There is an interesting observation which can be made concerning the result of these problems. Solutions of the systems given by Equations (5) and (12) which are differentiable in the interval $0<\xi<2b^2$ and $0<\xi<b$ exist only where $2b^2$ and b, respectively, are less than unity. That this should be true can be observed by referring to the original problems. These problems are associated with certain hyperbolic differential equations. The solutions of such equations are completely determined inside a certain domain of influence* when only the initial conditions and the boundary conditions at $x=0$ are stated. Thus if the curve $x_0=(1+bt)^2$ [in the first problem] or $x_0=1+bt$ [in the second] should lie within this domain then, in general, no continuous solution can exist which obeys the given conditions on $x=x_0(t)$. Physically this states that the support moves along the string at a speed greater than the wave propagation speed associated with the phenomenon under investigation. Thus the lack of existence of solutions outside a certain range of b is consistent with the physical facts.

A RECURRING THEOREM ON DETERMINANTS

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1. Introduction. This note concerns a theorem (Theorem I) on determinants **[0–21, 25–27]** of which proofs are being published again and again; on the other hand, the theorem is not as well known as it deserves to be. The theorem has arisen in many varied connections as is indicated by the titles of the papers quoted. Although it can be proved in a very simple manner, some of the proofs that have been given are very complicated. The theorem deals with determinants of matrices with a "dominant" main diagonal. Such matrices are particularly useful.

In what follows the theorem and several generalizations are discussed. A rather important application to estimating characteristic roots of general matrices with complex elements is mentioned. By applying these estimates to the matrices with a "dominant" main diagonal more general results are obtained.

2. Complex matrices. It will be convenient to denote by A_i the sum of the moduli of the non-diagonal terms of the ith row of a matrix $\mathbf{A}=(a_{ij})$.

THEOREM I. *If (a_{ik}) is an $n\times n$ matrix with complex elements such that*

$$|a_{ii}| > A_i, \qquad i = 1, \cdots, n \tag{1}$$

then $|a_{ik}| \neq 0$.

* In the x, t, plane.

Proof. Assume that $|a_{ik}|=0$. The system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= 0 \end{aligned} \tag{2}$$

then has a non-trivial solution $x_1, \cdots, x_n$. Let r be one of the indices for which $|x_i|$, $(i=1, \cdots, n)$, is maximum. Consider the rth equation in System (2). It implies

$$|a_{rr}||x_r| \leqq \sum_{k=1, k\neq r}^{n} |a_{rk}||x_k| \leqq |A_r||x_r|$$

which is in contradiction with (1). Hence $|a_{ik}|\neq 0$.

THEOREM II. *Let (a_{ik}) be an $n\times n$ matrix with complex elements such that*

$$|a_{ii}| \geqq A_i, \qquad i = 1, \cdots, n \tag{3}$$

with equality in at most $n-1$ cases. Assume further that the matrix cannot be transformed to a matrix of the form

$$\begin{pmatrix} P & U \\ 0 & Q \end{pmatrix}$$

by the same permutation of the rows and columns, where P and Q are square matrices and 0 *consists of zeros. It follows that the determinant $|a_{ik}|\neq 0$.*

Proof. The proof is similar to that for Theorem I. Assume, for example, that the first relation in (3) is not an equality. From this it follows that $|x_r|>|x_k|$ for at least one value of k. Hence the rth equation of (2) is in contradiction with (3), provided not all $a_{ri}=0$ for which $|x_r|>|x_i|$. If this, however, is the case, then the rth row contains $n-s$ zeros where s is the number of suffixes j for which $|x_j|=|x_r|$. All the s corresponding rows contain $n-s$ zeros in the same places. It follows that the matrix is of the form which was excluded.

3. Real matrices. If all the relations in (3) become equalities the theorem ceases to hold, as is shown by any real matrix (a_{ik}) with $a_{ik}\leqq 0$ for $i\neq k$ and $\sum_{k=1}^{n} a_{ik}=0$, $i=1, \cdots, n$.

In this connection the following result was established: **[6, 10, 14]**

THEOREM III. *Let (a_{ik}) be a real $n\times n$ matrix such that $a_{ii}\geqq 0$ and $a_{ik}\leqq 0$ for $i\neq k$. Assume in addition that*

$$a_{ii} \geqq A_i, \qquad i = 1, \cdots, n \tag{4}$$

and that the matrix is not of the type excluded in Theorem II. The determinant then vanishes if and only if $\sum_{k=1}^{n} a_{ik}=0$, $i=1, \cdots, n$.

Proof. The determinant obviously vanishes if $\sum_{k=1}^{n} a_{ik}=0$. Assume, conversely, that the determinant vanishes. Consider the System (2). From the arguments used for the proof of Theorem I and II it follows that $|x_1|=|x_2| = \cdots = |x_n|$. Any equation of the system (2) then implies $\sum_{k=1}^{n} a_{ik}=0$.

THEOREM IV. *If (a_{ik}) is a real $n\times n$ matrix such that*

$$a_{ii} > A_i, \qquad i = 1, \cdots, n \tag{5}$$

then $|a_{ik}|>0$.

Proof. This theorem can be proved by induction [12]. A different proof is obtained by using the fact that Theorem IV is obviously true if $a_{ik}=0$ for $i\neq k$. Using this and Theorem I a proof can be obtained by continuity arguments.

4. Generalizations.

THEOREM V. *If (a_{ik}) is an $n\times n$ matrix such that*

$$|a_{ii}||a_{kk}| > A_iA_k, \qquad i, k = 1, \cdots, n; i \neq k, \tag{6}$$

then $|a_{ik}|\neq 0$. [14, 23]

Proof. Note that the relations (6) imply $|a_{ii}|>A_i$ for all i but one. If these inequalities are satisfied for all i, the relations (1) hold. In this case the theorem is known. Assume, for example, that

$$|a_{11}| < A_1; \quad |a_{ii}| > A_i, \qquad i = 2, \cdots, n;$$
$$|a_{ii}||a_{kk}| > A_iA_k, \quad i, k = 1, \cdots, n; i \neq k.$$

Without loss of generality it may be assumed that $a_{11}=1$ so that the above relations can be replaced by

$$1 < A_1; \quad |a_{ii}| > A_i, \qquad i = 2, \cdots, n;$$
$$|a_{ii}| > A_1A_i, \qquad i = 2, \cdots, n;$$
$$|a_{ii}||a_{kk}| > A_iA_k, \quad i, k = 2, \cdots, n; i \neq k.$$

Multiply the first column of the matrix (a_{ik}) by A_1. It will be sufficient to prove that this new matrix is non-singular. Denote its elements by a'_{ik} the numbers corresponding to A_i by A'_i. The following inequalities hold:

$$|a'_{11}| = A'_1 = A_1; \qquad |a'_{ii}| > A'_i, \qquad i = 2, \cdots, n.$$

Since the matrix (a'_{ik}) satisfies (3), it follows that $|a'_{ik}|\neq 0$. Hence also $|a_{ik}|\neq 0$.

THEOREM VI. *Theorems I and II are best possible insofar as the inequalities involved cannot be replaced by weaker ones.*

Proof. Suppose that

$$|a_{11}| + \epsilon > A_1; \qquad |a_{ii}| > A_i, \qquad i = 2, \cdots, n,$$

where $\epsilon > 0$, but is arbitrarily small. The result follows because the matrix

$$\begin{pmatrix} \epsilon/2 & \epsilon \\ 1/2 & 1 \end{pmatrix}$$

for which these relations are satisfied is clearly singular.

5. Application. If Theorem II is applied to the characteristic determinant of any $n \times n$ matrix (a_{ik}) with complex coefficients it follows that the characteristic roots must lie inside the circles with centres a_{ii} and radii A_i [**9, 10, 14, 18, 22, 24, 25, 28–30**]. A boundary point can only be a characteristic root if it is also on the boundary of the $n-1$ other circles.

Similarly, the application of Theorem VI shows that the roots lie inside or on the boundary of a set of $n(n-1)/2$ Cassini ovals.

Now apply the circles in particular to a real matrix (a_{ik}) of the type considered in Theorems III and IV. These circles may pass through the origin, but otherwise lie entirely to the right of the imaginary axis. This gives

THEOREM VII. *All the non-zero characteristic roots of matrices with real elements which satisfy* (4) *or* (5) *have positive real parts.*

If none of the non-diagonal elements is positive it has been shown that the root with the smallest real part is real [**10**].

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