Worksheet 4

Spring 2016

MATH 222, Week 4: 2.1,2.2,2.3,2.5

Name: ________________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Problem 1. Compute $\int \frac{x}{\sqrt{x^2-1}}\, dx$

Solution 1.

You could solve this using trig sub or rational substitution, but u-sub turns out to be the best option. If we let $u = x^2 - 1$. Then $du = 2xdx$. Substituting we have:

$$\int \frac{x}{\sqrt{x^2-1}}\, dx = 1/2 \int \frac{1}{u^{1/2}} du = u^{1/2} = \sqrt{x^2+1} + C$$

□

Problem 2. Let a be some constant. Compute the following integral in two ways $\int \frac{1}{\sqrt{x^2-a^2}}\, dx$.

Solution 2.

The two options are trig sub or rational sub. We'll do both and compare answers. Option (a) is trig sub, letting $x = a\sec(\theta)$ or option (b) for rational sub with $x = aU(t)$.

(a) If we make the trig sub we have:

$$\int \frac{a\sec(\theta)\tan(\theta)}{a\tan(\theta)}\, d\theta = \int \sec(\theta)\, d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$$

Now we need to revert back to x using triangles! We know that $\sec(\theta) = x/a$ so if we draw our triangle we find that $\tan(\theta) = \frac{\sqrt{x^2-a^2}}{a}$. Substituting this back in we have:

$$\int \frac{x}{\sqrt{x^2-1}}\, dx = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

(b) Using rational substitution we have:

$$\int \frac{aU'(t)}{aV(t)}\, dt = \int \frac{1-1/t^2}{t-1/t} dt = \int 1/t \frac{(t-1/t)}{(t-1/t)}\, dt = \ln|t| + C$$

We now need to solve for t in terms of x. We know that $t = U + V$, $U = x/a$ and $V = \sqrt{U^2-1} = \sqrt{(x/a)^2-1}$. So our final solution is:

$$\int \frac{x}{\sqrt{x^2-1}}\, dx = \ln\left|\frac{x}{a} + \sqrt{(\frac{x}{a})^2 - 1}\right| + C$$

Notice that these are the same answer. What must we do to see they are the same?

□

Problem 3. Compute $\int \frac{(z+3)^2}{(40-6z-z^2)^{3/2}}\ dz$

Solution 3.

We must first complete the square:

$$40 - 6z - z^2 = -((z+3)^2 - 7^2))$$

Substituting this back in we have:

$$\int \frac{(z+3)^2}{(40-6z-z^2)^{3/2}}\ dz = \int \frac{(z+3)^2}{(49-(z+3)^2)^{3/2}}\ dz$$

Now if we let $z+3 = 7\sin(\theta)$. Then $dz = 7\cos(\theta)d\theta$. Substituting we have:

$$\int \frac{(z+3)^2}{(49-(z+3)^2)^{3/2}}\ dz = \int \frac{\sin^2(\theta)}{\cos^2(\theta)} d\theta = \int \tan^2(\theta)\ d\theta$$

We have to be slightly crafty here. We don't necessarily know how to take the antiderivative of $\tan^2(\theta)$ but we do for $\sec^2(\theta)$ and we know $\tan^2(\theta) = \sec^2(\theta) - 1$. So we have:

$$\int \tan^2(\theta)\ d\theta = \int \sec^2(\theta) - 1 d\theta = (\tan(\theta) + \theta) + C$$

Using our handy triangle again we find that $\tan(\theta) = \frac{z+3}{\sqrt{49-(z+3)^2}}$ and that $\theta = \arcsin(\frac{z+3}{7})$. Substituting this back in we have:

$$\int \frac{(z+3)^2}{(40-6z-z^2)^{3/2}}\ dz = \frac{z+3}{\sqrt{49-(z+3)^2}} + \arcsin\left(\frac{z+3}{7}\right) + C$$

□

Problem 4. Compute $\int \frac{e^x}{\sqrt{e^{2x}-1}}\ dx$. You may find problem 2 helpful.

Solution 4.

Once we make a u-sub this becomes an exact application of problem 2. Let $u = e^x$ then $du = udx$. Substituting we have:

$$\int \frac{e^x}{\sqrt{e^{2x}-1}}\ dx = \int \frac{1}{\sqrt{u^2-1}}\ du$$

Now we apply the formula we discovered in problem 2 and substitute back to find:

$$\int \frac{e^x}{\sqrt{e^{2x}-1}}\ dx = \ln|\sqrt{e^{2x}-1} + e^x| + C$$

□

Problem 5. (a) Compute $\int_0^\infty \frac{x}{\sqrt{1+x^2}}\ dx$

(b) Compute $\int_{-\infty}^0 \frac{x}{\sqrt{1+x^2}}\ dx$

(c) What does this say about $\int_{-\infty}^\infty \frac{x}{\sqrt{1+x^2}}\ dx$

Solution 5.

(a) We can use u-sub here. If we let $u = 1 + x^2$ then $du = 2xdx$. Substituting this in we have

$$\int_0^\infty \frac{x}{\sqrt{1+x^2}}\, dx = 1/2 \int_0^\infty \frac{1}{u^{1/2}}$$

Now we set up our improper integral as a limit:

$$1/2 \int_0^\infty \frac{1}{u^{1/2}} = \lim_{a\to\infty} 1/2 \int_0^a \frac{1}{u^{1/2}}$$

Antidifferentiating,

$$\lim_a \to \infty 1/2 \int_0^a \frac{1}{u^{1/2}} = \lim_{a\to\infty} u^{1/2}|_0^a = \lim_{a\to\infty} \sqrt{1+x^2}|_0^a = \lim_{a\to\infty} \sqrt{1+a^2} - \sqrt{1} = \infty$$

(b) We'll skip right to the limit of the antiderivative here because we already did the work in part (a). Just remember that now we are taking the limit as $b \to -\infty$ where $b \neq a$:

$$\lim_{b\to-\infty} \sqrt{1+x^2}|_b^0 = \lim_{b\to\infty} \sqrt{1} - \sqrt{1+b^2} = -\infty$$

(c) We know that

$$\int_{-\infty}^\infty \frac{x}{\sqrt{1+x^2}} dx = \lim_{b\to-\infty} \int_b^0 \frac{x}{\sqrt{x^2+1}} dx + \lim_{a\to\infty} \int_0^a \frac{x}{\sqrt{x^2+1}} dx = -\infty + \infty$$

So the limit diverges.

□

Problem 6. (a) Show that $\int_1^\infty \frac{dx}{x^2-4}$ is not a finite number.
(b) What answer do you get if you forget to account for the asymptote at $x = 2$?

Solution 6.

(a) To see this we only need to show that one of the pieces of the integral is infinite. Take:

$$\int_1^2 \frac{dx}{x^2-4}$$

There is an asymptote at $x = 2$ and so to compute this improper integral we need to take a limit as $a \to 2$ from below:

$$\int_1^2 \frac{dx}{x^2-4} = \lim_{a\to 2^-} \int_1^a \frac{dx}{x^2-4}$$

Using partial fractions we find the antiderivative:

$$\lim_{a\to 2^-}\int_1^a \frac{dx}{x^2-4} = \lim_{a\to 2^-}\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|_1^a$$

As $a \to 2$ we see that $\frac{x-2}{x+2} \to 0$ and so the log goes to $-\infty$. Hence

$$\lim_{a\to 2^-}\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|_1^a = -\infty$$

So this integral cannot be finite.

(b) If we forget about the asymptote we end up with

$$\lim_{a\to \infty}\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|_1^a$$

In this case as $a \to \infty$ $\frac{a-2}{a+2} \to 1$ and so the log goes to 0. Hence

$$\lim_{a\to \infty}\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|_1^a = \lim_{a\to \infty}\frac{1}{4}\ln\left|\frac{a-2}{a+2}\right| - \frac{1}{4}\ln|1/3| = 0 + \frac{1}{4}\ln(3)$$

We get a finite number.

□