Worksheet 5

Fall 2018

MATH 222, Week 5: Improper Integrals and Modeling with Differential Equations

Name: ________________________________

Some Rationalizing Substitution

Problem 1. Evaluate the following integrals:

(a) $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$

(b) $\displaystyle\int \frac{\sqrt{x+3}}{x^2}\,dx$

(c) $\displaystyle\int \frac{\sqrt{x}}{x^2-1}\,dx$

(d) $\displaystyle\int \frac{\sqrt{x}}{3x^{3/2}+2x+1}\,dx$

Solution 1. I'll work through one of these in great detail as an example, then leave the answers for the others.

(a) Here we want to eliminate the root so we set $u = \sqrt{x+4}$ i.e. $u^2 - 4 = x$ and $2u\,du = dx$. Making this substitution we have

$$\int \frac{2u^2}{u^2-4}\,du.$$

This is now a partial fractions problem. The degree of the top equals the degree of the bottom, so we must divide.

$$\int \frac{2u^2}{u^2-4}\,du = \int 2 + \frac{8}{(u-2)(u+2)}\,du.$$

Now if we work through the partial fraction decomposition of the second part of the integral we find

$$\int 2 + \frac{2}{u-2} - \frac{2}{u+2}\,du = 2u + 2\ln|u-2| - 2\ln|u+2| + C = 2\sqrt{x+4} + 2\ln|\sqrt{x+4}-2| - 2\ln|\sqrt{x+4}+2| + C.$$

(b)

(c) $\arctan(\sqrt{x}) + \frac{1}{x}\ln|1-\sqrt{x}| - \frac{1}{2}\ln|1+\sqrt{x}| + C$

(d) $\frac{2}{5}\log\left(\sqrt{x}+1\right) + \frac{2}{15}\log\left(3x-\sqrt{x}+1\right) - \frac{8\tan^{-1}\left(\frac{6\sqrt{x}-1}{\sqrt{11}}\right)}{15\sqrt{11}}$

□

Improper Integrals, we made it

Problem 2. (a) Compute $\int_0^\infty \frac{x}{\sqrt{1+x^2}}\,dx$

(b) Compute $\int_{-\infty}^0 \frac{x}{\sqrt{1+x^2}}\,dx$

(c) What does this say about $\int_{-\infty}^\infty \frac{x}{\sqrt{1+x^2}}\,dx$

Solution 2.

(a) It diverges.

(b) It diverges.
(c) This implies this integral must diverge.

□

Problem 3. (a) Show that $\int_1^\infty \frac{dx}{x^2-4}$ is not a finite number.

(b) What answer do you get if you forget to account for the asymptote at $x = 2$?

Solution 3.
(a) This is an improper integral of both type I and type II. To show it is not a finite number it is enough to show that $\int_1^2 \frac{dx}{x^2-4}$ does not converge. To do this you use the partial fraction decomposition to get that this integral equals,

$$\lim_{t\to 2^-} \frac{1}{2}\ln|x-2| - \frac{1}{2}\ln|x+2|\Big|_1^t.$$

Now when you evaluate at t and try to take the limit, this does not converge because $\lim_{t\to 2^-}\ln|x-2|$ does not exist, while all the other limits are well defined.
(b) If you forget the asymptote, you end up with

$$\lim_{t\to\infty} \frac{1}{2}\ln|x-2| - \frac{1}{2}\ln|x+2|\Big|_1^t.$$

In this case when you simplify the logs this limit actually exists,

$$\lim_{t\to\infty} \frac{1}{2}\ln\left|\frac{x-2}{x+2}\right|\Bigg|_1^t. = -\frac{1}{2}\ln\left|\frac{-1}{3}\right|.$$

So you would wrongly conclude this integral converges.

□

Problem 4. Evaluate the following integrals, in each, identify why the integral is improper. Try to decide ahead of time whether you expect the integral to diverge or converge.

(a) $\int_1^\infty \frac{1}{x^2}\,dx$
(b) $\int_1^\infty \frac{\ln(x)}{x^3}\,dx$
(c) $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$
(d) $\int_3^{10} \frac{1}{(x-9)^{1/3}}\,dx$
(e) $\int_0^4 \frac{1}{x^2+x-6}\,dx$
(f) $\int_0^1 \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

For the last three problems, what would happen if you forgot about the asymptote?

Solution 4. This was on the previous worksheet, check there for solutions. □

Problem 5. Evaluate the following integrals. What happens in the third integral? How is this different than what occurs in the fourth integral?

(a) $\displaystyle\int_0^2 \frac{x}{(x+1)(x-1)}\,dx$

(b) $\displaystyle\int_0^2 \frac{\sqrt{x}}{(x+1)(x-1)}\,dx$

(c) $\displaystyle\int_0^2 \frac{x-\sqrt{x}}{(x+1)(x-1)}\,dx$

(d) $\displaystyle\int_{-2}^3 \frac{1}{(x-2)(x+1)}\,dx$

Solution 5.

(a) This diverges.

(b) This also diverges.

(c) This integral converges to $\frac{\log(3)}{2} + \log\left(1+\sqrt{2}\right) - \tan^{-1}\left(\sqrt{2}\right)$. The key thing to note here is that although the two pieces of the integral might diverge, their difference may converge. So be careful when you're trying to split integrals up over sums.

(d) This diverges. It suffices to check one of the integrals when you break up this integral into two pieces because you break the integral up over bounds, as compared to over sums. See the WES worksheet 4.2 for more on this.

□

Bounds

The goal here is to become comfortable bounding functions with simpler functions to eventually compare.

Problem 6. (a) Find a value of a so that for all $x > a$

$$\frac{1}{x-2} > \frac{1}{(x-2)^2}.$$

(b) Find a value of a so that for all $x > a$

$$\frac{1}{2x-1} > \frac{1}{(2x-1)^2}.$$

(c) Find a value of a so that for all $x < a$ with $x > 2$

$$\frac{1}{\sqrt{x-2}} < \frac{1}{x-2}.$$

(d) True or false: for all $x > 0$

$$\frac{x}{\sqrt{1+x^2}} < \frac{1}{\sqrt{x}}.$$

Solution 6.

(a) This is equivalent to finding $x-2>1$ if I multiply both sides by $(x-2)^2$ which will always be positive. We then find that $x > 3$ is necessary and sufficient, so we can take $a = 3$.

(b) Using similar logic we need $x > 1$.

(c) $x > 3$.

(d) This is false because it reduces to the inequality $x^3 < 1 + x^2$ which is false for all $x > 0$, in particular it is false when $x = 2$ because $8 \not< 5$.

□

Problem 7. Conclude whether the following integrals converge or diverge without computing it explicitly. Then explicitly evaluate the integrals.

(a) $\displaystyle\int_4^\infty \frac{1}{x^3-x}\,dx$

(b) $\displaystyle\int_1^2 \frac{dt}{t\sqrt{t^2-1}}.$

Solution 7.

(a) For large values of x this will behave a lot like $\frac{1}{x^3}$ so we might guess this will converge. Unfortunately because we have $x^3 - x$ comparing this to $\frac{1}{x^3}$ won't exactly work. However, we can notice that for $x \geq 4$

$$\frac{1}{x^3 - x} < \frac{1}{x^2}.$$

This inequality is equivalent to $x^2 < x^3 - x = x(x^2 - 1)$ when we cross multiply. If we divide by x, this is equivalent to $x < x^2 - 1$ for $x \geq 4$ which is clearly true. If we wanted to be super rigorous, we could note that $x^2 - x - 1$ has roots $\frac{1 \pm \sqrt{5}}{2}$ which are both less than 4 and is positive when $x \geq 4$ so it can never become negative again because by the intermediate value theorem it would have to be 0 for that to happen.
Since $\int_4^\infty \frac{1}{x^2}\, dx$ converges, by comparison our integral converges. To solve this explicitly you have to do partial fractions, factoring the denominator as $x(x-1)(x+1)$. If you do this, you get

$$\ln(4) - \ln(15)/2.$$

□

Differential Equations, if we get there...

Problem 8. Find a solution to the initial value problem

$$\frac{dy}{dx} = e^y x^3$$
$$y(0) = 0$$

Problem 9. Find a solution to the initial value problem

$$\frac{dy}{dx} = y\sqrt{y^2 - 1}\cos(x)$$
$$y(0) = 1$$

Problem 10. Find the general solution to the differential equation

$$\frac{dy}{dx} = x^2 + y^2 x^2$$

Problem 11. Find the general solution to the differential equation (for $x \neq 0$)

$$x\frac{dy}{dx} = -y + x$$

Solution 8. We didn't get to this. □