WES Worksheet 4.1

Fall 2018

MATH 222, Week 4

Name: ______________________________

Solving Systems of Equations

I noticed that a lot of people had a good understanding of how to break up partial fractions problems, but found it a little trickier if you couldn't use the Heavyside method to find your constants A, B etc. So this section will focus on solving linear systems of equations. First I want to remind you of the following fact:

Fact 1. A system of linear equations containing n equations in n unknowns has a solution (for us it will usually be unique).

It could be interesting to think about this in terms of matrices, but if you want to talk more about that let me know. When you have your partial fraction decomposition

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \cdots + \frac{A_n}{(a_nx+b_n)}$$

This is the simplest case, but the idea will generalize. This gives you a system of n equations in n unknowns. How does this happen? Well, if you clear denominators you have

$$R(x) = A_1\frac{Q(x)}{(a_1x+b_1)} + \cdots + A_n\frac{Q(x)}{(a_nx+b_n)} \tag{1}$$

and remember each a_ix+b_i divides $Q(x)$ because this was the factorization of $Q(x)$ so each of these is a polynomial. Now this is an equality of polynomials. We know $R(x)$ is a polynomial with degree less than $Q(x)$ and $\deg Q(x) = n$, so $\deg(R(x)) < n$. This means we can write $R(x)$ as

$$R(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0.$$

Now we need the important following fact

Fact 2. If $f(x)$ and $g(x)$ are polynomials over the real numbers, with

$$f(x) = a_nx^n + \cdots + a_1x + a_0 \qquad \text{and} \qquad g(x) = b_nx^n + \cdots + a_1x + a_0$$

then $f(x) = g(x)$ if and only if we have $a_i = b_i$ for all $i = 0, \ldots, n$. So all the corresponding coefficients are equal.

If we apply this to (1), expanding the right hand side we get a system of n equations in n unknowns $A_1, \ldots, A_n$. Let's see this in practice.

Problem 3. Find and solve the system of equations corresponding to each of the following partial fraction integrals:

(a) $\dfrac{x^2}{(x-1)^3(x+2)}$ $= \frac{1}{54}\left(\frac{21-30x}{(x-1)^2} + 8\ln|x-1| - 8\ln|2+x|\right)$

(b) $\dfrac{x}{(x^2+1)(x-1)}$ $= \frac{\tan^{-1}(x)}{2} + \frac{1}{2}\ln|1-x| - \frac{1}{4}\ln|1+x^2|$

(c) $\dfrac{x}{(x+1)(x-2)(x-1)^2}$ $= \frac{1}{12}\left(\frac{6}{-1+x} - 9\ln|1-x| + 8\ln|2-x| + \ln|1+x|\right)$

(d) $\dfrac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}$ $\frac{1}{2}\left(\frac{-4+3x}{2+2x+x^2} - 5\tan^{-1}|1+x| + \ln|2+2x+x^2|\right)$

Irreducible Quadratics in Partial Fractions

In this section we will focus on solving partial fraction problems where the denominator contains an irreducible (over the real numbers) quadratic term.

(c) $: \frac{1}{32}\left(-\frac{6}{(x-1)^2}+\frac{18}{(x-1)}-\frac{4}{1+x}+\frac{11}{1+x^2}+\right.$
$4\tan^{-1}(x)+15\ln|x-1|+5\ln|x+1|-10\ln|x^2+1|)$

Problem 4. Evaluate the following integrals

(a) $\int \frac{1}{(x^2+1)(x-1)}\,dx = -\frac{\tan^{-1}(x)}{2}-\frac{1}{4}\ln\left|\frac{2+2(x-1)+}{(x-1)^2}\right|+\frac{1}{2}\ln|x-1|+C$

(b) $\int \frac{x}{(x^2+3)(x-2)^2}\,dx$

$= \frac{1}{98}\left(-\frac{28}{x-2}-8\sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right)-2\ln|2-x|+\ln|3+x^2|+C\right.$

(c) $\int \frac{2x^3+3x+1}{(x^2-1)^2(x^2+1)^2(x-1)}\,dx$

(d) $\int \frac{x^2}{(x^4-16)(x^2-4x+4)}\,dx$

(d) $= \frac{1}{128}\left(-\frac{8}{(x-2)^2}-\frac{4}{x-2}-3\ln|x-2|-\ln|x+2|\right.$
$+2\ln|x^2+4|)+C$

Rationalizing Substitution

In this section, we will focus on using partial fractions and some nice substitutions to solve problems with a linear term in a square root.

Problem 5. Evaluate the following integrals:

(a) $\int \frac{\sqrt{x+4}}{x}\,dx$ $\quad 2\sqrt{4+x}+2\ln|2-\sqrt{x+4}|-2\ln|2+\sqrt{4+x}|+C$

(b) $\int \frac{\sqrt{x+3}}{x^2}\,dx = -\frac{\sqrt{3+x}}{x}-\frac{\operatorname{Arctanh}\left(\frac{\sqrt{3+x}}{\sqrt{3}}\right)}{\sqrt{3}}$

$u=\sqrt{x+3} \quad u^2=x+3 \quad x=u^2-3$

(c) $\int \frac{\sqrt{x}}{x^2-1}\,dx = \arctan(\sqrt{x})+\frac{1}{2}\ln|1-\sqrt{x}|-\frac{1}{2}\ln|1+\sqrt{x}|$

(d) $\int \frac{\sqrt{x}}{3x^{3/2}+2x+1}\,dx$

$= -\frac{8\tan^{-1}\left(\frac{-1+6\sqrt{x}}{\sqrt{11}}\right)}{15\sqrt{11}}+\frac{2}{5}\ln|1+\sqrt{x}|+\frac{2}{15}\ln|1-\sqrt{x}+3x|$

More Partial Fractions

These problems will be slightly different than above, but are still partial fraction problems.

Problem 6. Evaluate the following integrals:

(a) $\int \sec(\theta)\,d\theta. = \ln|\sec\theta+\tan\theta|+C$

(b) $\int \frac{2}{e^z-e^{-z}}\,dz$. The integrand is also known as the hyperbolic cosecant. $= -\ln|\operatorname{csch}\theta+\coth\theta|+C$

(c) $\int \frac{\cos(x)}{(\sin(x)+2)^2(\sin(x)-1)(\sin(x)+2)^2}\,dx. = \frac{1}{162}\left(2\ln|1-\sin x|-2\ln|2+\sin x|+\frac{60+33\sin|x|+6\sin|x|^2}{(2+\sin x)^3}\right)$

(d) $\int \frac{2}{e^{-z}+e^z}\,dz$. The integrand is known as the hyperbolic secant.

$= 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right)$

$\tanh = \frac{e^{2z}-1}{e^{2z}+1} = \frac{e^z-e^{-z}}{e^z+e^{-z}}$