Worksheet 2

Fall 2018

MATH 222, Week 2: Integration by Parts and Trigonometric Integrals

Name: ________________________________

1 Integration by Parts without Trig Identities

Problem 1. Compute the following

(a) $\int \frac{1}{x}\ln(x)dx$

(b) $\int x\ln(x)\,dx$

(c) $\int \ln(x)dx$

What integration techniques did you use to solve each of these integrals and what lead you to choose them?

Solution 1.

(a) We can use u- sub here. Let $u = \ln(x)$, so $du = \frac{1}{x}dx$. When we sub back in we find

$$\int u\,du.$$

The anti-derivative is $u^2/2 + C$. Subbing back in we find the answer is $\frac{\ln(x)^2}{2} + C$.

(b) We need to do integration by parts here. Let $dv = x$ and $u = \ln(x)$. Then $du = \frac{1}{x}$ and $v = x^2/2$. Using the integration by parts formula:

$$\int x\ln(x)\,dx = \tfrac{x^2}{2}\ln(x) - \int \tfrac{x^2}{2x}\,dx$$

Simplifying we find the answer is $\frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$.

(c) Use integration by parts again with $u = \ln(x)$ and $dv = 1\,dx$. After applying the integration by parts formula we find the answer is

$$x\ln(x) - x + C.$$

□

Problem 2. (a) Compute $\int_0^\pi \cos(x)dx$ and $\int_0^\pi x^2\cos(x)dx$

(b) Show that:

$$\int x^n\cos(x)\,dx = x^n\sin(x) + nx^{n-1}\cos(x) - n(n-1)\int x^{n-2}\cos(x)\,dx$$

(Hint: The steps are very similar to what you did in part (a) for $\int_0^\pi x^2\cos(x)\,dx$).

(c) Use the identity you just proved and part (a) to compute $\int_0^\pi x^4\cos(x)\,dx$.

Solution 2.

(a) $\int_0^\pi \cos(x)dx = 0$ and $\int_0^\pi x^2\cos(x)dx = -2\pi$.

(b) When asked to verify a reduction formula, one can differentiate both sides and check equality. In this case we find

$$x^n\cos(x) = x^n\cos(x) + nx^{n-1}\sin(x) - nx^{n-1}\sin(x) + n(n-1)x^{n-2}\cos(x) - n(n-1)x^{n-2}\cos(x)$$

This is equal, so the identity is valid. You can re-derive this reduction formula using integration by parts.

(c) $\int_0^\pi x^4 \cos(x)\, dx = -4\pi(\pi^2 - 6)$.

$\square$

2 Integration by Parts with Trig Identities

Problem 3. There are some extra Integration by Parts problems that require some of the trig identities from the last worksheet. We worked through this problem in lecture, but it is worth slowly repeating yourselves:

(a) Compute $\int \cos^2(x)dx$.

(b) Compute $\int \cos^3(x)dx$.

(c) Compute $\int \cos^4(x)dx$.

(d) How do the previous three problems compare? Did you use the same idea for each or was there some difference? If so, how would you separate them? (Hint: How would you compute $\int \cos^5(x)dx$?)

(e) (**Challenge**) Try to find a recursive formula as in Problem 2 for $\int \cos^n(x)dx$ using IBP. That is let $J_n = \int \cos^n(x)dx$ express J_n in terms of smaller J_k (This is what we did in lecture, try to do it just using the previous parts, not your notes).

Solution 3.

(a) We use the trig identity $\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$,

$$\int \tfrac{1}{2}(\cos(2x) + 1)\, dx = \tfrac{x}{2} + \tfrac{1}{4}\sin(2x).$$

(b) In this case we can use a u-sub. If we let $u = \sin(x)$, then $du = \cos(x)$, this integral reduces to

$$\int (1 - u^2)\, du = u - \tfrac{u^3}{3} + C = \sin(x) - \tfrac{\sin(x)^3}{3} + C.$$

(c) We see an even power of cos, so we separate this as

$$\int \cos^2(x)\cos^2(x)\, dx = \int \tfrac{1}{2}(\cos(2x) + 1) \cdot \tfrac{1}{2}(\cos(2x) + 1)\, dx = \tfrac{1}{4}\int \cos^2(2x) + 2\cos(2x) + 1\, dx.$$

We once again use the identity $\cos^2(2x) = \frac{1}{2}(\cos(4x) + 1)$

$$\tfrac{1}{4}\int \tfrac{1}{2}(\cos(4x) + 1) + 2\cos(2x) + 1\, dx = \frac{3x}{8} + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x).$$

(d) We notice that for even powers we need a trig identity, but for odd powers we can use u-sub.

(e) I'm going to give the answer and not how to get there. This is a good problem to try to work out. Feel free to ask your TA.

$$\int \cos^n(x)\, dx = \tfrac{1}{n}\cos^{n-1}(x)\sin(x) + \tfrac{n-1}{n}\int \cos^{n-2}(x)\, dx.$$

$\square$

Problem 4. We will do the same problem as above, but for $\sin(x)$ this time:

(a) Compute $\int \sin^2(x)dx$

(b) Compute $\int \sin^3(x)dx$

(c) Compute $\int \sin^4(x)dx$

(d) How do the previous three problems compare? Did you use the same idea for each or was there some difference? If so, how would you separate them?

(e) How did problem 4 compare to problem 3? What differences and similarities were there? How would you try to explain some of the differences?

(f) (**Challenge**) Try to find a recursive formula as in Problem 2 and Problem 3 for $\int \sin^n(x)dx$ using IBP.

Solution 4. The solutions here are basically the same as above. I would encourage you to work it out for yourself using the previous problem. The check your answers you can consult www.wolframalpha.com. $\square$

Problem 5. Compute $\int \arctan(x)\ dx$.

Solution 5. We use integration by parts. We want to simplify this integral by getting rid of the arctan. So we let $u = \arctan(x)$ and $dv = 1$. With this choice we have $du = \frac{1}{1+x^2}$ and $v = x$. Applying the integration by parts formula

$$\int \arctan(x)\ dx = x\arctan(x) - \int \tfrac{x}{1+x^2}\ dx.$$

To solve this last integral, we can use u-sub or recognize it as the integral of $\frac{1}{2}\ln|1+x^2|$. So the final answer is

$$x\arctan(x) - \tfrac{1}{2}\ln|1+x^2|.$$

$\square$

3 Challenge Problem

Problem 6. Find $\int e^x\cos(x)dx$.

Solution 6. The key here is to double back on ourselves. No matter the choice of u or dv, this doesn't seem to get better. So we will let $u = \cos(x)$ and $dv = e^x$. Then $du = -\sin(x)$ and $v = e^x$. Applying integration by parts

$$\int e^x\cos(x)dx = e^x\cos(x) + \int e^x\sin(x)\ dx.$$

Now we apply integration by parts again, making sure we choose the trig function as u and the exponential as dv. Try to make the opposite choice and see what happens. When we apply integration by parts again we have

$$\int e^x\cos(x)dx = e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)\ dx.$$

If we let $I = \int e^x\cos(x)dx$, this is

$$I = e^x\cos(x) + e^x\sin(x) - I,$$

so if we add I to both sides we find

$$2I = e^x \cos(x) + e^x \sin(x).$$

Dividing by 2 the final answer is

$$I = \tfrac{1}{2}(e^x \cos(x) + e^x \sin(x)).$$

$\square$