1 Some Continuity

Consider the following functions:

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

and $g(x) = x^2$. Which of the following are continuous and why? What does this illustrate about compositions of functions?

(a) $g(x)$:
Continuous since it's a polynomial.

(b) $f(x)$:
Not continuous, there's a jump

(c) $g \circ f(x)$:
This is identically 1 so it's continuous.

(d) $f \circ g(x)$:
This is not continuous. There's a hole at $x = 0$. How could you explicitly write out this function? The goal of this problem is to illustrate that anything can happen when you compose a continuous function with a discontinuous function! In particular just because a function in a composition is not continuous it does NOT mean the composition is not continuous.

2 Trig Limits!

Evaluate the following limits assuming $\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{x\to 0} \frac{1-\cos(\theta)}{\theta^2} = 1/2$. Double angle identities will be helpful...

(a) $\lim_{x\to\infty} \frac{\cos(x)}{x^2+1}$
Use the sandwich theorem! You know $-1 \leq \cos(x) \leq 1$. We also know $x^2 + 1 > 0$ for any x so we can divide

through by it to find

$$\frac{-1}{x^2+1} \leq \frac{\cos(x)}{x^2+1} \leq \frac{1}{x^2+1}$$

Now if we apply the sandwich theorem we notice both the functions bounding $\frac{\cos(x)}{x^2+1}$ to go 0 as $x \to \infty$. Hence our answer must be 0.

(b) $\lim_{x\to 0} \frac{\tan(2x)}{\tan(\pi x)}$

Rewrite this as

$$\lim_{x\to 0} \frac{\sin(2x)}{\cos(2x)} \frac{\cos(\pi x)}{\sin(\pi x)}$$

Now the cos terms are fine, but we need to fix the sine terms. We can do this

$$\lim_{x\to 0} \frac{\sin(2x)}{2x\cos(2x)} \frac{\pi x\cos(\pi x)}{\sin(\pi x)} \cdot \frac{2x}{\pi x}$$

We can take this limit now using the limits we know

$$\lim_{x\to 0} \frac{\sin(2x)}{2x\cos(2x)} \frac{\pi x\cos(\pi x)}{\sin(\pi x)} \cdot \frac{2x}{\pi x} = \frac{2}{\pi}$$

(c) $\lim_{\theta\to 0} \sin(\theta)\cot(2\theta)$

When you see tangent or cotangent we almost always replace it with sin and cos. So

$$\lim_{\theta\to 0} \sin(\theta)\cot(2\theta) = \lim_{\theta\to 0} \frac{\sin(\theta)\cos(2\theta)}{\sin(2\theta)}$$

Now we apply the double angle identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$\lim_{\theta\to 0} \frac{\sin(\theta)\cos(2\theta)}{\sin(2\theta)} = \lim_{\theta\to 0} \frac{\sin(\theta)\cos(2\theta)}{2\sin(\theta)\cos(\theta)} = \lim_{\theta\to 0} \frac{\cos(2\theta)}{2\cos(\theta)} = \frac{1}{2}$$

(d) $\lim_{x\to 0} \frac{x+\tan(x)}{\sin(2x)}$

We distribute the denominator and rewrite tangent:

$$\lim_{x\to 0} \frac{x+\tan(x)}{\sin(2x)} = \lim_{x\to 0} \frac{x}{\sin(2x)} + \frac{\sin(x)}{\sin(2x)\cos(x)}$$

We now apply the double angle identity to the second summand and multiply by 2/2 in the first

$$\lim_{x\to 0} \frac{x}{\sin(2x)} + \frac{\sin(x)}{\sin(2x)\cos(x)} = \lim_{x\to 0} \frac{2x}{2\sin(2x)} + \frac{\sin(x)}{2\sin(x)\cos(x)\cos(x)} = \frac{1}{2} + \frac{1}{2} = 1$$

3 Derivatives!

Didn't do!

Differentiate the following using the definition of the derivative:

(a) $f(x) = 1/x$ for $x \neq 0$

(b) $f(x) = \sqrt{x+3}$

(c) $f(x) = 4 - \sqrt{x+3}$

(d) $f(x) = \sqrt[3]{x}$ (this should look familiar)

(e) $f(x) = x^{2/3}$