# Limits of Differences of Roots

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Hopefully Helpful Notes

## 1 Square Roots

I want to give you all a quick way of dealing with a limit of a difference of square roots like:

$$\lim_{x\to\infty} \sqrt{x^2-2x+1} - \sqrt{x^2+3x}$$

At first your intuition might tell you that the limit should be zero, but this will not always be the case. Intuitively you are right in a sense, these two terms will behave very similarly as $x$ gets large and hence they should go to a limit. However, where this limit settles is not obvious and requires some work.

There is a relatively standard way to approach these. The main issue here is that you are subtracting one of the terms and because everything is in square roots you can't cancel anything. If you were adding both of them you would know for sure the limit is infinity and if the square roots were gone the subtraction wouldn't matter as much. So our goal is to exploit these two observations using the difference of squares:

$$(a^2 - b^2) = (a-b)(a+b)$$

As long as $a + b \neq 0$ we can rewrite this as

$$(a-b) = \frac{a^2-b^2}{a+b}$$

Now if we let $a = \sqrt{x^2-2x+1}$ and $b = \sqrt{x^2+3x}$ we notice a few nice things. First $a^2 - b^2$ is not an issue because if we square both of the square roots, we can start to combine what's inside. Then we know we can find $\lim_{x\to\infty} a + b$ because we are adding the two square roots rather than subtracting them. This means we should be able to evaluate this limit

$$\lim_{x\to\infty} \sqrt{x^2-2x+1} - \sqrt{x^2+3x} = \lim_{x\to\infty} \frac{(x^2-2x+1)-(x^2+3x)}{\sqrt{x^2-2x+1}+\sqrt{x^2+3x}} = \lim_{x\to\infty} \frac{-5x+1}{\sqrt{x^2-2x+1}+\sqrt{x^2+3x}}$$

Now the numerator still has an $x$ in it, so we are not done yet. If the numerator did not have an $x$ the limit would be zero because the denominator would be growing incredibly large while the numerator remained constant. If this occurs, you can use a common and incredibly important trick to simplify the denominator. Factor out the highest degree term from the square roots. In this case it is $x^2$, so in our example

$$\sqrt{x^2-2x+1}+\sqrt{x^2+3x} = \sqrt{x^2(1-2/x+1/x^2)} + \sqrt{x^2(1+3/x)}$$

We know square roots split over products BUT NOT OVER SUMS! So

$$\sqrt{x^2(1-2/x+1/x^2)} + \sqrt{x^2(1+3/x)} = |x|\sqrt{1-2/x+1/x^2} + |x|\sqrt{1+3/x}$$

We are taking the limit as $x \to \infty$ so we are only concerned with positive values of $x$. Hence we know $|x| = x$, so our denominator is:

$$x(\sqrt{1-2/x+1/x^2} + \sqrt{1+3/x})$$

This simplification is incredibly nice because the terms on the inside go to 2. So we have essentially factored out the term responsible for making the denominator go to infinity. Plugging this back in

$$\lim_{x\to\infty} \frac{-5x+1}{\sqrt{x^2-2x+1}+\sqrt{x^2+3x}} = \lim_{x\to\infty} \frac{x(-5+1/x)}{x(\sqrt{1-2/x+1/x^2}+\sqrt{1+3/x})}$$

We see the $x$ term cancels! This is what we would expect from our intuition in the beginning. Both of these terms behave like $x$ when $x$ gets large, so the limit should settle somewhere. Now what is left over will be where the limit settles:

$$\lim_{x\to\infty} \frac{x(-5+1/x)}{x(\sqrt{1-2/x+1/x^2}+\sqrt{1+3/x})} = \lim_{x\to\infty} \frac{(-5+1/x)}{(\sqrt{1-2/x+1/x^2}+\sqrt{1+3/x})}$$

Both of these limits exist so from our limit rules we can take the limit of the numerator and denominator separately and it will equal our limit. If we let $x \to \infty$ all of the terms with an $x$ in the denominator go to 0, so we find that

$$\lim_{x\to\infty} \sqrt{x^2-2x+1}-\sqrt{x^2+3x} = \lim_{x\to\infty} \frac{(-5+1/x)}{(\sqrt{1-2/x+1/x^2}+\sqrt{1+3/x})} = -\frac{5}{2}$$

This general technique can be applied to any of the difference of square roots problems. It is tedious, but will give you the right result! Now we're going to tackle a slightly harder version.

## 2 Difference of Cube Roots

In the previous section we worked with the limit of the difference of square roots, but you can come across a similar problem but with cube roots (or really any root you'd like):

$$\lim_{x\to\infty} \sqrt[3]{x^3-2x^2+1}-\sqrt[3]{x^3+3x^2}$$

Let's take a similar example as above except with cube roots this time and see what happens. We have the same issue, we're subtracting two cube roots and we can't simplify because everything is contained in the roots. We once again observe that if the roots were gone and if we were adding the roots the limits would be much easier. However, in this case we can't just multiply by

$$\frac{\sqrt[3]{x^3-2x^2+1}+\sqrt[3]{x^3+3x^2}}{\sqrt[3]{x^3-2x^2+1}+\sqrt[3]{x^3+3x^2}}$$

This intuition is good because you notice this problem is related to the square root case, but this will not give you the desired cancelation. What we next think about is that we were using the difference of squares formula above, what if we try to exploit the difference of cubes formula here? Recall these from your first worksheet!

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

Just as above, if $a^2+ab+b^2 \neq 0$ we can rewrite this as

$$a-b = \frac{a^3-b^3}{a^2+ab+b^2}$$

Now this is what we want! If we let $a = \sqrt[3]{x^3-2x^2+1}$ and $b = \sqrt[3]{x^3+3x^2}$ this cubes both of the cube roots in the numerators and has a bunch of terms being ADDED in the denominator! Using this we find

$$\lim_{x\to\infty} \sqrt[3]{x^3-2x^2+1}-\sqrt[3]{x^3+3x^2} = \lim_{x\to\infty} \frac{x^3-2x^2+1-(x^3+3x^2)}{\sqrt[3]{(x^3-2x^2+1)^2}+\sqrt[3]{(x^3-2x^2+1)(x^3+3x^2)}+\sqrt[3]{(x^3+3x^2)^2}}$$

Now the denominator looks pretty terrible but the numerator is far nicer and we'll see it's not too terrible working with the denominator. Simplifying a little

$$\lim_{x\to\infty} \frac{-5x^2+1}{\sqrt[3]{x^6(1+\text{Terms that go to 0})}+\sqrt[3]{x^6(1+\text{Terms that go to 0})}+\sqrt[3]{x^6(1+\text{Terms that go to 0})}}$$

We do the same simplification trick in the denominator as above. When we multiply out these polynomials we pull out the leading term and the only number that will matter in the limit is the coefficient of the leading term because every other term will go to zero as $x \to \infty$. Explicitly for the first polynomial:

$$(x^3-2x^2+1)^2 = x^6-4x^5+4x^4+2x^3-4x^2+1 = x^6(1-4/x+4/x^2+2/x^3-4/x^4+1/x^6)$$

So in the limit every other term vanishes and we are left with the coefficient of $x^6$ along with $x^6$. Think about this for all of the other polynomials. Once again cube roots split over products and $\sqrt[3]{x^6}=x^2$, so we can rewrite the denominator as

$$\lim_{x\to\infty} \frac{-5x^2+1}{x^2\sqrt[3]{(1+\text{Terms that go to 0})}+x^2\sqrt[3]{(1+\text{Terms that go to 0})}+x^2\sqrt[3]{(1+\text{Terms that go to 0})}}$$

Now if we factor $x^2$ out of each of the terms in the denominator we have

$$\lim_{x\to\infty} \frac{x^2(-5+1/x^2)}{x^2(\sqrt[3]{(1+\text{Terms that go to 0})}+\sqrt[3]{(1+\text{Terms that go to 0})}+\sqrt[3]{(1+\text{Terms that go to 0})}}$$

Once again we see that the $x^2$ cancels! So we can take these limits separately and all of those terms that go to 0 as $x \to \infty$ vanish leaving us with

$$\frac{-5}{3}$$

In all we found

$$\lim_{x\to\infty} \sqrt[3]{x^3-2x^2+1}-\sqrt[3]{x^3+3x^2} = -\frac{5}{3}$$

This was harder than the square root case and you probably won't find one this complicated on your exams but I wanted to illustrate the general technique that will usually work for any limit of a difference of roots you see. Let me know if you have any other questions!