Worksheet 13

Spring 2016

MATH 222, Week 13: Vectors!

Name: ________________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Problem 1. Let $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Compute

(a) $||\vec{a}||$

(b) $2\vec{a}$

(c) $||2\vec{a}||^2$

(d) $\vec{a} + \vec{b}$

Solution 1.

See the last worksheet solutions. □

Problem 2. Let $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Which of the following expressions are nonsense? Evaluate the sensible ones.

(a) $2\vec{a} + \vec{b}$

(b) $\vec{a} + \vec{c}$

(c) $\vec{a} \cdot \vec{c}$

(d) $\vec{a} - 2\vec{b}$

(e) $t\vec{a}$ where t is a real number

(f) $\vec{a}\vec{b}$

(g) $\vec{a} + 5$

Solution 2.

These should be fairly straightforward! They are also from the textbook. If you have any questions feel free to ask your TA. □

Problem 3. Let $\vec{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find s and t real numbers so that $\begin{pmatrix} 3 \\ 5 \end{pmatrix} = s\vec{a} + t\vec{b}$. Could we represent any vector $\vec{c}$ as a linear combination of $\vec{a}$ and $\vec{b}$? What if $\vec{a} = \begin{pmatrix} 2 \\ 16 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -4 \\ -32 \end{pmatrix}$?

Solution 3.

We solve the system of linear equations

$$\begin{aligned} 2s - t &= 3 \\ 4s + t &= 5 \end{aligned}$$

This yields $s = 8/6$ and $t = 5 - 32/6 = -\frac{1}{3}$. We can represent any vector as a linear combination of $\vec{a}, \vec{b}$ because they are linearly independent. The second pair of vectors are not linearly independent because they are multiples of each other, hence they do nor form a basis. □

Problem 4. Find a parametric equation for the line that passes through the points $P = (1, 0, 2)$ and $Q = (3, 1, 4)$.

Solution 4.

Consider the line described by $\vec{P} + t\vec{PQ} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. We see that at time 0 this yields the point P and at time 1 this gives the point Q and hence it describes the desired line. □

Problem 5. Let $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find $\vec{a}^{\parallel}$ and $\vec{a}^{\perp}$ so that $\vec{a} = \vec{a}^{\parallel} + \vec{a}^{\perp}$, where $\vec{a}^{\parallel}$ is parallel to $\vec{b}$ and $\vec{a}^{\perp}$ is perpendicular to $\vec{b}$

Solution 5.

We know $a^{\parallel}$ is the orthogonal projection of a onto b. We know

$$\mathrm{Proj_b a} = \frac{a \cdot b}{b \cdot b}\vec{b} = \frac{1-2}{2}\vec{b} = \begin{pmatrix} -1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

Then $a^{\perp}$ will be $a - a^{\parallel} = \begin{pmatrix} 3/2 \\ 2 \\ -3/2 \end{pmatrix}$ □

Problem 6. Suppose that a merchant sells three types of goods in quantities q_1, q_2, q_3 and that ther merchant sells these goods at prices p_1, p_2, p_3 dollars per unit respectively. Suppose further that it costs the merchant c_i dollars to make one unit of the i^{th} good. If

$$\vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \qquad \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \qquad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

What is the significance of the quantity $\vec{q} \cdot (\vec{p} - \vec{c})$? Describe in words why the merchant cares if this quantity is positive or negative?

Solution 6.

This is precisely his profit. As this is his profit it makes sense that he wants it to be positive! $\square$