MATH 222 (Lectures 1,2, and 4) Fall 2015

**Midterm 1 Solutions** Student ID#: ____________________________

Circle your TA's name from the following list.

| | | | | |
|---|---|---|---|---|
| Carolyn Abbott | Tejas Bhojraj | Zachary Carter | Mohamed Abou Dbai | Ed Dewey |
| Jale Dinler | Di Fang | Bingyang Hu | Canberk Irimagzi | Chris Janjigian |
| Tao Ju | Ahmet Kabakulak | Dima Kuzmenko | Ethan McCarthy | Tung Nguyen |
| Jaeun Park | Adrian Tovar Lopez | Polly Yu | | |

Please inform your TA if you find any errors in the solutions.

| | Problem 1 | Problem 2 | Problem 3 | Problem 4 | Problem 5 | Problem 6 | Problem 7 |
|---|---|---|---|---|---|---|---|
| Score | | | | | | | |

## Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- On Problems 1, 2, and 3, only the answer will be graded.
- On Problems 4, 5, 6, and 7 you must show your work and we will grade the work and your justification, and not just the final answer.
- Each problem worth either 14 or 15 points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1-x^2}$. Note that we have provided some formulas on the next page to help with this.

### Formulas

You may freely quote any algebraic or trigonometric identity, as well as well as any of the following formulas or minor variants of those formulas.

- $\cos(\arcsin x) = \sqrt{1-x^2}$
- $\sec(\arctan x) = \sqrt{1+x^2}$.
- $\tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$.
- $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ \ln|x| + C & \text{when } n = -1 \end{cases}$
- $\int e^x dx = e^x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$.
- $\int \csc x dx = -\ln|\csc x + \cot x| + C$.
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.

1. For each statement below, CIRCLE true or false.

| (a) | | (b) | | (c) | | (d) | | (e) | |
|---|---|---|---|---|---|---|---|---|---|
| True | False | True | False | True | False | True | False | True | False |

(a) If $\frac{x}{7} = \cos\theta$ then $\tan\theta = \frac{\sqrt{49-x^2}}{x}$.

(b) $\int 3\sin^2(\theta)d\theta = \frac{\sin^3\theta}{\cos\theta} + C$

(c) $\frac{1+\sin(x)}{x^3} \geq \frac{1}{x^3}$ for all $x \geq 1$.

(d) $\int_2^\infty \frac{1}{x^2-9}dx$ is a finite number.

(e) $\int_3^\infty \frac{x-\sqrt{x}}{3x^3+11}dx$ is a finite number.

**Solution:**

(a) True.

(b) False.

(c) False.

(d) False.

(e) True.

2. On this page, only the answer will be graded.

(a) Compute $\int \sin^2(x) - \cos^2(x) dx$.

**Solution:**

$$\begin{aligned} \int \sin^2(x) - \cos^2(x)x &= \int \sin^2(x) - (1 - \sin^2(x))dx \\ &= \int (2\sin^2(x) - 1)dx \\ &= \int (1 - \cos(2x)) - 1)dx \\ &= - \int \cos(2x)dx \\ &= -\frac{1}{2}\sin(2x) + C \end{aligned}$$

(b) Compute $\int \frac{4}{(x-1)(3x+1)} dx$.

**Solution:** We rewrite this in the form:

$$\int \frac{4}{(x-1)(3x+1)} dx = \int \frac{1}{x-1} - \frac{3}{3x+1}$$

Solving using the method of equating coefficients yields $A = 1$ and $B = -3$.

(c) Compute $\int_{-3}^{\infty} \frac{1}{x^2+6x+10} dx$.

**Solution:**

$$\begin{aligned} \int_{-3}^{\infty} \frac{1}{x^2+6x+10} &= \lim_{b\to\infty} \int_{-3}^{b} \frac{1}{1+(x+3)^2} dx \\ &= \lim_{b\to\infty} [\arctan(x+3)]_{-3}^{b} \\ &= \lim_{b\to\infty} (\arctan(b+3) - \arctan(0)) = \pi/2. \end{aligned}$$

3. On this page, only the answer will be graded.

(a) Find a positive number $A$ such that $\int_{100}^{\infty} \frac{1}{x^2+73x-5} dx < A$.

**Solution:** Any $A$ bigger than .0075 will work.

(b) Compute $\int xe^{7x+1} dx$.

**Solution:** Let $f = x$ and $g' = e^{7x+1}$ so that $f' = 1$ and $g = \frac{1}{7}e^{7x+1}$. Then

$$\begin{aligned}\int xe^{7x+1} dx &= \int fg' \\ &= fg - \int f'g \\ &= \frac{x}{7}e^{7x+1} - \frac{1}{7}\int e^{7x+1} dx \\ &= \frac{x}{7}e^{7x+1} - \frac{1}{49}e^{7x+1} + C\end{aligned}$$

(c) Compute $\int \frac{1}{\sqrt{2x-x^2}} dx$.

**Solution:** Complete the square to get $2x - x^2 = 1 - (x-1)^2$. Then we get:

$$\int \frac{1}{\sqrt{2x-x^2}} dx = \int \frac{1}{\sqrt{1-(x-1)^2}} dx$$

Using $x - 1 = \sin\theta$ and $dx = \cos\theta d\theta$ this yields:

$$\begin{aligned}&= \int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} \\ &= \int 1 d\theta \\ &= \theta + C \\ &= \arcsin(x-1) + C.\end{aligned}$$

4. Compute $\displaystyle\int_1^\infty \frac{4x+3}{x(2x+1)(2x+3)}dx$ or explain why the integral does not exist. (You may freely use the formula $\frac{4x+3}{x(2x+1)(2x+3)} = \frac{1}{x} - \frac{1}{2x+1} - \frac{1}{2x+3}$.)

**Solution:** We compute:

$$\begin{aligned}
\int_1^\infty \frac{4x+3}{x(2x+1)(2x+3)}dx &= \int_1^\infty \frac{1}{x} - \frac{1}{2x+1} - \frac{1}{2x+3}dx \\
&= \lim_{b\to\infty} \left[\ln|x| - \frac{1}{2}\ln|2x+1| - \frac{1}{2}\ln|2x+3|\right]_1^b \\
&= \lim_{b\to\infty} \left[\ln \frac{|x|}{\sqrt{(2x+1)(2x+3)}}\right]_1^b \\
&= \lim_{b\to\infty} \left[\ln \frac{|x|}{\sqrt{4x^2+8x+3}}\right]_1^b \\
&= \lim_{b\to\infty} \ln \frac{b}{\sqrt{4b^2+8b+3}} - \ln\frac{1}{\sqrt{15}} \\
&= \ln(\frac{1}{\sqrt{4}}) - \ln(\frac{1}{15}) = \ln(\frac{\sqrt{15}}{2})
\end{aligned}$$

There are other equivalent answers.

5. Compute $\int (z+e^z)\sin(3z)dz$.

**Solution:** We compute this as the sum of two integrals:

$$\int (z+e^z)\sin(3z)dz = \int z\sin(3z)dz + \int e^z\sin(3z)dz$$

For $\int z\sin(3z)dz$ we "double back". First do integration by parts with $f = z$ so $f' = 1$ and $g' = \sin(3z)$ so $g = -\frac{1}{3}\cos(3z)$. And we get:

$$\begin{aligned}\int z\sin(3z)dz &= fg - \int f'g \\ &= -\frac{1}{3}z\cos(3z) + \frac{1}{3}\int \cos(3z)dz \\ &= -\frac{1}{3}z\cos(3z) + \frac{1}{9}\sin(3z) + C\end{aligned}$$

Then we let $I = \int e^z\sin(3z)dz$. We first integrate by parts with $f = \sin(3z)$ and $g' = e^z$. Then $f' = 3\cos(3z)$ and $g = e^z$, yielding:

$$\begin{aligned}I = \int e^z\sin(3z)dz &= fg - \int f'g \\ &= \sin(3z)e^z - 3\int \cos(3z)e^z dz\end{aligned}$$

We integrate by parts again, with $h = \cos(3z)$ and $k' = e^z$ so $h' = -3\sin(3z)$ and $k = e^z$:

$$\begin{aligned}&= \sin(3z)e^z - 3\int hk' \\ &= \sin(3z)e^z - 3\left(hk - \int h'k\right) \\ &= \sin(3z)e^z - 3\left(\cos(3z)e^z - \int(-3\sin(3z))e^z dz\right) \\ &= \sin(3z)e^z - 3\cos(3z)e^z - 9\int \sin(3z))e^z dz \\ &= \sin(3z)e^z - 3\cos(3z)e^z - 9I\end{aligned}$$

We thus have the equation:

$$I = \sin(3z)e^z - 3\cos(3z)e^z - 9I$$

which, after moving all of the $I$ terms to the left side, yields:

$$(1+9)\,I = \sin(3z)e^z - 3\cos(3z)e^z + C.$$

We thus obtain:

$$I = \frac{1}{10}\left(\sin(3z)e^z - 3\cos(3z)e^z\right) + C$$

Putting this together yields:

$$\int (z+e^z)\sin(3z)dz = -\frac{1}{3}z\cos(3z) + \frac{1}{9}\sin(3z) + \frac{1}{10}\left(\sin(3z)e^z - 3\cos(3z)e^z\right)$$

6. Compute $\int e^{-x}\sqrt{4-e^{2x}}dx$.

**Solution:** Set $z = e^x$ so that $dz = e^x dx$ and $dx = \frac{dz}{e^x} = \frac{dz}{z}$. Then we have:

$$\int e^{-x}\sqrt{4-e^{2x}}dx = \int z^{-2}\sqrt{4-z^2}dz$$

Now let $z = 2\sin\theta$ so that $dz = 2\cos\theta d\theta$ and we get:

$$\begin{aligned}
&= \int (2\sin\theta)^{-2}\sqrt{4-4\sin^2\theta}2\cos\theta d\theta \\
&= \int \frac{4\cos^2\theta}{4\sin^2\theta}d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}d\theta \\
&= \int (\cot^2\theta - 1)d\theta \\
&= \int \csc^2\theta - 1d\theta \\
&= -\cot\theta - \theta + C
\end{aligned}$$

Since $\sin(\theta) = \frac{z}{2}$ we get $\theta = \arcsin(\frac{z}{2})$ and $\cot(\theta) = \frac{\sqrt{4-z^2}}{z}$ yielding:

$$\begin{aligned}
&= -\frac{\sqrt{4-z^2}}{z} - \arcsin(\frac{z}{2}) + C \\
&= -\frac{\sqrt{4-e^{2x}}}{e^x} - \arcsin(\frac{e^x}{2}) + C
\end{aligned}$$

7. (a) For $n = 0, 1, \ldots$ let $I_n = \int x^n e^{13x+2} dx$. Derive a reduction formula for $I_n$.

(b) Let $J_n = \int x^5 (\ln x)^n dx$ for $n \geq 0$. This satisfies the reduction formula $J_n = (\ln x)^n \frac{x^6}{6} - \frac{n}{6} J_{n-1}$ for $n \geq 1$. Compute $J_2$.

**Solution:**

(a): Let $f = x^n$ so $f' = nx^{n-1}$ and let $g' = e^{13x+2}$ so that $g = \frac{1}{13} e^{13x+2}$ Then:

$$\begin{aligned} I_n = \int x^n e^{13x+2} dx &= \int f g' \\ &= fg - \int f' g \\ &= \frac{1}{13} x^n e^{13x+2} - \frac{n}{13} \int x^{n-1} e^{13x+2} dx \\ &= \frac{1}{13} x^n e^{13x+2} - \frac{n}{13} I_{n-1} \end{aligned}$$

(b): $J_0 = \int x^5 dx = \frac{x^6}{6} + C$. Then

$$J_1 = (\ln x)^1 \frac{x^6}{6} - \frac{1}{6} J_0 = \frac{1}{6} x \ln x - \frac{x^6}{36} + C.$$

Then

$$J_2 = (\ln x)^2 \frac{x^6}{6} - \frac{2}{6} J_1 = (\ln x)^2 \frac{x^6}{6} - \frac{1}{3} \left( \frac{1}{6} x \ln x - \frac{x^6}{36} \right) + C$$

If you simplify, you might get other, equivalent, answers.