MATH 222 (Lectures 1,2, and 4) Fall 2015

Practice Midterm 2.2 Solutions Student ID#: ____________________

Circle your TA's name from the following list.

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Please inform your TA if you find any errors in the solutions.

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7
Score							

Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- On Problems 1, 2, and 3, only the answer will be graded.
- On Problems 4, 5, 6, and 7 you must show your work and we will grade the work and your justification, and not just the final answer.
- Problem 3 is worth 10 points. All other problems worth 15 points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1-x^2}$. Note that we have provided some formulas on the next page to help with this.

Formulas

- $T_\infty e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $T_\infty \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
- $T_\infty \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
- $T_\infty \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$
- $T_\infty \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$
- $T_\infty (1+x)^b = \sum_{k=0}^{\infty} \binom{b}{k} x^k \quad \text{where} \quad \binom{b}{k} = \frac{b(b-1)(b-2)\cdots(b-k+1)}{k!}$

1. For each statement below, CIRCLE the correct answer. You do not need to show your work.

(a)		(b)		(c)		(d)	(e)
True	False	True	False	True	False	I II	I II
						III	III

True or false:

(a) $(x\cos(x) - x)$ is $o(x^5)$.

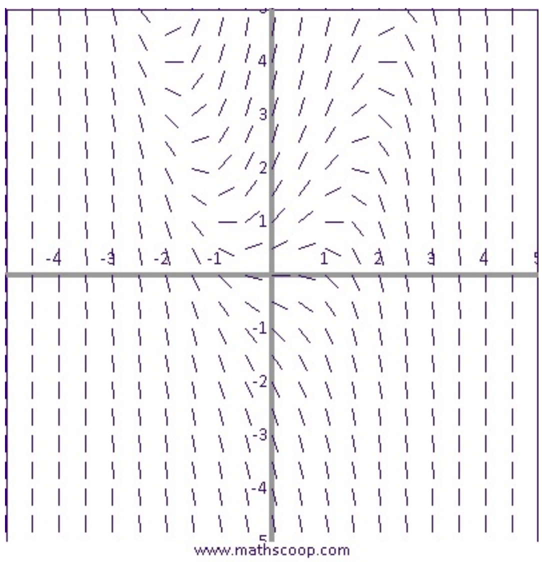
(b) If $f(x)$ is a degree 5 polynomial then $T_{15}f(x) = f(x)$.

(c) $R_4 \sin x = \sin x - (x - \frac{x^3}{3!})$

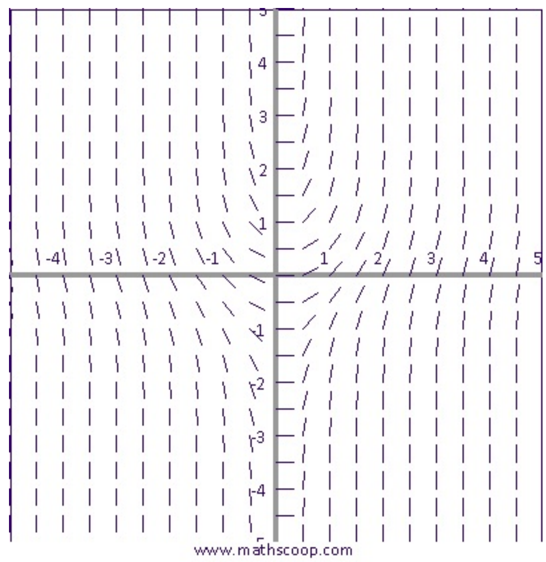
Below are three direction fields. The equations for *two* of those fields are given below. Match the equation to the appropriate direction field and record your answer on the previous page.

(d) $\frac{dy}{dx} = x^2 - y$

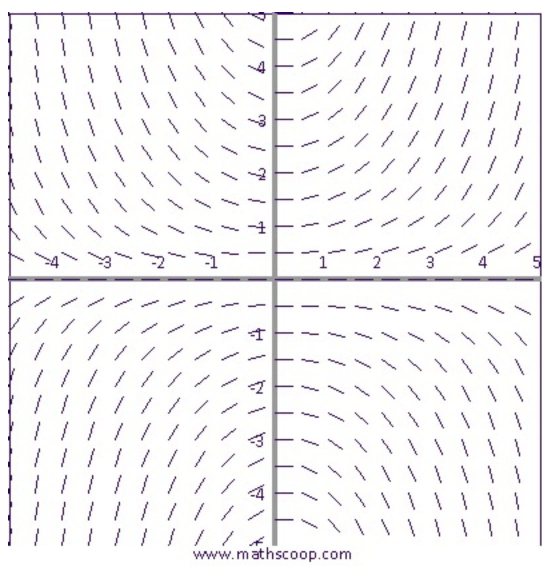
(e) $\frac{dy}{dx} = xy$



I.



II.



III.

Solution:

(a) False.
(b) True.
(c) True.
(d) I.
(e) III.

2. (a) Use Euler's method with step size $h = 0.1$ to estimate $y(0.1)$ where $y(x)$ satisfies

$$\frac{dy}{dx} = x + y \text{ and } y(0) = 1.$$

Solution:

x_k	y_k	$m_k = x_k + y_k$	$y_{k+1} = y_k + m_k h$
$x_0 = 0$	$y_0 = 1$	$0 + 1 = 1$	$1 + (1)(0.1) = 1.1$
$x_1 = 0.1$	$y_1 = 1.1$		

Thus $y(0.1) \approx 1.1$.

(b) Find $T_3\left(x^3 + \frac{1}{1+2x^2}\right)$.

Solution: $T_\infty(x^3 + \frac{1}{1+2x^2}) = T_\infty(x^3) + T_\infty(\frac{1}{1+2x^2}) = x^3 + (1 - 2x^2 + o(x^3))$. Thus

$$T_3\left(x^3 + \frac{1}{1+2x^2}\right) = 1 - 2x^2 + x^3$$

(c) Find $T_2^1(x^3)$.

Solution: We compute:

- $f(x) = x^3$
- $f'(x) = 3x^2$
- $f''(x) = 6x$

Thus:

- $f(1) = 1$
- $f'(1) = 3$
- $f''(1) = 6$

This yields:

$$\begin{aligned} T_2^1\left(x^3\right) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 1 + 3(x-1) + \frac{6}{2!}(x-1)^2. \end{aligned}$$

3. In the problem below: 1. Clearly define variables (including units!); 2. Set up the appropriate differential equation; and 3. Write down the appropriate initial condition. DO NOT SOLVE THE DIFFERENTIAL EQUATION.

   Ten thousand dollars is deposited in a bank account on January 1, 1990 with a nominal annual interest rate of 5% compounded continuously. No further deposits are made. Money is withdrawn continuously at a rate of \$4000 per year. We are interested in a function that models the amount of money left in the account.

   - **Variables (2pts):**
   - **Differential equation (6pts)**
   - **Initial condition (2pts):**

Solution:

- **Variables (2pts):** t = time in years since January 1, 1990. $M(t)$ = money in dollars in the bank account at time t.
- **Differential equation (6pts)** $\frac{dM}{dt} = .05M - 4000$.
- **Initial condition (2pts):** $M(0) = 10,000$.

4. Find a solution to each initial value problem.

(a)

$$\frac{dy}{dx} = 4x^3(y + e^{x^4}) \text{ and } y(0) = 1.$$

Solution:

$$\frac{dy}{dx} - 4x^3 y = 4x^3 e^{x^4}$$

This is linear with $a(x) = -4x^3$ and $k(x) = 4x^3 e^{x^4}$. Thus

$$m(x) = e^{\int -4x^3 dx} = e^{-x^4}.$$

We check that this satisfies the differential equation $\frac{dm}{dx} = m(x)a(x)$. We have

$$\frac{dm}{dx} = e^{-x^4} \cdot (-4x^3) \qquad \text{while} \qquad m(x)a(x) = e^{-x^4} \cdot (-4x^3).$$

Since these are equal, this passes the sanity check. We thus have:

$$\begin{aligned} y &= \frac{1}{m(x)} \int m(x)k(x)dx \\ &= \frac{1}{e^{-x^4}} \int e^{-x^4} \cdot 4x^3 e^{x^4} dx \\ &= e^{x^4} \int 4x^3 \cdot dx \\ &= e^{x^4}(x^4 + C) \end{aligned}$$

The initial condition $y(0) = 1$ gives $1 = e^0(0 + C) = C$ and thus $\boxed{y = e^{x^4}(x^4 + 1).}$

(b)

$$\frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \cos x \text{ and } y(0) = 0.$$

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{1-y^2}\cos x \\ \frac{dy}{\sqrt{1-y^2}} &= \cos x dx \\ \arcsin(y) &= \sin x + C \\ y &= \sin(\sin x + C) \end{aligned}$$

Solving initial condition gives $C = 0$ and we get

$$y = \sin(\sin x).$$

5. Let t stand for time in minutes from 12:00pm and let $B(t)$ denote the number of bacteria in a petri dish at time t. Assume that B satisfies $\frac{dB}{dt} = 50 \cdot B \cdot (1 - B)$. Also assume that at 12:00pm there were 2 bacteria in the dish. Compute $B(t)$.

Solution: This is a separable differential equation. So we set up and solve:

$$\int \frac{dB}{B(1-B)} = \int 50dt \text{ or } B = 0 \text{ or } B = 1$$

But $B = 0$ and $B = 1$ are impossible because they do not satisfy the initial condition. So we can throw them out and move on. Solving the partial fractions problem we get $\frac{1}{B(1-B)} = \frac{1}{B} + \frac{1}{1-B}$. Thus

$$\begin{aligned}
\int \frac{1}{B} + \frac{1}{1-B} &= \int 50dt \\
\ln|B| - \ln|1-B| &= 50t + C \\
\ln|\frac{B}{1-B}| &= 50t + C \\
|\frac{B}{1-B}| &= e^{50t+C} = e^C e^{50t} \\
\frac{B}{1-B} &= \pm e^C e^{50t}
\end{aligned}$$

At this point, it might be helpful to use our initial condition to solve for C. Since $B(0) = 2$ we get $\frac{2}{1-2} = \pm e^C e^0$ and thus $-2 = \pm e^C$ and so we choose the negative sign and let $C = \ln 2$.

$$\begin{aligned}
\frac{B}{1-B} &= -e^{\ln 2} e^{50t} = -2e^{50t} \\
B &= (1-B)(-2e^{50t}) = -2e^{50t} + 2Be^{50t} \\
B - 2Be^{50t} &= -2e^{50t} \\
B(1 - 2e^{50t}) &= -2e^{50t} \\
B &= \frac{-2e^{50t}}{1 - 2e^{50t}}
\end{aligned}$$

$\boxed{\text{Thus } B(t) = \frac{-2e^{50t}}{1-2e^{50t}}.}$

6. Let $f(x) = \sin(2x)$. Find n such that $|f(x) - T_n f(x)| \leq \frac{1}{100}$ for x in the range $-\frac{1}{2} \leq x \leq \frac{1}{2}$. *It may be helpful to know that* $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$ *and* $6! = 720$.

   Solution: For any n we have that $f^{(n+1)}(x)$ is either $\pm 2^{n+1} \sin 2x$ or $\pm 2^{n+1} \cos 2x$. In either case, this function is bounded by 2^{n+1} so we can choose $M = 2^{n+1}$ for any choice of n. Then we have a bound of the following form:

$$|f(x) - T_n f(x)| \leq \frac{M \cdot c^{n+1}}{(n+1)!} = \frac{2^{n+1} \cdot (\frac{1}{2})^{n+1}}{(n+1)!} = \frac{1}{(n+1)!}.$$

   So we need to choose n so that $\frac{1}{(n+1)!} \leq \frac{1}{100}$. Here we use the fact that $5! = 120$. Thus $n + 1 = 5$ and $n = 4$ will work.

7. Let $f(x)$ be a function satisfying the differential equation

$$f''(x) + 2e^{2x^2} - f(x) = 0$$

and also satisfying the initial conditions $f(0) = 0$ and $f'(0) = -1$. Compute $T_4 f(x)$.

Solution: We write $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4)$. By definition of the Taylor series of $f(x)$, we have that $a_0 = f(0)$ and $a_1 = f'(0)$. Thus $a_0 = 0$ and $a_1 = -1$. Thus

- $f(x) = 0 - x + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4)$.
- $f'(x) = -1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + o(x^3)$.
- $f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + o(x^2)$.
- $T_\infty 2e^{2x^2} = 2T_\infty e^{x^2} = 2(1 + 2x^2 + \frac{(2x^2)^2}{2!}) + o(x^4) = 2 + 4x^2 + 8x^4 + o(x^4)$

We compute

$$\begin{aligned}
0 &= f''(x) + 2e^{2x^2} - f(x) \\
&= (2a_2 + 6a_3 x + 12a_4 x^2 + o(x^2)) + (2 + 4x^2 + 8x^4 + o(x^4)) - (-x + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4)) \\
&= (2a_2 + 2 - 0) + (6a_3 + 0 + 1)x + (12a_4 + 4 - a_2)x^2 + o(x^2)
\end{aligned}$$

By equating coefficients we see that $0 = 2a_2 + 2$ and thus $a_2 = -1$. Also $6a_3 + 1 = 0$ and so $a_3 = -\frac{1}{6}$. And finally $12a_4 + 4 - a_2 = 0$ but $a_2 = -1$ and so $12a_4 + 5 = 0$ and so $a_4 = -\frac{5}{12}$. We conclude that

$$T_4 f(x) = -x - x^2 - \frac{1}{6}x^3 - \frac{5}{12}x^4$$