WES Worksheet 3.1

Fall 2018

MATH 222, Week 3

Name: ______________________________

Factoring polynomials

This section will focus on factoring polynomials. In general this is an incredibly difficult problem that algebraist's are extremely interested in. We will focus on more doable cases. There is a cool result that characterizes which rational numbers could appear as roots of a polynomial, it's called the rational roots theorem.

The **rational roots theorem** says that $(x - \frac{a}{b})$, where $\frac{a}{b}$ is a fraction in lowest terms, divides a polynomial $a_n x^n + \cdots + a_1 x + a_0$ of degree n precsiely when b divides a_n and a divides a_0. Why might this be the case?

Problem 1. What are the possible rational roots of the polynomial $x^3 - 6x^2 + 11x - 6$? ±1, ±3, ±2

Problem 2. Factor the polynomial $x^3 - 8x^2 + 21x - 18$. = $(x-3)^2(x-2)$

The Discriminant

The discriminant of a degree two polynomial $f(x) = ax^2 + bx + c$ is $\Delta = b^2 - 4ac$. We want to think about what the discriminant tells us about the polynomial $f(x)$.

(a) What can we say about $f(x)$ if $\Delta \geq 0$, i.e. how can we write $f(x)$? (Quadratic formula) We can factor

(b) What happens if $\Delta < 0$? What does this tell us about $f(x)$? Remember we are always working over the real numbers. We can't factor

(c) Can we factor $x^2 - 2x + 1$ in the real numbers? Yes $(x-1)^2$

(d) How about $x^2 + 2x + 7$? Can we factor it in the real numbers? Why or why not?
No $\Delta < 0$.

Partial fractions

Heaviside cover-up method

A rational function $f(x)$ is **really simple** if it has a proper rational expression $f(x) = \dfrac{p(x)}{q(x)}$ such that every root of $q(x)$ has order 1 and $q(x)$ has no factors $x^2 + bx + c$ with $b^2 - 4a < 0$. Every really simple rational function has a **partial fraction decomposition**:

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{(x - r_1)(x - r_2)\cdots(x - r_n)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_n}{x - r_n}$$

Think about why this is the case. This is most easily found using the **Heaviside cover-up method**:

$$\frac{p(x)}{(x-r_1)\cdots(x-r_{k-1})(x-r_k)(x-r_{k+1})\cdots(x-r_n)} = \frac{A_1}{x-r_1} + \cdots + \frac{A_{k-1}}{x-r_{k-1}} + \frac{A_k}{x-r_k} + \frac{A_{k+1}}{x-r_{k+1}} + \cdots + \frac{A_n}{x-r_n}$$

$$\frac{p(x)}{(x-r_1)\cdots(x-r_{k-1})(x-r_{k+1})\cdots(x-r_n)} = \frac{A_1(x-r_k)}{x-r_1} + \cdots + \frac{A_{k-1}(x-r_k)}{x-r_{k-1}} + A_k + \frac{A_{k+1}(x-r_k)}{x-r_{k+1}} + \cdots + \frac{A_n(x-r_k)}{(x-r_n)}$$

$$\frac{p(r_k)}{(r_k-r_1)\cdots(r_k-r_{k-1})(r_k-r_{k+1})\cdots(r_k-r_n)} = 0 + \cdots + 0 + A_k + 0 + \cdots + 0$$

So to find the constant above $x - r_k$ in our partial fraction decomposition we just need to plug r_k into the original polynomial and divide by the differences of r_k with all the other roots. Let's see this in practice.

Problem 3. Use long division and a partial fraction decomposition to find the anti-derivative

$$\int \frac{x^2+x+1}{x^2+2x-3}\, dx.$$

(b) $\frac{3/2}{1-x} + \frac{-2}{x} + \frac{1/2}{1+x}$

Problem 4. Find the partial fraction decomposition of each of the following:

(a) $\frac{1}{x^3-x}$ (b) $\frac{x+2}{x^3-x}$ (c) $\frac{x^2+1}{x^3+4x^2+x-6}$ (d) $\frac{2x^2+2x+2}{x^3+4x^2+x-6}$

(c) $\frac{-1/6}{(x-1)} + \frac{-5/3}{x-2} + \frac{5/2}{x+3}$

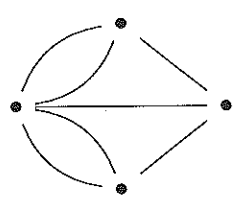
$= \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$ $= 2\left(\frac{-1/4}{x-1} + \frac{-1}{x+2} + \frac{7/4}{x+3}\right)$

Problem 5. Integrate all of the expressions in Problem 4 with respect to x using your partial fraction decomposition. logs

More on the Seven Bridges of Königsberg

The city of Könisberg in Prussia (not Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands - Kneiphof and Lomse - which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

Note that one cannot reach an island or mainland via something that is not a bridge or access any bridge without crossing to its other end. In 1736 Euler was able to prove that this problem has no solution. This is harder than finding a solution because you have to show that no possible path will work rather than just giving a path that does. My hint was to consider the picture of the bridges as a **graph**:



Where each dot, called a **vertex**, represents a land mass and each line, called an **edge**, represents a bridge. The goal is to start at some vertex and walk along each edge exactly once. After Euler's solution to this problem, such walks through graphs became known as **Eulerian Trails**. So what Euler proved is that there is no Eulerian Trail in this graph. He actually proved something more general. He proved exactly when an Eulerian Trail could exist in a graph and showed that this graph does not have the necessary and sufficient condition. I challenge you to think about what this necessary and sufficient condition might be.