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Laplacian spectral characterization of some graphs obtained by product operation

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ABSTRACT

A graph is said to be DLS, if there is no other non-isomorphic graph with the same Laplacian spectrum. Let G be a DLS graph. We show that $G \times K_r$ is DLS if G is disconnected. If G is connected, it is proved that $G \times K_r$ is DLS under certain conditions. Applying this result, we prove that $G \times K_r$ is DLS if G is a tree on n ($n \geqslant 5$) vertices or a unicyclic graph on n ($n > 6$) vertices.

1. Introduction

All graphs considered here are simple and undirected. For a graph G, let $A(G)$ be the adjacency matrix of G, let $D(G)$ be the diagonal matrix of vertex degrees of G. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. The eigenvalues of $L(G)$ are called the *Laplacian eigenvalues* of G. Since $L(G)$ is real, symmetric and positive semidefinite, the Laplacian eigenvalues of G are all nonnegative real numbers. The largest eigenvalue of $L(G)$ is called the *L-index* of G. It is well-known that the smallest Laplacian eigenvalue of G is always 0. The multiset of the eigenvalues of $L(G)$ is called the *Laplacian spectrum* of G. Two graphs are said to be *L-cospectral*, if they have the same Laplacian spectrum. A graph is said to be *determined by the Laplacian spectrum*, if there is no other non-isomorphic graph with the same Laplacian spectrum. We shall use "DLS" as an abbreviation for "determined by the Laplacian spectrum" in this paper.

For two disjoint graphs G and H, let $G \cup H$ denote the *disjoint union* of G and H, and mG denote the disjoint union of m copies of G. Let $\overline{G}$ denote the complement of G. The *product* of G and H, denoted by $G \times H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. Clearly, $\overline{G \times H} = \overline{G} \cup \overline{H}$. As usual, P_n, C_n and K_n stand for the path, the cycle and the complete graph on n vertices, respectively. In particular, K_1 stands for an isolated vertex. Let $K_{1,n-1}$ denote the star on n vertices.

Which graphs are determined by their spectra is a difficult problem in the theory of graph spectra. Only some graphs with special structures have been proved to be determined by their spectra [4,25,21,14,5]. Some DLS graphs can be obtained from the product of a DLS graph and an isolated vertex or a complete graph. Here we introduce some relevant results.

(a) Paths and cycles are DLS. The disjoint union of paths is DLS, and the disjoint union of cycles is also DLS (see [19]).
(b) The *multi-fan graph* $(P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_s}) \times K_1$ is DLS (see [12]).
(c) The wheel graph $C_n \times K_1$ is DLS when $n \neq 6$ (see [24]).
(d) The graph $C_n \times K_m$ is DLS when $n \neq 6$, and the graph $(P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_s}) \times K_m$ is DLS (see [10]).

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Let G be a DLS graph. We show that $G \times K_m$ is DLS when G is disconnected. If G is connected, it is proved that $G \times K_n$ is DLS under certain conditions. Applying this result, we prove that $G \times K_n$ is DLS if G is a tree on n ($n \geqslant 5$) vertices or a unicyclic graph on n ($n > 6$) vertices.

2. Preliminaries

In order to get our main results, some helpful lemmas are given in this section.

Lemma 2.1 ([12]). *Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n = 0$ and $\overline{\mu}_1 \geqslant \overline{\mu}_2 \geqslant \cdots \geqslant \overline{\mu}_n = 0$ be the Laplacian spectra of G and $\overline{G}$, respectively. Then $\mu_i + \overline{\mu}_{n-i} = n$ for any $i \in \{1, 2, \ldots, n-1\}$.*

Lemma 2.2 ([6]). *Let G be a connected graph on n vertices, the L-index of G is $\mu(G)$. Then $\mu(G) \leqslant n$, with equality if and only if $\overline{G}$ is disconnected.*

It is not difficult to obtain the following lemma from Lemma 2.1.

Lemma 2.3 ([24]). *Let G_1 and G_2 be graphs with n_1 and n_2 vertices, respectively. Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n_1} = 0$ and $\eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_{n_2} = 0$ be the Laplacian spectra of G_1 and G_2, respectively. Then $n_1 + n_2, \mu_1 + n_2, \mu_2 + n_2, \ldots, \mu_{n_1-1} + n_2, \eta_1 + n_1, \eta_2 + n_1, \ldots, \eta_{n_2-1} + n_1, 0$ are the Laplacian eigenvalues of graph $G_1 \times G_2$.*

Lemma 2.4 ([20]). *A graph G is DLS if and only if its complement $\overline{G}$ is DLS.*

Lemma 2.5 ([20]). *Let G be a graph on n vertices with L-index n. If G is DLS, then $G \cup mK_1$ is DLS for any positive integer m.*

The second smallest Laplacian eigenvalue of graph G is called the *algebraic connectivity* of G, denoted by $a(G)$. It is well-known that G is connected if and only if $a(G) > 0$. Let $\kappa(G)$ denote the vertex connectivity of G.

Lemma 2.6 ([9]). *Let G be a non-complete, connected graph on n vertices. Then $a(G) = \kappa(G) = 1$ if and only if $G = H \times K_1$, where H is a disconnected graph on $n-1$ vertices.*

Lemma 2.7 ([23]). *Let G be a connected graph with vertex set $V(G)$, let $\mu(G)$ be the L-index of G. Then*

$$\mu(G) \leqslant \max\{d(v) + \sqrt{d(v)m(v)} \mid v \in V(G)\},$$

where $d(v)$ is the degree of vertex v, $m(v)$ is the average degree of all neighbours of vertex v.

Lemma 2.8 ([10]). *Let G be a graph. For the Laplacian matrix, the following invariants of G can be obtained from the spectrum:*

(1) *the number of vertices;*
(2) *the number of edges;*
(3) *the number of components.*

Lemma 2.9. *Let f be a positive integer such that $f = f_0 + f_1 + \cdots + f_r$, where $r, f_0, f_1, \ldots, f_r$ are positive integers, and $\max\{f_0, f_1, \ldots, f_r\} \leqslant f - r - 2, f - r \geqslant 5$. Let $\Gamma = \sum_{i=0}^{r} \frac{f_i(f_i-1)}{2}$, then*

$$\Gamma \leqslant \frac{(f-r-2)(f-r-3)}{2} + 3,$$

with equality if and only if $\{f_0, f_1, \ldots, f_r\} = \{f-r-2, 3, 1, \ldots, 1\}$.

Proof. It is easy to see that

$$\Gamma = \sum_{i=0}^{r} \frac{f_i(f_i-1)}{2} = -\frac{1}{2}f + \frac{1}{2}\sum_{i=0}^{r} f_i^2.$$

Obviously Γ is maximal if and only if $\sum_{i=0}^{r} f_i^2$ is maximal. Without loss of generality, let $f_0 = \max\{f_0, f_1, \ldots, f_r\}$. First we will show that $\sum_{i=0}^{r} f_i^2$ is not maximal when $f_0 < f - r - 2$. If $f_0 < f - r - 2$, by $f_0 + f_1 + \cdots + f_r = f$, it is easy to see that there exists a positive integer j ($1 \leqslant j \leqslant r$) such that $f_j > 1$. If we replace f_0 and f_j with $f_0 + 1$ and $f_j - 1$ respectively, then the sum of the squares increases. So $\sum_{i=0}^{r} f_i^2$ is not maximal when $f_0 < f - r - 2$. If $f_0 = f - r - 2$, then $f_1 + f_2 + \cdots + f_r = r + 2$. It is not difficult to see that $\sum_{i=0}^{r} f_i^2$ is maximal if and only if $\{f_1, f_2, \ldots, f_r\} = \{3, 1, 1, \ldots, 1\}$. Hence we have $\Gamma = \sum_{i=0}^{r} \frac{f_i(f_i-1)}{2} \leqslant \frac{(f-r-2)(f-r-3)}{2} + 3$, with equality if and only if $\{f_0, f_1, \ldots, f_r\} = \{f-r-2, 3, 1, \ldots, 1\}$. □

Lemma 2.10. *Let G be a star. Then $G \times K_r$ is DLS for any positive integer r.*

Proof. Suppose that $G = K_{1,n-1}$. By Lemma 2.4 we know that $K_{1,n-1} \times K_r$ is DLS if and only if $K_{n-1} \cup (r+1)K_1$ is DLS. Since K_{n-1} is a DLS graph with L-index $n-1$, by Lemma 2.5, $K_{n-1} \cup (r+1)K_1$ is DLS. Hence $K_{1,n-1} \times K_r$ is DLS. $\square$

Let $K_n - e$ denote the graph obtained from complete graph K_n by deleting one edge. Let G be a graph L-cospectral with $K_n - e$. Lemma 2.8 implies that G is a connected graph on n vertices and $\frac{n(n-1)}{2} - 1$ edges. Hence $G = K_n - e$, i.e., $K_n - e$ is DLS.

Lemma 2.11. *Let $G = (K_2 \cup (n-3)K_1) \times K_1$ $(n \geqslant 6)$. Then $G \times K_r$ is DLS for any positive integer r.*

Proof. By Lemma 2.4 we know that $G \times K_r$ is DLS if and only if $(K_{n-1} - e) \cup (r+1)K_1$ is DLS. The L-index of $K_{n-1} - e$ is $n-1$. Since $K_{n-1} - e$ is DLS, by Lemma 2.5, $(K_{n-1} - e) \cup (r+1)K_1$ is DLS, i.e., $G \times K_r$ is DLS. $\square$

Lemma 2.12 ([10]). *The graph $C_n \times K_m$ is DLS when $n \neq 6$.*

Lemma 2.13 ([2]). *The graph $K_a \cup K_b$ $(b > 1)$ with $\frac{a}{b} > \frac{5}{3}$ is DLS.*

Lemma 2.14 ([6]). *Let G be a graph on n vertices, and the Laplacian eigenvalues of G are $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n = 0$. Then the number of spanning trees of G is $\frac{1}{n} \prod_{i=1}^{n-1} \mu_i$.*

3. Main results

For a disconnected DLS graph G, it is known that the product $G \times K_1$ is DLS (cf. [20, Proposition 4]). This property also holds if K_1 is replaced by a complete graph.

Theorem 3.1. *Let G be a disconnected DLS graph on n vertices, then $G \times K_m$ is DLS for any positive integer m.*

Proof. By Lemma 2.4 we know that $G \times K_m$ is DLS if and only if $\overline{G} \cup mK_1$ is DLS. Since G is disconnected, its complement $\overline{G}$ is connected. By Lemma 2.2, the L-index of $\overline{G}$ is n. Since G is DLS, by Lemma 2.4 we know that $\overline{G}$ is DLS. Lemma 2.5 implies that $\overline{G} \cup mK_1$ is DLS. Hence $G \times K_m$ is DLS. $\square$

Remark 3.1. Since the disjoint union of paths is DLS (see [19]), by Theorem 3.1, we know that graph $(P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_s}) \times K_m$ is DLS. In [10], this result is obtained by induction on m. Some disconnected DLS graphs can be found in [2,16,17,7,22].

For a connected graph G on n vertices and m edges, the quantity $m - n + 1$ is called the *cyclomatic number* of G. G is called a *unicyclic graph* if its cyclomatic number is 1.

Theorem 3.2. *Let G be a connected DLS graph on n vertices with cyclomatic number $c \leqslant n-5$, and $\overline{G}$ is connected. Let H be a graph that is L-cospectral with $G \times K_r$. Then one of the following holds:*

(a) *H is isomorphic to $G \times K_r$;*
(b) *$H = N \times 2K_1 \times K_{r-1}$, where N is a graph on $n-1$ vertices and $c+1$ edges. In this case, $n-2$ is a Laplacian eigenvalue of G, the algebraic connectivity of G is 1, and G has 1 as a Laplacian eigenvalue with multiplicity at least 2.*

Proof. Lemma 2.1 implies that $\overline{H}$ and $\overline{G} \cup rK_1$ are L-cospectral. Lemma 2.8 implies that $\overline{H}$ has $r+1$ components. Suppose that $\overline{H} = H_0 \cup H_1 \cup \cdots \cup H_r$, where H_i is a connected graph on n_i vertices and m_i edges $(i = 0, 1, \ldots, r)$. Without loss of generality, assume that $n_0 \geqslant n_1 \geqslant \cdots \geqslant n_r \geqslant 1$. Since the cyclomatic number of G is c, G has n vertices and $c + n - 1$ edges. So $\overline{G}$ has n vertices and $\frac{(n-1)(n-2)}{2} - c$ edges. By Lemma 2.8 we have

$$\sum_{i=0}^{r} n_i = n + r, \qquad \sum_{i=0}^{r} m_i = \frac{(n-1)(n-2)}{2} - c.$$

By $n_r \geqslant 1$, we have $n_0 \leqslant n$. So we can consider the following three cases.

Case 1. If $n_0 = n$, then $n_1 = n_2 = \cdots = n_r = 1$. Hence $\overline{H} = H_0 \cup rK_1$. Since $\overline{H}$ and $\overline{G} \cup rK_1$ are L-cospectral, $\overline{G}$ and H_0 are L-cospectral. Since G is DLS, by Lemma 2.4, $\overline{G}$ is also DLS. Hence $H_0 = \overline{G}$, $\overline{H} = \overline{G} \cup rK_1$. In this case, H is isomorphic to $G \times K_r$, i.e., part (a) holds.

Case 2. If $n_0 = n-1$, then $n_1 = 2$, $n_2 = n_3 = \cdots = n_r = 1$. Hence H_0 has $n-1$ vertices, $H_1 = K_2$, $H_2 = H_3 = \cdots = H_r = K_1$. Since $\overline{H} = H_0 \cup H_1 \cup \cdots \cup H_r$ and $\overline{G} \cup rK_1$ are L-cospectral, $H_0 \cup K_2$ and $\overline{G} \cup K_1$ are L-cospectral. By Lemma 2.8 we have $m_0 + 1 = \frac{(n-1)(n-2)}{2} - c$, $m_0 = \frac{(n-1)(n-2)}{2} - (c+1)$. Hence $\overline{H_0}$ has $n-1$ vertices and $c+1$ edges and $H = \overline{H_0} \times 2K_1 \times K_{r-1}$. Since $H_0 \cup K_2$ is L-cospectral with $\overline{G} \cup K_1$, H_0 and $\overline{G}$ have the same L-index, and 2 is a Laplacian eigenvalue of $\overline{G}$ (2 is a Laplacian eigenvalue of K_2). Lemma 2.1 implies that $n-2$ is a Laplacian eigenvalue of G. Note that $\overline{H_0}$ has $n-1$ vertices and $c+1$ edges. Since $c \leqslant n-5$, $\overline{H_0}$ has at least 3 components, i.e., $\overline{H_0}$ has 0 as a Laplacian eigenvalue with multiplicity at least 3. Lemma 2.1 implies that the L-index of H_0 is $n-1$, and its multiplicity is at least 2. Since $H_0 \cup K_2$ and $\overline{G} \cup K_1$ are L-cospectral, by Lemma 2.1, the algebraic connectivity of G is 1, and its multiplicity is at least 2. Hence part (b) holds.

Case 3. Suppose $n_0 \leqslant n-2$. Notice that $\frac{(n-1)(n-2)}{2} - c = \sum_{i=0}^{r} m_i \leqslant \sum_{i=0}^{r} \frac{n_i(n_i-1)}{2}$. By $0 \leqslant c \leqslant n-5$, we have $n \geqslant 5$. Lemma 2.9 implies that

$$\frac{(n-1)(n-2)}{2} - c = \sum_{i=0}^{r} m_i \leqslant \sum_{i=0}^{r} \frac{n_i(n_i-1)}{2} \leqslant \frac{(n-2)(n-3)}{2} + 3. \tag{1}$$

Since $c \leqslant n-5$, we have $\frac{(n-1)(n-2)}{2} - c \geqslant \frac{(n-2)(n-3)}{2} + 3$. Inequality (1) implies that

$$\frac{(n-1)(n-2)}{2} - c = \frac{(n-2)(n-3)}{2} + 3, \quad c = n-5. \tag{2}$$

By inequality (1) and Lemma 2.9, we have $n_0 = n-2$, $n_1 = 3$, $n_2 = n_3 = \cdots = n_r = 1$, and H_0 and H_1 are complete graphs. Since $\overline{H} = H_0 \cup H_1 \cup \cdots \cup H_r$ and $\overline{G} \cup rK_1$ are L-cospectral, $K_{n-2} \cup K_3$ and $\overline{G} \cup K_1$ are L-cospectral. If $n > 7$, by Lemma 2.13, $K_{n-2} \cup K_3$ and $\overline{G} \cup K_1$ are isomorphic, a contradiction. So we have $5 \leqslant n \leqslant 7$.

If $n = 5$, the Laplacian spectra of $K_{n-2} \cup K_3$ and $\overline{G} \cup K_1$ are both 3, 3, 3, 3, 0, 0. Lemma 2.14 implies that the number of spanning trees of $\overline{G}$ is $\frac{81}{5}$, a contradiction.

If $n = 6$, the Laplacian spectrum of $\overline{G}$ is 4, 4, 4, 3, 3, 0. Lemma 2.1 implies that the Laplacian spectrum of G is 3, 3, 2, 2, 2, 0. By Lemma 2.14, the number of spanning trees of G is 12. From Eq. (2) we have $c = n-5 = 1$. Hence G is a unicyclic graph on 6 vertices, the number of spanning trees of G is smaller than or equal to 6, a contradiction.

If $n = 7$, the Laplacian spectrum of $\overline{G}$ is 5, 5, 5, 5, 3, 3, 0. Lemma 2.14 implies that the number of spanning trees of $\overline{G}$ is $\frac{9\times 5^4}{7}$, a contradiction. □

For a connected graph G on n vertices, Lemma 2.2 implies that $\overline{G}$ is disconnected if and only if the L-index of G is n. Since the L-index of G is n if and only if G is the product of two graphs (cf. [24, Lemma 2.7]), $\overline{G}$ is connected if and only if G is not the product of two graphs. Clearly a DLS tree T has cyclomatic number 0, and $\overline{T}$ is connected if and only if T is not a star. A DLS unicyclic graph U has cyclomatic number 1, and $\overline{U}$ is connected if and only if $U \neq C_4$ or $(K_2 \cup (n-3)K_1) \times K_1$ $(n \geqslant 3)$. Note that almost all known connected DLS graphs are trees or unicyclic graphs (see [15,1,13,18,8,3,11]). So most known connected DLS graphs satisfy the conditions given in Theorem 3.2.

Let G and H be two L-cospectral graphs. We say that H is a *cospectral mate* of G, if H is not isomorphic to G. Obviously a graph G is DLS if and only if G has no cospectral mates.

Theorem 3.3. *Let G be a connected DLS graph on n vertices with cyclomatic number $c \leqslant n-5$, and $\overline{G}$ is connected. If $G \times K_1$ is DLS, then $G \times K_r$ is DLS for any positive integer r.*

Proof. Assume that $G \times K_r$ has a cospectral mate H. By Theorem 3.2 we have $H = N \times 2K_1 \times K_{r-1}$, where N is a graph on $n-1$ vertices. Lemma 2.3 implies that $G \times K_1$ and $N \times 2K_1$ are L-cospectral. Since $G \times K_1$ is DLS, we know that $N \times 2K_1$ is isomorphic to $G \times K_1$. So $\overline{N} \cup K_2$ is isomorphic to $\overline{G} \cup K_1$. By $\overline{G}$ is connected we have $\overline{G} = K_2$, a contradiction to G is connected. Hence $G \times K_r$ has no cospectral mates, i.e., $G \times K_r$ is DLS. □

Theorem 3.4. *Let G be a connected DLS graph on n vertices with cyclomatic number $c \leqslant n-5$, $\overline{G}$ is connected, the maximum degree of G is smaller than $\frac{n-2}{2}$. Then $G \times K_r$ is DLS for any positive integer r.*

Proof. If $G \times K_r$ has a cospectral mate, by Theorem 3.2, we know that $n-2$ is a Laplacian eigenvalue of G. Let $\mu(G)$ be the L-index of G, then $\mu(G) \geqslant n-2$. Since the maximum degree of G is smaller than $\frac{n-2}{2}$, by Lemma 2.7, we have $\mu_1 < n-2$, a contradiction. Hence $G \times K_r$ has no cospectral mates, i.e., $G \times K_r$ is DLS. □

Theorem 3.5. *Let G be a connected DLS graph on n vertices with cyclomatic number $c \leqslant n-5$, $\overline{G}$ is connected, the vertex connectivity $\kappa(G) = 1$. Then $G \times K_r$ is DLS for any positive integer r.*

Proof. Since $\overline{G}$ is connected, G is not a complete graph. Let $a(G)$ be the algebraic connectivity of G. If $G \times K_r$ has a cospectral mate, by Theorem 3.2, we have $a(G) = 1$. Since $\kappa(G) = 1$, by Lemma 2.6, G has a vertex v such that v is adjacent to every other vertex of G. In this case, $\overline{G}$ is disconnected, a contradiction to $\overline{G}$ is connected. Hence $G \times K_r$ has no cospectral mates, i.e., $G \times K_r$ is DLS. □

An ∞-graph, denoted by $G_{s,t}$, is a graph consisting of cycles C_s and C_t with just one vertex in common (see Fig. 1). Clearly an ∞-graph has cyclomatic number 2. If $G_{s,t}$ has no triangles, then it has at least 7 vertices and its complement is connected. It is known that an ∞-graph $G_{s,t}$ without triangles is DLS (cf. [21, Theorem 5.1]). Theorem 3.5 implies that $G_{s,t} \times K_r$ is DLS if $G_{s,t}$ has no triangles.

Corollary 3.6. *Let G be a DLS tree on n vertices and $n \geqslant 5$. Then $G \times K_r$ is DLS for any positive integer r.*

Proof. If $\overline{G}$ is disconnected, then $G = K_{1,n-1}$. By Lemma 2.10, $G \times K_r$ is DLS. If $\overline{G}$ is connected, by Theorem 3.5, $G \times K_r$ is DLS. □

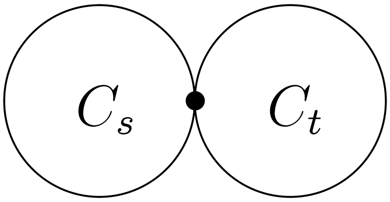

Fig. 1. The ∞-graph $G_{s,t}$.

Some DLS trees are given in [15,1,13,18].

Corollary 3.7. *Let G be a DLS unicyclic graph on n vertices and* $n \geqslant 6$. *Then* $G \times K_r$ *is DLS when G is not a cycle of order* 6.

Proof. If $\overline{G}$ is disconnected, by $n \geqslant 6$, we have $G = (K_2 \cup (n-3)K_1) \times K_1$. By Lemma 2.11, $G \times K_r$ is DLS. So we only need to consider the case that $\overline{G}$ is connected. If G is a cycle, by Lemma 2.12, $G \times K_r$ is DLS. If G is not a cycle, then the vertex connectivity of G is 1. By Theorem 3.5, $G \times K_r$ is DLS. □

Some DLS unicyclic graphs can be found in [8,3,11].

4. Some observations

Let G be a connected DLS graph on n vertices, and G satisfies the conditions given in Theorem 3.2. If $G \times K_r$ has a cospectral mate, by Theorems 3.2 and 3.5, the following facts hold.

(1) $n-2$ is a Laplacian eigenvalue of G.
(2) The algebraic connectivity of G is 1, and its multiplicity is at least 2.
(3) G is a 2-connected graph. (If G has a cut vertex, by Theorem 3.5, G is DLS.)
(4) Let $\mu(G)$ be the L-index of G, then $n-2 \leqslant \mu(G) < n$. (Since $\overline{G}$ is connected, by Lemma 2.2, we have $\mu(G) < n$.)

Most known connected DLS graphs do not satisfy the above four facts simultaneously (most known connected DLS graphs have cut vertices). In [24], Zhang et al. showed that wheel graph $C_6 \times K_1$ has a cospectral mate $(2K_2 \cup K_1) \times 2K_1$. Cycle C_6 is a DLS graph satisfying the conditions of Theorem 3.2. Graph $2K_2 \cup K_1$ has 5 vertices and 2 edges. The Laplacian eigenvalues of C_6 are 4, 3, 3, 1, 1, 0. Clearly $C_6 \times K_r$ and $(2K_2 \cup K_1) \times 2K_1 \times K_{r-1}$ satisfy the conditions of part (b) of Theorem 3.2.

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