De Jongh's Theorem for Intuitionistic Zermelo-Fraenkel Set Theory

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Preliminaries

Intuitionism and intuitionistic logic

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- Intuitionistic logic is a subsystem of classical logic.
- Thus allows to study intuitionistic systems that do not contradict classical mathematics/systems.

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Intuitionistic logic

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There is a continuum of *intermediate logics* J such that IPC $\subseteq$ J $\subseteq$ CPC, and CPC is maximally consistent.

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$$\text{ZF(C)} \supsetneq \text{IZF}$$

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$\cup\!\!\nmid$

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IZF impredicative, high proof-theoretic strength

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CZF predicative, low proof-theoretic strength

The Propositional Logic of IZF

Logics of theories

Definition

Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. We define the *propositional logic of* T:

$$\mathsf{L}(\mathsf{T}) = \{\varphi \mid \mathsf{T} \vdash \varphi^\sigma \text{ for all } \sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}\}$$

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  or: $\mathsf{IPC} \subsetneq \mathsf{L}(\mathrm{HA} + \mathrm{MP} + \mathrm{ECT}_0) \subsetneq \mathsf{CPC}$
- This means: Axioms can imply logical principles
  e.g., the *axiom of choice* implies *tertium non datur*.

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The propositional logic of Heyting arithmetic HA *is* $\mathbf{IPC}$*, i.e.,* $\mathbf{L}(\mathsf{HA}) = \mathbf{IPC}$.

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If $\mathbf{J}$ *is an intermediate logic that is complete with respect to finite frames, then* $\mathbf{L}(\mathsf{HA}(\mathbf{J})) = \mathbf{J}$.

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Theorem (P., 2018)

If J *is a Kripke-complete intermediate logic, then* $\mathsf{L}(\mathrm{BCZF}(\mathsf{J})) = \mathsf{J}$.

The de Jongh property for IZF

In this talk:

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If J *is an intermediate logic complete with respect to a class of finite trees, then* $\mathsf{L}(\mathsf{IZF}(\mathsf{J})) = \mathsf{J}$.

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Corollary

The propositional logic of IZF *is* **IPC**.

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We'll focus on the special case for IPC:

$$\mathsf{IPC} \vdash \varphi \text{ if and only if } \mathsf{IZF} \vdash \varphi^\sigma \text{ for all } \sigma$$

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But we don't have much time... so here's an example.

Blended models: Example

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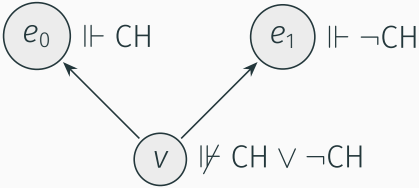

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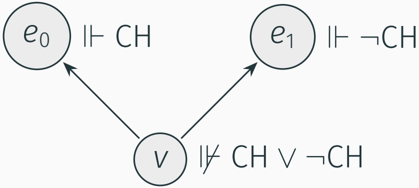

What's the domain at v?

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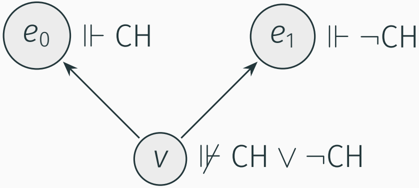

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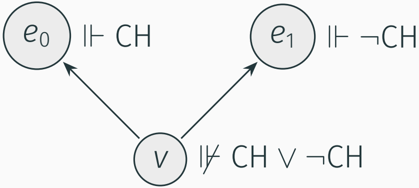

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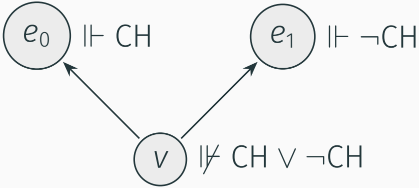

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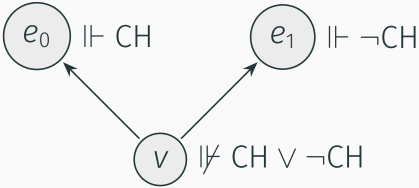

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Finally, we define the domain $\mathcal{D}_v$ at the node v to be the set

$$\mathcal{D}_v = \bigcup_{\alpha \in \mathrm{Ord}^M} \mathcal{D}_v^\alpha.$$

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- In fact, a bit more: Every set theory weaker than IZF has the de Jongh property with respect to every intermediate logic characterised by a class of finite trees.

Question

Is it the case that $\mathsf{L}(\mathrm{IZF}(\mathsf{J})) = \mathsf{J}$ *for all intermediate logics* J*?*

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(preprint available on my website)

Thank you! – Questions?

Robert Passmann

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Constructing blended models in 3 steps

Let $(K, \leq)$ be a finite tree, E_K be its set of end-nodes, and an assignment $e \mapsto M_e$ of end-nodes to models of set theory such that $\mathrm{Ord}^{M_{e_0}} = \mathrm{Ord}^{M_{e_1}}$.

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Then let $\mathcal{D}_e = f_e[M_e]$. Hence, each $\mathcal{D}_e$ is a set of functions $K^{\geq e} \to V$ (where $K^{\geq e} = \{e\}$). Moreover, for $\alpha \in \mathrm{Ord}^M$, let $\mathcal{D}_e^\alpha = f_e[(V_\alpha)^{M_e}]$. Then $\mathcal{D}_e^0 = \emptyset$ and it holds that

$$\bigcup_{\alpha \in \mathrm{Ord}^M} \mathcal{D}_e^\alpha = \mathcal{D}_e.$$

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By induction on $\alpha \in \mathrm{Ord}^{M_e}$ simultaneously for all $v \in K \setminus E_K$. Let $\mathcal{D}_v^\alpha$ consist of the functions $x : K^{\geq v} \to V$ such that the following properties hold:

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Transition between domains is by restriction $x \mapsto x \upharpoonright K^{\geq w}$.

Constructing blended models in 3 steps

Step 3. Defining the semantics

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Inductively define the forcing relation of the Kripke model:

1. $(K, \leq, \mathcal{D}), v \Vdash x \in y$ if and only if $x \in y(v)$,
2. $(K, \leq, \mathcal{D}), v \Vdash a = b$ if and only if $a = b$,
3. $(K, \leq, \mathcal{D}), v \Vdash \varphi \wedge \psi$ if and only if $(K, \leq, \mathcal{D}), v \Vdash \varphi$ and $(K, \leq, \mathcal{D}), v \Vdash \psi$,
4. $(K, \leq, \mathcal{D}), v \Vdash \varphi \vee \psi$ if and only if $(K, \leq, \mathcal{D}), v \Vdash \varphi$ or $(K, \leq, \mathcal{D}), v \Vdash \psi$,
5. $(K, \leq, \mathcal{D}), v \Vdash \varphi \rightarrow \psi$ if and only if for all $w \geq v$, $(K, \leq, \mathcal{D}), w \Vdash \varphi$ implies $(K, \leq, \mathcal{D}), w \Vdash \psi$,
6. $(K, \leq, \mathcal{D}), v \Vdash \perp$ holds never.
7. $(K, \leq, \mathcal{D}), v \Vdash \exists x\, \varphi(x, \bar{y})$ if and only if there is some $a \in D_v$ with $(K, \leq, \mathcal{D}), v \Vdash \varphi(a, \bar{y})$,
8. $(K, \leq, \mathcal{D}), v \Vdash \forall x\, \varphi(x, \bar{y})$ if and only if for all $w \geq v$ and $a \in D_w$ we have $(K, \leq, \mathcal{D}), w \Vdash \varphi(a, \bar{y})$.

Constructing blended models: IZF

Theorem (P.)

If K is finite, then the model $(K, \leq, \mathcal{D})$ satisfies IZF. *For arbitrary K, the model $(K, \leq, \mathcal{D})$ satisfies* $\mathrm{IZF} - \mathrm{Collection}$.

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Proof.

Check all axioms. Collection is the only axiom scheme that needs (?) finiteness. $\square$

Sketch of the proof

To show: If $\mathsf{IPC} \nvdash \varphi$, then there is σ such that $\mathrm{IZF} \nvdash \varphi^\sigma$.

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For this special case, consider finite splitting trees. Let $\{e_1, \ldots, e_n\}$ be the set of end-nodes. Let M be a countable transitive model of set theory, and take generic G_i for $1 \leq i \leq n$ such that $M[G_i] \vDash 2^{\aleph_0} = \aleph_i$. Let $(K, \leq, \mathcal{D})$ be the blended model obtained from $\{M[G_i] \mid 1 \leq i \leq n\}$.

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Let $v \in K$, and take

$$\rho_v = \bigwedge_{e_i \ngeq v} \neg(2^{\aleph_0} = \aleph_i).$$

Then $[\![\rho_v]\!] = K^{\geq v}$. Given any valuation V, let $\psi_p = \bigvee_{v \in V(p)} \rho_v$. Define the substitution σ by $p \mapsto \psi_p$.

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An easy induction shows that $K, \leq, V, v \Vdash \chi$ if and only if $(K, \leq, \mathcal{D}), v \Vdash \chi^\sigma$.

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An easy induction shows that $K, \leq, V, v \Vdash \chi$ if and only if $(K, \leq, \mathcal{D}), v \Vdash \chi^\sigma$. So, given a propositional formula φ such that **IPC** $\nvdash \varphi$, there is a finite splitting tree such that $K, \leq, V \nVdash \varphi$, and, $(K, \leq, \mathcal{D}) \nVdash \varphi^\sigma$.

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An easy induction shows that $K, \leq, V, v \Vdash \chi$ if and only if $(K, \leq, \mathcal{D}), v \Vdash \chi^\sigma$. So, given a propositional formula φ such that **IPC** $\nvdash \varphi$, there is a finite splitting tree such that $K, \leq, V \nVdash \varphi$, and, $(K, \leq, \mathcal{D}) \nVdash \varphi^\sigma$. Hence, IZF $\nvdash \varphi^\sigma$. $\square$