MATH 222 (Lectures 1,2, and 4) Fall 2015
Practice Midterm 1.1 Solutions

Circle your TA's name from the following list.

Carolyn Abbott	Tejas Bhojraj	Zachary Carter	Mohamed Abou Dbai	Ed Dewey
Jale Dinler	Di Fang	Bingyang Hu	Canberk Irimagzi	Chris Janjigian
Tao Ju	Ahmet Kabakulak	Dima Kuzmenko	Ethan McCarthy	Tung Nguyen
Jaeun Park	Adrian Tovar Lopez	Polly Yu		

Please inform your TA if you find any errors in the solutions.

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7
Score							

Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- On Problems 1, 2, and 3, only the answer will be graded.
- On Problems 4, 5, 6, and 7 you must show your work and we will grade the work and your justification, and not just the final answer.
- Each problem worth either 14 or 15 points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1-x^2}$. Note that we have provided some formulas on the next page to help with this.

Formulas

You may freely quote any algebraic or trigonometric identity, as well as well as any of the following formulas or minor variants of those formulas.

- $\cos(\arcsin x) = \sqrt{1 - x^2}$
- $\sec(\arctan x) = \sqrt{1 + x^2}$.
- $\tan(\operatorname{arcsec} x) = \sqrt{x^2 - 1}$.
- $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ \ln|x| + C & \text{when } n = -1 \end{cases}$
- $\int e^x dx = e^x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$.
- $\int \csc x dx = -\ln|\csc x + \cot x| + C$.
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.

1. For each statement below, CIRCLE true or false.

(a)		(b)		(c)		(d)		(e)	
True	False	True	False	True	False	True	False	True	False

(a) $\int_3^\infty \frac{x-\sqrt{x}}{3x^3+11}dx$ is a finite number.

(b) $\int_3^\infty \frac{1}{2x^2}dx \geq \int_3^\infty \frac{1}{x^2+3x}dx$.

(c) $\int \cos^2(5\theta+1)\sin(5\theta+1)d\theta = -\frac{1}{15}\cos^3(5\theta+1)+C$.

(d) $\int_3^{10} \frac{1}{\sqrt{x-3}}dx$ is a finite number.

(e) Let $I_n = \int x^n e^x dx$ then a reduction formula for these integrals is given by:

$$I_n = x^n e^x + (n-1)I_{n-1}.$$

Solution:

(a) True.

(b) False.

(c) True.

(d) True.

(e) False, it should be $I_n = x^n e^x - nI_{n-1}$.

2. On this page, only the answer will be graded.

   (a) Compute $\int \frac{7}{(t-1)(2t+5)} dt$.

   Solution: We rewrite this in the form:

$$\int \frac{7}{(t-1)(2t+5)} dt = \int \frac{A}{t-1} + \frac{B}{2t+5} dt$$

   Solving using the method of equating coefficients yields $7 = A(2t+5) + B(t-1)$. Hence $0 = 2A + B$ and $7 = 5A - B$. So we have $7 = 7A$ and hence $A = 1$ and $B = -2$.

$$\begin{aligned} &= \int \frac{1}{t-1} - \frac{2}{2t+5} dt \\ &= \ln|t-1| - \ln|2t+5| + C \end{aligned}$$

   (b) Compute $\int x \cos(4x) dx$.

   Solution: Let $f = x$ and $g' = \cos(4x)$ so that $f' = 1$ and $g = \frac{1}{4}\sin(4x)$. Then

$$\begin{aligned} \int x \cos(4x) dx &= \int fg' \\ &= fg - \int f'g \\ &= \frac{x}{4}\sin(4x) - \frac{1}{4}\int \sin(4x) dx \\ &= \frac{x}{4}\sin(4x) + \frac{1}{16}\cos(4x) + C \end{aligned}$$

   (c) Compute $\int \frac{x}{1+(x+3)^2} dx$.

   Solution: Let $z = x + 3$ so that $dx = dz$ and $x = z - 3$. Then we have

$$\begin{aligned} \int \frac{x}{1+(x+3)^2} dx &= \int \frac{z-3}{1+z^2} dx \\ &= \int \frac{z}{1+z^2} - \frac{3}{1+z^2} dz \\ &= \frac{1}{2}\ln|1+z^2| - 3\arctan(z) + C \\ &= \frac{1}{2}\ln|1+(x+3)^2| - 3\arctan(x+3) + C \end{aligned}$$

3. On this page, only the answer will be graded.

   (a) Find a such that $\frac{1}{x^2+x-5} < \frac{1}{x^2+2} < \frac{1}{x^2+1-x}$ for all $x > a$.

   Solution: To satisfy the inequalities we need $x - 5 > 2 > 1 - x$. For the first we need $x > 7$ and for the second we need $x > -1$. So $x > 7$ will work.

   (b) Compute $\int \sin^2(3x+1)dx$.

   Solution:

$$\begin{aligned}\int \sin^2(3x+1)dx &= \frac{1}{2}\int (1-\cos(6x+2))dx \\ &= \frac{1}{2}\left[x - \frac{1}{6}\sin(6x+2)\right] + C \\ &= \frac{1}{2}x - \frac{1}{12}\sin(6x+2) + C.\end{aligned}$$

   (c) Compute $\int_0^\pi \sin^7 x dx$. (You may use $A_n = \frac{n-1}{n}A_{n-2}$ where $A_n = \int_0^\pi \sin^n x dx$.)

   Solution: We first compute A_1 by hand and obtain

$$A_1 = \int_0^\pi \sin^1 x dx = [-\cos(x)]_0^\pi = -\cos(\pi)+\cos(0) = 1-(-1) = 2.$$

   Then we have:

   - $A_3 = \frac{3-1}{3}A_1 = \frac{2}{3}\cdot 2 = \frac{2\cdot 2}{3}$.
   - $A_5 = \frac{5-1}{5}A_3 = \frac{4\cdot 2\cdot 2}{5\cdot 3}$.
   - $A_7 = \frac{7-1}{7}A_5 = \frac{6\cdot 4\cdot 2\cdot 2}{7\cdot 5\cdot 3}$.

   So $A_7 = \int_0^\pi \sin^7 x dx = \frac{6\cdot 4\cdot 2\cdot 2}{7\cdot 5\cdot 3} = \frac{32}{35}$.

4. Compute $\int \frac{1}{(1+x^2)^2} dx$.

Solution: Let $x = \tan\theta$. Then $dx = \sec^2\theta d\theta$ and we get:

$$\begin{aligned}
\int \frac{1}{(1+x^2)^2} dx &= \int \frac{1}{(1+\tan^2\theta)^2} \sec^2\theta d\theta \\
&= \int \frac{1}{(\sec^2\theta)^2} \sec^2\theta d\theta \\
&= \int \frac{\sec^2\theta}{\sec^4\theta} d\theta \\
&= \int \frac{1}{\sec^2\theta} d\theta \\
&= \int \cos^2\theta d\theta \\
&= \frac{1}{2} \int (1+\cos(2\theta)) d\theta \\
&= \frac{1}{2} \left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C
\end{aligned}$$

Now to get back to the original variable x. We first use $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$= \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C$$

Then we use $\theta = \arctan(x)$ and use a "triangle argument" to get $\sin(\arctan x) = \frac{x}{\sqrt{x^2+1}}$ and $\cos(\arctan x) = \frac{1}{\sqrt{x^2+1}}$ yielding:

$$\begin{aligned}
&= \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}} + C \\
&= \frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)} + C
\end{aligned}$$

5. For $n = 0, 1, \dots$ let $I_n = \int \sec^n x dx$. Use integration by parts to derive a reduction formula for I_n.

Solution: We use integration by parts with $f = \sec^{n-2} x$ and $g' = \sec^2 x$. Then $f' = (n-2)\sec^{n-3} x(\sec x \tan x)$ and $g = \tan x$. We get:

$$
\begin{aligned}
I_n &= \int \sec^n x dx \\
&= \int \sec^{n-2} \sec^2 x dx \\
&= \int fg' \\
&= fg - \int f'g \\
&= (\sec^{n-2} x)(\tan x) - \int (n-2)\sec^{n-3} x(\sec x \tan x)(\tan x)dx \\
&= (\sec^{n-2} x)(\tan x) - (n-2)\int \sec^{n-2} x \tan^2 x dx \\
&= (\sec^{n-2} x)(\tan x) - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx \\
&= (\sec^{n-2} x)(\tan x) - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2} dx \\
&= (\sec^{n-2} x)(\tan x) - (n-2)I_n + (n-2)I_{n-2}
\end{aligned}
$$

Moving the $-(n-2)I_n$ terms from left to right, we thus get:

$$
\begin{aligned}
I_n + (n-2)I_n &= \sec^{n-2} x \tan x + (n-2)I_{n-2} \\
(n-1)I_n &= \sec^{n-2} x \tan x + (n-2)I_{n-2} \\
I_n &= \frac{1}{n-1}\left[\sec^{n-2} x \tan x + (n-2)I_{n-2}\right]
\end{aligned}
$$

6. Compute $\displaystyle\int \frac{e^{-x}dx}{1+e^{2x}}$.

Solution: First let's set $z = e^x$ so that $dz = e^x dx$ and $dx = dz/z$ We get:

$$\int \frac{e^{-x}dx}{1+e^{2x}} = \int \frac{dz}{z^2(1+z^2)}$$

Now use partial fractions. Rewrite $\frac{1}{z^2(1+z^2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{Cz+D}{1+z^2}$. Clearing denominators we get $1 = Az(1+z^2) + B(1+z^2) + (Cz+D)z^2 = (A+C)z^3 + (B+D)z^2 + (A)z + (B)$. Thus $B = 1$ and $A = 0$ and $D = -1$ and $C = 0$.

$$\begin{aligned}
&= \int \frac{1}{z^2} - \frac{1}{1+z^2} dz \\
&= -z^{-1} - \arctan(z) + C \\
&= -e^{-x} - \arctan(e^x) + C
\end{aligned}$$

7. Compute $\displaystyle\int_1^\infty \frac{1}{x(x^2+1)}dx$. (You may freely use the formula $\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$.)

Solution: We rewrite the integral as

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

We solve using the method of equating coefficients to obtain $1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A$ yielding $C = 0, A = 1$ and $B = -1$. We thus have:

$$\begin{aligned}
\int_1^\infty \frac{1}{x(x^2+1)}dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x(x^2+1)}dx \\
&= \lim_{b\to\infty} \int_1^b \frac{1}{x} + \frac{-x}{x^2+1}dx \\
&= \lim_{b\to\infty} \left[\ln|x| - \frac{1}{2}\ln|x^2+1|\right]_1^b \\
&= \lim_{b\to\infty} \left[\ln\left|\frac{x}{\sqrt{x^2+1}}\right|\right]_1^b \\
&= \lim_{b\to\infty} \left[\ln\left|\frac{b}{\sqrt{b^2+1}}\right| - \ln\left|\frac{1}{\sqrt{1^2+1}}\right|\right] \\
&= \lim_{b\to\infty} \left[\ln\left|\frac{b}{\sqrt{b^2+1}}\right|\right] - \ln(\frac{1}{\sqrt{2}}) \\
&= \left[\ln\lim_{b\to\infty}\left|\frac{b}{\sqrt{b^2+1}}\right|\right] - \ln(\frac{1}{\sqrt{2}}) \\
&= \left[\ln\lim_{b\to\infty}\left|\frac{b\cdot\frac{1}{b}}{\sqrt{b^2+1}\cdot\frac{1}{b}}\right|\right] - \ln(\frac{1}{\sqrt{2}}) \\
&= \left[\ln\lim_{b\to\infty}\left|\frac{1}{\sqrt{1+\frac{1}{b^2}}}\right|\right] - \ln(\frac{1}{\sqrt{2}}) \\
&= \left[\ln\left|\frac{1}{\sqrt{1+0}}\right|\right] - \ln(\frac{1}{\sqrt{2}}) \\
&= \ln(1) - \ln(\frac{1}{\sqrt{2}}) = -\ln(\frac{1}{\sqrt{2}}) \text{ or } \frac{1}{2}\ln(2).
\end{aligned}$$