Worksheet 9

Spring 2016

MATH 222, Week 9: Taylor Series!

Name: ________________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Problem 1. Write the following series in summation notation:
(a) $1 + x + x^2 + x^3 + x^4 + \cdots$
(b) $1 + x^2 + x^4 + x^6 + \cdots$
(c) $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ (This is an extremely important Taylor Series! Which one?)

Solution 1.

(a) $\sum_{i=0}^{\infty} x^i$

(b) $\sum_{i=0}^{\infty} x^{2i}$

(c) $\sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}$. This is the taylor series for $\cos(x)$.

$\square$

Problem 2. Compute the second order Taylor polynomial of $\sin(x^2)$ around 0 and use this to approximate $\sin(1/4)$. Note that the actual value is $\sin(1/4) \approx 0.247404$

Solution 2.

By definition $T_2(\sin(x^2)) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$ where $f(x) = \sin(x)$. So we need to find $f'(0)$ and $f''(0)$. $f'(0) = 2(0)\cos(0) = 0$ and $f''(0) = -2(0)\sin(0) + 2\cos(0) = 2$. Hence

$$T_2(\sin(x^2)) = \frac{2}{2}x^2 = x^2$$

This should make sense because we know $\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = 1$, so intuitively this tells us that close to 0 $\sin(x^2) \approx x^2$. To find $\sin(1/4)$ we let $x = 1/2$, and our approximation yields $\sin(1/4) \approx 1/4 = .25$ so it's not too far off. $\square$

Problem 3. Compute the degree two Taylor polynomial of the function $f(x) = e^{\tan(x)}$ around 0. Use this to estimate $e^{\tan(.1)}$. Note that the actual value is $e^{\tan(.1)} \approx 1.10554$.

Solution 3.

We do the same thing as above. $T_2(e^{tan(x)}) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$ where $f(x) = e^{tan(x)}$. $f'(0) = \sec^2(0)e^{\tan(0)} = 1$ and $f''(0) = \sec^4(0)e^{\tan(0)} + 2\sec^2(0)\tan(0)e^{\tan(0)} = 1$. Hence

$$T_2(e^{tan(x)}) = 1 + x + \frac{1}{2}x^2$$

So our approximation tells us that $e^{\tan(0.1)} \approx 1 + .1 + \frac{.01}{2} = 1.105$ which is pretty great and makes sense because .1 is pretty close to 0. $\square$

Problem 4. Find the second order Taylor polynomial around 0 for $f(x) = \int_0^x e^{-t^2} dt$ and use this to estimate $f(.1)$. This allows us to approximate this integral for different bounds! To show you how useful this is, try to think about taking the antiderivative of e^{-t^2}.

Solution 4.

We do the same thing as above. $T_2(f(x)) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$ where $f(x) = \int_0^x e^{-t^2} dt$. $f'(0) = e^0 = 1$ and $f''(0) = 0$. This uses the fundamental theorem of calculus. We also know $f(0) = 0$ so

$$T_2(f(x)) = x$$

So our approximation tells us that $f(.1) \approx .1$. The actual value is .99667 so this is pretty great. $\square$

Problem 5. Solve the following initial value problem exactly, then compute its degree two Taylor polynomial around zero and use this to compute an estimate for $y(.3)$. Then use Euler's method with step size $\Delta x = .1$ to estimate $y(.3)$.

$$\frac{dy}{dx} = -2xy$$

$y(0) = 1$. Just as a sanity check, the true value of $y(.3)$ is about .914.

Solution 5.

This is separable, once we separate it is pretty straight forward to solve and find

$$y = e^{-x^2}$$

To find the second order Taylor polynomial for this we just use the Taylor series for e^t and substitute $t = -x^2$.

$$T_2(e^{-x^2}) = 1 - x^2$$

So our approximation for $y(.3)$ is $1 - (.3)^2 = .91$. Which is pretty close. Now let's use Euler's Method. We know $y(0) = 1$, and that when $x = 0$ and $y = 1$ $m = 0$, so $y(.1) \approx y(0) + mh = 1$. We then continue to iterate. If we let $y = 1$ and $x = .1$, then $m = -.2$ and $y(.2) \approx y(.1) + mh = 1 + (-.2)(.1) = .98$. Now $y = .98$ and $x = .2$, then $m = -.196$ and $y(.3) \approx y(.2) + mh = .98 + (-.196)(.1) = .9604$. This is fairly close, but our step size could have been better. $\square$

Problem 6. Hasdrubal has designed a rocket. While proving mathematically that it won't explode, he used the approximation $e^{1/3} \approx 1 + \frac{1}{3} + \frac{1}{3^2 2!} + \frac{1}{3^3(3!)}$ If this approximation is off by more than $\frac{2}{4!}\left(\frac{1}{3}\right)^4$, the rocket might blow up. Convince Hasdrubal that it won't.

Solution 6.

See Next Worksheet $\square$

Problem 7. Find a bound for $R_n^0 \sin(3x)$ and use this to show that $T_n^0 \sin(3x) \to \sin(3x)$ for all x as $n \to \infty$.

Solution 7.

See Next Worksheet. $\square$