Review Sheet

Spring 2016

MATH 222, REVIEW

Name: ________________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

1 Topics

1.1 Chapter 3: First Order ODEs

(a) Separable differential equations
(b) Integrating Factors
(c) Direction Fields
(d) Euler's Method
(e) Word Problems: Mixing, Population

1.2 Chapter 4: Taylor Series

(a) Definition/computation of Taylor Series and Taylor Polynomials
(b) Special Taylor Series: $e^x, \cos(x), \sin(x), \frac{1}{1-x}, \ln(1+x)$. See p. 78.
(c) Remainder Term
(d) Lagrange's Remainder Theorem
(e) Little-oh Notation definition, properties and usages
(f) Computing Taylor series from other taylor series. See p. 88-93.

This is not necessarily a complete list. To make sure you hit everything consult your notes and the textbook. The following is a list of some more practice problems. For other practice problems see my website for old exams, your homeworks and the worksheets.

2 Practice Problems

ANY SOLUTIONS THAT AREN'T HERE ARE IN ONE OF THE WORKSHEETS

Problem 1. Find a general solution to the following differential equations and solve the initial value problem when possible:

(a) $\frac{dy}{dx} = e^y x^3$, $y(0) = 0$.

(b) $\frac{dy}{dx} = (1 + y^2)e^x$, $y(0) = 0$.

(c) $\frac{dy}{dx} = y\sqrt{y^2 - 1}\cos(x)$, $y(0) = 1$.

(d) $x\frac{dy}{dx} = -y + x$

(e) $\frac{1}{2x}\frac{dy}{dx} = y + e^{x^2}$

(f) $x\frac{dy}{dx} + 2y = \frac{\cos(x)}{x}$

Problem 2. Solve the following initial value problem exactly, then use Euler's method with step size of $\Delta x = .1$ to approximate $y(1.3)$. $y(1) = 0$ and

$$\frac{dy}{dx} = 1 + y$$

Problem 3. Use Euler's Method with step size 0.1 to approximate $\sin(0.3)$. For sanity: $\sin(0.3) \approx 0.29552$.

Solution.

We need a differential equation whose solution is $\sin(x)$. We don't want $y' = \cos(x)$ because although this is true it's not useful at all. However we note that $y'' = -y$ is satisfied by $\sin(x)$. Most initial conditions we could state would be fairly useless because we don't know the value of sin for many values. However we do know $y(0) = 0$ and $y'(0) = 1$. So we'll use these in Euler's Method. This also makes sense because we want to approximate $\sin(0.3)$ so now we need to take three steps.

The next cool step here is that we'll use $y''(x)$ to approximate $y'(x)$. This will be a double Euler's Method. So to start we note that $x = 0$ $y(0) = 0$ and $y'(0) = 1$ and so $y(0.1) \approx y(0) + .1y'(0) = .1$. Alright so we set $y(0.1) = .1$ and iterate again.

Except now we need $y'(0.1)$. To find this we'll approximate it with Euler's Method. We know $y'(0) = 1$ and $y''(0) = 0$. So Euler's Method tells us $y'(0.1) \approx 1 + .1(0) = 1$. Now using this to approximate $y(0.2)$ we have $y(0.2) \approx y(0.1) + .1(y'(0.1)) \approx .1 + (.1)(1) = .2$.

Now we do this one more time! We want to find $y'(0.2)$, so we'll apply Euler's Method. $y'(0.2) \approx y'(0.1) + (0.1)(y''(0.1)) \approx 1 + (0.1)(-0.1) = .99$. Using this we can approximate $y(0.3) \approx y(0.2) + (0.1)(y'(0.2)) \approx .2 + (0.1)(.99) = .299$. That's a pretty good estimate! This problem is far harder than something you'd see on an exam, but it's extremely instructive and really interesting! □

Problem 4. A tank begins with 100 litres of salt water in it. Fresh water is pumped in at a rate of twenty litres per minute and the mixed water is pumped out at a rate of ten litres per minute. If the tank initially has ten kilograms of salt in it, find an equation for the amount of salt left in the tank in kilograms as a function of time. Note that the volume of the water in the tank is changing.

Problem 5. You just graduated and you cash in at your graduation party! You made \$3500. You, being incredibly responsible, decide to place all of your money in a savings account where you earn interest on it compounded continuously at a rate of 5%. How much money do you have after 10 years? How long will it take for your investment to double in value? What interest rate would you need for your investment to double in 5 years?

Problem 6. A 100 litre vat of water begins with an algae concentration of 1,000 organisms per litre. Suppose that the algae naturally reproduce at a rate of five percent per minute and die at a rate of four percent per minute. If the vat is being drained at a rate of one litre per minute, what will the algae concentration be ten minutes from now? You should assume that the algae are uniformly distributed in the vat. Remember to define your variables with units.

Problem 7. Solve the following initial value problem exactly, then compute its degree two Taylor polynomial around zero and use this to compute an estimate for $y(.3)$. Then use Euler's method with step size $\Delta x = .1$ to estimate $y(.3)$.

$$\frac{dy}{dx} = -2xy$$

$y(0) = 1$. Just as a sanity check, the true value of $y(.3)$ is about .914.

Problem 8. State Lagrange's Remainder Theorem and why it is important/how it is used.

Problem 9. Recompute the taylor series for $\sin(x)$. Use this to find a taylor series for $\sin(3x)$. Find a bound for $R_n^0 \sin(3x)$ and use this to show that $T_n^0 \sin(3x) \to \sin(3x)$ for all x as $n \to \infty$.

Problem 10. Recompute the taylor series for $\frac{1}{1-t}$. Use this to find $T_{2n}\{\frac{1}{1+x^2}\}$. Use this to find $T_{2n+1}\{\arctan(x)\}$.

Solution.

$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$. Hence if we substitute $t \mapsto -x^2$ this is valid since $0 \mapsto 0$. So $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. Hence $T_{2n}\{\frac{1}{1+x^2}\} = 1 - x^2 + \cdots + (-1)^n x^{2n}$. Recall that $\displaystyle\int \frac{1}{1+x^2} = \arctan(x)$. Hence if we integrate this taylor polynomial we'll get the Taylor Polynomial for $\arctan(x)$. So

$$T_{2n+1}\{\arctan(x)\} = x - \frac{x^3}{3} + \cdots + (-1)^n x^{2n+1}/(2n+1)$$

You could try to compute this explicitly, but it wouldn't be much fun. □

Problem 11. Is it true that $\cos(x) - 1 + x^2/2 = o(x^4)$?

Problem 12. Show that $e^x - \sqrt{1+2x} = o(x)$.

Problem 13. Compute the Taylor Series for $\frac{1}{\sqrt{1-x}}$. Use this to find the Taylor series for $\frac{1}{\sqrt{1-x^2}}$ and the Taylor Series for $\arcsin(x)$.

Solution.

Repeat the process in Problem 10. □