Practice Final a Solutions (12/7 Version)
MATH 222 (Lectures 1,2, and 4) Fall 2015.

Name: ______________________________ Student ID#: ________________________

Circle your TAs name:

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	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5
Score					
	Problem 6	Problem 7	Problem 8	Problem 9	Problem 10
Score					

Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- Please read the instructions on every problem carefully.
- On Problems 1–4 only the answer will be graded.
- On Problems 5–10 you must show your work and we will grade the work and your justification, and not just the final answer.
- Each problem is worth ten points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1-x^2}$. Note that we have provided some formulas on the next page to help with this.

Formulas

You may freely quote any algebraic or trigonometric identity, as well as well as any of the following formulas or minor variants of those formulas.

Integrals

- $\cos(\arcsin x) = \sqrt{1-x^2}$
- $\sec(\arctan x) = \sqrt{1+x^2}$.
- $\tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$.
- $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ \ln|x| + C & \text{when } n = -1 \end{cases}$
- $\int e^x dx = e^x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$.
- $\int \csc x dx = -\ln|\csc x + \cot x| + C$.
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.

Taylor series

- $T_\infty e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $T_\infty \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
- $T_\infty \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
- $T_\infty \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$
- $T_\infty \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$
- $T_\infty (1+x)^b = \sum_{k=0}^{\infty} \binom{b}{k} x^k$ where $\binom{b}{k} = \frac{b(b-1)(b-2)\cdots(b-k+1)}{k!}$

Other

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

1. For each statement below, CIRCLE true or false. You do not need to show your work.

(a)		(b)		(c)		(d)		(e)	
True	False	True	False	True	False	True	False	True	False

(a) The integral $\int_1^\infty \frac{1}{\sqrt{x-3}} dx$ is finite.

(b) $\int_2^\infty \frac{1}{x^3+1} dx \leq \int_2^\infty \frac{1}{x^2+5} dx$.

(c) The series $\sum_{k=0}^\infty \frac{k^2+e^{-k}}{2^k+\sqrt{k}}$ converges.

(d) The series $\sum_{k=0}^\infty \frac{k!}{k^k}$ is finite.

(e) $\sin(x^2) - x^2$ is $o(x^4)$.

(a) False. This integral has two improprieties: a vertical asymptote at $x = 3$ and an infinite bound. We can rewrite it as:

$$\int_1^3 \frac{1}{\sqrt{x-3}} dx + \int_3^4 \frac{1}{\sqrt{x-3}} dx + \int_4^\infty \frac{1}{\sqrt{x-3}} dx.$$

The first are finite but the last one is not.

(b) True. A bigger denominator yields a smaller function and thus a smaller integral. And $x^3 + 1 \geq x^2 + 5$ for all $x \geq 2$.

(c) True. Try Limit Comparison Test with $\sum \frac{k^2}{2^k}$ followed by the Ratio Test.

(d) True, though this one's tricky. Try the Ratio Test.

(e) True. $\sin(x^2) = x^2 - \frac{x^6}{3!} + \cdots = x^2 + o(x^4)$. Thus $\sin(x^2) - x^2$ is $o(x^4)$.

2. On this page only the answer will be graded.

   (a) Compute $\int \frac{1}{x^2(x+1)}dx$.

   Solution: This is a partial fractions problem. We rewrite

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$

   Equating coefficients we get $1 = Ax(x+1)+B(x+1)+Cx^2 = (A+C)x^2+(A+B)x+B$. This yields the system of equations:

$$\begin{cases} 0 = A + C \\ 0 = A + B \\ 1 = B \end{cases}$$

   So $A = -1$ and $C = 1$ and we get

$$\int \frac{1}{x^2(x+1)}dx = \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx \quad = -\ln|x| - x^{-1} + \ln|x+1| + C.$$

   So our answer is $\boxed{-\ln|x| - x^{-1} + \ln|x+1| + C}$.

   (b) Compute $\int x^{2013}\ln(x)dx$.

   Solution Use integration by parts with $f = \ln(x)$ and $g' = x^{2013}$. Then $f' = \frac{1}{x}$ and $g = \frac{x^{2014}}{2014}$ and we get:

$$\begin{aligned}
\int x^{2013}\ln(x)dx &= \int fg' \\
&= fg - \int f'g \\
&= \frac{\ln(x)x^{2014}}{2014} - \int \frac{1}{x}\cdot\frac{x^{2014}}{2014}dx \\
&= \frac{\ln(x)x^{2014}}{2014} - \frac{1}{2014}\int x^{2013}dx \\
&= \frac{\ln(x)x^{2014}}{2014} - \frac{x^{2014}}{2014^2} + C
\end{aligned}$$

3. On this page only the answer will be graded.

(a) Compute $T_4(\frac{e^{x^4}}{\sqrt{1+x^3}})$

Solution: We start with the fact that $T_\infty e^{x^4}(1+x^3)^{-1/2} = T_\infty e^{x^4} \cdot T_\infty(1+x^3)^{-1/2}$. We know that $T_\infty e^u = 1+u+u^2+\dots$ which yields $T_\infty e^{x^4} = 1+x^4+o(x^4)$. We also know that $T_\infty(1+u)^{-1/2} = 1+\binom{-1/2}{1}u+\binom{-1/2}{2}u^2$ which yields $T_\infty(1+x^3)^{-1/2} = 1+(-1/2)x^3+\binom{-1/2}{2}x^6+\cdots = 1+(-1/2)x^3+o(x^4)$. We thus obtain:

$$T_\infty e^{x^4}(1+x^3)^{-1/2} = (1+x^4+o(x^4))(1-\frac{1}{2}x^3+o(x^4)) = 1-\frac{1}{2}x^3+x^4+o(x^4)$$

We conclude that $T_4 e^{x^4}(1+x^3)^{-1/2}$ equals $\boxed{1-\frac{1}{2}x^3+x^4}$.

(b) For which values of b does the Taylor series for $\frac{1}{3+2x^2}$ converge at $x=b$?

Solution Since $T_\infty \frac{1}{1+u} = \sum_{k=0}^\infty (-u)^k$, we see that

$$T_\infty \frac{1}{3+2x^2} = \frac{1}{3} T_\infty \frac{1}{1+\frac{2}{3}x^2} = \frac{1}{3}\sum_{k=0}^\infty (-\frac{2}{3}x^2)^k = \sum_{k=0}^\infty \frac{1}{3}\cdot(-\frac{2}{3}x^2)^k$$

Plugging in $x=b$ we get:

$$T_\infty \frac{1}{3+2x^2}|_{x=b} = \sum_{k=0}^\infty \frac{1}{3}\cdot(-\frac{2}{3}b^2)^k$$

This is a geometric series $\sum_{k=0}^\infty ar^k$ with $a=\frac{1}{3}$ and $r=-\frac{2}{3}b^2$. The problem only asks about convergence. The Geometric Series Test shows that a geometric series converges if and only if $|r|<1$. This series thus converges if and only if $|\frac{2}{3}b^2|<1$ which is equivalent to asking that $|b^2|<\frac{3}{2}$ or that $\boxed{|b|<\sqrt{\frac{3}{2}}}$.

4. On this page, only the answer will be graded.

(a) Let P be the plane through the points $(1,1,1),(3,2,0)$ and $(0,4,0)$. Compute the normal vector to P.

Solution: We first compute two vector lying in the plane: $\vec{a} = \begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}0\\4\\0\end{pmatrix} = \begin{pmatrix}3\\-2\\0\end{pmatrix}$. We then compute $\vec{a}\times\vec{b} = \begin{pmatrix}2\\1\\-1\end{pmatrix} \times \begin{pmatrix}3\\-2\\0\end{pmatrix} = \begin{pmatrix}-2\\-3\\-7\end{pmatrix}$.

Thus $\boxed{\begin{pmatrix}-2\\-3\\-7\end{pmatrix}}$ is our normal vector. Note that any scalar multiple of this vector would also be a correct answer.

(b) Let $\vec{a} = (1,2)$. Compute $\vec{v}^{\perp}$ with respect to $\vec{a}$ where $\vec{v} = (3,4)$.

We first compute $\vec{v}^{//} = \frac{\vec{v}\cdot\vec{a}}{||\vec{a}||^2}\vec{a} = \frac{1\cdot3+2\cdot4}{1^2+2^2}\vec{a} = \frac{11}{5}\begin{pmatrix}1\\2\end{pmatrix}$. Thus

$$\begin{aligned}\vec{v}^{\perp} &= \vec{v} = \vec{v}^{//}\\ &= \begin{pmatrix}3\\4\end{pmatrix} - \frac{11}{5}\begin{pmatrix}1\\2\end{pmatrix}\\ &= \begin{pmatrix}3-\frac{11}{5}\\4-\frac{22}{5}\end{pmatrix} = \begin{pmatrix}\frac{4}{5}\\-\frac{2}{5}\end{pmatrix}\end{aligned}$$

(c) At what point (x,y) does the line $\ell : \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1+2t\\3+4t\end{pmatrix}$ intersect the line $y = 3x+2$?

Solution: We plug in the values. Combining the equation $y = 3x+2$ with the parametric equation for ℓ we get:

$$\begin{pmatrix}x\\3x+2\end{pmatrix} = \begin{pmatrix}1+2t\\3+4t\end{pmatrix} \Rightarrow \begin{cases}x = 1+2t\\3x+2 = 3+4t\end{cases}$$

We then solve this system of equations. This yields $3(1+2t)+2 = 3+4t$ and thus $5+6t = 3+4t$ and so $2 = -2t$ and $t = -1$. It follows that $x = -1$ and $y = 3(-1)+2 = -1$. Note that this point is also on ℓ. $\boxed{\text{So the lines intersect at the point } (-1,-1).}$

5. On this page, you must show your work to receive full credit.

(a) Let $f(x) = x^3 \sin(x^2)$. Compute $f^{(409)}(0)$.

Solution: By definition, we have $T_\infty f(x) = \sum_{j=0} \frac{f^{(j)}(0)}{j!} x^j$. Since $T_\infty \sin(u) = \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k+1}}{(2k+1)!}$, we can comput $T_\infty f(x)$ another way as

$$\begin{aligned} T_\infty x^3 \sin(x^2) &= x^3 T_\infty \sin(x^2) \\ &= x^3 \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!} \\ &= x^3 \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+5}}{(2k+1)!} \end{aligned}$$

Relating these two expressions for $T_\infty f(x)$ we see that the coefficient of x^{409} must be equal. On the hand this coefficient is when $j = 409$ which is $\frac{f^{(409)}(0)}{409!}$. On the other hand, if $k = 101$ then we also get a coefficient of x^{409} and in that case it equals $(-1)^{101} \frac{1}{(2 \cdot 101+1)!} = -\frac{1}{203!}$. Thus we get:

$$\frac{f^{(409)}(0)}{409!} = -\frac{1}{203!}$$

and so $\boxed{f^{(409)}(0) = -\frac{409!}{203!}}$.

(b) Let t denote time in years since January 1, 2010 and let $S(t)$ denote the number of squirrels in Madison at time t. This squirrel population has a continuous birth rate of 8% and a natural continuous death rate of 2%. In addition, each year 250 squirrels are eaten by foxes and 150 squirrels are run over by cars. Write down a different equation for $S(t)$. DO NOT SOLVE THE DIFFERENTIAL EQUATION.

Solution: $\frac{dS}{dt} = (.08 - .02)S + (-250 - 150) = .06S - 400.$

6. On this page, you must show your work to receive full credit.

(a)

$$\frac{dy}{dx} - \frac{2x}{1+x^2}y = 1 + x^2 \qquad \textit{and} \qquad y(0) = 7.$$

Solution: This is a linear differential equation with $a(x) = -\frac{2x}{1+x^2}$. Thus

$$m(x) = e^{\int -\frac{2x}{1+x^2}dx} = e^{-\ln|1+x^2|} = (e^{\ln|1+x^2|})^{-1} = |1+x^2|^{-1} = (1+x^2)^{-1}.$$

We then get

$$\begin{aligned} y &= \frac{1}{m(x)} \cdot \int m(x)k(x)dx \\ &= (1+x^2)\int \frac{1}{1+x^2}\cdot(1+x^2)dx \\ &= (1+x^2)\int 1dx \\ &= (1+x^2)(x+C). \end{aligned}$$

Solving for the initial condition we get $7 = (1+0^2)(0+C)$ and thus $C = 7$ and our final answer is $\boxed{y = (1+x^2)(x+7).}$

(b)

$$\frac{dy}{dx} = \frac{e^x}{2y+1} \text{ and } y(0) = 1$$

Solution:

$$\begin{aligned} \int 2y + 1dy &= \int e^x dx \\ y^2 + y &= e^x + C \end{aligned}$$

Rearranging we get:

$$y^2 + y - (e^x + C) = 0$$

Use the quadratic formula to get two solutions:

$$y = \frac{-1 \pm \sqrt{1+4(e^x+C)}}{2}$$

Now solve for C:

$$1 = \frac{-1 \pm \sqrt{1+4(1+C)}}{2} \Rightarrow 2 = -1 \pm \sqrt{5+4C}$$

So we choose the $+$ sign and get $C = 1$. $\boxed{\text{The final answer is } y = \frac{-1+\sqrt{1+4(e^x+1)}}{2}.}$

7. On this page, you must show your work to receive full credit.

$$\text{Compute} \int \sqrt{2x - x^2} dx.$$

For your final answer, you should simplify any expression that combines a trigonometric and inverse trigonometric function (e.g. $\cos(\arcsin x) = \sqrt{1 - x^2}$).

Solution We complete the square to get $2x - x^2 = 1 - (x-1)^2$ and we have the integral:

$$\int \sqrt{2x - x^2} dx = \int \sqrt{1 - (x-1)^2} dx$$

We perform the trig substitution $x - 1 = \sin\theta$ and $dx = \cos\theta d\theta$ to get

$$= \int \sqrt{1 - \sin^2\theta} \cos\theta d\theta$$
$$= \int \cos^2\theta d\theta$$

We now use the double angle formula to get:

$$= \frac{1}{2} \int (1 + \cos(2\theta)) d\theta$$
$$= \frac{1}{2}(\theta + \frac{1}{2}\sin(2\theta)) + C$$

Now we need to replace the θ's by the original variables. We have $\theta = \arcsin(x-1)$. For $\sin(2\theta)$ we use the double angle formula $\sin(2\theta) = 2\sin\theta\cos\theta = 2\sin(\arcsin(x-1))\cos(\arcsin(x-1)) = 2(x-1)\sqrt{1-(x-1)^2}$. Plugging this all in yields:

$$= \frac{1}{2}\left(\arcsin(x-1) + (x-1)\sqrt{1-(x-1)^2}\right) + C.$$

8. On this page, you must show your work and justify your answer to receive full credit. Let $a > 0$ be a real number, and let $I_n = \int x^n e^{ax} dx$. Derive a reduction formula for I_n.

   Solution: We have:

$$I_n = \int x^n e^{ax} dx$$

   Using integration by parts with $f = x^n$ and $g' = e^{ax}$ we get $f' = nx^{n-1}$ and $g = \frac{1}{a}e^{ax}$. This yields:

$$\begin{aligned}
&= \int fg' \\
&= fg - \int f'g \\
&= \frac{1}{a}x^n e^{ax} - \int (nx^{n-1})(\frac{1}{a}e^{ax})dx \\
&= \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx \\
&= \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}
\end{aligned}$$

   Thus our reduction formula is $I_n = \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}$.

9. On this page, you must show your work and justify your answer to receive full credit. Let $f(x) = e^{-2x} + x^5$.

$$\text{Find } B \text{ so that } |R_2 f(x)| \leq B \text{ for all } -2 \leq x \leq 2.$$

If using the Error Bound Formula, you must clearly indicate your chosen value for M and explain why this choice of M is valid for the desired range.

Solution: We will use the Error Bound Formula with $c = 2$ and $n = 2$. We must compute M and for this we will compute $f^{(3)}(x)$. We have:

- $f'(x) = (-2)e^{-2x} + 5x^4$
- $f''(x) = (-2)^2 e^{-2x} + 20x^3$
- $f^{(3)}(x) = (-2)^3 e^{-2x} + 60x^2$.

We wish to find M where $M \geq |f^{(3)}(x)|$ for all $-2 \leq x \leq 2$. By the Triangle Inequality $|(-2)^3 e^{-2x} + 60x^2| \leq |(-2)^3 e^{-2x}| + |60x^2| = 8e^{-2x} + 60x^2$. Since e^{-2x} is a decreasing function, the maximum of $8e^{-2x}$ occurs at the left endpoint, and is given by $8e^{-2(-2)} = 8e^4$. Since $60x^2$ is an increasing function its maximal value occurs at the right endpoint, and is $60 \cdot 2^2 = 240$. Thus we have:

$$|f^{(3)}(x)| \leq 8e^4 + 240 \text{ for all } -2 \leq x \leq 2.$$

So we choose $M = 8e^4 + 240$.

Using the Error Bound Formula, we then obtain that

$$|R_2 f(x)| \leq \frac{Mc^{n+1}}{(n+1)!} = \frac{(8e^4 + 240)2^3}{3!} \text{ for all } -2 \leq x \leq 2.$$

So $\boxed{B = \frac{(8e^4+240)2^3}{3!}}$ is our final answer.

10. On this page, you must show your work and justify your answer to receive full credit. You must clearly state any convergence test that you use and explicitly verify each hypothesis for that test.

For which values of b does the series $\sum_{k=1}^{\infty} \frac{e^{kb}+\sqrt{k}}{k!}$ converge?

Solution: Our solution will have two cases depending on the value of b.

Case 1: $b > 0$ We first assume $b > 0$ so that e^{kb} is an exponential growth function in k and thus e^{kb} is the term which grows the fastest in the numerator. We define $a_k = \frac{e^{kb}+\sqrt{k}}{k!}$ and $c_k = \frac{e^{kb}}{k!}$ and we will apply the Limit Comparison Theorem to these two series. We check the hypotheses:

- $a_k = \frac{e^{kb}+\sqrt{k}}{k!}$ and $c_k = \frac{e^{kb}}{k!}$ are positive for all k since the numerator and denominator are built from positive numbers for all k.
- We check that $\lim_{k\to\infty} \frac{a_k}{c_k} = \lim_{k\to\infty} \frac{e^{kb}+\sqrt{k}}{k!} \cdot \frac{k!}{e^{kb}} = \lim_{k\to\infty} \frac{e^{kb}+\sqrt{k}}{e^{kb}} = \lim_{k\to\infty} 1+\frac{\sqrt{k}}{e^{kb}} = 1+0 = 1$ since exponential growth function grows faster than any polynomial function. Since this limit is a positive real number, it satisfies the second hypothesis.

We may thus apply the Limit Comparison Theorem to conclude that either both series converge or both diverge.

We next apply the Ratio Test to $\sum_{k=1} \infty c_k$. We compute $L = \lim_{k\to\infty} |\frac{c_{k+1}}{c_k}| = \lim_{k\to\infty} |\frac{e^{(k+1)b}}{(k+1)!} \cdot \frac{k!}{e^{kb}}| = \lim_{k\to\infty} \frac{e^b}{k+1} = 0$. Since $L = 0 < 1$, the Ratio Test implies that $\sum_{k=1} \infty c_k$ converges. Thus the original series converges as well whenever $b > 0$.

Case 2: $b \leq 0$ We now assume $b \leq 0$ so that e^{kb} does not grow with k. Then $\sqrt{k}$ is the term which grows the fastest in the numerator. We define $d_k = \frac{\sqrt{k}}{k!}$ and we will apply the Limit Comparison Theorem to these two series. We check the hypotheses:

- $a_k = \frac{e^{kb}+\sqrt{k}}{k!}$ and $d_k = \frac{\sqrt{k}}{k!}$ are positive for all k since the numerator and denominator are built from positive numbers for all k.
- We check that $\lim_{k\to\infty} \frac{a_k}{d_k} = \lim_{k\to\infty} \frac{e^{kb}+\sqrt{k}}{k!} \cdot \frac{k!}{\sqrt{k}} = \lim_{k\to\infty} \frac{e^{kb}+\sqrt{k}}{\sqrt{k}} = \lim_{k\to\infty} \frac{e^{kb}}{\sqrt{k}}+1 = 0 + 1$ since $\sqrt{k}$ grows faster than e^{kb} when $b \leq 0$. Since this limit is a positive real number, it satisfies the second hypothesis.

We may thus apply the Limit Comparison Theorem to conclude that either both series converge or both diverge.

Finally, we apply the Ratio Test to $\sum_{k=1}^{\infty} d_k$. We compute $L = \lim_{k\to\infty} |\frac{d_{k+1}}{d_k}| = \lim_{k\to\infty} |\frac{\sqrt{k+1}}{(k+1)!} \cdot \frac{k!}{\sqrt{k}}| = \lim_{k\to\infty} \frac{\sqrt{k+1}}{\sqrt{k}} \frac{1}{k+1} = 1 \cdot 0 = 0$. Since $L = 0 < 1$, the Ratio Test implies that $\sum_{k=1}^{\infty} d_k$ converges. We concdlue that the original series also converges whenver $b \leq 0$.

$\boxed{\text{Thus in conclusion, our original series converges for all values of } b.}$

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