MATH 222 (Lectures 1,2, and 4) Fall 2015

**Practice Midterm 2.1 Solutions** Student ID#: ____________________________

Circle your TA's name from the following list.

| | | | | |
|---|---|---|---|---|
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Please inform your TA if you find any errors in the solutions.

| | Problem 1 | Problem 2 | Problem 3 | Problem 4 | Problem 5 | Problem 6 | Problem 7 |
|---|---|---|---|---|---|---|---|
| Score | | | | | | | |

## Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- On Problems 1, 2, and 3 only the answer will be graded.
- On Problems 4, 5, 6, and 7 you must show your work and we will grade the work and your justification, and not just the final answer. Limited partial credit will be available.
- Problem 3 is worth 10points. All other problems worth 15 points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write $\cos(\arcsin x) = \sqrt{1-x^2}$. Note that we have provided some formulas on the next page to help with this.

## Formulas

- $T_\infty e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $T_\infty \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
- $T_\infty \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
- $T_\infty \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$
- $T_\infty \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$
- $T_\infty (1+x)^b = \sum_{k=0}^{\infty} \binom{b}{k} x^k$ where $\binom{b}{k} = \frac{b(b-1)(b-2)\cdots(b-k+1)}{k!}$

1. On this page are three True/False statements. On the following page you will be asked to match direction fields to their defining equations. CIRCLE the correct answers below.

| (a) | | (b) | | (c) | | (d) | | | (e) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| True | False | True | False | True | False | I | II | III | I | II | III |

True or false:

(a) The function $(x^2 + x^3)^2$ is $o(x^3)$.

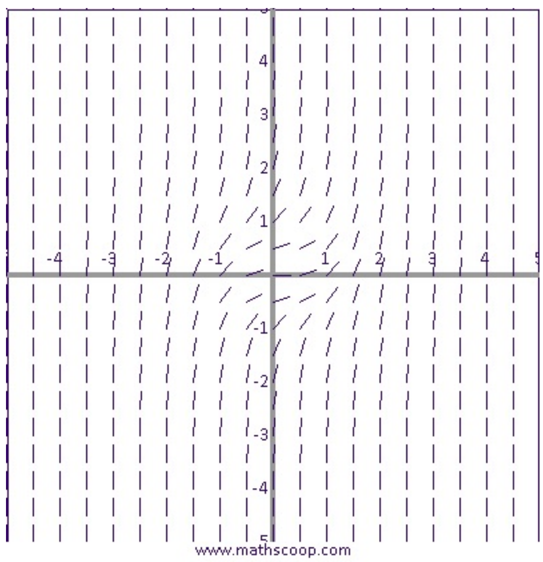
(b) $R_4 e^x = e^x - (1 + x + x^2 + x^3 + x^4)$

(c) Let $f(x)$ and $g(x)$ be functions whose Taylor series exist. Then for any $n$ we have $T_n(f(x)g(x)) = (T_n f(x))(T_n g(x))$.

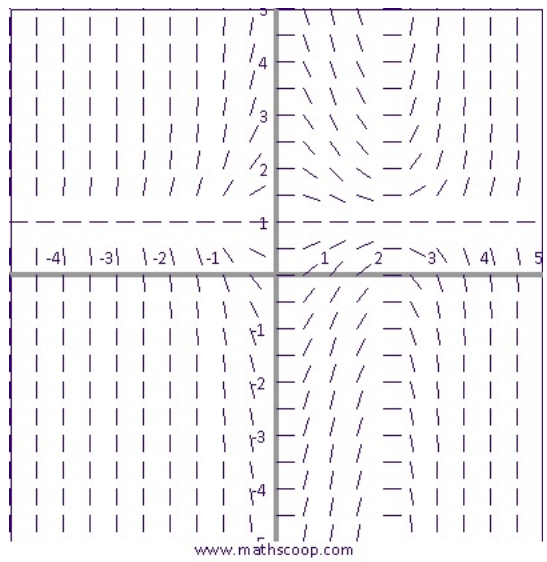
Below are three direction fields. The equations for *two* of those fields are given below. Match the equation to the appropriate direction field and record your answer on the previous page.

(d) $\frac{dy}{dx} = x(x-2)(1-y)$

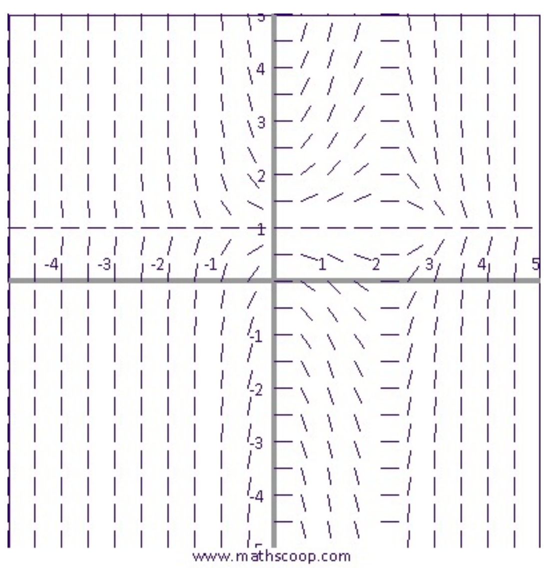
(e) $\frac{dy}{dx} = x^2 + y^2$



I.



II.



III.

**Solution:**

(a) True.
(b) False.
(c) False.
(d) III
(e) I

2. (a) Use Euler's method with step size $h = 0.1$ to approximate $y(1.1)$ where $y(x)$ is the solution of

$$\frac{dy}{dx} = -2y + 3x \text{ and } y(1.0) = 0.$$

**Solution:**

| $x_k$ | $y_k$ | $m_k = -2y_k + 3x_k$ | $y_{k+1} = y_k + m_k h$ |
|---|---|---|---|
| $x_0 = 1$ | $y_0 = 0$ | $-2 \cdot (0) + 3 \cdot (1)$ | $0 + 3(0.1) = 0.3$ |
| $x_1 = 1.1$ | $y_1 = 0.3$ | | |

So we have that $\boxed{y(1.1) \approx 0.3.}$

(b) Find $T_1^2(\arctan x)$. *You may find it helpful to recall that* $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$. *You do not need to simplify any values of* arctan *in your answer.*

**Solution:** We compute:

- $f(x) = \arctan x$
- $f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

Thus:

- $f(2) = \arctan 2$
- $f'(2) = \frac{1}{1+2^2} = \frac{1}{5}$

This yields:

$$\begin{aligned} T_2^2(\arctan x) &= f(2) + f'(2)(x-2) \\ &= \arctan(2) + \frac{1}{5}(x-2) \end{aligned}$$

(c) Find $T_2$ of $\sqrt{1+2x}$ (your final answer must not contain any binomial expression $\binom{a}{b}$).

**Solution:** We have $T_2\sqrt{1+u} = T_2(1+u)^{1/2} = 1 + \binom{1/2}{1}u + \binom{1/2}{2}u^2$. Substituting $u = 2x$ yields

$$T_2\sqrt{1+2x} = 1 + \binom{1/2}{1}(2x) + \binom{1/2}{2}(2x)^2.$$

Lastly we need to rewrite the binomial expressions. We have $\binom{1/2}{1} = \frac{(1/2)}{1} = (1/2)$ and $\binom{1/2}{2} = \frac{(\frac{1}{2})\cdot(\frac{1}{2}-1)}{2!} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$. Thus:

$$T_2\sqrt{1+2x} = 1 + (1/2)(2x) + (-1/8)4x^2 = 1 + x - \frac{1}{2}x^2.$$

3. In the problem below: 1. Clearly define variables (including units!); 2. Set up the appropriate differential equation; and 3. Write down the appropriate initial condition. DO NOT SOLVE THE DIFFERENTIAL EQUATION.

   Start with a full 50 quart vat containing a 8% (by volume) solution of vinegar, at 10:00am. A solution of 5% vinegar flows in at a rate of 2 quarts per minute. The solution is kept thoroughly mixed and drawn off at a rate of 3 quarts per minute. We are interested in a function describing the **total amount of vinegar in the vat at a given time**.

   - **Variables (2pts):**

   - **Differential equation (5pts)**

   - **Initial condition (3pts):**

   **Solution:**

   - **Variables:** $t$ = time in minutes from 10:00am and $A(t)$ = amount of vinegar in the vat at time $t$, in quarts.
   - **Differential equation:** $\frac{dA}{dt} = .1 - \frac{3A}{50-t}$
   - **Initial condition:** $A(0) = 4$

4. Find a solution to each initial value problem.

(a)

$$\frac{dy}{dx} - e^x y^2 = e^x \text{ and } y(0) = 0.$$

**Solution:** Rewriting, we get the separable differential equation:

$$\frac{dy}{dx} = e^x(1 + y^2)$$

After separating variables we get:

$$\begin{aligned}\int \frac{dy}{1+y^2} &= \int e^x dx \\ \arctan(y) &= e^x + C \\ y &= \tan(e^x + C)\end{aligned}$$

Plugging in $y(0) = 0$ gives:

$$0 = \tan(e^0 + C)$$

and thus $C = -1$ yielding $\boxed{y = \tan(e^x - 1).}$

(b)

$$(1 + x^2)\frac{dy}{dx} + 2xy = 3(1 + x^2) \text{ and } y(1) = \frac{5}{2}.$$

**Solution:** Dividing through by $(1+x^2)$ yields $\frac{dy}{dx} + y\frac{2x}{1+x^2} = 3$. This is a then a linear differential equation with $a(x) = \frac{2x}{1+x^2}$ and $k(x) = 3$. Since $\int \frac{2x}{1+x^2} = \ln|1 + x^2| + C$ we get $m(x) = e^{\ln|1+x^2|} = |1 + x^2| = 1 + x^2$. We then get:

$$\begin{aligned}y &= \frac{1}{1+x^2}\int (1 + x^2)(3)dx \\ &= \frac{1}{1+x^2}\int 3 + 3x^2 dx \\ &= \frac{1}{1+x^2}[3x + x^3 + C].\end{aligned}$$

Plugging in $y(1) = \frac{5}{2}$ gives:

$$\frac{5}{2} = \frac{1}{2}[3 + 1 + C]$$

and thus $C = 1$ yielding $\boxed{y = \frac{1}{1+x^2}[3x + x^3 + 1].}$

5. We have a vat containing a mixture of acid and water. Let:

   - $t$ stand for time in minutes from 12:00pm
   - $A(t)$ denote the total amount of acid in the vat at time $t$
   - $V(t)$ denote the total volume of liquid in the vat at time $t$.

   Assume that

   $$\frac{dA}{dt} = 3 - \frac{2A}{20-t} \qquad \text{and} \qquad A(0) = 0 \qquad \text{and} \qquad V(t) = 1000 - 50t.$$

   What is the <u>*concentration*</u> (%) of acid in the vat after 10 minutes?

   **Solution:** We want to know $A(10)/V(10)$. The differential equation $\frac{dA}{dt} + \frac{2A}{20-t} = 3$ is linear with $a(t) = \frac{2}{20-t}$ and $k(t) = 3$. The integrating factor is

   $$m(t) = e^{\int a(t)dt} = e^{-2\ln|20-t|} = |20-t|^{-2}$$

   Note that the initial value is at $t = 0$ and $20 - 0$ is positive, so we can replace $|20-t|^{-2}$ by $(20-t)^{-2}$ near $t = 0$.

   We check that this is the right integrating factor by checking whether $\frac{dm}{dt} = m(t)a(t)$. So we check:

   - $\frac{dm}{dt} = -2(20-t)^{-3} \cdot (-1) = 2(20-t)^{-3}$ and
   - $m(t)a(t) = (20-t)^{-2} \cdot \frac{2}{20-t} = 2(20-t)^{-3}$.

   We then get

   $$\begin{aligned} A &= \frac{1}{m(t)} \int m(t)k(t)dt \\ &= (20-t)^2 \int (20-t)^{-2} \cdot 3 \\ &= (20-t)^2 \cdot [3(20-t)^{-1} + C] \\ &= 3(20-t) + C(20-t)^2. \end{aligned}$$

   We use the initial condition at $t = 0$ to get:

   $$0 = 3(20-0) + C(20-0)^2 = 60 + 400C \text{ and thus } C = -\frac{60}{400} = -\frac{3}{20}$$

   Our equation is:

   $$A(t) = 3(20-t) - \frac{3}{20}(20-t)^2$$

   We compute

   $$\frac{A(10)}{V(10)} = \frac{3(20-10) - \frac{3}{20}(20-10)^2}{1000-50(10)} = \frac{30 - \frac{300}{20}}{500} = \frac{15}{500} = \frac{3}{100}$$

   $\boxed{\text{After ten-minutes, the concentration of acid in the vat is 3\% or } \frac{3}{100}}$.

6. Let $f(x) = e^{5-2x}$. Find a number $B$ such that $|f(x) - T_4 f(x)| \le B$ for all $x$ in the range $-1 \le x \le 1$. You must justify your answer.

**Solution:** We want to use the error-bound formula for $|R_4 f(x)|$. We have $f(x) = e^{5-2x}$ and $c = 1$ and $n = 4$. But we need to find $M$ which requires bounding $f^{(5)}(x)$. So we compute:

- $f'(x) = -2e^{5-2x}$
- $f^{(2)}(x) = 2^2 e^{5-2x}$
- $f^{(3)}(x) = -2^3 e^{5-2x}$
- $f^{(4)}(x) = 2^4 e^{5-2x}$
- $f^{(5)}(x) = -2^5 e^{5-2x}$

We want to maximize $|f^{(5)}(x)| = 2^5 |e^{5-2x}|$ on the range $-1 \le x \le 1$. This will be largest when the exponent $5 - 2x$ is largest, which will happen when $x = -1$ and $5 - 2x = 7$. Thus

$$|f^{(5)}(x)| \le 2^5 \cdot e^7 \text{ for all } -1 \le x \le 1.$$

We choose $M = 2^5 e^7$. We then apply the error bound formula to obtain:

$$|R_4 f(x)| = |f(x) - T_4 f(x)| \le \frac{Mc^{n+1}}{n!} = \frac{(2^5 e^7) \cdot 1^{n+1}}{5!} \text{ for all } -1 \le x \le 1.$$

$\boxed{\text{Thus } B = \frac{2^5 e^7}{5!} \text{ is our answer.}}$

7. Let $f(x)$ be a function satisfying the differential equation

$$f''(x) + \sin(2x^2) - f(x) = 0$$

and also satisfying the initial conditions $f(0) = 2$ and $f'(0) = 0$. Compute $T_4 f(x)$.

*Note: it is essential that you use notation correctly in your answer, as part of what we are testing is whether you understand what the notation means.*

**Solution:** We write $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4)$. By definition of the Taylor series of $f(x)$, we have that $a_0 = f(0)$ and $a_1 = f'(0)$. Thus $a_0 = 2$ and $a_1 = 0$. Thus

$$f(x) = 2 + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4).$$

We then compute:

$$f'(x) = 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + o(x^3),$$

and also

$$f''(x) = 2a_2 + 6a_3 x + 12a_3 x^4 + o(x^2).$$

Note that

$$T_\infty \sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \dots$$

By the substitution method with $u = 2x^2$ (which is a polynomial), we then get:

$$T_\infty \sin(2x^2) = (2x^2) - \frac{(2x^2)^3}{3!} + \cdots = 2x^2 + o(x^4)$$

(NB: we only track terms of degree at most 4 because we are interested in computing $T_4 f(x)$.)

We then compute

$$\begin{aligned}
0 &= f''(x) + \sin(2x^2) - f(x) \\
&= (2a_2 + 6a_3 x + 12a_4 x^2 + o(x^2)) + (2x^2 - \frac{8x^6}{3!} + o(x^5)) - (2 + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4)) \\
&= (2a_2 - 2) + (6a_3)x + (12a_4 + 2 - a_2)x^2 + o(x^2)
\end{aligned}$$

By equating coefficients we see that $0 = 2a_2 - 2$ and thus $a_2 = 1$. Also $a_3 = 0$. And $12a_4 + 2 - a_2 = 0$, but $a_2 = 1$ and so $12a_4 = -1$ and thus $a_4 = -\frac{1}{12}$.

$\boxed{\text{We conclude that } T_3 f(x) = 2 + x^2 - \frac{1}{12}x^4.}$