2

Matrices and Graphs

2.1 Basic Concepts

A *graph* G (*simple graph*) consists of a finite set $V = \{a, b, c, \ldots\}$ of elements called *vertices* (*points*) together with a prescribed set E of *unordered* pairs of *distinct* vertices of V. (The set E is necessarily finite.) The number n of elements in the finite set V is called the *order* of the graph G. Every unordered pair α of vertices a and b in E is called an *edge* (*line*) of the graph G, written

$$\alpha = \{a, b\} = \{b, a\}.$$

We call a and b the *endpoints* of α. Two vertices on the same edge or two distinct edges with a common vertex are *adjacent*. Also, an edge and a vertex are *incident* with one another if the vertex is contained in the edge. Those vertices incident with no edge are *isolated*. A *complete graph* is one in which all possible pairs of vertices are edges. Let G be a graph and let K be the complete graph with the same vertex set V. Then the *complement* $\overline{G}$ of G is the graph with vertex set V and with edge set equal to the set of edges of K minus those of G.

A *subgraph* of a graph G consists of a subset V' of V and a subset E' of E that themselves form a graph. If E' contains all edges of G both of whose endpoints belong to V', then the subgraph is called an *induced subgraph* and is denoted by $G(V')$. A *spanning subgraph* of G has the same vertex set as G. Two graphs G and G' are *isomorphic* provided there exists a 1-1 correspondence between their vertex sets that preserves adjacency. Two complete graphs with the same order are isomorphic, and we denote a complete graph of order n by K_n.

If the definition of a graph is altered to allow a pair of vertices to form more than one distinct edge, then the structure is called a *multigraph*. Its

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edges are called *multiedges* (*multilines*) and the number of distinct edges of the form $\{a, b\}$ is called the *multiplicity* $m\{a, b\}$ of the edge $\{a, b\}$. The further generalization by allowing *loops*, edges of the form $\{a, a\}$ making a vertex adjacent to itself, results in a *general graph*. For both multigraphs and general graphs we require that the edge sets be finite. Terms such as *order*, *endpoints*, *adjacent*, *incident*, *isolated*, etc. carry over directly to multigraphs and general graphs.

Let G be a multigraph. Then the *degree* (*valency*) of a vertex in G is the number of edges incident with the vertex. Since each edge of G has two distinct endpoints, the sum of the degrees of the vertices of G is twice the number of its edges. The graph G is *regular* if all vertices have the same degree. If there are precisely k edges incident with each vertex of a graph, then we say that the graph is *regular of degree* k. A regular graph of degree 3 is called *cubic*.

One may ask for the number of graphs of a specified order n. This number has been determined in a certain sense. But the answer is far from elementary and we refer the reader to Harary and Palmer[1973] for a discussion of a variety of problems dealing with graphical enumeration.

Exercises

1. Prove there are as many graphs of order n with k edges as there are with $\binom{n}{2} - k$ edges. Determine the number of graphs of order at most 5.
2. Prove that a graph always has two distinct vertices with the same degree. Show by example that this need not hold for multigraphs.
3. Prove that a cubic graph has an even number of vertices.

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2.2 The Adjacency Matrix of a Graph

Let G denote a general graph of order n with vertex set

$$V = \{a_1, a_2, \ldots, a_n\}.$$

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We let a_{ij} equal the multiplicity $m\{a_i, a_j\}$ of the edges of the form $\{a_i, a_j\}$. This means, of course, that $a_{ij} = 0$ if there are no edges of the form $\{a_i, a_j\}$. Also, $m\{a_i, a_i\}$ equals the number of loops at vertex a_i. The resulting matrix

$$A = [a_{ij}], \quad (i, j, = 1, 2, \ldots, n)$$

of order n is called the *adjacency matrix* of G. The matrix A characterizes G.

We note that A is a symmetric matrix with nonnegative integral elements. The trace of A denotes the number of loops in G. If G is a multigraph, then the trace of A is zero and the sum of line i of A equals the degree of vertex a_i. If G is a graph, then A is a symmetric (0,1)-matrix of trace zero.

The concept of graph isomorphism has a direct interpretation in terms of the adjacency matrix of the graph. Thus let G and G' denote two general graphs of order n and let the adjacency matrices of these graphs be denoted by A and A', respectively. Then the general graphs G and G' are isomorphic if and only if A is transformable into A' by simultaneous permutations of the lines of A. Thus G and G' are isomorphic if and only if there exists a permutation matrix P of order n such that

$$PAP^T = A'.$$

Let G be a general graph. A sequence of m successively adjacent edges

$$\{a_0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{m-1}, a_m\}, (m > 0)$$

is called a *walk* of *length* m, and is also denoted by

$$a_0 \to a_1 \to a_2 \to \cdots \to a_{m-1} \to a_m$$

and by

$$a_0, a_1, a_2, \ldots, a_{m-1}, a_m.$$

The vertices a_0 and a_m are the *endpoints* of the walk. The walk is *closed* or *open* according as $a_0 = a_m$ or $a_0 \neq a_m$. A walk with distinct edges is called a *trail.* A walk with distinct edges and in addition distinct vertices (except, possibly, $a_0 = a_m$) is called a *chain.* A closed chain is called a *cycle.* Notice that in a graph a cycle must contain at least 3 edges. But in a general graph a loop or a pair of multiple edges form a cycle.

Let us now form

$$A^2 = \left[\sum_{t=1}^{n} a_{it}a_{tj}\right], \quad (i, j = 1, 2, \ldots, n). \tag{2.1}$$

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Then (2.1) implies that the element in the (i, j) position of A^2 equals the number of walks of length 2 with a_i and a_j as endpoints. In general, the element in the (i, j) position of A^k equals the number of walks of length k with a_i and a_j as endpoints. The numbers for closed walks appear on the main diagonal of A^k.

Let G be the complete graph K_n of order n. We determine the number of walks of length k in K_n with a_i and a_j as endpoints. The adjacency matrix of K_n is

$$A = J - I.$$

We know that $J^e = n^{e-1}J$ so that

$$A^k = \left[n^{k-1} - \binom{k}{1} n^{k-2} + \binom{k}{2} n^{k-3} - \cdots + (-1)^{k-1}\binom{k}{k-1}\right] J + (-1)^k I.$$

But

$$(n-1)^k = n^k - \binom{k}{1} n^{k-1} + \binom{k}{2} n^{k-2} - \cdots + (-1)^{k-1}\binom{k}{k-1} n + (-1)^k$$

and hence we have

$$A^k = \left(\frac{(n-1)^k - (-1)^k}{n}\right) J + (-1)^k I.$$

We return to the general graph G and its adjacency matrix A. The polynomial

$$f(\lambda) = \det(\lambda I - A)$$

is called the *characteristic polynomial* of G. The collection of the n eigenvalues of A is called the *spectrum* of G. Since A is symmetric the spectrum of G consists of n real numbers.

Suppose that G and G' are isomorphic general graphs. Then we have noted that there exists a permutation matrix P such that the adjacency matrices A and A' of G and G', respectively, satisfy

$$PAP^T = A'.$$

But the transpose of a permutation matrix is equal to its inverse. Thus A and A' are similar matrices and hence G and G' have the same spectrum. Two nonisomorphic general graphs G and G' with the same spectrum are called *cospectral*. We exhibit in Figure 2.1 two pairs of cospectral graphs of orders 5 and 6 with characteristic polynomials $f(\lambda) = (\lambda - 2)(\lambda + 2)\lambda^3$ and $f(\lambda) = (\lambda - 1)(\lambda + 1)^2(\lambda^3 - \lambda^2 - 5\lambda + 1)$, respectively.

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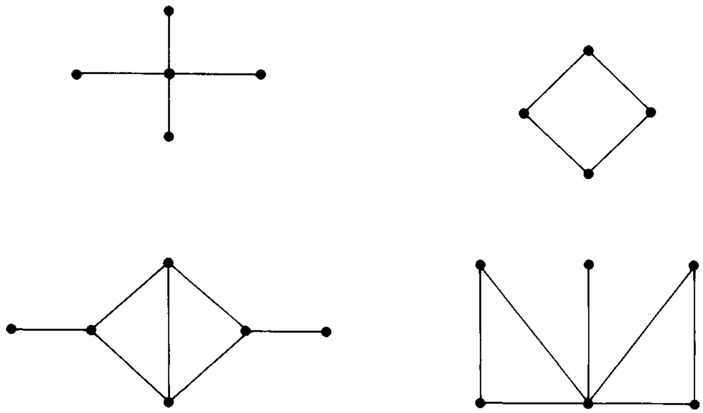

Figure 2.1. Two pairs of cospectral graphs.

A general graph G is *connected* provided that every pair of vertices a and b is joined by a walk with a and b as endpoints. A vertex is regarded as trivially connected to itself. Otherwise, the general graph is *disconnected.* Connectivity between vertices is reflexive, symmetric, and transitive. Hence connectivity defines an equivalence relation on the vertices of G and yields a partition

$$V_1 \cup V_2 \cup \cdots \cup V_t,$$

of the vertices of G. The induced subgraphs $G(V_1)$,$G(V_2), \ldots, G(V_t)$ of G formed by taking the vertices in an equivalence class and the edges incident to them are called the *connected components* of G. For most problems concerning G it suffices to study only the connected components of G.

Connectivity has a direct interpretation in terms of the adjacency matrix A of G. Thus we may simultaneously permute the lines of A so that A is transformed into a direct sum of the form

$$A_1 \oplus A_2 \oplus \cdots \oplus A_t,$$

where A_i is the adjacency matrix of the connected component $G(V_i)$, $(i = 1, 2, \ldots, t)$.

Let G be a connected general graph. The length of the shortest walk between two vertices a and b is the *distance* $d(a, b)$ between a and b in G. A vertex is regarded as distance 0 from itself. The maximum value of the distance function over all pairs of vertices is called the *diameter* of G.

Theorem 2.2.1. *Let G be a connected general graph of diameter d. Then G has at least $d+1$ distinct eigenvalues in its spectrum.*

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Proof. Let a and b be vertices with $d(a,b) = d$ and let

$$a_0 \to a_1 \to a_2 \to \cdots \to a_d$$

be a walk of length equal to the diameter d. Then for each $i = 1, 2, \ldots, d$ there is at least one walk of length i and no shorter walk that joins a_0 to a_i. Thus A^i has a nonzero entry in the position determined by a_0 and a_i, whereas $I, A, A^2, \ldots, A^{i-1}$ each have zeros in this position. We conclude that A^i is not a linear combination of $I, A, A^2, \ldots, A^{i-1}$. Hence the minimum polynomial of A is of degree at least $d+1$. But since A is a real symmetric matrix it is similar to a diagonal matrix and consequently the zeros of its minimal polynomial are distinct. □

Exercises

1. Prove that the complement of a disconnected graph is a connected graph.
2. Determine the spectrum of the complete graph K_n of order n.
3. Show that k is an eigenvalue of a regular graph of degree k.
4. Let G be a graph of order n. Suppose that G is regular of degree k and let $\lambda_1 = k, \lambda_2, \ldots, \lambda_n$ be the spectrum of G. Prove that the spectrum of the complement of G is $n-1-k, -1-\lambda_2, \ldots, -1-\lambda_n$.
5. Let $f(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$ be the characteristic polynomial of a graph G of order n. Prove that c_1 equals 0, c_2 equals -1 times the number of edges of G, and c_3 equals -2 times the number of cycles of length 3 of G (a cycle of length 3 in a graph is sometimes called a *triangle*).
6. Let $K_{1,n-1}$ be the graph of order n whose vertices have degrees $n-1, 1, \ldots, 1$, respectively. ($K_{1,n-1}$ is the *star* of order n.) Prove that the spectrum of $K_{1,n-1}$ is $\pm\sqrt{n-1}, 0, \ldots, 0$.
7. Prove that there does not exist a connected graph which is cospectral with the star $K_{1,n-1}$.
8. Let G be a connected graph of order n which is regular of degree 2. The edges of G thus form a cycle of length n, and G is sometimes called a *cycle graph* of order n. Determine the spectrum of G.

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2.3 The Incidence Matrix of a Graph

Let G be a general graph of order n with vertices $a_1, a_2, \ldots, a_n$ and edges $\alpha_1, \alpha_2, \ldots, \alpha_m$. We set $a_{ij} = 1$ if vertex a_j is on edge α_i and we set $a_{ij} = 0$ otherwise. The resulting (0,1)-matrix

$$A = [a_{ij}], \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

of size m by n is called the *incidence matrix* of G. The matrix A is in fact the conventional incidence matrix in which the edges are regarded as subsets of vertices. Each row of A contains at least one 1 and not more than two 1's. The rows with a single 1 in A correspond to the edges in G that are loops. Identical rows in A correspond to multiple edges in G.

The simple row structure of the matrix A is misleading because A describes the full complexity of the general graph G. For example, there is no computationally effective procedure known for the determination of the minimal number of columns in A with the property that these columns of A collectively contain at least one 1 in each of the m rows of A. In terms of G this quantity is the minimal number of vertices in G that touch all edges.

The incidence matrix and the adjacency matrix of a multigraph are related in the following way.

Theorem 2.3.1. *Let G be a multigraph of order n. Let A be the incidence matrix of G and let B be the adjacency matrix of G. Then*

$$A^T A = D + B,$$

where D is a diagonal matrix of order n whose diagonal entry d_i is the degree of the vertex a_i of G, $(i = 1, 2, \ldots, n)$.

Proof. The inner product of columns i and j of A $(i \neq j)$ equals the multiplicity $m\{a_i, a_j\}$ of the edge $\{a_i, a_j\}$. The inner product of column i of A with itself equals the degree of the vertex a_i. □

Now let G denote a graph of order n with vertices $a_1, a_2, \ldots, a_n$ and edges $\alpha_1, \alpha_2, \ldots, \alpha_m$. We assign to each of the edges of G one of the two possible orientations and thereby transform G into a graph in which each of the edges of G is assigned a direction. We set $a_{ij} = 1$ if a_j is the "initial" vertex of α_i, we set $a_{ij} = -1$ if a_j is the "terminal" vertex of α_i and we set $a_{ij} = 0$ if a_j is not an endpoint of α_i. The resulting $(0, 1, -1)$-matrix

$$A = [a_{ij}], \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

of size m by n is called the *oriented incidence matrix* of G. Each row of A contains exactly two nonzero entries, one of which is 1 and the other -1.

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We note that in the notation of Theorem 2.3.1 the oriented incidence matrix satisfies

$$A^T A = D - B. \tag{2.2}$$

The matrix $A^T A$ in (2.2) is called the *Laplacian matrix* (also called the *admittance matrix*) of G. It follows from (2.2) that the Laplacian matrix is independent of the particular orientation assigned to G. The Laplacian matrix will be discussed further in section 2.5.

The oriented incidence matrix is used to determine the number of connected components of G.

Theorem 2.3.2. *Let G be a graph of order n and let t denote the number of connected components of G. Then the oriented incidence matrix A of G has rank $n - t$. In fact, each matrix obtained from A by deleting t columns, one corresponding to a vertex of each component, has rank $n - t$. A submatrix A' of A of order $n-1$ has rank $n-t$ if and only if the spanning subgraph G' of G whose edges are those corresponding to the rows of A' has t connected components.*

Proof. Let the connected components of G be denoted by

$$G(V_1), G(V_2), \ldots, G(V_t).$$

Then we may label the vertices and edges of G so that the oriented incidence matrix A is a direct sum of the form

$$A_1 \oplus A_2 \oplus \cdots \oplus A_t,$$

where A_i displays the vertices and edges in $G(V_i), (i = 1, 2, \ldots, t)$. Let $G(V_i)$ contain n_i vertices. We prove that the rank of A_i equals $n_i - 1$. The conclusion then follows by addition.

Let β_j denote the column of A_i corresponding to the vertex a_j of $G(V_i)$. Since each row of A_i contains exactly one 1 and one -1, it follows that the sum of the columns of A_i is the zero vector. Hence the rank of A_i is at most $n_i - 1$. Suppose then that we have a linear relation $\sum b_j \beta_j = 0$, where the summation is over all columns of A_i and not all the coefficients are zero. Let us suppose that column β_k has $b_k \neq 0$. This column has nonzero entries in those rows corresponding to the edges incident with a_k. For each such row there is just one other column β_l with a nonzero entry in that row. In order for the dependency to hold we must have $b_k = b_l$. Hence if $b_k \neq 0$, then $b_l = b_k$ for all vertices a_l adjacent to a_k. Since $G(V_i)$ is connected it follows that all of the coefficients b_j are equal and the linear relation is merely a multiple of our earlier relation $\sum \beta_j = 0$. Hence the rank of $G(V_i)$ is $n_i - 1$, and deleting any column of A_i results in a matrix of rank $n_i - 1$. Finally we observe that the last conclusion of the theorem follows by applying the earlier conclusions to G'. □

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A matrix A with integral elements is *totally unimodular* if every square submatrix of A has determinant $0, 1,$ or -1. It follows at once that a totally unimodular matrix is a $(0, 1, -1)$-matrix.

The following theorem is due to Hoffman and Kruskal[1956].

Theorem 2.3.3. *Let A be an m by n matrix whose rows are partitioned into two disjoint sets B and C and suppose that the following four properties hold:*

(i) Every entry of A is $0, 1,$ or -1.
(ii) Every column of A contains at most two nonzero entries.
(iii) If two nonzero entries in a column of A have the same sign, then the row of one is in B and the row of the other is in C.
(iv) If two nonzero entries in a column of A have opposite signs, then the rows of both are in B or in C.

Then the matrix A is totally unimodular.

Proof. An arbitrary submatrix of A also satisfies the hypothesis of the theorem. Hence it suffices to prove that an arbitrary square matrix A satisfying the hypotheses of the theorem has $\det(A)$ equal to 0,1, or -1. The proof is by induction on n. For $n = 1$ the theorem follows trivially from (i). Suppose that every column of A has two nonzero entries. Then the sum of the rows in B equals the sum of the rows in C and $\det(A) = 0$. This assertion is also valid in case $B = \emptyset$ or $C = \emptyset$. Also if some column of A has all 0's, then $\det(A) = 0$. Hence we are left with the case in which some column of A has exactly one nonzero entry. We expand $\det(A)$ by this column and apply the induction hypothesis. □

The preceding result implies the following theorem of Poincaré[1901].

Corollary 2.3.4. *The oriented incidence matrix A of a graph G is totally unimodular.*

Proof. We apply Theorem 2.3.3 to the matrix A^T with $C = \emptyset$. □

A square $(0, 1, -1)$-matrix is *Eulerian* provided all line sums are even integers. Camion[1965] (see also Padberg[1976]) has established the following theorem giving a necessary and sufficient condition for total unimodularity which we state without proof.

Theorem 2.3.5. *A $(0, 1, -1)$-matrix A of size m by n is totally unimodular if and only if the sum of the elements in each Eulerian submatrix is a multiple of 4.*

Totally unimodular matrices are intimately related to a special class of *matroids* (see White[1986]) called *unimodular matroids* (also called *regular matroids*). These are matroids which can be coordinatized by a matrix A over the rational field for which A is totally unimodular. Unimodular

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matroids were characterized by Tutte[1958] (see also Gerards[1989]) in a very striking theorem. Another striking characterization of unimodular matroids was obtained by Seymour[1980]. The characterizations of Tutte and of Seymour are in terms of the linear dependence structure of the columns of the matrix A.

A *tree* is a connected graph that contains no cycle. We will assume a familiarity with a few of the most elementary properties of trees (Brualdi [1977] or Wilson[1972]). Let T be a graph of order n. Then the following statements are equivalent: (1) T is a tree; (2) T contains no cycles and has exactly $n-1$ edges; (3) T is connected and has exactly $n-1$ edges; (4) each pair of distinct vertices of T is joined by exactly one chain.

Let T be a tree of order n with vertices $a_1, a_2, \ldots, a_n$ and edges $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. We suppose that the edges of T have been oriented. Let (s_i, t_i), $(i = 1, 2, \ldots, l)$ be l ordered pairs of vertices of T. We set $m_{ij} = 1$ if the unique chain γ in T joining s_i and t_i uses the edge α_j in its assigned direction, we set $m_{ij} = -1$ if the chain γ uses the edge α_j in the direction opposite to its assigned direction, and we set $m_{ij} = 0$ if γ does not use the edge α_j. The resulting $(0, 1, -1)$-matrix

$$M = [m_{ij}], \quad (i = 1, 2, \ldots, l;\ j = 1, 2, \ldots, n-1)$$

of size l by $n-1$ is called a *network matrix* (Tutte[1965]). If we delete the column of the network matrix A corresponding to the arc α_k of G, the resulting matrix is a network matrix for the tree obtained from T by contracting the edge α_k, that is by deleting the arc α_k and identifying its two endpoints. It follows that submatrices of network matrices are also network matrices.

Theorem 2.3.6. *A network matrix M corresponding to the oriented tree T is a totally unimodular matrix.*

Proof. We continue with the notation in the preceding paragraph. Let G be the graph with vertices $a_1, a_2, \ldots, a_n$ and edges $\{s_i, t_i\}, (i = 1, 2, \ldots, l)$. We orient each edge $\{s_i, t_j\}$ from s_i to t_j. Let A be the l by n oriented incidence matrix of G, and let B be the $n-1$ by n oriented incidence matrix of T. Let A' and B' result from A and B, respectively, by deleting the last column (the column corresponding to vertex a_n in each case). From the definitions of the matrices involved we obtain the relation $MB = A$. Hence $MB' = A'$, and since by Theorem 2.3.2 B' is invertible, the relation $M = A'B'^{-1}$ holds. By Theorem 2.3.3 the matrix

$$\begin{bmatrix} I_{n-1} \\ B' \\ A' \end{bmatrix}$$

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is totally unimodular. It follows that the matrix

$$\begin{bmatrix} I_{n-1} \\ B' \\ A' \end{bmatrix} B'^{-1} = \begin{bmatrix} B'^{-1} \\ I_{n-1} \\ A'B'^{-1} \end{bmatrix}$$

has the property that all of its submatrices of order $n-1$ have determinants equal to one of $0, 1$ and -1. This implies that the matrix $M = A'B'^{-1}$ is totally unimodular. □

Seymour's characterization of unimodular matroids can be restated in matrix terms. In this characterization the totally unimodular matrices

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix}, \tag{2.3}$$

and

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{2.4}$$

have an exceptional role.

The following theorem of Seymour[1982] asserts that a totally unimodular matrix which is not a network matrix, the transpose of a network matrix, or one of the two exceptional matrices above admits a "diagonal decomposition" into smaller totally unimodular matrices.

Theorem 2.3.7. *Let A be a totally unimodular matrix. Then one of the following properties holds:*

(i) A is a network matrix or the transpose of a network matrix;

(ii) A can be obtained from one of the two exceptional matrices above by line permutations and by the multiplication of some of their lines by -1;

(iii) The lines of A can be permuted to obtain a matrix

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where B_{11} and B_{22} are totally unimodular matrices, and either
(a) $\operatorname{rank}(B_{12}) + \operatorname{rank}(B_{21}) = 0$
and B_{11} and B_{22} each have at least one line, or

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(b) $\text{rank}(B_{12}) + \text{rank}(B_{21}) = 1$
and B_{11} *and* B_{22} *each have at least two lines, or*
(c) $\text{rank}(B_{12}) + \text{rank}(B_{21}) = 2$
and B_{11} *and* B_{22} *each have at least six lines.*

The above theorem implies that it is possible to construct the entire class of totally unimodular network matrices from the class of network matrices and the two exceptional matrices (2.3) and (2.4). Neither of these exceptional matrices is a network matrix or the transpose of a network matrix. If the matrix A satisfies *(iii)* above, then it does not necessarily follow that A is totally unimodular. However, a result of Brylawski[1975] gives a list of conditions such that a matrix A which satisfies *(iii)* is totally unimodular and such that every totally unimodular matrix can be constructed in the manner of *(iii)* starting from the network matrices, the transposes of the network matrices and the two exceptional matrices. These conditions are difficult to state and we refer the reader to the original paper by Brylawski. A consequence of Seymour's characterizations of unimodular matroids and totally unimodular matrices is the existence of an algorithm to determine whether a $(0, 1, -1)$-matrix is totally unimodular whose number of steps is bounded by a polynomial function in the number of lines of the matrix (see Schrijver[1986]).

Exercises

1. Verify that the matrices (2.3) and (2.4) are totally unimodular.
2. Prove that each nonsingular submatrix of a totally unimodular matrix has an integral inverse.
3. Let A be a totally unimodular matrix of size m by n. Let b be a matrix of size m by 1 each of whose elements is an integer. Prove that the consistent equation $Ax = b$ has an integral solution.
4. Let A be a totally unimodular matrix and let B be a nonsingular submatrix of A of order k. Prove that for each nonzero $(0, 1, -1)$-vector y of size k, the greatest common divisor of the elements of the vector yB equals 1. [Indeed Chandrasekaran has shown that this property characterizes totally unimodular matrices (see Schrijver[1986]).]
5. Let A be a nonsingular $(0, 1, -1)$-matrix and suppose that $|\det(A)| \neq 1$. Prove that A has a square submatrix B with $|\det(B)| = 2$.

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2.4 Line Graphs

Let G denote a graph of order n on m edges. The *line graph* $L(G)$ of G is the graph whose vertices are the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G have a vertex in common. The line graph $L(G)$ is of order m.

Theorem 2.4.1. *Let A be the incidence matrix of a graph G on m edges and let B_L be the adjacency matrix of the line graph $L(G)$. Then*

$$AA^T = 2I_m + B_L. \tag{2.5}$$

Proof. For $i \neq j$ the entry in the (i,j) position of B_L is 1 if the edges α_i and α_j of G have a vertex in common and 0 otherwise. But the same conclusion holds for the entry in the (i,j) position of AA^T. The main diagonal elements are as indicated. □

Theorem 2.4.1 implies a severe restriction on the spectrum of a line graph.

Theorem 2.4.2. *If λ is an eigenvalue of the line graph $L(G)$, then $\lambda \geq -2$. If G has more edges than vertices, then $\lambda = -2$ is an eigenvalue of $L(G)$.*

Proof. The symmetric matrix AA^T is positive semidefinite and hence its eigenvalues are nonnegative. But if α is an eigenvector of B_L associated with the eigenvalue λ, then (2.5) implies

$$AA^T\alpha = (2+\lambda)\alpha$$

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so that $\lambda \geq -2$. If G has more edges than vertices, then AA^T is singular and hence 0 is an eigenvalue of AA^T. □

If the graph G is regular of degree k, then the number of its edges is $m = nk/2$ and the line graph $L(G)$ is regular of degree $2(k-1)$. The following theorem of Sachs[1967] shows that in this case the characteristic polynomials of G and $L(G)$ are related in an elementary way.

Theorem 2.4.3. *Let G be a graph of order n which is regular of degree k on m edges. Let $f(\lambda)$ and $g(\lambda)$ be the characteristic polynomials of G and $L(G)$, respectively. Then*

$$g(\lambda) = (\lambda+2)^{m-n} f(\lambda+2-k).$$

Proof. We recall that two matrix products of the form XY and YX have the same collection of eigenvalues apart from zero eigenvalues. Thus the incidence matrix A of G satisfies

$$\det(\lambda I_m - AA^T) = \lambda^{m-n} \det(\lambda I_n - A^T A). \tag{2.6}$$

We let B and B_L denote the adjacency matrices of G and $L(G)$, respectively. Then using Theorems 2.3.1 and 2.4.1, we obtain

$$\begin{aligned} \det(\lambda I_m - B_L) &= \det((\lambda+2)I_m - AA^T) \\ &= (\lambda+2)^{m-n} \det((\lambda+2)I_n - A^T A) \\ &= (\lambda+2)^{m-n} \det((\lambda+2-k)I_n - B). \end{aligned}$$ □

The study of the spectral properties of line graphs was initiated by A.J. Hoffman and extensively investigated by him and his associates over a period of many years. The condition $\lambda \geq -2$ of Theorem 2.4.2 imposes severe restrictions on the spectrum of $L(G)$. But graphs other than line graphs exist that also satisfy this requirement. Generalized line graphs were introduced by Hoffman[1970,1977] as a class of graphs more general than line graphs that satisfy $\lambda \geq -2$. We briefly discuss these contributions.

Let t be a nonnegative integer. The *cocktail party graph* $CP(t)$ of order $2t$ is the graph with vertices $b_1, b_2, \ldots, b_{2t}$ in which each pair of distinct vertices form an edge with the exception of the pairs $\{b_1, b_2\}, \{b_3, b_4\}, \ldots, \{b_{2t-1}, b_{2t}\}$. The cocktail party graph $CP(t)$ can be obtained from the complete graph K_{2t} of order $2t$ by deleting t edges no two of which are adjacent. If $t = 0$, then the cocktail party graph is a graph with no vertices. Now let G be a graph of order n with vertices $a_1, a_2, \ldots, a_n$, and let $k_1, k_2, \ldots, k_n$ be an n-tuple of nonnegative integers. The *generalized line graph* $L(G; k_1, k_2, \ldots, k_n)$ is the graph of order $n + 2(k_1 + k_2 + \cdots + k_n)$ defined as follows. We begin with the line graph $L(G)$ and disjoint cocktail party graphs $CP(k_1), CP(k_2), \ldots, CP(k_n)$. Each vertex of $L(G)$ is an

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edge $\alpha = \{a_i, a_j\}$ of G and we put edges between α and each vertex of $CP(k_i)$ and of $CP(k_j)$. If λ is an eigenvalue of a generalized line graph, then $\lambda \geq -2$.

Theorem 2.4.4. *If G is a connected graph of order greater than 36 for which each eigenvalue λ satisfies the condition $\lambda \geq -2$, then G is a generalized line graph.*

The following theorem of Hoffman and Ray-Chaudhuri characterizes regular graphs satisfying $\lambda \geq -2$.

Theorem 2.4.5. *If G is a regular connected graph of order greater than 28 for which each eigenvalue λ satisfies the condition $\lambda \geq -2$, then G is either a line graph or a cocktail party graph.*

Finally, we mention the important paper by Cameron, Goethals, Seidel and Shult [1976] in which the above two theorems are obtained by appealing to the classical root systems.

Exercises

1. Determine the spectrum of the line graph $L(K_n)$.
2. The complement of the line graph of K_5 is a cubic graph of order 10 and is known as the *Petersen graph.* Determine the spectrum of the Petersen graph.
3. Find an example of a graph of order 4 which is not isomorphic to the line graph of any graph. Deduce that the Petersen graph is not isomorphic to a line graph.
4. Show that the spectrum of the cocktail party graph $CP(t)$ of order $2t$ is $2t-2$, 0 (with multiplicity t), and -2 (with multiplicity $t-1$).
5. Let B be the adjacency matrix of a generalized line graph G. Determine a matrix N such that $NN^T = 2I + B$, and then deduce that $\lambda \geq -2$ for each eigenvalue λ of G.
6. Let A be the incidence matrix of a tree of order n. Prove that the rank of A equals $n-1$.
7. Let G be a connected graph of order n on n edges and let A be the incidence matrix of G. The graph G has a unique cycle γ. Prove that A has rank n if γ has odd length and that A has rank $n-1$ if γ has even length. Deduce that the incidence matrix of a connected graph of order n has rank n if it has an odd length cycle and has rank $n-1$ otherwise.
8. Let G be a connected graph of order n. Prove that the multiplicity of 0 as an eigenvalue of the line graph $L(G)$ equals $m-n$ if G has an odd length cycle and equals $m-n+1$ otherwise.

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2.5 The Laplacian Matrix of a Graph

Let G denote a graph of order n with vertices $a_1, a_2, \ldots, a_n$ and edges $\alpha_1, \alpha_2, \ldots, \alpha_m$. Let A be the oriented incidence matrix of G of size m by n, and let B be the adjacency matrix of G. Recall that the Laplacian matrix of G is the matrix of order n

$$F = A^T A = D - B$$

where D is a diagonal matrix of order n whose diagonal entry d_i is the degree of the vertex a_i of $G, (i = 1, 2, \ldots, n)$. By Theorem 2.3.2 the matrix A, and hence the Laplacian matrix F has rank at most equal to $n-1$. Thus the matrix F is a singular matrix. A *spanning tree* T of G is a spanning subgraph of G which forms a tree. Every connected graph contains a spanning tree. Let U be a subset of the edges of G. Then we denote by $\langle U \rangle$ the subgraph of G consisting of the edges of U and all vertices of G incident with at least one edge of U. The following lemma is an immediate consequence of Theorem 2.3.2.

Lemma 2.5.1. *Let U be an $(n-1)$-subset of edges of the connected graph G of order n. Let A_U denote a submatrix of order $n-1$ of the oriented incidence matrix A of G consisting of the intersection of the $n-1$ rows of A corresponding to the edges of U and any set of $n-1$ columns of A. Then A_U is nonsingular if and only if $\langle U \rangle$ is a spanning tree of G.*

The *complexity* of a graph G of order n is the number of spanning trees of G. We denote the complexity of G by $c(G)$. In case G is disconnected we have $c(G) = 0$.

Lemma 2.5.2. *Let A be the oriented incidence matrix of a graph G of order n. Then the adjugate of the Laplacian matrix*

$$F = A^T A = D - B$$

is a multiple of J.

Proof. If G is disconnected, then $\text{rank}\,(F) = \text{rank}\,(A) < n-1$ and hence $\text{adj}\,(F) = O$.

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If G is connected, then $\operatorname{rank}(F) = n-1$. But since

$$F\operatorname{adj}(F) = \det(F)I = O$$

it follows that each column of $\operatorname{adj}(F)$ is in the kernel of F. But this kernel is a one-dimensional space spanned by the vector $e_n = (1,1,\ldots,1)^T$. Hence each column of $\operatorname{adj}(F)$ is a multiple of u. But $F = A^TA$ is symmetric and this implies that $\operatorname{adj}(F)$ is also symmetric. Hence it follows that $\operatorname{adj}(A)$ is a multiple of J. □

We now obtain a classical formula.

Theorem 2.5.3. *In the above notation we have*

$$\operatorname{adj}(F) = c(G)J.$$

Proof. By Lemma 2.5.2 we need only show that one cofactor of F is equal to $c(G)$. Let A_0 denote the matrix obtained from A by removing the last column of A. It follows that $\det(A_0^TA_0)$ is a cofactor of F. Now let A_U denote a submatrix of order $n-1$ of A_0 whose rows correspond to the edges in an $(n-1)$-subset U of the edges of G. Then by the Binet-Cauchy theorem we have

$$\det(A_0^TA_0) = \sum \det(A_U^T)\det(A_U),$$

where the summation is over all possible choices of U. By Lemma 2.5.1 we have that A_U is nonsingular if and only if $\langle U\rangle$ is a spanning tree of G, and in this case by Corollary 2.3.4 we have $\det(A_U) = \pm 1$. But $\det(A_U) = \det(A_U^T)$ so that $\det(A_0^TA_0) = c(G)$ and the conclusion follows. □

For the complete graph K_n of order n we have $F = nI - J$, and an easy calculation (cf. Exercise 1, Sec. 1.3) yields the famous Cayley formula [1889]

$$c(K_n) = n^{n-2}$$

for the number of labeled trees of order n.

Theorem 2.5.3 may be formulated in an even more elegant form (Temperley [1964]).

Theorem 2.5.4. *The complexity of a graph G of order n is given by the formula*

$$c(G) = n^{-2}\det(F+J).$$

Proof. We have $J^2 = nJ$ and $FJ = O$ so that

$$(F+J)(nI-J) = nF.$$

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We take the adjugate of both sides and use Theorem 2.5.3 to obtain

$$n^{n-2}J\mathrm{adj}\,(F+J) = n^{n-1}c(G)J.$$

We now multiply by $F+J$ and this gives

$$\det(F+J)J = n^2c(G)J,$$

as desired. □

Now let $(x_1, x_2, \ldots, x_n)^T$ be a real n-vector. Then

$$x^TFx = x^TA^TAx = \sum_{\alpha_t=\{a_i,a_j\}} (x_i - x_j)^2 \tag{2.7}$$

where the summation is over all m edges $\alpha_t = \{a_i, a_j\}$ of G. The matrix F is a positive semidefinite symmetric matrix. Moreover, 0 is an eigenvalue of F with corresponding eigenvector $e_n = (1, 1, \ldots, 1)^T$. Let $\mu = \mu(G)$ denote the second smallest eigenvalue of F. In case $n = 1$ we define μ to be 0. We have from Theorem 2.3.2 that $\mu \geq 0$ with equality if and only if G is a disconnected graph. Fiedler[1973] defined μ to be the *algebraic connectivity* of the graph G. The algebraic connectivity of the complete graph K_n of order n is n (cf. Exercise 2, Sec. 2.2).

Let U be the set of all real n-tuples x such that $x^Tx = 1$ and $x^Te_n = 0$. From the theory of symmetric matrices (see, e.g., Horn and Johnson[1987]) we obtain the characterization

$$\mu = \min\{x^TFx | x \in U\} \tag{2.8}$$

for the algebraic connectivity of the graph G. It follows from equations (2.7) and (2.8) that if G' is a spanning subgraph of G then $\mu(G') \leq \mu(G)$. Thus for graphs with the same set of vertices the algebraic connectivity is a nondecreasing function of the edges.

Theorem 2.5.5. *Let G_t be a graph of order n which is obtained from the graph G by removing the vertex a_t and all edges incident with a_t. Then*

$$\mu(G_t) \geq \mu(G) - 1.$$

Proof. Let F_t be the Laplacian matrix of G_t. First suppose that a_t is adjacent to all other vertices in G. Then the matrix $F_t + I$ is a principal submatrix of order $n-1$ of the Laplacian matrix F of G. By the interlacing inequalities for the eigenvalues of symmetric matrices we have

$$\mu(G_t) + 1 \geq \mu(G).$$

In the general case we use the fact that the algebraic connectivity is a nondecreasing function of the edges. □

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There are two standard ways to measure the extent to which a graph is connected. The *vertex connectivity* of the graph G is the smallest number of vertices whose removal from G, along with each edge incident with at least one of the removed vertices, leaves either a disconnected graph or a graph with a single vertex. Thus if G is not a complete graph, the vertex connectivity equals $n-p$ where p is the largest order of a disconnected induced subgraph of G. The *edge connectivity* of G is the smallest number of edges whose removal from G leaves a disconnected graph or a graph with one vertex. The edge connectivity is no greater than the minimum degree of the vertices of G. The vertex and edge connectivities of the complete graph K_n equal $n-1$.

Theorem 2.5.6. *Let G be a graph of order n which is not complete. The algebraic connectivity $\mu(G)$, the vertex connectivity $v(G)$ and the edge connectivity $e(G)$ of G satisfy*

$$\mu(G) \leq v(G) \leq e(G).$$

Proof. Since G is not complete there is a disconnected graph G^* which results from G by the removal of $v(G)$ vertices. Then $\mu(G^*) = 0$ and by repeated application of Theorem 2.5.5 we get $\mu(G^*) \geq \mu(G) - v(G)$. Now let $k = e(G)$ and let $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}$ be a set of k edges whose removal from G results in a disconnected graph. This disconnected graph has exactly two connected components G_1 and G_2, and each of the removed edges joins a vertex of G_1 to a vertex of G_2. Let x_j be the vertex of α_{i_j} which belongs to $G_1, (j = 1, 2, \ldots, k)$. Notice that the vertices $x_1, x_2, \ldots, x_k$ are not necessarily distinct. If the removal of the vertices $x_1, x_2, \ldots, x_k$ disconnects G, then $v(G) \leq k = e(G)$. Otherwise $x_1, x_2, \ldots, x_k$ are the only vertices of G_1 and it follows that each vertex x_i has degree at most k. Hence each vertex x_i has degree exactly k. We now delete all the vertices adjacent to x_i and disconnect the graph G. Hence $v(G) \leq k = e(G)$. □

We now assume that the graph G is connected and hence that the algebraic connectivity μ of G is positive. The following theorem of Fiedler[1975] shows that an eigenvector x of the Laplacian matrix corresponding to its eigenvalue μ contains easily accessible information about the graph G.

Theorem 2.5.7. *Let G be a connected graph of order n with vertices $a_1, a_2, \ldots, a_n$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of the Laplacian matrix F of G corresponding to the eigenvalue μ. Let r be a nonnegative number, and define*

$$V_r = \{a_i | x_i + r \geq 0, 1 \leq i \leq n\}.$$

Then the induced subgraph $G(V_r)$ of G with vertex set V_r is connected.

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Proof. We first prove the theorem under the assumption that $r = 0$. In this case $V_0 = \{a_i | x_i \geq 0, 1 \leq i \leq n\}$. Suppose to the contrary that the induced subgraph $G(V_0)$ is disconnected. We simultaneously permute the lines of F to obtain

$$\begin{bmatrix} F_1 & O & R \\ O & F_2 & S \\ R^T & S^T & F' \end{bmatrix}, \tag{2.9}$$

where F_1 corresponds to a connected component of $G(V_0)$ and F_2 corresponds to the remaining components of $G(V_0)$. We write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x' \end{bmatrix}$$

to conform to the partition of F in (2.9). The elements of the vectors x_1 and x_2 are nonnegative, and the elements of the vector x' are negative. The matrix

$$\begin{bmatrix} F_1 & O \\ O & F_2 \end{bmatrix} \tag{2.10}$$

is a principal submatrix of F. Since 0 is a simple eigenvalue of F, it follows from the interlacing inequalities for symmetric matrices that the multiplicity of 0 as an eigenvalue of the matrix (2.10) is at most 1. This implies that we may assume that the matrix F_1 is nonsingular. The equation $Fx = \mu x$ implies that

$$(F_1 - \mu I)x_1 = -Rx'. \tag{2.11}$$

The eigenvalues of $F_1 - \mu I$ are nonnegative and hence $F_1 - \mu I$ is a positive semidefinite matrix. The sign patterns of the matrices and vectors involved imply that the elements of the vector $-Rx'$ are nonpositive. Multiplying equation (2.11) on the left by x_1^T we obtain

$$x_1^T (F_1 - \mu I)x_1 = -x_1^T Rx' \leq 0.$$

Since the matrix $F_1 - \mu I$ is positive semidefinite, this implies that

$$(F_1 - \mu I)x_1 = O.$$

By (2.11) we also have $Rx' = O$. Since R has nonpositive entries and x' has negative entries, we have $R = O$, contradicting the hypothesis that G is a connected graph.

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For general nonnegative numbers r we replace the vector x by $x + re_n$ and then proceed as above. □

A very similar proof can be given for the following: Let r be a nonpositive number and define

$$V_r = \{a_i | x_i + r \leq 0, 1 \leq i \leq n\}.$$

Then the induced subgraph $G(V_r)$ of G is connected. More general results can be found in Fiedler[1975]. The algebraic connectivity of trees is studied in Grone and Merris[1987] and in Fiedler[1990]. A survey of the eigenvalues of the Laplacian matrix of graphs is given by Mohar[1988]. A more general survey of the Laplacian is given by Grone[1991].

Exercises

1. Determine the complexity of the Petersen graph.
2. Determine the algebraic connectivity of the star $K_{1,n-1}$ of order n.
3. Let G be a graph of order n and let d denote the smallest degree of a vertex of G. Prove that $\mu(G) \leq dn/(n-1)$ (Fiedler[1973]).
4. Let G be a connected graph of order n which is regular of degree k. Let the spectrum of G be $\lambda_1 = k, \lambda_2, \ldots, \lambda_n$. Use Theorem 2.5.4 to show that the complexity $c(G)$ satisfies

$$c(G) = \frac{1}{n} \prod_{i=2}^{n} (k - \lambda_i)$$

(Biggs[1974]).

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2.6 Matchings

A graph G is called *bipartite* provided that its vertices may be partitioned into two subsets X and Y such that every edge of G is of the form $\{a, b\}$ where a is in X and b is in Y. We call $\{X, Y\}$ a *bipartition* of G. A connected bipartite graph has a unique bipartition. In a bipartite graph the vertices may be colored red and blue in such a way that each edge of the graph has a red endpoint and a blue endpoint. Trees are simple instances of bipartite graphs.

Let A denote the adjacency matrix of a bipartite graph G with bipartition $\{X, Y\}$ where X is an m-set and Y is an n-set. Then we may write A in the following special form:

$$A = \begin{bmatrix} O & B \\ B^T & O \end{bmatrix}, \tag{2.12}$$

where B is a (0,1)-matrix of size m by n which specifies the adjacencies between the vertices of X and the vertices of Y. Without loss of generality we may select the notation so that $m \le n$. The matrix B characterizes the bipartite graph G. We note the paradoxical nature of combinatorial representations. A general graph is characterized by a very special (0,1)-matrix called its incidence matrix, whereas a very special graph called a bipartite graph is characterized by an arbitrary (0,1)-matrix.

A *matching* M in the bipartite graph G is a subset of its edges no two of which are adjacent. A matching defines a one-to-one correspondence between a subset X' of X and a subset Y' of Y such that corresponding vertices of X' and Y' are joined by an edge of the matching. The cardinality $|M|$ of the matching M is the common cardinality of X' and Y'. A matching in G with cardinality t corresponds in the matrix B of (2.12) to a set of t 1's with no two of the 1's on the same line. In A it corresponds to a set of $2t$ symmetrically placed 1's with no two of the 1's on the same line. The cardinality of a matching cannot exceed m. The number, possibly zero, of matchings having cardinality m is given by $\operatorname{per}(B)$. The König theorem (Theorem 1.2.1) applied to the matrix B yields: *The maximum cardinality of a matching in the bipartite graph G equals the minimum cardinality of a set S of vertices such that each edge of G is incident with at least one vertex in S.*

Now let G be a general graph of order n with vertex set V. The concept of a *matching* may be carried over to the general graph G without any change in the definition. In investigating the largest cardinality of a matching we may assume without loss of generality that each edge of G has multiplicity 1. However, the possible presence of loops in G adds to the generality of the situation. Let A be the adjacency matrix of order n of G. Let M be a matching in G with cardinality t, and suppose that $p \ge 0$ of the edges of M are loops. The cardinality of the set of endpoints of the edges of M

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is denoted by $\#(M)$. It follows that $\#(M) = 2t - p$. In the matrix A, M corresponds to a set of $2t-p$ symmetrically placed 1's with no two of the 1's on the same line. Each of the edges of M which is not a loop corresponds to two symmetrically placed 1's. Each of the loops in M corresponds to a single 1 on the main diagonal of A.

We now turn to the fundamental minimax theorem for matchings in general graphs which extends the König theorem for bipartite graphs. This theorem in its original form is due to Tutte[1947]. It was later extended by Berge[1958]. The modification presented here was mentioned in Brualdi[1976] and allows for the presence of loops in a matching. The proof we give is based on Anderson[1971], Brualdi[1971], Gallai[1963] and Mader[1973].

If S is a subset of the vertex set V of the general graph G of order n, we define $\mathcal{C}(G;S)$ to be the set of connected components of the induced subgraph $G(V-S)$ which have an odd number of vertices and no loops. The cardinality of $\mathcal{C}(G;S)$ is denoted by $p(G;S)$. We also define the function $f(G;S)$ by

$$f(G;S) = n - p(G;S) + |S|. \tag{2.13}$$

Theorem 2.6.1. *Let G be a general graph of order n whose vertex set is V. The maximum cardinality of the set of endpoints of a matching in G equals the minimum value of $f(G;S)$ over all subsets S of the vertex set V:*

$$\max\{\#(M) : M \text{ a matching in } G\} = \min\{f(G;S) : S \subseteq V\} \tag{2.14}$$

Proof. Let M be a matching in G and let S be a subset of V. We first show that $\#(M) \leq f(G;S)$. Let $G(W)$ be a component of $G(V-S)$ which belongs to $\mathcal{C}(G;S)$. Then $|W|$ is odd and at most $(|W|-1)/2$ edges in M are edges of $G(W)$. An edge of G which has one of its endpoints in W has its other endpoint in $W \cup S$ and this implies that there are at least $p(G;S) - |S|$ vertices of G which are not incident with any edge in M. Hence

$$\#(M) \leq n - (p(G;S) - |S|) = f(G;S). \tag{2.15}$$

Since (2.15) holds for each matching M and each set S of vertices, the value of the expression on the left in (2.14) is at most equal to the value of the expression on the right.

We now denote the value of the expression on the right in (2.14) by m and prove by induction on m that G has a matching M with $\#(M) = m$. If $m = 0$, we may take M to be empty. Now let $m \geq 1$ and let $\mathcal{S}$ denote the collection of maximal subsets T of the vertex set V for which

$$n - p(G;T) + |T| = m. \tag{2.16}$$

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We choose T in $\mathcal{S}$ and first show that each component of $G(V-T)$ with an even number of vertices has a loop. Suppose to the contrary that $G(W)$ is a component of $G(V-T)$ such that $|W|$ is even and $G(W)$ has no loops. Let x be a vertex in W. Then $p(G; T \cup \{x\}) \geq p(G; T) + 1$ and hence by (2.16)

$$n - p(G; T \cup \{x\}) + |T \cup \{x\}| \leq m. \tag{2.17}$$

It follows that equality holds in (2.17) and we contradict the choice of T in $\mathcal{S}$.

Now let $G(W)$ be a component of $G(V-T)$ which has at least one loop. We show that there is a matching M_W in $G(W)$ satisfying $\#(M_W) = |W|$. Let $\{x, x\}$ be a loop in $G(W)$ and consider the graph $G(W-\{x\})$. Suppose there is a subset U of $W - \{x\}$ satisfying $p(G(W - \{x\}); U) \geq |U| + 1$. Then

$$p(G(W); U \cup \{x\}) \geq |U \cup \{x\}| \tag{2.18}$$

and using (2.16) and (2.18) we obtain

$$\begin{aligned} & n - p(G; T \cup U \cup \{x\}) + |T \cup U \cup \{x\}| \\ = {} & n - p(G; T) + |T| + |U \cup \{x\}| - p(G(W); U \cup \{x\}) \leq m. \end{aligned}$$

Again equality holds and we contradict the choice of T in $\mathcal{S}$. Thus for all subsets U of $W - \{x\}$,

$$p(G(W - \{x\}); U) \leq |U|,$$

and hence

$$|W - \{x\}| - p(G(W - \{x\}); U) + |U| \geq |W - \{x\}|.$$

Since $|W - \{x\}| < m$, it follows from the induction hypothesis that $G(W - \{x\})$ has a matching M' with $\#(M') = |W - \{x\}|$. Then $M_W = M' \cup \{\{x, x\}\}$ is a matching of $G(W)$ with $\#(M_W) = |W|$.

We now deal exclusively with the components in $\mathcal{C}(G; T)$ and the edges between vertices of these components and the vertices of T. We distinguish two cases.

Case 1. Either $T \neq \emptyset$ or $\mathcal{C}(G; T)$ contains at least two components with more than one vertex.

Let $(G(W_i) : i \in I)$ be the components in $\mathcal{C}(G; T)$ satisfying $|W_i| > 1$, where we have indexed these components by some set I. The assumptions of this case imply that $|W_i| - 1 < m$ for all $i \in I$. Let j be any element in I and let z be any vertex in W_j. We next show that $G(W_j - \{z\})$ has

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a matching $M_j(z)$ satisfying $\#(M_j(z)) = |W_j| - 1$. If not, then by the induction hypothesis there exists $S \subseteq W_j - \{z\}$ such that

$$p(G(W_j - \{z\}); S) \geq |S| + 1.$$

The fact that $|W_j - \{z\}|$ is an even number now implies that

$$p(G(W_j - \{z\}) : S) \geq |S| + 2.$$

We then use (2.16) to calculate that

$$\begin{aligned} p(G; T \cup S \cup \{z\}) &= p(G;T) - 1 + p(G(W_j - \{z\}); S) \\ &\geq n + |T| - m - 1 + |S| + 2 = n + |S \cup T \cup \{z\}| - m. \end{aligned}$$

Once again we have contradicted the choice of T in $\mathcal{S}$, and we have established the existence of the matching $M_j(z)$ in $G(W_j - \{z\})$.

We now define a bipartite graph G^* with bipartition $\{S, I\}$. If $s \in S$ and $i \in I$, then $\{s, x\}$ is an edge of G^* if and only if there is an edge in G joining s and *some* vertex in W_i. We prove that G^* has a matching whose cardinality equals $|S|$. If not then by the König theorem there exist $X \subseteq S$ and $J \subseteq I$ such that $|J| < |X|$ and no edge in G^* has one endpoint in X and the other endpoint in $I - J$. We then conclude that

$$p(G; T - X) \geq p(G;T) - |J|.$$

Hence

$$n - p(G; T - X) + |T - X| \leq n - p(G;T) + |J| + |T| - |X| < m,$$

which contradicts the definition of m. Thus G^* has a matching whose cardinality equals $|S|$. This means that there exists a subset K of I with $|K| = |S|$ and vertices z_i in $W_i, (i \in K)$ such that G has a matching M' whose set of endpoints is $S \cup \{z_i : i \in K\}$. For each $i \in I - K$, let z_i be any chosen vertex in W_i. Then

$$M = M' \cup \cup_{i \in I} M_i(z_i) \cup \cup_W M_W,$$

where the last union is over those components $G(W)$ of $G(V - T)$ which contain a loop, is a matching in G with $\#(M) = m$.

Case 2. $T = \emptyset$ (that is, $\mathcal{S} = \{\emptyset\}$) and $\mathcal{C}(G; \emptyset)$ has at most one component with more than one vertex.

In this case, $m = |V| - p(G; \emptyset)$ and $\mathcal{C}(\mathcal{G}; \emptyset)$ is the set of components of G with an odd number of vertices and no loops. In addition for all nonempty subsets S of V

$$m + 1 \leq |V| - p(G; S) + |S|.$$

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Let $G(U)$ be a component of G such that the number of vertices of $G(U)$ is an odd number greater than 1 and $G(U)$ has no loops [by the assumptions of this case $G(U)$, if it exists, is unique]. We show that $G(U)$ has a matching M^* with $\#(M^*) = |U| - 1$. We choose an edge $\{x, y\}$ in $G(U)$. Let U' be obtained from U by removing the vertices x and y and let $S' \subseteq U'$. We let $S = S' \cup \{x, y\}$ and calculate

$$\begin{aligned} m + 1 &\leq |V| - p(G; S) + |S| \\ &= |V| - (p(G(U'); S') + p(G; \emptyset) - 1) + |S'| + 2 \\ &= (|V| - p(G; \emptyset) + 1) + |S'| - p(G(U'); S') + 2 \\ &= m + 1 + |S'| - p(G(U'); S') + 2. \end{aligned}$$

Hence

$$p(G(U'); S') \leq |S'| + 2,$$

and since $|U'|$ is odd,

$$p(G(U'); S') \leq |S'| + 1.$$

Thus for all subsets S' of U',

$$|U'| - p(G(U'); S') + |S'| \geq |U'| - 1.$$

We now apply the induction hypothesis and obtain a matching M' in $G(U')$ with $\#(M') = |U'| - 1$. Then $M^* = M' \cup \{\{x, y\}\}$ is a matching in $G(U)$ with $\#(M^*) = |U| - 1$. Now

$$M = M^* \cup \cup_W M_W,$$

where the last union is over those components $G(W)$ of G which have a loop, is a matching in G satisfying $\#(M) = m$. □

Let G be a general graph of order n with vertex set V. A matching M with $\#(M) = n$ has the property that every vertex of G is an endpoint of an edge in M and is called a *perfect matching* or 1-*factor* of G. It follows from Theorem 2.6.1 that G has a perfect matching if and only if

$$p(G; S) \leq |S|, \text{ for all } S \subseteq V.$$

Now let A be a symmetric (0,1)-matrix of order n and let G be the general graph whose adjacency matrix is A. A perfect matching M in G corresponds to a set of n symmetrically placed 1's in A with no two of the 1's on the same line of A. Thus the above special case of Theorem 2.6.1 gives necessary and sufficient conditions for there to exist a symmetric permutation matrix P with $P \leq A$. We briefly describe these conditions in terms of the matrix A. Let S be a set of vertices of G with $|S| = k$. The

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adjacency matrix of the subgraph $G(V-S)$ is a principal submatrix A' of A of order $n-k$. The connected components of $G(V-S)$ correspond to certain "connected" principal submatrices $A_1, A_2, \ldots, A_t$ of A'. The condition (2.6) asserts that the number of submatrices $A_1, A_2, \ldots, A_t$ which are of odd order and have zero trace is at most k.

A (0,1)-matrix P of size m by n is a *subpermutation matrix of rank* r provided P has exactly r 1's, and no two 1's of P are on the same line of P. Let A be the adjacency matrix of a general graph G of order n. A subpermutation matrix P of rank r which is in addition symmetric corresponds to a matching in G of r edges. Hence Theorem 2.6.1 characterizes the maximum integer r such that A can be written in the form

$$A = P + X$$

where P is a symmetric subpermutation matrix of rank r and X is a nonnegative matrix.

We next consider expressions of the form

$$A = P_1 + P_2 + \cdots + P_l \tag{2.19}$$

where $P_1, P_2, \ldots, P_l$ are symmetric permutation matrices of arbitrary ranks, and we seek to minimize the value of l in (2.19). Any l for which we have a decomposition of the form (2.19) is at least equal to the maximum line sum k of A. A theorem of Vizing[1964] concerning graphs asserts the existence of a decomposition (2.19) in which $l = k+1$. We state this theorem in terms of matrices in a somewhat more general form to allow the presence of 1's on the main diagonal.

Theorem 2.6.2. *Let A be a symmetric $(0,1)$-matrix of order n, and let k be the maximum number of 1's in the off-diagonal positions of the lines of A. Then there exist symmetric subpermutation matrices $P_1, P_2, \ldots, P_{k+1}$ such that*

$$A = P_1 + P_2 + \cdots + P_{k+1}. \tag{2.20}$$

Proof. We first show that if the theorem is true in the case that the matrix A has zero trace, then it is true in general. Let A' be a symmetric (0,1)-matrix having at least one 1 on its main diagonal, and let k be the maximum number of 1's in the off-diagonal positions of the lines of A'. Let A be the matrix obtained from A by replacing the 1's on the main diagonal of A' with 0's. Suppose that there are subpermutation matrices $P_1, P_2, \ldots, P_{k+1}$ satisfying (2.20). Since the maximum number of 1's in a line of A is k, for each $i = 1, 2, \ldots, n$ at least one of the matrices $P_1, P_2, \ldots, P_{k+1}$ has only 0's in line i. It follows that we may replace certain 0's on the main diagonals of $P_1, P_2, \ldots, P_{k+1}$ and obtain symmetric subpermutation matrices $P'_1, P'_2, \ldots, P'_{k+1}$ satisfying $A' = P'_1 + P'_2 + \cdots + P'_{k+1}$.

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We now prove the theorem under the added assumption that A has zero trace. Let G be the graph of order n whose adjacency matrix is A. The maximum degree of a vertex of G is k. Let σ be a function which assigns to each edge of G an integer from the set $\{1, 2, \ldots, k+1\}$. We think of σ as assigning a *color* to each edge of G from a set $\{1, 2, \ldots, k+1\}$ of $k+1$ colors. We call σ a $(k+1)$-*edge coloring* provided adjacent edges are always assigned different colors. Let F_i be the set of edges of G that are assigned color i by the edge coloring σ, and let P_i be the adjacency matrix of the spanning subgraph of G whose set of edges is F_i $(i = 1, 2, \ldots, k+1)$. Then each P_i is a subpermutation matrix and $A = P_1 + P_2 + \cdots + P_{k+1}$. To complete the proof we show that a graph G has a $(k+1)$-edge coloring if k is the maximum degree of its vertices. The proof is by induction on the number of edges of G. Let $\alpha_1 = \{a, b_1\}$ be an edge of G. It suffices to show that if there is a $(k+1)$-edge coloring for G with the edge α_1 deleted, then there is a $(k+1)$-edge coloring for G.

Let σ be a $(k+1)$-edge coloring for the graph G' obtained by deleting the edge α_1 of G. There is a color t which is not assigned to any edge of G' which is incident with a. There is also a color t_1 which is not assigned to any edge incident with b_1. If $t = t_1$, then we may assign the color t to α_1 and thereby obtain a $(k+1)$-edge coloring of G. We now assume that there is an edge $\alpha_2 = \{a, b_2\}$ with color t_1. We remove the color t_1 from α_2 and assign the color t_1 to α_1. Let G_{t,t_1} be the subgraph of G consisting of those edges assigned colors t or t_1 and the vertices incident to these edges. The vertices a and b_1 belong to the same connected component C_1 of G_{t,t_1}. Suppose that b_2 is not a vertex of C_1. We then switch the colors t and t_1 on the edges of C_1. Now there is no edge of color t_1 incident to a or to b_2, and we may assign the color t_1 to the edge α_2 and obtain a $(k+1)$-edge coloring of G. We therefore assume that b_2 is a vertex of C_1. In particular, there is an edge with color t incident with b_2.

There is now a color t_2 different from t and t_1 which is not assigned to any edge incident with b_2. If the color t_2 were not assigned to some edge incident with a, we could assign t_2 to α_2 and obtain a $(k+1)$-edge coloring of G. We therefore proceed under the assumption that there is an edge $\alpha_3 = \{a, b_3\}$ which is assigned the color t_2. Arguing as above we may assume that a, b_2 and b_3 all belong to the same component of the subgraph G_{t,t_2} of G determined by the colors t and t_2. In particular, there is an edge with color t incident with b_3. We continue in this fashion and obtain a sequence of edges $\alpha_1 = \{a, b_1\}, \alpha_2 = \{a, b_2\}, \ldots, \alpha_k = \{a, b_k\}$ where after reassigning colors the edge α_i has color t_i $(i = 1, 2, \ldots, k-1)$, edge α_k has no color assigned to it, and there is a color t_k different from colors t and t_{k-1} which is not assigned to any edge incident with b_k. We choose k to be the first integer such that $t_k = t_j$ for some $j < k-1$. The vertices a, b_j and b_{j+1} belong to the same connected component C_j of the subgraph G_{t,t_j} of G. Since there is no edge of color t at a and since

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there is no edge of color t_j at b_{j+1}, the component C_j consists of the vertices and edges of a chain

$$a \to b_j \to \cdots \to b_{j+1}.$$

Since $t_k = t_j$ and there is no edge incident with b_k which is assigned the color t_k, this chain cannot contain the vertex b_k. Thus b_k is a vertex of a connected component C^* of G_{t,t_j} different from C_j. We next switch the colors t and $t_j = t_k$ on the edges of C^*. Now there is no edge incident with b_k which is assigned the color t. We assign the color t to the edge $\alpha_k = \{a, b_k\}$ and thereby obtain a $(k+1)$-coloring of G. □

In the proof of Theorem 2.6.2 we have shown that a graph has a $(k+1)$-edge coloring if each of its vertices has degree at most equal to k. This conclusion need not hold for multigraphs. For example, the multigraph obtained by doubling each edge of the complete graph K_3 has six edges each pair of which are adjacent. Hence it has no 5-edge coloring. The smallest number t such that a multigraph G has a t-edge coloring is called the *chromatic index* of G. Thus Vizing's theorem asserts that the chromatic index of a graph for which the maximal degree of a vertex is k equals k or $k+1$.

Vizing[1965] generalized Theorem 2.6.2 to include multigraphs. We state this theorem without proof in the language of matrices.

Theorem 2.6.3. *Let A be a symmetric nonnegative integral matrix of order n. Let k be the maximum sum of the off-diagonal entries in the lines of A, and let m be the maximum element in all of A. Then there exist symmetric subpermutation matrices $P_1, P_2, \ldots, P_{k+m}$ such that*

$$A = P_1 + P_2 + \cdots + P_{k+m}.$$

A theorem of Shannon[1949] sometimes gives a better result than Theorem 2.6.3.

Theorem 2.6.4. *Let A be a symmetric nonnegative integral matrix of order n. Let k be the maximum sum of the off-diagonal entries in the lines of A, and let l be the largest element on the main diagonal of A. Then there exist symmetric subpermutation matrices $P_1, P_2, \ldots, P_t$ with $t = k + \max\{\lceil k/2 \rceil, l\}$ such that*

$$A = P_1 + P_2 + \cdots + P_t.$$

Exercises

1. Prove the theorem of Petersen[1891]: Let G be a connected cubic graph such that each subgraph obtained from G by removing an edge is connected. Then G has a perfect matching.

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2. Show that the chromatic index of the Petersen graph is 4.
3. Determine the chromatic index of the complete graph K_n, that is, the smallest number of symmetric subpermutation matrices into which the matrix $J_n - I_n$ can be decomposed.
4. Show that J_n can be decomposed into n symmetric permutation matrices.
5. Let G be a graph which is regular of degree k and suppose that k is a positive even integer. Prove that G can be decomposed into $k/2$ spanning subgraphs each of which is a regular graph of degree 2.

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