Consider the following:

- an $n \times n$ matrix L (not necessarily Laplacian, or even, real),
- the Hermitian part of L, $H_L = (L + L^*)/2$,
- the $n \times 1$ all ones vector $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$,
- two orthonormal bases $\{\mathrm{q}_1, \mathrm{q}_2, \dots, \mathrm{q}_{n-1}\}$ and $\{\hat{\mathrm{q}}_1, \hat{\mathrm{q}}_2, \dots, \hat{\mathrm{q}}_{n-1}\}$ of the subspace $\Omega = \mathrm{span}\{\mathbf{1}\}^{\perp}$,
- the $n \times (n-1)$ matrices $Q = \begin{bmatrix} \mathrm{q}_1 & | & \mathrm{q}_2 & | & \cdots & | & \mathrm{q}_{n-1} \end{bmatrix}$ and $\hat{Q} = \begin{bmatrix} \hat{\mathrm{q}}_1 & | & \hat{\mathrm{q}}_2 & | & \cdots & | & \hat{\mathrm{q}}_{n-1} \end{bmatrix}$ with orthonormal columns,
- the $n \times n$ unitary matrices $V = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right) & | & Q \end{bmatrix}$ and $\hat{V} = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right) & | & \hat{Q} \end{bmatrix}$.

(Indeed, we have

$$V^* \cdot V = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right)^* \\ ----- \\ Q^* \end{bmatrix} \cdot \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right) & | & Q \end{bmatrix} = \begin{bmatrix} (\mathbf{1}^* \cdot \mathbf{1})/n & \left(\mathbf{1}/\sqrt{n}\right)^* \cdot Q \\ Q^* \cdot \left(\mathbf{1}/\sqrt{n}\right) & Q^* \cdot Q \end{bmatrix} = \begin{bmatrix} 1 & \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & I_{n-1} \end{bmatrix} = I_n,$$

and similarly, $\hat{V}^* \cdot \hat{V} = I_n$.)

It is easy to verify that

$$\hat{V} \cdot V^* \cdot \mathbf{1} = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right) & | & \hat{Q} \end{bmatrix} \cdot \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right)^* \\ ----- \\ Q^* \end{bmatrix} \cdot \mathbf{1} = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right) & | & \hat{Q} \end{bmatrix} \cdot \begin{bmatrix} n/\sqrt{n} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{1} = (\text{similarly}) = V \cdot \hat{V}^* \cdot \mathbf{1}.$$

Moreover,

$$V^* \cdot Q = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right)^* \\ ----- \\ Q^* \end{bmatrix} \cdot Q = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right)^* \cdot Q \\ ------ \\ Q^* \cdot Q \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \\ --------- \\ I_{n-1} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right)^* \cdot \hat{Q} \\ ------ \\ \hat{Q}^* \cdot \hat{Q} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{1}/\sqrt{n}\right)^* \\ ----- \\ \hat{Q}^* \end{bmatrix} \cdot \hat{Q} = \hat{V}^* \cdot \hat{Q}.$$

As a consequence, $\hat{Q} = \left(\hat{V} \cdot V^*\right) \cdot Q$, where the $n \times n$ matrix $\hat{V} \cdot V^*$ is unitary (as product of unitary matrices).

Now, let's consider the $(n-1)\times(n-1)$ Hermitian matrices

$$Q^* \cdot H_L \cdot Q \quad \text{and} \quad \hat{Q}^* \cdot H_L \cdot \hat{Q} = \left(\left(\hat{V}\cdot V^*\right)\cdot Q\right)^* \cdot H_L \cdot \left(\left(\hat{V}\cdot V^*\right)\cdot Q\right) = Q^* \cdot \left(\hat{V}\cdot V^*\right)^* \cdot H_L \cdot \left(\hat{V}\cdot V^*\right)\cdot Q .$$

Suppose that μ_0 is an eigenvalue of $Q^* \cdot H_L \cdot Q$, which is NOT an eigenvalue of H_L, with a corresponding eigenvector $x_0 \neq 0$. Then we have

$$\begin{aligned}
\left(Q^* \cdot H_L \cdot Q\right)\cdot x_0 = \mu_0 \cdot x_0 &\Leftrightarrow \left(Q^* \cdot H_L \cdot Q - Q^* \cdot (\mu_0 I_n)\cdot Q\right)\cdot x_0 = 0 \\
&\Leftrightarrow Q^* \cdot \left(H_L - \mu_0 I_n\right)\cdot Q \cdot x_0 = 0 \quad (Q\cdot x_0 \neq 0 \text{ due to indepedence of columns of } Q) \\
&\Leftrightarrow \left(H_L - \mu_0 I_n\right)\cdot Q \cdot x_0 \ \text{ is a non-zero vector in } \mathrm{Ker}(Q^*) \quad (\mu_0 \text{ is not an eigv of } H_L).
\end{aligned}$$

Consider the $(n-1)\times(n-1)$ matrix $S=[s_{i,j}]$ whose entries satisfy the relations (recall that the columns of Q and $\hat{Q}$ are bases of the same subspace)

$$\mathrm{q}_j = s_{1,j}\hat{\mathrm{q}}_1 + s_{2,j}\hat{\mathrm{q}}_2 + \cdots + s_{\mathrm{n}-1,j}\hat{\mathrm{q}}_{n-1}, \qquad j = 1,2,\ldots,n-1,$$

or equivalently,

$$\left[\begin{array}{c|c|c|c} \mathrm{q}_1 & \mathrm{q}_2 & \cdots & \mathrm{q}_{n-1} \end{array}\right] = \left[\begin{array}{c|c|c|c} \hat{\mathrm{q}}_1 & \hat{\mathrm{q}}_2 & \cdots & \hat{\mathrm{q}}_{n-1} \end{array}\right] \cdot \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n-1} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{\mathrm{n}-1,1} & s_{n-1,2} & \cdots & s_{n-1,\mathrm{n}-1} \end{bmatrix} \Leftrightarrow Q = \hat{Q}\cdot S.$$

Then, it follows that

$$\begin{aligned}
\left(Q^* \cdot H_L \cdot Q\right)\cdot x_0 = \mu_0 \cdot x_0 &\Leftrightarrow \left(H_L - \mu_0 I_n\right)\cdot \hat{Q}\cdot (S\cdot x_0) \ \text{ is a non-zero scalar multiple of the vector } \mathbf{1} \\
&\Leftrightarrow \hat{Q}^* \cdot \left(H_L - \mu_0 I_n\right)\cdot \hat{Q}\cdot (S\cdot x_0) \ = 0 \\
&\Leftrightarrow \hat{Q}^* \cdot H_L \cdot \hat{Q}\cdot (S\cdot x_0) = \mu_0 \cdot (S\cdot x_0) .
\end{aligned}$$

Since $\mathrm{rank}(Q) - \mathrm{rank}(\hat{Q}) = n-1$, the matrix S is invertible, and hence, $S\cdot x_0 \neq 0$.

Thus, we conclude that the Hermitian matrices $Q^* \cdot H_L \cdot Q$ and $\hat{Q}^* \cdot H_L \cdot \hat{Q}$ have exactly the same eigenvalues in $\mathbb{C}\backslash\sigma(H_L)$ with the same (algebraic and geometric) multiplicities.