal.

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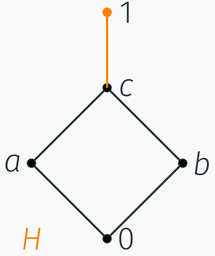

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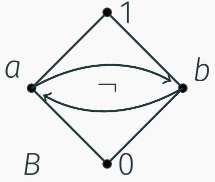
Proof.
$\varphi \to \psi \vee \psi \to \varphi \in \mathsf{L}(M^{(H)}) \setminus \mathsf{L}(H)$. $\square$

Illoyal models: Transposition twist

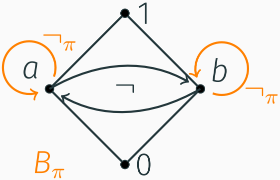
Start from an atomic Boolean algebra B.



Illoyal models: Transposition twist

Start from an atomic Boolean algebra B and twist its negation using a transposition $\pi : \mathrm{At}(B) \to \mathrm{At}(B), a \mapsto b$ to obtain B_π:

$\neg_\pi(\bigvee X) := \bigvee\{\pi(t) \in \mathrm{At}(B) \mid t \notin X\}$



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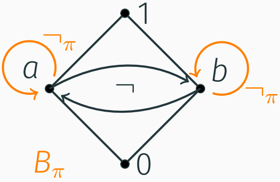

Observation

The negation $\neg_\pi$ satisfies the rule of contraposition: $x \leq y$ implies $\neg_\pi y \leq \neg_\pi x$.

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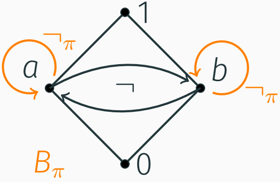

Theorem

$M^{(B_\pi)}$ is a model of $\mathrm{NFF} - \mathrm{ZF}$.

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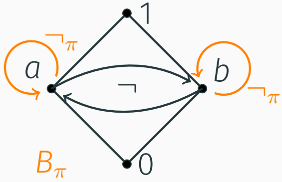

Observation

$\neg(p \wedge \neg p) \notin \mathsf{L}(B_\pi)$, *i.e.*, $\mathsf{L}(B_\pi) \subsetneq \mathsf{CPC}$.

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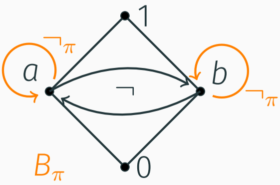

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Proof.
Calculation: $\neg_\pi(a \wedge \neg_\pi a) = \neg_\pi(a \wedge a) = \neg_\pi a = a$. $\square$

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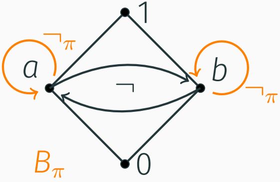

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$\mathsf{L}(M^{(B_\pi)}) = \mathsf{CPC}.$

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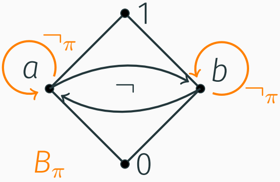

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Proof.
There is an automorphism f of B with $f(a) = b$. Using this and our main tool, one can show that if $x \in \mathsf{ran}(\llbracket\cdot\rrbracket^{(B_\pi)} \upharpoonright \mathrm{Sent})$, then $\neg_\pi x = \neg x$. Hence, $\mathsf{ran}(\llbracket\cdot\rrbracket^{(B_\pi)} \upharpoonright \mathrm{Sent})$ is a Boolean algebra. $\square$

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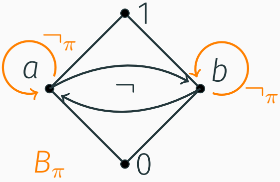

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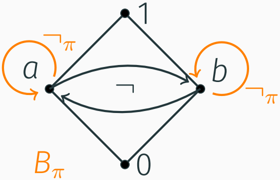

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$\mathsf{L}(B_\pi) \subsetneq \mathsf{CPC} = \mathsf{L}(M^{(B_\pi)})$. $\square$

Illoyal models: Maximal twist

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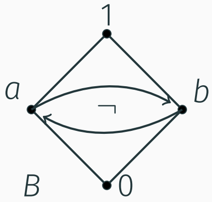
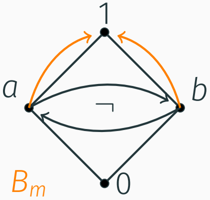


Illoyal models: Maximal twist

Start from an atomic Boolean algebra B and define a maximal negation:

$$\neg_m x = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1. \end{cases}$$



Illoyal models: Maximal twist

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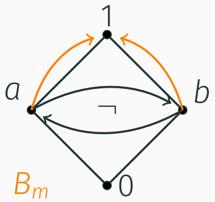
$$\neg_m x = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1. \end{cases}$$



Observation

The negation $\neg_m$ satisfies the rule of contraposition: $x \leq y$ implies $\neg_m y \leq \neg_m x$.

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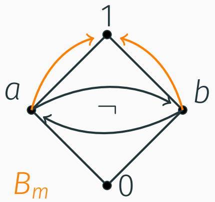

Theorem
$M^{(B_\pi)}$ is a model of NFF − ZF.

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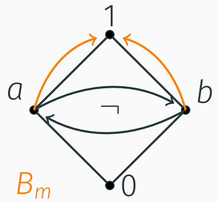

Observation

$(p \wedge \neg p) \to q \notin \mathsf{L}(B_\pi)$, *i.e.*, $\mathsf{L}(B_\pi) \subsetneq \mathsf{CPC}$.

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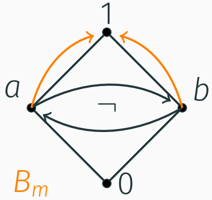

Observation

$(p \wedge \neg p) \to q \notin \mathbf{L}(B_\pi)$, *i.e.*, $\mathbf{L}(B_\pi) \subsetneq \mathbf{CPC}$.

Proof.
Calculation:

$(a \wedge \neg_m a) \to b = (a \wedge 1) \to b = a \to b = b.$ $\square$

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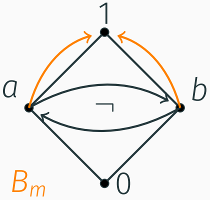

Observation

$\mathsf{L}(M^{(B_\pi)}) = \mathsf{CPC}$.

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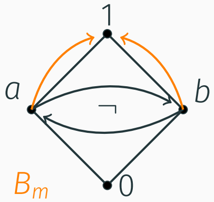

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Observation

$\mathsf{L}(M^{(B_\pi)}) = \mathsf{CPC}$.

Proof.
Every non-trivial element of B_m is moved by an automorphism. By our main tool, it follows that $\mathrm{ran}(\llbracket\cdot\rrbracket^{B_m} \restriction \mathrm{Sent}) = \{0, 1\}$, and that's a Boolean algebra. $\square$

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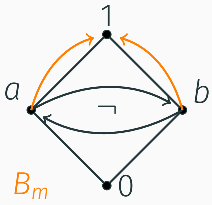

Theorem
The model $M^{(B_m)}$ is not loyal.

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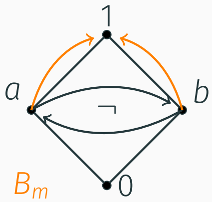

$$\neg_m x = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1. \end{cases}$$

Theorem
The model $M^{(B_m)}$ is not loyal.

Proof.
$\mathsf{L}(B_m) \subsetneq \mathsf{CPC} = \mathsf{L}(M^{(B_m)})$. $\square$

Conclusions & Future Work

- The propositional logics of the algebra and of the model can be quite different (e.g., a model satisfying *ex falso* whose algebra doesn't)
- **Future work:** A more systematic study!

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Constructing illoyal algebra-valued models of set theory

Benedikt Löwe, Robert Passmann and Sourav Tarafder

(paper available on, e.g., my website)

Thank you! – Questions?

Robert Passmann

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