MATH 222(002) Spring 2018
Worksheet 1.2 September 7 Name: ______________________

Solve the following problems.

(1) Consider the integral $I_n = \int e^{nx}\cos(x)\,dx$. Integrate by parts and solve for I_n.

Solution: This is a "doubling-back" question. Let $f = \cos x$ and $g' = e^{nx}\,dx$, so $f' = -\sin x\,dx$ and $g = \frac{1}{n}e^{nx}$. Then

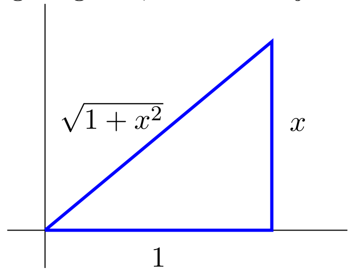
$$\begin{aligned}
I_n &= \int \underbrace{e^{nx}}_{g'}\underbrace{\cos x}_{f}\,dx \\
&= \underbrace{\cos x}_{f}\underbrace{\frac{1}{n}e^{nx}}_{g} - \int \underbrace{\frac{1}{n}e^{nx}}_{g}\underbrace{(-\sin x)}_{f'}\,dx \\
&= \frac{1}{n}e^{nx}\cos x + \frac{1}{n}\int e^{nx}\sin x\,dx.
\end{aligned}$$

Integrate by parts once more, with $f = \sin x$, $g' = e^{nx}\,dx$, so $f' = \cos x$ and $g = \frac{1}{n}e^{nx}$:

$$\begin{aligned}
I_n &= \frac{1}{n}e^{nx}\cos x + \frac{1}{n}\left[\underbrace{\sin x}_{f}\underbrace{\frac{1}{n}e^{nx}}_{g} - \int \underbrace{\frac{1}{n}e^{nx}}_{g}\underbrace{\cos x}_{f'}\,dx\right] \\
&= \frac{1}{n}e^{nx}\cos x + \frac{1}{n^2}e^{nx}\sin x - \frac{1}{n^2}\underbrace{\int e^{nx}\cos x\,dx}_{I_n} \\
I_n &= \frac{1}{n}e^{nx}\cos x + \frac{1}{n^2}e^{nx}\sin x - \frac{1}{n^2}I_n \\
I_n + \frac{1}{n^2}I_n &= \frac{1}{n}e^{nx}\cos x + \frac{1}{n^2}e^{nx}\sin x \\
I_n\left(1 + \frac{1}{n^2}\right) &= \frac{1}{n}e^{nx}\cos x + \frac{1}{n^2}e^{nx}\sin x \\
I_n &= \left(1 + \frac{1}{n^2}\right)^{-1}\left(\frac{1}{n}e^{nx}\cos x + \frac{1}{n^2}e^{nx}\sin x\right)
\end{aligned}$$

(2) (a) If $\theta = \arctan(x)$, what is $\cos(\theta)$? Hint: draw a right angle triangle.

Solution: If $\theta = \arctan(x)$, then $\tan\theta = \frac{x}{1}$. We can draw a right-angle triangle with the opposite side having length x, and the adjacent side having lenght 1:



Completing the picture: the hypotenuse has length $\sqrt{1+x^2}$ by Pythagoras' theorem. So $\cos\theta$ is given by "adjacent over hypotenuse", i.e. $\cos\theta = \dfrac{1}{\sqrt{1+x^2}}$.

(b) Show that the derivative of $y = \arctan(x)$ is $\frac{dy}{dx} = \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

Solution: Rewrite $\tan y = x$, and then differentiate:

$$\begin{aligned}
\frac{d}{dx}(x) &= \frac{d}{dx}(\tan y) \\
1 &= \sec^2 y \frac{dy}{dx} \qquad \text{(multiply both sides by } \cos^2 y\text{)} \\
\cos^2 y &= \frac{dy}{dx} \\
= \cos^2(\arctan x) &= \frac{dy}{dx}.
\end{aligned}$$

If $\theta = \arctan x$, by part (a), we see that $\cos^2(\arctan x) = \cos^2\theta = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$. Thus,

$$\frac{dy}{dx} = \cos^2(\arctan x) = \frac{1}{1+x^2}.$$

(c) Integrate $\displaystyle\int x \arctan(x)\,dx$.

Solution: Let $f = \arctan x$ and $g' = x\,dx$. So $f' = \frac{1}{1+x^2}\,dx$ and $g = \frac{x^2}{2}$. Hence,

$$\begin{aligned}
\int \underbrace{x}_{g'} \underbrace{\arctan x}_{f}\,dx &= \underbrace{\arctan x}_{f} \underbrace{\frac{x^2}{2}}_{g} - \int \underbrace{\frac{x^2}{2}}_{g} \underbrace{\frac{1}{1+x^2}}_{f'}\,dx \\
&= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\
&= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+(1-1)}{1+x^2}\,dx \\
&= \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx \\
&= \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C \\
&= \frac{x^2}{2}\arctan x + \frac{1}{2}\arctan x - \frac{x}{2} + C.
\end{aligned}$$

(3) For $m, n = 1, 2, 3, \ldots$, consider the integral $I_{m,n} = \displaystyle\int x^m (\ln x)^n\,dx$.

(a) Integrate $I_{m,n}$ by parts once to find a reduction formula in terms of $I_{m,(n-1)}$.

Solution: Set $f = (\ln x)^n$ and $g' = x^m\,dx$, so $f' = n(\ln x)^{n-1}\frac{1}{x}$ and $g = \frac{x^{m+1}}{m+1}$.

$$\begin{aligned}
I_{m,n} &= \int x^m (\ln x)^n\,dx \\
&= \underbrace{(\ln x)^n}_{f} \underbrace{\frac{x^{m+1}}{m+1}}_{g} - \int \underbrace{\frac{x^{m+1}}{m+1}}_{g} \underbrace{n(\ln x)^{n-1}\frac{1}{x}}_{f'}\,dx \\
&= \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\,dx \\
&= \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}I_{m,n-1}.
\end{aligned}$$

(b) Now consider the definite integral $A_{m,n} = \displaystyle\int_0^1 x^m (\ln x)^n\,dx$.
Write down the reduction formula for $A_{m,n}$ in terms of $A_{m,(n-1)}$. You may use as a fact that $\displaystyle\lim_{x\to 0^+} (\ln x)^n x^m = 0$.

Solution:

$$\begin{aligned}
A_{m,n} &= \frac{1}{m+1}x^{m+1}(\ln x)^n\Big|_{x=0}^{1} - \frac{n}{m+1}A_{m,n-1} \\
&= \frac{1}{m+1}\left(1\cdot(\underbrace{\ln 1}_{=0})^n - \lim_{x\to 0^+} x^{m+1}(\ln x)^n\right) - \frac{n}{m+1}A_{m,n-1} \\
&= -\frac{n}{m+1}A_{m,n-1}.
\end{aligned}$$

(c) Using part (b), write down a formula for $A_{m,n}$ in terms of $A_{m,0}$.

Solution:

$$\begin{aligned}
A_{m,n} &= -\frac{n}{m+1}A_{m,n-1} \\
&= \left(-\frac{n}{m+1}\right)\left(-\frac{n-1}{m+1}\right)A_{m,n-2} \\
&= \left(-\frac{n}{m+1}\right)\left(-\frac{n-1}{m+1}\right)\left(-\frac{n-2}{m+1}\right)A_{m,n-3} \\
&\qquad\vdots \\
&= \left(-\frac{n}{m+1}\right)\left(-\frac{n-1}{m+1}\right)\cdots\left(-\frac{n-(n-1)}{m+1}\right)A_{m,n-n} \\
&= \left(-\frac{n}{m+1}\right)\left(-\frac{n-1}{m+1}\right)\cdots\left(-\frac{1}{m+1}\right)A_{m,0} \\
&= \left(\frac{-1}{m+1}\right)^n n!\cdot A_{m,0},
\end{aligned}$$

where $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.

(d) Evaluate $A_{m,0} = \int_0^1 x^m\,dx$. What is $A_{m,n}$?

Solution:

$$A_{m,0} = \int_0^1 x^m\,dx = \frac{x^{m+1}}{m+1}\Big|_{x=0}^{1} = \frac{1}{m+1}.$$

Therefore,

$$A_{m,n} = \left(\frac{-1}{m+1}\right)^n n! A_{m,0} = \left(\frac{-1}{m+1}\right)^n n! \frac{1}{m+1} = \frac{(-1)^n n!}{(m+1)^{n+1}}.$$