

Worksheet 5

Spring 2016

MATH 222, Week 5: 2.5,3.1,3.2,3.3

Name: _____

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Determine the convergence or divergence of the following integrals without computing them. Then compute them explicitly.

Problem 1. $\int_4^{\infty} \frac{1}{x-2} dx.$

Solution 1.

Notice that $x - 2 < x$. This implies that $\frac{1}{x-2} > \frac{1}{x}$. We don't have to worry about negatives because of the interval of integration. The comparison test tells us that this integral diverges because $\frac{1}{x}$ does.

To see this explicitly, we set up the limit and antidifferentiate:

$$\lim_{a \rightarrow \infty} \int_4^a \frac{1}{x-2} dx = \lim_{a \rightarrow \infty} \ln|x-2| \Big|_4^a = \infty$$

□

Problem 2. $\int_4^{\infty} \frac{1}{x^3 - x} dx.$

Solution 2.

We want to compare this to something. Intuitively we would guess this behaves like x^3 as x gets really big so this should converge. We'll use the limit comparison test with $\frac{1}{x^3}$

$$\lim_{x \rightarrow \infty} \frac{1/x^3}{1/(x^3 - x)} = \lim_{x \rightarrow \infty} \frac{x^3 - x}{x^3} = 1$$

As the limit is greater than zero and is constant we know that convergence of one integral is equivalent to convergence of the other. Thus our integral should converge.

To see this explicitly we'll use partial fractions to integrate.

$$\lim_{a \rightarrow \infty} \int_4^a \frac{1}{x(x-1)(x+1)} = \lim_{a \rightarrow \infty} -\ln|x| + \frac{1}{2} \ln|1-x^2| \Big|_4^a = \lim_{a \rightarrow \infty} \frac{1}{2} \ln|(1-x^2)/x^2| \Big|_4^a = \ln(4) + \frac{1}{2} \ln(15)$$

□

Problem 3. $\int_1^{\infty} \frac{dt}{1 + e^{2t}}.$

Solution 3.

We expect this to converge as e^{2x} becomes massive as $x \rightarrow \infty$. We know that $\frac{1}{e^{2x}}$ converges. We know $1 + e^{2x} > e^{2x}$ this implies that $\frac{1}{1 + e^{2x}} < \frac{1}{e^{2x}}$. So by the comparison test our integral will converge.

To see this explicitly we can make a u -sub. Let $u = e^x$, $du = e^x$. Then our integral becomes:

$$\lim_{a \rightarrow \infty} \int_*^* \frac{1}{u(1+u^2)} du = \lim_{a \rightarrow \infty} \ln|u| - (1/2) \ln|1+u^2| \Big|_*^* = \lim_{a \rightarrow \infty} (1/2) \ln \left(\frac{e^{2x}}{1+e^{2x}} \right) \Big|_1^a = -(1/2) \ln \left(\frac{e^2}{1+e^2} \right)$$

□

Problem 4. $\int_0^\infty \frac{\sin(x^2)}{x^2} dx$. Don't worry about computing this explicitly. Sketch the function in the integrand though and look at what happens at $x = 0$?

Solution 4.

We know that $-1 \leq \sin(x^2) \leq 1$, and so $\frac{\sin(x^2)}{x^2} \leq \frac{1}{x^2}$. By the comparison test our integral converges. To compute it explicitly:

$$\lim_{a \rightarrow 0} \lim_{b \rightarrow \infty} \int_a^b \frac{\sin(x^2)}{x^2} dx$$

You could use integration by parts to solve this integral. Letting $u = \sin(x^2)$ and $v' = 1/(x^2)$. After one iteration you have:

$$\sin(x^2)/x^2 + \int \frac{2x \cos(x^2)}{x} dx = -\sin(x^2)/x + \int 2 \cos(x^2)$$

So we have:

$$\lim_{a \rightarrow 0} \lim_{b \rightarrow \infty} -\sin(x^2)/x \Big|_a^b + \int_a^b 2 \cos(x^2) dx = 2 \int_0^\infty \cos(x^2) dx$$

To find this we recognize this as a Fresnel integral, and when integrated from 0 to infinity it equals $\sqrt{\pi/2}$. You're not expected to know this, but I'm including it just in case anyone is curious! So in all

$$\int_0^\infty \frac{\sin(x^2)}{x^2} dx = \sqrt{\pi/2}$$

□

Problem 5. Find a solution to the initial value problem:

$$\frac{dy}{dx} = e^y x^3$$

With initial value $y(0) = 0$.

Problem 6. Find a solution to the initial value problem:

$$\frac{dy}{dx} = y \sqrt{y^2 - 1} \cos(x)$$

With initial value $y(0) = 1$.

Problem 7. Find the general solution to the differential equation

$$\frac{dy}{dx} = x^2 + y^2 x^2$$