Name: ANSWER KEY

Give an answer. No calculators are allowed.

1. Find the following limit:

$$\lim_{x\to 0} \frac{\tan(2x^2)}{x}$$

   Solution: Notice that

$$\lim_{x\to 0} \frac{\tan(2x^2)}{x} = \lim_{x\to 0} \frac{\sin(2x^2)}{\cos(2x^2)x} = \lim_{x\to 0} \frac{2x\sin(2x^2)}{2x^2\cos(2x^2)} = \lim_{x\to 0} \frac{2x}{\cos(2x^2)} = 0$$

2. Differentiate:

   (a) $y = \sin^3(4x^2 + e^x)$

   Solution: Apply the chain rule:

$$\frac{dy}{dx} = 3\sin^2(4x^2 + e^x)\cdot\cos(4x^2 + e^x)\cdot(8x + e^x)$$

   (b) $y = \ln(x^3 + 5)$

   Solution: Recall the formula for the derivative of the natural log:

$$\frac{dy}{dx} = \frac{3x^2}{x^3 + 5}$$

3. Integrate:

   (a) $\int x\sin(x^2)dx$

   Solution: You could do this two ways. One of which is using u-substitution with $u = x^2$. Or you could notice that $\frac{d}{dx}(-\cos(x^2)) = 2x\sin(x^2)$ and then divide by 2 to get what you want:

$$\int x\sin(x^2)dx = \frac{-\cos(x^2)}{2} + C$$

   Don't forget the constant!

(b) $\displaystyle\int \frac{\sqrt{x}-1}{\sqrt{x}}dx$

Solution: It's much easier if you notice that:

$$\frac{\sqrt{x}-1}{\sqrt{x}} = 1 - \frac{1}{\sqrt{x}}$$

This clears up the integral:

$$\int 1 - \frac{1}{\sqrt{x}} = x - 2\sqrt{x} + C$$

(c) $\displaystyle\int_{-1}^{2} 2\sec^2(x)dx$

Solution: This is tricky. Remember that in order to apply the fundamental theorem of calculus you need the function you're integrating to be continuous on the entire interval of integration. Unfortunately, $\sec^2(x)$ is not continuous at $x = \pi/2$ as $\cos(\pi/2) = 0$. So you can't apply the fundamental theorem of calculus here. If you followed this path instinctually you probably got something like $2\tan(2) - 2\tan(-1)$. Be careful of this! The integral doesn't converge.

4. What is the integral representation of the volume you get when you rotate the region bounded by $y = 4 - x^2$ and the x-axis around the x-axis?

Solution: We want to slice, estimate and integrate. You usually want to draw a picture of this first. To find where the curve intersects the x-axis you set it equal to zero. Solving $4 - x^2 = 0$ yields $x = \pm 2$. These are then our bounds of integration.

It will be easiest to slice vertically, which means our integral will be with respect to x. For a given x value, each circle that we want to integrate has radius $4 - x^2$. So the integral we desire is:

$$\int_{-2}^{2} \pi(4 - x^2)^2 \, dx$$

We only want the integral representation, so we're done!

5. Find $x \geq 0$ so that the function $F(x)$ is maximized:

$$F(x) = \int_0^x -t^2 + 2t + 8$$

Solution: This actually looks far harder than it is. To maximize any well defined function we want to take the derivative and set it equal to zero. Applying the Fundamental Theorem of Calculus, we have:

$$\frac{dF}{dx} = -x^2 + 2x + 8 = (-x + 4)(x + 2)$$

We set this equal to zero and solve to find the possible critical values: $x = 4, -2$.

We were asked to find $x \geq 0$ such that this is maximized so $x = 4$ must be the answer. However, to make sure this is a maximum we can apply the second derivative test $F''(x) = -2x + 2$. We check $F''(4) = -6 < 0$ so F is concave down here and thus we have a maximum as desired. What's the other way to check that a critical value is a (local) maximum?

The longer way to do this is by first taking the antiderivative, then differentiating it. You would arrive at the same derivative, it would just take a few more steps.