# Worksheet 3

Fall 2016

MATH 221, Week 3

**Name:** ________________________________

## 1 Absolute Value Refresher

Find all the solutions to the following equations:

$$|2x - 1| = 5$$

This is equivalent to finding $2x - 1 = 5$ and $2x - 1 = -5$. Solving both of these equalities we find $x = 3, -2$ are the solutions.

$$|x^2 - 4x - 5| < 7$$

We have to solve two inequalities here, $x^2 - 4x - 5 < 7$ and $x^2 - 4x - 5 > -7$. Solving the first is far easier as it is equivalent to $x^2 - 4x - 12 < 0$. This factors nicely and we have $(x + 2)(x - 6) < 0$. This is equivalent to $x$ being in the interval $-2 < x < 6$.

For the second inequality we need to solve for $x$ such that $x^2 - 4x + 2 > 0$. This does not factor nicely so we have to apply the quadratic formula to find the roots $x = 2 \pm \sqrt{2}$. We need to look at a numberline with these values and see where $x^2 - 4x + 2 > 0$ or less than zero. We find that this inequality ($x^2 - 4x + 2 > 0$) is equivalent to $x$ being in the interval $x < 2 - \sqrt{2}$ or $x > 2 + \sqrt{2}$. Hence our answer is the overlap of the intervals we found in the two parts, i.e. $x$ must be in the intervals:

$$(-2, 2 - \sqrt{2}) \cup (2 + \sqrt{2}, 6)$$

$$|2x - 1| = |4x + 9|$$

To solve this we need to solve $2x - 1 = 4x + 9$ and $2x - 1 = -(4x + 9)$. I claim this covers all possible cases because if $-(2x - 1) = 4x + 9$ is equivalent to solving $2x - 1 = -(4x + 9)$, and a similar statement applies for $-(2x + 1) = -(4x + 9)$. Solving these two equalities we find $x = -5$ and $x = -4/3$

(Hint: For the third problem in particular, think about what the absolute value sign does and how you could think about the function in different pieces. It may help to graph these two lines to get a geometric intuition.)

## 2 Rigorous Limits

Use the $\epsilon - \delta$ definition to prove the following limits. We will do every proof below assuming that we were given $\epsilon = 1$. I.e. we want to find $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < 1$ for the appropriate $x \to c$ and $L$ the limit (notice you could replace 1 throughout the argument with $\epsilon$):

(a) $\lim_{x\to 3} x = 3$

We want to find $\delta > 0$ so that $0 < |x-3| < \delta \implies |x-3| < \epsilon$. There are no simplifications needed, take $\delta = \epsilon = 1$.

(b) $\lim_{x\to 1} 2x + 4 = 6$

We want to find $\delta > 0$ so that $0 < |x-1| < \delta \implies |2x-4| < \epsilon = 1$. If we factor out a 2 form this inequality we find that

$$|2x-4| < 1 \iff 2|x-2| < 1$$

However $|x-2| < \delta$. Hence $2\delta < 1 \implies 2|x-2| < 1$ since $2|x-2| < 2\delta < 1$. So let $\delta = 1/2$.

(c) $\lim_{x\to 2} x^2 = 4$

We want to find $\delta > 0$ so that $0 < |x-2| < \delta \implies |x^2-4| < \epsilon = 1$. This is precisely what we did in class the other day, so we find $\delta = \sqrt{5}-2$ works. You can also find this in the book

(d) $\lim_{x\to 2} x^2 - 4 = 0$

The goal here was to notice that this is the exact same problem as the last one. Hence our answer is the same $\delta = \sqrt{5}-2$. Think about why this makes sense!

(e) $\lim_{x\to 1} \frac{2-x}{4-x} = \frac{1}{3}$

This problem is meant as a challenge, it's much harder than anything you'd see on an exam. We want to find $\delta > 0$ so that $0 < |x-1| < \delta \implies |\frac{2-x}{4-x} - 1/3| < \epsilon = 1$.

We can start manipulating the absolute values, we know

$$|\frac{2-x}{4-x} - 1/3| < 1 \iff -1 < \frac{2-x}{4-x} - 1/3 < 1 \iff -2/3 < \frac{2-x}{4-x} < 4/3$$

Now we want to clear the denominator, but we have to be careful to make sure that it is not negative and is nonzero. Hence we can specify at this point that $\delta < 1$. If we do this, it forces $|x-1| < 1$ so $4-x > 0$. The idea is that we can make $x$ close enough to 1 for $4-x$ to not be an issue. Now if we do this we can multiply through by $4-x$ and the inequalities will not change

$$-2/3 < \frac{2-x}{4-x} < 4/3 \iff -2/3(4-x) < 2-x < 4/3(4-x)$$

It's important to notice that this is true only so long as $\delta < 1$. Now we examine each separately

$$2-x < 16/3 - 4/3x \iff x/3 < 16/3 - 2 \iff x < 10 \iff x-1 < 9 \impliedby |x-1| < 9$$

The other inequality

$$-8/3 + (2/3)x < 2-x \iff -14/3 < -5/3x \iff 14/5 > x \iff 9/5 > x-1 \impliedby |x-1| < 9/5$$

So we want to take $\delta = \min\{9/5, 9, 1\} = 1$ because we can't forget the restriction we placed on $\delta$ in the beginning. If this is the case then $|x-1| < 9/5$ and $|x-1| < 9$. This implies the desired result by following the trail of implications backwards.

(f) $\lim_{x\to 3}\sqrt{x+6} = 9$

We want to find $\delta > 0$ so that $0 < |x-3| < \delta \implies |\sqrt{x+6}-9| < \epsilon = 1$. Notice that this is how you should start all of these problems, to make sure you know exactly what you're looking for!

Now we do the usual

$$|\sqrt{x+6}-9| < \epsilon = 1 \iff 8 < \sqrt{x+6} < 10$$

Everything here is positive, so we can square everything and the equality will stay the same because the function $x^2$ is increasing so if $a < b$ are positive then $a^2 < b^2$. Notice we could have issues if $a$ were negative, for example we know $-5 < 1$, but $(-5)^2 = 25$ is definitely not less than 1. So you have to be careful here. We also notice that we always need $x+6$ to be positive for our function to be well defined so we specify now that $0 < \delta < 1$ (see the book for more details) this guarantees that $|x-3| < \delta < 1$ so that $x > 2$ i.e. it is positive and so $x+6$ is positive as well.

$$8 < \sqrt{x+6} < 10 \impliedby 64 < x+6 < 100$$

If we subtract 9 from everything

$$64 < x+6 < 100 \iff 55 < x-3 < 91$$

Considering both of these inequalities separately we notice that

$$x-3 < 91 \impliedby |x-3| < 91$$

And also that

$$3-x < -55 \impliedby |x-3| < -55$$

This last inequality is impossible, so we are left with $|x-3| < 91$ as the only necessary. We can't forget our choice from the beginning though that forced $x+6$ to be positive, hence we take $\delta = \min\{91, 1\} = 1$. So we can take $\delta = 1$.

# 3 Different Limits

Compute the following limits (please show work):

(a) $\lim_{x\to 1^-}\frac{x^2+2x-3}{x^2-1}$

If we factor the top and bottom we find $x^2+2x-3 = (x+3)(x-1)$ and $x^2-1 = (x+1)(x-1)$. So we can cancel the $x-1$ and we find

$$\lim_{x\to 1^-}\frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1^-}\frac{x+3}{x+1} = 2$$

(b) $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1}$

Notice the right and left hand limits both exist here so this limit is equal to the left hand limit computed above 2.

(c) $\lim_{x\to\infty} \frac{1}{x} = 0$

(d) $\lim_{x\to\infty} \frac{x^2+3}{x^2+4} = 1$

To see this consider using our rules after you divide through by $x^2$ in the numerator and denominator.

(e) $\lim_{x\to-\infty} \frac{x^3+2x^2-2}{x-2x^2}$

This limit does not exist, in particular it goes to $-\infty$ as $x \to -\infty$

(f) $\lim_{x\to\infty} \frac{16x^{29}}{2534x^{12}\times 33245x^7+14x^3}$

This limit also does not exist, it goes to infinity. This is because the numerator grows far faster than the denominator as the degree of the polynomial is larger.

# 4 To Think About

Use the limit you computed in 3(c) to show that $\lim_{x\to\infty} x$ does not exist. (Hint: Suppose $\lim_{x\to\infty} x = L$ does exist. Then apply the limit properties to $\lim_{x\to\infty} x \cdot \frac{1}{x}$).

Following the hint if $\lim_{x\to\infty} = L$ then our limit rules tell us that $\lim_{x\to\infty} x \cdot \frac{1}{x} = \lim_{x\to\infty} x \cdot \lim_{x\to\infty} \frac{1}{x} = L \cdot 0 = 0$. However $x \cdot 1/x = 1$. So this implies that $1 = 0$ which is impossible.

True or false (if true, provide an example; if false, provide a reason why)

(a) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x) + g(x)$ could still exist.

(b) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x)g(x)$ could still exist.

(c) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x)/g(x)$ could still exist.

(d) If $\lim_{x\to a} f(x)$ does not exist and $\lim_{x\to a} g(x)$ does exist, then $\lim_{x\to a} f(x)/g(x)$ could still exist.

(a) is false because using our limit rules if $\lim_{x\to a} f(x) + g(x)$ exists then $\lim_{x\to a}(f(x) + g(x)) - \lim_{x\to a} f(x) = \lim_{x\to a}(f(x) + g(x)) - f(x) = \lim_{x\to a} g(x)$ would exist which is impossible.

(b) is true, consider $a = 0$, $f(x) = x$ and $g(x) = 1/x$. Then $f(x)g(x) = 1$ so this limit exists, but $\lim_{x\to 0} \frac{1}{x}$ does not exist. Think about why this example works.

(c) is true, take the same example as in (b). The quotient is then $x^2$ which has a limit. Once again think about why this does not contradict our limit rules. What is special about $f$ and $g$?

(d) is false because if $\lim_{x\to a} f(x)/g(x)$ exists and $\lim_{x\to a} g(x)$ exists by the product rule for limits $\lim_{x\to a} f(x)/g(x)\cdot$

$\lim_{x\to a} g(x) = \lim_{x\to a} \frac{f(x)g(x)}{g(x)} = \lim_{x\to a} f(x)$ would exist. However we know this is impossible. Think about what is different in this example compared to (c).