1

Incidence Matrices

1.1 Fundamental Concepts

Let

$$A = [a_{ij}], \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

be a matrix of m rows and n columns. We say that A is of *size* m by n, and we also refer to A as an m by n matrix. In the case that $m = n$ then the matrix is square of *order* n. It is always assumed that the entries of the matrix are elements of some underlying field F. Evidently A is composed of m row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ over F and n column vectors $\beta_1, \beta_2, \ldots, \beta_n$ over F, and we write

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}.$$

It is convenient to refer to either a row or a column of the matrix as a *line* of the matrix. We use the notation A^T for the transpose of the matrix A. We always designate a zero matrix by O, a matrix with every entry equal to 1 by J, and the identity matrix of order n by I. In order to emphasize the size of these matrices we sometimes include subscripts. Thus $J_{m,n}$ denotes the all 1's matrix of size m by n, and this is abbreviated to J_n if $m = n$. The notations $O_{m,n}$, O_n and I_n have similar meanings. In displaying a matrix we often use $*$ to designate a submatrix of no particular concern. The $n!$ permutation matrices of order n are obtained from I_n by arbitrary permutations of the lines of I_n. A permutation matrix P of order n satisfies the matrix equations

$$PP^T = P^T P = I_n.$$

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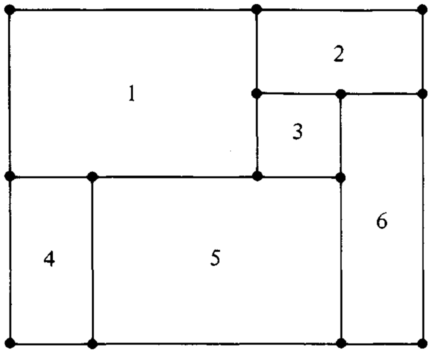

Figure 1.1

In most of our discussions the underlying field F is the field of real numbers or the subfield of the rational numbers. Indeed we will be greatly concerned with matrices whose entries consist exclusively of the integers 0 and 1. Such matrices are referred to as (0,1)-*matrices*. and they play a fundamental role in combinatorial mathematics.

We illustrate this point by an example that reformulates an elementary problem in geometry in terms of (0,1)-matrices. Let a rectangle R in the plane be of integral height m and of integral length n. Let all of R be partitioned into t smaller rectangles. Each of these rectangles is also required to have integral height and integral length. We number these smaller rectangles in an arbitrary manner $1, 2, \ldots, t$. An example with $m = 4$, $n = 5$ and $t = 6$ is illustrated in Figure 1.1.

We associate with the partitioned rectangle of Figure 1.1 the following two (0,1)-matrices of sizes 4 by 6 and 6 by 5, respectively:

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$Y = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The number of 1's in column i of X is equal to the height of rectangle i, and all of the 1's in column i occur consecutively. The topmost and bottommost 1's in column i of X locate the position of rectangle i with respect to the

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top and bottom horizontal lines of the full rectangle. The matrix Y behaves in much the same way but with respect to rows. Thus the number of 1's in row j of Y is equal to the length of rectangle j, and all the 1's in row j of Y occur consecutively. The first and last 1's in row j of Y locate the position of rectangle j with respect to the left and right vertical lines of the full rectangle. It follows from the definition of matrix multiplication that the product of the matrices X and Y satisfies

$$XY = J. \tag{1.1}$$

The matrix J in (1.1) is the matrix of 1's of size 4 by 5.

This state of affairs is valid in the general case. Thus the partitioning of a rectangle in the manner described is precisely equivalent to a matrix equation of the general form (1.1) with the following constraints. The matrices X and Y are (0,1)-matrices of sizes m by t and t by n, respectively, and J is the matrix of 1's of size m by n. The 1's in the columns of X and the 1's in the rows of Y are required to occur consecutively. If the original problem is further restricted so that all rectangles involved are squares, then we must require in addition that $m = n$ and that the sum of column i of X is equal to the sum of row i of Y, $(i = 1, 2, \ldots, t)$.

We next describe in more general terms the basic link between (0,1)-matrices and a wide variety of combinatorial problems. Let

$$X = \{x_1, x_2, \ldots, x_n\}$$

be a nonempty set of n elements. We call X an *n-set*. Now let $X_1, X_2, \ldots,$ X_m be m not necessarily distinct subsets of the n-set X. We refer to this collection of subsets of an n-set as a *configuration* of subsets. Vast areas of modern combinatorics are concerned with the structure of such configurations. We set $a_{ij} = 1$ if $x_j \in X_i$, and we set $a_{ij} = 0$ if $x_j \notin X_i$. The resulting (0,1)-matrix

$$A = [a_{ij}], \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

of size m by n is the *incidence matrix* for the configuration of subsets $X_1, X_2, \ldots, X_m$ of the n-set X. The 1's in row α_i of A display the elements in the subset X_i, and the 1's in column β_j display the occurrences of the element x_j among the subsets. Thus the lines of A give us a complete description of the subsets and the occurrences of the elements within these subsets. This representation of our configuration in terms of the (0,1)-matrix A is of the utmost importance because it allows us to apply the powerful techniques of matrix theory to the particular problem under investigation.

Let A be a (0,1)-matrix of size m by n. The *complement* C of the incidence matrix A is obtained from A by interchanging the roles of the 0's and 1's and satisfies the matrix equation

$$A + C = J.$$

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We note that the matrices O and J of size m by n are complementary and correspond to the configurations with the empty set repeated m times and the full n-set repeated m times, respectively. A second incidence matrix associated with the (0,1)-matrix A of size m by n is the transposed matrix A^T of size n by m. The configuration of subsets associated with a transposed incidence matrix is called the *dual* of the configuration.

Suppose that we have subsets $X_1, X_2, \ldots, X_m$ of an n-set X and subsets $Y_1, Y_2, \ldots, Y_m$ of an n-set Y. Then these two configurations of subsets are regarded as the same or *isomorphic* provided that we may relabel the subsets $X_1, X_2, \ldots, X_m$ and the elements of the n-set X so that the resulting configuration coincides with the configuration $Y_1, Y_2, \ldots, Y_m$ of the n-set Y. This means that our original configurations are the same apart from the notation in which they are written.

The above isomorphism concept for configurations of subsets has a direct interpretation in terms of the incidence matrices that represent the configurations. Thus suppose that A and B are two (0,1)-matrices of size m by n that represent incidence matrices for subsets $X_1, X_2, \ldots, X_m$ of an n-set X and for subsets $Y_1, Y_2, \ldots, Y_m$ of an n-set Y, respectively. Then these configurations of subsets are isomorphic if and only if A is transformable to B by line permutations. In other words the configurations of subsets are isomorphic if and only if there exist permutation matrices P and Q of orders m and n, respectively, such that

$$PAQ = B.$$

In many combinatorial investigations we are often primarily concerned with those properties of a (0,1)-matrix that remain invariant under arbitrary permutations of the lines of the matrix. The reason for this is now apparent because such properties of the matrix become invariants of isomorphic configurations.

If two configurations of subsets are isomorphic, then their associated incidence matrices of size m by n are required to satisfy a number of necessary conditions. Thus the row sums including multiplicities are the same for both matrices and similarly for the column sums. The ranks of the two matrices must also coincide. The incidence matrices may be tested for invariants like these. But thereafter it may still be an open question as to whether or not the given configurations are isomorphic. Suppose, for example, that A is a (0,1)-matrix of order n such that all of the line sums of A are equal to the positive integer k. We may ask if the configuration associated with this incidence matrix is isomorphic to its dual. This will be the case if and only if there exist permutation matrices P and Q of order n such that

$$PAQ = A^T.$$

We now look at some examples of the isomorphism problem which are

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readily solvable. Suppose that the configuration is represented by a permutation matrix of order n. Then clearly the configuration may be represented equally well by the identity matrix I_n, and thus any two such configurations are isomorphic.

Suppose next that the configuration is represented by a (0,1)-matrix A of order n such that all of the line sums are equal to 2. This restriction on A allows us to replace A under line permutations by a direct sum of the form

$$A_1 \oplus A_2 \oplus \cdots \oplus A_e.$$

Each of the components of this direct sum has line sums equal to 2, and is "fully indecomposable" in the sense that it cannot be further decomposed by line permutations into a direct sum. Each fully indecomposable component is itself normalized by line permutations so that the 1's appear on the main diagonal and in the positions directly above the main diagonal with an additional 1 in the lower left corner. For example, a normalized fully indecomposable component of order 5 is given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have constructed a canonical form for A in the sense that if

$$B_1 \oplus B_2 \oplus \cdots \oplus B_f$$

is a second decomposition for A, then we have $e = f$ and the A_i are equal to the B_j in some ordering. The essential reasoning behind this is as follows. We first label all of the 1's in A from 1 to $2n$ in an arbitrary manner. Under line permutations two labeled 1's in a line always remain within the same line. Consider the component A_1 and its two labeled 1's in the (1,1) and (1,2) positions of A_1. These labeled 1's occur in some row of a component of the second decomposition, say in component B_i. But then the labeled 1 in the (2,2) position of A_1 also occurs in B_i and similarly for the labeled 1 in the (2,3) position of A_1. In this way we see that all of the labeled 1's in A_1 occur in B_i. But then A_1 and B_i are equal because both matrices are fully indecomposable and normalized. We may then identify the labeled 1's in the component A_2 with the labeled 1's in another component B_j of the second decomposition and so on.

The above canonical form for A implies that we have devised a straightforward procedure for deciding the isomorphism problem for configurations whose incidence matrices have all of their line sums equal to 2. The situation for (0,1)-matrices that have all of their line sums equal to 3 is vastly more

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complicated. Such matrices may already possess a highly intricate internal structure.

In the study of configurations of subsets two broad categories of problems emerge. The one deals with the structure of very general configurations, and the other deals with the structure of much more restricted configurations. In the present chapter we will see illustrations of both types of problems. We will begin with the proof of a minimax theorem that holds for an arbitrary (0,1)-matrix. Later we will also discuss certain (0,1)-matrices of such a severely restricted form that their very existence is open to question.

1.2 A Minimax Theorem

We now prove the fundamental minimax theorem of König[1936]. This theorem has a long history and many ramifications which are described in detail in the book by Mirsky[1971]. The theorem deals exclusively with properties of a (0,1)-matrix that remain invariant under arbitrary permutations of the lines of the matrix.

Theorem 1.2.1. *Let A be a $(0,1)$-matrix of size m by n. The minimal number of lines in A that cover all of the 1's in A is equal to the maximal number of 1's in A with no two of the 1's on a line.*

Proof. We use induction on the number of lines in A. The theorem is valid in case that $m = 1$ or $n = 1$. Hence we take $m > 1$ and $n > 1$. We let ρ' equal the minimal number of lines in A that cover all of the 1's in A, and we let ρ equal the maximal number of 1's in A with no two of the 1's on a line. We may conclude at once from the definitions of ρ and ρ' that $\rho \leq \rho'$. Thus it suffices to prove that $\rho \geq \rho'$. A minimal covering of the 1's of A is called *proper* provided that it does not consist of all m rows of A or of all n columns of A. The proof of the theorem splits into two cases.

In the first case we assume that A does not have a proper covering. It follows that we must have $\rho' = \min\{m, n\}$. We permute the lines of A so that the matrix has a 1 in the (1,1) position. We delete row 1 and column 1 of the permuted matrix and denote the resulting matrix of size $m - 1$ by $n - 1$ by A'. The matrix A' cannot have a covering composed of fewer than $\rho' - 1 = \min\{m - 1, n - 1\}$ lines because such a covering of A' plus the two deleted lines would yield a proper covering for A. We now apply the induction hypothesis to A' and this allows us to conclude that A' has $\rho' - 1$ 1's with no two of the 1's on a line. But then A has ρ' 1's with no two of the 1's on a line and it follows that $\rho \geq \rho'$.

In the alternative case we assume that A has a proper covering composed of e rows and f columns where $\rho' = e + f$. We permute lines of A so that

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these e rows and f columns occupy the initial positions. Then our permuted matrix assumes the following form

$$\begin{bmatrix} * & A_1 \\ A_2 & O \end{bmatrix}.$$

In this decomposition O is the zero matrix of size $m-e$ by $n-f$. The matrix A_1 has e rows and cannot be covered by fewer than e lines and the matrix A_2 has f columns and cannot be covered by fewer than f lines. This is the case because otherwise we contradict the fact that $\rho' = e+f$ is the minimal number of lines in A that cover all of the 1's on A. We may apply the induction hypothesis to both A_1 and A_2 and this allows us to conclude that $\rho \geq \rho'$. □

The maximal number of 1's in the (0,1)-matrix A with no two of the 1's on a line is called the *term rank* of A. We denote this basic invariant of A by

$$\rho = \rho(A).$$

We next investigate some important applications of the König theorem. Let $X_1, X_2, \ldots, X_m$ be m not necessarily distinct subsets of an n-set X. Let

$$D = (a_1, a_2, \ldots, a_m)$$

be an ordered sequence of m distinct elements of X and suppose that

$$a_i \in X_i, (i = 1, 2, \ldots, m).$$

Then the element a_i *represents* the set X_i, and we say that our configuration of subsets has a *system of distinct representatives* (abbreviated SDR). We call D an SDR for the ordered sequence of subsets $(X_1, X_2, \ldots, X_m)$. The definition of SDR requires $a_i \neq a_j$ whenever $i \neq j$, but X_i and X_j need not be distinct as subsets of X.

The following theorem of P. Hall[1935] gives necessary and sufficient conditions for the existence of an SDR. We derive the Hall theorem from the König theorem. We remark that one may also reverse the procedure and derive the König theorem from the Hall theorem (Ryser[1963]).

Theorem 1.2.2. *The subsets $X_1, X_2, \ldots, X_m$ of an n-set X have an SDR if and only if the set union $X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_k}$ contains at least k elements for $k = 1, 2, \ldots, m$ and for all k-subsets $\{i_1, i_2, \ldots, i_k\}$ of the integers $1, 2, \ldots, m$.*

Proof. The necessity of the condition is clear because if a set union $X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_k}$ contains fewer than k elements then it is not possible to select an SDR for these subsets.

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We now prove the reverse implication. Let A be the (0,1)-matrix of size m by n which is the incidence matrix for our configuration of subsets. Suppose that A does not have the maximal possible term rank m. Then by the König theorem we may cover the 1's in A with e rows and f columns, where $e + f < m$. We permute the lines of A so that these e rows and f columns occupy the initial positions. Then our permuted A assumes the form

$$\begin{bmatrix} * & A_1 \\ A_2 & O \end{bmatrix}.$$

In this decomposition O is the zero matrix of size $m - e$ by $n - f$. The matrix A_2 of size $m - e$ by f has $m - e > f$. But then the last $m - e$ rows of the displayed matrix correspond to subsets of X whose union contains fewer than $m-e$ elements, and this is contrary to our hypothesis. Hence the matrix A is of term rank m, and this in turn implies that our configuration of subsets has an SDR. □

Let $A = [a_{ij}]$ be a matrix of size m by n with elements in a field F and suppose that $m \leq n$. Then the *permanent* of A is defined by

$$\operatorname{per}(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{mi_m},$$

where the summation extends over all the m-permutations $(i_1, i_2, \ldots, i_m)$ of the integers $1, 2, \ldots, n$. Thus $\operatorname{per}(A)$ is the sum of all possible products of m elements of A with the property that the elements in each of the products lie on different lines of A. This scalar valued function of the matrix A occurs throughout the combinatorial literature in connection with various enumeration and extremal problems. We remark that $\operatorname{per}(A)$ remains invariant under arbitrary permutations of the lines of A. Furthermore, in the case of square matrices $\operatorname{per}(A)$ is the same as the determinant function apart from a factor ± 1 preceding each of the products in the summation. In the case of square matrices certain determinantal laws have direct analogues for permanents. In particular, the Laplace expansion for determinants has a simple counterpart for permanents. But the basic multiplicative law of determinants

$$\det(AB) = \det(A)\det(B)$$

is flagrantly false for permanents. Similarly, the permanent function is in general greatly altered by the addition of a multiple of one row of a matrix to another. These facts tend to severely restrict the computational procedures available for the evaluation of permanents.

We return to the (0,1)-matrix A of size m by n, and we now assume that $m \leq n$. Then it follows directly from the definition of $\operatorname{per}(A)$ that

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per$(A)>0$ if and only if A is of term rank m. The following theorem is also a direct consequence of the terminology involved.

Theorem 1.2.3. *Let A be the incidence matrix for m subsets $X_1, X_2, \ldots,$ X_m of an n-set X and suppose that $m \leq n$. Then the number of distinct SDR's for this configuration of subsets is* per(A). □

The permanent function is studied more thoroughly in Chapter 7.

We have characterized a configuration of subsets by means of a (0,1)-matrix. The choice of the integers 0 and 1 is particularly judicious in many situations, and this is already exemplified by Theorem 1.2.3. But the configuration could also be characterized by a $(1, -1)$-matrix or for that matter by a more general matrix whose individual entries possess or fail to possess a certain property. For example, the following assertion is entirely equivalent to our formulation of the König theorem. *Let A be a matrix of size m by n with elements from a field F. The minimal number of lines in A that cover all of the nonzero elements of A is equal to the maximal number of nonzero elements in A with no two of the nonzero elements on a line.* In what follows we apply the König theorem to nonnegative real matrices.

A matrix of order n is called *doubly stochastic* provided that its entries are nonnegative real numbers and all of its line sums are equal to 1. The $n!$ permutation matrices of order n as well as the matrix of order n with every entry equal to $1/n$ are simple instances of doubly stochastic matrices. The following theorem on doubly stochastic matrices is due to Birkhoff[1946].

Theorem 1.2.4. *A nonnegative real matrix A of order n is doubly stochastic if and only if there exist permutation matrices $P_1, P_2, \ldots, P_t$ and positive real numbers $c_1, c_2, \ldots, c_t$ such that*

$$A = c_1P_1 + c_2P_2 + \cdots + c_tP_t \tag{1.2}$$

and

$$c_1 + c_2 + \cdots + c_t = 1. \tag{1.3}$$

Proof. If the nonnegative matrix A satisfies (1.2) and (1.3) then

$$AJ = JA = J$$

and A is doubly stochastic.

We now prove the reverse implication. We assert that the doubly stochastic matrix A has n positive entries with no two of the positive entries on a line. For if this were not the case, then by the König theorem we could cover all of the positive entries in A with e rows and f columns, where $e+f<n$. But then since all of the line sums of A are equal to 1, it follows that $n \leq e+f<n$, and this is a contradiction. Now let P_1 be the permutation matrix of order n with 1's in the same positions as those occupied by

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the n positive entries of A. Let c_1 be the smallest of these n positive entries. Then $A - c_1P_1$ is a scalar multiple of a doubly stochastic matrix, and at least one more 0 appears in $A - c_1P_1$ than in A. Hence we may iterate the argument on $A - c_1P_1$ and eventually obtain the desired decomposition (1.2). We now multiply (1.2) by J and this immediately implies (1.3). □

Corollary 1.2.5. *Let A be a $(0,1)$-matrix of order n such that all of the line sums of A are equal to the positive integer k. Then there exist permutation matrices $P_1, P_2, \ldots, P_k$ such that*

$$A = P_1 + P_2 + \cdots + P_k.$$

Proof. The (0,1)-matrix A is a scalar multiple of a doubly stochastic matrix. This means that the same arguments used in the proof of Theorem 1.2.4 may be applied directly to the matrix A. But we now have each $c_i = 1$ and the entire process comes to an automatic termination in k steps. □

Corollary 1.2.5 has the following amusing interpretation. A dance is attended by n boys and n girls. Each boy has been previously introduced to exactly k girls and each girl has been previously introduced to exactly k boys. No further introductions are allowed. Is it possible to pair the boys and the girls so that the boy and girl of each pair have been previously introduced? We number the boys $1, 2, \ldots, n$ in an arbitrary manner and similarly for the girls. Then we let $A = [a_{ij}]$ denote the (0,1)-matrix of order n defined by $a_{ij} = 1$ provided boy j has been previously introduced to girl i and by $a_{ij} = 0$ in the alternative situation. Then A satisfies all of the requirements of Corollary 1.2.5, and each of the k permutation matrices P_i gives us a desired pairing of boys and girls. The totality of all of the permitted pairings of boys and girls is equal to per(A). But it should be noted that per(A) depends not only on n and k, but also on detailed information involving the structure of the previous introductions.

Exercises

1. Derive Theorem 1.2.1 from Theorem 1.2.2.
2. Suppose in Theorem 1.2.2 the set union $X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_k}$ always contains at least $k+1$ elements. Let x be any element of X_1. Show that the sets $X_1, X_2, \ldots, X_m$ have an SDR with the property that x represents X_1.
3. Let A be a (0,1)-matrix of order n satisfying the equation $A + A^T = J - I$. Prove that the term rank of A is at least $n - 1$.
4. Let A be an m by n (0,1)-matrix. Suppose that there exist a positive integer p such that each row of A contains at least p 1's and each column of A contains at most p 1's. Prove that per$(A) > 0$.
5. Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_m$ be two partitions of the n-set X into m subsets. Prove that there exists a permutation $j_1, j_2, \ldots, j_m$ of $\{1, 2, \ldots, m\}$ such that

$$X_i \cap Y_{j_i} \neq \emptyset, \quad (i = 1, 2, \ldots, m)$$

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if and only if each union of k of the sets $X_1, X_2, \ldots, X_m$ contains at most k of the sets $Y_1, Y_2, \ldots, Y_m$, $(k = 1, 2, \ldots, m)$.

6. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two *monotone* real vectors:

$$x_1 \geq x_2 \geq \cdots \geq x_n; \qquad y_1 \geq y_2 \geq \cdots \geq y_n.$$

Assume that there exists a doubly stochastic matrix S of order n such that $x = yS$. Prove that

$$x_1 + \cdots + x_k \leq y_1 + \cdots + y_k, \quad (k = 1, 2, \ldots, n)$$

with equality for $k = n$. (The vector x is said to be *majorized* by y.)

7. Prove that the product of two doubly stochastic matrices is a doubly stochastic matrix.
8. Let A be a doubly stochastic matrix of order n. Let A' be a matrix of order $n-1$ obtained by deleting the row and column of a positive element of A. Prove that $\operatorname{per}(A) > 0$.

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1.3 Set Intersections

We return to the m not necessarily distinct subsets $X_1, X_2, \ldots, X_m$ of an n-set X. Up to now we have discussed in some detail the formal structure of the (0,1)-matrix A of size m by n which is the incidence matrix for this configuration of subsets. In what follows the algebraic properties of the matrix A will begin to play a much more dominant role.

We are now concerned with the cardinalities of the set intersections $X_i \cap X_j$, and in order to study this concept we multiply the above matrix A by its transpose. We thereby obtain the matrix equation

$$AA^T = S. \tag{1.4}$$

The matrix S of (1.4) is a symmetric matrix of order m with nonnegative integral elements. The element s_{ij} in the (i, j) position of S records the cardinality of the set intersection $X_i \cap X_j$, namely,

$$s_{ij} = |X_i \cap X_j|, \quad (i, j = 1, 2, \ldots, m).$$

The main diagonal elements of S display the cardinalities of the m subsets $X_1, X_2, \ldots, X_m$. It should be noted that all of this information on the

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cardinalities of the set intersections is exhibited in (1.4) in an exceedingly compact form.

We mention next two variants of (1.4). We may reverse the order of multiplication of the matrix A and its transpose and this yields the matrix equation

$$A^T A = T. \tag{1.5}$$

The matrix T of (1.5) is a symmetric matrix of order n with nonnegative integral elements. The element t_{ij} in the (i, j) position of T records the number of times that the elements x_i and x_j occur among the subsets $X_1, X_2, \ldots, X_m$. The main diagonal elements of T display the totality of the occurrences of each of the n elements among the m subsets. The matrix T may also be regarded as recording the cardinalities of the set intersections of the dual configuration.

Our second variant of (1.4) involves the complement C of the incidence matrix A. We may multiply A by the transpose of C and this yields the matrix equation

$$AC^T = W. \tag{1.6}$$

This matrix equation differs noticeably from the two preceding equations. The matrix W need no longer be symmetric. The element w_{ij} in the (i, j) position of W records the cardinality of the set difference $X_i - X_j$. (The set difference $X_i - X_j$ is the set of all elements in X_i but not in X_j.) The matrix W has 0's in the m main diagonal positions.

We recall that for a matrix A with real elements we have

$$\operatorname{rank}(A) = \operatorname{rank}(AA^T).$$

Hence the matrices A and S of (1.4) satisfy

$$\operatorname{rank}(S) = \operatorname{rank}(A) \leq m, n. \tag{1.7}$$

Thus it follows from (1.7) that if S is nonsingular, then we must have

$$m \leq n. \tag{1.8}$$

The inequality (1.8) is of interest because it tells us that the algebraic requirement of the nonsingularity of S automatically imposes a constraint between the two integral parameters m and n. In many investigations the extremal configurations with $m = n$ are especially significant. For the dual configuration it follows that if T is nonsingular, then we must have $n \leq m$.

We make no attempt now to study the matrix equation (1.4) and its variants (1.5) and (1.6) in their full generality. We look at a very special case of (1.4) and show that this already leads us to important unanswered questions.

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Let t be a positive integer and suppose that A is a (0,1)-matrix of size m by n that satisfies the matrix equation

$$AA^T = tI + J. \tag{1.9}$$

Thus in (1.9) we have selected our symmetric matrix S of (1.4) in a particularly simple form, namely with $t+1$ in the m main diagonal positions and with 1's in all other positions. In order to evaluate the determinant of $tI + J$ we first subtract column 1 from all other columns and we then add the last $m-1$ rows to the first row. This tells us that

$$\det(tI + J) = (t+m)t^{m-1} \neq 0.$$

Thus the matrix $tI + J$ is nonsingular, and by (1.8) we may conclude that $m \leq n$.

Now suppose that $m = n$. We show that in this case the incidence matrix A possesses a number of remarkable symmetries. Since the matrix A is square of order n we may apply the multiplicative law of determinants to the matrix equation (1.9). Thus we have

$$\det(AA^T) = \det(A)\det(A^T) = (\det(A))^2 = (t+n)t^{n-1}$$

and

$$\det(A) = \pm(t+n)^{1/2}t^{(n-1)/2}. \tag{1.10}$$

Since A is a (0,1)-matrix it follows that the expression on the right side of (1.10) is of necessity an integer. It also follows from (1.9) that all of the row sums of A are equal to $t+1$. Thus we may write

$$AJ = (t+1)J. \tag{1.11}$$

But A is a nonsingular matrix and hence the inverse of A satisfies

$$A^{-1}J = (t+1)^{-1}J.$$

Moreover, it follows from (1.9) that

$$AA^TJ = tJ + J^2 = (t+n)J$$

and hence

$$A^TJ = (t+1)^{-1}(t+n)J.$$

We next take transposes of both sides of this equation and obtain

$$JA = (t+1)^{-1}(t+n)J. \tag{1.12}$$

The multiplication of (1.12) by J implies

$$JAJ = n(t+1)^{-1}(t+n)J.$$

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But from (1.11) we also have

$$JAJ = n(t+1)J,$$

whence it follows that

$$n = t^2 + t + 1. \tag{1.13}$$

This additional relation between n and t allows us to write (1.10) in the form

$$\det(A) = \pm(t+1)t^{t(t+1)/2},$$

and we now see that our formula for $\det(A)$ is, indeed, an integer. We may substitute (1.13) into (1.12) and with (1.11) obtain

$$AJ = JA = (t+1)J. \tag{1.14}$$

The equations of (1.14) tell us that all of the line sums of A are equal to $t+1$.

We next investigate the matrix product $A^T A$ and note that

$$A^T A = A^{-1}(AA^T)A = A^{-1}(tI + J)A = tI + A^{-1}JA = tI + J.$$

A matrix of order n with real elements is called *normal* provided that it commutes under multiplication with its transpose. It follows that our matrix A is normal and satisfies

$$AA^T = A^T A = tI + J. \tag{1.15}$$

We may also readily verify that the complement C of A satisfies

$$AC^T = C^T A = t(J - I)$$

and

$$CC^T = C^T C = tI + t(t-1)J.$$

We now discuss some specific solutions of the matrix equation (1.15). We have shown that the order n of A satisfies (1.13) so that there is only a single integer parameter t involved. For the case in which $t = 1$ it follows readily that all solutions of (1.15) are given by the (0,1)-matrices of order 3 with all line sums equal to 2. These six matrices yield a single configuration in the sense of isomorphism.

The configuration associated with a solution of (1.15) for $t > 1$ is called

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a *finite projective plane* of *order* t. We exhibit the incidence matrix for the projective plane of order 2:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}. \tag{1.16}$$

This is the "smallest" finite projective plane, and it is easy to verify that the projective plane of order 2 is unique in the sense of isomorphism. We remark in passing that the incidence matrix A of (1.16) possesses a most unusual property:

$$\text{per}(A) = |\det(A)| = 24.$$

Thus all of the 24 permutations that contribute to $\det(A)$ are of the same sign.

Finite projective planes have been constructed for all orders t that are equal to the power of a prime number. No planes have as yet been constructed for any other orders, but they are known to be impossible for infinitely many values of t. For a long time the smallest undecided case was $t = 10$. Notice that the associated incidence structure is already of order 111. Using sophisticated computer calculations, Lam, Thiel and Swierzc[1989] have recently concluded that there is no finite projective plane of order 10. The smallest order for which nonisomorphic planes exist is $t = 9$.

One of the major unsolved problems in combinatorics is the determination of the precise range of values of t for which projective planes of order t exist. The determination of the number of nonisomorphic solutions for a general t appears to be well beyond the range of present day techniques. These extremal configurations are of the utmost importance and have many ramifications. They and their generalizations will be studied in some detail in the sequel to this book, *Combinatorial Matrix Classes.*

We next consider a finite projective plane whose associated incidence matrix is symmetric. The proof of the following theorem illustrates the effective use of matrix algebra techniques.

Theorem 1.3.1. *Let a finite projective plane* Π *be such that its associated incidence matrix* A *is symmetric. Suppose further that the order* t *of* Π *is not equal to an integral square. Then the incidence matrix* A *of* Π *contains exactly* $t+1$ *1's on its main diagonal.*

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Proof. We first recall the following fundamental property concerning the eigenvalues (characteristic roots) of a matrix. Let A be a matrix of order n with elements in a field F and let the n eigenvalues of A be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $f(A)$ be an arbitrary polynomial in the matrix A. Then the n eigenvalues of $f(A)$ are $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$.

Since the incidence matrix A of Π is symmetric it follows that we may write (1.15) in the form

$$A^2 = tI + J. \tag{1.17}$$

The characteristic polynomial $f(\lambda)$ of $tI + J$ equals

$$f(\lambda) = \det(\lambda I - (tI + J)) = (\lambda - (t+1)^2)(\lambda - t)^{t^2+t}. \tag{1.18}$$

The calculation of $f(\lambda)$ in (1.18) is much the same as the one carried out earlier for $\det(tI + J)$. Thus we see that the $n = t^2 + t + 1$ eigenvalues of $tI + J$ are $(t+1)^2$ of multiplicity 1 and t of multiplicity $t^2 + t$. By (1.17) and the property concerning eigenvalues quoted at the outset of the proof it follows that the n eigenvalues of A are either $t + 1$ or else $-(t+1)$ of multiplicity 1, and $\pm\sqrt{t}$ of appropriate multiplicities. Let u denote the column vector of n 1's. The matrix A has all its row sums equal to $t + 1$ so that

$$Au = (t+1)u. \tag{1.19}$$

Equation (1.19) tells us that u is an eigenvector of A with associated eigenvalue $t + 1$, and thus $-(t+1)$ does not arise as an eigenvalue of A.

The *trace* of a matrix of order n is the sum of the n main diagonal elements of the matrix and this in turn is equal to the sum of the n eigenvalues of the matrix. Thus there exists an integer e determined by the multiplicities of the eigenvalues $\pm\sqrt{t}$ of our incidence matrix A such that we may write

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n = t + 1 + e\sqrt{t}.$$

We know that A is a (0,1)-matrix so that $\operatorname{tr}(A)$ is an integer. But now using for the first time our hypothesis that t is not equal to an integral square it follows that we must have $e = 0$. □

We note that the incidence matrix A of (1.16) for the projective plane of order 2 is symmetric. Consequently we now see that it is no accident that exactly three 1's appear on its main diagonal.

Exercises

1. Show that the determinant of the matrix $tI + aJ$ of order n equals $t^{n-1}(t + an)$.
2. Show that the n eigenvalues of the matrix $tI + aJ$ of order n are t with multiplicity $n - 1$ and $t + an$.

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3. Let A be an m by n (0,1)-matrix which satisfies the matrix equation $AA^T = tI + aJ$ where $t \neq 0$. Prove that $n \geq m$.
4. Let A be a (0,1)-matrix of order n which satisfies the matrix equation $AA^T = tI + aJ$. Generalize the argument given in the text for $a = 1$ to prove that A is a normal matrix.
5. Verify that the projective plane of order 2 is unique in the sense of isomorphism.
6. Verify that the incidence matrix A of the projective plane of order 2 satisfies $\operatorname{per}(A) = |\det(A)| = 24$.
7. Determine a formula for the permanent of the matrix $tI + aJ$ of order n in terms of derangement numbers D_k. (D_k is the number of permutations of $\{1, 2, \ldots, k\}$ which have no fixed point.)
8. Let S denote a nonzero symmetric matrix of order $m \geq 2$ with nonnegative integral elements and with 0's in all of the main diagonal positions. Prove that there exists a diagonal matrix D of order m, an integer n and a (0,1)-matrix A of size m by n such that $AA^T = D + S$. Indeed show that a matrix A can be found with all column sums equal to 2.

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1.4 Applications

We now apply the terminology and concepts of the preceding sections to prove several elementary theorems. The results are appealing in their simplicity and give us additional insight into the structure of (0,1)-matrices. We recall that a submatrix of order m of a matrix A of order n is called *principal* provided that the submatrix is obtained from A by deleting $n-m$ rows and $n-m$ columns of A with both sets of deleted rows and columns numbered identically $i_1, i_2, \ldots, i_{n-m}$. This definition of principal submatrix is equivalent to the assertion that the submatrix may be placed in the upper left corner of A by simultaneous permutations of the lines of A.

Theorem 1.4.1. *Let A be a $(0,1)$-matrix of order n and suppose that A contains no column of 0's. Then A contains a principal submatrix which is a permutation matrix.*

Proof. The proof is by induction on n. The result is certainly valid in case $n = 1$ so that we may assume that $n > 1$. Let A contain e columns with column sums equal to 1 and $n - e$ columns with column sums greater than 1. We simultaneously permute the lines of A so that the e columns with column sum equal to 1 are the initial columns of the permuted matrix. We designate the permuted matrix by A' and note that it suffices to prove the theorem for A'. Let A_1 denote the principal submatrix of order e in the

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upper left corner of A' and let A_2 denote the principal submatrix of order $n-e$ in the lower right corner of A'.

In the event that A_1 is empty we delete a row of A_2 and its corresponding column. We then apply the induction hypothesis to the submatrix of order $n-1$ and the result follows. Now suppose that A_1 is not empty and that A_1 contains a row of 0's. We now delete this row in A' and its corresponding column and once again apply the induction hypothesis to the submatrix of order $n-1$. There remains the alternative case in which A_1 contains no row of 0's. Then A_1 has all of its row sums greater than or equal to 1 and all of its column sums less than or equal to 1. This state of affairs now implies that A_1 has all of its line sums equal to 1. Thus A_1 itself is the required principal submatrix of A'. □

We now use the preceding theorem to characterize the (0,1)-matrices of order n whose permanents are equal to 1 (Brualdi[1966]).

Theorem 1.4.2. *Let A be a $(0,1)$-matrix of order n. Then* $\operatorname{per}(A) = 1$ *if and only if the lines of A may be permuted to yield a triangular matrix with 1's in the n main diagonal positions and with 0's above the main diagonal.*

Proof. The proof is immediate in case A is permutable to triangular form. We use induction on n for the reverse implication. The result is obvious for $n = 1$. Since $\operatorname{per}(A) = 1$ we may permute the lines of A so that n 1's appear on the main diagonal of the matrix. We designate the permuted matrix by A' and suppose that A' has all of its row sums greater than 1. Then the transpose of the matrix $A' - I$ satisfies the requirements of Theorem 1.4.1 and hence contains a principal submatrix which is a permutation matrix. But then it follows that $\operatorname{per}(A) = \operatorname{per}(A') > 1$ and this is a contradiction. Hence A contains a row with a single 1. Thus we may permute the lines of A so that row 1 of the matrix contains a 1 in the (1,1) position and 0's elsewhere. We now delete the first row and column of this matrix and apply the induction hypothesis to this submatrix of order $n-1$. □

A *triangle* of a (0,1)-matrix A is a submatrix of A of order 3 such that all of the line sums of the submatrix are equal to 2. The following theorem of Ryser[1969] deals with the set intersections of configurations whose incidence matrices contain no triangles.

Theorem 1.4.3. *Let A be a $(0,1)$-matrix of size m by n. Suppose that A contains no triangles and that every element of AA^T is positive. Then A contains a column of m 1's.*

Proof. The proof is by induction on m. The result is valid for both $m = 1$ and $m = 2$ so that we may assume that $m \geq 3$. We delete row 1 of A and apply the induction hypothesis to the submatrix of A consisting of the last $m-1$ rows of A. This submatrix contains a column of $m-1$ 1's. Then

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either A contains a column of m 1's and we are done, or else A contains a column with a 0 in the first position and with 1's in the remaining $m-1$ positions. We repeat the argument on A with row 2 of A deleted. Then either A contains a column of m 1's and we are done, or else A contains a column with a 0 in the second position and with 1's in the remaining $m-1$ positions. We finally repeat the argument a third time on A with row 3 of A deleted. But now A cannot contain a column with a 0 in the third position and with 1's in the remaining positions because such a column yields a triangle within A. Hence the matrix A contains a column of m 1's as desired. □

An extensive literature in the combinatorial geometry of convex sets is concerned with "Helly type" theorems (Hadwiger et al.[1964]). The following elementary proposition affords a good illustration of a Helly type theorem. *Let there be given a finite number of closed intervals on the real line with the property that every pair of the intervals has a point in common. Then all of the intervals have a point in common.*

We show that the above proposition is actually a special case of Theorem 1.4.3. Let the closed intervals be labeled $X_1, X_2, \ldots, X_m$ and let the endpoints of these intervals occur at the following points on the real line

$$e_1 < e_2 < \cdots < e_n.$$

We now form the incidence matrix A of size m by n of intervals versus endpoints. Thus we set $a_{ij} = 1$ if the point e_j is contained in the interval X_i and we set $a_{ij} = 0$ in the contrary case. This incidence matrix has a very special form, namely, the 1's in each row occur consecutively. Now the 1's in every submatrix also occur consecutively in each of the rows of the submatrix and hence A contains no triangles. Furthermore, the pairwise intersection property of the intervals implies that every element of AA^T is positive. But then Theorem 1.4.3 asserts that the matrix A contains a column of m 1's and this means that all of the intervals have a point in common.

We digress and consider in somewhat oversimplified form a problem from archaeology (Kendall[1969] and Shuchat[1984]). Suppose that we have a set of *graves* $G_1, G_2, \ldots, G_m$ and a set of *artifacts* (or aspects of artifacts) $a_1, a_2, \ldots, a_n$ collected from these graves. We form the incidence matrix of size m by n of graves versus artifacts in the usual way. Suppose that it is possible for us to permute the rows of A so that the 1's in each column occur consecutively. Then such a permutation of the rows of A determines a chronology of the graves and this in turn assigns a sequence date to each artifact. An incidence matrix in which the 1's in each column occur consecutively is called a *Petrie matrix* in honor of Flinders Petrie, a noted English Egyptologist. We have encountered Petrie matrices (or their transposes) in

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our discussion of the Helly type theorem as well as in the rectangle partitioning problem exemplified by Figure 1.1.

Our next theorem deals directly with set intersections and yields a considerable refinement of the inequality (1.8) for configurations related to finite projective planes.

Theorem 1.4.4. *Let A be a $(0,1)$-matrix of size $m = t^2 + t + 1$ by n. Suppose that A contains no column of 0's and that A satisfies the matrix equation*

$$AA^T = tI + J \ (t \geq 2). \tag{1.20}$$

Then the only possible values of n occur for $n = t^2 + t + 1$ and for $n = t^3 + t^2 + t + 1$. The first case yields a projective plane of order t and the second case yields the unique configuration in which A contains a column of 1's.

Proof. We first note that the assumption that A contain no column of 0's is a natural one because such columns can be adjoined to A without affecting the general form of the matrix equation (1.20).

We suppose next that A contains a column of 1's. Then it follows from (1.20) that all of the remaining column sums of A are equal to 1 and hence A has a totality of

$$n = t(t^2 + t + 1) + 1 = t^3 + t^2 + t + 1$$

columns. The matrix A is unique apart from column permutations.

We now deal with the case in which A does not contain a column of 1's. We denote the sum of column 1 of A by s. We permute the rows of A so that the s 1's in column 1 of A occupy the initial positions in column 1, and we then permute the remaining columns of A so that the $t + 1$ 1's in row 1 occupy the initial positions in row 1. We designate the resulting matrix by A'. Then by (1.20) the first $t + 1$ columns of A' contain exactly one 1 in each of rows $2,3,\ldots,m$. Hence the total number of 1's in the first $t + 1$ columns of A' is equal to

$$(t + 1) + (t^2 + t) = (t + 1)^2.$$

Now by construction row $s + 1$ of A' has a 0 in the initial position. But by (1.20) row $s + 1$ of A' has inner product 1 with each of rows $1, 2, \ldots, s$ of A'. Since the s 1's in column 1 of A' occur in the initial positions, it follows from (1.20) that row $s + 1$ of A' contains at least s 1's. But row $s + 1$ contains exactly $t + 1$ 1's and hence

$$s \leq t + 1. \tag{1.21}$$

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The argument applied to column 1 of A holds for an arbitrary column of A, and hence every column of A satisfies (1.21). We have noted that the total number of 1's in the first $t+1$ columns of A' is equal to $(t+1)^2$, and this in conjunction with (1.21) tells us that $s = t+1$. But then all of the column sums of A are equal to $t+1$, and hence all of the line sums of A are equal to $t+1$. This means that A is a square and $m = n$. □

Our concluding theorem in this chapter involves an application of (0,1)-matrices to number theory. We study the following integral matrix B of order n:

$$B = [b_{ij}] = [(i,j)], \quad (i,j = 1,2,\ldots,n) \tag{1.22}$$

where (i,j) denotes the positive greatest common divisor of the integers i and j.

Let m be a positive integer and let $\phi(m)$ denote the Euler ϕ-function of m. We recall that $\phi(m)$ is defined as the number of positive integers less than or equal to m and relatively prime to m. We also recall that

$$m = \sum_{d|m} \phi(d), \tag{1.23}$$

where the summation extends over all of the positive divisors d of m.

We now prove a classical theorem of Smith[1876] using the techniques of Frobenius[1879].

Theorem 1.4.5. *The determinant of the matrix B of* (1.22) *satisfies*

$$\det(B) = \prod_{i=1}^{n} \phi(i). \tag{1.24}$$

Proof. Let $A = [a_{ij}]$ be the (0,1)-matrix of order n defined by the relationships $a_{ij} = 1$ if j divides i and $a_{ij} = 0$ if j does not divide i $(i,j = 1,2,\ldots,n)$. We define the diagonal matrix

$$\Phi = \text{diag}[\phi(1), \phi(2), \ldots, \phi(n)]$$

of order n whose main diagonal elements are $\phi(1), \phi(2), \ldots, \phi(n)$. Then

$$A\Phi A^T = [a_{ij}]\Phi[a_{ji}] = [a_{ij}\phi(j)][a_{ji}] = \left[\sum_{t=1}^{n} a_{it}\phi(t)a_{jt}\right].$$

The definition of the (0,1)-matrix A implies that

$$\sum_{t=1}^{n} a_{it}\phi(t)a_{jt} = \sum \phi(d_{ij}),$$

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where d_{ij} ranges over all of the positive common divisors of i and j. But then by (1.23) we have

$$\sum \phi(d_{ij}) = (i, j),$$

whence

$$A\Phi A^T = B.$$

The (0,1)-matrix A of order n is triangular with 1's in the n main diagonal positions. Since the determinant function is multiplicative it follows that

$$\det(B) = \det(\Phi).$$

□

Exercises

1. Let A be a (0,1)-matrix of order n and suppose that $\text{per}(A) = 2$. Show that there exists an integer $k \geq 2$ and a square submatrix B of A whose lines can be permuted to obtain a (0,1)-matrix with 1's exactly in the positions $(1,1), (2,2), \ldots, (k,k), (1,2), \ldots, (k-1,k), (k,1)$.
2. Deduce that the matrix B of Smith in (1.22) is a positive definite matrix.
3. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct positive integers. Let $A = [a_{ij}]$ be the *greatest common divisor matrix* for X defined by $a_{ij} = (x_i, x_j), (i, j = 1, 2, \ldots, n)$. If X is *factor closed* in the sense that each positive integral divisor of an element in X is also in X, then generalize the argument in the proof of Theorem 1.4.5 to evaluate the determinant of A. Prove that the matrix A is positive definite for all X (Beslin and Ligh[1989]).

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