Worksheet 11

Spring 2016

MATH 222, Week 11: Sequences!

Name: ______________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Problem 1. Find

$$\lim_{n\to\infty} \frac{n^2+n+1}{3n^2-n-2}$$

Solution 1.

The limit is $\frac{1}{3}$. Remember the limit of the quotient of two polynomials of the same degree is the quotient of the leading coefficients. □

Problem 2. Find an example of a sequence a_n which is bounded but not convergent.

Solution 2.

Let $a_n = (-1)^n$. It never converges but it is bounded. □

Problem 3. Let $a_n = (-1)^n$ for $n = 1, 2, 3, \ldots$.

(a) Does a_n converge? i.e. does $\lim_{n\to\infty} a_n = L$ for some real number L? If it exists, what is it?
(b) Let $f(x) = x^2$. Does $f(a_n)$ converge? If so, what is $\lim_{n\to\infty} f(a_n)$?
(c) Try to state in words what (a) and (b) illustrate.

Solution 3.

(a) No see the last problem.
(b) $f(a_n) = (-1)^{2n} = 1$, so $\lim_{n\to\infty} f(a_n) = 1$.
(c) This illustrates that even though a sequence may not converge a function applied to the sequence can still converge.

□

Problem 4. Let's try to think of the last problem in the opposite direction. Let $a_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$.

(a) Does a_n converge? If so, what is $\lim_{n\to\infty} a_n$?
(b) Define a function on the interval from 0 to 1 by

$$f(x) = \begin{cases} (-1)^{\frac{1}{x}} & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

Does $f(a_n)$ converge? If so, what is $\lim_{n\to\infty} f(a_n)$?

(c) Does $f(\lim_{n\to\infty} a_n)$ exist, and if so what is it?
(d) Try to state in words what (a) and (b) illustrate.

Solution 4.

(a) Yes it does. We see $\lim_{n\to\infty} \frac{1}{n} = 0$.
(b) No it does not. $f(a_n) = (-1)^n$ which we know does not converge.
(c) We see $f(\lim_{n\to\infty} a_n) = f(0) = 0$.
(d) This shows that even if we have a convergent sequence, if f is not continuous $f(a_n)$ may not converge.

$\square$

Problem 5. We've essentially been playing with examples that lead to an interesting question that I want you to try to answer. Under what circumstances is it true that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$?

Solution 5.

The answer to this is in the book. The equality will hold if f is continuous. To better understand this I definitely recommend checking the book! $\square$