K-Inductive Proof

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For our base case refer to Dr. Cameron's proof that the traditional imploding star Laplacian becomes the identity under the Q transform. Symbolically, if G is an imploding star, then $Q^T L(G)Q = I$, and thus the numerical range of $Q^T LQ$ is $\{1\}$. For our inductive step, we take the inductive statement

$$P(k) := \text{The k- imploding star has a singleton numerical range of } \{k\}$$

Assuming we have two 1 - imploding stars on a graph with n vertices, we see that one vertex, the one which is being "imploded" upon, has in-degree $k-1$, and all others have out degree 1. We know that for the definition of the algebraic connectivity, α and the similarly defined β in [**?**] Wu, that for two graphs A and B

$$\alpha(A) + \alpha(B) \leq \alpha(A \cup B) \leq \beta(A \cup B) \leq \beta(A) + \beta(B)$$

We take an medium strength inductive hypothesis that $P(G_{k-1})$and $P(G_1)$, where G_k is the $k-1$ imploding star and G_1 is the traditional 1 imploding star, imploding on vertex m. We consider both G_k and G_1 to be graphs on n vertices. We also assume that $k < n$, and deal with the case that $k = n$ (the complete graph) separately. Let us consider the outdegrees of each of these graphs.

For G_1 we have out degree 0 on vertex m, and out degree 1 on all other vertices, which all point to vertex m. In G_{k-1}, we have $k-1$ vertices with outdegree $k-2$ each, and we call these the vertices imploded upon. The other $n-(k-1)$ vertices will have outdegree $k-1$ each, and we will call these the imploding vertices. We specify without loss of generality that the vertex imploded upon, m, is an imploding vertex in G_{k-1}.

Now we consider $G_z = G_{k-1} \cup G_1$. We see that each vertex in G_{k-1} except m now has an additional edge pointing toward m in G_z, which it did not have before, since we specified that m is an imploding vertex in G_{k-1}. Thus, we see that in G_z each imploding vertex now has outdegree $k-2+1 = k-1$, and there are now $n-(k-1)-1 = n-k$ imploding vertices. Furthermore, we see that since m is a vertex imploded upon in G_1, it must be a vertex imploded on in G_z since $E(G_1) \subseteq E(G_z)$, so G now has $k-1+1 = k$ vertices imploded upon each with outdegree $k-2+1 = k-1$. We see, as we would expect, that the number of imploding vertices $n-k$ plus the number of vertices imploded upon k sum to n.

Knowing that G_z has k vertices imploded upon and $n-k$ imploding vertices characterizes it as G_k. Revisiting out inequalities, we know that in the case of $NR(G_1) = \{1\}, NR(G_{k-1} = \{k-1\}$. Further, since α and β are the real endpoints of the numerical range by [**?**] Cameron?, we see that $\alpha(G_1) = \min NR(G_1) = \min(\text{Re}\{1\}) = 1$ and $\beta(NR(G_1)) = \max \text{Re}(\{1\}) = 1$, and also that $\alpha(G_{k-1}) = \min NR(G_{k-1}) = \min(\text{Re}\{k-1\}) = k-1$ and $\beta(NR(G_{k-1})) = \max \text{Re}(\{k-1\}) = k-1$. Applying this to our inequalities, we see

$$
\begin{aligned}
\alpha(G_1) + \alpha(G_{k-1}) \leq \alpha(G_1 \cup G_{k-1}) \leq \beta(G_1) + \beta(G_{k-1}) \leq \beta(G_1) + \beta(G_{k-1}) \\
1 + k - 1 \leq \alpha(G_k) \leq \beta(G_k) \leq 1 + k - 1 \\
k \leq \alpha(G_k) \leq \beta(G_k) \leq k \\
\implies \alpha(G_k) = \beta(G_k) = k
\end{aligned}
$$

Using the fact that the numerical range is bounded above and below by α and β, we see that for all $x \in \text{NR}(Q^T L(G_k) Q)$.

$$
\begin{aligned}
\alpha(G_k) \leq \text{Re}(x) \leq \beta(G_k) \\
k \leq \text{Re}(x) \leq k
\end{aligned}
$$

which implies that $Re(x) = k$ for all $x \in \text{NR}(Q^T L(G_k) Q)$.

If we take the matrix in Frobenius normal form, based on the characterization above, we konw that it can be decomposed into two blocks. The first block, which is upper triangular, comes from the imploding vertices, all of which have outdegree and diagonal element k. The second block, which is square comes from the vertices being imploded upon, all of which have diagonal element/ outdegree $k-1$, and -1 elsewhere along the row. This block has eigenvalues k and 0. Thus, $\sigma((Q^T L(G_k) Q)) \subseteq \mathbb{R}$.

Since we have determined that the real values of the numerical range are restricted to the singleton $\{k\}$, we know that the numerical range must either be a line or a point in the complex plane. Using the contrapositive of Theorem 7 of Psarrakos, we know that if λ is not an eigenvalue of A, then λ is not a corner of A. Since we have no complex eigenvalues, we can conclude we have no complex points in our Numerical Range, so it must be a single real point rather than a line. Thus, $\text{Re}(x) = x$ for $x \in \text{NR}(Q^T L(G_k) Q)$. This allows us to apply our bound on the real part of the numerical range to the entire numerical range, so we see $k \leq x \geq k$, which implies that $x = k$ for all $x \in \text{NR}(Q^T L(G_k) Q)$, so $\text{NR}(Q^T L(G_k) Q) = \{k\}$. We have shown that $P(1) \wedge P(k-1) \implies P(k)$, and the proof of $P(k)$ for all $kin\mathbb{N}$ is complete by induction.