WES Worksheet 4.2

Fall 2018

MATH 222, Week 4

Name: ______________________________

1 Improper Integrals, A gentle introduction

Problem 1. Evaluate the following integrals

(a) $\int_0^1 \ln(t)\, dt$ $= -1$

(b) $\int_{10}^{\infty} \frac{dx}{x^2-4}$ $= \frac{1/4}{(x-2)} + \frac{1/4}{(x+2)} = \frac{1}{4}\ln\left|\frac{3}{2}\right|$

(c) $\int_1^{\infty} \frac{1}{x^2}\, dx$ $= 1$

(d) $\int_1^{\infty} \frac{\ln(x)}{x^3}\, dx$ $= 1/4$

2 An important difference

There are two ways that we usually "split up" integrals. One is **over sums** and the other is **over bounds**. When we split an integral over sums we use the following

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

When we split an integral over bounds we use the following property of integrals

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

Where here we have that $a \le b \le c$. These are both completely valid ways to split up an integral, but there are important differences that arise when concluding whether an improper integral diverges of converges.

If you split an integral over bounds, to conclude the original integral diverges, it is enough to show that one of the pieces diverges. For example, try to show the following diverges using this line of argument:

Problem 2. $\displaystyle\int_{-1}^{2} \frac{1}{(x-1)(x+3)}\, dx$ diverges.

However, if you split an integral over sums it is possible that all of the pieces diverge, while the original integral converges. This is because you are still working on the same interval so you can have cancellation. When you split an integral over bounds, if you get multiple infinities, this means you did not simplify enough. To see this explicitly, do the following problem:

Problem 3. Evaluate the following integrals. What happens in the third integral? How is this different than what occurs in the fourth integral?

(a) $\displaystyle\int_0^2 \frac{x}{(x+1)(x-1)}\, dx$ diverges

(b) $\displaystyle\int_0^2 \frac{\sqrt{x}}{(x+1)(x-1)}\, dx$ diverges

(c) $\displaystyle\int_0^2 \frac{x-\sqrt{x}}{(x+1)(x-1)}\, dx$ $= \arctan(\sqrt{2}) + \frac{\ln(3)}{2} + \ln|1+\sqrt{2}|,$

(d) $\displaystyle\int_{-1}^{2} \frac{1}{(x-1)(x+3)}\, dx$ diverges

3 More Problems

Problem 4. Calculate the following integrals or determine they do not exist

(a) $\int_1^\infty \frac{2}{1-4x^2}\,dx$ $= -\ln|3|/2$

(b) $\int_0^\infty \frac{4\arctan(x)}{1+x^2}\,dx$ $= \frac{\pi^2}{2}$

(c) $\int_0^2 \frac{x+1}{\sqrt{4-x^2}}\,dx$ $= \frac{4+\pi}{2}$

(d) $\int_0^1 3x\ln(2x)\,dx.$ $= \frac{3}{4}(-1+\ln(4))$

Problem 5. Determine for which r the following integrals exist. Take care with the limits

(a) $\int_1^\infty \frac{1}{x^r}\,dx$ $r>1$

(b) $\int_0^1 \frac{1}{x^r}\,dx$. For which r is this integral even "improper"? Improper for $r > 0$. Converges $r<1$.

(c) $\int_0^\infty e^{-rx}\,dx.$ $= \frac{e^{-rx}}{rx}\Big|_0^\infty$ $r \neq 0$

Problem 6. For each of the following functions, find a simpler function that bounds it from about or below (I will specify) on the given interval.

(a) $\frac{1}{(x+1)^2}$ on the interval $[1,\infty)$. Bound it from above. $\leq \frac{1}{x^2}$

(b) $\frac{1}{\ln(x)}$ on the interval $[1,\infty)$. Bound it from below. $\geq \frac{1}{x}$

(c) $\frac{3+\sin(x)}{x^3}\,dx$ on the interval $[1,\infty)$. Bound it from above.

(b) $\frac{1}{\ln(x)} > \frac{1}{x} \Rightarrow x > \ln(x) \Rightarrow e^x > x$ true $x>1$

(c) $\leq \frac{4}{x^3}$.

$2 = (Au+B)(u+1)(u-1) + C(u^2+1)(u-1) + D(u^2+1)(u+1)$

$\Rightarrow 2 = u^3(A+C+D) + u^2(-A+A+B-C+D)$

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(5) $\int_0^2 \frac{x}{(x+1)(x-1)}$ $=$

$\lim_{t\to 1}\int_0^t \frac{x}{(x+1)(x-1)} + \lim_{s\to 1}\int_s^2 \frac{x}{(x+1)(x-1)}$ $= \int \frac{1/2}{(x+1)} + \frac{1/2}{(x-1)}$

$= \frac{A}{x+1} + \frac{B}{x-1}$

$= \lim_{t\to 1} \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln|x-1|\Big|_0^t$

Diverges.

$u^2 = x$
$2u = dx$

$= \int \frac{2}{(u^2+1)(u+1)(u-1)}\,du = \int$