MATH 222(001) Fall 2018
Worksheet 6 October 8– Solutions

Please inform your TA if you find any errors in the solutions.

1. (a) True or false? $\displaystyle\int \frac{1}{1+x^2}\,dx = \ln|1+x^2| + C$.
Solution: False. Don't forget that if $u = 1+x^2$, then $du = 2x\,dx$. The correct answer is $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$.

(b) True or false? $\dfrac{1}{x^2} \le \dfrac{1+\sin x\cos x}{x^2}$ for $x \ge 1$.
Solution: False:

$$\frac{1+\sin x\cos x}{x^2} = \frac{1}{x^2} + \frac{\sin x\cos x}{x^2},$$

and for $x \ge 1$, we have $\sin x\cos x = \frac{1}{2}\sin(2x)$ wave between positive values and negative values.

(c) True or false? $\dfrac{1}{x^2} \le \dfrac{1+\sin x\cos x}{x^2}$ for $0 \le x \le \frac{\pi}{2}$.
Solution: True:

$$\frac{1+\sin x\cos x}{x^2} = \frac{1}{x^2} + \frac{\sin x\cos x}{x^2} = \frac{1}{x^2} + \frac{\sin(2x)}{2x^2}.$$

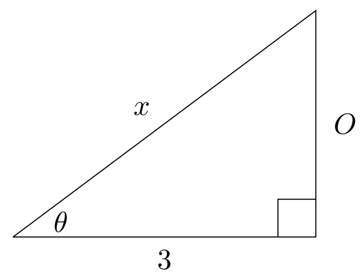
For $0 \le x \le \frac{\pi}{2}$ or $0 \le 2x \le \pi$, we have $\sin(2x) \ge 0$.

(d) True or false? $x^x \ge 1$ for $0 < x < 1$.
Solution: False. Take log of both sides: $\ln 1 = 0$, and

$$\ln(x^x) = x\ln x < 0 = \ln 1,$$

since $\ln x < 0$ for $0 < x < 1$.

(e) If $x = \sqrt{2}\sec\theta$, what is $\tan\theta$ equal to in terms of x?
Solution: We have the following triangle:



Using Pythagoras' theorem, we know that $O = \sqrt{x^2 - 2}$, so

$$\tan\theta = \frac{\sqrt{x^2-2}}{\sqrt{2}}.$$

(f) Given the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,dx,$$

write down a reduction formula for $\displaystyle\int_0^{\frac{\pi}{4}} \cos^n x\,dx$.

Solution:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \cos^n x\,dx &= \frac{1}{n}\sin x\cos^{n-1}x\Big|_{x=0}^{\frac{\pi}{4}} + \frac{n-1}{n}\int_0^{\frac{\pi}{4}}\cos^{n-2}x\,dx \\
&= \frac{1}{n}\sin\frac{\pi}{4}\cos^{n-1}\frac{\pi}{4} - \frac{1}{n}\sin(0)\cos^{n-1}(0)\frac{n-1}{n}\int_0^{\frac{\pi}{4}}\cos^{n-2}x\,dx \\
&= \frac{1}{n}\left(\frac{1}{\sqrt{2}}\right)^{1+n-1} + \frac{n-1}{n}\int_0^{\frac{\pi}{4}}\cos^{n-2}x\,dx \\
&= \frac{1}{n\sqrt{2}^n} + \frac{n-1}{n}\int_0^{\frac{\pi}{4}}\cos^{n-2}x\,dx.
\end{aligned}$$

2. Compute $\int x\ln(x)dx$.

Solution:

$$\begin{aligned}
\int x\ln(x)dx &= \int \underbrace{\ln(x)}_{\text{F(x)}}\ \underbrace{xdx}_{\text{G'(x)dx}} \\
&= \underbrace{\frac{x^2}{2}}_{\text{G(x)}}\ \underbrace{\ln(x)}_{\text{F(x)}} - \int \underbrace{\frac{x^2}{2}}_{\text{G(x)}}\ \underbrace{\frac{1}{x}dx}_{\text{F'(x)dx}} \\
&= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int xdx \\
&= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C
\end{aligned}$$

3. Compute $\int \frac{1}{y\sqrt{1-y^2}}dy$.

Solution:

$$\begin{aligned}
\int \frac{1}{y\sqrt{1-y^2}}dy &= \int \frac{-\sin(\theta)d\theta}{\cos(\theta)\sqrt{1-\cos^2(\theta)}} \qquad\qquad y=\cos(\theta) \quad dy=-\sin(\theta)d\theta \\
&= \int \frac{-d\theta}{\cos(\theta)} \\
&= -\int \sec(\theta)d\theta \\
&= -\ln|\sec(\theta)+\tan(\theta)|+C \\
&= -\ln|\sec(\arccos(y))+\tan(\arccos(y))|+C \\
&= -\ln|\frac{1}{\cos(\arccos(y))}+\frac{\sin(\arccos(y))}{\cos(\arccos(y))}|+C \\
&= -\ln|\frac{1}{y}+\frac{\sqrt{1-y^2}}{y}|+C
\end{aligned}$$

4. Compute $\int \frac{dx}{\sqrt{1-e^{2x}}}$.

Solution:

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-e^{2x}}} &= \int \frac{du}{u\sqrt{1-u^2}} && u = e^x \quad \frac{du}{u} = dx \\
&= \int \frac{\cos(\theta)}{\sin(\theta)\sqrt{1-\sin^2(\theta)}} && u = \sin(\theta) \quad du = \cos(\theta)d\theta \\
&= \int \csc(\theta)d\theta \\
&= -\ln|\csc(\theta)+\cot(\theta)| + C \\
&= -\ln|\csc(\arcsin(u)) + \cot(\arcsin(u))| + C \\
&= -\ln|\frac{1}{u} + \frac{\sqrt{1-u^2}}{u}| + C \\
&= \ln|\frac{e^x}{1+\sqrt{1-e^{2x}}}| + C \\
&= x - \ln(1+\sqrt{1-e^{2x}}) + C
\end{aligned}$$

5. Compute $\int_0^1 \ln(2t+1)dt$.

Solution:

$$\begin{aligned}
\int_0^1 \ln(2t+1)dt &= \frac{1}{2}\int_{t=0}^{t=1} \underbrace{\ln(u)}_{\text{F(u)}} \underbrace{du}_{\text{G'(u)du}} && u = 2t+1 \quad \frac{1}{2}du = dt \\
&= \underbrace{u}_{\text{G(u)}} \underbrace{\ln(u)}_{\text{F(u)}} |_{t=0}^{t=1} - \int_{t=0}^{t=1} \underbrace{u}_{\text{G(u)}} \underbrace{\frac{1}{u}du}_{\text{F'(u)du}}
\end{aligned}$$

6. Compute $\int (\cos(x)+\sin(x))^2\, dx$.

Solution:

$$\begin{aligned}\int (\cos(x)+\sin(x))^2\,dx &= \int \cos^2(x)+2\sin(x)\cos(x)+\sin^2(x)dx\\ &= \int 1+\sin(2x)dx\\ &= x-\frac{1}{2}\cos(2x)+C\end{aligned}$$

7. Determine whether

$$\int_3^\infty \frac{4-x}{2x^2+2x-4}\,dx$$

coverges or diverges. If it converges, what does it converge to?

Solution: We recognize this as a partial fractions problem and can rewrite it as

$$\lim_{t\to\infty}\int_3^t \frac{1}{2x-2}-\frac{1}{x+2}\,dx.$$

If we integrate we have

$$\lim_{t\to\infty} \ln|2x-2|-\ln|x+2|\Big|_3^t.$$

Now we have to combine these logs in order to take the limit as $t\to\infty$. If we do this we have

$$\lim_{t\to\infty}\ln\left|\frac{2t-2}{t+2}\right|-\ln\left|\frac{6-2}{5}\right| = \ln(2)-\ln\left|\frac{6-2}{5}\right|.$$

8. Show that $\int_1^\infty \frac{dx}{x^2-4}$ is not a finite number. What answer do you get if you forget that the integrand has an asymptote at 2 and fail to split the integral up there?

Solution: You can use partial fractions to compute that

$$\int \frac{dx}{x^2-4} = \frac{1}{4}\ln|\frac{x-2}{x+2}|+C$$

If we write

$$\int_1^\infty \frac{dx}{x^2-4} = \int_1^2 \frac{dx}{x^2-4}+\int_2^3 \frac{dx}{x^2-4}+\int_3^\infty \frac{dx}{x^2-4}$$

it suffices to show that one of these integrals is infinite. For example

$$\begin{aligned}\int_1^2 \frac{dx}{x^2-4} &= \lim_{A\uparrow 2}\int_1^2 \frac{dx}{x^2-4}\\ &= \lim_{A\uparrow 2}\left[\frac{1}{4}\ln|\frac{x-2}{x+2}|\right]_1^A\\ &= \lim_{A\uparrow 2}\frac{1}{4}\ln|\frac{A-2}{A+2}| - \frac{1}{4}\ln|\frac{1}{2}|\\ &= -\infty\end{aligned}$$

Now we will see what happens if we forget that the integrand has an asymptote at 2.

$$\lim_{A\to\infty}\frac{1}{4}\ln|\frac{A-2}{A+2}| - \frac{1}{4}\ln|\frac{-1}{3}| = \frac{1}{4}\ln(3)$$

9. Compute $\displaystyle\int \frac{p+2}{p^2-1}dp$

Solution: We start by computing the partial fraction decomposition for $\frac{p+2}{p^2-1}$:

$$\frac{p+2}{p^2-1} = \frac{p+2}{(p+1)(p-1)} = \frac{A}{p+1} + \frac{B}{p-1}$$

Using the Heaviside trick we get $A = -1/2$ and $B = 3/2$.

$$\frac{p+2}{p^2-1} = \frac{-1/2}{p+1} + \frac{3/2}{p-1}$$

Integrating this gives

$$-\frac{1}{2}\ln|p+1| + \frac{3}{2}\ln|p-1| + c$$

10. Compute $\int e^x \sin(x)dx$.

Solution:

$$\begin{aligned}
\int e^x \sin(x)dx &= \int \underbrace{e^x}_{\text{F(x)}} \underbrace{\sin(x)dx}_{\text{G'(x)dx}} \\
&= \underbrace{e^x}_{\text{F(x)}} \underbrace{(-\cos(x))}_{\text{G(x)}} - \int \underbrace{-\cos(x)}_{\text{G(x)}} \underbrace{e^x dx}_{\text{F'(x)dx}} + C \\
&= -e^x \cos(x) + \int e^x \cos(x)dx + C \\
&= -e^x \cos(x) + \int \underbrace{e^x}_{\text{F(x)}} \underbrace{\cos(x)dx}_{\text{G'(x)dx}} + C \\
&= -e^x \cos(x) + \left[\underbrace{e^x}_{\text{F(x)}} \underbrace{\sin(x)}_{\text{G(x)}} - \int \underbrace{\sin(x)}_{\text{G(x)}} \underbrace{e^x dx}_{\text{F'(x)dx}} \right] + C \\
&= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)dx + C \\
2\int e^x \sin(x)dx &= e^x \left(\sin(x) - \cos(x)\right) + C \\
&= e^x \left(\sin(x) - \cos(x)\right) + C
\end{aligned}$$

11. Compute $\int \sec^2(x)\tan(x)dx$.

Solution:

$$\begin{aligned}
\int \sec^2(x)\tan(x)dx &= \int udu & u &= \sec(x) \\
&= \frac{u^2}{2} + C & du &= \sec(x)\tan(x)dx \\
&= \frac{1}{2}\sec^2(x) + C
\end{aligned}$$

12. Find a solution to the initial value problem

$$\begin{aligned}\frac{dy}{dx} &= y\sqrt{y^2-1}\cos(x)\\ y(0) &= 1\end{aligned}$$

Solution: First, we can observe that one solution to this problem is given by $y(x) = 1$. We can find another solution by separating variables.

$$\begin{aligned}\frac{dy}{dx} &= y\sqrt{y^2-1}\cos(x)\\ \frac{dy}{y\sqrt{y^2-1}} &= \cos(x)dx\\ \int \frac{dy}{y\sqrt{y^2-1}} &= \int \cos(x)dx\\ \operatorname{arcsec}(y) &= \sin(x) + C\\ y &= \sec(\sin(x) + C)\end{aligned}$$

Substituting in the initial condition $y(0) = 1$ we find that

$$1 = y(0) = \sec(C)$$

So we may take, for example, $C = 0$. Our final solution is then either of $y(x) = 1$ or $y(x) = \sec(\sin(x))$.

13. Determine whether

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

converges or diverges.

Solution: For $x \gg 0$, the integrand behaves a lot like $\frac{1}{x}$, so we will attempt to show this integral diverges. To do this we will note that $\frac{1}{2} < 1 - e^{-x}$ for any $x \geq 1$. Really all we need is that this inequality holds for some $a \geq 1$, but it is true on the whole interval.

This implies that

$$\frac{1/2}{x} < \frac{1-e^{-x}}{x}$$

Both of these functions are continuous on $[1, \infty)$. We know $\displaystyle\int_1^\infty \frac{1/2}{x}\,dx$ diverges so by comparison so does our original integral.

14. Compute $\displaystyle\int \frac{x^3}{x^2+2}dx$.

Solution: Notice that the degree of the numerator is greater than or equal to the degree of the denominator, so we need to do some kind of polynomial division to get this into a form we can work with. We compute

$$
\begin{array}{r|l}
 & \;\; x \\
\cline{2-2}
x^2+2 & \;\; x^3 \\
 & -x^3-2x \\
\cline{2-2}
 & \;-2x
\end{array}
$$

and therefore

$$
\begin{aligned}
\int \frac{x^3}{x^2+2}dx &= \int x - \frac{2x}{x^2+2}dx \\
&= \frac{x^2}{2} - \int \frac{1}{u}du && u = x^2+2 \quad du = 2xdx \\
&= \frac{x^2}{2} - \ln|u| + C \\
&= \frac{x^2}{2} - \ln(x^2+2) + C
\end{aligned}
$$

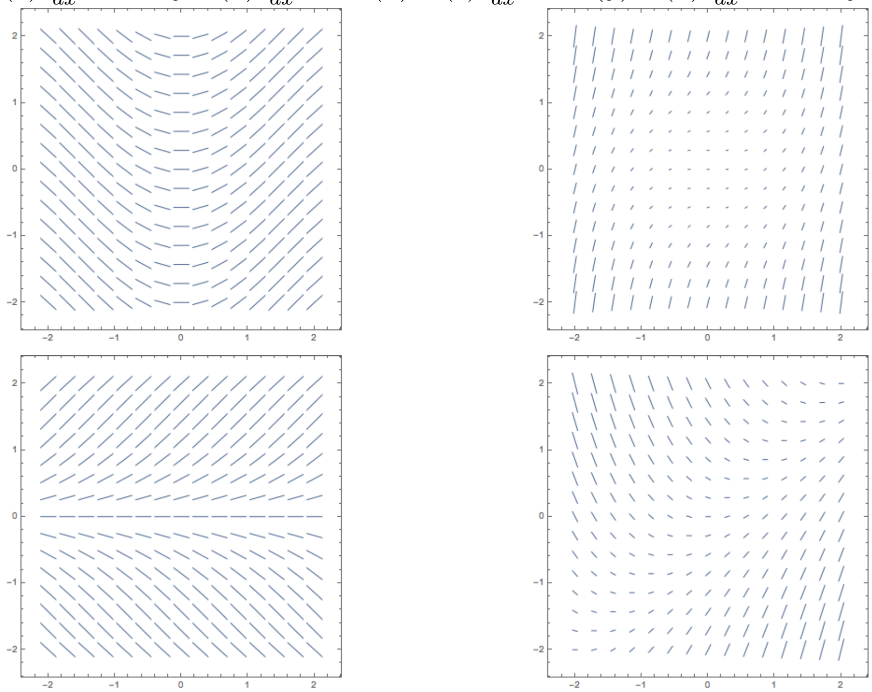
15. Compute $\int \cos^4(x)dx$.

Solution:

$$
\begin{aligned}
\int \cos^4(x)dx &= \int \left(\cos^2(x)\right)^2 dx \\
&= \frac{1}{4}\int (1+\cos(2x))^2 dx \\
&= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)dx \\
&= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\,dx \\
&= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C \\
&= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}
$$

16. Identify which of the following differential equations are associated to each of the following direction fields:

(a) $\frac{dy}{dx} = x - y$ (b) $\frac{dy}{dx} = \sin(x)$ (c) $\frac{dy}{dx} = \sin(y)$ (d) $\frac{dy}{dx} = x^2 + y^2$



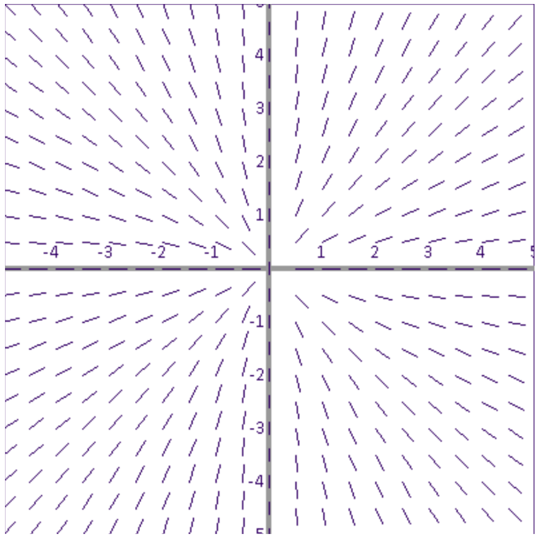
Solution:

$(b)\frac{dy}{dx} = \sin(x)$ (d) $\frac{dy}{dx} = x^2 + y^2$

$(c)\frac{dy}{dx} = \sin(y)$ (a) $\frac{dy}{dx} = x - y$

17. Circle the differential equation that corresponds to the slope field shown below.

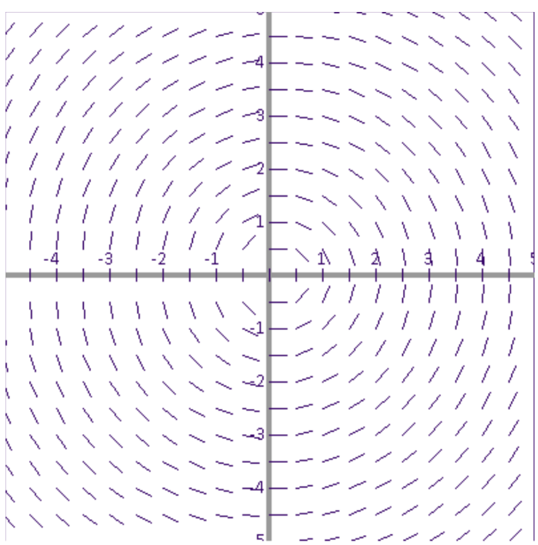
$y' = y/x$ $y' = \sin(x)$ $y' = x + y$ $y' = -x/y$



Solution: $y' = y/x$

18. Circle the differential equation that corresponds to the slope field shown below.

$y' = y/x$ $\qquad$ $y' = \sin(x)$ $\qquad$ $y' = x + y$ $\qquad$ $y' = -x/y$



Solution: $y' = -x/y$

19. Consider a continuous function $f(x)$:

$$f(x) = \begin{cases} 2x^{3/2} & \text{if } 0 \le x \le 4 \\ 4x & \text{if } 4 \le x < \infty \end{cases}$$

For what p values is the integral $\displaystyle\int_0^\infty x^p f(x)\,dx$ convergent? For what p values is the integral divergent?

Solution: First, we split the integral over two pieces of the domain of integration:

$$\int_0^\infty x^p f(x)\,dx = \int_0^4 x^p f(x)\,dx + \int_4^\infty x^p f(x)\,dx.$$

The first integral is

$$\int_0^4 x^p f(x)\,dx = 2\int_0^4 x^{p+\frac{3}{2}}\,dx,$$

which, by the p-test, converges if $p + \frac{3}{2} > -1$ and diverges if $p + \frac{3}{2} \le -1$. In other words, this integral converges if $p > -\frac{5}{2} = -2.5$ and diverges if $p \le -\frac{5}{2} = -2.5$.

The second integral is

$$\int_4^\infty x^p f(x)\,dx = 4\int_0^4 x^{p+1}\,dx,$$

which, by the p-test, converges if $p + 1 < -1$ and diverges if $p + 1 \ge -1$. In other words, this integral converges if $p < -2$ and diverges if $p \ge -2$.

In order for the initial integral $\displaystyle\int_0^\infty x^p f(x)\,dx$, we need the integrals over each piece of the domain to be convergent, which only occurs if $-2.5 < p < -2$. For any other p values, the integral diverges. (If $p \le 2.5$, the integral $\displaystyle\int_0^4 x^p f(x)\,dx$ diverges. If $-2 \le p$, the integral $\displaystyle\int_4^\infty x^p f(x)\,dx$ diverges.)