**Final a**
MATH 222 (Lectures 1,2, and 4) Fall 2015.

Name: ______________________________ Student ID#: ________________________

Circle your TAs name:

| | | | | |
|---|---|---|---|---|
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| | Problem 1 | Problem 2 | Problem 3 | Problem 4 | Problem 5 |
|---|---|---|---|---|---|
| Score | | | | | |
| | Problem 6 | Problem 7 | Problem 8 | Problem 9 | Problem 10 |
| Score | | | | | |

## Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- Please read the instructions on every problem carefully.
- On Problems 1–4 only the answer will be graded.
- On Problems 5–10 you must show your work and we will grade the work and your justification, and not just the final answer.
- Each problem is worth ten points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1-x^2}$. Note that we have provided some formulas on the next page to help with this.

## Formulas

You may freely quote any algebraic or trigonometric identity, as well as well as any of the following formulas or minor variants of those formulas.

## Integrals

- $\cos(\arcsin x) = \sqrt{1-x^2}$
- $\sec(\arctan x) = \sqrt{1+x^2}$.
- $\tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$.
- $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ \ln|x| + C & \text{when } n = -1 \end{cases}$
- $\int e^x dx = e^x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$.
- $\int \csc x dx = -\ln|\csc x + \cot x| + C$.
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.

## Taylor series

- $T_\infty e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $T_\infty \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
- $T_\infty \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
- $T_\infty \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$
- $T_\infty \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$
- $T_\infty (1+x)^b = \sum_{k=0}^{\infty} \binom{b}{k} x^k$ where $\binom{b}{k} = \frac{b(b-1)(b-2)\cdots(b-k+1)}{k!}$

## Other

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

1. For each statement below, CIRCLE true or false. You do not need to show your work.

| (a) | | (b) | | (c) | | (d) | | (e) | |
|---|---|---|---|---|---|---|---|---|---|
| True | False | True | False | True | False | True | False | True | False |

(a) The series $\sum_{k=0}^{\infty} \frac{2^{2k+1}}{3^k}$ converges.

(b) The series $\sum_{k=0}^{\infty} \frac{k^2+e^{-k}}{\sqrt{k}+e^k}$ converges.

(c) The integral $\int_3^{\infty} \frac{1}{x} - \frac{1}{x-1} dx$ converges.

(d) $\int_e^{\infty} \frac{1}{x^2 \ln(x)} dx \leq \int_e^{\infty} \frac{1}{x^2} dx$

(e) $e^{x^2} - (1 + x^2)$ is $o(x^4)$.

(a) False

(b) True

(c) True

(d) True

(e) False

2. On this page only the answer will be graded.

(a) Compute $\int \frac{1}{(x+2)(x+3)}dx$.

**Solution:** We rewrite this as $\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$. Clearing denomoinators yields:

$$1 = A(x+3) + B(x+2) = (A+B)X + (3A+2B) \Rightarrow \begin{cases} 0 & = A+B \\ 1 & = 3A+2B \end{cases}$$

and thus $A = 1$ and $B = -1$. We thus have:

$$\begin{aligned} \int \frac{1}{(x+2)(x+3)}dx &= \int \frac{1}{x+2} - \frac{1}{x+3}dx \\ &= \ln|x+2| - \ln|x+3| + C. \end{aligned}$$

(b) Let $f(x) = x^2 \sin(x^3)$. Compute $f^{(605)}(0)$.

**Solution:** On the one hand $T_\infty f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}x^j$. We can also compute:

$$T_\infty f(x) T_\infty x^2 \cdot T_\infty \sin(x^3) = x^2 \cdot \sum_{k=0}^{\infty}(-1)^k \frac{(x^3)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{6k+5}}{(2k+1)!}$$

If we consider the $x^{605}$ coefficient from each expression we obtain:

$$\frac{f^{(605)}(0)}{605!} = \frac{1}{201!}$$

and thus $\boxed{f^{(605)}(0) = \frac{605!}{201!}}$.

3. On this page only the answer will be graded.

(a) Compute $T_4(\sqrt{1-x^3}\cdot\cos(x^2))$.

**Solution:** We write $T_\infty\sqrt{1-x^3} = (1+\binom{1/2}{1}(-x^3)+o(x^4))$ and $T_\infty\cos(x^2) = 1-\frac{x^4}{2}+o(x^4)$ and thus $T_\infty\left(\sqrt{1-x^3}\cos(x^2)\right) = (1+\binom{1/2}{1}(-x^3)+o(x^4))(1-\frac{x^4}{2}+o(x^4)) = 1-\frac{1}{2}x^3-\frac{1}{2}x^4+o(x^4)$. And thus $\boxed{T_4\left(\sqrt{1-x^3}\cos(x^2)\right) = 1-\frac{1}{2}x^3-\frac{1}{2}x^4.}$

(b) For which values of $b$ does the Taylor series for $\frac{x}{3-5x^4}$ converge at $x=b$?

**Solution:** We have that

$$\begin{aligned}
T_\infty\frac{x}{3-5x^4} &= \frac{x}{3}T_\infty\frac{1}{1-\frac{5}{3}x^4}\\
&= \frac{x}{3}\cdot\sum_{k=0}^{\infty}(\frac{5}{3}x^4)^k\\
&= \cdot\sum_{k=0}^{\infty}\frac{x}{3}(\frac{5}{3}x^4)^k
\end{aligned}$$

This is a geometric series $\sum_{k=0}^{\infty}ar^k$ with $a=\frac{x}{3}$ and $r=(\frac{5}{3}x^4)$. By the Geometric Series Test it thus converges when $|r|<1$ or when $|\frac{5}{3}x^4|<1$ which is when $\boxed{|x|<\sqrt[4]{\frac{3}{5}}}$.

4. On this page, only the answer will be graded.

(a) Let $P$ be the plane through the points $(1,1,1),(3,2,0)$ and $(0,4,0)$. Compute a nonzero normal vector to $P$.

**Solution:** We compute two vector on the plane as $\begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}0\\4\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\-1\end{pmatrix}$. Their cross product gives a normal vector to the plane, which is

$$\begin{pmatrix}2\\1\\-1\end{pmatrix} \times \begin{pmatrix}-1\\3\\-1\end{pmatrix} = \begin{pmatrix}2\\3\\7\end{pmatrix}$$

Any scalar multiple of this vector would also be an acceptable answer.

(b) Compute the angle between $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-3\end{pmatrix}$. You do not need to simplify any arccos or arcsin value that arises in your answer.

**Solution:** We use the formula $\vec{a}\cdot\vec{b} = ||\vec{a}||||\vec{b}||\cos\theta$. This yields:

$$1\cdot 2 + 1\cdot(-3) = \sqrt{1^2+1^2}\sqrt{2^2+(-3)^2}\cos\theta$$

Simplifying we get $-1 = \sqrt{2}\sqrt{13}\cos\theta$ so that $\boxed{\theta = \arccos(\frac{-1}{\sqrt{26}}).}$

(c) Where do the lines $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1+3s\\2-6s\end{pmatrix}$ and $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4t\\1-5t\end{pmatrix}$ intersect?

**Solution:** We solve the system of equations: $\begin{cases}1+3s & = 4t\\ 2-6s & -1-5t\end{cases}$ which has solution $s=1$ and $t=1$. Using $s=1$ we get the intersection point is $(1+3(1), 2-6(1)) = (4,-4)$. So the intersection point is $\boxed{(4,-4)}$. Note that this also the point given by $t=1$ which is $(4(1), 1-5(1))$.

5. Partial credit is available on this page.

(a) Let $I_n = \int x^{20}(\ln x)^n dx$ for $n = 0, 1, \ldots$. Derive a reduction formula for $I_n$.

**Solution:** We use integration by parts to find the solution. We let $f = (\ln x)^n$ so that $f' = n(\ln x)^{n-1}\frac{1}{x}$ and we let $g' = x^{20}$ so that $g = \frac{x^{21}}{21}$. We then have:

$$\begin{aligned}
I_n &= \int x^{20}(\ln x)^n dx \\
&= \int fg' \\
&= fg - \int f'g \\
&= \frac{x^{21}(\ln x)^n}{21} - \int n(\ln x)^{n-1}\frac{1}{x}\frac{x^{21}}{21}dx \\
&= \frac{x^{21}(\ln x)^n}{21} - \frac{n}{21}\int x^{20}(\ln x)^{n-1}dx \\
&= \frac{x^{21}(\ln x)^n}{21} - \frac{n}{21}I_{n-1}
\end{aligned}$$

So we get $\boxed{I_n = \frac{x^{21}(\ln x)^n}{21} - \frac{n}{21}I_{n-1}.}$

(b) On January 1, 2010 there were 10,000 squirrels living in Madison. Let $t$ denote time in years since January 1, 2010 and $S(t)$ denote the number of squirrels in Madison at time $t$. This squirrel population has a continuous birth rate of 8% and a natural continuous death rate of 2%. In addition, each year 250 squirrels are eaten by foxes and 150 squirrels are run over by cars. Write down a differential equation for $S(t)$. (You do not need to define variables or give the initial condition. DO NOT SOLVE THE DIFFERENTIAL EQUATION.)

**Solution:** $\boxed{dS/dt = .06S - 400.}$

6. On this page, you must show your work to receive full credit. Find a solution to each initial value problem.

(a)

$$\frac{dy}{dx} = x(2y + 2xe^{x^2}) \qquad \text{and} \qquad y(0) = 13.$$

**Solution:** We rewrite this as $\frac{dy}{dx} - (2x)y = 2x^2e^{x^2}$ which is a linear differential equation. We get $m(x) = e^{\int -2x\ dx} = e^{-x^2}$. Then we have:

$$\begin{aligned} y &= \frac{1}{m(x)} \cdot \int m(x)k(x)dx \\ &= e^{x^2} \int e^{-x^2} \cdot 2x^2e^{x^2}\,dx \\ &= e^{x^2} \int 2x^2 dx \\ &= e^{x^2}(\frac{2}{3}x^3 + C). \end{aligned}$$

Plugging in $y(0) = 13$ we get $13 = 1(0+C)$ so $C = 13$ and our answer is $\boxed{y = e^{x^2}(\frac{2}{3}x^3 + 13).}$

(b)

$$\frac{dy}{dx} = \frac{\sin x}{y+3} \qquad \text{and} \qquad y(0) = -2$$

**Solution:** Separating variables yields:

$$\begin{aligned} (y+3)dy &= \sin x\ dx \\ \int (y+3)dy &= \int \sin x\ dx \\ \frac{1}{2}y^2 + 3y &= -\cos x + C \\ \frac{1}{2}y^2 + 3y + (\cos x - C) &= 0. \end{aligned}$$

We use the quadratic formula to solve for $y$ and obtain

$$y = -3 \pm \sqrt{9 - 2(\cos x - C)}$$

Plugging in the initial condition shows that $-2 = -3 \pm \sqrt{9 - 2(\cos(0) - C)}$ and thus we choose the positive sign and let $C = -3$ yielding a final answer of $\boxed{y = -3 + \sqrt{9 - 2(\cos x + 3)}.}$

7. On this page, you must show your work to receive full credit.

$$\text{Compute } \int \sqrt{-3+4x-x^2}\ dx.$$

**Solution:** Complete the square to obtain $\int \sqrt{-3+4x-x^2}\ dx = \int \sqrt{1-(x-2)^2}dx$. We then compute:

$$\int \sqrt{-3+4x-x^2}dx = \int \sqrt{1-(x-2)^2}dx$$

We perform the trig substitution $x-2 = \sin\theta$ and $dx = \cos\theta d\theta$ to get

$$\begin{aligned} &= \int \sqrt{1-\sin^2\theta}\cos\theta d\theta \\ &= \int \cos^2\theta d\theta \end{aligned}$$

We now use the double angle formula to get:

$$\begin{aligned} &= \frac{1}{2}\int (1+\cos(2\theta))d\theta \\ &= \frac{1}{2}(\theta + \frac{1}{2}\sin(2\theta)) + C \end{aligned}$$

Now we need to replace the $\theta$'s by the original variables. We have $\theta = \arcsin(x-2)$. For $\sin(2\theta)$ we use the double angle formula $\sin(2\theta) = 2\sin\theta\cos\theta = 2\sin(\arcsin(x-2))\cos(\arcsin(x-2)) = 2(x-2)\sqrt{1-(x-2)^2}$. Plugging this all in yields:

$$= \frac{1}{2}\left(\arcsin(x-2) + (x-2)\sqrt{1-(x-2)^2}\right) + C.$$

8. On this page, you must show your work to receive full credit.

Compute $\displaystyle\int_1^{\infty} \frac{8x+6}{x(2x+1)(2x+3)}dx$ or explain why the integral does not exist. (You may freely use the formula $\frac{8x+6}{x(2x+1)(2x+3)} = \frac{2}{x} - \frac{2}{2x+1} - \frac{2}{2x+3}$.)

**Solution:**

$$\begin{aligned}
\int_1^{\infty} \frac{8x+6}{x(2x+1)(2x+3)}dx &= \int_1^{\infty} \frac{2}{x} - \frac{2}{2x+1} - \frac{2}{2x+3}dx \\
&= \lim_{b\to\infty} \left[2\ln|x| - \ln|2x+1| - \ln|2x+3|\right]_1^b \\
&= \lim_{b\to\infty} \left[\ln|\frac{x^2}{(2x+1)(2x+3)}\right]_1^b \\
&\lim_{b\to\infty} \ln|\frac{b^2}{(2b+1)(2b+3)}| - \ln\frac{1}{3\cdot 5} \\
&= \ln(1/4) - \ln(1/15)
\end{aligned}$$

There are several other equivalent answers like $\ln(15/4)$.

9. On this page, you must show your work and justify your answer to receive full credit. This problem is about the remainder term for $f(x) = x - 500x^3 + x^5$.

   (a) Compute $R_2 f(x)$.

   **Solution:** We compute $f'(x) = 1 - 1500x^2 + 5x^4$ and $f''(x) = -3000x + 20x^3$ and $f^{(3)}(x) = -3000 + 60x^2$. Thus $f(0) = 0)$ and $f'(0) = 1$. We then obtain $T_2 f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 0 + x + 0 = x$. Thus

   $$R_2 f(x) = f(x) - T_2 f(x) = (x - 500x^3 + x^5) - x = -500x^3 + x^5.$$

   (b) Find $M$ so that $M \geq |f^{(3)}(x)|$ for all $-2 \leq x \leq 2$. *Explain why your choice of $M$ is valid.*

   **Solution:** Multiple potential answers here. $f^{(3)}(x) = -3000 + 60x^2$. Note that $60x^2$ is always positive but it is never bigger than 3000 on the range $-2 \leq x \leq 2$. Thus the max of $|f^{(3)}(x)|$ on the rang $-2 \leq x \leq 2$ occurs at $x = 0$ and is 3000.

   Alternately, you could use the triangle inequality to observe that $|f^{(3)}(x)| = |-3000 + 60x^2| \leq |-3000| + |60x^2| = 3000 + 60x^2$. Since the max of $60x^2$ occurs at $x = \pm 2$ this would yield $M = 3240$.

   (c) Use your answer in part (b) to find $B$ so that $|R_2 f(x)| \leq B$ for all $-2 \leq x \leq 2$.

   **Solution:** This will depend on the answer in $B$ but will be $B = \frac{M2^3}{3!}$ for whichever $M$ you found.

10. On this page, you must show your work and justify your answer to receive full credit. Justifying your answer must include: clearly stating any convergence test that you use and explicitly verifying each hypothesis for that test.

For which values of $x$ does the series $\sum_{k=0}^{\infty} \frac{e^{kx}+k^2}{10^k+\sqrt{k}}$ converge?

**Solution:** Our solution will have two cases depending on the value of $x$.

**Case 1:** $x > 0$ We first assume $x > 0$ so that $e^{kx}$ is an exponential growth function in $k$ and thus $e^{kx}$ is the term which grows the fastest in the numerator. We define $a_k = \frac{e^{kx}+k^2}{10^k+\sqrt{k}}$ and $c_k = \frac{e^{kx}}{10^k}$ and we will apply the Limit Comparison Theorem to these two series. We check the hypotheses:

- $a_k = \frac{e^{kx}+k^2}{10^k+\sqrt{k}}$ and $c_k = \frac{e^{kx}}{10^k}$ are positive for all $k$ since the numerator and denominator are built from positive numbers for all $k$.
- We check that $\lim_{k\to\infty} \frac{a_k}{c_k} = \lim_{k\to\infty} \frac{e^{kx}+k^2}{10^k+\sqrt{k}} \cdot \frac{\frac{1}{e^{kx}}}{\frac{1}{10^k}} = \lim_{k\to\infty} \frac{1+(k^2/e^{kx})}{1+(\sqrt{k}/10^k)} = \frac{1+0}{1+0}$ since exponential growth function grows faster than any polynomial function. Since this limit is a positive real number, it satisfies the second hypothesis.

We may thus apply the Limit Comparison Theorem to conclude that either both series converge or both diverge.

We next observe that $\sum_{k=0}^{\infty} c_k$ is a geometric series $\sum_{k=0}^{\infty} (\frac{e^x}{10})^k$ with $r = \frac{e^x}{10}$. By the Geometric Series Test, this converges if and only if $|r| < 1$ which is when $x < \ln 10$. Thus for $x > 0$, the original series converges if and only if $x < \ln 10$.

**Case 2:** $x \leq 0$ We now assume $x \leq 0$ so that $e^{kx}$ does not grow with $k$. Then $k^2$ is the term which grows the fastest in the numerator. We define $d_k = \frac{k^2}{10^k}$ and we will apply the Limit Comparison Theorem to these two series. We check the hypotheses:

- $a_k = \frac{e^{kx}+k^2}{10^k+\sqrt{k}}$ and $d_k = \frac{k^2}{10^k}$ are positive for all $k$ since the numerator and denominator are built from positive numbers for all $k$.
- We check that $\lim_{k\to\infty} \frac{a_k}{d_k} = \lim_{k\to\infty} \frac{e^{kx}+k^2}{10^k+\sqrt{k}} \cdot \frac{\frac{1}{k^2}}{\frac{1}{10^k}} = \lim_{k\to\infty} \frac{(e^{kx}/k^2+1}{1+(\sqrt{k}/10^k)} = \frac{0+1}{1+0}$ since $e^{kx}$ is either constant or an exponential decay function. Since this limit is a positive real number, it satisfies the second hypothesis.

We may thus apply the Limit Comparison Theorem to conclude that either both series converge or both diverge.

Finally, we apply the Ratio Test to $\sum_{k=1}^{\infty} d_k$. We compute

$$L = \lim_{k\to\infty} |\frac{d_{k+1}}{d_k}| = \lim_{k\to\infty} |\frac{(k+1)^2}{10^{k+1}} \cdot \frac{10^k}{k^2}| = \lim_{k\to\infty} \frac{(k+1)^2}{k^2} \cdot \frac{1}{10} = 1 \cdot \frac{1}{10} = \frac{1}{10}.$$

Since $L = \frac{1}{10} < 1$, the Ratio Test implies that $\sum_{k=1}^{\infty} d_k$ converges. We conclude that the original series also converges whenver $x \leq 0$.

$\boxed{\text{Thus in conclusion, our original series converges for all } x < \ln 10.}$