

Spectral Characterization of Dominance Graph

Measure Functions

```
In[1]:= eigM[e_, s_] := Module[{k, perm},
    perm = Permutations[e];
    Min[Table[Max[Abs[perm[[k]] - s]], {k, 1, Length[perm]}]]
];

subD[u_, v_] := Module[{n, q, r},
    n = Length[u];
    q = RandomReal[{-1, 1}, {n, n}];
    q[[All, 1]] = v;
    {q, r} = QRDecomposition[q];
    q = ConjugateTranspose[q];
    Norm[Normalize[Conjugate[u]].q[[All, 2 ;; n]]]
];

vecM[v_, s_] := Module[{i, j, perm},
    perm = Permutations[v];
    Min[Table[Max[Table[subD[perm[[i]][[All, j]], s[[All, j]]], {j, 1, Length[v]}]],
        {i, 1, Length[perm]}]]
];

rankM[e_, v_] := Module[{k, n, s1, s2},
    n = Length[e];
    s1 = Table[k, {k, n - 1, 0, -1}];
    s2 = IdentityMatrix[n];
    For[k = 2, k ≤ n, k++,
        s2[[All, k]] = s2[[All, k]] + s2[[All, k - 1]];
    ];
    eigM[N[e], s1] + vecM[N[v], s2]
];
```

Examples

```
In[5]:= s = {{1, 1, 1, 1, 1, 1}, {0, 1, 1, 1, 1, 1}, {0, 0, 1, 1, 1, 1},
    {0, 0, 0, 1, 1, 1}, {0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 0, 1}};
```

Permuted Dominance Graph

```
In[6]:= a = {{0, 1, 1, 1, 1, 1}, {0, 0, 1, 0, 1, 1}, {0, 0, 0, 0, 1, 1},
           {0, 1, 1, 0, 1, 1}, {0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0}};
d = {{5, 0, 0, 0, 0, 0}, {0, 3, 0, 0, 0, 0}, {0, 0, 2, 0, 0, 0},
      {0, 0, 0, 4, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0}};
l = d - a;
{e, v} = Eigensystem[l];
v = Transpose[v];
eigM[N[e], {5, 4, 3, 2, 1, 0}]
vecM[N[v], s]
rankM[e, v]
```

Out[11]= 0.

Out[12]= 2.84495×10^{-16}

Out[13]= 2.6258×10^{-16}

Dominance + Perturbation

```
In[14]:= a = {{0, 1, 1, 1, 1, 1}, {0, 0, 0, 1, 1, 1}, {1, 0, 0, 1, 1, 1},
           {0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0}};
d = {{5, 0, 0, 0, 0, 0}, {0, 3, 0, 0, 0, 0}, {0, 0, 4, 0, 0, 0},
      {0, 0, 0, 2, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0}};
l = d - a;
{e, v} = Eigensystem[l];
v = Transpose[v];
eigM[N[e], {5, 4, 3, 2, 1, 0}]
vecM[N[v], s]
rankM[e, v]
```

Out[19]= 0.618034

Out[20]= 0.525731

Out[21]= 1.14377

Perturbed Random Graph

```
In[22]:= a = {{0, 1, 0, 0, 1, 1}, {0, 0, 1, 1, 0, 0}, {0, 1, 0, 0, 0, 0},
             {1, 0, 0, 0, 0, 1}, {0, 0, 0, 1, 0, 0}, {1, 1, 1, 0, 1, 0}};
d = {{3, 0, 0, 0, 0, 0}, {0, 2, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0},
     {0, 0, 0, 2, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 4}};
l = d - a;
{e, v} = Eigensystem[l];
v = Transpose[v];
eigM[N[e], {5, 4, 3, 2, 1, 0}]
vecM[N[v], s]
rankM[e, v]
```

Out[27]= 1.70205

Out[28]= 0.905628

Out[29]= 2.60768

Nearly Disconnected

```
In[30]:= a = {{0, 1, 1, 1, 0, 0}, {0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
             {0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0}};
d = {{3, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
     {0, 0, 0, 2, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0}};
l = d - a;
{e, v} = Eigensystem[l];
v = Transpose[v];
eigM[N[e], {5, 4, 3, 2, 1, 0}]
vecM[N[v], s]
rankM[e, v]
```

Out[35]= 2.

Out[36]= 0.74162

Out[37]= 2.74162

Random

```
In[38]:= a = {{0, 1, 0, 0, 0, 1}, {0, 0, 1, 1, 0, 0}, {0, 1, 0, 0, 0, 0},
             {1, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0}, {0, 1, 1, 0, 0, 0}};
d = {{2, 0, 0, 0, 0, 0}, {0, 2, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0},
     {0, 0, 0, 2, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 2}};
l = d - a;
{e, v} = Eigensystem[l];
v = Transpose[v];
eigM[N[e], {5, 4, 3, 2, 1, 0}]
vecM[N[v], s]
rankM[e, v]
```

Out[43]= 2.64575

Out[44]= 0.912871

Out[45]= 3.55862

Cyclic

```
In[46]:= a = {{0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0},
             {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1}, {1, 0, 0, 0, 0, 0}};
d = IdentityMatrix[6];
l = d - a;
{e, v} = Eigensystem[l];
v = Transpose[v];
eigM[N[e], {5, 4, 3, 2, 1, 0}]
vecM[N[v], s]
rankM[e, v]
```

Out[51]= 3.

Out[52]= 0.983192

Out[53]= 3.98319

Completely Connected

```
In[54]:= a = {{0, 1, 1, 1, 1, 1}, {1, 0, 1, 1, 1, 1}, {1, 1, 0, 1, 1, 1},
             {1, 1, 1, 0, 1, 1}, {1, 1, 1, 1, 0, 1}, {1, 1, 1, 1, 1, 0}};
d = 5 * IdentityMatrix[6];
l = d - a;
{e, v} = Eigensystem[l];
v = Transpose[v];
eigM[N[e], {5, 4, 3, 2, 1, 0}]
vecM[N[v], s]
rankM[e, v]
```

Out[59]= 5.

Out[60]= 0.948683

Out[61]= 5.94868

Empty Graph

```
In[62]:= a = ConstantArray[0, {6, 6}];
d = ConstantArray[0, {6, 6}];
l = d - a;
{e, v} = Eigensystem[l];
v = Transpose[v];
eigM[N[e], {5, 4, 3, 2, 1, 0}]
vecM[N[v], s]
rankM[e, v]
```

Out[67]= 5.

Out[68]= 0.912871

Out[69]= 5.91287