The de Jongh property for bounded CZF set theory

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Intuitionism and intuitionistic logic

- **Brouwer** proposed *intuitionistic mathematics* on a philosophical basis: mathematics as mental constructions.
- His intuitionistic mathematics rejects some classical theorems and proves new ones ("classical" example: all functions are continuous).
- **Heyting** formalised the rules of reasoning behind intuitionistic mathematics as *intuitionistic logic*.
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ZF(C)

Set Theory: classical, intuitionistic, constructive

ZF(C) satisfies classical logic

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$$\rotatebox{90}{$\subsetneq$}$$

$$\mathrm{IZF}$$

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\mathrm{BCZF} &
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Constructive set theory: axioms

- Extensionality
- Pairing
- Union
- Empty set
- Infinity
- Bounded separation schema
- Strong collection schema
- Subset collection schema
- $\in$-induction

Motivation: de Jongh's theorem

Theorem (de Jongh, 1970)

The propositional logic $\mathsf{L}(\mathsf{HA})$ *of Heyting arithmetic* HA *is intuitionistic propositional logic* IPC.

Or, equivalently, we have for every propositional formula φ:

$$\mathsf{IPC} \vdash \varphi \text{ if and only if } \mathsf{HA} \vdash \varphi^\sigma \text{ for every substitution } \sigma.$$

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Propositional logics of theories

Definition

Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. We define the *propositional logic of* T:

$$\mathsf{L}(\mathsf{T}) = \{\varphi \mid \mathsf{T} \vdash \varphi^\sigma \text{ for all } \sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}\}$$

The de Jongh property I

We can now restate de Jongh's theorem…

Theorem (de Jongh, 1970)

$\mathbf{L}(\mathsf{HA}) = \mathbf{IPC}$

…and easily generalise this property…

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A theory T has *the de Jongh property* if $\mathbf{L}(\mathsf{T}) = \mathbf{IPC}$.

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The de Jongh property II

…even further:

Given an intermediate logic $\mathbf{J}$ (i.e., $\mathbf{IPC} \subseteq \mathbf{J} \subseteq \mathbf{CPC}$), consider the closure $\mathsf{T}(\mathbf{J})$ of T under $\mathbf{J}$. We then say:

Definition

T has *the de Jongh property with respect to* $\mathbf{J}$ if $\mathbf{L}(\mathsf{T}(\mathbf{J})) = \mathbf{J}$.

Theorem (de Jongh, Verbrugge, Visser, 2010)

HA *has the de Jongh property with respect to every intermediate logic that has the finite frame property.*

In this talk: Analyse this property for set theory!

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The de Jongh property for set theories: main result

For this work, we restrict our attention to BCZF, i.e., CZF with the collection schemes restricted to bounded formulas.

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The theory BCZF *has the de Jongh property with respect to every intermediate logic that can be characterised by a class of Kripke frames.*

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Models for BCZF

We will use a construction of Kripke models for BCZF due to Iemhoff.

A sketch:

1. Start with a Kripke frame $(K, \leq)$ (i.e., partial order),
2. construct a sound assignment $\mathcal{M} : K \to V$ of models of ZF set theory such that $\mathcal{M}_v \subseteq \mathcal{M}_w$ for all $v \leq w$, and this inclusion is an $\in$-homomorphism,
3. use this sound assignment as the domains of the Kripke model $K(\mathcal{M})$.

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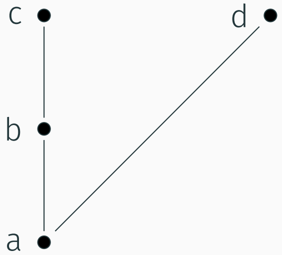

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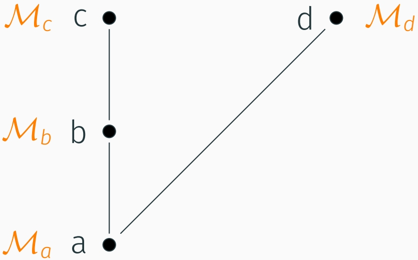

Models for BCZF: Forcing

We then define the forcing relation as expected:

- $K(\mathcal{M}), v \Vdash \varphi \wedge \psi$ iff $K(\mathcal{M}), v \Vdash \varphi$ and $K(\mathcal{M}), v \Vdash \psi$,
- $K(\mathcal{M}), v \Vdash \varphi \vee \psi$ iff $K(\mathcal{M}), v \Vdash \varphi$ or $K(\mathcal{M}), v \Vdash \psi$,
- $K(\mathcal{M}), v \Vdash \varphi \to \psi$ iff for all $w \geq v$, $K(\mathcal{M}), w \Vdash \varphi$ implies $K(\mathcal{M}), w \Vdash \psi$,
- $K(\mathcal{M}), v \Vdash \bot$ holds never.
- $K(\mathcal{M}), v \Vdash a \in b$ iff $\mathcal{M}_v \vDash a \in b$,
- $K(\mathcal{M}), v \Vdash a = b$ iff $a = b$,
- $K(\mathcal{M}), v \Vdash \exists x \varphi(x, \bar{y})$ iff there is some $a \in D_v$ with $K(\mathcal{M}), v \Vdash \varphi(a, \bar{y})$,
- $K(\mathcal{M}), v \Vdash \forall x \varphi(x, \bar{y})$ iff for all $w \geq v$ and $a \in D_w$ we have $K(\mathcal{M}), w \Vdash \varphi(a, \bar{y})$.

Theorem (Iemhoff, 2010)

For every Kripke frame K, and every sound assignment $\mathcal{M}$, we have $K(\mathcal{M}) \Vdash$ BCZF.

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Proof Sketch I

Theorem (P.)

The theory BCZF *has the de Jongh property with respect to every intermediate logic that can be characterised by a class of Kripke frames.*

Sketch of the proof. We need to show that

$$\mathsf{J} \vdash \varphi \text{ if and only if } \mathrm{BCZF}(\mathsf{J}) \vdash \varphi^\sigma \text{ for all } \sigma : \mathrm{Prop} \to \mathcal{L}_\in^{\mathrm{sent}}.$$

We only need to prove the direction from right to left. By contraposition: Assume $\mathsf{J} \nvdash \varphi$, then there is some Kripke frame K, and a valuation V on K such that $K, V \nVdash \varphi$. Without loss of generality, $V(p) = \emptyset$ for all p that do not appear in φ.

If we can find an assignment of models $\mathcal{M}$ and a collection of sentences ψ_i in the language of set theory such that $K, V, v \Vdash p_i$ if and only if $K(\mathcal{M}), v \Vdash \psi_i$, the proof can be finished by induction: $K(\mathcal{M}) \nVdash \varphi^\sigma$.

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Proof Sketch II

Left to do: Find $\mathcal{M}$ and a collection of sentences ψ_i in the language of set theory such that $K, V, v \Vdash p_i$ if and only if $K(\mathcal{M}), v \Vdash \psi_i$.

Crucial observations:

1. Σ_1-formulas are evaluated locally in Kripke semantics,
2. the constructible universe $\mathbb{L}$ is absolute between models of ZFC set theory, and,
3. formulas relativised to $\mathbb{L}$, i.e., $\psi^{\mathbb{L}}$ behave like Δ_0-formulas.

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We will use the following sentences ψ_i:

There is an injection from $\aleph^{\mathrm{L}}_{i+2}$ to $\mathcal{P}(\aleph^{\mathrm{L}}_i)$.

Formally:

$$\begin{aligned}\exists x \exists y \exists g ((x = \aleph_{i+2})^{\mathrm{L}} &\wedge (y = \aleph_i)^{\mathrm{L}} \\ &\wedge g \text{ “is an injective function”} \\ &\wedge \mathrm{dom}(g) = x \\ &\wedge \forall \alpha \in x \forall z \in g(\alpha)\ z \in y)\end{aligned}$$

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Proof Sketch IV: Constructing the assignment

Proposition (Friedman, Fuchino and Sakai, 2017)

Let $A \in \mathrm{L}$ such that $A \subseteq \omega$. Then there is a generic extension $\mathrm{L}[G^A]$ such that for all $i \in \omega$, we have $\mathrm{L}[G^A] \vDash \psi_i$ if and only if $i \in A$.

For every $v \in K$, let $\mathcal{M}_v$ be the model $\mathrm{L}[G^{V^{-1}(v)}]$. (Code the propositional letters as natural numbers.)

Then, we have:

$$\begin{aligned} K, V, v \Vdash p_i &\text{ if and only if } \mathcal{M}_v \vDash \psi_i \\ &\text{ if and only if } K(\mathcal{M}), v \Vdash \psi_i. \end{aligned}$$

Now, let σ be the assignment $p_i \mapsto \psi_i$. We finish the proof by an induction on subformulas χ of φ showing that:

$$K, V, v \Vdash \chi \text{ if and only if } K(\mathcal{M}), v \vDash \chi^\sigma.$$

As $K, V, v \nVdash \varphi$, have $K(\mathcal{M}), v \nvDash \chi^\sigma$. $\square$

Proof Sketch IV: Constructing the assignment

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Let $A \in \mathrm{L}$ such that $A \subseteq \omega$. Then there is a generic extension $\mathrm{L}[G^A]$ such that for all $i \in \omega$, we have $\mathrm{L}[G^A] \vDash \psi_i$ if and only if $i \in A$.

For every $v \in K$, let $\mathcal{M}_v$ be the model $\mathrm{L}[G^{V^{-1}(v)}]$. (Code the propositional letters as natural numbers.)

Then, we have:

$$K, V, v \Vdash p_i \text{ if and only if } \mathcal{M}_v \vDash \psi_i$$
$$\text{if and only if } K(\mathcal{M}), v \Vdash \psi_i.$$

Now, let σ be the assignment $p_i \mapsto \psi_i$. We finish the proof by an induction on subformulas χ of φ showing that:

$$K, V, v \Vdash \chi \text{ if and only if } K(\mathcal{M}), v \vDash \chi^\sigma.$$

As $K, V, v \nVdash \varphi$, have $K(\mathcal{M}), v \nvDash \chi^\sigma$. $\square$

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Discussion: CZF or IZF?

This exact proof cannot easily be strengthened to full CZF or even IZF:

As soon as non-trivial generic extensions are involved, the axiom of exponentiation (a constructive consequence of full collection) fails in the models we use above.

There is even a model forcing the negation of exponentiation.

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Discussion: model-theoretic vs. proof-theoretic perspective

The de Jongh property is a *proof-theoretic* property. There's a *model-theoretic* side to the discussion above.

We call an algebra-valued model $M^{(A)}$:

faithful if for every $a \in A$, there is a set-theoretical sentence φ such that $[\![\varphi]\!]^A = a$, and,

loyal if the propositional logic of the algebra-valued model is the same as the propositional logic of the algebra, i.e., $\mathbf{L}(M^{(A)}) = \mathbf{L}(A)$.

The argument above shows that the class of models above has a certain degree of faithfulness (see paper).

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Conclusions & Future Work

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- We have shown that BCZF has the de Jongh property with respect to all Kripke-complete logics.
- We have a preliminary result on IZF.

Future Work:

- Further explore the connection between proof-theoretic and model-theoretic point of view.
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The de Jongh property for bounded constructive Zermelo-Fraenkel set theory

(paper available on, e.g., my website)

Thank you! – Questions?

Robert Passmann

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`http://robertpassmann.github.io`