# Worksheet 12

Spring 2016

MATH 222, Week 12: Series! (and vectors maybe...)

**Name:** ____________________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

**Problem 1.** Let $a_n = \frac{1}{n^2-n}$ and $S_N = \sum_{n=2}^{N} a_n$.

(a) Use one of your convergence tests to conclude that this series converges.
(b) Now we'll find what it converges to. Use partial fractions to rewrite $a_n$
(c) Use part(a) to write out $S_2, S_3, S_4$ explicitly and notice how terms cancel. Generalize this to find a formula for $S_N$.
(d) Compute $\sum_{n=2}^{\infty} a_n$ i.e. $\lim_{N\to\infty} S_N$.

**Solution 1.**

(a) We can use the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Consider the limit

$$\lim_{n\to\infty} \frac{n^2}{n^2-n} = 1 > 0$$

Using the limit comparison test we know that because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges that $\lim_{n\to\infty} \frac{n^2}{n^2-n}$ converges.

(b) We see $a_n = \frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1}$.

(c) $S_2 = 1 - \frac{1}{2}$, $S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$ and

$$S_4 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

We see that this is a telescoping series and in general $S_N = 1 - \frac{1}{N}$.

(d) So $\sum_{n=2}^{\infty} a_n = \lim_{N\to\infty} 1 - \frac{1}{N} = 1$, so the series equals 1.

□

**Problem 2.** If $x > 2$, use the geometric series formula to find $\sum_{n=0}^{\infty} \frac{2^{n+1}}{x^n}$

**Solution 2.**

$\sum_{n=0}^{\infty} \frac{2^{n+1}}{x^n} = 2\sum_{n=0}^{\infty} \left(\frac{2}{x}\right)^n$. When $x > 2$ we have $|2/x| < 1$ and so the series converges and it is a geometric series so

$$2\sum_{n=0}^{\infty} \left(\frac{2}{x}\right)^n = \frac{2}{1-2/x}$$

□

**Problem 3.** Using convergence tests determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$
(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$
(c) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
(d) $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$
(e) $\sum_{n=1}^{\infty} \sin(n)$
(f) $\sum_{n=1}^{\infty} \frac{5^k}{3^k+4^k}$

**Solution 3.**

(a) We can use the integral test.

$$\int_1^{\infty} xe^{-x^2} dx = \lim_{a\to\infty} -e^{-x^2}/2\bigg|_1^a = 0 + \frac{1}{2e}$$

As this converges we conclude the series converges.

(b) If we use the alternating series test we conclude this series converges.

(c) If we use the ratio test

$$\lim_{n\to\infty} \left|\frac{2^{n+1}n!}{2^n(n+1)!}\right| = 0 < 1$$

By the ratio test this converges.

(d) We can use the direct comparison test, comparing to $a_n = \frac{1}{3^n}$ because

$$\frac{1}{2+3^n} < \frac{1}{3^n}$$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a geometric series and converges, so our series converges.

(e) By the divergence test this diverges because $\lim_{n\to\infty} \sin(n)$ does not exist and so this series cannot converge. We could also use the integral test.

(f) This diverges by the divergence test. To see a more explicit solution this was a quiz problem, check those solutions!

□

**Problem 4.** Let $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Compute

(a) $||\vec{a}|| = \sqrt{1+4+4} = 3$

(b) $2\vec{a} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$

(c) $||2\vec{a}||^2 = (4+16+16) = 36$

(d) $\vec{a} + \vec{b} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$