Worksheet 3

Spring 2016

MATH 222, Week 3: I.10,I.11,I.12

Name: ________________________________

Problem 1. Compute $\int \frac{dx}{(x^2-4)(x^2+1)^2}$
Solution 1.

We first notice that $x^2 - 4 = (x+2)(x-2)$, unfortunately we have no way or decomposing x^2+1. So we begin our partial fraction decomposition:

$$\frac{1}{(x^2-4)(x^2+1)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C_1x + D_1}{x^2+1} + \frac{C_2x+D_2}{(x^2+1)^2}$$

Multiplying to get a common denominator we now have:

$$A(x^2+1)^2(x-2) + B(x+2)(x^2+1)^2 + (C_1x+D_1)(x+2)(x-2)(x^2+1) + (C_2x+D_2)(x+2)(x-2) = 1$$

If we multiplied this out we would have six equations with six unknowns, but let's use a trick. We know this equation must hold for every x value, so in particular it must hold for any x value I select. If I take $x = -2$ then I see:

$$A(25)(-4) = 1$$

Every other term vanishes, so if there is a solution A must equal $-1/100$. Similarly if I take $x = 2$:

$$B(4)(25) = 1$$

So $B = 1/100$. Now unfortunately there isn't an apparent x value I can take to easily find C_1 and D_1, but now I can substitute these value in for A and B, multiply out and solve:

$$1/25 - 4D_1 - 4D_2 + (-4C_1 - 4C_2)x + (2/25 - 3D_1 + D_2)x^2 + (-3C_1 + C_2)x^3 + (1/25 + D_1)x^4 + C_1x^5 = 1$$

This forces $C_1 = 0$, and so $C_2 = 0$ as well. Then we can see $D_1 = -1/25$ and $D_2 = -1/5$. Plugging this back in we now want to compute the integral:

$$\int \frac{-1/100}{x+2} + \frac{1/100}{x-2} + \frac{-1/25}{x^2+1} + \frac{-1/5}{(x^2+1)^2}dx$$

We know how to do this:

$$\int \frac{-1/100}{x+2} + \frac{1/100}{x-2} + \frac{-1/25}{x^2+1} + \frac{-1/5}{(x^2+1)^2}dx = \frac{1}{100}\left(\frac{-10x}{1+x^2} - 14\tan^{-1}(x) + \ln(2-x) - \ln(2+x)\right) + C$$

$\square$

Problem 2. (a) Compute $\int_2^4 \frac{1}{x^2}\,dx$

(b) Compute $\int_2^4 \frac{1}{x(x-h)}\,dx$ where h is any positive number.

(c) What happens as $h \to 0$ in the integral for part (b)? How is this related to part (a)?

Solution 2.

(a) $\int_2^4 \frac{1}{x^2}\,dx = \frac{-1}{x}|_2^4 = \frac{-1}{4} + \frac{1}{2} = \frac{1}{4}$

(b) Using partial fractions we can write:

$$\frac{1}{x(x-h)} = \frac{A}{x} + \frac{B}{x-h}$$

Finding a common denomiator and equating the numerators we have $A(x-h)+Bx = 1$. If we let $x = 0$ we find that $A = -1/h$ and letting $x = h \neq 0$ we have $B = 1/h$. So now we substitute and integrate:

$$\int_2^4 \frac{-1/h}{x} + \frac{1/h}{x-h}\,dx = \frac{\ln(x-h) - \ln(x)}{h}|_2^4 = \frac{\ln(4-h) - \ln(2-h) - \ln(4) + \ln(2)}{h}$$

As $h \to 0$ we recognize this as the definition for the derivative of $\ln(x)$ with a slight modification, replacing h with $-h$ and so the integral goes to $-\frac{d}{dx}\ln(x) = -1/x$ as we saw in part (a).

□

Problem 3. Compute $\int \frac{1}{x^2+a^2}\,dx$

Solution 3.

We know that the antiderivative of $\frac{1}{x^2+1}$ is $\tan^{-1}(x)$. So if we mate a substitution and let $t = x/a$ then $dx = adt$ and we notice by multiplying in the numerator and denominator by $1/a^2$:

$$\frac{1}{x^2+a^2} = \frac{1/a^2}{(x/a)^2+1}$$

We can then make our substitution and solve:

$$\int \frac{1/a}{t^2+1}dt = \frac{\tan^{-1}(t)}{a} + C = \frac{\tan^{-1}(x/a)}{a} + C$$

□

Problem 4. Use trig substitution to eliminate the root in $\sqrt{1-4x-2x^2}$

Solution 4.

We must first complete the square. Notice $1-4x-2x^2 = -2(x^2+2x-1/2) = -2((x+1)^2-3/2) = 2(3/2-(x+1)^2)$. Now whenever we see $a-x^2$ we should think sin or cos substitution, either would work. Let's set $x+1 = \sqrt{3/2}\sin(\theta)$. Then we have:

$$\sqrt{2}\sqrt{3/2 - 3/2\sin(\theta)^2} = \sqrt{2}\sqrt{3/2}\sqrt{1-\sin(\theta)^2} = \sqrt{3}\cos(\theta)$$

□

Problem 5. Compute $\int \frac{1}{\sqrt{2x-x^2}}\, dx$

Solution 5.

We first must complete the square. $2x - x^2 = -(x^2 - 2x) = -((x-1)^2 - 1) = 1 - (x-1)^2$. So if we let $(x-1) = \sin(\theta)$ then $dx = \cos(\theta)$ and our integral becomes:

$$\int \frac{\cos(\theta)}{\cos(\theta)} d\theta = \int 1\ d\theta = \theta + C = \arcsin(x-1) + C$$

$\square$

Problem 6. Compute $\int t^3(3t^2-4)^{3/2}\, dt$

Solution 6.

We could do this with trig sub, but it is so much easier if we use u-substitution. First let $u = t^2$, then $du = 2tdt$ so we have:

$$\int t^3(3t^2-4)^{3/2}\ dt = 1/2 \int u(3u-4)^{3/2}\ du$$

Now if we let $s = 3u - 4$ we find that $ds = 3du$, so substituting we have:

$$1/2 \int u(3u-4)^{3/2}\ du = 1/18 \int (s+4)(s)^{3/2}\ ds = 1/18 \int s^{5/2} + 4s^{3/2}\ ds$$

Now if we integrate we have:

$$1/18 \int s^{5/2} + 4s^{3/2}\ ds = 1/18(2/7s^{7/2} + 8/5s^{5/2}) + C$$

Going all the way back to t we have:

$$\int t^3(3t^2-4)^{3/2}\ dt = 1/63(3t^2-4)^{7/2} + 8/90(3t^2-4)^{5/2} + C$$

$\square$