Worksheet 8

Spring 2016

MATH 222, Week 8: Applications of Differential Equations!

Name: ________________________________

You aren't necessarily expected to finish the entire worksheet in discussion. There are a lot of problems to supplement your homework and general problem bank for studying.

Problem 1. You just graduated and you cash in at your graduation party! You made \$3500. You, being incredibly responsible, decide to place all of your money in a savings account where you earn interest on it compounded continuously at a rate of 5%. How much money do you have after 10 years? How long will it take for your investment to double in value? What interest rate would you need for your investment to double in 5 years?

Solution 1.

Let $y(t)$ be the money in your bank account after t years. Then the word problem tells us

$$\frac{dy}{dt} = .05y$$

and $y(0) = 3500$. Solving we find

$$y(t) = 3500e^{.05t}$$

So after 10 years $y(10) = 5770.52$. Our money will double at the time t when

$$3500e^{.05t} = 7000$$

So as we know the doubling time is independent of the amount of money, it is always $\ln(2)/r$ where r is your interest rate. Hence $\ln(2)/.05 = 13.86$ years. Similarly we want the doubling time to be one year, so we need $\ln(2)/r = 1$, that is $r = \ln(2) = .693$. $\square$

Problem 2. A 100 litre tank is filled with water infested with dangerous bacteria. Clean water is pumped in and infected water is pumped out at a rate of 10 litres per minute, but the bacteria population reproduces at a rate of two percent per minute. Assume that the bacteria are always perfectly uniformly mixed in the water. If the tank begins with a bacteria concentration of one percent at what time will the bacteria population be half of its present value?

Solution 2.

Let $y(t)$ be the amount of bacteria in the water measured in litres. From the above we know the bacteria population is increasing by $.02y(t)$ litres of bacteria per minute. We also pump out bacteria infested water so we lose bacteria there. The amount of bacteria we lose will be exactly the number of gallons of water pumped out, times the concentration of bacteria in that water. As we are assuming uniform distribution the concentration is exactly

$$y(t)/V(t)$$

Where $V(t)$ is the volume of water in the tank. Since we are pumping water in as fast as we are draining it out, $V(t) = 100$ is constant. We will see below that this is not always the case. We also have to consider whether we gain any bacteria from the incoming water. As with the outgoing water we would gain exactly the number of gallons of water pumped in times the concentration of bacteria in that water. The water is clean, though, so the concentration is zero. Hence this does not add any bacteria. In all we have shown,

$$\frac{dy}{dt} = .02y - \frac{y}{100}(10) = -.08y$$

And the initial number of litres of bacteria is exactly 1 as we begin with a bacteria concentration of one percent of 100 litres. Thus $y(0) = 1$. This is in a very familiar form, just like the first problem. We know the solution to this differential equation is

$$y = e^{-.08t}$$

So we not want to find when the bacteria population will have half its current value, i.e. for what t does

$$\frac{1}{2}(1) = e^{-.08t}$$

Solving this equation we find $t = \ln(2)/.08 = 8.6646$ minutes. This should also look familiar. When we have decay instead of increasing interest the more reasonable question to ask is about half-life rather than doubling time! $\square$

Problem 3. A tank begins with 100 litres of salt water in it. Fresh water is pumped in at a rate of twenty litres per minute and the mixed water is pumped out at a rate of ten litres per minute. If the tank initially has ten kilograms of salt in it, find an equation for the amount of salt left in the tank in kilograms as a function of time. Note that the volume of the water in the tank is changing.

Solution 3.

This is similar to the last problem except now the volume in the tank is changing. If we let $y(t)$ be the amount of salt in kilograms in the tank after t minutes then we know $\frac{dy}{dt}$ will be negative (concentration of salt in water)(water leaving the To find the concentration we need to divide the amount of salt $y(t)$ by the volume of water in the tank $V(t)$. Every minute we gain 20 litres of water and lose 10 and we started with 100 litres so $V(t) = 100 + 10t$. You could also solve for this using a differential equation $\frac{dV}{dt} = 10$ with $V(0) = 100$. As water flows out at a rate of 10 litres per minute we have

$$\frac{dy}{dt} = -\frac{y}{100 + 10t}(10) = -\frac{y}{10 + t}$$

This is a separable equation, dividing through by y and integrating we find

$$\ln(y) = -\ln|10 + t| + C_1$$

Solving for y

$$y = \frac{C_2}{10 + t}$$

We know $y(0) = 10$ so

$$10 = \frac{C_2}{10}$$

Hence $C_2 = 100$ and the solution to our initial value problem is

$$y = \frac{100}{10 + t}$$

For a sanity check we note that if $t = 0$ then $y = 10$ as we would want and the amount of salt decreases as $t \to \infty$

as we would also expect. $\square$

Problem 4. A 100 litre vat of water begins with an algae concentration of 1,000 organisms per litre. Suppose that the algae naturally reproduce at a rate of five percent per minute and die at a rate of four percent per minute. If the vat is being drained at a rate of one litre per minute, what will the algae concentration be ten minutes from now? You should assume that the algae are uniformly distributed in the vat. Remember to define your variables with units.

Solution 4.

We break this down piece by piece. If we let $y(t)$ be the population of algae in litres. We know the algae naturally reproduce at 5% so this leads to a positive $.05y$ in our differential equation. They die at a rate of 4% to account for this we pick up a $-.04y$. The vat is drained at a rate of one litre per minute. How much algae do we lose from this? Well we lose $y(t)/V(t)$. This time $V(t) = 100 - t$. To find the initial amount of bacteria we know the concentration is 1000 organisms per litre and we have a 100 litre vat so we get 100000 organisms. $y(0) = 100000$. In all our differential equation is

$$\frac{dy}{dt} = 0.05y - 0.04y - \frac{y}{100-t} = 0.01y - \frac{y}{100-t} = y(0.01 - \frac{1}{100-t})$$

With $y(0) = 100,000$. Separating and integrating we find

$$\ln(y) = 0.01t + \ln|100-t| + C_1$$

Solving for y

$$y = C_2 e^{0.01t}(100-t)$$

Letting $y(0) = 100,000$ we have

$$100,000 = C_2(100)$$

So $C_2 = 1000$. In all solution to our initial value problem is

$$y = 1000e^{0.01t}(100-t)$$

Now we can find $y(10) = 1000e^{\cdot 1}$. $\square$