Worksheet 3 Solutions

2018

MATH 222, Week 3: Trig Sub and Partial Fractions

Name: ___________________________________

Integrals of Products of Sine and Cosine

Problem 1. Use trig identities and what we learned in class to calculate the following integrals.

(a) $\int \sin^2(x)\cos^3(x)\,dx$
(b) $\int \sin^3(x)\cos^3(x)\,dx$
(c) $\int \sin^3(x)\cos^{2018}(x)\,dx$.

Solution 1.

(a) This is an odd and even power problem, so we let $u = \sin(x)$ so $du = \cos(x)dx$ and we have

$$\int u^2(1-u^2)\,du = u^3/3 - u^5/5 + C = \sin(x)^3/3 - \sin(x)^5/5 + C$$

(b) This problem has tow odd powers, which is also okay. So we let $u = \sin(x)$ and $du = \cos(x)dx$, then this becomes

$$\int u^3(1-u^2)du = u^4/4 - u^5/5 + C = \sin(x)^4/4 - \sin(x)^5/5 + C.$$

(c) We have an even and an odd here so we want to get rid of the odd power. To do this, we choose $u = \cos(x)$ so $du = -\sin(x)dx$ and we have

$$\int (1-u^2)u^{2018}\,du = u^{2019}/2019 - u^{2021}/2021 + C = \cos(x)^{2019}/2019 - \cos(x)^{2021}/2021 + C.$$

□

There's more to life than sine and cosine!

Problem 2. Calculate these integrals using trig identities:

(a) $\int \sec(x)\,dx$
(b) $\int \tan(x)\,dx$
(c) $\int \tan^5(x)\sec^{2017}(x)\,dx$.
(d) $\int \sec^6(x)\tan^{2018}(x)\,dx$
(e) (**Challenge**) $\int \sec^3(x)\,dx$
(f) (**Challenge**) $\int \frac{\sec^3(x)}{\tan(x)}\,dx$
(g) (**Challenge**) $\int \sec(x)\tan^2(x)\,dx$ (Hint: Solving parts (a) and (e) can help)

Solution 2.

(a) The key here is to rewrite $\sec(x) = \dfrac{\cos(x)}{\cos^2(x)}$. Then if we let $u = \sin(x)$ we can recognize this integral as

$$\int \frac{1}{1-u^2}\, du.$$

We then use partial fractions to complete this problem, we can rewrite this as

$$\int \frac{1/2}{1-u} + \frac{1/2}{1+u}\, du = -1/2 \ln|1-u| + 1/2 \ln|1+u| + C.$$

Now if we use some log identities and convert back to x we have

$$1/2 \ln|\frac{(1+u)^2}{1-u^2}| + C = \ln|\frac{1-\sin(x)}{\cos(x)}| + C = \ln|\sec(x)+\tan(x)| + C.$$

(b) We can realize this as

$$\int \frac{\sin(x)}{\cos(x)}\, dx$$

so if we let $u = \cos(x)$, $du = -\sin(x)dx$, and we have

$$-\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos(x)| + C.$$

(c) Here we see an odd power of both tan and sec so we want to let $u = \sec(x)$ so $du = \sec(x)\tan(x)dx$ and the integral becomes

$$\int (u^2-1)^2 u^{2016} du.$$

From here if you expand and simplify you get $\frac{u^{2021}}{2021} - \frac{2u^{2019}}{2019} + \frac{u^{2017}}{2017} + C$ and if you put $\sec(x)$ in for every u you get your answer in terms of x.

(d) This is the same as the previous problem except now you have two even powers, so you want to choose $u = \tan(x)$. If you work through this as in the previous problem you should get

$$\frac{u^{2027}}{2027} + \frac{4u^{2025}}{2025} + \frac{6u^{2023}}{2023} + \frac{4u^{2021}}{2021} + \frac{u^{2019}}{2019} + C.$$

Plugging back $u = \tan(x)$ gives you your answer.

$\square$

Trig Substitution

Problem 3. Choose the best trig substitution (when all else fails) in each case:

(a) $\int \sqrt{a^2 - x^2}\, dx$: $x = a\sin\theta$ **or** $x = a\sec\theta$ **or** $x = a\tan\theta$

(b) $\int \sqrt{x^2 - a^2}\, dx$: $x = a\sin\theta$ **or** $x = a\sec\theta$ **or** $x = a\tan\theta$

(c) $\int \sqrt{a^2 + x^2}\, dx$: $x = a\sin\theta$ **or** $x = a\sec\theta$ **or** $x = a\tan\theta$

Solution 3.

(a) $x = a\sin\theta$
(b) $x = a\sec\theta$
(c) $x = a\tan\theta$

$\square$

Problem 4. Use trig substitution to eliminate the root in $\sqrt{1-4x-2x^2}$. (Hint: Complete the square)
Solution 4. Following the hint we complete the square to rewrite

$$1-4x-2x^2 = -2(x^2+2x-1/2) = -2((x+1)^2-3/2) = 3-2(x+1)^2.$$

So if we let $x+1 = \dfrac{\sqrt{3}\sin(\theta)}{\sqrt{2}}$ this simplifies to

$$\sqrt{3-3\sin(\theta)^2} = \sqrt{3}\cos(\theta).$$

$\square$

Problem 5. If we have $x = \sin(\theta)$, then what is $\cos(\theta)$ and $\tan(\theta)$ in terms of x? What if $x = \cos(\theta)$, what is $\sin(\theta)$ and $\tan(\theta)$ in terms of x? This will be useful when doing trig sub. (Hint: Use Triangles!)
Solution 5. We have to set up a triangle to solve this. We will just do the first problem. If $x = \sin(\theta)$, then if we place θ as the corner of a right triangle, from SOH CAH TOA we know we can label the opposite as x and the hypotenuse as 1. This tells us that the adjacent side is $\sqrt{1-x^2}$ from the pythagorean theorem. Then using SOH CAH TOA we can get all the other trig functions in terms of x. I would encourage you to actually draw the triangle. So $\cos(\theta) = A/H = \sqrt{1-x^2}$ and $\tan(\theta) = O/A = \frac{x}{\sqrt{1-x^2}}$. $\square$

Problem 6. Compute the following integrals using trig sub (use Problems 3,4 and 5 as your guide):
(a) $\int t\sqrt{1-t^2}\,dt$
(b) $\int \frac{1}{\sqrt{2x-x^2}}\,dx$
(c) $\int t^3(3t^2-4)^{3/2}\,dt$
(d) $\int \frac{ds}{(2+s^2)^{3/2}}\,ds$
(e) $\int \frac{e^t}{\sqrt{4-e^{2t}}}\,dt$
(f) $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$
(g) $\int e^{4x}\sqrt{1-e^{2x}}\,dx$.
(h) (**Semi-Challenge**) $\int \frac{1}{(1+x^2)\sqrt{\arctan(x)^2-1}}\,dx$
(i) (**Challenge**) $\int \sqrt{\frac{x}{1-x^3}}\,dx$. Hint: First make a usual kind of substitution.
(j) (**Semi-Challenge**) $\int \frac{x}{\sqrt{2x^2-4x-7}}\,dx$.

1 Partial Fractions

THESE ARE ALL ON THE NEXT WORKSHEET, CHECK THERE FOR SOLUTIONS
Problem 7. Fill in the appropriate numerators and denominators in the following partial fraction decomposition. (The first is an example.)
(a) $\dfrac{f(x)}{(x-1)(x-2)} = \left[\quad \dfrac{A}{x-1} \quad + \quad \dfrac{B}{x-2} \quad \right]$

(b) $\dfrac{f(x)}{(x-1)(x-2)(x-\pi)} = \Big[\quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad \Big]$

(c) $\dfrac{f(x)}{(x+1)^3} = \Big[\quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad \Big]$

(d) $\dfrac{f(x)}{(x-1)(x^2+1)} = \Big[\quad \underline{\qquad\qquad} \quad + \quad \underline{\qquad\qquad} \quad \Big]$

Problem 8. Compute $\int \frac{dx}{(x^2-4)(x^2+1)^2}$

Problem 9. (a) Compute $\int_2^4 \frac{1}{x^2}\, dx$

(b) Compute $\int_2^4 \frac{1}{x(x-h)}\, dx$ where h is any positive number.

(c) What happens as $h \to 0$ in the integral for part (b)? How is this related to part (a)?

Problem 10. Compute $\int \frac{1}{x^2-a^2}\, dx$