MATH 222 (002 and 004) Fall 2013
Practice Final Solutions

Circle your TA's name from the following list.

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Please inform your TA if you find any errors in the solutions.

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6
Score						
	Problem 7	Problem 8	Problem 9	Problem 10	Problem 11	Problem 12
Score						

Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- You must show all of your work, except on Problem 1.
- All problems graded out of 10.
- No calculators, books, or notes (except for those notes on your 3×5 notecard.)

Note: Everything on this page will appear on the actual exam as well.

Formulas

- $\cos(\arcsin x) = \sqrt{1 - x^2}$
- $\sec(\arctan x) = \sqrt{1 + x^2}$.
- $\tan(\operatorname{arcsec} x) = \sqrt{x^2 - 1}$.
- $\csc(\arcsin x) = \frac{1}{x}$
- $\cot(\arcsin x) = \frac{\sqrt{1-x^2}}{x}$

Bound for remainder term

If f is a $n+1$ differentiable function on an interval containing $x = 0$ and if we have a constant M_n such that

$$\left|f^{(n+1)}(t)\right| \leq \text{ for all } t \text{ between } 0 \text{ and } x$$

then

$$|R_n f(x)| \leq \frac{M_n |x|^{n+1}}{(n+1)!}$$

1. For each statement below, CIRCLE true or false. You do not need to show your work.

(a)		(b)		(c)		(d)		(e)	
True	False	True	False	True	False	True	False	True	False

(a) $\int_3^\infty \frac{1}{e^{3x}} dx \geq \int_3^\infty \frac{1}{e^{x^2}} dx.$

(b) $(x^2 + x^3)^2 = o(x^3).$

(c) $\sum_{n=1}^\infty \frac{1}{n^4+5}$ is a finite number.

(d) Let $\vec{\mathbf{a}}$ and $\mathbf{b}$ be any space vectors and let t be any number. Then $\mathbf{a} \cdot (\mathbf{b} + t\mathbf{a}) = \mathbf{a} \cdot \mathbf{b} + ||t\mathbf{a}||^2$.

(e) $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 0 \end{pmatrix}.$

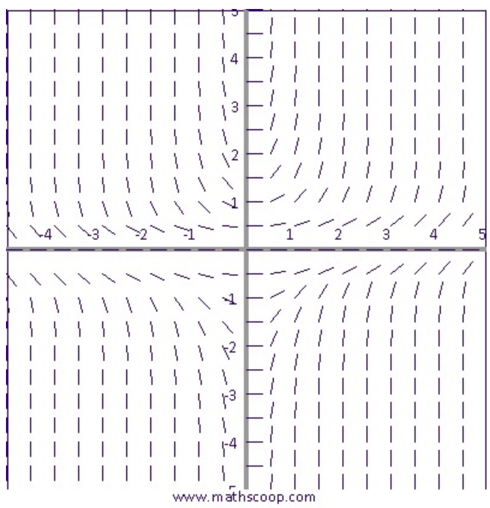
Solution:

(a) True.
(b) True.
(c) True.
(d) False.
(e) False.

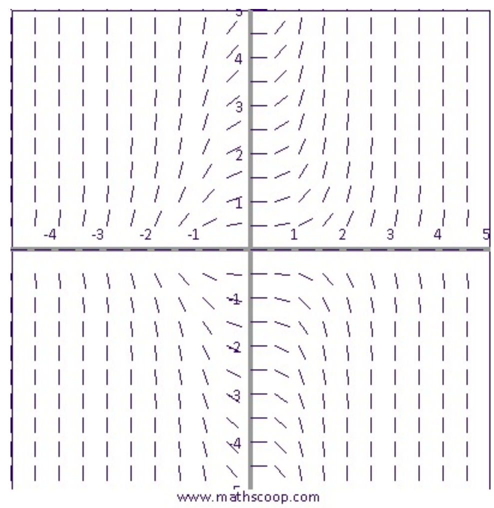
2. Below are four direction fields and two equations. Match the equation to the appropriate direction field. (5 points each).

(a) $\frac{dy}{dx} = \sin x + \cos y$. Answer: ________

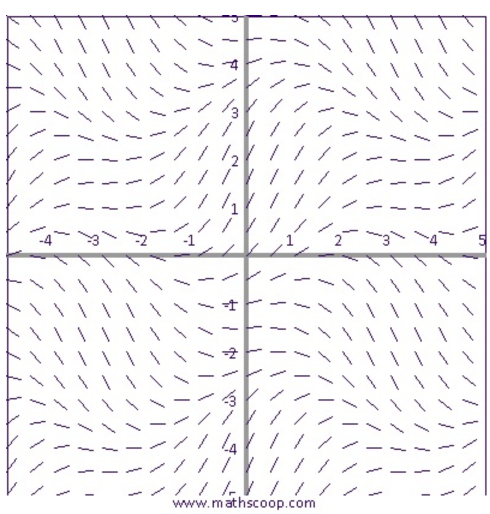
(b) $\frac{dy}{dx} = xy^2$. Answer: ________



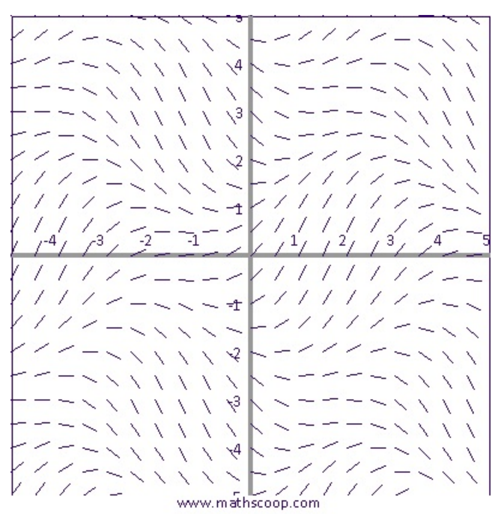
I.



II.



III.



IV.

Solution:

(a) IV

(b) I

3. Below you will find a number of mathematical expressions. Circle those which are *nonsense*. For instance, writing $\begin{pmatrix} 1 \\ 0 \end{pmatrix}^5$ is nonsense since we cannot raise a vector to the fifth power. Let

$$\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -2 \\ 11 \\ 6 \end{pmatrix}, \text{ and } \mathbf{d} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

(a) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

(b) $\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$

(c) $3\mathbf{a} + 5\mathbf{b} - \mathbf{d}$

(d) $\mathbf{a} \times \mathbf{a}$

(e) $\frac{(\mathbf{a} \cdot \mathbf{b})^2}{||\mathbf{a}||}\mathbf{d}$.

Solution:

(a) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ Well defined.

(b) $\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$ Nonsense.

(c) $3\mathbf{a} + 5\mathbf{b} - \mathbf{d}$ Nonsense.

(d) $\mathbf{a} \times \mathbf{a}$ Well defined.

(e) $\frac{(\mathbf{a} \cdot \mathbf{b})^2}{||\mathbf{a}||}\mathbf{d}$. Well defined.

4. Compute $\int \frac{5x-1}{(x+5)(x^2+1)}dx$.

Solution: We rewrite this as:

$$\frac{5x-1}{(x+5)(x^2+1)} = \frac{A}{x+5} + \frac{Bx+C}{x^2+1}$$

We use the method of equating coefficients to determine A, B and C. This gives

$$\begin{aligned} 5x - 1 &= A(x^2+1) + (Bx+C)(x+5) \\ &= Ax^2 + A + Bx^2 + Cx + 5Bx + 5C \\ &= (A+B)x^2 + (C+5B)x + A + 5C. \end{aligned}$$

So we get the system of equations:

$$\begin{cases} 0 & = A + B \\ 5 & = C + 5B \\ -1 & = A + 5C \end{cases}$$

The first equation yields $B = -A$ so it reduces to the system of equations

$$\begin{cases} 5 & = C - 5A \\ -1 & = A + 5C \end{cases}$$

Solving this yields $A = -1$ and $C = 0$ and hence $B = 1$. So we have

$$\begin{aligned} \int \frac{5x-1}{(x+5)(x^2+1)}dx &= \int \frac{-1}{x+5} + \frac{x}{x^2+1}dx \\ &= -\ln|x+5| + \frac{1}{2}\ln|x^2+1| + C \end{aligned}$$

Our final answer is $\boxed{-\ln|x+5| + \frac{1}{2}\ln|x^2+1| + C}$.

5. Compute $\int \cos^3(5\theta + 1)d\theta$.

Solution: We have

$$\int \cos^3(5\theta + 1)d\theta = \int (1 - \sin^2(5\theta + 1)) \cos(5\theta + 1)d\theta$$

$u = \sin(5\theta + 1)$ so $du = 5\cos(5\theta + 1)d\theta$ and

$$\begin{aligned} &= \int (1 - u^2)\frac{du}{5} \\ &= \frac{u}{5} - \frac{u^3}{15} + C \\ &= \frac{1}{5}\sin(5\theta + 1) - \frac{1}{15}\sin^3(5\theta + 1) + C \end{aligned}$$

The final answer is $\boxed{\frac{1}{5}\sin(5\theta + 1) - \frac{1}{15}\sin^3(5\theta + 1) + C}$.

6. Compute $\int_0^\infty x^2e^{-x}dx$. *(Note: The original copy of the practice exam had xe^{-x} instead of x^2e^{-x}. The problems are similar, but I like this one more.)*

Solution: We have

$$\int_0^\infty x^2e^{-x}dx = \lim_{b\to\infty}\int_0^b x^2e^{-x}dx$$

We use integration by parts with $f = x^2$ and $g' = e^{-x}$ so that $f' = 2x$ and $g = -e^{-x}$:

$$\begin{aligned}
&= \lim_{b\to\infty}[x^2(-e^{-x})]_0^b - \int_0^b 2x(-e^{-x})dx \\
&= \lim_{b\to\infty} -[x^2e^{-x}]_0^b + 2\int_0^b xe^{-x}dx
\end{aligned}$$

We then use integration by parts again, with $h = x$ and $k' = e^{-x}$ so that $h' = 1$ and $k = -e^{-x}$ yielding:

$$\begin{aligned}
&= \lim_{b\to\infty} -[x^2e^{-x}]_0^b + 2\left([x(-e^{-x})]_0^b - \int_0^b 1(-e^{-x})dx\right) \\
&= \lim_{b\to\infty} -[x^2e^{-x}]_0^b + 2\left([x(-e^{-x})]_0^b + \int_0^b e^{-x}dx\right) \\
&= \lim_{b\to\infty} -[x^2e^{-x}]_0^b - 2[x(e^{-x})]_0^b + 2[-e^{-x}]_0^b \\
&= \lim_{b\to\infty} -[b^2e^{-b} - 0] - 2[b(e^{-b}) - 0] + 2[-e^{-b} + 1]
\end{aligned}$$

Since $\frac{b^2}{e^b}$ and $\frac{b}{e^b}$ both go to 0 as $b \to \infty$, we then have:

$$= 0 - 0 - 2[0 - 0] + 2[0 + 1] = 2$$

So the final answer is $\boxed{\int_0^\infty x^2e^{-x}dx = 2.}$

7. The squirrel population in Madison has a continuous birth rate of 8% and a natural continuous death rate of 3%. In addition, each year 300 squirrels are eaten by foxes and 100 squirrels are run over by cars. There were $10,000$ squirrels in Madison on January 1, 2010. We are interested in explicitly finding a function S that models the **squirrel population in Madison at a given time**. Use the following space to work out your answer, and record the various parts of the problem at the bottom of the page.

- **Variables:**

- **Differential equation and initial condition for S:**

- **Solution for P satisfying initial conditions:**

Solution: Answers below. The differential equation is obtained by

$$\frac{dS}{dt} = .08S - .03S - 300 - 100 = .05S - 400.$$

The solution is as follows:

$$\begin{aligned}
\frac{dS}{dt} &= .05S - 400 \\
\int \frac{dS}{.05S - 400} &= \int dt && \text{or } .05S = 400 \\
20 \ln |.05S - 400| &= t + C && \text{or } .05S = 400 \\
\ln |.05S - 400| &= \frac{t}{20} + \frac{C}{20} && \text{or } .05S = 400 \\
|.05S - 400| &= e^{\frac{t}{20} + \frac{C}{20}} = e^{C/20} e^{t/20} && \text{or } .05S = 400 \\
.05S - 400 &= \pm e^{C/20} e^{t/20} && \text{or } .05S = 400
\end{aligned}$$

Changing constants, we can rewrite $\pm e^{C/20}$ as a new constant A, where the solution $.05S = 400$ gets absorbed by the case $A = 0$:

$$\begin{aligned}
.05S - 400 &= Ae^{t/20} \\
.05S &= Ae^{t/20} + 400 \\
S &= 20Ae^{t/20} + 8,000
\end{aligned}$$

Solving the initial value yields

$$10,000 = S(0) = 20Ae^{0/20} + 8,000 = 20A + 8,000$$

so $A = 100$.

- **Variables:** t is time in years since January 1, 2010. $S(t)$ is squirrel population in Madison at time t.
- **Differential equation and initial condition for S:**

$$\frac{dS}{dt} = .05S - 400 \text{ and } S(0) = 10,000.$$

- **Solution for P satisfying initial conditions:** $2,000e^{t/20} + 8,000$

8. Compute a solution to the initial value problem

$$\frac{dy}{dx} = \frac{x + xy^2}{2y} \quad \text{and} \quad y(0) = \sqrt{e^2 - 1}$$

Solution: This is a separable differential equation.

$$\begin{aligned} \frac{dy}{dx} &= x\frac{1+y^2}{2y} \\ \int \frac{2ydy}{1+y^2} &= \int xdx \end{aligned}$$

Since $1 + y^2$ cannot equal 0, we do not need to worry about division by 0.

$$\begin{aligned} \ln|1+y^2| &= \frac{x^2}{2} + C \\ |1+y^2| &= e^C e^{\frac{x^2}{2}} \\ 1+y^2 &= \pm e^C e^{\frac{x^2}{2}} \\ y^2 &= \pm e^C e^{\frac{x^2}{2}} - 1 \\ y &= \sqrt{\pm e^C e^{\frac{x^2}{2}} - 1} \end{aligned}$$

Now to solve for C we use the initial condition

$$y(0) = \sqrt{e^2 - 1} = \sqrt{\pm e^C e^{\frac{0^2}{2}} - 1} = \sqrt{\pm e^C - 1}$$

So we choose $C = 2$ and the positive branch (i.e. the + from the $\pm$) yielding. This yields our final answer $\boxed{y = \sqrt{e^2 \cdot e^{\frac{x^2}{2}} - 1}.}$

9. Find the Taylor polynomial of degree 14 at $x = 0$ (i.e. find T_{14}) of the function $f(x) = \frac{10x^4}{(1-x^5)^2}$. Remember to use notation correctly!

Solution: Since $\frac{10x^4}{(1-x^5)^2} = 2\frac{d}{dx}\frac{1}{1-x^5}$ we have:

$$\begin{aligned}
T_\infty \frac{10x^4}{(1-x^5)^2} &= T_\infty 2\frac{d}{dx}\frac{1}{1-x^5} \\
&= 2\frac{d}{dx} T_\infty \frac{1}{1-x^5} \\
&= 2\frac{d}{dx}\left(1 + x^5 + x^{10} + x^{15} + o(x^{15})\right) \\
&= 2\left(5x^4 + 10x^9 + 15x^{14} + o(x^{14})\right) \\
&= 10x^4 + 20x^9 + 30x^{14} + o(x^{14})
\end{aligned}$$

So $\boxed{T_{14}f(x) = 10x^4 + 20x^9 + 30x^{14}.}$

10. Does $\sum_{n=1}^{\infty} \frac{e^{-n}+n+\sqrt{n}}{e^{n}+n^{-3}+\sqrt[3]{n}}$ converge? You must justify your answer.

Solution: We first use the limit comparison test, comparing with $\sum_{n=1}^{\infty} \frac{n}{e^n}$. We check that the limit comparison test applies:

$$\begin{aligned}
\lim_{n\to\infty} \frac{e^{-n}+n+\sqrt{n}}{e^{n}+n^{-3}+\sqrt[3]{n}} \cdot \frac{e^n}{n} &= \lim_{n\to\infty} \left(\frac{\frac{1}{ne^n}}{\frac{1}{ne^n}}\right) \frac{e^{-n}+n+\sqrt{n}}{e^{n}+n^{-3}+\sqrt[3]{n}} \cdot \frac{e^n}{n} \\
&= \lim_{n\to\infty} \frac{\frac{e^{-n}}{n}+\frac{n}{n}+\frac{\sqrt{n}}{n}}{\frac{e^n}{e^n}+\frac{n^{-3}}{e^n}+\frac{\sqrt[3]{n}}{e^n}} \cdot \frac{\frac{e^n}{e^n}}{\frac{n}{n}} \\
&= \frac{0+1+0}{1+0+0} \cdot \frac{1}{1} = 1
\end{aligned}$$

Since this limit converges to a positive number, the limit comparison test applies. So the original series converges if and only if the new series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges. To check this, we apply the ratio test:

$$L = \lim_{n\to\infty} \left|\frac{(n+1)/e^{n+1}}{n/e^n}\right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{e} = 1 \cdot \frac{1}{e} < 1.$$

Since $L < 1$, the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges and hence

$\boxed{\text{the original series also converges.}}$

11. Imagine that you have a function $f(x)$ that satisfies

$$|f^{(n+1)}(x)| \leq (n+1)$$

for all n. Show that the Taylor series $T_\infty f(x)$ converges to $f(x)$ for any value of x. You should make use of the "bound for the remainder term" on the second page of this exam.

Solution: For some fixed x we want to show that $T_\infty f(x)$ converges to $f(x)$. It is equivalent to show that for our fixed x we have that $\lim_{n\to\infty} |R_n f(x)| = 0$. We use the Bound on the Remainder Term, with $M = n+1$, to obtain

$$\begin{aligned}
\lim_{n\to\infty} |R_n f(x)| &<= \lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{(n+1)!} \\
&= |x| \lim_{n\to\infty} \frac{|x|^n}{n!} \\
&= 0
\end{aligned}$$

where the last limit equals 0 because "factorial beats exponential".

12. Let $\mathcal{P}$ be the plane spanned by the points $(5,0,0)$, $(4,2,0)$ and $(0,10,6)$. Use the space below to compute the following:

(a) (4 points). The normal vector $\mathbf{n}$ to $\mathcal{P}$.

(b) (3 points). An equation (standard or parametric is fine) for $\mathcal{P}$.

(c) (3 points). Does the point $A = (2,6,11)$ line in $\mathcal{P}$?

Solution: To compute the normal vector, we first compute two vectors lying in $\mathcal{P}$:

$$\mathbf{a} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ 10 \\ 6 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \\ 6 \end{pmatrix}$$

We get a normal vector to our plane by taking the cross product of $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned} \mathbf{n} &= \mathbf{a} \times \mathbf{b} \\ &= \det \begin{pmatrix} \mathbf{i} & -1 & -5 \\ \mathbf{j} & 2 & 10 \\ \mathbf{k} & 0 & 6 \end{pmatrix} \\ &= -(-10\mathbf{k} + 0\mathbf{i} - 6\mathbf{j}) + (12\mathbf{i} - 10\mathbf{k} + 0\mathbf{j}) \\ &= 12\mathbf{i} + 6\mathbf{j} + 0\mathbf{k} = \begin{pmatrix} 12 \\ 6 \\ 0 \end{pmatrix} \end{aligned}$$

So $\boxed{n = \begin{pmatrix} 12 \\ 6 \\ 0 \end{pmatrix}.}$

Based on the normal vector, we know that a standard equation for $\mathcal{P}$ has the form $12x + 6y + 0z = c$ for some constant c. Plugging in the point $(5, 0, 0)$ yields $12(5) + 6(0) + 0(0) = c$ so $c = 60$ and our equation is $\boxed{12x + 6y = 60}$.

To check if $A = (2, 6, 11)$ lies on $\mathcal{P}$ we plug into our equation, yielding $12(2) + 6(6) \stackrel{?}{=} 60$. Since this is true, $\boxed{\text{we see that } A \text{ lies on } \mathcal{P}.}$