9

Combinatorial Matrix Algebra

9.1 The Determinant

We begin this chapter by examining in more detail the definition of the determinant function from a combinatorial point of view.

Let

$$A = [a_{ij}], \quad (i, j = 1, 2, \ldots, n)$$

be a matrix of order n. Then the determinant of A is defined by the formula

$$\det(A) = \sum_{\pi} (\operatorname{sign} \pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \tag{9.1}$$

where the summation extends over all permutations π of $\{1, 2, \ldots, n\}$. Suppose that the permutation π consists of k permutation cycles of sizes $\ell_1, \ell_2, \ldots, \ell_k$, respectively, where $\ell_1 + \ell_2 + \cdots + \ell_k = n$. Then $\operatorname{sign} \pi$ can be computed by

$$\operatorname{sign} \pi = (-1)^{\ell_1 - 1 + \ell_2 - 1 + \cdots + \ell_k - 1} = (-1)^{n-k} = (-1)^n (-1)^k. \tag{9.2}$$

Let D_n be the *complete digraph* of order n with vertices $\{1, 2, \ldots, n\}$ in which each ordered pair (i, j) of vertices forms an arc of D_n. We assign to each arc (i, j) of D_n the weight a_{ij} and thereby obtain a weighted digraph. The *weight of a directed cycle*

$$\gamma : i_1 \to i_2 \to \cdots \to i_t \to i_1$$

is defined to be

$$-a_{i_1 i_2} \cdots a_{i_{t-1} i_t} a_{i_t i_1},$$

the negative of the product of the weights of its arcs.

Let π be a permutation of $\{1, 2, \ldots, n\}$. The *permutation digraph* $D(\pi)$ is the digraph with vertices $\{1, 2, \ldots, n\}$ and with the n arcs

$$\{(i, \pi(i)) : i = 1, 2, \ldots, n\}.$$

The digraph $D(\pi)$ is a spanning subdigraph of the complete digraph D_n. The directed cycles of $D(\pi)$ are in one-to-one correspondence with the permutation cycles of π and the arc sets of these directed cycles partition the set of arcs of $D(\pi)$. The *weight* $\mathrm{wt}(D(\pi))$ *of the permutation digraph* $D(\pi)$ is defined to be the product of the weights of its directed cycles. Hence if π has k permutation cycles,

$$\mathrm{wt}\,(D(\pi)) = (-1)^k a_{1\pi(1)} a_{2\pi(2)} \cdots a_{\pi(n)}.$$

Using (9.1) and (9.2) we obtain

$$\det(-A) = \sum \mathrm{wt}\,(D(\pi)), \tag{9.3}$$

where the summation extends over all permutation digraphs of order n. Let σ denote a permutation of a subset X of $\{1, 2, \ldots, n\}$. The digraph $D(\sigma)$ is a permutation digraph with vertex set X and is a (not necessarily spanning) subdigraph of D_n with weight equal to the product of the weights of its cycles. (If $X = \emptyset$, the weight is defined to be 1.) Since $\det(I_n - A)$ is the sum of the determinants of all principal submatrices of $-A$, we also have

$$\det(I_n - A) = \sum \mathrm{wt}\,(D(\sigma)), \tag{9.4}$$

where the summation extends over all permutation digraphs whose vertices form a subset of $\{1, 2, \ldots, n\}$.

Similar formulas hold for the permanent. If we define the weight of a directed cycle to be the product of the weights of its arcs, and define the weight $\mathrm{wt}'(D(\sigma))$ of a permutation digraph to be the product of the weights of its directed cycles, then we have

$$\mathrm{per}(A) = \sum \mathrm{wt}'(D(\pi)),$$

and

$$\mathrm{per}(I_n + A) = \sum \mathrm{wt}'(D(\sigma)).$$

We remark that the fact that the determinant of a matrix is equal to the determinant of its transpose is a direct consequence of the formula (9.3) for the determinant. This is because assigning weights to the arcs of D_n using the elements of A^T is equivalent to assigning weights using the elements of A and then reversing the direction of all arcs. The resulting involution on the permutation digraphs of D_n is weight-preserving and hence $\det(A) = \det(A^T)$.

Exercises

1. Let A be the tridiagonal matrix of order n

$$\left[\begin{array}{cccccc} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{array}\right].$$

   Verify that

$$\det(A) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} a^{n-2k}(-bc)^k.$$

2. Let A and B be two matrices of order n. Prove that

$$\det(AB) = \det(A)\det(B).$$

3. Prove that the determinant of a matrix with two identical rows equals 0.

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9.2 The Formal Incidence Matrix

Let

$$X = \{x_1, x_2, \ldots, x_n\}$$

be a nonempty set of n elements, and let

$$X_1, X_2, \ldots, X_m \tag{9.5}$$

be a configuration of m not necessarily distinct subsets of the n-set X. We recall from section 1.1 that the incidence matrix for these subsets is the $(0,1)$-matrix

$$A = [a_{ij}], \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

of size m by n with the property that row i of A displays the subset X_i and column j displays the occurrences of the element x_j among the subsets. The matrix A may be regarded as a matrix over any field F, although we generally think of A as a matrix over the rational or real field.

A more general incidence matrix for our configuration may be obtained by replacing the 1's in A by not necessarily identical nonzero elements of the field F. Thus the nonzero elements of A now play the role of the identity element of F. In this generality every m by n matrix A over F may serve as an incidence matrix of some configuration of m subsets of the n-set X. In the discussion that follows A is an arbitrary matrix over F of size m by n.

Let

$$Z = [z_{ij}], \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

be a matrix of size m by n whose elements are mn independent indeterminates z_{ij} over the field F. The Hadamard product

$$M = A * Z = [a_{ij}z_{ij}], \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

is called a *formal incidence matrix* of the configuration (9.5). The elements of M belong to the polynomial ring

$$F^* = F[z_{11}, z_{12}, \ldots, z_{mn}],$$

and the nonzero elements of M are independent indeterminates over F.

We now call an m by n matrix

$$M = [m_{ij}], \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

a *generic matrix*, with respect to the field F, provided its nonzero elements are independent indeterminates over the field F. Every generic matrix with respect to F may serve as a formal incidence matrix. In that which follows we frame our discussion in terms of generic matrices with respect to a field F and work in the polynomial ring F^* obtained by adjoining the nonzero elements of M to F.

The term rank of the (0,1)-matrix A has been defined as the maximal number of 1's in A with no two of the 1's on a line. This important combinatorial invariant is equal to an algebraic invariant of the formal incidence matrix $A * Z$. The rank of $A * Z$ equals the maximal order of a square submatrix with a nonzero determinant. A square submatrix of $A * Z$ of order r has a nonzero determinant if and only if the corresponding submatrix of A has term rank r. Thus the term rank of A equals the rank of $A * Z$. (The term rank of an arbitrary matrix is defined as the maximal number of its nonzero elements with no two of the elements on a line.) The above argument implies the following observation of Edmonds[1967].

Theorem 9.2.1. *The term rank of a generic matrix equals its rank.*

The next theorem, already evident in the work of Frobenius[1912,1917] (see also Ryser[1973] and Schneider[1977]), shows that the combinatorial property of full indecomposability of a matrix also has an algebraic characterization. We first prove two simple lemmas about polynomials.

Lemma 9.2.2. *Let $u_1, u_2, \ldots, u_k$ be k independent indeterminates over a field F. Let $p(u_1, u_2, \ldots, u_k)$ be a polynomial in $F[u_1, u_2, \ldots, u_k]$ which is linear in each of the indeterminates $u_1, u_2, \ldots, u_k$. Suppose that there is a factorization*

$$p(u_1, u_2, \ldots, u_k) = p_1(u_1, u_2, \ldots, u_k)p_2(u_1, u_2, \ldots, u_k)$$

where $p_1(u_1, u_2, \ldots, u_k)$ and $p_2(u_1, u_2, \ldots, u_k)$ are polynomials of positive degree in $F(u_1, u_2, \ldots, u_k)$. Then there is a partition

$$\{u_1, u_2, \ldots, u_k\} = \{u_{i_1}, \ldots, u_{i_r}\} \cup \{u_{j_1}, \ldots, u_{j_{k-r}}\}$$

of the set of indeterminates into two nonempty sets such that p_1 is a polynomial in the indeterminates $\{u_{i_1}, \ldots, u_{i_r}\}$, p_2 is a polynomial in the indeterminates $\{u_{j_1}, \ldots, u_{j_{k-r}}\}$ and each of the polynomials p_1 and p_2 is linear in its indeterminates.

Proof. For each $i = 1, 2, \ldots, k$ we may regard the polynomials p, p_1 and p_2 as polynomials over the integral domain $F[u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k]$ in the indeterminate u_i. These polynomials so regarded are linear polynomials and the conclusions follow. $\square$

Let $M = [m_{ij}]$ be a generic matrix of order n. A polynomial with coefficients in F of the form

$$f(M) = \sum c_\pi m_{1\pi(1)} m_{2\pi(2)} \cdots m_{n\pi(n)}$$

where the summation is over all permutations π of $\{1, 2, \ldots, n\}$ is called a *generalized matrix function* over F. A nonzero generalized matrix function is linear in each of its indeterminates and is homogeneous of degree n. The determinant and the permanent are examples of generalized matrix functions.

Lemma 9.2.3. *Let $M = [m_{ij}]$ be a generic matrix of order n and let $f(M)$ be a nonzero generalized matrix function. Suppose that there is a factorization in F^* of the form*

$$f(M) = pq$$

where p and q are polynomials in the nonzero indeterminates m_{ij} with positive degrees r and $n - r$, respectively. Then there are complementary square submatrices M_r and M_{n-r} of M of orders r and $n - r$, respectively, such that p is a generalized matrix function of M_r and q is a generalized matrix function of M_{n-r}.

Proof. It follows from Lemma 9.2.2 that p and q are homogeneous polynomials of degree r and $n-r$, respectively, in distinct indeterminates. Let p_i be a typical nonzero term that occurs in p. Then, apart from a scalar factor, p_i is a product of r elements of M no two from the same line, and hence a product of r elements no two from the same line from a submatrix M_r of M of order r. It follows from the factorization $f = pq$ that a typical nonzero term q_j that occurs in q is, apart from a scalar, a product of $n-r$ elements no two on a line from the complementary submatrix M_{n-r}. Hence q is a generalized matrix function of M_{n-r}. Returning to p we now also see that p is a generalized matrix function of M_r. □

Theorem 9.2.4. *Let M be a generic matrix of order n. Then M is fully indecomposable if and only if* $\det(M)$ *is an irreducible polynomial in the polynomial ring F^*.*

Proof. First assume that M is not fully indecomposable. Then there exist positive integers r and $n-r$ and permutation matrices P and Q of order n such that

$$PMQ = \begin{bmatrix} M_r & O \\ * & M_{n-r} \end{bmatrix}$$

where M_r and M_{n-r} are square matrices of order r and $n-r$, respectively. Hence

$$\det(M) = \pm \det(M_r)\det(M_{n-r})$$

and it follows that $\det(M)$ is not an irreducible polynomial.

Now assume that M is fully indecomposable. Suppose that $\det(M)$ is reducible and thus admits a factorization

$$\det(M) = pq \tag{9.6}$$

into two polynomials p and q of positive degree. By Lemma 9.2.3 there exist complementary submatrices M_r and M_{n-r} of M of orders r and $n-r$, respectively, such that p is a generalized matrix function of M_r and q is a generalized matrix function of M_{n-r}. Let P and Q be permutation matrices such that

$$PMQ = \begin{bmatrix} M_r & * \\ * & M_{n-r} \end{bmatrix}. \tag{9.7}$$

Since M is fully indecomposable it follows (see Theorem 4.2.2) that each nonzero element of M appears in $\det(M)$. Since p and q are generalized matrix functions of M_r and M_{n-r}, respectively, it now follows from the factorization (9.6) that the asterisks in (9.7) correspond to zero matrices and this contradicts the assumption that M is fully indecomposable. □

Theorem 9.2.4 remains true if we replace the determinant function with the permanent function. Almost no change is required in its proof.

It was established in Theorem 4.2.6 that the fully indecomposable components of a matrix of order n with term rank equal to n are uniquely determined to within arbitrary permutations of their lines. As noted by Ryser[1973] this fact follows from the algebraic characterization of full indecomposability given in Theorem 9.2.4. The following corollary is equivalent to the uniqueness conclusion of Theorem 4.2.6.

Corollary 9.2.5. *Let M be a generic matrix of order n such that* $\det(M) \neq 0$. *Then the set of fully indecomposable components of M is uniquely determined apart from arbitrary permutations of the lines of each component.*

Proof. Suppose that $E_1, E_2, \ldots, E_r$ and $F_1, F_2, \ldots, F_s$ are two sets of fully indecomposable components of M. Then by Theorem 9.2.4 we have that

$$\det(M) = \pm p_1 p_2 \cdots p_r = \pm q_1 q_2 \cdots q_s$$

where the p_i and the q_j are irreducible polynomials equal to the determinant of appropriate submatrices of M. Since F^* is a unique factorization domain, we conclude that $r = s$ and that the p_i and the q_j are the same apart from order and scalar factors. Hence the fully indecomposable components are the same apart from order and line permutations. □

The property of irreducibility of a matrix also admits an algebraic characterization. We now let

$$Y = \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{bmatrix}$$

be the diagonal matrix of order n whose main diagonal elements are independent indeterminates over the field F. The polynomial $\det(A+Y)$ is a polynomial in the polynomial ring $F^\star = F[y_1, y_2, \ldots, y_n]$ which is linear in each of the indeterminates $y_1, y_2, \ldots, y_n$. If f is a polynomial in $F^\star$ which is linear in the indeterminates, then for $\alpha \subseteq \{1, 2, \ldots, n\}$ the coefficient of $\prod_{i\in\alpha} y_i$ in f is denoted by f_α. The following lemma and theorem is from Schneider[1977].

Lemma 9.2.6. *Let $f(y_1, y_2, \ldots, y_n)$ be a polynomial in $F^\star$ which is linear in each of the indeterminates. Suppose that for some integer r with $1 \leq r < n$*

$$f(y_1, y_2, \ldots, y_n) = p(y_1, \ldots, y_r) q(y_{r+1}, \ldots, y_n)$$

where p and q are polynomials that are linear in the indeterminates which they contain. Then

$$f_{\emptyset} f_{\{1,2,\ldots,n\}} = f_{\{1,\ldots,r\}} f_{\{r+1,\ldots,n\}}.$$

Proof. We have

$$f_{\{1,2,\ldots,n\}} = p_{\{1,\ldots,r\}} q_{\{r+1,\ldots,n\}} \qquad \text{and} \qquad f_{\emptyset} = p_{\emptyset} q_{\emptyset}$$

and

$$f_{\{1,\ldots,r\}} = p_{\{1,\ldots,r\}} q_{\emptyset} \qquad \text{and} \qquad f_{\{r+1,\ldots,n\}} = p_{\emptyset} q_{\{r+1,\ldots,n\}}. \qquad \square$$

Theorem 9.2.7. *Let A be a matrix of order n over the field F. Then A is irreducible if and only if* $\det(A+Y)$ *is an irreducible polynomial in the polynomial ring $F^\star$.*

Proof. The polynomial $\det(A+Y)$ contains the nonzero term $y_1 y_2 \cdots y_n$ and hence is a nonzero polynomial. First assume that A is reducible. Then there exist positive integers r and $n-r$ and a permutation matrix P of order n such that

$$P(A+Y)P^T = \begin{bmatrix} A_1 + Y_1 & O \\ * & A_2 + Y_2 \end{bmatrix}$$

where A_1 and A_2 are square matrices of order r and $n-r$, respectively, and Y_1 and Y_2 are diagonal matrices the union of whose diagonal elements is $\{y_1, y_2, \ldots, y_n\}$. The factorization

$$\det(A+Y) = \det(A_1 + Y_1)\det(A_2 + Y_2)$$

of $\det(A+Y)$ into two nonconstant polynomials in $F^\star$ implies that the polynomial $\det(A+Y)$ is reducible.

Now assume that the matrix $A = [a_{ij}]$ is irreducible. Suppose that $\det(A+Y)$ is reducible. Then it follows from Lemma 9.2.2 that there is a partition of the set of indeterminates $\{y_1, y_2, \ldots, y_n\}$ into two nonempty sets U and V and a factorization

$$\det(A+Y) = pq \tag{9.8}$$

where p is a polynomial in the indeterminates of U of positive degree r and q is a polynomial in the indeterminates of V of positive degree $n-r$. The polynomials p and q are linear in the indeterminates of U and V, respectively. Without loss of generality we assume that

$$U = \{y_1, \ldots, y_r\} \qquad \text{and} \qquad V = \{y_{r+1}, \ldots, y_n\}$$

and hence that

$$p = p(y_1, \ldots, y_r) \qquad \text{and} \qquad q = q(y_{r+1}, \ldots, y_n).$$

We now consider the digraph $D_0(A)$ associated with the off-diagonal elements of A. The set of vertices of $D_0(A)$ is taken to be $\{y_1, y_2, \ldots, y_n\}$ and there is an arc from y_i to y_j if and only if $i \neq j$ and $a_{ij} \neq 0$. Since A is irreducible, the digraph $D_0(A)$ is strongly connected. It follows that there is a directed cycle γ containing at least one vertex from $\{y_1, y_2, \ldots, y_r\}$ and at least one vertex from $\{y_{r+1}, y_{r+2}, \ldots, y_n\}$. We choose such a cycle γ of minimal length. Let α be the set of vertices of γ, and let $\alpha_1 = \alpha \cap \{y_1, y_2, \ldots, y_r\}$ and $\alpha_2 = \alpha \cap \{y_{r+1}, y_{r+2}, \ldots, y_n\}$. The minimality of γ implies that the principal submatrix of $A + Y$ determined by the indices in α satisfies

$$\det(A[\alpha] + Y[\alpha]) = \det(A[\alpha_1] + Y[\alpha_1]) \det(A[\alpha_2] + Y[\alpha_2]) + c \qquad (9.9)$$

where c is a nonzero scalar. Let

$$\det(A[\alpha] + Y[\alpha]) = \sum_{\beta \subseteq \alpha} c_\beta \prod_{i \in \beta} y_i.$$

It follows from (9.8) that

$$\det(A[\alpha] + Y[\alpha]) = p'q'$$

where p' is the coefficient of $\prod_{i \in \{1,2,\ldots,r\} - \alpha_1} y_i$ in p and q' is the coefficient of $\prod_{j \in \{r+1,r+2,\ldots,n\} - \alpha_2} y_j$ in q. By Lemma 9.2.6 we have

$$c_\emptyset = c_\emptyset c_\alpha = c_{\alpha_1} c_{\alpha_2}.$$

By (9.9) we may also apply Lemma 9.2.6 to $\det(A[\alpha] + Y[\alpha]) - c$ and conclude that

$$c_\emptyset - c = c_{\alpha_1} c_{\alpha_2}.$$

Combining these last two equations we obtain $c = 0$, a contradiction. Hence $\det(A + Y)$ is irreducible. □

Ryser[1975] has obtained a different algebraic characterization of irreducible matrices.

The following corollary gives another algebraic characterization of fully indecomposable matrices. It is a direct consequence of Theorem 9.2.7 and Theorem 4.2.3.

Corollary 9.2.8. *Let A be a matrix of order n over F and assume that the main diagonal of A contains no 0's. Then A is a fully indecomposable matrix if and only if* $\det(A + Y)$ *is an irreducible polynomial in $F^\star$.*

We now show how the fundamental theorem of Hall[1935](Theorem 1.2.1) on systems of distinct representatives (SDR's) can be proved algebraically. The proof is taken from Edmonds[1967] (see also Mirsky[1971]). As already noted in section 1.2, the fundamental minimax theorem of König[1936]

(Theorem 1.2.1) can be derived from Hall's theorem. A direct algebraic proof of König's theorem is given at the end of this section.

Theorem 9.2.9. *The subsets $X_1, X_2, \ldots, X_m$ of the n-set X have an SDR if and only if the set union $X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_k}$ contains at least k elements for $k = 1, 2, \ldots, m$ and for all k-subsets $\{i_1, i_2, \ldots, i_k\}$ of the integers $1, 2, \ldots, m$.*

Proof. The necessity part of the theorem is obvious, and we turn to the sufficiency of Hall's condition for an SDR. Let M be a generic matrix of size m by n associated with the given subsets of X. The existence of an SDR is equivalent to M having term rank equal to m. By Theorem 9.2.1 it suffices to show that the rows of M are linearly independent over the polynomial ring F^* (equivalently, its quotient field) obtained by adjoining to the field F the nonzero elements of M.

Suppose that the rows of M are linearly dependent. We choose a minimal set of linearly dependent rows, which we assume without loss of generality are the first k rows of M. By Hall's condition M has no zero row and hence $k > 1$. The submatrix of M consisting of its first k rows has rank $k-1$ and hence contains $k-1$ linearly independent columns. Without loss of generality we assume that

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

where M_1 is a k by $k-1$ matrix of rank $k-1$. The rows of M_1 are linearly dependent and hence there exists a row vector $u = (u_1, u_2, \ldots, u_k)$ whose elements are polynomials in the indeterminates of M_1 such that

$$uM_1 = 0.$$

Since the columns of M_2 are linear combinations of the columns of M_1, we also have

$$u \begin{bmatrix} M_1 & M_2 \end{bmatrix} = 0.$$

Since the first k rows of M form a minimal linearly dependent set of rows, each element of u is different from 0. Let

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

be any column of M_2. Then

$$u_1c_1 + u_2c_2 + \cdots + u_kc_k = 0.$$

Since the u_i are polynomials in the indeterminates of M_1 and since the nonzero elements of M are independent indeterminates, we now conclude that each of the c_i's equals 0 and hence that $M_2 = O$. This means that $A_1 \cup A_2 \cup \cdots \cup A_k$ contains at most $k-1$ elements, contradicting Hall's condition. □

We conclude this section by giving the proof of Kung[1984] of the König theorem. This proof makes use of a classical determinantal identity of Jacobi[1841]. We include a proof of Jacobi's identity. The *adjugate* of a square matrix A is denoted by $\mathrm{adj}(A)$.

Lemma 9.2.10. *Let A be a nonsingular matrix of order n with elements in a field F. Let B be a submatrix of order r of $\mathrm{adj}(A)$ and let C be the complementary submatrix of order $n-r$ in A. Then*

$$\det(B) = \det(A)^{r-1} \det(C).$$

In particular, B is nonsingular if and only if C is nonsingular.

Proof. Without loss of generality we assume that A and $\mathrm{adj}(A)$ are in the form

$$A = \begin{bmatrix} * & * \\ * & C \end{bmatrix}, \ \mathrm{adj}(A) = \begin{bmatrix} B & F \\ * & * \end{bmatrix}.$$

We now form

$$\begin{bmatrix} B & F \\ O & I_{n-r} \end{bmatrix} A = \begin{bmatrix} \det(A) I_r & O \\ * & C \end{bmatrix}.$$

Taking determinants we obtain

$$\det(B)\det(A) = \det(A)^r \det(C).$$

□

We now prove the theorem of König in the following equivalent form.

Theorem 9.2.11. *Let M be a generic matrix of size m by n. Then the minimal number of lines in M that cover all the indeterminates of M equals the maximal number of indeterminates of M with no two on the same line.*

Proof. By Theorem 9.2.1 the rank of M equals its term rank, and we let this number be r. Let N be a submatrix of M of order and rank r. Without loss of generality we may assume that N occurs in the upper left corner of M. We permute the first r rows and the first r columns of M so that M assumes the form

$$M = \begin{bmatrix} * & * & M_1 \\ * & U & O \\ M_2 & O & O \end{bmatrix},$$

where the matrix M_1 of size e by $n-r$ has nonzero rows and the matrix M_2 of size $m-r$ by f has nonzero columns.

We first deal with certain degeneracies. In case $r = \min\{m, n\}$ the theorem is clearly valid so that we may take

$$r < \min\{m, n\}.$$

In case $e = 0$ we have zero columns in M and we may cover M with the first r columns. A similar situation holds for $f = 0$. Consider the case in which $e = r$. Suppose that $f > 0$. There is some submatrix of N of order and rank $r-1$ within its last $r-1$ rows. This submatrix along with a nonzero element in column 1 of M_2 and a nonzero element in an appropriate row of M_1 implies that M has rank greater than r. Thus if $e = r$ then $f = 0$, and similarly if $f = r$ then $e = 0$. Hence we may now assume that all nine blocks displayed in our decomposition of M are actually present.

We next write $\mathrm{adj}(N)$ in the form

$$\mathrm{adj}(N) = \begin{bmatrix} W & * \\ * & * \end{bmatrix}$$

where the matrix W is of size e by f. We assert that

$$W = [w_{ij}] = O.$$

Suppose that some $w_{ij} \neq 0$. Now $\mathrm{adj}(N)$ is nonsingular, and by Lemma 9.2.10, $w_{ij} \neq 0$ implies that the submatrix N_{ij} of order $r-1$ of N in the rows complementary to i and the columns complementary to j is nonsingular. But now we may take $r-1$ nonzero elements no two on a line in N_{ij} and add two nonzero elements from M_1 and M_2. We thus increase the term rank of M and this is a contradiction. Hence $W = O$.

We note that

$$e + f \leq r.$$

For if $e + f > r$ then the nonsingular matrix $\mathrm{adj}(N)$ of order r has a zero submatrix of e by f with $e + f > r$, and this contradicts the nonsingularity of $\mathrm{adj}(N)$.

We now prove that

$$\mathrm{rank}(U) \leq r - (e + f).$$

We deny this and suppose that

$$\mathrm{rank}(U) \geq g \text{ where } g = r - (e + f) + 1.$$

This means that U contains a nonsingular submatrix U' of order g. We return to the nonsingular matrix $\mathrm{adj}(N)$. The complementary matrix of U' in $\mathrm{adj}(N)$ is a matrix V of order $e + f - 1$. But by Lemma 9.2.10 V is nonsingular and at the same time contains a zero submatrix of size e by f. This is again a contradiction.

The theorem now follows by induction on the (term) rank of M. By the induction hypothesis we may cover U with $r-(e+f)$ lines and hence M with r lines. □

We remark that Perfect[1966] has given an algebraic proof of a symmetrized form of Theorem 9.2.11 due to Dulmage and Mendelsohn[1958]. A generalization of Theorem 9.2.4 to mixed matrices, matrices whose nonzero elements are either indeterminates or scalars, has been given by Murota [1989]. Hartfiel and Loewy[1984] show that a singular mixed matrix contains a submatrix with special properties.

Exercises

1. Let A and B be fully indecomposable matrices of order n with elements from a field F. Let $M = A*Z$ and $N = B*Z$ be formal incidence matrices associated with A and B, respectively. Suppose that $\det(M) = c\det(N) \neq 0$ where c is a scalar in F. Use Lemma 7.5.3 and prove that there exist diagonal matrices D_1 and D_2 with elements from F such that $D_1AD_2 = B$ (Ryser[1973]).
2. Let A be an m by n matrix. Let r and s be integers with $1 \leq r \leq m$ and $1 \leq s \leq n$. Assume that the r by n submatrix of A determined by the first r rows has term rank r and that the m by s submatrix determined by the first s columns has term rank s. Prove that there exists a set T of nonzero elements of A, no two on the same line, such that T contains one element from each of the first r rows of A and one element from each of the first s columns of A (Dulmage and Mendelsohn[1958] and Perfect[1966]).

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9.3 The Formal Intersection Matrix

We again let

$$\{x_1, x_2, \ldots, x_n\} \tag{9.10}$$

be a nonempty set of n elements. Now we consider two configurations

$$X_1, X_2, \ldots, X_m \tag{9.11}$$

and

$$Y_1, Y_2, \ldots, Y_p \tag{9.12}$$

of subsets of (9.10). Let F be a field. Let A be a general m by n incidence matrix for the subsets (9.11) and let B be a general p by n incidence matrix for the subsets (9.12). Thus the elements of A and B come from the field F, and there is a nonzero element in position (i, j) if and only if x_j is a member of the ith set of the configuration. We now regard the elements $x_1, x_2, \ldots, x_n$ as indeterminates over the field F and consider the diagonal matrix

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} \tag{9.13}$$

of order n. We then form the matrix product

$$Y = Y(x_1, x_2, \ldots, x_n) = AXB^T. \tag{9.14}$$

The matrix Y is of size m by p and we know the structure of Y quite explicitly. Indeed the matrix Y has in its (i, j) position a linear form

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

in the indeterminates $x_1, x_2, \ldots, x_n$ in which the coefficient c_k of x_k is nonzero if and only if x_k is in the intersection $X_i \cap Y_j$. The matrix Y is a *formal intersection matrix* of the configurations (9.11) and (9.12).

Two special cases of (9.14) are of considerable importance in their own right. We set $p = m$ and assume that the configurations (9.11) and (9.12) are identical. Then $B = A$ and (9.14) becomes

$$Y = AXA^T. \tag{9.15}$$

The matrix Y in (9.15) is now a symmetric matrix of order m and has in its (i, j) position a linear form in the indeterminates in the set intersection

$$X_i \cap X_j$$

in which each of these indeterminates appears with a nonzero coefficient. In this case Y is called a *formal symmetric intersection matrix* of the configuration (9.11).

Now let (9.12) be the complementary configuration

$$\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_m$$

whose sets are the complements of the sets of the configuration (9.11). The matrix Y now has in its (i, j) position a linear form in the indeterminates in the set difference

$$X_i - X_j$$

in which each of these indeterminates appears with a nonzero coefficient. In this case we call Y a *formal set difference matrix* of the configuration (9.11).

Now suppose that A and B are the (0,1)-adjacency matrices of the configurations (9.11) and (9.12), respectively. Then the element in the (i, j) position of Y is

$$\sum_{x_k \in X_i \cap X_j} x_k,$$

the sum of the indeterminates in the intersection of X_i and X_j. If we set

$$x_1 = x_2 = \cdots = x_n = 1,$$

then we obtain the basic equation

$$Y(1, 1, \ldots, 1) = AB^T, \tag{9.16}$$

which reveals the cardinalities of the set intersections $X_i \cap X_j$. If Y is the formal symmetric intersection matrix of the configuration, then (9.16) reveals the cardinalities of the set intersections $X_i \cap X_j$. If Y is the formal set difference matrix, then (9.16) reveals the cardinalities of the set differences $X_i - X_j$.

The basic matrix equations (9.14), (9.15) and (9.16) allow one to apply the methods of matrix theory to the study of the intersection pattern of a configuration of sets. The vast area of combinatorial designs is primarily concerned with these equations. We make no attempt in this volume to study this equation in general or to develop the basic theory of combinatorial designs. These topics will be included in a subsequent volume to be entitled *Combinatorial Matrix Classes.* We briefly discuss some applications of these matrix equations.

An application of standard theorems on rank to the formal symmetric intersection matrix Y in (9.15) yields that

$$\begin{aligned}\operatorname{rank}(AA^T) &= \operatorname{rank}(Y(1,1,\ldots,1)) \leq \operatorname{rank}(Y(x_1,x_2,\ldots,x_n)) \\ &\leq \operatorname{rank}(A) = \operatorname{rank}(AA^T).\end{aligned}$$

Hence in the case of a formal symmetric intersection matrix we have

$$\operatorname{rank}(Y) = \operatorname{rank}(A).$$

The following two theorems are from Ryser[1972].

Theorem 9.3.1. *Let $X_1, X_2, \ldots, X_m$ be a configuration of subsets of the n-set*

$$\{x_1, x_2, \ldots, x_n\}$$

and let A be the m by n incidence matrix for this configuration. Assume that the number of distinct nonempty set intersections

$$X_i \cap X_j, \quad (i \neq j)$$

is strictly less than n. Then there exists an integral nonzero diagonal matrix D such that

$$ADA^T = E$$

is a diagonal matrix.

Proof. We consider the formal symmetric intersection matrix

$$AXA^T = Y.$$

Let t denote the number of distinct nonzero elements occurring in the positions of Y not on the main diagonal. The assumption in the theorem implies that $t < n$. We equate to zero these t nonzero elements and this gives us a homogeneous system of t equations in the n unknowns $x_1, x_2, \ldots, x_n$. This system of equations has a nonzero rational solution and hence an integral solution $d_1, d_2, \ldots, d_n$ with at least one d_i different from zero. The desired matrix D is the diagonal matrix whose main diagonal elements are $d_1, d_2, \ldots, d_n$. □

Theorem 9.3.2. *Let A be a $(0,1)$-matrix of order n. Assume that A satisfies the matrix equation*

$$ADA^T = E \tag{9.17}$$

where D and E are complex diagonal matrices and E is nonsingular. Then A is a permutation matrix of order n.

Proof. The assumption that E is nonsingular implies that both A and D are nonsingular matrices. The equation (9.17) implies that we may write

$$ADA^TE^{-1} = DA^TE^{-1}A = I_n.$$

Hence it follows that

$$A^TE^{-1}A = D^{-1}. \tag{9.18}$$

Let the elements on the main diagonal of D and E be $d_1, d_2, \ldots, d_n$ and $e_1, e_2, \ldots, e_n$, respectively. Then inspecting the main diagonal of (9.18) and using the fact that A is a (0,1)-matrix we see that

$$A^T \begin{bmatrix} e_1^{-1} \\ e_2^{-1} \\ \vdots \\ e_n^{-1} \end{bmatrix} = \begin{bmatrix} d_1^{-1} \\ d_2^{-1} \\ \vdots \\ d_n^{-1} \end{bmatrix}. \tag{9.19}$$

We now multiply (9.19) by AD and this gives

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Thus each of the row sums of A equals 1. Because A is a nonsingular (0,1)-matrix, A is a permutation matrix. □

We now consider set differences and first prove a theorem of Marica and Schönheim[1969].

Theorem 9.3.3. *Let $X_1, X_2, \ldots, X_m$ be a configuration of m distinct subsets of $\{x_1, x_2, \ldots, x_n\}$. Then the number of distinct differences*

$$X_i - X_j, \quad (i, j = 1, 2, \ldots, m)$$

is at least m.

Proof. We prove the theorem by induction on m. If $m \leq 2$, the theorem clearly holds. Now let $m \geq 3$. We denote our configuration by $\mathcal{C}$, and we let

$$\Delta(\mathcal{C}) = \{X_i - X_j : i, j = 1, 2, \ldots, m\}$$

be the collection of differences of the configuration $\mathcal{C}$. Let

$$k = \min_{i \neq j}\{|X_i \cap X_j|\}.$$

Without loss of generality we assume that $X_1 \cap X_2 = F$ where $|F| = k$. We partition $\mathcal{C}$ into three configurations:

(i) $\mathcal{C}_1$ consists of those X_i satisfying $F \not\subseteq X_i$.

The remaining sets of the configuration $\mathcal{C}$ contain F and for these sets we write $X_i' = X_i - F$.

(ii) $\mathcal{C}_2$ consists of those X_i not in $\mathcal{C}_1$ for which

$$X_i' \cap X_j' \neq \emptyset \text{ for all } X_j'.$$

(iii) $\mathcal{C}_3$ consists of those X_i not in $\mathcal{C}_1$ for which

$$X_i' \cap X_j' = \emptyset \text{ for some } X_j'.$$

The configuration $\mathcal{C}_3$ contains $t \geq 2$ sets, since X_1 and X_2 are in $\mathcal{C}_3$.

Let X_i be a set in $\mathcal{C}_3$ and let X_j be a set not in $\mathcal{C}_1$ satisfying $X_i' \cap X_j' = \emptyset$. Then X_j is also in $\mathcal{C}_3$. Since $X_i - X_j = X_i$ it follows that for each set X_i in $\mathcal{C}_3$, X_i' is a difference in $\Delta(\mathcal{C}_3)$. We now show that no such X_i' is a difference of the configuration $\mathcal{C}_1 \cup \mathcal{C}_2$ consisting of the sets in $\mathcal{C}_1$ and the sets in $\mathcal{C}_2$. It follows from the definition of $\mathcal{C}_2$ that each set in $\mathcal{C}_2$ has a nonempty intersection with X_i'. Since a set in $\mathcal{C}_1$ does not contain F but has at least k elements in common with X_i, it follows that each set in $\mathcal{C}_1$ also has a nonempty intersection with X_i'. Therefore the difference of two sets in $\mathcal{C}_1 \cup \mathcal{C}_2$ cannot equal X_i'. We thus have the inequality

$$|\Delta(\mathcal{C})| \geq |\Delta(\mathcal{C}_1 \cup \mathcal{C}_2)| + t.$$

Applying the induction hypothesis to the configuration $\mathcal{C}_1 \cup \mathcal{C}_2$ of $m - t \leq m - 2$ sets we complete the proof. □

The following theorem of Ryser[1984] demonstrates how an assumption on the rank of a formal set intersection matrix implies a stronger conclusion than that given in Theorem 9.3.3.

Theorem 9.3.4. *Let*

$$Y = Y(x_1, x_2, \ldots, x_n) = AXB^T$$

be a formal set intersection matrix of size m by p, and assume that the rank of Y equals m. Then Y contains m distinct nonzero elements, one in each of the m rows of Y.

Proof. Let R_i denote the set consisting of the nonzero elements in row i of the matrix $Y, (i = 1, 2, \ldots, m)$. The theorem asserts that the sets $R_1, R_2, \ldots, R_m$ have a system of distinct representatives. By Theorem 9.2.9 it suffices to prove that every k rows of Y contain in their union at least k distinct nonzero elements.

Assume to the contrary that there is an integer k with $1 \le k \le m$ and a k by p matrix Y_k of k rows of Y with the property that Y_k contains fewer than k distinct nonzero elements. Let the distinct nonzero elements in Y_k be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_s$ where $s < k$. Suppose that the element α_i of Y_k occurs exactly e_{ij} times in row j of Y_k. We consider the system of s linear homogeneous equations

$$e_{i1}z_1 + e_{i2}z_2 + \cdots + e_{ik}z_k = 0, \quad (i = 1, 2, \ldots, s) \tag{9.20}$$

in the unknowns $z_1, z_2, \ldots, z_k$. Since $s < k$ this system has an integral solution with not all of the $z_j = 0$. We let $z_1, z_2, \ldots, z_k$ denote such a solution and we multiply (9.20) by α_i and obtain

$$e_{i1}z_1\alpha_i + e_{i2}z_2\alpha_i + \cdots + e_{ik}z_k\alpha_i = 0, \quad (i = 1, 2, \ldots, s). \tag{9.21}$$

We return to the matrix Y_k and multiply row j of Y_k by z_j, $(i = 1, 2, \ldots, k)$. We call the resulting matrix Z_k. It follows from (9.21) that the sum of all of the elements of Z_k equals 0. Because each α_i is a sum of certain of the indeterminates $x_1, x_2, \ldots, x_n$, all terms of the form $z_j x_i$ in Z_k involving a particular indeterminate x_i must also sum to 0.

We now consider the location of these terms within Z_k. It follows from the nature of the formal set intersection matrix that the element x_i is confined to a certain submatrix of Y and appears in every position of that submatrix. (The possibility that x_i does not appear in Y is not excluded.) An entirely similar situation holds concerning the location of all terms of the form $z_j x_i$ in Z_k involving a particular indeterminate x_i. Now we have noted that all such terms within Z_k sum to 0. The location of these terms within Z_k implies that all such terms within a particular column of Z_k must also sum to 0. Hence all of the column sums of Z_k are 0. This implies that there is a dependence relation among the rows of Y_k and this contradicts the hypothesis that Y is of rank m. □

Daykin and Lovász[1976] have refined Theorem 9.3.3 by proving that if in the formal set difference matrix $Y = AX(J - A)^T$ of order $m \ge 2$, the incidence matrix A has distinct rows, then Y has m distinct nondiagonal elements with no two of the nondiagonal elements on a line. Ahlswede and Daykin[1979] (see also Baston[1982]) have extended Theorem 9.3.3 to include more general intersection matrices.

Ryser[1982] has studied the determinant and characteristic polynomial of the formal symmetric intersection matrix. In addition, Ryser[1973] has solved the "inverse problem" for the formal intersection matrix.

Theorem 9.3.5. *Let Y be a matrix of order $n \ge 3$, and assume that each element of Y is a linear form in the n indeterminates $x_1, x_2, \ldots, x_n$ with respect to a field F. Let X be the diagonal matrix of order n whose elements* *on the main diagonal are $x_1, x_2, \ldots, x_n$. Assume that the determinant of Y satisfies*

$$\det(Y) = cx_1x_2\cdots x_n$$

where c is a nonzero element of F. Assume further that each element of Y^{-1} is a linear form in $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$ with respect to the field F. Then there exist matrices A and B of order n with elements in F such that

$$Y = AXB.$$

Exercise

1. Let $Y = Y(x_1, x_2, \ldots, x_n)$ be a matrix of size m by p such that every element of Y is a linear form in the indeterminates $x_1, x_2, \ldots, x_n$ over a field F. Assume that the rank of the matrix $Y(0, \ldots, 0, x_i, 0, \ldots, 0)$ equals 0 or 1 for each $i = 1, 2, \ldots, n$. Prove that Y is a formal intersection matrix, that is, prove that there exist matrices A and B of sizes m by n and p by n, respectively, with elements from the field F such that $Y = AXB^T$ (Ryser[1973]).

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9.4 MacMahon's Master Theorem

Let $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ be a matrix of order n over a field F. As in section 9.2 we let

$$Y = \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{bmatrix}$$

be the diagonal matrix of order n whose main diagonal elements are independent indeterminates over the field F, and we let $F^\star$ denote the polynomial ring $F[y_1, y_2, \ldots, y_n]$. The *Master Theorem for Permutations* of MacMahon[1915] identifies the coefficients $A(m_1, m_2, \ldots, m_n)$ in the expansion

$$\det(I_n - AY)^{-1} = \sum_{(m_1,m_2,\ldots,m_n)} A(m_1, m_2, \ldots, m_n) y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n} \tag{9.22}$$

where the summation extends over all n-tuples $(m_1, m_2, \ldots, m_n)$ of nonnegative integers. The combinatorial proof of the Master Theorem given below is due to Foata[1965] (see also Cartier and Foata[1969]), as it is described by Zeilberger[1985].

Theorem 9.4.1. *The coefficient $A(m_1, m_2, \ldots, m_n)$ in (9.22) equals the coefficent of $y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$ in the product*

$$\prod_{i=1}^{n} (a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n)^{m_i}. \tag{9.23}$$

Proof. We consider general digraphs whose vertex sets are $\{1, 2, \ldots, n\}$, and we denote by $\mathcal{D}$ the collection of all general digraphs D of order n for which each vertex has the same indegree as outdegree. We assume that the arcs of D with the same initial vertex have been linearly ordered. We denote by $\mathcal{H}$ the collection of all digraphs H of order n for which each vertex has the same indegree as outdegree and this common value is either 0 or 1 (H consists of a number of pairwise disjoint directed cycles and hence is a permutation digraph on a subset of $\{1, 2, \ldots, n\}$).

The weight of an arc (i, j) is defined by

$$\mathrm{wt}(i, j) = a_{ij}y_j.$$

The weight $\mathrm{wt}(D)$ of a digraph D in $\mathcal{D}$ is defined to be the product of the weights of its arcs. The weight $\mathrm{wt}(H)$ of a digraph H in $\mathcal{H}$ is defined to be

$$(-1)^{c(H)}(\text{the product of the weights of its arcs}),$$

where $c(H)$ equals the number of directed cycles of H. Notice that a digraph in $\mathcal{H}$ is also in $\mathcal{D}$, and thus the computation of its weight depends on whether it is being regarded as a member of $\mathcal{D}$ or of $\mathcal{H}$. We define

$$\mathrm{wt}\mathcal{D} = \sum_{D\in\mathcal{D}} \mathrm{wt}(D)$$

and

$$\mathrm{wt}\mathcal{H} = \sum_{H\in\mathcal{H}} \mathrm{wt}(H).$$

We consider the cartesian product

$$\mathcal{G} = \mathcal{D} \times \mathcal{H} = \{(D, H) : D \in \mathcal{D}, H \in \mathcal{H}\}$$

and define the weight of the pair (D, H) by

$$\mathrm{wt}(D, H) = \mathrm{wt}(D)\mathrm{wt}(H).$$

The weight of $\mathcal{G}$ is

$$\mathrm{wt}\mathcal{G} = \sum_{(D,H)\in\mathcal{G}} \mathrm{wt}(D, H) = (\mathrm{wt}\mathcal{D})(\mathrm{wt}\mathcal{H}).$$

It follows from section 9.1 that

$$\mathrm{wt}\mathcal{H} = \det(I_n - AY).$$

Let $B(m_1, m_2, \ldots, m_n)$ denote the coefficient of $y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$ in the product (9.23). We now show that

$$\mathrm{wt}\mathcal{D} = \sum_{(m_1,m_2,\ldots,m_n)} B(m_1, m_2, \ldots, m_n) y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}, \tag{9.24}$$

where the summation is over all n-tuples of nonnegative integers $(m_1, m_2, \ldots, m_n)$.

Let $\mathcal{D}_{(m_1,m_2,\ldots,m_n)}$ be the subset of $\mathcal{D}$ consisting of those general digraphs in which vertex i has outdegree (and hence indegree) $m_i, (i = 1, 2, \ldots, n)$. Each term in the expanded product (9.23) is the weight of a general digraph with outdegrees $m_1, m_2, \ldots, m_n$. In order that the general digraph have indegrees equal to $m_1, m_2, \ldots, m_n$ as well, we take only those terms containing $y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$. It follows that the weight of $\mathcal{D}_{(m_1,m_2,\ldots,m_n)}$ equals

$$B(m_1, m_2, \ldots, m_n) y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}.$$

Therefore (9.24) holds and hence

$$\mathrm{wt}\mathcal{G} = \left(\sum_{(m_1,m_2,\ldots,m_n)} B(m_1, m_2, \ldots, m_n) y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n} \right) \det(I_n - AY).$$

To complete the proof we show that $\mathrm{wt}\mathcal{G} = 1$.

We now define a mapping

$$\sigma : \mathcal{G} - \{(\emptyset, \emptyset)\} \to \mathcal{G} - \{(\emptyset, \emptyset)\},$$

where $\emptyset$ denotes the digraph with vertices $\{1, 2, \ldots, n\}$ and with an empty set of edges. Given a pair $(D, H) \neq (\emptyset, \emptyset)$, we determine the first vertex u whose outdegree in either D or H is positive. Beginning at that vertex u

we walk along the arcs of D, always choosing the topmost arc, until one of the following occurs:

(i) We encounter a previously visited vertex (and thus have located a directed cycle γ of D).
(ii) We encounter a vertex which has positive outdegree in H (and thus is a vertex on a directed cycle δ of H).

We note that if u is a vertex with positive outdegree in H then we are immediately in case (ii). We also note that cases (i) and (ii) cannot occur simultaneously. If case (i) occurs, we remove γ from D and put it in H. If case (ii) occurs, we remove δ from H and put it in D in such a way that each arc of γ is put in front of (in the linear order) those with the same initial vertex. Let (D', H') be the pair obtained in this way. Then D' is in $\mathcal{D}$ and H' is in $\mathcal{H}$, and hence (D', H') is in $\mathcal{G}$. Moreover, since the number of directed cycles in H' differs from the number in H by one, it follows that $\mathrm{wt}(D', H') = -\mathrm{wt}(D, H)$. We define $\sigma(D, H) = (D', H')$ and note that $\sigma(D', H') = (D, H)$. Thus $\sigma : \mathcal{G} - \{(\emptyset, \emptyset)\} \to \mathcal{G} - \{(\emptyset, \emptyset)\}$ is an involution which is sign-reversing on the weight. It follows that

$$\mathrm{wt}\mathcal{G} = \mathrm{wt}(\emptyset, \emptyset) = 1,$$

and the proof of the theorem is complete. □

By Theorem 9.4.1 the coefficient $B(m_1, m_2, \ldots, m_n)$ of $y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$ in the product (9.23) equals $A(m_1, m_2, \ldots, m_n)$. We now show that this coefficient can be expressed in terms of permanents. Let $\mathcal{R}$ denote the set of all nonnegative integral matrices $R = [r_{ij}]$ of order n with row sum vector and column sum vector equal to $(m_1, m_2, \ldots, m_n)$. Thinking of r_{ij} as the number of times the y_j term is selected from the factor $(a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n)^{m_i}$ in (9.23), we see that

$$B(m_1, m_2, \ldots, m_n) = \sum_{R \in \mathcal{R}} \left(\prod_{i=1}^{n} \frac{m_i!}{r_{i1}! r_{i2}! \cdots r_{in}!} \right) \prod_{i,j=1}^{n} a_{ij}^{r_{ij}}. \qquad (9.25)$$

The quantity on the right-hand side of (9.25) is easily seen to be equal to

$$\frac{1}{m_1! m_2! \cdots m_n!} \mathrm{per}_{(m_1, m_2, \ldots, m_n)}(A)$$

where $\mathrm{per}_{(m_1, m_2, \ldots, m_n)}(A)$ equals the permanent of the matrix of order $m_1 + m_2 + \cdots + m_n$ obtained from A by replacing each element a_{ij} by the constant matrix of size m_i by m_j each of whose elements equals a_{ij}. We thus obtain the following identity of Vere-Jones[1984].

Corollary 9.4.2.

$$\det(I_n - AY)^{-1} = \sum \frac{1}{m_1! m_2! \cdots m_n!} per_{(m_1, m_2, \ldots, m_n)}(A) y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$$

where the summation extends over all n-tuples $(m_1, m_2, \ldots, m_n)$ *of non-negative integers.*

We conclude this section by presenting an expansion for the determinant of a matrix in terms of the permanents of its principal submatrices and an expansion for the permanent in terms of the determinants of its principal submatrices.

Let S be a subset of the set $\{1, 2, \ldots, n\}$ and as usual let $A[S]$ denote the principal submatrix of A with rows and columns indexed by the elements of S. The set of ordered partitions of S into nonempty sets is denoted by Λ_S. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_j)$ be in Λ_S. The number j of nonempty parts of α is denoted by p_α. We define $\det_\alpha(A)$ and $\mathrm{per}_\alpha(A)$ by

$$\det_\alpha(A) = \prod_{i=1}^{p_\alpha} \det(A[\alpha_i])$$

and

$$\mathrm{per}_\alpha(A) = \prod_{i=1}^{p_\alpha} \mathrm{per}(A[\alpha_i]).$$

The following identity is from Chu[1989].

Theorem 9.4.3. *The matrix A and the diagonal matrix Y of order n satisfy*

$$\det(A+Y) = \sum_{S \subseteq \{1,2,\ldots,n\}} \sum_{\alpha \in \Lambda_S} (-1)^{|S|+p_\alpha} \left(\prod_{i \in \overline{S}} y_i \right) \mathrm{per}_\alpha(A). \qquad (9.26)$$

Proof. The equation (9.26) asserts that for each subset S of $\{1, 2, \ldots, n\}$ we have

$$\det(A[S]) = \sum_{\alpha \in \Lambda_S} (-1)^{|S|+p_\alpha} \mathrm{per}_\alpha(A). \qquad (9.27)$$

Clearly it suffices to prove (9.27) in the case that $S = \{1, 2, \ldots, n\}$, that is to prove that

$$\det(A) = \sum_{\alpha \in \Lambda_{\{1,2,\ldots,n\}}} (-1)^{n+p_\alpha} \mathrm{per}_\alpha(A). \qquad (9.28)$$

Let σ be a permutation of $, 2, \ldots, n$ and suppose that σ has k permutation cycles in its cycle decomposition. We verify (9.28) by showing that the coefficient $c(\sigma)$ of

$$a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

in the expression on the right-hand side is the same as its coefficient sign $\sigma = (-1)^{n+k}$ in $\det(A)$.

Let $\alpha \in \Lambda_{\{1,2,\ldots,n\}}$. If there is some cycle of σ which is not wholly contained in a part of α, then $\mathrm{per}_\alpha(A)$ does not contribute to the coefficient $c(\sigma)$. Therefore

$$c(\sigma) = \sum_{j=0}^{k} (-1)^{n+j} p(k,j)$$

where $p(k,j)$ equals the number of partitions of a set of k distinct elements (the k cycles of σ) into j distinguishable boxes (the parts of α) with no empty box. It follows from the inclusion-exclusion principle (see, e.g., Brualdi[1977] or Stanley[1986]) that

$$p(k,j) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k.$$

Hence

$$\begin{aligned} c(\sigma) &= \sum_{j=0}^{k} (-1)^{n+j} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k \\ &= \sum_{i=0}^{k} (-1)^{n-i}\, i^k \sum_{j=0}^{k} \binom{j}{i} \\ &= \sum_{i=0}^{k} (-1)^{n-i}\, i^k \binom{k+1}{i+1} = (-1)^{n+k}. \end{aligned}$$

The penultimate equation follows from a basic identity for the binomial coefficients. The last equation follows from the identity for divided differences (see, e.g., Stanley[1986]):

$$\sum_{i=0}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} f(i) = 0$$

for a polynomial of degree at most k, in particular for the polynomial $f(x) = (x-1)^k$. □

The next identity, obtained by interchanging the determinant and permanent in (9.26), can be proved in a similar way.

Theorem 9.4.4. *The matrix A and diagonal matrix Y of order n satisfy*

$$\mathrm{per}\,(A+Y) = \sum_{S \subseteq \{1,2,\ldots,n\}} \sum_{\alpha \in \Lambda_S} (-1)^{|S|+p_\alpha} \left(\prod_{i \in \overline{S}} y_i \right) \det_\alpha(A). \tag{9.29}$$

The following recurrence relation involving both the determinant and the permanent was established by Muir[1897].

Theorem 9.4.5. *Let A be a matrix of order n. Then*

$$\sum_{S\subseteq\{1,2,\dots,n\}} (-1)^{|S|}\ \mathrm{per}\,(A[\overline{S}])\det(A[S]) = 0. \tag{9.30}$$

Proof. Using the identity (9.27) in the left-hand side of (9.30), we obtain

$$\sum_{S\subseteq\{1,2,\dots,n\}} \sum_{\alpha\in\Lambda_S} (-1)^{p_\alpha}\mathrm{per}(A[\overline{S}])\mathrm{per}_\alpha(A)$$

$$= \sum_{\alpha\in\Lambda_{\{1,2,\dots,n\}}} (-1)^{p_\alpha}\mathrm{per}_\alpha(A) + \sum_{S\subset\{1,2,\dots,n\}} \sum_{\alpha\in\Lambda_S} (-1)^{p_\alpha}\mathrm{per}(A[\overline{S}])\mathrm{per}_\alpha(A)$$

$$= \sum_{\alpha\in\Lambda_{\{1,2,\dots,n\}}} (-1)^{p_\alpha}\mathrm{per}_\alpha(A) + \sum_{\alpha\in\Lambda_{\{1,2,\dots,n\}}} (-1)^{p_\alpha-1}\mathrm{per}_\alpha(A) = 0.$$

□

Further identities for $\mathrm{per}(I_n - AY)^{-1}$ and $\det(I_n - AY)^{-1}$ and their relation to MacMahon's master theorem can be found in Chu[1989]. We also mention the identity of Vere-Jones[1988] for arbitrary powers of $\det(I_n - AY)$.

Exercises

1. Prove Theorem 9.4.4.
2. Let A and B be two matrices of order n. Prove that

$$\mathrm{per}(A)\det(B) = \sum_\sigma \mathrm{sign}(\sigma)\det(A*(P_\sigma B))$$

where the sum extends over all permutations σ of $\{1,2,\dots,n\}$ and P_σ denotes the permutation matrix of order n corresponding to σ (Muir[1882]).

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9.5 The Formal Adjacency Matrix

Let G denote a graph of order n with vertex set

$$V = \{1, 2, \ldots, n\}. \tag{9.31}$$

As defined in section 2.2 the adjacency matrix $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ of G is a symmetric (0,1)-matrix of order n of trace zero. Row i of A displays the vertices which are adjacent to vertex i. We now let

$$Z = [z_{ij}], \quad (i, j = 1, 2, \ldots, n)$$

be a skew-symmetric matrix of order n whose elements $z_{ij}, (1 \leq i < j \leq n)$ above the main diagonal are independent indeterminates over the field F. Since Z is skew-symmetric we have $z_{ii} = 0, (i = 1, 2, \ldots, n)$ and $z_{ij} = -z_{ji}, (i, j = 1, 2, \ldots, n)$. The Hadamard product

$$M = A * Z = [a_{ij} z_{ij}], \quad (i, j = 1, 2, \ldots, n)$$

is called the *formal adjacency matrix* of the graph G. The nonzero elements of the skew-symmetric matrix M above the main diagonal are independent indeterminates over F. We call such a matrix a *generic skew-symmetric matrix* with respect to the field F. Every generic skew-symmetric matrix is the formal adjacency matrix of a graph. We let F^* denote the polynomial ring obtained by adjoining the nonzero elements above the main diagonal of M to F.

If G is a bipartite graph, the formal adjacency matrix of G can be taken in the form

$$M = \begin{bmatrix} O & M_1 \\ -M_1^T & O \end{bmatrix}. \tag{9.32}$$

The matrix M_1 is a generic matrix of size k by l for some integers k and l with $k + l = n$. Conversely, a generic matrix M_1 determines a formal adjacency matrix of a bipartite graph by means of the equation (9.32).

Let $M = [m_{ij}]\ (i, j = 1, 2, \ldots, n)$ be a generic skew-symmetric matrix of order n. Then $M^T = -M$ and hence $\det(M) = (-1)^n \det(M)$. Hence if n is odd then $\det(M) = 0$ and M is a singular matrix. We henceforth assume that n is even.

Let K_n denote the complete graph with vertex set (9.31). Let $\alpha = \{i, j\}$ be an edge of K_n where $i < j$. We define the *weight* of α by

$$\text{wt}\,\alpha = \text{wt}\{i, j\} = m_{ij}, \quad (1 \leq i < j \leq n).$$

Let

$$L = \{\{i_1, i_2\}, \{i_3, i_4\}, \ldots, \{i_{n-1}, i_n\}\}$$

be a perfect matching (1-factor) of K_n. To standardize the notation, we assume that

$$i_1 < i_2, i_3 < i_4, \ldots, i_{n-1} < i_n; i_1 < i_3 < \cdots < i_{n-1}.$$

To the perfect matching L we let correspond the permutation

$$\pi_L = (i_1, i_2, i_3, i_4, \ldots, i_{n-1}, i_n)$$

of $\{1, 2, \ldots, n\}$. The weight of the perfect matching (1-factor) L is defined to be

$$\text{wt}(L) = (\text{sign}\pi_L) \prod_{\alpha \in L} \text{wt}(\alpha),$$

the signed product of the weights of its edges. Since M is skew-symmetric, the weight of a perfect matching L depends only on L and not on the order in which the vertices are listed in the edges or the order in which its edges are listed. We denote the set of perfect matchings in K_n by $\mathcal{F}$. The size of the set $\mathcal{F}$ is $1 \cdot 3 \cdot 5 \cdots (n-1)$.

The classical definition of the *pfaffian* of the matrix M is equivalent to the following:

$$\text{pf}(M) = \sum_{L \in \mathcal{F}} \text{wt}(L), \tag{9.33}$$

the sum of the weights of the perfect matchings of K_n. The pfaffian of M is a linear form in the indeterminates m_{ij} that appear in M and is homogeneous of degree $n/2$. Since the weight of a perfect matching in $\mathcal{F}$ is zero if it contains an edge not belonging to the graph G, it suffices in (9.33) to sum only over the perfect matchings in G. The following theorem is now a consequence of the definitions involved.

Theorem 9.5.1. *The graph G has a perfect matching if and only if the pfaffian of its formal adjacency matrix is not identically zero.*

Pfaffians occupy an obscure position in matrix theory. The pfaffian of the skew-symmetric matrix $M = [m_{ij}]$ of order $n = 4$ is

$$m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23},$$

the three terms corresponding, respectively, to the three perfect matchings of K_n

$$\{\{1,2\},\{3,4\}\},\{\{1,3\},\{2,4\}\},\{\{1,4\},\{2,3\}\}.$$

Unlike determinants, pfaffians lack a simple multiplication formula. Their usefulness in combinatorial problems is demonstrated by the original proof of Tutte[1947] of his theorem for the existence of a perfect matching in a graph (cf. Theorem 2.6.1) and the solution by Kasteleyn[1961] of the dimer problem.

We now present the proof of Kasteleyn[1967] of the theorem of Cayley[1854] that the determinant of a skew-symmetric matrix is the square of its pfaffian.

Theorem 9.5.2. *The skew-symmetric matrix M of even order n satisfies*

$$\det(M) = (\mathrm{pf}(M))^2.$$

Proof. Let $\mathcal{I}$ denote the set of permutations of $\{1,2,\ldots,n\}$ which contain at least one (permutation) cycle of odd length, and let $\mathcal{J}$ denote the set of permutations of $\{1,2,\ldots,n\}$ each of whose cycles has even length. Since n is even, (9.3) implies that

$$\det(M) = \sum_{\pi\in\mathcal{I}} \mathrm{wt}\,(D(\pi)) + \sum_{\pi\in\mathcal{J}} \mathrm{wt}\,(D(\pi)). \tag{9.34}$$

We show that the first summation in (9.34) equals zero and the second summation equals $(\mathrm{pf}(M))^2$.

We define a mapping

$$\sigma : \mathcal{I} \to \mathcal{I}$$

as follows. Let π be a permutation in $\mathcal{I}$, and consider the cycle of π of odd length which contains the smallest integer $i \in \{1,2,\ldots,n\}$. Let $\sigma(\pi)$ be the permutation obtained from π by reversing the direction of that cycle. Then $\sigma(\pi)$ is in $\mathcal{I}$. Moreover, since M is skew-symmetric, it follows that

$$\mathrm{wt}\,(D(\pi)) = -\mathrm{wt}\,(D(\sigma(\pi))).$$

We have $\sigma(\sigma(\pi)) = \pi$, and hence σ is an involution on $\mathcal{I}$ which is sign-reversing on the weight. [Note that if the cycle defined above has length one, then $\sigma(\pi) = \pi$ and $\mathrm{wt}\ D((\pi)) = 0$ follows since the main diagonal elements of M are all equal to zero.] We conclude that

$$\sum_{\pi\in\mathcal{I}} \mathrm{wt}\,(D(\pi)) = 0.$$

We now define a mapping

$$\tau : \mathcal{F}\times\mathcal{F} \to \mathcal{J}$$

for which

$$\text{wt}\,(L_1)\text{wt}\,(L_2) = \text{wt}\,(\tau((L_1, L_2))), (L_1, L_2 \in \mathcal{F}).$$

Let L_1 and L_2 be perfect matchings in $\mathcal{F}$. We define a multigraph $G(L_1, L_2)$ with vertex set $\{1, 2, \ldots, n\}$ in which $\{i, j\}$ is an edge of multiplicity 1 or 2 according as $\{i, j\}$ is an edge of exactly one or both of L_1 and L_2. Each vertex of the multigraph $G(L_1, L_2)$ has degree equal to 2, and, since L_1 and L_2 are perfect matchings, $G(L_1, L_2)$ decomposes into $t \geq 1$ cycles γ_j of even length $k_j, (j = 1, 2, \ldots, t)$ no two of which pass through the same vertex. This same collection of cycles results from 2^t pairs (L_1, L_2) of perfect matchings in $\mathcal{F}$. Each of the cycles γ_j can be oriented in two ways and this gives us 2^t permutations σ in $\mathcal{J}$ each of which has sign equal to $(-1)^t$. Each permutation in $\mathcal{J}$ arises in this way and this gives us a one-to-one corrrespondence τ between $\mathcal{F} \times \mathcal{F}$ and $\mathcal{J}$. It follows from the definition of weight of perfect matchings and the definition of weight of permutation digraphs that

$$(\text{wt}\,F_1)(\text{wt}\,F_2) = \pm\text{wt}\,(D(\sigma)).$$

However, by an appropriate choice of the listing of the edges of L_1 and L_2 and an appropriate choice of the listing of the vertices in their edges, we obtain that

$$\pi(L_1)\pi(L_2)^{-1} = \sigma$$

and hence

$$\text{sign}(\pi(L_1))\text{sign}(\pi(L_2)) = \text{sign}\,\sigma.$$

Therefore

$$\text{wt}\,(F_1)\text{wt}\,(F_2) = \text{wt}\,(D(\sigma))$$

and hence $(\text{pf}(M))^2 = \det(M)$. □

We now turn to Tutte's algebraic proof of the theorem characterizing graphs with a perfect matching. We first derive some consequences of Lemma 9.2.10, the identity of Jacobi, for a skew-symmetric matrix $M = [m_{ij}]$ of even order n. Let $M_{i,j}$ denote the matrix obtained from M by deleting row i and column j and let $C_{i,j} = (-1)^{i+j}\det(M_{j,i})$ be the cofactor of the element m_{ij} of $M, (i, j = 1, 2, \ldots, n)$. Each principal submatrix of M is a skew-symmetric matrix. Let $M(i_1, i_2, \ldots, i_k)$ denote the principal submatrix of M obtained by deleting rows and columns $i_1, i_2, \ldots, i_k$. If k is odd, then $\det(M(i_1, i_2, \ldots, i_k)) = 0$. If k is even, then $\det(M(i_1, i_2, \ldots, i_k)) = (\text{pf}(M(i_1, i_2, \ldots, i_k)))^2$. We note that $M_{i,i} = M(i)$. It follows from Lemma 9.2.10 and the skew-symmetry of M that for $i \neq j$,

$$\det(M)\det(M(i,j)) = \det(M(i))\det(M(j)) - C_{i,j}C_{j,i} = (C_{i,j})^2, \quad (9.35)$$

and

$$(\det(M))^3 \det(M(i,j,k,l)) = \det \begin{bmatrix} 0 & C_{i,j} & C_{i,k} & C_{i,l} \\ -C_{i,j} & 0 & C_{j,k} & C_{j,l} \\ -C_{i,k} & -C_{j,k} & 0 & C_{k,l} \\ -C_{i,l} & -C_{j,l} & -C_{k,l} & 0 \end{bmatrix}$$

$$= (C_{i,j}C_{k,l} - C_{i,k}C_{j,l} + C_{i,l}C_{j,k})^2, (i,j,k,l \text{ distinct}). \tag{9.36}$$

Applying (9.35) and (9.36), we obtain

$$\begin{aligned} \mathrm{pf}(M)\mathrm{pf}(M(i,j,k,l)) = {} & \pm\mathrm{pf}(M(i,j))\mathrm{pf}(M(k,l)) \\ & \pm\mathrm{pf}(M(i,k))\mathrm{pf}(M(j,l)) \pm \mathrm{pf}(M(i,l))\mathrm{pf}(M(j,k)), \end{aligned} \tag{9.37}$$

for all distinct i, j, k and l. The actual choice of signs in (9.37) is of no consequence in what follows.

Next we recall some notation from Chapter 2. If S is a subset of the set $V = \{1, 2, \ldots, n\}$ of vertices of the graph G of order n, then $G(V-S)$ is the induced subgraph obtained from G by removing the vertices in S and all incident edges. The set of connected components of $G(V-S)$ with an odd number of vertices is denoted by $\mathcal{C}(G; S)$ and the cardinality of $\mathcal{C}(G; S)$ is $p(G; S)$.

Theorem 9.5.3. *The graph G has a perfect matching if and only if*

$$p(G; S) \le |S|, \text{ for all } S \subseteq V. \tag{9.38}$$

Proof. The condition (9.38) is clearly a necessary condition for the existence of a perfect matching. Now assume that G does not have a perfect matching. We use Lemma 9.2.10, the identity of Jacobi, to show that there exists a set S_0 of vertices for which (9.38) does not hold. If n is odd, we may choose $S_0 = \emptyset$. We now suppose that n is even. By Theorems 9.5.1 and 9.5.2 the formal adjacency matrix $M = [m_{ij}]$ of G satisfies

$$\det(M) = \mathrm{pf}(M) = 0.$$

If $\{i, j\}$ is an edge of G, then the graph $G(V - \{i, j\})$ cannot have a perfect matching and hence $\mathrm{pf}(M(i,j)) = 0$. A vertex k of G is called a *singularity* of G provided that for all vertices $i \neq k$ the graph $G(V - \{i, k\})$ does not have a perfect matching. Suppose that there is a chain i, j, k joining distinct vertices i and j such that j is not a singularity. Then there exists a vertex l different from i, j and k such that $G(V - \{j, l\})$ has a perfect matching. Then it follows from (9.37) that

$$\mathrm{pf}(M(i,k))\mathrm{pf}(M(j,l)) = 0.$$

Since $\mathrm{pf}(M(j,l)) \neq 0$, we must have $\mathrm{pf}(M(i,k)) = 0$. Hence the graph $G(V - \{i,k\})$ does not have a perfect matching. We now conclude that if i and j are distinct vertices of G which are joined by a chain each of whose interior vertices is *not* a singularity, then the graph $G(V - \{i,j\})$ does not have a perfect matching.

Suppose that there are distinct vertices i and j such that $\{i,j\}$ is not an edge of G but $G(V - \{i,j\})$ does not have a perfect matching. Adding the edge $\{i,j\}$ to G we obtain a graph which, like G, does not have a perfect matching. It follows that there exists a graph G' of order n of which G is a (spanning) subgraph such that (i) G' does not have a perfect matching, and (ii) $G(V - \{i,j\})$ has a perfect matching if and only if $\{i,j\}$ is not an edge of $G', i \neq j$.

For each set S of vertices we have

$$p(G,S) \geq p(G',S).$$

Hence it suffices to show there exists a set S_0 of vertices for which

$$p(G',S_0) > |S|. \tag{9.39}$$

Let S_0 be the set consisting of those vertices which are singularities of G'. Then each pair of vertices, at least one of which is in S_0, forms an edge of G'. Moreover, it follows from the above arguments that the connected components of the graph $G(V - S_0)$ are complete graphs. Since G' does not have a perfect matching (9.39) holds, and the proof of the theorem is complete. □

Lovász[1979] has observed that Theorem 9.5.1 provides a random algorithm for deciding whether a graph of even order has a perfect matching. Let the number of edges of G be m. If a randomly generated m-tuple of real numbers does not cause the pfaffian of the formal adjacency matrix of G to vanish, then with probability one G has a perfect matching. Pla[1965] (see also Little and Pla[1972]) and Gibson[1972] have used pfaffians in order to count the number of perfect matchings of a graph. Identities involving pfaffians are given in Heymans[1969] and Lieb[1968].

Exercises

1. Verify that the weight of a perfect matching depends neither on the order in which the vertices are listed in the edges nor on the order in which the edges are listed.
2. Let $\alpha = (a_1, a_2, \ldots, a_p)^T$ and $\beta = (b_1, b_2, \ldots, b_p)^T$ be two real p-vectors. Define their *wedge product* $\alpha \wedge \beta$ to be the skew-symmetric matrix of order p for which the element in position (i,j) equals $a_i b_j - a_j b_i, (1 \leq i,j \leq p)$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m$ be $2m$ real $2n$-vectors. Prove that

$$\mathrm{pf}\left(\sum_{k=1}^{m} x_k(\alpha_k \wedge \beta_k)\right)$$

is linear in each of the variables $x_1, x_2, \ldots, x_m$.

3. (Continuation of Exercise 2) Prove that there exist n pairs of vectors $\{\alpha_k, \beta_k\}$ whose union is a linearly independent set if and only if

$$\det\left(\sum_{k=1}^{m} x_k(\alpha_k \wedge \beta_k)\right)$$

is not identically 0 in the variables $x_1, x_2, \ldots, x_m$ (Lovász[1979]).

4. Prove that the square of

$$\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

equals

$$\det\begin{bmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{bmatrix},$$

where $s_k = x_1^k + x_2^k + \cdots + x_n^k, (k \geq 0)$.

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9.6 The Formal Laplacian Matrix

In this section we consider digraphs D of order n with vertex set

$$V = \{1, 2, \ldots, n\}$$

and we assume that D has no loops. The possible lengths of directed cycles in such a digraph D are $2, 3, \ldots, n$. A *spanning arborescence*[1] of D is a subdigraph D' of D with vertex set V with the two properties: (i) D' has no directed cycles and (ii) there exists a vertex u such that every vertex different from u has outdegree equal to one in D'. The vertex u is called the *root* of D' and we say that D' is *rooted at* u. The collection of spanning arborescences of D which are rooted at u is denoted by $\Lambda_u(D)$.

Let G be a graph with vertex set V, and let $\overleftrightarrow{G}$ denote the digraph obtained from G by replacing each edge $\{i, j\}$ with the two oppositely directed arcs (i, j) and (j, i). The adjacency matrix of the graph G equals the adjacency matrix of the digraph $\overleftrightarrow{G}$. Let u be a vertex in V. The spanning arborescences of $\overleftrightarrow{G}$ rooted at u are obtained from the spanning trees T of G by orienting the edges of T toward the vertex u. The number of spanning arborescences of $\overleftrightarrow{G}$ rooted at u is thus the number $c(G)$ of spanning trees of G (the complexity of G) and hence does not depend on the choice of vertex u. In section 2.5 we showed that the adjugate of the Laplacian matrix of G is a constant matrix with each element equal to the complexity of G.

Let

$$Z = [z_{ij}], \quad (i, j = 1, 2, \ldots, n)$$

be a matrix of order n whose elements are independent indeterminates over the field F. Let

$$A = [a_{ij}], \quad (i, j = 1, 2, \ldots, n)$$

be the adjacency matrix of the digraph D of order n. Since D has no loops, each main diagonal element of A is zero. The Hadamard product

$$M = [m_{ij}] = A * Z = [a_{ij} z_{ij}]$$

is called the *formal adjacency matrix of the digraph* D. Any generic matrix of order n with main diagonal elements equal to zero may serve as the

[1] In section 3.7 we defined a spanning directed tree with root r of a digraph D. In a spanning arborescence with root u the arcs are directed toward the root; in a spanning directed tree with root r the arcs are directed away from the root.

formal adjacency matrix of a digraph D of order n with no loops. The matrix

$$L(D) = \begin{bmatrix} \sum_{j\neq 1} m_{1j} & -m_{12} & \cdots & -m_{1n} \\ -m_{21} & \sum_{j\neq 2} m_{2j} & \cdots & -m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & -m_{n2} & \cdots & \sum_{j\neq n} m_{nj} \end{bmatrix}$$

is the *formal Laplacian matrix of the digraph* D. The determinant of the formal Laplacian is identically zero. The matrix obtained from $L(D)$ by setting each of the nonzero elements $m_{ij}, (i \neq j)$ equal to 1 is the *Laplacian matrix* of the digraph D. If D is the digraph $\overleftrightarrow{G}$ then the Laplacian matrix of D is the Laplacian matrix of G as defined in section 2.3.

The *matrix tree theorem* asserts that the cofactor of the uth diagonal element of the Laplacian matrix of D equals the number of spanning arborescences of D which are rooted at vertex u. This theorem was apparently first proved by Borchardt[1860] and independently by Tutte[1948]. In its more general form it gives the weight of the arborescences of D which are rooted at u. Combinatorial proofs of the theorem have been given by Orlin[1978] and Chaiken[1982]. These proofs use the inclusion-exclusion principle. We follow the combinatorial proof of Zeilberger[1985].

Let $\overleftrightarrow{K_n}$ denote the complete digraph (with no loops) of order n. The digraph D of order n is a spanning subdigraph of $\overleftrightarrow{K_n}$. Let $M = [m_{ij}]$ be the formal adjacency matrix of order n of the digraph D. The *weight of an arc* (i, j) of $\overleftrightarrow{K_n}$ is defined to be

$$\mathrm{wt}(i,j) = m_{ij}.$$

Thus if (i, j) is not an arc of D then its weight is equal to zero. The *weight* $\mathrm{wt}(H)$ *of a subdigraph* H of $\overleftrightarrow{K_n}$ is defined to be the product of the weights of its arcs. If H is not a subdigraph of D its weight is equal to zero.

Theorem 9.6.1. *The determinant of the principal submatrix $L(D)(u)$ of the formal Laplacian matrix $L(D)$ obtained by deleting row u and column u equals*

$$\sum_{T\in\Lambda_u(\overleftrightarrow{K_n})} \mathrm{wt}\,(T) = \sum_{T\in\Lambda_u(D)} \mathrm{wt}\,(T),$$

the sum of the weights of the spanning arborescences of D which are rooted at u.

Proof. Without loss of generality we assume that $u = n$. Let $\mathcal{D}$ be the set of pairs (D_1, D_2) of digraphs D_1 and D_2 such that D_1 is a (possibly empty) set of vertex-disjoint directed cycles of length at least 2 whose vertices form a subset X of $\{1,2,\ldots,n-1\}$, and D_2 is a digraph with vertex set $V=\{1,2,\ldots,n\}$ and with $n-1-|X|$ arcs, one issuing from each vertex in $\overline{X}=\{1,2,\ldots,n-1\}-X$ (the arcs may terminate at any vertex in V). The weight of (D_1,D_2) is defined by

$$\mathrm{wt}(D_1,D_2)=(-1)^k\mathrm{wt}(D_1)\mathrm{wt}(D_2),$$

where k is the number of directed cycles of D_1. It follows from formula (9.3) applied to $L(D)(n)$ that

$$\det(L(D)(n))=\mathrm{wt}\mathcal{D}=\sum_{(D_1,D_2)\in\mathcal{D}}\mathrm{wt}(D_1,D_2).$$

Let $\mathcal{D}_0$ be the subset of $\mathcal{D}$ consisting of those pairs (D_1,D_2) for which at least one of D_1 and D_2 has a directed cycle. We define a mapping $\sigma:\mathcal{D}_0\to\mathcal{D}_0$ as follows. Let (D_1,D_2) be in $\mathcal{D}_0$. Let γ be the directed cycle containing the smallest numbered vertex of all directed cycles in D_1 and D_2. If γ is a directed cycle of D_1 we remove the arcs of γ from D_1 and put them in D_2 resulting in digraphs D_1' and D_2' for which (D_1',D_2') is in $\mathcal{D}_0$. If γ is a directed cycle in D_2 then we move the arcs of γ from D_2 to D_1 and obtain a pair (D_1',D_2') in $\mathcal{D}_0$. We define $\sigma(D_1,D_2)=(D_1',D_2')$ and observe that σ is an involution which is sign-reversing on the weight. Hence

$$\sum_{(D_1,D_2)\in\mathcal{D}_0}\mathrm{wt}(D_1,D_2)=0$$

and therefore

$$\det(L(D)(n))=\sum_{(D_1,D_2)\in\mathcal{D}-\mathcal{D}_0}\mathrm{wt}(D_1,D_2).$$

But a pair (D_1,D_2) belongs to $\mathcal{D}-\mathcal{D}_0$ if and only if D_1 is an empty graph ($X=\emptyset$) and D_2 is an arborescence rooted at vertex n. Moreover, for such (D_1,D_2) we have

$$\mathrm{wt}(D_1,D_2)=\mathrm{wt}(D_2),$$

and the theorem now follows. □

A combinatorial interpretation of the determinant of each square submatrix of the formal Laplacian matrix has been given by Chen[1976] and Chaiken[1982].

Exercises

1. Prove that a digraph has a spanning arborescence if and only if it satisfies the following property: For each pair of vertices a and b there is a vertex c (possibly c equals a or b) such that there are directed walks from a to c and from b to c.

2. Let D be a digraph of order n which has no directed cycles. Prove that all spanning arborescences of D (if any) have the same root.
3. (Continuation of Exercise 2) Let D be a digraph of order n which has no directed cycles. Let u be a vertex of D such that there is a spanning arborescence of D rooted at u. Determine the sum of the weights of the spanning arborescences of D rooted at u. Show that the number of spanning arborescences of D which are rooted at u equals the product of the outdegrees of all vertices different from u.

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9.7 Polynomial Identities

In this section we let R denote a commutative ring with identity 1 and consider the ring $M_n(R)$ of all matrices of order n with elements in R. Let $x_1, x_2, \ldots, x_k$ be k independent *noncommuting* indeterminates over R, and let

$$f(x_1, x_2, \ldots, x_k)$$

be a polynomial in the polynomial ring $R[x_1, x_2, \ldots, x_k]$. The polynomial $f(x_1, x_2, \ldots, x_k)$ is called an *identity* of $M_n(R)$ provided that

$$f(A_1, A_2, \ldots, A_k) = O$$

for all matrices $A_1, A_2, \ldots, A_k$ in $M_n(R)$. The general theory of polynomial identities is developed in Rowen[1980]. Our goal in this section is to show how combinatorial arguments can be used to prove three important identities of $M_n(R)$. One of these is a polynomial identity known as the *standard polynomial identity* of $M_n(R)$. Another is a trace identity known as the *fundamental trace identity* of $M_n(R)$. The third is the identity known as the *Cayley–Hamilton theorem.* Although the Cayley–Hamilton theorem asserts that a certain polynomial of one variable x yields a zero matrix upon replacing x by a matrix A, the coefficients of the polynomial depend on the elements of A and thus the polynomial is not a polynomial identity as defined above.

Let $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ be a matrix in $M_n(R)$, and let x be an indeterminate over R. The *characteristic polynomial* of A is the monic polynomial in $R[x]$ of degree n given by

$$\chi_A(x) = \det(xI_n - A) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_k x^{n-k} + \cdots + \sigma_{n-1}x + \sigma_n,$$

where σ_k equals the sum of the determinants of all the principal submatrices of $-A$ of order $k, (k = 1, 2, \ldots, n)$. Rutherford[1964] first discovered a combinatorial proof of the Cayley–Hamilton theorem. This proof was rediscovered by Straubing[1983] and given an exposition by Zeilberger[1985] and Brualdi[1990].

Theorem 9.7.1. *The matrix A of order n satisfies its characteristic polynomial $\chi_A(x)$, that is,*

$$A^n + \sigma_1 A^{n-1} + \cdots + \sigma_k A^{n-k} + \cdots + \sigma_{n-1}A + \sigma_n I_n = O. \tag{9.40}$$

Proof. We prove (9.40) by showing that each element of the matrix in the left-hand side equals 0. We assign the weight

$$\mathrm{wt}(i, j) = a_{ij}, \quad (i, j = 1, 2, \ldots, n)$$

to the arcs (i, j) of the complete digraph D_n of order n, and, as in section 9.1, we use these weights to assign weights to the directed cycles of D_n and to the permutation digraphs whose vertices form a subset of $\{1, 2, \ldots, n\}$. We also define the weight of a walk to be the product of the weights of its arcs.

It follows from (9.3) [see also (9.4)] that σ_k equals the sum of the weights of all permutation digraphs whose vertices form a subset of size k of $\{1, 2, \ldots, n\}$. The element in the (i, j) position of A^{n-k} equals the sum of the weights of all walks in D_n of length $n - k$ from vertex i to vertex j. Let Ω_{ij} be the set of all ordered pairs (γ, π) for which γ is a walk in D_n from i to j of length at most n, π is a collection of vertex disjoint directed cycles in D_n and the number of arcs of γ plus the number of arcs of π equals n. We assign a weight to each pair (γ, π) in Ω_{ij} by

$$\mathrm{wt}(\gamma, \pi) = (-1)^t \times (\text{product of all arc weights of } \gamma \text{ and of } \pi),$$

where t is the number of directed cycles in π. The element in position (i, j) of the matrix on the left-hand side of equation (9.40) equals

$$\mathrm{wt}(\Omega_{ij}) = \sum_{(\gamma,\pi)\in\Omega_{ij}} \mathrm{wt}(\gamma, \pi).$$

Hence the Cayley–Hamilton theorem asserts that

$$\mathrm{wt}(\Omega_{ij}) = 0, \quad (i, j = 1, 2, \ldots, n). \tag{9.41}$$

Let i and j be integers with $1 \le i, j \le n$. We define a mapping $\tau : \Omega_{ij} \to \Omega_{ij}$ as follows. Let (γ, π) be in Ω_{ij}. Since the total number of arcs in γ and π equals the number n of vertices in D_n, either there is a vertex of γ which is also a vertex of one of the directed cycles in π or the directed walk γ contains a repeated vertex and hence "contains" a directed cycle (possibly both). We walk along γ until we first arrive at a vertex u which has been previously visited or is a vertex of a directed cycle in π. (Note that these two events cannot occur simultaneously.) In the first instance, γ contains a directed cycle γ_0 whose vertices are disjoint from the vertices of the directed cycles in π; we then "remove" γ_0 from γ and include γ_0 in π. In the second instance, π contains a directed cycle π_0 one of whose vertices is a vertex of γ; we remove π_0 from π and join it to γ. In each instance we obtain a pair (γ', π') which belongs to Ω_{ij}. We define $\tau(\gamma, \pi) = (\gamma', \pi')$. The mapping τ is sign-reversing on weight and we have $\tau(\gamma', \pi') = (\gamma, \pi)$. Hence τ is a sign-reversing involution on Ω_{ij} and hence (9.41) holds. □

We next turn to the fundamental trace identity. The *standard polynomial* of degree k is the polynomial in $R[x_1, x_2, \ldots, x_k]$

$$[x_1, x_2, \ldots, x_k]_k = \sum_{\pi} (\mathrm{sign}\pi) x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(k)}$$

where the summation extends over all permutations π of $\{1, 2, \ldots, k\}$. The standard polynomials satisfy the relations

$$[x_1, x_2, \ldots, x_k]_k = \sum_{i=1}^{k} (-1)^{i-1} x_i [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k]_{k-1}, \quad (9.42)$$

and

$$[x_1, x_2, \ldots, x_k]_k = \sum_{i=1}^{k} (-1)^{k-i} [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k]_{k-1} x_i, \quad (9.43)$$

where we adopt the convention that $[\emptyset]_0 = 1$. We have

$$[x_1, x_2]_2 = x_1 x_2 - x_2 x_1$$

and hence

$$\mathrm{tr}([A_1, A_2]_2) = \mathrm{tr}(A_1 A_2 - A_2 A_1) = 0.$$

Thus $\mathrm{tr}[x_1, x_2]_2$ is a trace identity of $M_n(R)$ for all $n \ge 1$. More generally, it follows from the above relations that

$$\mathrm{tr}[x_1, x_2, \ldots, x_k]_k, \quad (k \ge 2)$$

is a trace identity of $M_n(R)$ for all $n \ge 1$. These identities are sometimes called the *trivial trace identities* since they hold for all n. The following trace identity, which is not trivial in the sense just described, goes back to Frobenius[1896]. It was also discovered by Lew[1966] and Procesi[1976]. We give the combinatorial proof of Laue[1988], which is also presented in Brualdi[1990]. For $n = 2$ and $k = 3$ this identity asserts that if A_1, A_2, A_3 are matrices of order 2 with elements in R, then

$$\operatorname{tr}(A_1A_2A_3) + \operatorname{tr}(A_1A_3A_2) - \operatorname{tr}(A_1)\operatorname{tr}(A_2A_3) - \operatorname{tr}(A_2)\operatorname{tr}(A_1A_3)$$
$$-\operatorname{tr}(A_3)\operatorname{tr}(A_1A_2) + \operatorname{tr}(A_1)\operatorname{tr}(A_2)\operatorname{tr}(A_3) = 0.$$

Theorem 9.7.2. *Let k and n be integers with $k > n$ and let $A_1, A_2, \ldots, A_k$ be matrices in $M_n(R)$. Then*

$$\sum_{\pi} \operatorname{sign}(\pi) \prod_{\pi_i} \operatorname{tr}\left(\prod_{p\in\pi_i} A_p\right) = O, \tag{9.44}$$

where the summation extends over all permutations π of $\{1, 2, \ldots, k\}$.

Proof. First we clarify the expression on the left hand side in (9.44). The first product is over all cycles π_i of the permutation π. The second product is over all elements p of the cycle π_i taken in the cyclical order of π_i.

The ring $M_n(R)$ is actually an R-module, and the function on the left-hand side of (9.44) is an R-multilinear function of its arguments $A_1A_2, \ldots, A_k$. Hence the validity of (9.44) for all matrices in $M_n(R)$ follows from its validity for all matrices chosen from the standard basis $\{E_{ij} : i, j = 1, 2, \ldots, n\}$ of the R-module $M_n(R)$, where E_{ij} denote the matrix in $M_n(R)$ whose only nonzero element is a 1 in position (i, j).

Let $A_t = E_{i_t j_t}, (t = 1, 2, \ldots, k)$. We define a general digraph D with vertex set $V = \{1, 2, \ldots, n\}$ in which (i, j) is an arc of multiplicity r provided (i, j) occurs r times among the elements $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$. The general digraph D has n vertices and $k > n$ arcs. Let π be a permutation of $\{1, 2, \ldots, k\}$ and let π_i be a cycle of π. Then $\operatorname{tr}(\prod_{p\in\pi_i} A_p)$ equals 1 if the arcs $(i_p, j_p), p \in \pi_i$, taken in the cyclical order of π_i, determine a directed closed trail of D and equals 0 otherwise. Hence

$$\prod_{\pi_i} \operatorname{tr}\left(\prod_{p\in\pi_i} A_p\right)$$

equals 1 if the cycles π_i of π determine a partition θ of the arcs of D into directed closed trails, and equals 0 otherwise. (We note that each permutation π of $\{1, 2, \ldots, k\}$ determines a permutation of the arcs of D, but only some of the permutations determine a partition of the arcs into directed closed trails.) It now follows that (9.44) is equivalent to the following combinatorial statement:

(∗) If D is a general digraph of order n with $k > n$ arcs, then the number of partitions of the arcs of D into directed closed trails which correspond to even permutations of $\{1, 2, \ldots, k\}$ equals the number of partitions which correspond to odd permutations of $\{1, 2, \ldots, k\}$.

We now verify $(*)$. Since D has more arcs than vertices, there are two arcs α and β with the same initial vertex u (α and β might also have the same terminal vertex).

Let Θ be the set of partitions of the arcs of D into directed closed trails. We define a mapping $\sigma : \Theta \to \Theta$ as follows. Let θ be a partition in Θ. First suppose that α and β are in the same directed closed trail $\theta_{\alpha,\beta}$ of θ. We may assume that the arc α is written first in $\theta_{\alpha,\beta}$:

$$\alpha, \cdots, \gamma, \beta, \cdots, \delta.$$

We then replace θ in Θ with the two directed closed trails

$$\alpha, \cdots, \gamma$$

and

$$\beta, \cdots, \delta,$$

resulting in a partition θ' in Θ. Now suppose that the arcs α and β are in different directed closed trails θ_α and θ_β of θ. We may assume that α is written first in θ_α and β is written first in θ_β. We then follow θ_α with θ_β resulting in a closed directed trail $\theta_{\alpha,\beta}$. In this case we let θ' be the partition in Θ obtained by replacing θ_α and θ_β with $\theta_{\alpha,\beta}$. We define $\sigma(\theta)$ to be θ'. It follows from the construction that σ is a sign-reversing involution on Θ and $(*)$ holds. □

Procesi[1976] proved that every trace identity is a consequence of the trace identity (9.44) and the trivial trace identities.

We now turn to the theorem of Amitsur and Levitzki[1950,1951]. This theorem asserts that the standard polynomial of degree $2n$

$$[x_1, x_2, \ldots, x_{2n}]_{2n}$$

is a polynomial identity of $M_n(R)$. This implies that matrices of order n satisfy a weakened form of the commutative law. We follow the combinatorial proof of Swan[1963,1969], an exposition of which is also given in Bollobás[1979].

Theorem 9.7.3. *For all matrices $A_1, A_2, \ldots, A_{2n}$ in $M_n(R)$, we have*

$$[A_1, A_2, \ldots, A_{2n}]_{2n} = O. \tag{9.45}$$

Proof. As in the proof of (9.44) we need only verify (9.45) for matrices $A_1, A_2, \ldots, A_{2n}$ chosen from the standard basis $\{E_{ij} : i, j = 1, 2, \ldots n\}$ of

$M_n(R)$. Let $A_t = E_{i_t j_t}, (t = 1, 2, \ldots, 2n)$. Let D be the general digraph with vertex set $V = \{1, 2, \ldots, n\}$ and the $2n$ arcs

$$\alpha_t = (i_t, j_t), \quad (t = 1, 2, \ldots, 2n).$$

Let π be a permutation of $\{1, 2, \ldots, 2n\}$. We have

$$A_{\pi(1)} A_{\pi(2)} \cdots A_{\pi(2n)} = E_{i_{\pi(1)} i_{\pi(2n)}} \tag{9.46}$$

if

$$\alpha_{\pi(1)}, \alpha_{\pi(2)}, \ldots, \alpha_{\pi(2n)} \tag{9.47}$$

is a directed trail in D from vertex $i_{\pi(1)}$ to vertex $j_{\pi(2n)}$. Such a trail includes each arc of D exactly once and is called an *Eulerian trail.* The product on the left-hand side of (9.46) equals a zero matrix if (9.47) is not an Eulerian trail. Each Eulerian trail in D is a permutation of the $2n$ arcs $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ of D. It is convenient to identify the Eulerian trail (9.47) with the permutation π of $\{1, 2, \ldots, 2n\}$. Thus each Eulerian trail has a *sign* which is equal to the sign of the permutation π. Let Γ_{uv} denote the set of permutations π which are Eulerian trails from vertex u to vertex v. Let

$$\epsilon(D : u, v) = \sum_{\pi \in \Gamma_{uv}} \operatorname{sign} \pi.$$

It then follows that (9.45) is equivalent to the combinatorial statement:

($*$) For each pair of vertices u and v of the directed multigraph D of order n with $2n$ arcs,

$$\epsilon(D : u, v) = 0.$$

By introducing a new vertex w and the arcs (w, u) and (v, w) from w to u and v to w, respectively, we see that it suffices to prove ($*$) under the additional assumption that $v = u$. In addition it suffices to verify ($*$) for any one vertex u.

We verify ($*$) by induction on n. If $n = 1$ then ($*$) holds. Now suppose that $n > 1$. We assume that for each vertex of D the indegree equals the outdegree, for otherwise Γ_{uu} is empty and the assertion holds trivially. We distinguish three cases.

Case 1. There is a vertex $a \neq u$ with indegree and outdegree equal to 1. Let (a_1, a) be the unique arc entering a and let (a, a_2) be the unique arc issuing from a. If $a_1 = a_2$, the assertion follows by applying the induction hypothesis to the digraph obtained by removing the vertex a and the arcs (a_1, a) and (a, a_2). Now suppose that $a_1 \neq a_2$. We may assume that $u \neq a_2$. Let $\alpha_1 = (a_2, b_1), \alpha_2 = (a_2, b_2), \ldots, \alpha_t = (a_2, b_t)$ be the arcs issuing from a_2. Let D_i be the general digraph obtained from D by removing the vertex

a and the arcs $(a_1, a), (a, a_2)$ and (a_2, b_i) and adding the arc $(a_1, b_i), (i = 1, 2, \ldots, t)$. By the induction hypothesis $\epsilon(D_i : u, u) = 0$ and hence

$$\epsilon(D : u, u) = \sum_{i=1}^{t} \epsilon(D_i : u, u) = 0.$$

Case 2. There is a loop at a vertex $a \neq u$ whose indegree and outdegree equal 2. Let the arcs incident with a be $(a, a), (b, a)$ and (a, c). Let D' be the general digraph obtained from D by removing the vertex a and its incident arcs and adding the arc (b, c). Applying the induction hypothesis we obtain

$$\epsilon(D : u, u) = \epsilon(D' : u, u) = 0.$$

Since the total number of arcs of D is $2n$, if Cases 1 and 2 do not apply, then the following case applies.

Case 3. Either each vertex has indegree and outdegree equal to 2, *or u has indegree and outdegree equal to* 1, *there is a vertex with indegree and outdegree equal to* 3 *and every other vertex has indegree and outdegree equal to* 2. Since Case 2 does not apply, there exist vertices a and b such that $\alpha = (a, b)$ is an arc and a and b have indegree and outdegree equal to 2. Let the two arcs entering a be $\alpha_1 = (c_1, a)$ $\alpha_2 = (c_2, a)$. Let D_i be the general digraph obtained from D removing the arcs α and α_i and adding the arcs (c_i, b) and $(b, b), (i = 1, 2)$. Each Eulerian trail from u to u in D is an Eulerian trail in exactly one of D_1 and D_2. However, the general digraphs D_1 and D_2 contain Eulerian trails that do not correspond to Eulerian trails in D. Let the two arcs of D leaving b be $\beta_1 = (b, d_1)$ and $\beta_2 = (b, d_2)$. Let D'_i be the general digraph obtained from D by removing the arcs α and β_i and adding the arcs (b, b) and $(a, c_i), (i = 1, 2)$. It can be verified that the Eulerian trails from u to u in D_1 and D_2 that do not arise from Eulerian trails in D correspond exactly to the Eulerian trails in D'_1 and D'_2 from u to u. Moreover, we have

$$\epsilon(D : u, u) = \epsilon(D_1 : u, u) + \epsilon(D_2 : u, u) - \epsilon(D'_1 : u, u) - \epsilon(D'_2 : u, u).$$

The general digraphs D_1 and D_2 satisfy the requirements of Case 1 and the general digraphs D'_1 and D'_2 satisfy the requirements of Case 2. Hence $\epsilon(D : u, u) = 0$ in this case also. □

Razmyslov[1974] has shown that the Amitsur–Levitzki theorem can be deduced from the Cayley–Hamilton theorem and indeed that every polynomial identity (in the case of characteristic zero) is a "consequence" of the Cayley–Hamilton theorem. Amitsur and Levitzki have shown that $M_n(R)$

does not satisfy the standard identity of degree $2n-1$ and that every multilinear identity of $M_n(R)$ (again in the case of characteristic zero) of degree at most $2n$ is a multiple of the standard identity of degree $2n$.

Exercises

1. Verify the relations (9.42) and (9.43).
2. Write out (9.44) in the case that $n=3$ and $k=4$.
3. Let the $2n-1$ matrices $A_1, A_2, \ldots, A_{2n-1}$ of order n be defined by: $A_1 = E_{11}, A_2 = E_{12}, A_3 = E_{22}, A_4 = E_{23}, \ldots, A_{2n-1} = E_{nn}$. Let π be a permutation of $\{1, 2, \ldots, 2n-1\}$. Prove that $A_{\pi(1)}A_{\pi(2)} \cdots A_{\pi(2n-1)}$ equals E_{nn} if π is the identity permutation and O otherwise.
4. Prove that the standard polynomial of degree $2n-1$ is not a polynomial identity for $M_n(R)$.
5. (Newton's formulas) Let A be a matrix of order n and let σ_k be the coefficient of x^{n-k} in the characteristic polynomial of A. Prove that

$$(-1)^k k\sigma_k = \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-i} \operatorname{tr}(A^i), \quad (1 \le i \le k).$$

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9.8 Generic Nilpotent Matrices

Let

$$M = [m_{ij}], \quad (i, j = 1, 2, \ldots, n)$$

be a generic matrix of order n whose nonzero elements are independent indeterminates over a field F, and let $D(M)$ be the digraph corresponding to M. The vertex set of $D(M)$ is $V = \{1, 2, \ldots, n\}$ and there is an arc (i, j) from vertex i to vertex j if and only if m_{ij} is an indeterminate. We assign to each arc (i, j) the *weight*

$$\mathrm{wt}(i, j) = m_{ij}.$$

The matrix M is *nilpotent* provided that there exists a positive integer k such that M^k is a zero matrix. The smallest such integer k is the *nilpotent index* of M. The matrix M is nilpotent if and only if each of its eigenvalues is equal to zero.

The digraph of the nilpotent generic matrix M does not have any directed cycles and thus $D(M)$ is an *acyclic digraph* of order n. It follows from Lemma 3.2.3 that without loss of generality we may assume that each arc of $D(M)$ is of the form (i, j) where $1 \leq i < j \leq n$. With this assumption the matrix M has 0's on and below its main diagonal and is a *strictly upper triangular matrix.* For each positive integer k the element in position (i, j) of M^k equals the sum of the weights of all directed walks from i to j of length k. Because $D(M)$ is acyclic there exists a nonnegative integer r equal to the length of its longest path (directed chain). The acyclicity of $D(M)$ implies there is no directed walk of length $r + 1$ or greater in $D(M)$ and hence M^{r+1} is a zero matrix. Because M is a generic matrix, M^r is not a zero matrix and we conclude that the nilpotent index of M is $r + 1$.

We remark that a nilpotent matrix with elements from the field F may have a digraph which is not acyclic. For example, the matrix

$$A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

is nilpotent of index 2 and the digraph $D(A)$ is the complete digraph of order 2.

Let A be a nilpotent matrix of order n. The Jordan canonical form

JCF(A) of A is a direct sum of (nilpotent) *Jordan blocks* of order $t \geq 1$ of the form

$$B_t = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The ordering of the Jordan blocks in the Jordan canonical form can be chosen arbitrarily. In order that the Jordan canonical form of A be unique, we assume that the Jordan blocks are ordered from largest to smallest. Thus there exists a partition

$$jp(A) = (n_1, n_2, \ldots, n_s)$$

of n into $s \geq 1$ parts such that

$$\text{JCF}(A) = B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_s}. \tag{9.48}$$

[Recall that $(n_1, n_2, \ldots, n_s)$ is a partition of n means that $n = n_1 + n_2 + \cdots + n_s$ and $n_1 \geq n_2 \geq \cdots \geq n_s \geq 1$. In section 6.5 a partition is regarded as an infinite nonincreasing sequence of nonnegative integers with only a finite number of nonzero terms. In the partition $(n_1, n_2, \ldots, n_s)$ we have suppressed the trailing terms equal to zero.] As in section 6.5 the set of all partitions of n is denoted by P_n. We call the partition $jp(A)$ the *Jordan partition* of the nilpotent matrix A. The *length* s of the Jordan partition equals the *nullity* of A, equivalently s equals the number of linearly independent eigenvectors for the eigenvalue 0. The order n_1 of the largest Jordan block equals the nilpotent index of A. Thus the order of the largest Jordan block of the Jordan canonical form of a generic nilpotent matrix M equals $r + 1$ where r is the length of the longest path in $D(M)$. We shall show how the entire Jordan partition of M can be determined by the digraph $D(M)$.

Let D be a digraph of order n with vertex set V. Let k be a positive integer. A *k-path* of D is a set X of vertices which can be partitioned into k (possibly empty) sets $X_1, X_2, \ldots, X_k$ such that each X_i is the set of vertices of a path of D. (We allow a trivial path consisting of a single vertex.) The largest number of vertices in a k-path of D is called the *k-path number* of D and is denoted by $p_k(D)$. We define $p_0(D) = 0$ and note that there is a smallest positive integer $s \leq n$ such that

$$0 = p_0(D) < p_1(D) < \cdots < p_s(D) = \cdots = p_n(D) = \cdots = n.$$

The sequence

$$p(D) = (p_0(D), p_1(D), \ldots, p_n(D))$$

is called the *path-number sequence* of D. It is sometimes convenient to regard the path-number sequence as an infinite sequence all but a finite number of terms of which equal n. In the notation of section 6.5 the path-number sequence $p(D)$ belongs to T_n. We call the integer s the *width* of D and denote it by width(D). If $k \leq$ width(D) then a k-path can be partitioned into k but no fewer paths of D. Notice that $p_1(D)$ equals 1 plus the length of the longest path in D. Hence if M is a generic nilpotent matrix, then $p_1(D(M))$ equals the nilpotent index of M.

If D is the digraph corresponding to the Jordan canonical form matrix in (9.48), then we see that the width of D equals the number s of Jordan blocks and

$$p_k(D) = n_1 + n_2 + \cdots + n_k, \quad (k = 0, 1, \ldots, s).$$

Hence

$$n_k = p_k(D) - p_{k-1}(D), \quad (k = 1, 2, \ldots, s).$$

Gansner[1981] and Saks[1986] showed that these equations hold between the Jordan partition of a generic nilpotent matrix and the k-path numbers of its digraph. In order to derive their results we first recall some basic facts concerning the Jordan invariants of a nilpotent matrix A of order n.

Let λ be an indeterminate over the field F. The greatest common divisor of the determinants of the submatrices of order k of $\lambda I_n - A$ is called the kth *determinantal divisor* of A (strictly speaking, of $\lambda I_n - A$) and is denoted by $d_k(A : \lambda), (k = 1, 2, \ldots, n)$. We define $d_0(A : \lambda) = 1$. There exists a positive integer $t \leq n$ and integers $\ell_0, \ell_1, \ldots, \ell_n$ satisfying

$$0 = \ell_0 < \ell_1 < \cdots < \ell_t = \ell_{t+1} = \cdots = \ell_n = n$$

such that

$$d_{n-k}(A : \lambda) = \lambda^{n-\ell_k}, \quad (k = 0, 1, \ldots, n).$$

We call the sequence

$$d(A) = (\ell_0, \ell_1, \ldots, \ell_n)$$

the *divisor sequence* of A. With the convention established above $d(A)$ belongs to T_n.

The polynomials

$$d_i(A : \lambda)/d_{i-1}(A : \lambda) = \lambda^{\ell_{n-i+1} - \ell_{n-i}}, \quad (i = n - t + 1, n - t + 2, \ldots, n)$$

are the elementary divisors, equivalently the invariant factors, of A. The

Jordan partition of A equals the difference sequence (as defined in section 6.5) of the divisor sequence

$$\delta d(A) = (\ell_1 - \ell_0, \ell_2 - \ell_1, \ldots, \ell_t - \ell_{t-1})$$

where the difference sequence has been truncated by omitting the trailing terms equal to 0. The nullity of A is equal to t, the number of nonconstant elementary divisors of A. Thus the Jordan partition of A is determined by the divisor sequence of A. We shall show that the divisor sequence of a generic nilpotent matrix M equals the path-number sequence of its digraph, and hence that the Jordan partition is the difference sequence of the path-number sequence. First we prove three lemmas.

Lemma 9.8.1. *Let $A = [a_{ij}]$ be a matrix of order n each of whose main diagonal elements equals zero. Let k be a positive integer with $k \leq$ width($D(M)$) and let X be a k-path of $D(A)$ of size r. Then there is a submatrix of A of order $r - k$ with term rank equal to $r - k$.*

Proof. Since $k \leq \text{width}(D(M))$ the set X can be partitioned into exactly k paths γ_t joining a vertex i_t to a vertex $j_t, (t = 1, 2, \ldots, k)$. Let B be the principal submatrix of order r of A determined by the rows and columns whose indices lie in X. Let B' be the submatrix of B obtained by deleting rows $j_1, j_2, \ldots, j_k$ and columns $i_1, i_2, \ldots, i_k$. Then B' is a submatrix of order $r - k$ of A and B' has term rank equal to $r - k$. □

Lemma 9.8.2. *Let T be a strictly upper triangular matrix of order n. Let α and β be subsets of $\{1, 2, \ldots, n\}$ of size r, and suppose that the submatrix $T[\alpha, \beta]$ of T determined by the rows with index in α and columns with index in β has term rank r. Then the complementary submatrix $T[\overline{\alpha}, \overline{\beta}]$ of order $n - r$ is a strictly upper triangular matrix.*

Proof. Let

$$\alpha = \{i_1, i_2, \ldots, i_r\}, \qquad \overline{\alpha} = \{k_1, k_2, \ldots, k_{n-r}\}$$

and

$$\beta = \{j_1, j_2, \ldots, j_r\}, \qquad \overline{\beta} = \{l_1, l_2, \ldots, l_{n-r}\}.$$

(The elements of each of the sets are assumed to be listed in strictly increasing order.)

If there exists an integer k with $j_k \leq i_k$, then it follows from the assumption that T is strictly upper triangular that $T[\alpha, \beta]$ has a zero submatrix of size $(r - k + 1)$ by k contradicting the assumption that $T[\alpha, \beta]$ has term rank r. Hence

$$j_k > i_k, \quad (k = 1, 2, \ldots, r).$$

This implies that

$$k_i \geq l_i, \quad (i = 1, 2, \ldots, n-r)$$

and $T[\overline{\alpha}, \overline{\beta}]$ is a strictly upper triangular matrix. □

Lemma 9.8.3. *Let T, α and β satisfy the assumptions of Lemma 9.8.2. Let X be a set of r nonzero elements of $T[\alpha, \beta]$ with no two on the same line. Then the arcs of the digraph $D(T)$ corresponding to the elements of X can be partitioned into $e \leq n-r$ pairwise vertex disjoint paths each of which joins a vertex in $\overline{\beta}$ to a vertex in $\overline{\alpha}$. The vertices on these paths form an $(n-r)$-path of $D(T)$ of size $e+r$.*

Proof. By Lemma 9.8.2 $T[\overline{\alpha}, \overline{\beta}]$ is a strictly upper triangular matrix. Let T' and $T[\overline{\alpha}, \overline{\beta}]'$ be the matrices obtained from T and $T[\overline{\alpha}, \overline{\beta}]$, respectively, by replacing the 0's on the main diagonal of $T[\overline{\alpha}, \overline{\beta}]$ with 1's. Let X' be the union of X and the set of diagonal positions of $T[\overline{\alpha}, \overline{\beta}]$. The elements of X' correspond to a set Γ of n arcs of the digraph $D(T')$, one entering each vertex and one issuing from each vertex. The arcs of Γ can be partitioned into directed cycles $\gamma_1, \gamma_2, \ldots, \gamma_t$. Since T is strictly upper triangular $D(T)$ has has no directed cycles, and hence the removal of the $n-r$ arcs corresponding to the main diagonal elements of $T[\overline{\alpha}, \overline{\beta}]'$ results in a collection of pairwise vertex disjoint paths of $D(T)$. The removal of each arc increases the number of paths by at most one, and the conclusions now follow. □

We now prove the theorem of Gansner[1981] and Saks[1986].

Theorem 9.8.4. *Let M be a generic nilpotent matrix of order n. Then the divisor sequence $d(M)$ of M equals the path-number sequence $p(D(M))$ of the digraph $D(M)$. Hence the Jordan partition of M equals the difference sequence $\delta p(D(M))$.*

Proof. Let k be an integer with $k \leq \text{width}(D(M))$. The theorem asserts that the degree $n - \ell_k$ of the determinantal divisor $d_{n-k}(M : \lambda)$ of M satisfies

$$n - \ell_k = n - p_k(D(M)). \tag{9.49}$$

It follows from Lemma 9.8.1 that there is a submatrix of M of order $p_k(D(M)) - k$ with a nonzero term in its determinant expansion and hence a submatrix C of order $n-k$ of $\lambda I_n - M$ whose determinant expansion contains a nonzero term of degree $n - p_k(D(M))$. Since M is generic, $\det(C)$ contains a nonzero term of degree $n - p_k(D(M))$. Hence

$$n - \ell_k \leq n - p_k(D(M)).$$

Now let $B = (\lambda I_n - M)[\alpha, \beta]$ be a submatrix of order $n-k$ of $\lambda I_n - M$ such that $\det(B)$ has a nonzero term of degree $n - \ell_k$. Then there exists a set γ of size $n - \ell_k$ with $\gamma \subseteq \alpha \cap \beta$ such that there is a nonzero term in the determinant expansion of the submatrix $M[\alpha-\gamma,\beta-\gamma]$ of order $\ell_k - k$ of the strictly upper triangular matrix $M[\overline{\gamma},\overline{\gamma}]$ of order ℓ_k. We apply Lemma 9.8.3 where T is the strictly upper triangular matrix $M[\overline{\gamma},\overline{\gamma}]$ of order ℓ_k and $r = \ell_k - k$ and conclude that there exists an e-path X of size $\ell_k - k + e$ in $D(M)$ where $e \leq k$. There are $(k-e)+(n-\ell_k)$ vertices not on this e-path. Adding any $k-e$ of them to X we obtain a k-path of size ℓ_k. Hence $p_k(D(M)) \geq \ell_k$ and thus

$$n - p_k(D(M)) \leq n - \ell_k.$$

We now conclude that $\ell_k = p_k(D(M))$ for all $k = 0,1,\ldots,n$. □

Corollary 9.8.5. *The path-number sequence of an acyclic digraph is a convex sequence.*

Let A be a nilpotent matrix of order n. It is a direct consequence of the Jordan canonical form that the conjugate $(\mathrm{jp}(A))^*$ of the Jordan partition $\mathrm{jp}(A) = (n_1, n_2, \ldots, n_s)$ of A is the partition

$$(m_1, m_2, \ldots, m_{n_1})$$

of n in which $m_1 + m_2 + \cdots + m_k$ equals the nullity of A^k, $(k = 1, 2, \ldots, n_1)$. (Note that $A^s = O$ for $s \geq n_1$.) For a generic nilpotent matrix M of order n we have by Theorem 9.8.4 that the difference sequence of the conjugate of the path-number sequence equals the sequence of nullities of powers of M.

For a matrix which is not a generic nilpotent matrix the path-number sequence need not equal the divisor sequence. This is because of the possibility of cancellation of terms in taking determinants. The general situation is discussed in Brualdi[1987] and also in Brualdi[1985] where the above ideas are used to derive the Jordan partition of the tensor product of two matrices in terms of the Jordan partition of the individual matrices.

We conclude this section by showing how the existence of the Jordan canonical form can be derived using some of the ideas of the theory of digraphs. The idea for this derivation goes back to Turnbull and Aitken[1932] and the derivation was given an exposition in Brualdi[1987].

Let A be a complex matrix of order n. The Jordan canonical form of A is a matrix $\mathrm{JCF}(A)$ similar to A which is a direct sum of Jordan blocks

$$aI_t + B_t = \begin{bmatrix} a & 1 & 0 & \cdots & 0 & 0 \\ 0 & a & 1 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 1 \\ 0 & 0 & 0 & \cdots & 0 & a \end{bmatrix}$$

each having a unique eigenvalue a. The existence of the Jordan canonical form can be easily reduced to consideration of nilpotent matrices in the following way.

By Jacobi's theorem there is an upper triangular matrix T which is similar to A. The eigenvalues of A are the n diagonal elements of T and T can be chosen so that equal eigenvalues of A occur consecutively on the main diagonal of T. Suppose that A has two unequal eigenvalues. Then we may write

$$T = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix}$$

where T_1 is an upper triangular matrix each of whose elements on the main diagonal equals a constant a and T_2 is an upper triangular matrix with no element on its main diagonal equal to a. The matrix T is similar to the matrix obtained from T by replacing X by a zero matrix. This can be seen as follows. Adding a multiple h of column i of T to column j and then the multiple $-h$ of row j to row i replaces T with a matrix similar to T. By means of *elementary similarities* of this type we may make each element of X equal to 0. We do this row by row, starting with the last row of X, and treating each element in the current row beginning with its first. We may now treat T_2 as we did T and continue like this until we arrive at a matrix T' which is similar to T (and hence to A) where T' is a direct sum of upper triangular matrices T_i' each with a constant main diagonal. The reduction of A to Jordan canonical form will be complete once the matrices T_i' have been reduced to Jordan canonical form. Thus we need only consider upper triangular matrices with a constant main diagonal. By subtracting a multiple of an identity matrix we may assume that the main diagonal elements equal 0. Therefore we have left to show that a nilpotent upper triangular matrix N of order n has a Jordan canonical form. We use induction on n.

If $n = 1$ then N is in Jordan canonical form. If $n = 2$, then

$$N = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

and N can be brought to one of the two Jordan canonical forms

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (b = 0), \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (b \neq 0)$$

by a similarity transformation. Now suppose $n > 2$. We apply the induction hypothesis to the leading principal submatrix of order $n-1$ of N, and conclude that N is similar to a matrix

$$N' = \left[\begin{array}{cccc} & & & * \\ & N_1 & & \vdots \\ & & & * \\ 0 & \cdots & 0 & 0 \end{array}\right], \tag{9.50}$$

where N_1 is a matrix of order $n-1$ in Jordan canonical form. Each row of the matrix (9.50) contains at most two nonzero elements.

We now show that by similarity transformations we can eliminate all nonzero elements in the last column of (9.50) which are in a row containing a 1 of N_1. Suppose that there is a nonzero element h in row i of the last column, and that row i of N_1 contains a 1. Since N_1 is in Jordan canonical form this 1 is in column $i+1$. The elementary similarity transformation which adds $-h$ times column $i+1$ to column n and h times row n to row $i+1$ replaces h by 0 and does not alter any other element of the matrix (9.50). Hence we may apply a finite number of similarity transformations of this kind and obtain a matrix N' of the form (9.50) where N_1 is a matrix in Jordan canonical form and each row of N' contains at most one nonzero element.

If the last column of N' contains only 0's, then N' is in Jordan canonical form. Suppose now that there is at least one nonzero element in column n of N'. We show that by similarity transformations we can eliminate all but one nonzero element in column n without changing any other element of N'. We consider the digraph $D(N')$ with vertex set $\{1, 2, \ldots, n\}$. This digraph consists of $t \geq 0$ vertex disjoint paths $\gamma_1, \ldots, \gamma_t$, and $s \geq 1$ paths $\pi_1, \ldots, \pi_s$ which are vertex disjoint except for the fact that they all terminate at vertex n. Moreover the set of t paths and the set of s paths have no vertex in common. If $t > 0$, then the inductive assumption applies to the principal submatrix of N' obtained by deleting the rows and columns corresponding to the vertices of γ_1. We now assume that $t = 0$ so that $D(N')$ consists of paths $\pi_1, \pi_2, \ldots, \pi_s$ of lengths $\ell_1, \ell_2, \ldots, \ell_s$, respectively, all terminating at vertex n. We now show that by a sequence of similarity transformations we can eliminate all the nonzero elements in column n of N' except for one corresponding to an arc of a longest path, again without changing any other element of N'. Let h be a nonzero element in the last column n of N corresponding to the last arc of a longest path π_k. We may simultaneously permute rows and columns if necessary and assume that π_k corresponds to the last block in the Jordan canonical form N_1. The similarity transformation which multiplies the last row by h and the last column by h^{-1} allows us to assume that $h = 1$. Let

$$\pi_k : i_0 \to \cdots \to i_{\ell_k - 1} \to i_{\ell_k} = n.$$

Let

$$\pi_r : j_0 \to \cdots \to j_{\ell_r -1} \to j_{\ell_r} = n$$

be any other path with $r \neq k$. Let p be the nonzero element in position $(j_{\ell_r -1}, n)$ of N'. We successively perform the following elementary similarity transformations:

(i) Add $-p$ times row $i_{\ell_k -1}$ to row $i_{\ell_r -1}$ and add p times column $i_{\ell_r -1}$ to column $i_{\ell_k -1}$.

(ii) Add $-p$ times row $i_{\ell_k -2}$ to row $j_{\ell_r -2}$ and add p times column $j_{\ell_r -2}$ to column $i_{\ell_k -2}$.

$\vdots$

(ℓ_r) Add $-p$ times row $i_{\ell_k - \ell_r}$ to row j_0 and add p times column j_0 to column $i_{\ell_k - \ell_r}$.

The result of this sequence of similarity transformations is to replace p with 0 and leave all other elements of N' unchanged. We may repeat until all the nonzero elements in column n of N', other than the nonzero element corresponding to the arc of the longest path π_k, have been replaced with 0. The digraph of the resulting matrix N'' consists of the path π_k and the paths π'_r obtained by deleting the last arc of $\pi_r, (r \neq k)$. It follows that the matrix N'' is similar to our given matrix A and is in Jordan canonical form with Jordan blocks of sizes $\ell_1, \ldots, \ell_{k-1}, \ell_k + 1, \ell_{k+1}, \ldots, \ell_s$.

In closing we remark that the above argument only establishes the existence (and not the uniqueness) of the Jordan canonical form.

Exercises

1. Construct an acyclic digraph D which does not have a 2-path γ of maximal size $p_2(D)$ such that γ can be partitioned into two 1-paths one of which has size $p_1(D)$. Thus a 2-path (and more generally a k-path) cannot always be obtained by successively choosing 1-paths of maximal size (Stanley[1980]).
2. Show by example that Theorem 9.8.4 does not hold in general for nilpotent matrices.
3. Let T be a tree of order n and let T^* be the digraph obtained by assigning a direction to each edge of T. Let A be the (nilpotent) adjacency matrix of T^*. Prove that the Jordan partition of A equals the difference sequence $\delta p(T^*)$ (Brualdi[1987]).
4. Determine the Jordan canonical form of the matrix

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

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