Worksheet 7

Spring 2016

MATH 222, Week 7: 3.3, 3.5, 3.7, 3.8, 3.10

Name: ___________________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Problem 1. Find the general solution to the differential equation:

$$\frac{dy}{dx} = \frac{1}{e^y\sqrt{1-x^2}}$$

Solution 1.

This is separable. If we multiply both sides by e^y and integrate

$$\int e^y dy = \int \frac{1}{\sqrt{1-x^2}} dx$$

We have

$$e^y = \arcsin(x) + C$$

Taking the ln of both sides, the general solution is

$$y = \ln(\arcsin(x) + C)$$

Notice that at least when we multiply through by e^y we do not lose any obvious solutions. □

Problem 2. Find a solution to the initial value problem:

$$\frac{dy}{dx} = \sqrt{1-y^2}\sec^2(x)$$

With initial value $y(0) = 0$.

Solution 2.

This is separable again. If we divide both sides by $\sqrt{1-y^2}$ and integrate we have

$$\int \frac{1}{\sqrt{1-y^2}} dy = \int \sec^2(x) dx$$

So $\arcsin(y) = \tan(x) + C$. We solve for C using our initial value, $y(0) = 0$.

$$\arcsin(0) = \tan(0) + C$$

So $C = 0$ works. If we take the sin of both sides the solution to our initial value problem is

$$y = \sin(\tan(x))$$

Notice that when we divide through by $\sqrt{1-y^2}$ we make the assumption that $y \neq \pm 1$. These would both be valid solutions for the general solution, but not for our initial value problem. □

Problem 3. Find a solution to the initial value problem:

$$\frac{dy}{dx} = (1+y^2)e^x$$

With initial value $y(0) = 0$.

Solution 3.

This is separable so we divide by $1+y^2$ and integrate

$$\int \frac{1}{1+y^2} dy = \int e^x dx$$

And we have $\arctan(y) = e^x + C$. Solving for C

$$\arctan(0) = e^0 + C$$

So $C = -1$ works. Applying tan to both sides a solution to our initial value problem is

$$\tan(e^x - 1)$$

Notice that if we were solving for the general solution, when we divide through by $1+y^2$ we do not lose any obvious solutions because it is strictly positive in $\mathbb{R}$. □

Problem 4. Find the general solution to the differential equation (for $x \neq 0$):

$$x\frac{dy}{dx} = -y + x$$

Solution 4.

Unfortunately this is not separable, but we can put it in a recognizable form

$$\frac{dy}{dx} + \frac{1}{x}y = 1$$

Now we have to find our integrating factor which we know will be $e^{A(x)}$ where $A(x) = \int \frac{1}{x} dx = \ln(x)$. So our integrating factor is $e^{\ln(x)} = x$. What this tells us is that our equation was actually already in the form we desired:

$$\frac{d}{dx}(xy) = x$$

Integrating both sides we have

$$xy = \frac{x^2}{2} + C$$

Dividing through by x the general solution is

$$y = \frac{x^2 + C}{2x}$$

Notice that when we divide through by x here we use the assumption that $x \neq 0$. $\square$

Problem 5. Find the general solution to the differential equation

$$\frac{1}{2x}\frac{dy}{dx} = y + e^{x^2}$$

Solution 5.

Once again we can get this into a recognizable form

$$\frac{dy}{dx} - 2xy = 2xe^{x^2}$$

Now our integrating factor is $e^{A(x)}$ where $A(x) = \int -2xdx = -x^2$. Multiplying through we have

$$e^{-x^2}\frac{dy}{dx} - 2xe^{-2x}y = 2x$$

As usual this is of the form

$$\frac{d}{dx}(e^{-x^2}y) = 2x$$

Integrating we have

$$e^{-x^2}y = x^2 + C$$

Multiplying both sides by e^{x^2}

$$y = e^{x^2}x^2 + Ce^{x^2}$$

$\square$

Problem 6. Find a solution to the initial value problem

$$\cos(x)\frac{dy}{dx} = 1 - \sin(x)y$$

With initial value $y(0) = 1$.

Solution 6.

We can put this in a recognizable form

$$\frac{dy}{dx} + \tan(x)y = \sec(x)$$

Our integrating factor is then $e^{\int \tan(x)dx} = e^{-\ln(\cos(x))} = \sec(x)$. Multiplying through we have

$$\sec(x)\frac{dy}{dx} + \sec(x)\tan(x)y = \sec^2(x)$$

As usual this can be rewritten as

$$\frac{d}{dx}(\sec(x)y) = \sec^2(x)$$

Integrating both sides we have

$$\sec(x)y = \tan(x) + C$$

Solving for our initial value

$$\sec(0)(1) = \tan(0) + C$$

So $C = 1$ and the solution to our initial value problem is

$$y = \frac{\tan(x)}{\sec(x)} + \frac{1}{\sec(x)} = \sin(x) + \cos(x)$$

Notice the general solution is $y = \sin(x) + C\cos(x)$ $\square$