Worksheet 5

Fall 2018

MATH 222, Week 5: Improper Integrals and Modeling with Differential Equations

Name: _____

Some Rationalizing Substitution

Problem 1. Evaluate the following integrals:

(a) $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$

(b) $\displaystyle\int \frac{\sqrt{x+3}}{x^2}\,dx$

(c) $\displaystyle\int \frac{\sqrt{x}}{x^2-1}\,dx$

(d) $\displaystyle\int \frac{\sqrt{x}}{3x^{3/2}+2x+1}\,dx$

Improper Integrals, we made it

Problem 2. (a) Compute $\int_0^\infty \frac{x}{\sqrt{1+x^2}}\,dx$

(b) Compute $\int_{-\infty}^0 \frac{x}{\sqrt{1+x^2}}\,dx$

(c) What does this say about $\int_{-\infty}^\infty \frac{x}{\sqrt{1+x^2}}\,dx$

Problem 3. (a) Show that $\int_1^\infty \frac{dx}{x^2-4}$ is not a finite number.

(b) What answer do you get if you forget to account for the asymptote at $x = 2$?

Problem 4. Evaluate the following integrals, in each, identify why the integral is improper. Try to decide ahead of time whether you expect the integral to diverge or converge.

(a) $\int_1^\infty \frac{1}{x^2}\,dx$

(b) $\int_1^\infty \frac{\ln(x)}{x^3}\,dx$

(c) $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

(d) $\int_3^{10} \frac{1}{(x-9)^{1/3}}\,dx$

(e) $\int_0^4 \frac{1}{x^2+x-6}\,dx$

(f) $\int_0^1 \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

For the last three problems, what would happen if you forgot about the asymptote?

Problem 5. Evaluate the following integrals. What happens in the third integral? How is this different than what occurs in the fourth integral?

(a) $\displaystyle\int_0^2 \frac{x}{(x+1)(x-1)}\,dx$

(b) $\displaystyle\int_0^2 \frac{\sqrt{x}}{(x+1)(x-1)}\,dx$

(c) $\displaystyle\int_0^2 \frac{x-\sqrt{x}}{(x+1)(x-1)}\,dx$

(d) $\displaystyle\int_{-2}^3 \frac{1}{(x-2)(x+1)}\,dx$

Bounds

The goal here is to become comfortable bounding functions with simpler functions to eventually compare.

Problem 6. (a) Find a value of a so that for all $x > a$

$$\frac{1}{x-2} > \frac{1}{(x-2)^2}.$$

(b) Find a value of a so that for all $x > a$

$$\frac{1}{2x-1} > \frac{1}{(2x-1)^2}.$$

(c) Find a value of a so that for all $x < a$ with $x > 2$

$$\frac{1}{\sqrt{x-2}} < \frac{1}{x-2}.$$

(d) True or false: for all $x > 0$

$$\frac{x}{\sqrt{1+x^2}} < \frac{1}{\sqrt{x}}.$$

Problem 7. Conclude whether the following integrals converge or diverge without computing it explicitly. Then explicitly evaluate the integrals.

(a) $\displaystyle\int_4^\infty \frac{1}{x^3-x}\,dx$

(b) $\displaystyle\int_1^2 \frac{dt}{t\sqrt{t^2-1}}.$

Differential Equations, if we get there...

Problem 8. Find a solution to the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= e^y x^3 \\ y(0) &= 0 \end{aligned}$$

Problem 9. Find a solution to the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= y\sqrt{y^2-1}\cos(x) \\ y(0) &= 1 \end{aligned}$$

Problem 10. Find the general solution to the differential equation

$$\frac{dy}{dx} = x^2 + y^2x^2$$

Problem 11. Find the general solution to the differential equation (for $x \neq 0$)

$$x\frac{dy}{dx} = -y + x$$