**Final a**
MATH 222 (Lectures 1,2, and 4) Fall 2015.

Name: ______________________________ Student ID#: ________________________

Circle your TAs name:

| | | | | |
|---|---|---|---|---|
| Carolyn Abbott | Tejas Bhojraj | Zachary Carter | Mohamed Abou Dbai | Ed Dewey |
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| Tao Ju | Ahmet Kabakulak | Dima Kuzmenko | Ethan McCarthy | Tung Nguyen |
| Jaeun Park | Adrian Tovar Lopez | Polly Yu | | |

| | Problem 1 | Problem 2 | Problem 3 | Problem 4 | Problem 5 |
|---|---|---|---|---|---|
| Score | | | | | |
| | Problem 6 | Problem 7 | Problem 8 | Problem 9 | Problem 10 |
| Score | | | | | |

## Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- Please read the instructions on every problem carefully.
- On Problems 1–4 only the answer will be graded.
- On Problems 5–10 you must show your work and we will grade the work and your justification, and not just the final answer.
- Each problem is worth ten points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1-x^2}$. Note that we have provided some formulas on the next page to help with this.

## Formulas

You may freely quote any algebraic or trigonometric identity, as well as well as any of the following formulas or minor variants of those formulas.

## Integrals

- $\cos(\arcsin x) = \sqrt{1-x^2}$
- $\sec(\arctan x) = \sqrt{1+x^2}$.
- $\tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$.
- $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ \ln|x| + C & \text{when } n = -1 \end{cases}$
- $\int e^x dx = e^x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$.
- $\int \csc x dx = -\ln|\csc x + \cot x| + C$.
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.

## Taylor series

- $T_\infty e^x = \sum_{k=0}^\infty \frac{x^k}{k!}$
- $T_\infty \sin x = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
- $T_\infty \cos x = \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{(2k)!}$
- $T_\infty \frac{1}{1-x} = \sum_{k=0}^\infty x^k$
- $T_\infty \frac{1}{1+x} = \sum_{k=0}^\infty (-1)^k x^k$
- $T_\infty (1+x)^b = \sum_{k=0}^\infty \binom{b}{k} x^k$ where $\binom{b}{k} = \frac{b(b-1)(b-2)\cdots(b-k+1)}{k!}$

## Other

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

1. For each statement below, CIRCLE true or false. You do not need to show your work.

| (a) | | (b) | | (c) | | (d) | | (e) | |
|---|---|---|---|---|---|---|---|---|---|
| True | False | True | False | True | False | True | False | True | False |

(a) The series $\sum_{k=0}^{\infty} \frac{2^{2k+1}}{3^k}$ converges.

(b) The series $\sum_{k=0}^{\infty} \frac{k^2+e^{-k}}{\sqrt{k}+e^k}$ converges.

(c) The integral $\int_3^{\infty} \frac{1}{x} - \frac{1}{x-1} dx$ converges.

(d) $\int_e^{\infty} \frac{1}{x^2 \ln(x)} dx \leq \int_e^{\infty} \frac{1}{x^2} dx$.

(e) $e^{x^2} - (1 + x^2)$ is $o(x^4)$.

2. On this page only the answer will be graded.

(a) Compute $\int \frac{1}{(x+2)(x+3)} dx$.

(b) Let $f(x) = x^2 \sin(x^3)$. Compute $f^{(605)}(0)$.

3. On this page only the answer will be graded.

(a) Compute $T_4(\sqrt{1-x^3} \cdot \cos(x^2))$.

(b) For which values of $b$ does the Taylor series for $\frac{x}{3-5x^4}$ converge at $x = b$?

4. On this page, only the answer will be graded.

(a) Let $P$ be the plane through the points $(1,1,1),(3,2,0)$ and $(0,4,0)$. Compute a nonzero normal vector to $P$.

(b) Compute the angle between $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-3\end{pmatrix}$. You do not need to simplify any arccos or arcsin value that arises in your answer.

(c) Where do the lines $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1+3s\\2-6s\end{pmatrix}$ and $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4t\\1-5t\end{pmatrix}$ intersect?

5. Partial credit is available on this page.

(a) Let $I_n = \int x^{20}(\ln x)^n dx$ for $n = 0, 1, \ldots$. Derive a reduction formula for $I_n$.

(b) On January 1, 2010 there were 10,000 squirrels living in Madison. Let $t$ denote time in years since January 1, 2010 and $S(t)$ denote the number of squirrels in Madison at time $t$. This squirrel population has a continuous birth rate of 8% and a natural continuous death rate of 2%. In addition, each year 250 squirrels are eaten by foxes and 150 squirrels are run over by cars. Write down a differential equation for $S(t)$. (You do not need to define variables or give the initial condition. DO NOT SOLVE THE DIFFERENTIAL EQUATION.)

6. On this page, you must show your work to receive full credit. Find a solution to each initial value problem.

(a)

$$\frac{dy}{dx} = x(2y + 2xe^{x^2}) \qquad \text{and} \qquad y(0) = 13.$$

(b)

$$\frac{dy}{dx} = \frac{\sin x}{y+3} \qquad \text{and} \qquad y(0) = -2$$

7. On this page, you must show your work to receive full credit.

$$\text{Compute } \int \sqrt{-3+4x-x^2}\; dx.$$

8. On this page, you must show your work to receive full credit.

Compute $\displaystyle\int_1^\infty \frac{8x+6}{x(2x+1)(2x+3)}\,dx$ or explain why the integral does not exist. (You may freely use the formula $\frac{8x+6}{x(2x+1)(2x+3)} = \frac{2}{x} - \frac{2}{2x+1} - \frac{2}{2x+3}$.)

9. On this page, you must show your work and justify your answer to receive full credit. This problem is about the remainder term for $f(x) = x - 500x^3 + x^5$.

   (a) Compute $R_2 f(x)$.

   (b) Find $M$ so that $M \geq |f^{(3)}(x)|$ for all $-2 \leq x \leq 2$. *Explain why your choice of $M$ is valid.*

   (c) Use your answer in part (b) to find $B$ so that $|R_2 f(x)| \leq B$ for all $-2 \leq x \leq 2$.

10. On this page, you must show your work and justify your answer to receive full credit. Justifying your answer must include: clearly stating any convergence test that you use and explicitly verifying each hypothesis for that test.

For which values of $x$ does the series $\sum_{k=0}^{\infty} \frac{e^{kx}+k^2}{10^k+\sqrt{k}}$ converge?

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