Worksheet 4 Solutions

2018

MATH 222, Week 4: Trig Sub with Bounds, Partial Fractions and

Name: ___________________________________

Trig Sub with Bounds

Problem 1. The point of the following questions is to get comfortable with changing bounds in a definite integral when making trig sub.

(a) Say we have a definite integral and we make a trig sub $x = 2\sin(\theta)$. We want to evaluate our integral from $x = 1$ to $x = 2$ what would the new bounds be in terms of θ after the trig sub?

(b) If $x = a\sin(\theta)$ and I want to evaluate my integral from $x = -a/2$ to $x = 0$, what would my new bounds be in terms of θ?

(c) Same question as above but what if we make the trig sub $x = 3\tan(\theta)$ and we want to evaluate our integral from $x = -3$ to $x = 0$?

(d) Same question as above but what if we make the trig sub $x = 2\sec(\theta)$ and we want to evaluate our integral from $x = 2\sqrt{2}$ to $x = 4$?

Solution 1.

(a) We'll walk through this for part (a), the rest are the same process. In this case we want to find θ so that $\sin(\theta) = 1/2$ and $\sin(\theta) = 1$. This is where knowing your unit circle is very helpful. If you think about it the angles you desire are exactly $\theta = \pi/6$ and $\theta = \pi/2$.

(b) $\theta = -\pi/6$ and $\theta = 0$.

(c) $\theta = -\pi/4$ to $\theta = 0$.

(d) $\theta = \pi/4$ to $\theta = \pi/3$.

$\square$

Problem 2. Evaluate the following integrals. For each, make a trig sub, change the bounds and evaluate, do not convert back to x. Be mindful of which bounds you choose for θ.

(a) $\int_2^4 \frac{x^2-4}{x}\,dx$

(b) $\int_1^2 \frac{4-x^2}{x^2}\,dx$

(c) $\int_2^4 \frac{1}{\sqrt{x^2+2}}\,dx$

(d) $\int_0^1 t\sqrt{1-t^2}\,dt$

(e) $\int_0^1 e^{4x}\sqrt{1-e^{2x}}\,dx$.

(f) For problems (d) and (e) you can consult your previous worksheet and check if your answer agrees with the answer you would get if you converted back to x and evaluated at the given bounds.

Solution 2.

(a) For this problem we actually don't need to make a trig sub. The point of this being on here was so that you might realize there's an easier way. If you divide the x into the numerator this becomes

$$\int_2^4 x - 4/x \; dx = \frac{x^2}{2} - 4\ln|x| \Big|_2^4 = 6 - 4\ln(2).$$

I would also encourage you to try this with trig sub. If you do, the correct sub is $x = 2\sec(\theta)$ and you should get the same answer.

(b) Once again you don't need trig sub here. If you just distribute the denominator into the numerator you have

$$\int_1^2 \frac{4}{x^2} - 1 \; dx = \frac{-4}{x} - x \Big|_1^2 = \ln(16) - \frac{3}{2}.$$

If you were to use trig sub, the correct sub here would be $x = 2\sin(\theta)$.

(c) In this problem the easiest route is trig sub. If you let $x = \sqrt{2}\tan(\theta)$ then $dx = \sqrt{2}\sec^2(\theta)$. Our bounds also change, but in this case there are not convenient values of θ, so we will compute the indefinite integral first, then convert back to x and plug in the bounds. For the mean time, to relay to the reader that we know the bounds change, we will place stars where the bounds are. The integral becomes

$$\int_*^* \frac{1}{\sqrt{2\tan^2(\theta) + 2}} \sqrt{2}\sec^2(\theta) \; d\theta.$$

Simplifying and using the trig identity $\tan^2(\theta) + 1 = \sec^2(\theta)$ we have

$$\int_*^* \frac{1}{\sqrt{2\tan^2(\theta) + 2}} \sqrt{2}\sec^2(\theta) \; d\theta.$$

After some further simplification we have

$$\int_*^* \sec(\theta) \; d\theta = \ln|\sec(\theta) + \tan(\theta)|.$$

We know $x = \sqrt{2}\tan(\theta)$, and $\sec(\theta) = \sqrt{1 + \tan^2(\theta)} = \sqrt{1 + \frac{x^2}{2}}$, so in terms of x this is

$$\ln\left|\sqrt{1 + \frac{x^2}{2}} + \frac{x}{\sqrt{2}}\right| \Bigg|_2^4.$$

If you evaluate at the bounds, you get your final answer.

(d) If you let $t = \sin(\theta)$ then $dt = \cos(\theta)$. Here you can easily change your bounds to be in terms of θ much like in the first problem. We can take $\theta = 0$ to $\theta = \pi/2$, and this becomes

$$\int_0^{\pi/2} \sin(\theta)\cos^2(\theta) \; d\theta.$$

Now if we let $u = \cos(\theta)$ and $du = -\sin(\theta)d\theta$ our bounds change once again to $u = 1$ to $u = 0$, this becomes

$$\int_1^0 -u^2 \; du = \int_0^1 u^2 \; du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}.$$

The benefit of changing the bounds as we go is that we never have to go through the process of converting back.

(e) This integral is impossible as stated because if you try to take $x > 0$ the integrand does not exist. You can compute the anti-derivative using a u-sub and some trig sub. But once you try to evaluate the bounds you run into an issue.

$\square$

Partial Fractions

Problem 3. Fill in the appropriate numerators and denominators in the following partial fraction decomposition. (The first is an example. We will assume the degree of $f(x)$ is smaller than the degree of the denominator.)

(a) $\dfrac{f(x)}{(x-1)(x-2)} = \left[\quad \dfrac{A}{x-1} \quad + \quad \dfrac{B}{x-2} \quad \right]$

(b) $\dfrac{f(x)}{(x-1)(x-2)(x-\pi)} = \left[\quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad \right]$

(c) $\dfrac{f(x)}{(x+1)^3} = \left[\quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad \right]$

(d) $\dfrac{f(x)}{(x-1)(x^2+1)} = \left[\quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad \right]$

Solution 3.

(a) $\dfrac{f(x)}{(x-1)(x-2)(x-\pi)} = f(x)\left[\quad \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-\pi} \quad \right]$

(b) $\dfrac{f(x)}{(x+1)^3} = f(x)\left[\quad \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} \quad \right]$

(c) $\dfrac{f(x)}{(x-1)(x^2+1)} = f(x)\left[\quad \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1} \quad \right]$

(d) $\dfrac{f(x)}{(x^2+1)^2} = f(x)\left[\quad \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2} \quad \right]$

$\square$

Problem 4. Evaluate the following integrals

(a) $\int \frac{1}{x^2-4}\, dx$

(b) $\int \frac{1}{2+e^{2t}}\, dt$

(c) $\int \frac{x^3}{x^2+2}\, dx$

(d) $\int \frac{x^2+2x}{x^3-8x^2-7x}\, dx$

(e) $\int \frac{dx}{(x+1)^3(x-2)^2}$

(f) $\int \frac{dx}{(x^2+1)(x-2)}$

(g) $\int \frac{dx}{(x^2-4)(x^2+1)^2}$

(h) (**Challenge**) $\int \frac{2x^3+3x+1}{(x^2-1)^2(x^2+1)^2(x-1)}\, dx$

Solution 4. I will give the solutions to these without many details except for one problem because all the others are the same using the decompositions from problem 3. For more detailed solutions feel free to ask your TA

(a) In this case we can decompose this as

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}.$$

We clear denominators and have $1 = A(x+2) + B(x-2)$, this implies $A = 1/4$ and $B = -1/4$. Plugging this back in we have

$$\int \frac{1/4}{x-2} + \frac{1/4}{x+2}\,dx = 1/4\ln|x-2| + 1/4\ln|x+2| + C.$$

(b) The answer here is $\frac{t}{2} - \frac{1}{4}\ln|2+e^{2t}| + C$.

(c) $\frac{x^2}{2} - \ln|x^2+2|$.

(d) $\frac{1}{46}\left(23+6\sqrt{23}\right)\log\left(-x+\sqrt{23}+4\right) + \frac{1}{46}\left(23-6\sqrt{23}\right)\log\left(x+\sqrt{23}-4\right)$

(e) $\frac{1}{54}\left(\frac{3\left(2x^2+x-4\right)}{(x-2)^2(x+1)^2} + \frac{6\left(2x^2+x-4\right)}{(x-2)(x+1)^3} - \frac{3(4x+1)}{(x-2)(x+1)^2} - \frac{2}{x-2} + \frac{2}{x+1}\right)$

(f) $-\frac{1}{10}\log\left((x-2)^2+4(x-2)+5\right) + \frac{1}{5}\log(x-2) - \frac{2}{5}\tan^{-1}(x)$

(g) $\frac{1}{100}\left(-\frac{10x}{x^2+1} + \log(2-x) - \log(x+2) - 14\tan^{-1}(x)\right)$.

(h) $\frac{1}{64}\left(\frac{4-4x}{x^2+1} - 6\log\left(x^2+1\right) + \frac{12}{x-1} + \frac{2}{x+1} - \frac{2}{(x-1)^2} + 19\log(x-1) - 7\log(x+1) - 16\tan^{-1}(x)\right)$.

□

Problem 5. (a) Compute $\int_2^4 \frac{1}{x^2}\,dx$

(b) Compute $\int_2^4 \frac{1}{x(x-h)}\,dx$ where h is any positive real number.

(c) What happens as $h \to 0$ in the integral for part (b)? How is this related to part (a)?

Solution 5.

(a) $\int_2^4 \frac{1}{x^2}\,dx = \frac{-1}{x}|_2^4 = \frac{-1}{4} + \frac{1}{2} = \frac{1}{4}$

(b) We first assume h is not in $[2,4]$ otherwise, this will diverge. Using partial fractions we can write:

$$\frac{1}{x(x-h)} = \frac{A}{x} + \frac{B}{x-h}$$

Finding a common denominator and equating the numerators we have $A(x-h) + Bx = 1$. If we let $x = 0$ we find that $A = -1/h$ and letting $x = h \neq 0$ we have $B = 1/h$. So now we substitute and integrate:

$$\int_2^4 \frac{-1/h}{x} + \frac{1/h}{x-h}\,dx = \frac{\ln(x-h)-\ln(x)}{h}|_2^4 = \frac{\ln(4-h)-\ln(2-h)-\ln(4)+\ln(2)}{h}$$

As $h \to 0$ we recognize this as the definition for the derivative of $\ln(x)$ with a slight modification, replacing h with $-h$ and so the integral goes to $-\frac{d}{dx}\ln(x) = -1/x$ as we saw in part (a).

□

Problem 6. Compute $\int \frac{1}{x^2-a^2}\,dx$ where a is any real number.

Solution 6. We can factor the denominator as $(x-a)(x+a)$. If we assume $a \neq 0$, we then have the partial fraction decomposition

$$\frac{1}{x^2-a^2} = \frac{1/(2a)}{x-a} - \frac{1/(2a)}{x+a}$$

If we integrate this, we have

$$\frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$$

Notice that if $a = 0$ this is not well defined, so we handle this case separately. However, here we are just taking the integral of $\frac{1}{x^2}$ which is $-\frac{1}{x} + C$. □

Strategy for Integration

Problem 7. For each of the following integrals, indicate which integration technique(s) should/could be used to evaluate the integral. There could be more than one answer, or multiple techniques combined. You do not need to evaluate the integrals.

(a) $\int \frac{\tan^3(x)}{\cos^3(x)}\,dx$

(b) $\int \frac{x^5+1}{x^3-3x^2-10x}\,dx$

(c) $\int x^8 \sin(x)\,dx$

(d) $\int \sec^3(x)\,dx$

(e) $\int \frac{x}{\sqrt{4-x^2}}\,dx$

(f) $\int \ln(x)\,dx$

(g) $\int \cos^2(x)\sin^3(x)\,dx$

(h) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

Solution 7.

(a) u-sub for $u = \sec(x)$.
(b) Partial fractions
(c) Integration by parts
(d) Integration by parts
(e) Trig sub
(f) Integration by parts
(g) u-sub.
(h) u-sub and then trig sub.

□

Improper Integrals, if we get there...

Problem 8. Evaluate the following integrals

(a) $\int_1^\infty \frac{1}{x^2}\, dx$

(b) $\int_1^\infty \frac{\ln(x)}{x^3}\, dx$

(c) $\int_{-\infty}^\infty \frac{x}{x^2+1}\, dx$

(d) $\int_3^{10} \frac{1}{(x-9)^{1/3}}\, dx$

(e) $\int_0^4 \frac{1}{x^2+x-6}\, dx$

(f) $\int_0^1 \frac{e^x}{\sqrt{1-e^{2x}}}\, dx$

For the last three problems, what would happen if you forgot about the asymptote?

Solution 8. I will set up two of these that illustrate how to do the others. I will give answers for all.

(a) In this case we have to take the limit as $t \to \infty$,

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = \lim_{t\to\infty} -\frac{1}{x}\bigg|_1^\infty = \lim_{t\to\infty} -\frac{1}{t} + 1 = 1.$$

(b) 1/4

(c) Diverges

(d) This is an improper integral of Type II because it is disccontinuous, so we have to take the limit as we approach the bad point $x = 9$,

$$\lim_{t\to 9-} \int_3^t \frac{1}{(x-9)^{1/3}}\, dx + \lim_{s\to 9+} \int_s^{10} \frac{1}{(x-9)^{1/3}}\, dx.$$

Now the antiderivative of $(x-9)^{-1/3}$ is $\frac{3}{2}(x-9)^{2/3}$. If you use this and evaluate your limits, you realize that this antiderivative is continuous as 0 and so we can just plug 9 in to find the final answer is

$$\frac{3}{2}(1 + 6^{2/3}).$$

(e) Divergent

(f) If we make the u-sub $u = e^x$ then $du = e^x dx$ and we have

$$\lim_{t\to 1} \int_t^e \frac{1}{\sqrt{1-u^2}}\, du = \lim_{t\to 1} \arcsin(u)\bigg|_t^e = \arcsin(e) - \arcsin(1) = \arcsin(e) - \pi/2.$$

If you forget about the asymptote in part (e) you will wrongly conclude the integral converges. In (d) and (f) you will not have an issue because the antiderivative happens to be continuous at the problem points, but you have to be careful about this because otherwise you will get the right answer for the wrong reasons. $\square$