Practice Final b
MATH 222 (Lectures 1,2, and 4) Fall 2015.

Name: ______________________________ Student ID#: ________________________

Circle your TAs name:

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	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5
Score					
	Problem 6	Problem 7	Problem 8	Problem 9	Problem 10
Score					

Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- Please read the instructions on every problem carefully.
- On Problems 1–4 only the answer will be graded.
- On Problems 5–10 you must show your work and we will grade the work and your justification, and not just the final answer.
- Each problem is worth ten points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1 - x^2}$. Note that we have provided some formulas on the next page to help with this.

Formulas

You may freely quote any algebraic or trigonometric identity, as well as well as any of the following formulas or minor variants of those formulas.

Integrals

- $\cos(\arcsin x) = \sqrt{1-x^2}$
- $\sec(\arctan x) = \sqrt{1+x^2}$.
- $\tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$.
- $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ \ln|x| + C & \text{when } n = -1 \end{cases}$
- $\int e^x dx = e^x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$.
- $\int \csc x dx = -\ln|\csc x + \cot x| + C$.
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.

Taylor series

- $T_\infty e^x = \sum_{k=0}^\infty \frac{x^k}{k!}$
- $T_\infty \sin x = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
- $T_\infty \cos x = \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{(2k)!}$
- $T_\infty \frac{1}{1-x} = \sum_{k=0}^\infty x^k$
- $T_\infty \frac{1}{1+x} = \sum_{k=0}^\infty (-1)^k x^k$
- $T_\infty (1+x)^b = \sum_{k=0}^\infty \binom{b}{k} x^k$ where $\binom{b}{k} = \frac{b(b-1)(b-2)\cdots(b-k+1)}{k!}$

Other

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

1. For each statement below, CIRCLE true or false. You do not need to show your work.

(a)		(b)		(c)		(d)		(e)	
True	False	True	False	True	False	True	False	True	False

(a) The series $\sum_{k=1}^{\infty} \frac{\cos^k(1/k)}{2^{2k+3}}$ converges.

(b) The series $\sum_{k=1}(1+\frac{1}{k})^k$ converges.

(c) The integral $\int_0^1 \frac{1}{\sqrt{x}} + \frac{1}{(x-2)^2}$ converges.

(d) The integral $\int_1^{\infty} \frac{1}{x\ln(x)} dx$ converges.

(e) If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ then $\vec{b} = \vec{c}$.

(a) True.

(b) False.

(c) True

(d) False

(e) False.

2. On this page only the answer will be graded.

(a) Compute $\int \frac{1}{\sqrt{4-(x+2)^2}}dx$.

Solution: Let $x+2 = 2\sin\theta$. Then $dx = 2\cos\theta d\theta$ and we get

$$\begin{aligned}
\int \frac{1}{\sqrt{4-(x+2)^2}}dx &= \int \frac{1}{\sqrt{4-4\sin^2\theta}}2\cos\theta d\theta \\
&= \int \frac{2\cos\theta d\theta}{2\cos\theta} \\
&= \int d\theta \\
&= \theta + C
\end{aligned}$$

Since $x+2 = 2\sin\theta$ we get $\frac{x+2}{2} = \sin\theta$ and $\theta = \arcsin(\frac{x+2}{2})$:

$$= \arcsin(\tfrac{x+2}{2}) + C$$

(b) Compute $\int \frac{x^2}{x(x-1)}dx$.

Solution: This is a partial fractions problem, though it's kind of a trick problem since you can cancel an x from the numerator and denominator and get:

$$\begin{aligned}
\int \frac{x^2}{x(x-1)}dx &= \int \frac{x}{x-1)}dx \\
&= \int \frac{(x-1)+1}{x-1}dx \\
&= \int 1 + \frac{1}{x-1}dx \\
&= x + \ln|x-1| + C.
\end{aligned}$$

3. On this page only the answer will be graded.

(a) Compute $T_3(\sin x \cos x)$.

Solution: We have

$$\begin{aligned} T_\infty(\sin x \cos x) &= T_\infty \sin x T_\infty \cos x \\ &= (x - \frac{x^3}{3!} + o(x^3))(1 - \frac{x^2}{2!} + o(x^3)) \\ &= x - \frac{x^3}{2!} - \frac{x^3}{3!} + o(x^3) \\ &= x - (\frac{1}{2!} + \frac{1}{3!})x^3 + o(x^3) \end{aligned}$$

Thus $\boxed{T_3(\sin x \cos x) = x - (\frac{1}{2!} + \frac{1}{3!})x^3 = x - \frac{2}{3}x^3.}$

(b) For which values of x does $\sum_{k=0}^{\infty} \frac{2^{2k+3}}{x^k}$ converge?

Solution: Note that $\sum_{k=0}^{\infty} \frac{2^{2k+3}}{x^k} = \sum_{k=0}^{\infty} 2^3 \left(\frac{2^2}{x}\right)^k$ which is a geometric series $\sum_{k=0}^{\infty} ar^k$ with $a = 2^3$ and $r = \frac{2^2}{x} = \frac{4}{x}$. By the Geometric Series test we know that this series converges if and only if $|r| < 1$. This is equivalent to asking that $|\frac{4}{x}| < 1$ which is equivalent to $|x| > 4$. Thus the series converges if and only if $|x| > 4$.

4. On this page, only the answer will be graded.

(a) Let ℓ and m be the lines $\ell : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+3s \\ 2+4s \end{pmatrix}$ and $m : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+t \\ 2t \end{pmatrix}$. Compute the angle between ℓ and m. You do not need to simplify any arccos value.

Solution: The direction vectors for the lines are $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. To compute the angle between these, we use the dot product formula $a \cdot b = ||a|| ||b|| \cos\theta$. This yields

$$3 + 8 = 5\sqrt{5}\cos\theta$$

Thus $\cos\theta = \frac{11}{5\sqrt{5}}$ and so $\boxed{\theta = \arccos\left(\frac{11}{5\sqrt{5}}\right)}$.

(b) Compute $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 6 \\ -5 \end{pmatrix}$.

Solution: $\begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$.

(c) Compute the distance from the point $(1, 1, 0)$ to the plane $x + 2y + 3z = 0$.

Solution: The normal vector to the plane is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Let ℓ be the line in that direction and containing $(1, 1, 0)$ so that ℓ is the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+t \\ 1+2t \\ 3t \end{pmatrix}$$

The intersection of this line and the plane is obtained by plugging these coordinates into the equation for the plane. This yields:

$$(1+t) + 2(1+2t) + 3(3t) = 0$$

Rewriting gives the equation $3 + 14t = 0$ and thus $t = -\frac{3}{14}$. This yields the point $(\frac{11}{14}, \frac{8}{14}, -\frac{9}{14})$ on the line ℓ. You can check that this is also on the plane a $\frac{11}{14} + 2(\frac{8}{14}) + 3(\frac{-9}{14}) = 0$.

The distance from $(1, 1, 0)$ to $(\frac{11}{14}, \frac{8}{14}, -\frac{9}{14})$ is $\boxed{\sqrt{(\frac{3}{14})^2 + (\frac{6}{14})^2 + (\frac{9}{14})^2}}$. You could leave this unsimplified or note that it simplifies to $\frac{3}{\sqrt{14}}$.

5. On this page, you must show your work to receive full credit.

   (a) Compute $T_2(\frac{1}{\cos x})$.

   Solution: Let $f(x) = \frac{1}{\cos x} = \sec x$. we will compute this Taylor polynomial by han.d $f'(x) = \sec x \tan x$ and $f''(x) = \sec^3 x + \sec^2 x \tan^2 x$. Plugging in $x = 0$ we get $f(0) = 1$; $f'(0) = 0$ and $f''(0) = 1$. Thus $\boxed{T_2(\frac{1}{\cos x}) = 1 + \frac{x^2}{2!}.}$

   (b) A tank begins with 100 litres of salt water in it at 10:00am. Fresh water is pumped in at a rate of twenty litres per minute and the mixed water is pumped out at a rate of ten litres per minute. Let t stand for time in minutes from 10:00am and let $A(t)$ for the amount of salt in the tank, in kilograms, at time t. Write down a differential equation for $A(t)$. DO NOT SOLVE THE DIFFERENTIAL EQUATION.

Solution: $\frac{dA}{dt} = -\frac{10A}{100+10t}$

6. On this page, you must show your work to receive full credit.

(a)
$$\frac{dy}{dx} = \frac{x + xy^2}{2y} \qquad \text{and} \qquad y(0) = e.$$

Solution:

$$\begin{aligned} \frac{dy}{dx} &= x\frac{1+y^2}{2y} \\ \frac{2y}{1+y^2}dy &= xdx \\ \ln(1+y^2) &= \frac{x^2}{2} + C \\ 1 + y^2 &= e^{x^2/2+C} \end{aligned}$$

Since $1 + e^2 = e^{0^2/2+C}$ we see that $\ln(1 + e^2) = C$ and

$$y = \pm\sqrt{e^{x^2/2+\ln(1+e^2)} - 1}$$

We choose the positive sign to ensure that this satisfies the initial condition, yielding:

$$\boxed{y = \sqrt{e^{x^2/2+\ln(1+e^2)} - 1}.}$$

(b)
$$x\frac{dy}{dx} = y + x^2e^x \qquad \text{and} \qquad y(1) = 10 + e.$$

NOTE: The original version of this problem had a typo. It should have been x^2e^x not xe^x.

We rewrite this as $\frac{dy}{dx} - \frac{1}{x}y = xe^x$. This yields $m(x) = e^{-\ln|x|} = |x|^{-1} = x^{-1}$ since $x = |x|$ near the initial value $x = 1$. We then get

$$y = x\int x^{-1} \cdot xe^x dx = x\int e^x = x(e^x + C)$$

Using the initial condition we get $10 + e = 1(e + C)$ and thus $C = 10$ yielding: $\boxed{y = x(e^x + 10).}$

7. On this page, you must show your work to receive full credit.

$$\text{Compute } \int x\arctan(x) + \frac{2}{1-x^2}dx.$$

Solution: We solve this as two separate integrals. The first is

$$\int x\arctan(x)dx = \int fg'$$

Where $f = \arctan x$ and $g' = x$ so $f' = \frac{1}{1+x^2}$ and $g = \frac{x^2}{2}$:

$$\begin{aligned}
&= fg - \int f'g \\
&= \frac{1}{2}x^2 \arctan x - \int \frac{x^2}{2(1+x^2)}dx \\
&= \frac{1}{2}x^2 \arctan x - \int \left(\frac{1}{2} - \frac{1}{2(1+x^2)}\right) dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{x}{2} + \frac{1}{2}\arctan(x) + C
\end{aligned}$$

For the second integral we do partial fractions. We rewrtie $\frac{2}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$. Equating coefficeints yields $2 = A(1-x) + B(1+x) = (-A+B)x + (A+B)$. Solving this yileds $A = B = 1$. So we compute

$$\begin{aligned}
\int \frac{2}{1-x^2}dx &= \int \frac{1}{1+x} + \frac{1}{1-x}dx \\
&= \ln|1+x| - \ln|1-x| + C
\end{aligned}$$

Combining these yields

$$\boxed{\int x\arctan(x) + \frac{2}{1-x^2}dx = \frac{1}{2}x^2 \arctan x - \frac{x}{2} + \frac{1}{2}\arctan(x) + \ln|1+x| - \ln|1-x| + C}$$

8. On this page, you must show your work and justify your answer to receive full credit.

Compute $\int_1^\infty \frac{8}{x^3-4x}dx$ or explain why the integral does not exist. You may freely use the formula $\frac{8}{x^3-4x} = -\frac{2}{x} + \frac{1}{x+2} + \frac{1}{x-2}$.

Solution: This integral is not finite (in other words, it does not exist). We write the integral as

$$\int_1^\infty \frac{8}{x^3-4x}dx = \int_1^2 \frac{8}{x^3-4x}dx + \int_2^3 \frac{8}{x^3-4x}dx + \int_3^\infty \frac{8}{x^3-4x}dx.$$

We will show that the first of these integrals is not finite. We have:

$$\begin{aligned}
\int_1^2 \frac{8}{x^3-4x}dx &= \int_1^2 -\frac{2}{x} + \frac{1}{x+2} + \frac{1}{x-2}dx \\
&= \lim_{b\to 2-} \int_1^2 -\frac{2}{x} + \frac{1}{x+2} + \frac{1}{x-2} \\
&= \lim_{b\to 2-} \left(-2\ln|x| + \ln|x+2| + \ln|x-2|\right) \\
&= \left(-2\ln|2| + \ln|4| + \ln|0|\right) = -\infty
\end{aligned}$$

$\boxed{\text{So this integral does not exist.}}$

9. On this page, you must show your work and justify your answer to receive full credit.

   We have a 300 gallon tank that begins full of milk containing 2% butterfat. Let t stand for time in minutes from 10:00am and let $A(t)$ stand for the total amount of butterfat in gallons in the vat. Starting at 10:00am, $A(t)$ satisfies the differential equation:

$$\frac{dA}{dt} = .1 - \frac{15A}{100 - 5t}.$$

   Compute A(t).

   Solution: Our initial condition is $A(0) = 6$. Rewriting slightly, this is a linear differential equation

$$\frac{dA}{dt} + \frac{3A}{20 - t} = \frac{1}{10}.$$

   We compute $m(t) = e^{\int \frac{3}{20-t}dx} = e^{-3\ln|20-t|} = |20 - t|^{-3} = (20 - t)^{-3}$ near the initial value $t = 0$. We then have

$$\begin{aligned} A(t) &= \frac{1}{m(t)} \int m(t)k(t)dt \\ &= (20 - t)^3 \int (20 - t)^{-3} \frac{1}{10} dt \\ &= (20 - t)^3 \cdot \frac{1}{10} \cdot \left(\frac{(20 - t)^{-2}}{2} + C \right) \\ &= \frac{20 - t}{20} + \frac{C(20 - t)^3}{10} \end{aligned}$$

   Since $A(0) = 6$ we get $6 = \frac{20-0}{20} + \frac{C(20-0)^3}{10} = 1 + \frac{C \cdot 20^3}{10}$ and so $5 = 800C$ and so $C = \frac{1}{160}$.

   This yields a final answer of: $\boxed{A(t) = \frac{20-t}{20} + \frac{(20-t)^3}{1600}}$

10. On this page, you must show your work and justify your answer to receive full credit. Justifying your answer must include: clearly stating any convergence test that you use and explicitly verifying each hypothesis for that test.

$$\text{Does } \sum_{k=0}^{\infty} \frac{e^{-k} + k + \sqrt{k}}{e^k + k^3 + \sqrt[3]{k}} \text{ converge?}$$

Solution We first use the Limit Comparison Theorem. For this, we set $a_k = \frac{e^{-k}+k+\sqrt{k}}{e^k+k^3+\sqrt[3]{k}}$ to be the terms of the original series and west $b_k = \frac{k}{e^k}$. We observe that these satisfy both hypotheses, namely:

- a_k and b_k are both nonnegative; this is because a_k is the sum and fractions of non-negative terms, and so is b_k.
- $$\begin{aligned} \lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{e^{-k} + k + \sqrt{k}}{e^k + k^3 + \sqrt[3]{k}} \cdot \frac{e^k}{k} \\ &= \lim_{k\to\infty} \frac{1 + ke^k + \sqrt{k}e^k}{ke^k + k^4 + k^{4/3}} \\ &= \lim_{k\to\infty} \frac{ke^k}{ke^k} \left(\frac{\frac{1}{ke^k} + 1 + \frac{1}{\sqrt{k}}}{1 + \frac{k^3}{e^k} + \frac{k^{1/3}}{e^k}} \right) \\ &= 1 \left(\frac{0+1+0}{1+0+0} \right) = 1 \end{aligned}$$

  where we use the fact that exponential beats polynomial for the limits in the denominator. Since $0 < 1 < \infty$, this satisfies the second hypothesis.

Thus a_k and b_k satisfy the hypotheses of the Limit Comparison Theorem. So we apply the theorem and conclude that either both series converge or both series diverge.

We now consider the simpler series $\sum_{k=0}^{\infty} \frac{k}{e^k}$. We can use the Ratio Test here (the Integral Test would work too!). We compute

$$L = \lim_{k\to\infty} |\frac{a_{k+1}}{a_k}| = \lim_{k\to\infty} \frac{(k+1)e^k}{e^{k+1}k} = \lim_{k\to\infty} \frac{(k+1)}{k \cdot e} = \frac{1}{e}.$$

Since $L = \frac{1}{e} < 1$ we conclude that this series converges by the Ratio Test. It then follows that the original series $\sum_{k=0}^{\infty} \frac{e^{-k}+k+\sqrt{k}}{e^k+k^3+\sqrt[3]{k}}$ also converges.

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