Worksheet 1 Solutions

Fall 2018

MATH 222, Week 1: Review

Name: ________________________________

1 Trig Identity Review

Problem 1. Use the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to show that $\tan^2(\theta) + 1 = \sec^2(\theta)$.

Solution 1. Divide the given identity by $\cos^2(\theta)$ to get

$$\left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 + 1 = \frac{1}{\cos(\theta)^2}.$$

We can recognize this as the exact identity we wished to derive. $\square$

Problem 2. (a) Circle the correct answer:

$2\sin(\theta)\cos(\theta) =$	$\sin(2\theta)$	$\cos(2\theta)$
$\cos^2(\theta) - \sin^2(\theta) =$	$\sin(2\theta)$	$\cos(2\theta)$

(b) Using part (a) and $\sin^2(\theta) + \cos^2(\theta) = 1$, prove the following half angle formulas:

(a) $\cos^2(\theta) = \frac{1}{2}(\cos(2\theta) + 1)$. There's a very similar identity for $\sin^2(\theta)$ that could be useful later on.

(b) $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

Solution 2.

(a) Recall that

$$2\sin(\theta)\cos(\theta) = \sin(2\theta)$$

and

$$\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$$

You aren't meant to derive these in any way, they are here as a reminder because they will be very important in the weeks to come.

(b) We have to do some work with the given trig identities. We start with the classic $\sin^2(\theta) + \cos^2(\theta) = 1$. Our hope is to find an alternative expression for $\cos^2(\theta)$ so we will isolate it

$$\cos^2(\theta) = 1 - \sin^2(\theta).$$

Now from the above second identity we know $\sin^2(\theta) = \cos^2(\theta) - \cos(2\theta)$, substituting this in we find

$$\cos^2(\theta) = 1 - (\cos^2(\theta) - \cos(2\theta)) = 1 + \cos(2\theta) - \cos^2(\theta).$$

If we add $\cos^2(\theta)$ to both sides we find

$$2\cos^2(\theta) = 1 + \cos(2\theta).$$

Dividing by 2 gives the desired identity.

(c) Whenever we see tan it is usually a good idea to try to express it in terms of cos and sin because we have more experience with these. So let's start there

$$\tan(2\theta) = \tfrac{\sin(2\theta)}{\cos(2\theta)}.$$

We want to get rid of the 2θ. From part (a) we can replace both the numerator and denominator as

$$\tfrac{\sin(2\theta)}{\cos(2\theta)} = \tfrac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)}.$$

Now the 2θ is gone and we have some algebraic simplification to do. If we factor a $\cos^2(\theta)$ out of the numerator and denominator (equivalently you could say we divide the numerator and denominator by $\cos^2(\theta)$) we have

$$\tfrac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)} = \tfrac{2\frac{\sin(\theta)}{\cos(\theta)}}{1-\frac{\sin^2(\theta)}{\cos^2(\theta)}}$$

But this is exactly the desired identity.

$\square$

2 Integration and Fundamental Theorem of Calculus Review

Problem 3. For each of the following, state whether the object is a function or a number. If it is a function, state what variable it is a function of.

(a) $\int_e^x e^{\sec^2(\ln(t))} dt$

(b) $\int \arcsin(x) dx$

(c) $\int_1^3 \arctan(s) ds$

(d) $\int_0^{\pi t} \arccos(x) dx$

(e) $\int_x^{x^2} \cos^3(t) \sin^2(t)\, dt$

(f) $\int_t^t f(x)\, dx$

Solution 3.
(a) This is a function of x.
(b) This is a function of x (one of many really).
(c) This is a number.
(d) This is a function of t.
(e) This is a function of x.
(f) This is a function of t but it's also a number. (It's the constant 0 function.)

$\square$

Problem 4. Compute $\int_0^x (\int_0^t \cos(s) ds) dt$.

Solution 4. We work from inside out. First we focus on the integral $\int_0^t \cos(s) ds = \sin(s)|_0^t = \sin(t)$ because $\sin(0) = 0$. If we plug this back in we now wish to find

$$\int_0^x \sin(t) dt.$$

Recall that the antiderivative of $\sin(t)$ is $-\cos(t)$, so

$$\int_0^x \sin(t) dt = -\cos(t)|_0^x = 1 - \cos(x).$$

$\square$

Problem 5. Define $f(x) = \int_x^{x^2} e^{t^3} dt$. Compute $f'(x)$. Hint: Split the integral into $\int_x^1$ and $\int_1^{x^2}$ and use the Fundamental Theorem of Calculus.

Solution 5.

$$\begin{aligned} f(x) &= \int_1^{x^2} e^{t^3} dt + \int_x^1 e^{t^3} dt \\ &= \int_1^{x^2} e^{t^3} dt - \int_1^x e^{t^3} dt \\ f'(x) &= e^{(x^2)^3}(2x) - e^{x^3} \end{aligned}$$

$\square$

Problem 6. Let a be any fixed real constant. Compute $\frac{d}{dx}\int_{x^3}^{a} \ln(t)\ dt$. (Hint: Fundamental Theorem of Calculus).

Solution 6. Suppose that we have a function $F(t)$ that is the antiderivative of $\ln t$, that choose $F(t)$ so that $F'(t) = \ln(t)$, then by the fundamental theorem of calculus

$$\begin{aligned} \frac{d}{dx}\int_{x^3}^{a} \ln t\ dt &= \frac{d}{dx}\left[F(t)\right]_{t=x^3}^{t=a} \\ &= \frac{d}{dx}\left(F(a) - F(x^3)\right) \\ &= 0 - F'(x^3)2x^2 \\ &= -3x^2 \ln x^3 \end{aligned}$$

This is because we know $F'(t) = \ln(t)$. Here we essentially derive the other part of the fundamental theorem of calculus. The key is, you never need to know $F(t)$, we just need to know that $F'(t) = \ln(t)$. $\square$

3 Challenge Problem

Problem 7. Compute $\int \sin^2(\theta)\cos^2(\theta)\ d\theta$. There are at least two ways to approach this.

Solution 7.

$$\begin{aligned} \int \sin^2(\theta)\cos^2(\theta)\ d\theta &= \int (\sin(\theta)\cos(\theta))^2\ d\theta \\ &= \int \tfrac{1}{4}\sin(2\theta)^2\ d\theta. \end{aligned}$$

Where here use use the half angle identity for $\sin(2\theta)$ from problem 2. From here we have to use integration by parts, let $u = \sin(2\theta)^2$ and $dv = 1$. This means $du = 4\sin(2\theta)\cos(2\theta)$ and $v = \theta$. Notice that $du = 2\sin(4\theta)$ from problem 2 again. Now this means we have

$$\int \tfrac{1}{4}\sin(2\theta)^2\ d\theta = \tfrac{1}{4}\left[\theta\sin(2\theta)^2 - 2\int \theta\sin(4\theta)\ d\theta\right]$$

We again apply integration by parts. Let $u = \theta$ and $dv = \sin(4\theta)$, then $du = 1$ and $v = -\frac{1}{4}\cos(4\theta)$. Using the integration by parts formula we have

$$\int \theta\sin(4\theta)\ d\theta = -\tfrac{\theta}{4}\cos(4\theta) + \tfrac{1}{4}\int \cos(4\theta)\ d\theta.$$

We can compute this final integral, we find

$$\tfrac{1}{4}\int \cos(4\theta)\ d\theta = \tfrac{1}{16}\sin(4\theta) + C$$

Plugging all of this into the original equation we have

$$\int \theta \sin(4\theta)\ d\theta = \frac{1}{4}\left[\theta \sin(2\theta)^2 - 2\left(-\frac{\theta}{4}\cos(4\theta) + \cdot\frac{1}{16}\sin(4\theta)\right) + C\right]$$

If we expand and simplify this becomes

$$\frac{\theta}{4}\sin(2\theta)^2 + \frac{\theta}{8}\cos(4\theta) - \frac{1}{32}\sin(4\theta) + C$$

Although this is correct, there are further simplifications we can make. Using the half-angle identities from problem 2 again we can express $\cos(4\theta) = \cos^2(2\theta) - \sin^2(2\theta)$ so we have,

$$\frac{\theta}{4}\sin(2\theta)^2 + \frac{\theta}{8}(\cos^2(2\theta) - \sin^2(2\theta)) - \frac{1}{32}\sin(4\theta) + C$$

Now if we expand this out this simplifies to

$$\frac{\theta}{8}(2\sin(2\theta)^2 - \sin(2\theta)^2 + \cos(2\theta)^2) - \frac{1}{32}\sin(4\theta) + C$$

But $\sin(2\theta)^2 + \cos(2\theta)^2 = 1$ so this expression actually equals

$$\frac{\theta}{8} - \frac{1}{32}\sin(4\theta) + C.$$

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