MATH 222 (Lectures 1,2, and 4) Fall 2015

Practice Midterm 1.2 Solutions

Circle your TA's name from the following list.

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Please inform your TA if you find any errors in the solutions.

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7
Score							

Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- On Problems 1, 2, and 3, only the answer will be graded.
- On Problems 4, 5, 6, and 7 you must show your work and we will grade the work and your justification, and not just the final answer.
- Each problem worth either 14 or 15 points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1-x^2}$. Note that we have provided some formulas on the next page to help with this.

Formulas

You may freely quote any algebraic or trigonometric identity, as well as well as any of the following formulas or minor variants of those formulas.

- $\cos(\arcsin x) = \sqrt{1-x^2}$
- $\sec(\arctan x) = \sqrt{1+x^2}$.
- $\tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$.
- $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ \ln|x| + C & \text{when } n = -1 \end{cases}$
- $\int e^x dx = e^x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$.
- $\int \csc x dx = -\ln|\csc x + \cot x| + C$.
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.

1. For each statement below, CIRCLE true or false.

(a)		(b)		(c)		(d)		(e)	
True	False	True	False	True	False	True	False	True	False

(a) $\int \frac{1}{(x-1)^2} dx = \ln |(x-1)^2| + C$.

(b) $\int_1^\infty \frac{t^2}{t^3+e^t} dt$ is a finite number.

(c) $\int_0^2 \sqrt{4-x^2} dx > 4$.

(d) $\int_0^\infty \frac{x+1}{x^{2/3}+x^5} dx$ is a finite number.

(e) If $I_n = \int \sec^n x dx$ then $I_2 = \tan(x) + C$.

Solution:

(a) False.
(b) True.
(c) False.
(d) True.
(e) True.

2. On this page, only the answer will be graded.

(a) Compute $\int \frac{dx}{3+7x^2}$.

Solution: We want $3+7x^2 = 3+3z^2$ so we substitute $x = \sqrt{\frac{3}{7}}z$. This yields $dx = \sqrt{\frac{3}{7}}dz$ This yields:

$$\begin{aligned}
\int \frac{dx}{\sqrt{3+7x^2}}dx &= \int \frac{\sqrt{\frac{3}{7}}dz}{3+3z^2} \\
&= \frac{\sqrt{3}}{3\sqrt{7}} \int \frac{dz}{1+z^2} \\
&= \frac{\sqrt{3}}{3\sqrt{7}} \arctan(z) + C \\
&= \frac{\sqrt{3}}{3\sqrt{7}} \arctan(\frac{\sqrt{7}}{\sqrt{3}}x) + C
\end{aligned}$$

(b) Compute $\int x \ln(4x)dx$.

Solution: Let $f = \ln(4x)$ and $g' = x$ so that $f' = \frac{1}{x}$ and $g = \frac{x^2}{2}$. Then

$$\begin{aligned}
\int x\ln(4x)dx &= \int fg' \\
&= fg - \int f'g \\
&= \ln(4x)\cdot\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx \\
&= \ln(4x)\cdot\frac{x^2}{2} - \frac{1}{2}\int xdx \\
&= \frac{1}{2}x^2\ln(4x) - \frac{x^2}{4} + C
\end{aligned}$$

(c) Compute $\int \frac{5}{(2x-1)(x+2)}dx$.

Solution: We rewrite this in the form:

$$\frac{5}{(2x-1)(x+2)}dx = \frac{A}{2x-1} + \frac{B}{(x+2)}.$$

Clearing denominators we get $5 = A(x+2) + B(2x-1) = (A+2B)x + (2A-B)$. We thus have $0 = A+2B$ and $5 = 2A-B$ which yields $A = 2$ and $B = -1$.

$$\begin{aligned}\int \frac{5}{(2x-1)(x+2)}dx &= \int \frac{2}{2x-1} - \frac{1}{x+2}dx \\ &= \ln|2x-1| - \ln|x+2| + C\end{aligned}$$

3. On this page, only the answer will be graded.

(a) Compute $\int(1+\sin(3\theta))^2 d\theta$.

Solution:

$$\begin{aligned}\int(1+\sin(3\theta))^2 d\theta &= \int 1+2\sin(3\theta)+\sin^2(3\theta)d\theta \\ &= \int 1+2\sin(3\theta)+\frac{1}{2}\left(1-\cos(6\theta)\right)d\theta \\ &= \int \frac{3}{2}+2\sin(3\theta)-\frac{1}{2}\cos(6\theta)d\theta \\ &= \frac{3\theta}{2}-\frac{2\cos(3\theta)}{3}-\frac{1}{12}\sin(6\theta)+C\end{aligned}$$

(b) Consider the improper integral $\int_a^\infty \frac{1}{x(x+1)(3x-11)(2x-57)}dx$. Find some $a>0$ such that this improper integral equals a finite number.

Solution: We need a to be sufficiently large so that it cuts out all of the vertical asymptotes. These vertical asymptotes lie at $x=0, x=-1, x=11/3, x=57/2$. So any $a>57/2$ will work.

(c) Compute $\int x^{2013}\ln(x)dx$.

Solution: Use integration by parts with $f=\ln(x)$ and $g'=x^{2013}$. Then $f'=\frac{1}{x}$ and $g=\frac{x^{2014}}{2014}$ and we get:

$$\begin{aligned}\int x^{2013}\ln(x)dx &= \int fg' \\ &= fg-\int f'g \\ &= \frac{\ln(x)x^{2014}}{2014}-\int\frac{1}{x}\cdot\frac{x^{2014}}{2014}dx \\ &= \frac{\ln(x)x^{2014}}{2014}-\frac{1}{2014}\int x^{2013}dx \\ &= \frac{\ln(x)x^{2014}}{2014}-\frac{x^{2014}}{2014^2}+C\end{aligned}$$

4. Compute $\int \frac{x^3+x^2}{x^2+2}dx$.

Solution: Notice that the degree of the numerator is greater than or equal to the degree of the denominator, so we need to do some kind of polynomial division to get this into a form we can work with. We compute

$$\frac{x^3+x^2}{x^2+2} = x+1-\frac{2x+2}{x^2+2}$$

We then have:

$$\begin{aligned}\int \frac{x^3+x^2}{x^2+2}dx &= \int x+1-\frac{2x+2}{x^2+2}dx \\ &= \frac{x^2}{2}+x-\int \frac{2}{x^2+2}dx - \int \frac{2x}{x^2+2}dx\end{aligned}$$

Now we solve the other integrals separately.

$$\begin{aligned}\int \frac{2}{x^2+2}dx & \int \frac{2}{2z^2+2}(\sqrt{2}dz) \qquad\qquad \sqrt{2}z = x \text{ and } \sqrt{2}dz = dx \\ &= \sqrt{2}\int \frac{1}{z^2+1}dz \\ &= \sqrt{2}\arctan(z)+C \\ &= \sqrt{2}\arctan(\frac{1}{\sqrt{2}}x)+C\end{aligned}$$

For the other integral we get:

$$\begin{aligned}\int \frac{2x}{x^2+2}dx &= \int \frac{1}{u}du \qquad\qquad u = x^2+2 \\ &= \ln|u|+C \\ &= \ln(x^2+2)+C\end{aligned}$$

Putting this all together, our final answer is:

$$\frac{x^2}{2}+x-\sqrt{2}\arctan(\frac{1}{\sqrt{2}}x)-\ln(x^2+2)+C$$

5. Compute $\int \frac{1}{t\ln^4(t)\sqrt{\ln^2(t)-1}}dt$.

Solution:

$$
\begin{aligned}
\int \frac{1}{t\ln^4(t)\sqrt{\ln^2(t)-1}}dt &= \int \frac{1}{u^4\sqrt{u^2-1}}du && u=\ln(t) \quad du=\frac{1}{t}dt \\
&= \int \frac{\sec(\theta)\tan(\theta)}{\sec^4(\theta)\sqrt{\sec^2(\theta)-1}}d\theta && u=\sec(\theta) \quad du=\sec(\theta)\tan(\theta)d\theta \\
&= \int d\theta \\
&= \int \frac{d\theta}{\sec^3\theta} \\
&= \int \cos^3\theta d\theta \\
&= \int \cos\theta(1-\sin^2\theta)d\theta && w=\sin\theta \quad dw=\cos\theta d\theta \\
&= \int (1-w^2)dw \\
&= w-\frac{w^3}{3}+C \\
&= \sin\theta-\frac{\sin^3\theta}{3}+C
\end{aligned}
$$

Since $u=\sec\theta$ we can use a triangle picture to get $\sin\theta=\frac{\sqrt{u^2-1}}{u}$

$$
\begin{aligned}
&= \frac{\sqrt{u^2-1}}{u}-\frac{1}{3}\left(\frac{\sqrt{u^2-1}}{u}\right)^3+C \\
&= \frac{\sqrt{\ln(t)^2-1}}{\ln(t)}-\frac{1}{3}\left(\frac{\sqrt{\ln(t)^2-1}}{\ln(t)}\right)^3+C
\end{aligned}
$$

6. Find *positive* numbers A and B so that $A \leq \int_1^\infty \frac{1+\sin^2(x)}{x^3+x}dx \leq B$. Justify your answer.

Solution: Solutions:

Lower bound: We can make the fraction smaller by making the numerator smaller and the denominator bigger. We have $1 \leq 1+\sin^2(x)$ for all x and $x^3+x \geq 2x^3$ for all $x \geq 1$. Thus

$$\int_1^\infty \frac{1}{2x^3} \leq \int_1^\infty \frac{1+\sin^2(x)}{x^3+x}dx$$

We compute that

$$\begin{aligned} A = \int_1^\infty \frac{1}{2x^3} &= \frac{1}{2}\lim_{b\to\infty}\int_1^b \frac{1}{x^3}dx \\ &= \frac{1}{2}\lim_{b\to\infty}[-\frac{1}{2}x^{-2}]_1^b \\ &= \frac{1}{2}\lim_{b\to\infty}[-\frac{1}{2}b^{-2}+\frac{1}{2}] \\ &= \frac{1}{4}. \end{aligned}$$

Upper bound: We can make the fraction bigger by making the numerator bigger and the denominator smaller. We have $1+\sin^2(x) \leq 2$ for all x and $x^3+x \geq x^3$ for all $x \geq 0$. Thus

$$B = \int_1^\infty \frac{1+\sin^2(x)}{x^3+x}dx \leq \int_1^\infty \frac{2}{x^3}dx.$$

By the above computation, we see that $\int_1^\infty \frac{2}{x^3}dx = 4\cdot\int_1^\infty \frac{1}{2x^3} = 4\cdot\frac{1}{4} = 1$. We thus have $A = \frac{1}{4}$ and $B = 1$.

7. Let $I_n = \int \sin^n x dx$. Use the reduction formula $I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$ to compute

$$\int_0^{2\pi} \sin^8 x dx.$$

Solution: First we compute a version of the reduction formula for the definite integrals $A_n = \int_0^{2\pi} \sin^n x dx$. We obtain this by adding bounds to the given reduction formula, so that we get:

$$\begin{aligned}
A_n &= \left[-\frac{1}{n}\sin^{n-1} x \cos x\right]_0^{2\pi} + \frac{n-1}{n}A_{n-2} \\
&= \left[-\frac{1}{n}\sin^{n-1}(2\pi)\cos(2\pi) + \frac{1}{n}\sin^{n-1}(0)\cos(0)\right] + \frac{n-1}{n}A_{n-2} \\
&= [0] + \frac{n-1}{n}A_{n-2} \\
&= \frac{n-1}{n}A_{n-2}
\end{aligned}$$

We first compute A_0 and then we will apply this new reduction formula repeatedly.

$$A_0 = \int_0^{2\pi} \sin^0 x dx = \int_0^{2\pi} 1 dx = [x]_0^{2\pi} = 2\pi.$$

Now we compute A_2, A_4 and A_6 based on A_0 and on our formula $A_n = \frac{n-1}{n}A_{n-2}$.

- $A_2 = \frac{1}{2}A_0 = \pi$.
- $A_4 = \frac{3}{4}A_2 = \frac{3}{4}\pi$.
- $A_6 = \frac{5}{6}A_4 = \frac{3\cdot 5}{4\cdot 6}\pi$.
- $A_8 = \frac{7}{8}A_6 = \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}\pi$.

So we conclude that

$$A_8 = \int_0^{2\pi} \sin^8 x dx = \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}\pi = \frac{35}{64}\pi.$$