THE PROPOSITIONAL LOGIC OF MODELS OF SET THEORY

COLLOQUIUM LOGICUM 2018

Robert Passmann

September 15, 2018

ILLC, Universiteit van Amsterdam

WHAT ARE WE DOING AND WHY ARE WE DOING IT?

Model constructions for non-classical set theories are often built on top of (Heyting) algebras or Kripke frames.

Our main question here:

> *How much does the propositional logic of these different model constructions reflect the logic of their underlying Heyting algebras or Kripke frames?*

WHAT ARE WE DOING AND WHY ARE WE DOING IT?

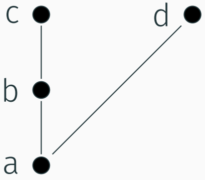
Model constructions for non-classical set theories are often built on top of (Heyting) algebras or Kripke frames.

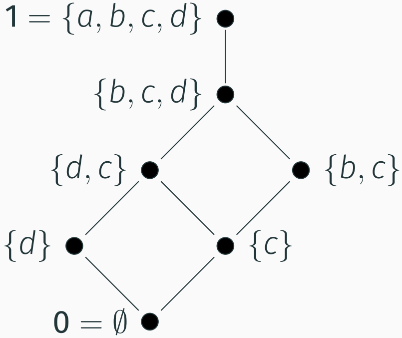
Our main question here:

> *How much does the propositional logic of these different model constructions reflect the logic of their underlying Heyting algebras or Kripke frames?*

HEYTING ALGEBRAS AND KRIPKE FRAMES

Heyting algebras and Kripke frames are *semantics for intuitionistic propositional logic* IPC.





LOGICS OF HEYTING ALGEBRAS AND KRIPKE FRAMES

A Heyting algebra is a bounded lattice $(H, \wedge, \vee, \mathbf{0}, \mathbf{1})$ with an implication operation $\rightarrow$ such that $c \wedge a \leq b$ is equivalent to $c \leq a \rightarrow b$ for all $a, b, c \in H$.

Given any Heyting algebra $\mathsf{H} = (H, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$, we can define its propositional logic:

$$\mathsf{L}(\mathsf{H}) = \{\varphi \in \mathcal{L}^{\mathrm{Prop}} \mid [\![\varphi]\!]_H^V = \mathbf{1} \text{ for all } V : \mathrm{Prop} \rightarrow H\}.$$

Similarly, given a Kripke frame (partial order) $\mathsf{K} = (K, \leq)$, its propositional logic is:

$$\mathsf{L}(\mathsf{K}) = \{\varphi \in \mathcal{L}^{\mathrm{Prop}} \mid \mathsf{K}, V, v \Vdash \varphi \text{ for all } v \in K \text{ and persistent } V : K \rightarrow \mathcal{P}(\mathrm{Prop})\}$$

LOGICS OF HEYTING ALGEBRAS AND KRIPKE FRAMES

A Heyting algebra is a bounded lattice $(H, \wedge, \vee, \mathbf{0}, \mathbf{1})$ with an implication operation $\to$ such that $c \wedge a \leq b$ is equivalent to $c \leq a \to b$ for all $a, b, c \in H$.

Given any Heyting algebra $\mathsf{H} = (H, \wedge, \vee, \to, \mathbf{0}, \mathbf{1})$, we can define its propositional logic:

$$\mathsf{L}(\mathsf{H}) = \{\varphi \in \mathcal{L}^{\mathrm{Prop}} \mid [\![\varphi]\!]_H^V = \mathbf{1} \text{ for all } V : \mathrm{Prop} \to H\}.$$

Similarly, given a Kripke frame (partial order) $\mathsf{K} = (K, \leq)$, its propositional logic is:

$$\mathsf{L}(\mathsf{K}) = \{\varphi \in \mathcal{L}^{\mathrm{Prop}} \mid \mathsf{K}, V, v \Vdash \varphi \text{ for all } v \in K \text{ and persistent } V : K \to \mathcal{P}(\mathrm{Prop})\}$$

A Heyting algebra is a bounded lattice $(H, \wedge, \vee, \mathbf{0}, \mathbf{1})$ with an implication operation $\to$ such that $c \wedge a \leq b$ is equivalent to $c \leq a \to b$ for all $a, b, c \in H$.

Given any Heyting algebra $\mathsf{H} = (H, \wedge, \vee, \to, \mathbf{0}, \mathbf{1})$, we can define its propositional logic:

$$\mathsf{L}(\mathsf{H}) = \{\varphi \in \mathcal{L}^{\mathrm{Prop}} \mid [\![\varphi]\!]_H^V = \mathbf{1} \text{ for all } V : \mathrm{Prop} \to H\}.$$

Similarly, given a Kripke frame (partial order) $\mathsf{K} = (K, \leq)$, its propositional logic is:

$$\mathsf{L}(\mathsf{K}) = \{\varphi \in \mathcal{L}^{\mathrm{Prop}} \mid \mathsf{K}, V, v \Vdash \varphi \text{ for all } v \in K \text{ and persistent } V : K \to \mathcal{P}(\mathrm{Prop})\}$$

MODELS OF SET THEORY

Given a Heyting algebra $\mathbf{H}$, an $\mathbf{H}$-valued structure (A, e, m) consists of a set A with a Heyting-valued equality $e : A \times A \to H$ and a Heyting-valued set-membership relation $m : A \times A \to H$.

Then extend the interpretation $[\![\cdot]\!]^A$ of terms and sentences in the language $\mathcal{L}_\in(A)$ with:

$$[\![a = b]\!]^A = e(a, b),$$
$$[\![a \in b]\!]^A = m(a, b).$$

Note that we can interpret Kripke models as Heyting-valued models.

MODELS OF SET THEORY

Given a Heyting algebra $\mathbf{H}$, an $\mathbf{H}$-valued structure (A, e, m) consists of a set A with a Heyting-valued equality $e : A \times A \to H$ and a Heyting-valued set-membership relation $m : A \times A \to H$.

Then extend the interpretation $\llbracket \cdot \rrbracket^A$ of terms and sentences in the language $\mathcal{L}_\in(A)$ with:

$$\llbracket a = b \rrbracket^A = e(a, b),$$
$$\llbracket a \in b \rrbracket^A = m(a, b).$$

Note that we can interpret Kripke models as Heyting-valued models.

Given a Heyting algebra $\mathbf{H}$, an $\mathbf{H}$-valued structure (A, e, m) consists of a set A with a Heyting-valued equality $e : A \times A \to H$ and a Heyting-valued set-membership relation $m : A \times A \to H$.

Then extend the interpretation $[\![\cdot]\!]^A$ of terms and sentences in the language $\mathcal{L}_{\in}(A)$ with:

$$[\![a = b]\!]^A = e(a, b),$$
$$[\![a \in b]\!]^A = m(a, b).$$

Note that we can interpret Kripke models as Heyting-valued models.

How can Heyting structures reflect the structure of their underlying Heyting algebras?

Notation. If $\mathcal{C}$ is a class of Heyting structures, let $\mathcal{H}_{\mathcal{C}}$ be the class of all underlying Heyting algebras of $\mathcal{C}$.

Definition

We call a class $\mathcal{C}$ of Heyting structures *faithful to* $H \in \mathcal{H}_{\mathcal{C}}$ if for every finite collection $\{h_i \in H \mid i < n\}$, there is some H-structure $A \in \mathcal{C}$ and a collection of $\mathcal{L}$-sentences $\{\varphi_i \mid i < n\}$ such that $[\![\varphi_i]\!]^A = h_i$ holds for all $i < n$. We call $\mathcal{C}$ *faithful* if it is faithful to every $H \in \mathcal{H}_{\mathcal{C}}$.

How can Heyting structures reflect the structure of their underlying Heyting algebras?

Notation. If $\mathcal{C}$ is a class of Heyting structures, let $\mathcal{H}_{\mathcal{C}}$ be the class of all underlying Heyting algebras of $\mathcal{C}$.

Definition

We call a class $\mathcal{C}$ of Heyting structures *faithful to* $H \in \mathcal{H}_{\mathcal{C}}$ if for every finite collection $\{h_i \in H \mid i < n\}$, there is some H-structure $A \in \mathcal{C}$ and a collection of $\mathcal{L}$-sentences $\{\varphi_i \mid i < n\}$ such that $[\![\varphi_i]\!]^A = h_i$ holds for all $i < n$. We call $\mathcal{C}$ *faithful* if it is faithful to every $H \in \mathcal{H}_{\mathcal{C}}$.

How can Heyting structures reflect the structure of their underlying Heyting algebras?

Notation. If $\mathcal{C}$ is a class of Heyting structures, let $\mathcal{H}_{\mathcal{C}}$ be the class of all underlying Heyting algebras of $\mathcal{C}$.

Definition

We call a class $\mathcal{C}$ of Heyting structures *faithful to* $H \in \mathcal{H}_{\mathcal{C}}$ if for every finite collection $\{h_i \in H \mid i < n\}$, there is some H-structure $A \in \mathcal{C}$ and a collection of $\mathcal{L}$-sentences $\{\varphi_i \mid i < n\}$ such that $[\![\varphi_i]\!]^A = h_i$ holds for all $i < n$. We call $\mathcal{C}$ *faithful* if it is faithful to every $H \in \mathcal{H}_{\mathcal{C}}$.

Definition

The propositional logic $\mathsf{L}(\mathcal{C})$ of $\mathcal{C}$ consists of all propositional formulas φ such that for all $C \in \mathcal{C}$ and all substitutions $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$ we have that $C \vDash \varphi^\sigma$.

Fact. We always have $\mathsf{IPC} \subseteq \mathsf{L}(\mathcal{C}) \subseteq \mathsf{CPC}$, i.e., $\mathsf{L}(\mathcal{C})$ is an *intermediate logic.*

Definition

We say that $\mathcal{C}$ *is loyal* if $\mathsf{L}(\mathcal{C}) = \mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Proposition

If $\mathcal{C}$ is faithful, then $\mathcal{C}$ is loyal.

FAITHFULNESS & LOYALTY

Definition

The propositional logic $\mathsf{L}(\mathcal{C})$ of $\mathcal{C}$ consists of all propositional formulas φ such that for all $C \in \mathcal{C}$ and all substitutions $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$ we have that $C \vDash \varphi^{\sigma}$.

Fact. We always have $\mathsf{IPC} \subseteq \mathsf{L}(\mathcal{C}) \subseteq \mathsf{CPC}$, i.e., $\mathsf{L}(\mathcal{C})$ is an *intermediate logic*.

Definition

We say that $\mathcal{C}$ *is loyal* if $\mathsf{L}(\mathcal{C}) = \mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Proposition

If $\mathcal{C}$ is faithful, then $\mathcal{C}$ is loyal.

Definition

The propositional logic $\mathsf{L}(\mathcal{C})$ of $\mathcal{C}$ consists of all propositional formulas φ such that for all $C \in \mathcal{C}$ and all substitutions $\sigma : \mathrm{Prop} \to \mathcal{L}^{\mathrm{sent}}$ we have that $C \vDash \varphi^\sigma$.

Fact. We always have $\mathsf{IPC} \subseteq \mathsf{L}(\mathcal{C}) \subseteq \mathsf{CPC}$, i.e., $\mathsf{L}(\mathcal{C})$ is an *intermediate logic*.

Definition

We say that $\mathcal{C}$ *is loyal* if $\mathsf{L}(\mathcal{C}) = \mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Proposition

If $\mathcal{C}$ is faithful, then $\mathcal{C}$ is loyal.

Definition

The propositional logic $\mathsf{L}(\mathcal{C})$ of $\mathcal{C}$ consists of all propositional formulas φ such that for all $C \in \mathcal{C}$ and all substitutions $\sigma : \mathrm{Prop} \to \mathcal{L}^{\mathrm{sent}}$ we have that $C \vDash \varphi^\sigma$.

Fact. We always have $\mathsf{IPC} \subseteq \mathsf{L}(\mathcal{C}) \subseteq \mathsf{CPC}$, i.e., $\mathsf{L}(\mathcal{C})$ is an *intermediate logic*.

Definition

We say that $\mathcal{C}$ *is loyal* if $\mathsf{L}(\mathcal{C}) = \mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Proposition

If $\mathcal{C}$ is faithful, then $\mathcal{C}$ is loyal.

Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. The *propositional logic* $\mathsf{L}(\mathrm{T})$ is the set of all propositional formulas φ such that for all substitutions $\sigma : \mathrm{Prop} \to \mathcal{L}^{\mathrm{sent}}$ we have $\mathrm{T} \vdash \varphi^{\sigma}$.

Definition

The *de Jongh property for a theory* T *with respect to an intermediate logic* J is the statement $\mathsf{L}(\mathrm{T}(\mathrm{J})) = \mathrm{J}$.

The de Jongh property for HA is **De Jongh's Theorem**.

Proposition

If $\mathcal{C}$ is loyal and $\mathcal{C} \vDash \mathrm{T}$, then T *has the de Jongh property with respect to* $\mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. The *propositional logic* $\mathsf{L}(\mathsf{T})$ is the set of all propositional formulas φ such that for all substitutions $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$ we have $\mathsf{T} \vdash \varphi^\sigma$.

Definition

The *de Jongh property for a theory* T *with respect to an intermediate logic* J is the statement $\mathsf{L}(\mathsf{T}(\mathsf{J})) = \mathsf{J}$.

The de Jongh property for HA is **De Jongh's Theorem**.

Proposition

If $\mathcal{C}$ is loyal and $\mathcal{C} \vDash \mathsf{T}$, then T *has the de Jongh property with respect to* $\mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. The *propositional logic* $\mathsf{L}(\mathsf{T})$ is the set of all propositional formulas φ such that for all substitutions $\sigma : \mathrm{Prop} \to \mathcal{L}^{\mathrm{sent}}$ we have $\mathsf{T} \vdash \varphi^\sigma$.

Definition

The *de Jongh property for a theory* T *with respect to an intermediate logic* J is the statement $\mathsf{L}(\mathsf{T}(\mathsf{J})) = \mathsf{J}$.

The de Jongh property for HA is **De Jongh's Theorem**.

Proposition

If $\mathcal{C}$ is loyal and $\mathcal{C} \vDash \mathsf{T}$, then T *has the de Jongh property with respect to* $\mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. The *propositional logic* $\mathsf{L}(\mathsf{T})$ is the set of all propositional formulas φ such that for all substitutions $\sigma : \mathrm{Prop} \to \mathcal{L}^{\mathrm{sent}}$ we have $\mathsf{T} \vdash \varphi^{\sigma}$.

Definition

The *de Jongh property for a theory* T *with respect to an intermediate logic* J is the statement $\mathsf{L}(\mathsf{T}(\mathsf{J})) = \mathsf{J}$.

The de Jongh property for HA is **De Jongh's Theorem**.

Proposition

If $\mathcal{C}$ is loyal and $\mathcal{C} \models \mathsf{T}$, then T *has the de Jongh property with respect to* $\mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. The *propositional logic* $\mathsf{L}(\mathsf{T})$ is the set of all propositional formulas φ such that for all substitutions $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$ we have $\mathsf{T} \vdash \varphi^\sigma$.

Definition

The *de Jongh property for a theory* T *with respect to an intermediate logic* J is the statement $\mathsf{L}(\mathsf{T}(\mathsf{J})) = \mathsf{J}$.

The de Jongh property for HA is **De Jongh's Theorem**.

Proposition

If $\mathcal{C}$ is loyal and $\mathcal{C} \vDash \mathsf{T}$, then T *has the de Jongh property with respect to* $\mathsf{L}(\mathcal{H}_{\mathcal{C}})$.

Definition

An *Iemhoff model* $K(\mathcal{M})$ consists of a Kripke frame $(K, \leq)$ and a *sound* assignment $\mathcal{M}$ of nodes of K to transitive models of ZF set theory.

Let CZF* be the theory CZF^{-c} + Bounded Strong Collection + Set-bounded Subset Collection.

Theorem (Iemhoff)

Let $K(\mathcal{M})$ be an Iemhoff model. Then $K(\mathcal{M}) \Vdash$ CZF.*

Definition

An *Iemhoff model* $K(\mathcal{M})$ consists of a Kripke frame $(K, \leq)$ and a *sound* assignment $\mathcal{M}$ of nodes of K to transitive models of ZF set theory.

Let CZF^* be the theory CZF^{-c} + Bounded Strong Collection + Set-bounded Subset Collection.

Theorem (Iemhoff)

Let $K(\mathcal{M})$ be an Iemhoff model. Then $K(\mathcal{M}) \Vdash \mathrm{CZF}^*$.

Definition

An *Iemhoff model* $K(\mathcal{M})$ consists of a Kripke frame $(K, \leq)$ and a *sound* assignment $\mathcal{M}$ of nodes of K to transitive models of ZF set theory.

Let CZF* be the theory CZF^{-c} + Bounded Strong Collection + Set-bounded Subset Collection.

Theorem (Iemhoff)

Let $K(\mathcal{M})$ be an Iemhoff model. Then $K(\mathcal{M}) \Vdash$ CZF.*

Theorem (P.)

The class of Iemhoff models is faithful, and therefore loyal.

Corollary

The theory CZF has the de Jongh property with respect to every logic characterised by a class of Kripke frames.*

Theorem (P.)

Iemhoff models that involve forcing non-trivially do not satisfy the axiom of exponentiation.

Theorem (P.)

The class of Iemhoff models is faithful, and therefore loyal.

Corollary

The theory CZF* *has the de Jongh property with respect to every logic characterised by a class of Kripke frames.*

Theorem (P.)

Iemhoff models that involve forcing non-trivially do not satisfy the axiom of exponentiation.

Theorem (P.)

The class of Iemhoff models is faithful, and therefore loyal.

Corollary

The theory CZF* *has the de Jongh property with respect to every logic characterised by a class of Kripke frames.*

Theorem (P.)

Iemhoff models that involve forcing non-trivially do not satisfy the axiom of exponentiation.

Definition

Let H be a complete Heyting algebra and $M \vDash$ ZFC and inductively define $M^{(H)} = \bigcup_{\alpha \in \mathrm{Ord}^M} M_\alpha^{(H)}$ (within M) to be the class of H-valued sets. This is a generalisation of the well-known Boolean valued models for set theory.

Theorem

If H is a complete Heyting algebra, then $M^{(H)} \vDash$ IZF.

If B is a complete Boolean algebra, then $M^{(B)} \vDash$ ZFC.

Definition

Let H be a complete Heyting algebra and $M \models$ ZFC and inductively define $M^{(H)} = \bigcup_{\alpha \in \mathrm{Ord}^M} M_\alpha^{(H)}$ (within M) to be the class of H-valued sets. This is a generalisation of the well-known Boolean valued models for set theory.

Theorem

If H is a complete Heyting algebra, then $M^{(H)} \models$ IZF.

If B is a complete Boolean algebra, then $M^{(B)} \models$ ZFC.

Theorem (P.)

The class of Heyting-valued models that are based on a finite Heyting algebra is not loyal. Indeed, the propositional logic of this class contains KC $(= \mathsf{IPC} + \neg\neg p \vee \neg p)$.

Theorem (Löwe, P. & Tarafder)

The class of Heyting-valued models is not faithful. In particular, it is not faithful to any Heyting algebra with a non-trivial automorphism.

Observation

The class of Boolean-valued models is loyal, but not faithful.

Theorem (P.)

The class of Heyting-valued models that are based on a finite Heyting algebra is not loyal. Indeed, the propositional logic of this class contains KC *(*$=$ IPC $+ \neg\neg p \vee \neg p$*).*

Theorem (Löwe, P. & Tarafder)

The class of Heyting-valued models is not faithful. In particular, it is not faithful to any Heyting algebra with a non-trivial automorphism.

Observation

The class of Boolean-valued models is loyal, but not faithful.

Theorem (P.)

The class of Heyting-valued models that are based on a finite Heyting algebra is not loyal. Indeed, the propositional logic of this class contains KC *(=* IPC $+ \neg\neg p \vee \neg p$*).*

Theorem (Löwe, P. & Tarafder)

The class of Heyting-valued models is not faithful. In particular, it is not faithful to any Heyting algebra with a non-trivial automorphism.

Observation

The class of Boolean-valued models is loyal, but not faithful.

Theorem (Galeotti & P.)

*The class of Heyting-valued models that are based on a finite **linear** Heyting algebra is faithful, and therefore loyal.*

Corollary

The theory IZF *has the de Jongh property with respect to* LC $(= \mathsf{IPC} + p \to q \vee q \to p)$.

Theorem (Galeotti & P.)

*The class of Heyting-valued models that are based on a finite **linear** Heyting algebra is faithful, and therefore loyal.*

Corollary

The theory IZF *has the de Jongh property with respect to* LC *(*$= \mathsf{IPC} + p \to q \vee q \to p$*).*

OPEN QUESTIONS AND FURTHER RESEARCH

- Give a complete characterisation of all Heyting-valued models in terms of loyalty and faithfulness.
- Extend the analysis to other model constructions (such as modal set theories or Boolean (Heyting?) ultrapowers, topological models, Topos theoretic semantics etc.) and other theories.
- What happens if we replace Heyting algebras by weaker (or different) structures? (see also Löwe, P. & Tarafder 2018)
- Are there interesting higher versions of faithfulness in connection to infinitary logics?

OPEN QUESTIONS AND FURTHER RESEARCH

- Give a complete characterisation of all Heyting-valued models in terms of loyalty and faithfulness.
- Extend the analysis to other model constructions (such as modal set theories or Boolean (Heyting?) ultrapowers, topological models, Topos theoretic semantics etc.) and other theories.
- What happens if we replace Heyting algebras by weaker (or different) structures? (see also Löwe, P. & Tarafder 2018)
- Are there interesting higher versions of faithfulness in connection to infinitary logics?

OPEN QUESTIONS AND FURTHER RESEARCH

- Give a complete characterisation of all Heyting-valued models in terms of loyalty and faithfulness.
- Extend the analysis to other model constructions (such as modal set theories or Boolean (Heyting?) ultrapowers, topological models, Topos theoretic semantics etc.) and other theories.
- What happens if we replace Heyting algebras by weaker (or different) structures? (see also Löwe, P. & Tarafder 2018)
- Are there interesting higher versions of faithfulness in connection to infinitary logics?

OPEN QUESTIONS AND FURTHER RESEARCH

- Give a complete characterisation of all Heyting-valued models in terms of loyalty and faithfulness.
- Extend the analysis to other model constructions (such as modal set theories or Boolean (Heyting?) ultrapowers, topological models, Topos theoretic semantics etc.) and other theories.
- What happens if we replace Heyting algebras by weaker (or different) structures? (see also Löwe, P. & Tarafder 2018)
- Are there interesting higher versions of faithfulness in connection to infinitary logics?

Dick de Jongh. "The maximality of the intuitionistic predicate calculus with respect to Heyting's arithmetic". In: *The Journal of Symbolic Logic* 35.4 (1970), p. 606.

Rosalie Iemhoff. "Kripke models for subtheories of CZF". In: *Archive for Mathematical Logic* 49.2 (2010), pp. 147–167.

- Dick de Jongh, Rineke Verbrugge, and Albert Visser. "Intermediate Logics and the de Jongh property". In: *Archive for Mathematical Logic* 50.1 (Feb. 2011), pp. 197–213.
- Benedikt Löwe, Robert Passmann, and Sourav Tarafder. *Constructing illoyal algebra-valued models of set theory*. In preparation. 2018.
- Robert Passmann. "Loyalty and Faithfulness of Model Constructions for Constructive Set Theory". Master of Logic Thesis (MoL) Series MoL-2018-03. MA thesis. ILLC, University of Amsterdam, 2018.

Thank you!

Robert Passmann
ILLC, Universiteit van Amsterdam
robertpassmann@posteo.de