Quiz 8

Spring 2016

MATH 222-004

Name: ________________________________

For full credit please explain all of your answers. **No calculators** are allowed.

Problem 1. Find the second degree Taylor Polynomial $T_2\{\cos(2x)\}$ and bound the error $|\cos(2x) - T_2\{\cos(2x)\}| = |R_2\{\cos(2x)\}|$ for $|x| < 1$.

Solution 1.

We know $T_2 f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$. If you used the taylor series for $\cos(t)$ below and substituted $2x$ for t that is completely valid and you'll get the right answer. We'll go through and do it to see that we get the same answer. $f(0) = 1$. $f'(x) = -2\sin(2x)$, so $f'(0) = 0$ and $f''(x) = -4\cos(2x)$, so $f''(0) = -4$. Plugging this in we find

$$T_2 \cos(2x) = 1 - 2x^2$$

The faster way was to make a substitution. Now to bound the error we recall Lagrange's Remainder theorem which tell us

$$R_2\{\cos(2x)\} = \frac{\cos^{(3)}(\xi)}{3!}x^3$$

We know $|x| < 1$ from the problem and we know $|\cos(2\xi)| < 1$ for any value of ξ. We just have to be careful because every time we differentiate we pick up a multiple of 2 via the chain rule. You could compute this explicitly and see that $\cos^{(3)}(x) = 8\sin(2x)$ so we've picked up three multiples of two. Or you could just notice what we discussed in the last sentence. Either is completely acceptable! Using this information we find

$$|R_2\{\cos(2x)\}| \leq \frac{8}{3!}$$

$\square$

Problem 2. Now find the Taylor series $T_\infty \cos(2x)$ by substituting, adding, multiplying or applying long division and/or differentiating the known Taylor series: $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$. (Memorize/write this down for the exam).

Solution 2.

We can just substitute as discussed above so

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}$$

Make sure that you are raising both 2 and x to the $2n$. For a sanity check you can see the first few terms agree with what you computed above. We can heuristically justify this as well because the only difference between the derivatives of $\cos(x)$ and $\cos(2x)$ are that you pick up a multiple of 2 every time you differentiate $\cos(2x)$. So if you look at the $2n$ term of $\cos(2x)$ it should differ from $\cos(x)$ by a multiple of 2^{2n} which is exactly what happens.

$\square$