Rankability and the SVD

Rankable and Unrankable Graphs

We use the examples given in Figure 5.1 of “The rankability of data” by Anderson, Chartier, and Langville. For each example the corresponding adjacency matrix is formed and the singular values are displayed in the table below.

Adjacency Matrix

In[1]:=
```
(* Dominance Graph *)
A1 = {{0, 1, 1, 1, 1, 1}, {0, 0, 1, 1, 1, 1}, {0, 0, 0, 1, 1, 1},
   {0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0}};
(* Dominance with small perturbation *)
A2 = {{0, 1, 1, 1, 1, 1}, {0, 0, 0, 1, 1, 1}, {1, 0, 0, 1, 1, 1},
   {0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0}};
(* Tree *)
A3 =
  {{0, 1, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 1, 0, 0}, {0, 0, 0, 0, 0, 1, 1}, {0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}};
(* Completely Connected *)
A4 = {{0, 1, 1, 1, 1, 1}, {1, 0, 1, 1, 1, 1}, {1, 1, 0, 1, 1, 1},
   {1, 1, 1, 0, 1, 1}, {1, 1, 1, 1, 0, 1}, {1, 1, 1, 1, 1, 0}};
(* Cyclic *)
A5 = {{0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0},
   {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1}, {1, 0, 0, 0, 0, 0}};
(* Completely Decomposable *)
A6 = {{0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0}, {1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1}, {0, 0, 0, 1, 0, 0}};
```

SVD

In[7]:=
```
{u1, s1, v1} = SingularValueDecomposition[N[A1]];
{u2, s2, v2} = SingularValueDecomposition[N[A2]];
{u3, s3, v3} = SingularValueDecomposition[N[A3]];
{u4, s4, v4} = SingularValueDecomposition[N[A4]];
{u5, s5, v5} = SingularValueDecomposition[N[A5]];
{u6, s6, v6} = SingularValueDecomposition[N[A6]];
S = {Diagonal[s1], Diagonal[s2],
   Diagonal[s3], Diagonal[s4], Diagonal[s5], Diagonal[s6]};
```

Table of Singular Values

In[14]:=
```
name = {"Dominance Graph", "Dominance with small perturbation",
   "Tree", "Completely Connected", "Cyclic", "Completely Decomposable"};
res = Table[{name[[i]], S[[i]]}, {i, 1, 6}];
MatrixForm[res]
```

Out[16]//MatrixForm=

Dominance Graph	{3.51334, 1.20362, 0.763521, 0.594351, 0.521109, 0
Dominance with small perturbation	{3.42194, 1.27676, 1., 0.628391, 0.515115, 0.}
Tree	{1.41421, 1.41421, 1.41421, 0., 0., 0., 0.}
Completely Connected	{5., 1., 1., 1., 1., 1.}
Cyclic	{1., 1., 1., 1., 1., 1.}
Completely Decomposable	{1., 1., 1., 1., 1., 1.}

Remarks

It is interesting to note that the unrankable graphs never have a singular value smaller than 1, whereas the rankable graphs all have a smallest singular value of 0. Furthermore, the dominance graph and perturbed dominance graph both have a largest singular value that is 3 times greater than the next largest singular value. Finally, the cyclic graph and completely decomposable graph both have the same SVD. This makes sense as the completely decomposable graph is simply two disjoint cyclic graphs. Furthermore, you can see the completely decomposable is block diagonal with each block the adjoint of a smaller cyclic graph.

In[17]:=
```
MatrixForm[A6]
```

Out[17]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$