Graph Laplacian and the Rankability of Data

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Weighted Directed Graphs

Let $\Gamma = (V, E, w)$ be a weighted directed graph (simple).

- The out degree of vertex i is defined by $d_i^{out} = \sum_{j \in V} w_{ij}$.
- Γ is weakly connected if replacing all of its edges with undirected edges produces a connected undirected graph.
- Γ is strongly connected if for any pair of vertices $i \neq j$, there is a path from i to j and a path from j to i.

Isolated Vertices and Subgraphs

Let $\Gamma' = (V', E', w')$ be an induced subgraph of Γ.

- Vertex i is isolated if $w_{ij} = 0$ for all $j \in V$.
- Vertex i is quasi-isolated if $\sum_{j \in V} w_{ij} = 0$.
- Subgraph Γ' is isolated if $w_{ij} = 0$ for all $i \in V'$ and $j \notin V'$.
- Subgraph Γ' is quasi-isolated if $\sum_{j \in V \setminus V'} w_{ij} = 0$ for all $i \in V'$.

Graph Laplace Operator

The Laplace operator Δ of the graph Γ is defined by

$$\Delta f(i) = \begin{cases} f(i)d_i^{out} - \sum_j w_{ij} f(j) & \text{if } d_i^{out} \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

where f is a complex valued function on V.

Reduced Graph Laplace Operator

Let $V_R \subseteq V$ be the set of vertices that are not quasi-isolated. The reduced Laplace operator Δ_R is defined by

$$\Delta_R f(i) = f(i) d_i^{out} - \sum_{j \in V_R} w_{ij} f(j), \ i \in V_R,$$

where f is a complex valued function on V_R and d_i^{out} is the out degree of vertex i in Γ.

Frobenius Normal Form

The Frobenius normal form of the Laplace operator:

$$\Delta = \begin{pmatrix} \Delta_1 & \Delta_{12} & \cdots & \Delta_{1z} \\ 0 & \Delta_2 & \cdots & \Delta_{2z} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_z \end{pmatrix},$$

where $\Delta_1, \ldots, \Delta_z$ are square matrices corresponding to the strongly connected components of Γ.

Basic Properties

Let $\sigma(\Delta)$ denote the set of eigenvalues of the Laplace operator Δ. Then,

- $0 \in \sigma(\Delta)$,
- $\sigma(\Delta)$ is symmetric with respect to the real axis,
- $\sigma(\Delta) = \sigma(\Delta_R) \cup \{0, \text{ repeated } |V \setminus V_R| \text{ times}\}$,
- $\sigma(\Delta) = \bigcup_i^z \sigma(\Delta_i)$,
- $\sigma(\Delta)$ is the union of the spectra of the Laplace operator reduced to the weakly connected components of Γ.

Spanning Tree

A graph Γ is said to have a spanning tree if there exists a vertex from which all other vertices can be reached following directed edges.

Proposition

Every graph Γ contains at least one isolated strongly connected component. Furthermore, Γ contains exactly one isolated strongly connected component if and only if Γ contains a spanning Tree.

Isolated Subgraph

Theorem

Let Γ be a directed graph with non-negative weights, and let Γ_i, $1 \leq i \leq z$, be its strongly connected components. Then, zero is an eigenvalue (in fact a simple eigenvalue) of Δ_i if and only if Γ_i is isolated.

Corollary

For directed graphs Γ with non-negative weights, the following are equivalent:

i. *The multiplicity (algebraic) of the zero eigenvalue of Δ is equal to k.*
ii. *There exist k isolated strongly connected components in Γ.*
iii. *The minimum number of directed trees needed to span the whole graph is equal to k.*

Spectral Characterization

Theorem

Let Γ be a directed graph on n-vertices with non-negative weights. Then, $\sigma(\Delta) = \{d_1^{out}, \ldots, d_n^{out}\}$ if and only if Γ is acyclic.

Corollary

Let Γ be a directed graph on n-vertices with binary weights. Then, $\sigma(\Delta) = \{d_1^{out}, \ldots, d_n^{out}\}$, where $d_i^{out} = n - i$ for all $i = 1, \ldots, n$, if and only if Γ is a perfect dominance graph.

Spectral Distance

Let A have eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\tilde{A}$ have eigenvalues $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$.

- The spectral variation of $\tilde{A}$ with respect to A is

$$\mathrm{sv}_A(\tilde{A}) = \max_i \min j \left|\tilde{\lambda}_i - \lambda_j\right|.$$

- The Hausdorff distance between the eigenvalues of A and $\tilde{A}$ is

$$\mathrm{hd}(A, \tilde{A}) = \max\{sv_A(\tilde{A}), \mathrm{sv}_{\tilde{A}}(A)\}$$

- The matching distance between the eigenvalues of A and $\tilde{A}$ is

$$\mathrm{md}(A, \tilde{A}) = \min_\pi \max_i \left|\tilde{\lambda}_{\pi(i)} - \lambda_i\right|$$

Rankability Measure

Example

Graph	*Hausdorff*	*Matching*
DominanceGraph	0.0	0.0
PerturbedDominanceGraph	0.62	0.62
PerturbedRandomGraph	1.23	1.70
NearlyDisconnected	2.0	2.0
Random	2.65	2.65
Cyclic	3.0	3.0
CompletelyConnected	3.0	5.0
EmptyGraph	5.0	5.0