

Worksheet 10

Spring 2016

MATH 222, Week 10: Taylor Series and Little-oh Notation

Name: _____

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Problem 1. Solve the following initial value problem exactly, then compute its degree two Taylor polynomial around zero and use this to compute an estimate for $y(.3)$. Then use Euler's method with step size $\Delta x = .1$ to estimate $y(.3)$.

$$\frac{dy}{dx} = -2xy$$

$y(0) = 1$. Just as a sanity check, the true value of $y(.3)$ is about .914.

Solution 1.

See Worksheet 9 Solutions. □

Problem 2. Hasdrubal has designed a rocket. While proving mathematically that it won't explode, he used the approximation $e^{1/3} \approx 1 + \frac{1}{3} + \frac{1}{3^2 2!} + \frac{1}{3^3 (3!)}$. If this approximation is off by more than $\frac{2}{4!} \left(\frac{1}{3}\right)^4$, the rocket might blow up. Convince Hasdrubal that it won't.

Solution 2.

He's using the third order Taylor polynomial about $x = 0$ to approximate $e^{1/3}$. So this reduces to showing that $R_3(e^x) < \frac{2}{4!} \left(\frac{1}{3}\right)^4$ when $x = 1/3$. We use Lagrange's Remainder Theorem, when $x = 1/3$ we have

$$|R_3(e^{1/3})| = \frac{f^4(\xi)}{4!} |x|^4 = \frac{e^{\xi}}{4!} \left(\frac{1}{3}\right)^4 \leq \frac{e^{1/3}}{4!} \left(\frac{1}{3}\right)^4$$

We then notice that $e^{1/3} < 2$, we could use the Taylor series for e^x when $x = 1/3$ compared to the Taylor series for $\frac{1}{1-x}$ when $x = 1/2$ to prove this. This gives us the result we desire. □

Problem 3. Find a bound for $R_n^0 \sin(3x)$ and use this to show that $T_n^0 \sin(3x) \rightarrow \sin(3x)$ for all x as $n \rightarrow \infty$.

Solution 3.

This is extremely similar to the quiz problem, so for details I'd suggest checking that out. We use Lagrange's Remainder theorem and the fact that every time we differentiate we pick up a multiple of 3 to find

$$|R_n \sin(3x)| \leq \frac{|x|^{n+1} 3^{n+1}}{(n+1)!}$$

As we've discussed in class, for large values of n , factorials are SO much larger than exponentials. Hence as $n \rightarrow \infty$ $R_n \sin(3x) \rightarrow 0$. Thus $T_n \sin(3x) \rightarrow \sin(3x)$. □

Problem 4. Find a bound on $R_n^0 e^{2x}$ and use this to show that for every x , $T_n^0 e^{2x} \rightarrow e^{2x}$ as $n \rightarrow \infty$.

Solution 4.

Here we use Lagrange's Theorem and the fact that e^{2x} is an increasing function so to maximize it we just plug in the largest possible value of x . Lagrange's Theorem tells us

$$R_n e^{2x} = \frac{2^{n+1} e^{2\xi}}{(n+1)!} x^{n+1}$$

Where ξ is between 0 and x . If we want to bound this as mentioned above we know we can plug in the largest possible value of ξ . If x is negative we can take ξ as 0 and if x is positive we can let $\xi = x$. In either case we get a constant, denote it by C , then $|R_n e^{2x}| \leq \frac{2^{n+1} C |x|^{n+1}}{(n+1)!}$. As noted in the last problem exponentials grow slower than factorials so this goes to 0 as $n \rightarrow \infty$ and hence our Taylor polynomials converge to e^{2x} . \square

Problem 5. First, define what it means if we write $h(x) = o(x^n)$. Then finish the following rules concerning little-oh notation:

We say $h(x) = o(x^n)$ if $\lim_{x \rightarrow 0} \frac{h(x)}{x^n} = 0$. Intuitively this means that $h(x)$ goes to 0 faster than x^n .

(a) $x^n \cdot o(x^m) = o(x^{n+m})$

(b) $o(x^n) \cdot o(x^m) = o(x^{n+m})$

(c) When does $x^m = o(x^n)$? Answer: If $n < m$

(d) When does $o(x^n) + o(x^m) = o(x^n)$ Answer: If $n < m$

(e) Given any constant C , $o(Cx^n) = o(x^n)$

Problem 6. Is it true that $\sin(x) - x + \frac{x^3}{3} = o(x^4)$?

Solution 5.

Yes. $\sin(x) = x - \frac{x^3}{3} + o(x^4)$ using a Taylor series expansion. Hence $\sin(x) - x + \frac{x^3}{3} = o(x^4)$. \square

Problem 7. Show that $e^x - \sqrt{1+2x} = o(x)$.

Solution 6.

Take the Taylor series expansions for both series and subtract them from each other. $e^x = 1 + x + o(x)$ and $\sqrt{1+2x} = 1 + x + o(x)$. Hence their difference is $o(x) - o(x) = o(x)$. Remember you can't cancel two $o(x)$ terms because they may be vastly different—as they are in this case. All you know for sure is that they are still $o(x)$. \square