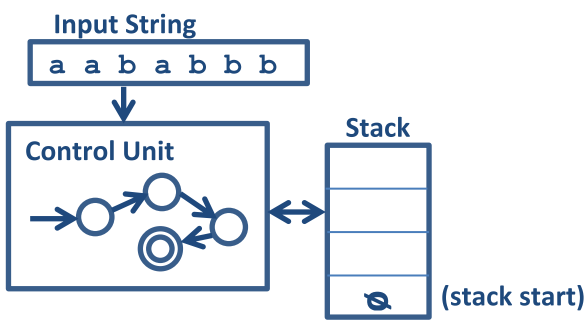
10 - Pushdown Automata

We have seen that *finite automata* are limited in that they are only capable of accepting *regular languages*. Such languages (and machines) are useful for lexical scanning, as we have seen. But for parsing, we found it useful to build a machine capable of recognizing a *context-free language*. Finite automata aren't adequate for this task because they have no "memory", beyond their states, which are finite in number.

Pushdown Automata (PDA) add a **stack** to the existing notion of a finite state machine, giving them an infinite (albeit simple) memory mechanism:



A Nondeterministic Pushdown Acceptor (NPDA) is defined by the septuple:

- Q – a finite set of states
- Σ – an input alphabet
- Γ – a stack alphabet
- δ – transitions $Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow Q \times \Gamma^*$
- s – initial state $\in Q$
- ø – start stack symbol $\in \Gamma$
- F – set of final states $\subseteq Q$

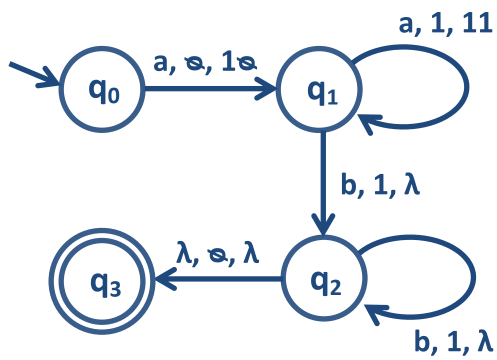
For example, transition δ (q,a,c) = (q',w) would mean:

1. the machine is currently in state q
2. consume (read in) the next symbol a from the input string
3. pop the symbol at the top of the stack
4. move to state q'
5. push a new string onto the top of the stack

Example 1 – Construct a NPDA for the language { $a^n b^n$; $n \geq 1$ }

$Q = \{q_0, q_1, q_2, q_3\}$
$\Sigma = \{a,b\}$
$\Gamma = \{\varnothing,1\}$
$z = \varnothing$
$s = q_0$
$F = \{q_3\}$

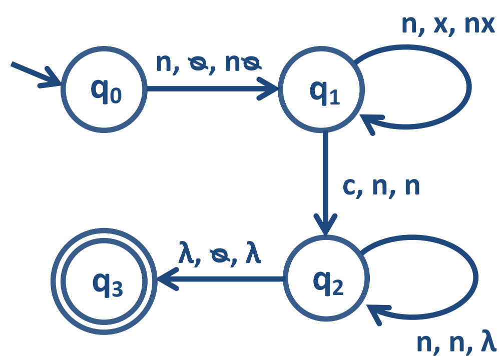
$\delta(q_0,a,\varnothing) = \{ (q_1,1\varnothing) \}$
$\delta(q_1,a,1) = \{ (q_1,11) \}$
$\delta(q_1,b,1) = \{ (q_2,\lambda) \}$
$\delta(q_2,b,1) = \{ (q_2,\lambda) \}$
$\delta(q_2,\lambda,\varnothing) = \{ (q_3,\lambda) \}$



Example 2 – Construct a NPDA for the language { wcw^R ; $w \in \{a,b\}^+$ }

$Q = \{q_0, q_1, q_2, q_3\}$
$\Sigma = \{a,b,c\}$
$\Gamma = \{\varnothing,a,b,c\}$
$z = \varnothing$
$s = q_0$
$F = \{q_3\}$

$\delta(q_0,n,\varnothing) = \{ (q_1,n\varnothing) \}$ $n \in \{a,b\}$
$\delta(q_1,n,x) = \{ (q_1,nx) \}$ $n,x \in \{a,b\}$
$\delta(q_1,c,n) = \{ (q_2,n) \}$ $n \in \{a,b\}$
$\delta(q_2,n,n) = \{ (q_2,\lambda) \}$ $n \in \{a,b\}$
$\delta(q_2,\lambda,\varnothing) = \{ (q_3,\lambda) \}$



NPDAs accept all strings for which there is *some* path to an “accept” state.

The distinction between “deterministic” and “non-deterministic” PDAs is a bit different than it is for FAs.   PDAs are only non-deterministic if there is more than one choice for a given scenario. Note that in a PDA, the input string isn’t the only factor in a state transition; the symbol at the top of the stack also plays a role. By this definition, ex. 1 (above) is deterministic. Finally, unlike FAs, “deterministic” and “non-deterministic” PDAs aren’t equivalent. NPDAs have more expressive power than DPDAs.

We will focus on NPDAs, because they are equivalent to CFGs.

Instantaneous Description (ID) ⊢

It is often useful to illustrate *specific* transitions in a PDA. A convenient notation for doing this uses the "**⊢**" symbol, and shows the remaining unread part of the input string, and the stack content. For example:

$$(q_1, \text{aaabb}, \text{bx}) \vdash (q_2, \text{aabb}, \text{yx})$$

Here, the transition is from state q_1 to q_2. The first "a" was read from the input string. The symbol "b" at the top of the stack was replaced with "y".

In example 1 above, the series of transitions in accepting the input string "aabb", using the ID notation, would be:

$$(q_0,\text{aabb},\oslash) \vdash (q_1,\text{abb},1\oslash) \vdash (q_1,\text{bb},11\oslash) \vdash (q_2,\text{b},1\oslash) \vdash (q_2,\lambda,\oslash) \vdash (q_3,\lambda,\lambda)$$

Building a NPDA for a CFG

This is done most easily if the CFG is in *Greibach Normal Form* (GNF), which requires all of the CFG rules to be in the following form:

$A \rightarrow t\ B$

or $A \rightarrow t$ where t is a terminal, and B is one or more non-terminals

Converting a CFG to GNF can sometimes be tricky, but in many cases it can be easily done by inspection. We won't learn how to convert a CFG to GNF in general, just some simple cases.

The general approach for building the NPDA is then as follows:

1. there are three (3) states: q_0, q_1, and q_2
2. there is a transition from q_0 to q_1 which pushes S onto the stack
3. each grammar rule $A \rightarrow t\ B$ then becomes a transition in the PDA:
    - the transition is from state q_1 back to itself
    - consume the terminal t, for stack symbol A
    - push non-terminals B (if any) onto the stack
    - for example: $\delta(q_1,t,A) = (q_1,B)$
4. a final transition is made to the accept state q_2

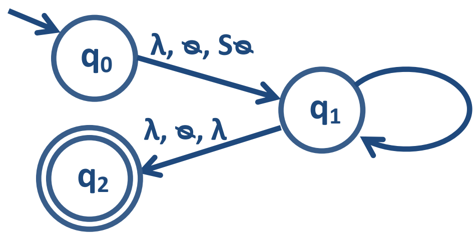
Example 3 – Build a NPDA for the CFG $S \rightarrow aSbb \mid a$

First, convert the CFG to GNF. Here, we can simply create new non-terminals for the b's in the first rule. The resulting GNF form is:

$$S \rightarrow aSXY \mid a$$
$$X \rightarrow b$$
$$Y \rightarrow b$$

Using the construction described on the previous page, the resulting transition rules are as follows:

$\delta(q_0,\lambda,\oslash) = \{ (q_1,S\oslash) \}$
$\delta(q_1,a,S) = \{ (q_1,SXY), (q_1,\lambda) \}$
$\delta(q_1,b,X) = \{ (q_1,\lambda) \}$
$\delta(q_1,b,Y) = \{ (q_1,\lambda) \}$
$\delta(q_1,\lambda,\oslash) = \{ (q_2,\lambda) \}$



The series of transitions that accept the string "aabb", are:

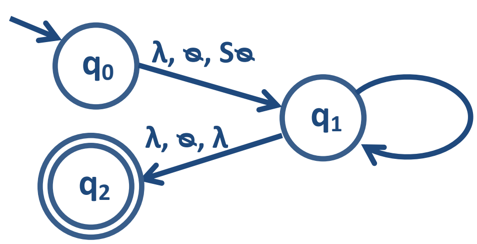
$(q_0,aabb,\oslash) \vdash (q_1,aabb,S\oslash) \vdash (q_1,abb,SXY\oslash) \vdash (q_1,bb,XY\oslash) \vdash (q_1,b,Y\oslash) \vdash (q_1,\lambda,\oslash) \vdash (q_2,\lambda,\lambda)$

Example 4 – Build a NPDA for the following CFG:

$$S \rightarrow aA$$
$$A \rightarrow aABC \mid bB \mid a$$
$$B \rightarrow b$$
$$C \rightarrow c$$

Since the grammar is already in GNF, we can proceed as previously:

$\delta(q_0,\lambda,\oslash) = \{ (q_1,S\oslash) \}$
$\delta(q_1,a,S) = \{ (q_1,A) \}$
$\delta(q_1,a,A) = \{ (q_1,ABC), (q_1,\lambda \}$
$\delta(q_1,b,A) = \{ (q_1,B) \}$
$\delta(q_1,b,B) = \{ (q_1,\lambda) \}$
$\delta(q_1,c,C) = \{ (q_1,\lambda) \}$
$\delta(q_1,\lambda,\oslash) = \{ (q_2,\lambda) \}$



And an example series of transitions that accept the string "aaabc", are:

$(q_0,aaabc,\oslash) \vdash (q_1,aaabc,S\oslash) \vdash (q_1,aabc,A\oslash) \vdash (q_1,abc,ABC\oslash)$
$\vdash (q_1,bc,BC\oslash) \vdash (q_1,c,C\oslash) \vdash (q1,\lambda,\oslash) \vdash (q_2,\lambda,\lambda)$