# 4

# Matrices and Bipartite Graphs

## 4.1 Basic Facts

Bipartite graphs are defined in section 2.6. A multigraph $G$ is *bipartite* provided that its vertices may be partitioned into two subsets $X$ and $Y$ such that every edge of $G$ is of the form $\{a,b\}$ where $a$ is in $X$ and $b$ is in $Y$. The pair $\{X,Y\}$ is called a *bipartition* of $G$. If $G$ is connected its bipartition is unique.

The bipartite multigraph $G$ is characterized by an $m$ by $n$ nonnegative integral matrix

$$B = [b_{ij}], (i = 1,2,\ldots,m; j = 1,2,\ldots,n),$$

where $m$ is the number of vertices in $X$ and $n$ is the number in $Y$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The element $b_{ij}$ equals the multiplicity $m\{x_i, y_j\}$ of the edges of the form $\{x_i, y_j\}$. The adjacency matrix of $G$ is the $m+n$ by $m+n$ matrix

$$\begin{bmatrix} O & B \\ B^T & O \end{bmatrix}. \tag{4.1}$$

We call $B$ the *reduced adjacency matrix* of the bipartite multigraph $G$. Every $m$ by $n$ nonnegative integral matrix is the reduced adjacency matrix of some bipartite multigraph.

We begin with two elementary but fundamental characterizations of bipartite multigraphs.

**Theorem 4.1.1.** *A multigraph $G$ is bipartite if and only if every cycle of $G$ has even length.*

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*Proof.* The definition of a bipartite graph implies at once that every cycle has even length.

It suffices to prove the converse proposition for a connected component $G'$ of $G$. We select an arbitrary vertex $a$ in $G'$. Let $X$ be the set of vertices of $G'$ whose distance from $a$ is even, and let $Y$ be the set of vertices of $G'$ whose distance from $a$ is odd. Let $p$ and $q$ be two vertices in $X$. We show that $G'$ does not contain an edge of the form $\{p,q\}$. Let $a \to \cdots \to p$ and $a \to \cdots \to q$ denote walks of minimal length from $a$ to $p$ and $a$ to $q$, respectively. Let $b$ be the last common vertex in these two walks. Then the walks $b \to \cdots \to p$ and $b \to \cdots \to q$ are both of even length or both of odd length. But an edge of the form $\{p,q\}$ implies the existence of a cycle of odd length

$$b \to \cdots \to p \to q \to \cdots \to b,$$

contrary to hypothesis. In the same way one shows that two distinct vertices in $Y$ are not connected by an edge. Hence $G'$ is bipartite. □

**Theorem 4.1.2.** *Let $G$ be a multigraph and let $A$ be the incidence matrix of $G$. Then $G$ is bipartite if and only if $A$ is totally unimodular.*

*Proof.* Let $G$ be bipartite. Then $G$ has a bipartition $\{X,Y\}$, and this implies that $A^T$ satisfies the requirements of Theorem 2.3.3. Hence $A$ is totally unimodular.

Now let $A$ be totally unimodular and suppose that $G$ is not bipartite. By Theorem 4.1.1 $G$ has a cycle of odd length $r$. But this implies that $A$ contains a submatrix $A'$ of order $r$ such that $\det(A') = \pm 2$. This contradicts the hypothesis that $A$ is totally unimodular. □

Let $G$ be a bipartite graph with bipartition $\{X,Y\}$ where $X$ is an $m$-set and $Y$ is an $n$-set. If for each $x$ in $X$ and each $y$ in $Y$, $G$ contains exactly one edge of the form $\{x,y\}$, then $G$ is called a *complete bipartite graph* and is denoted by $K_{m,n}$.

Let $K_n$ denote the complete graph of order $n$. Let $G_1, G_2, \ldots, G_r$ denote complete bipartite subgraphs of $K_n$. Suppose that the graphs $G_1, G_2, \ldots, G_r$ are edge disjoint and between them contain all of the edges of $K_n$. Then we say that $G_1, G_2, \ldots, G_r$ form a *decomposition* of $K_n$. It is easy to construct decompositions of $K_n$ for which $r = n-1$ and $G_1, G_2, \ldots, G_{n-1}$ is $K_{1,n-1}, K_{1,n-2}, \ldots, K_{1,1}$. The following theorem of Graham and Pollak[1971] tells us that it is not possible to form a decomposition of $K_n$ into complete bipartite subgraphs with $r < n-1$. The short proof is due to Peck[1984].

**Theorem 4.1.3.** *Let the complete graph $K_n$ of order $n$ have a decomposition $G_1, G_2, \ldots, G_r$ into complete bipartite subgraphs. Then $r \geq n-1$.*

*Proof.* Let $G_i'$ be the spanning subgraph of $K_n$ with the same set of edges as $G_i$ $(i = 1, 2, \ldots, r)$. The conditions of the theorem imply that we may write

$$J - I = \sum_{i=1}^{r} A_i', \tag{4.2}$$

where $A_i'$ is the adjacency matrix of $G_i'$. Let $A_i$ be the adjacency matrix of $G_i$ $(i = 1, 2, \ldots, r)$. Then $A_i$ is a principal submatrix of $A_i'$ and contains all the nonzero entries of $A_i'$. The matrix $A_i$, and hence the matrix $A_i'$, is of rank 2 because in the special form (4.1) we have $B$ equal to a matrix of all 1's. We now replace all of the 1's in $A_i$ corresponding to the 1's in $B^T$ with 0's. The resulting matrix $A_i''$ is clearly of rank 1. Furthermore the matrix

$$Q_i = A_i' - 2A_i''$$

is skew-symmetric. We may now write (4.2) in the form

$$I + Q = J - 2\sum_{i=1}^{r} A_i''$$

where $Q$ is a skew-symmetric matrix of order $n$. A real skew-symmetric matrix has pure imaginary eigenvalues so that we may conclude that $I + Q$ is nonsingular. But the rank of a sum of matrices does not exceed the sum of the ranks and hence it follows that $1 + r \geq n$. □

Another proof of Theorem 4.1.3 is given by Tverberg[1982]. A third proof is indicated in the exercises.

## Exercises

1. Let $B$ be the $m$ by $n$ reduced adjacency matrix of a bipartite graph $G$. Prove that $G$ is connected if and only if there do not exist permutation matrices $P$ and $Q$ such that

$$PBQ = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix}$$

where $B_1$ is a $p$ by $q$ matrix for some nonnegative integers $p$ and $q$ satisfying $1 \leq p + q \leq m + n - 1$.

2. Let $A$ be a tournament matrix of order $n$. Prove that the rank of $A$ is at least equal to $n - 1$. (Hint: Consider the matrix $N$ of size $n$ by $n + 1$ obtained from $A$ by adjoining a column of 1's and show that only the zero vector is in the left null space of $N$.)
3. Let $G_1, G_2, \ldots, G_r$ be a decomposition of the complete graph $K_n$ into complete bipartite graphs. For each $G_i$ direct the edges from the vertices in one set of its bipartition to the vertices in the other set of its bipartition. The result is a tournament of order $n$ and hence a tournament matrix $A$ of order $n$. Obtain from the decomposition $G_1, G_2, \ldots, G_r$ a factorization $A = CD$ of $A$ into an

$n$ by $r$ (0,1)-matrix $C$ and an $r$ by $n$ (0,1)-matrix $D$. Now use Exercise 2 to obtain an alternative proof of Theorem 4.1.3 (de Caen and Hoffman[1989]).

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## 4.2 Fully Indecomposable Matrices

In this section we deal primarily with $m$ by $n$ (0,1)-matrices. The definitions and results apply to arbitrary matrices upon replacing each nonzero element with a 1.

Let $A$ be an $m$ by $n$ (0,1)-matrix. The *term rank* $\rho = \rho(A)$ of $A$ is defined in section 1.2 to be the maximal number of 1's of $A$ with no two of the 1's on a line. A *line cover* of $A$ is a collection of lines of $A$ which together contain all the 1's of $A$. By Theorem 1.2.1 the minimal number of lines in a line cover is equal to the term rank of $A$. A line cover with the minimal number of lines is called a *minimum line cover* of $A$. We denote the set of minimum line covers of $A$ by $\mathcal{L} = \mathcal{L}(\mathcal{A})$. An *essential line* of $A$ is a line which belongs to every minimum line cover. An essential line may be either an *essential row* or an *essential column.*

Since the term rank of $A$ is $\rho$, we may permute the lines of $A$ so that there are 1's in the first $\rho$ positions on the main diagonal. The permuted $A$ assumes the form

$$\begin{bmatrix} A' & A_{12} \\ A_{21} & O \end{bmatrix} \tag{4.3}$$

where $A'$ is a $\rho$ by $\rho$ matrix with 1's everywhere on its main diagonal. Without loss of generality we assume that $A$ has the form (4.3). For each $i = 1, 2, \ldots, \rho$ each minimum line cover of $A$ contains either row $i$ or column $i$, but not both. Thus row $i$ and column $i$ cannot both be essential lines of $A, (i = 1, 2, \ldots, \rho)$. Let $r$ be the number of essential rows of $A$ and let $s$ be the number of essential columns. Let $t = \rho - r - s$. Then $r$, $s$ and $t$

are nonnegative integers summing to $\rho$. We now simultaneously permute the first $\rho$ rows and the first $\rho$ columns of $A$ so that the resulting matrix assumes the form

$$\begin{array}{c c} & \begin{array}{cccc} r & s & t & \end{array} \\ \begin{array}{c} r \\ s \\ t \\ \\ \end{array} & \left[\begin{array}{cccc} A_1 & X & Y & Z \\ * & A_2 & * & * \\ * & S & A_3 & * \\ * & T & * & O \end{array}\right] \end{array}, \tag{4.4}$$

where rows $1, 2, \ldots, r$ are the essential rows and columns $r+1, r+2, \ldots, r+s$ are the essential columns of the permuted $A$. For each $i = r+s+1, r+s+2, \ldots, \rho$ there is a minimum line cover of the permuted $A$ in (4.4) which does not contain row $i$ (and thus contains column $i$) and a minimum line cover which does not contain column $i$ (and thus contains row $i$). It now follows that all the submatrices marked with a $*$ in (4.4) are zero matrices. We summarize these conclusions in the following theorem (Dulmage and Mendelsohn[1958] and Brualdi[1966]).

**Theorem 4.2.1.** *Let $A$ be an $m$ by $n$ $(0,1)$-matrix with term rank equal to $\rho$. Then there is a permutation matrix $P$ of order $m$ and a permutation matrix $Q$ of order $n$ such that*

$$PAQ = \left[\begin{array}{cccc} A_1 & X & Y & Z \\ O & A_2 & O & O \\ O & S & A_3 & O \\ O & T & O & O \end{array}\right]. \tag{4.5}$$

*The matrices $A_1, A_2$ and $A_3$ are square, possibly vacuous, matrices with 1's everywhere on their main diagonals, and the sum of their orders is $\rho$. The essential rows of the matrix in (4.5) are those rows which meet $A_1$, and the essential columns are those columns which meet $A_2$.*

It follows from the description of the matrix (4.5) given in the statement of Theorem 4.2.1 that the only minimum line cover of the matrix

$$\left[\begin{array}{cc} A_1 & Z \end{array}\right]$$

is the line cover of all rows. Also the only minimum line cover of the matrix

$$\left[\begin{array}{c} A_2 \\ T \end{array}\right]$$

is the line cover of all columns. The matrix $A_3$ in (4.5), if not vacuous, has at least two minimum line covers. These are the line cover of all rows and the line cover of all columns. Whether or not there are other minimum line covers depends on the additional structure of $A_3$. This leads us to a study of matrices whose only minimum line covers are the all rows cover and the all columns cover. Such matrices are necessarily square.

We now assume that $A$ is a (0,1)-matrix of order $n$. The matrix $A$ is *partly decomposable* provided there exists an integer $k$ with $1 \leq k \leq n-1$ such that $A$ has a $k$ by $n-k$ zero submatrix. This means that we may permute the lines of $A$ to obtain a matrix of the form

$$\begin{bmatrix} B & O \\ D & C \end{bmatrix}$$

where the zero matrix $O$ is of size $k$ by $n-k$. The matrices $B$ and $C$ are square matrices of orders $k$ and $n-k$, respectively. The partly decomposable matrix $A$ has a line cover consisting of $n-k \geq 1$ rows and $k \geq 1$ columns. This line cover may or may not be a minimum line cover since $A$ may have a $p$ by $q$ zero submatrix with $p+q > n$. But it follows that *the square matrix $A$ is partly decomposable if and only if it has a minimum line cover other than the all rows cover and the all columns cover.* The square matrix $A$ is *fully indecomposable*[1] provided it is not partly decomposable. The property of full indecomposability is not affected by arbitrary line permutations. If $n = 1$, then $A$ is fully indecomposable if and only if $A$ is not the zero matrix of order 1. Each line of a fully indecomposable matrix of order $n \geq 2$ has at least two 1's. It follows from Theorem 1.2.1 that the fully indecomposable matrix $A$ of order $n$ has term rank equal to $n$. But even more is true. If $n \geq 2$ and we choose any entry of $A$ and delete its row and column from $A$, the resulting matrix $B$ of order $n-1$ has term rank $\rho(B)$ equal to $n-1$. For, if $\rho(B) < n-1$, then by Theorem 1.2.1 $B$ and hence $A$ would have a $p$ by $q$ zero submatrix for some positive integers $p$ and $q$ with $p+q = (n-1)+1 = n$. Conversely, if $A$ is a (0,1)-matrix of order $n$ and every submatrix of order $n-1$ has term rank equal to $n-1$, then $A$ is fully indecomposable. This characterization of fully indecomposable matrices (Marcus and Minc[1963] and Brualdi[1966]) is formulated in the next theorem. A collection of $n$ elements of $A$ (or the positions of those elements) is called a *diagonal* of $A$ provided no two of the elements belong to the same row or column of $A$. A *nonzero diagonal* of $A$ is a diagonal not containing any 0's. If $G$ is the bipartite graph whose reduced adjacency matrix is $A$, then the nonzero diagonals of $A$ are in one-to-one correspondence with the perfect matchings of $G$.

[1] One may wonder why the adverbs "partly" and "fully" are being used here. The reason is that the terminology "decomposable" and "indecomposable" is sometimes used in place of "reducible" and "irreducible." A reducible matrix of order $n$ also has a $k$ by $n-k$ zero submatrix for some integer $k$, but the row indices and column indices of the zero submatrix are disjoint sets. Thus reducibility is a more severe restriction on a matrix than that of part decomposability.

**Theorem 4.2.2.** *Let $A$ be a $(0,1)$-matrix of order $n \geq 2$. Then $A$ is fully indecomposable if and only if every $1$ of $A$ belongs to a nonzero diagonal and every $0$ of $A$ belongs to a diagonal all of whose other elements equal $1$.*

There is a close connection between fully indecomposable matrices and the irreducible matrices of Chapter 3 (see Brualdi[1979] and Brualdi and Hedrick[1979]).

**Theorem 4.2.3.** *Let $A$ be a $(0,1)$-matrix of order $n$. Let $A^{\#}$ be the matrix obtained from $A$ by replacing each entry on the main diagonal with a $1$. Then $A$ is irreducible if and only if $A^{\#}$ is fully indecomposable.*

*Proof.* We know that $A^{\#}$ is fully indecomposable if and only if it does not have a $k$ by $n-k$ zero submatrix for any integer $k$ with $1 \leq k \leq n-1$. It is a consequence of the definition of irreducibility that $A$ is irreducible if and only if each $k$ by $n-k$ zero submatrix of $A$ with $1 \leq k \leq n-1$ contains a 0 from the main diagonal of $A$. Since $A^{\#}$ is obtained from $A$ by replacing the 0's that occur on the main diagonal with 1's, the theorem follows. □

The conclusion of Theorem 4.2.3 can also be formulated as: *The square matrix $A$ is irreducible if and only if $A+I$ is fully indecomposable.* The following characterization of fully indecomposable matrices is given in Brualdi, Parter and Schneider[1966].

**Corollary 4.2.4.** *Let $A$ be a $(0,1)$-matrix of order $n$. Then $A$ is fully indecomposable if and only if there exist permutation matrices $P$ and $Q$ of order $n$ such that $PAQ$ has all $1$'s on its main diagonal and $PAQ$ is irreducible.*

*Proof.* Assume that $A$ is fully indecomposable. The term rank of $A$ equals $n$ and thus there exist permutation matrices $P$ and $Q$ of order $n$ such that all diagonal elements of $PAQ$ equal 1. The matrix $PAQ$ is also fully indecomposable and it follows from Theorem 4.2.3 that $PAQ$ is irreducible. The converse proposition is derived in a very similar way. □

We now continue with the study of matrices for which the all rows cover and the all columns cover are minimum line covers (but for which there may be other minimum line covers). By Theorem 1.2.1 this is equivalent to the study of (0,1)-matrices of order $n$ with term rank equal to $n$. First we prove the following preliminary result.

**Lemma 4.2.5.** *Let $A$ be a $(0,1)$-matrix of order $n$ of the form*

$$\begin{bmatrix} X & Z \\ O & Y \end{bmatrix}$$

*where $X$ and $Y$ are square matrices of orders $k$ and $l$, respectively. Then there does not exist a nonzero diagonal of $A$ which contains a $1$ of $Z$.*

*Proof.* Consider a 1 of $Z$ lying in row $i$ and column $j$ of $A$. Let $B$ be the matrix of order $n-1$ obtained from $A$ by deleting row $i$ and column $j$. Then $B$ has a line cover consisting of $k-1$ rows and $l-1$ columns where $(k-1)+(l-1)=n-2$. By Theorem 1.2.1 $\rho(B) \leq n-2$, and the conclusion follows. □

The following theorem is contained in Dulmage and Mendelsohn[1958] and Brualdi[1966].

**Theorem 4.2.6.** *Let $A$ be a $(0,1)$-matrix of order $n$ with term rank $\rho(A)$ equal to $n$. Then there exist permutation matrices $P$ and $Q$ of order $n$ and an integer $t \geq 1$ such that $PAQ$ has the form*

$$\left[\begin{array}{cccc} B_1 & B_{12} & \cdots & B_{1t} \\ O & B_2 & \cdots & B_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_t \end{array}\right] \tag{4.6}$$

*where $B_1, B_2, \ldots, B_t$ are square fully indecomposable matrices. The matrices $B_1, B_2, \ldots, B_t$ that occur as diagonal blocks in (4.6) are uniquely determined to within arbitrary permutations of their lines, but their ordering in (4.6) is not necessarily unique.*

*Proof.* Because $\rho(A) = n$, we may permute the lines of $A$ so that the resulting matrix $B$ has only 1's on its main diagonal. We now apply Theorem 3.2.4 to $B$. According to that theorem there exists an integer $t \geq 1$ such that the lines of $B$ can be *simultaneously* permuted to obtain a matrix of the form

$$\left[\begin{array}{cccc} A_1 & A_{12} & \cdots & A_{1t} \\ O & A_2 & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_t \end{array}\right] \tag{4.7}$$

where $A_1, A_2, \ldots, A_t$ are square irreducible matrices which are uniquely determined to within simultaneous permutations of their lines. Each of the matrices $A_i$ in (4.7) has only 1's on its main diagonal. By Theorem 4.2.3 the matrices $A_1, A_2, \ldots, A_t$ are fully indecomposable. It follows from Theorem 4.2.2 that each 1 of the matrix (4.7) which belongs to one of the matrices $A_1, A_2, \ldots, A_t$ is part of a nonzero diagonal, and from Lemma 4.2.5 that any other 1 does not belong to a nonzero diagonal. Hence the nonzero diagonals of the matrix (4.7) are the same as those of the matrix

$$B' = \begin{bmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_t \end{bmatrix}, \tag{4.8}$$

and every 1 of $B'$ belongs to a nonzero diagonal.

Suppose that the lines of $A$ could also be permuted to give

$$C = \begin{bmatrix} C_1 & C_{12} & \cdots & C_{1s} \\ O & C_2 & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & C_s \end{bmatrix} \tag{4.9}$$

where $C_1, C_2, \ldots, C_s$ are fully indecomposable matrices. Arguing as above the nonzero diagonals of $C$ are the same as those of

$$C' = \begin{bmatrix} C_1 & O & \cdots & O \\ O & C_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & C_s \end{bmatrix} \tag{4.10}$$

and every 1 of $C$ belongs to a nonzero diagonal. Thus $B'$ and $C'$ are both obtained by replacing with 0's all 1's of $A$ that do not belong to a nonzero diagonal and then permuting the lines of $A$. Therefore $C'$ can be obtained from $B'$ by permuting its lines. Since the matrices $A_1, A_2, \ldots, A_t$ and $C_1, C_2, \ldots, C_s$ are fully indecomposable, we conclude that $s = t$ and that there exists a permutation $i_1, i_2, \ldots, i_t$ of $1, 2, \ldots, t$ such that $C_{i_j}$ can be obtained from $A_j$ by line permutations for each $j = 1, 2, \ldots, t$. The ordering of the diagonal blocks in (4.6) is not unique, if, for instance, the matrix $A$ is a direct sum of two fully indecomposable matrices of different orders. □

Let $A$ be a (0,1)-matrix of order $n$ with $\rho(A) = n$. The matrices $B_1, B_2, \ldots, B_t$ that occur as diagonal blocks in (4.6) are called the *fully indecomposable components* of $A$. By Theorem 4.2.6 the fully indecomposable components of $A$ are uniquely determined to within permutations of their lines. As demonstrated in the proof of Theorem 4.2.6, the fully indecomposable components of $A$ are the irreducible components of any matrix obtained from $A$ by permuting lines so that there are no 0's on the main diagonal. Notice that the matrix $A$ is fully indecomposable if and only if it has exactly one fully indecomposable component.

A (0,1)-matrix $A$ of order $n$ has *total support* provided each of its 1's belongs to a nonzero diagonal. If $A = O$, then $A$ has total support. If

$A \neq O$, then it follows from Lemma 4.2.5 and Theorem 4.2.6 that $A$ *has total support if and only if there are permutation matrices* $P$ *and* $Q$ *of order* $n$ *such that* $PAQ$ *is a direct sum of fully indecomposable matrices.*

Fully indecomposable matrices can be characterized within the set of matrices with total support by using bipartite graphs.

**Theorem 4.2.7.** *Let* $A$ *be a nonzero* $(0,1)$*-matrix of order* $n$ *with total support, and let* $G$ *be the bipartite graph whose reduced adjacency matrix is* $A$*. Then* $A$ *is fully indecomposable if and only if* $G$ *is connected.*

*Proof.* If $A$ is not fully indecomposable, then there are permutation matrices $P$ and $Q$ such that $PAQ$ is a direct sum of two or more fully indecomposable matrices, and $G$ is not connected.

Conversely, suppose that $G$ is not connected. Then there are permutation matrices $R$ and $S$ such that

$$RAS = \begin{bmatrix} A' & O \\ O & A'' \end{bmatrix}$$

where $A'$ is a $p$ by $q$ matrix for some nonnegative integers $p$ and $q$ with $1 \leq p+q \leq 2n-1$. Without loss of generality we assume that $p \leq q$. Then $A$ has a line cover consisting of $p$ rows and $n-q$ columns where $p+(n-q) = n-(q-p)$. Since $A$ has total support and $A \neq O$, we have $\rho(A) = n$ and it follows that $p = q$. But then $A$ has a zero submatrix of size $p$ by $n-p$ and $A$ is not fully indecomposable. □

We close this section by applying Theorem 3.3.8 to obtain an inductive structure of Hartfiel[1975] for fully indecomposable matrices.

**Theorem 4.2.8.** *Let* $A$ *be a fully indecomposable* $(0,1)$*-matrix of order* $n \geq 2$*. Then there exist permutation matrices* $P$ *and* $Q$ *of order* $n$ *and an integer* $m \geq 2$ *such that* $PAQ$ *has the form*

$$\begin{bmatrix} A_1 & O & \cdots & O & E_1 \\ E_2 & A_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{m-1} & O \\ O & O & \cdots & E_m & A_m \end{bmatrix} \tag{4.11}$$

*where* $A_1, A_2, \ldots, A_m$ *are fully indecomposable matrices and the matrices* $E_1, E_2, \ldots, E_m$ *each contain at least one* $1$.

*Proof.* By Corollary 4.2.4 we may permute the lines of $A$ so that the resulting matrix $B$ has all 1's on its main diagonal and is irreducible. By Theorem 3.3.8 the lines of $B$ can be simultaneously permuted to obtain the form $(4.11)$ where

$m \geq 2$, the $A_i$ are irreducible matrices with all 1's on their main diagonals, and the $E_i$ each contain at least one 1. By Theorem 4.2.3 the matrices $A_i$ are fully indecomposable and the theorem follows. □

## Exercises

1. Let $A$ be an $m$ by $n$ (0,1)-matrix with $m < n$, and assume that each row of $A$ is an essential row. Prove that there exist a positive integer $p$ and permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} B_{11} & B_{12} & B_{13} & \cdots & B_{p1} & B_{10} \\ B_{21} & B_{22} & B_{23} & \cdots & B_{p2} & O \\ O & B_{32} & B_{33} & \cdots & B_{3p} & O \\ O & O & B_{43} & \cdots & B_{4p} & O \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & \cdots & B_{pp} & O \end{bmatrix}$$

where $B_{11}, B_{22}, \ldots, B_{pp}$ are square matrices with 1's everywhere on their main diagonals and $B_{10}, B_{21}, B_{32}, \ldots, B_{p,p-1}$ have no zero rows (Brualdi[1966]).
2. Let $n \geq 2$. Prove that the permanent of a fully indecomposable (0,1)-matrix of order $n$ is at least 2 and characterize those fully indecomposable matrices whose permanent equals 2.
3. Prove that the product of two fully indecomposable (0,1)-matrices of the same order is a fully indecomposable matrix (Lewin[1971]).
4. Let $A$ be a fully indecomposable (0,1)-matrix of order $n \geq 2$. Prove that $A^{n-1}$ is a positive matrix and then deduce that $A$ is a primitive matrix with exponent at most equal to $n-1$ (Lewin[1971]).
5. Find an example of a fully indecomposable (0,1)-matrix of order $n$ whose exponent equals $n-1$.
6. Let $A$ be a fully indecomposable (0,1)-matrix of order $n$. Prove that there is a doubly stochastic matrix $D$ of order $n$ such that the matrix obtained from $D$ by replacing each positive element with a 1 equals $A$.

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## 4.3 Nearly Decomposable Matrices

We continue to frame our discussion in terms of (0,1)-matrices with the understanding that the 1's can be replaced by arbitrary nonzero numbers.

Let $A$ be a fully indecomposable (0,1)-matrix. The matrix $A$ is called *nearly decomposable* provided whenever a 1 of $A$ is replaced with a 0, the resulting matrix is partly decomposable. Thus the nearly decomposable matrices are the "minimal" fully indecomposable matrices. Two examples of nearly decomposable matrices are

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{4.12}$$

The relationship between fully indecomposable matrices and irreducible matrices as described in Theorem 4.2.3 and Corollary 4.2.4 only partially extends to nearly decomposable matrices and nearly reducible matrices.

**Theorem 4.3.1.** *Let $A$ be a $(0,1)$-matrix of order $n$. If each element on the main diagonal of $A$ is 0 and $A+I$ is nearly decomposable, then $A$ is nearly reducible. If $A$ is nearly reducible, then each element on the main diagonal of $A$ is 0 and $A+I$ is fully indecomposable, but $A+I$ need not be nearly decomposable.*

*Proof.* Assume that $A$ has all 0's on its main diagonal. By Theorem 4.2.3 $A$ is irreducible if and only if the matrix $B = A+I$ is fully indecomposable. Suppose that $A+I$ is nearly decomposable. Let $A'$ be a matrix obtained from $A$ by replacing a 1 with a 0. If $A'$ is irreducible, then by Theorem 4.2.3 $A'+I$ is fully indecomposable and we contradict the near decomposability of $A+I$. Hence $A$ is nearly reducible.

Now suppose that $A$ is nearly reducible. Then each element on the main diagonal of $A$ is 0, and the matrix $B = A + I$ is fully indecomposable. By Theorem 4.2.3 the replacement of an off-diagonal 1 of $B$ with a 0 results in a partly decomposable matrix. The nearly decomposable matrix $A_2$ in (4.12) shows that it may be possible to replace a 1 on the main diagonal of $B$ with a 0 and obtain a fully indecomposable matrix. □

The fact that $A + I$ need not be nearly decomposable if $A$ is nearly reducible prevents in general theorems about nearly reducible matrices and nearly decomposable matrices from being directly obtainable from one another. We can, however, use the inductive structure of nearly reducible matrices given in Theorem 3.3.4 to obtain an inductive structure for nearly decomposable matrices. First we prove two lemmas.

**Lemma 4.3.2.** *Let $B$ be a $(0,1)$-matrix having the form*

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & \\ 1 & 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 1 & \cdots & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & F_1 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 1 & \\ \hline & & & F_2 & & & B_1 \end{array}\right] \tag{4.13}$$

*where $B_1$ is a fully indecomposable matrix, $F_1$ has a 1 in its first row and $F_2$ has a 1 in its last column. Then $B$ is a fully indecomposable matrix.*

*Proof.* By Theorem 4.2.7 it suffices to show that $B$ has total support and that the bipartite graph $G$ whose reduced adjacency matrix is $B$ is connected. It is a direct consequence of Theorem 4.2.2 that each 1 of $B$ belongs to a nonzero diagonal. By Theorem 4.2.7 the bipartite graph $G_1$ whose reduced adjacency matrix is $B_1$ is connected. The bipartite graph $G$ is obtained from $G_1$ by attaching a chain whose endpoints are two vertices of $G_1$ (and possibly some additional edges). Hence $G$ is connected as well. □

**Lemma 4.3.3.** *Assume that in Lemma 4.3.2 the matrix $B$ is nearly decomposable. Then the matrix $B_1$ in (4.13) is nearly decomposable and $F_1$ and $F_2$ each contain exactly one 1. Let the unique 1 in $F_1$ belong to column $j$ of $F_1$, and let the unique 1 in $F_2$ belong to row $i$ of $F_2$. If the order of $B_1$ is at least 2, then element in position $(i,j)$ of $B_1$ is a 0.*

*Proof.* If the replacement of some 1 of $B_1$ with a 0 results in a fully indecomposable matrix, then by Lemma 4.3.2 the replacement of that 1 in $B$ with a 0 also results in a fully indecomposable matrix. It follows that

$B_1$ is nearly decomposable. Lemma 4.3.2 also implies that $F_1$ and $F_2$ each contain exactly one 1.

Now assume that the order of $B_1$ is at least 2. Let $G_1$ be the connected bipartite graph whose reduced adjacency matrix is $B_1$. Suppose that the element in the position $(i,j)$ of $B_1$ equals 1. Let the matrices $B'$ and $B'_1$ be obtained from $B$ and $B_1$, respectively, by replacing this 1 with a 0. Since the order of $B_1$ is at least 2, $B'_1$ is not a zero matrix. Each nonzero diagonal of $B_1$ extends to a nonzero diagonal of $B$ by including the leading 1's on the main diagonal of $B$ which are displayed in (4.13). Since $B_1$ is fully indecomposable, $B_1$ has a nonzero diagonal which includes the 1 in position $(i,j)$. Consider such a nonzero diagonal of $B_1$. Removing the 1 in position $(i,j)$ and including the 1's of $F_1$ and $F_2$ as well as the 1's below the main diagonal of $B$ which are displayed in (4.13) results in a nonzero diagonal of $B$. It follows from these considerations that the matrix $B'$ has total support. The bipartite graph $G'$ whose reduced adjacency matrix is $B'$ is connected since it is obtained from the connected graph $G_1$ by replacing an edge with a chain joining its two endpoints. We now apply Theorem 4.2.7 to $B'$ and conclude that $B'$ is fully indecomposable, contradicting the near decomposability assumption of $B$. Hence the element in the position $(i,j)$ of $B_1$ equals 0. □

The following inductive structure for a nearly decomposable matrix is due to Hartfiel[1970]. It is a simplification of an inductive structure obtained by Sinkhorn and Knopp[1969].

**Theorem 4.3.4.** *Let $A$ be a nearly decomposable $(0,1)$-matrix of order $n \geq 2$. Then there exist permutation matrices $P$ and $Q$ of order $n$ and an integer $m$ with $1 \leq m \leq n-1$ such that $PAQ$ has the form (4.13) where $B_1$ is a nearly decomposable matrix of order $m$. The matrix $F_1$ contains a unique 1 and it belongs to its first row and column $j$ for some $j$ with $1 \leq j \leq m$. The matrix $F_2$ contains a unique 1 and it belongs to its last column and row $i$ for some $i$ with $1 \leq i \leq m$. If $m \geq 2$, then $m \neq 2$ and the element in position $(i,j)$ of $B_1$ is 0.*

*Proof.* The matrix $A$ is fully indecomposable and thus has term rank equal to $n$. We permute the lines of $A$ and obtain a nearly decomposable matrix $B$ all of whose diagonal elements equal 1. By Theorem 4.3.1 the matrix $B - I$ is nearly reducible. By Theorem 3.3.4 there is a permutation matrix $R$ of order $n$ such that

$$R(B-I)R^T = RBR^T - I = \left[\begin{array}{cccccc|c} 0 & 0 & 0 & \cdots & 0 & 0 & \\ 1 & 0 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 0 & \cdots & 0 & 0 & F_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ & & & F_2 & & & A_1 \end{array}\right]$$

where $A_1$ is a nearly reducible matrix of order $m$ for some integer $m$ with $1 \leq m \leq n-1$. The matrix $F_1$ contains a single 1 and it belongs to its first row and column $j$ where $1 \leq j \leq m$. The matrix $F_2$ contains a single 1 and it belongs to its last column and row $i$ where $1 \leq i \leq m$. The element in position $(i, j)$ of $A_1$ is 0. Hence $RBR^T$ has the form (4.13) with $B_1 = A_1 + I$. By Theorem 4.3.1, $B_1$ is fully indecomposable. Since $B$ is nearly decomposable, we may apply Lemma 4.3.3 and conclude that $B_1$ is nearly decomposable and that the element in position $(i, j)$ of $B_1$ is 0 if $m \geq 2$. Finally we note that if $m \geq 2$, then $m \neq 2$, since no nearly decomposable matrix of order 2 contains a 0. □

We remark that a matrix of the form (4.13) satisfying the conclusions of Theorem 4.3.4 need not be nearly decomposable. The matrix

$$B_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

is a nearly decomposable matrix (an easy way to see this is to notice that each 1 belongs to a line which contains exactly two 1's). However the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

is not nearly decomposable. This is because replacing the 1's in positions (3,2) and (4,3) positions with 0's results in a fully indecomposable matrix.

The matrix $B_1$ that occurs in the inductive structure of nearly decomposable matrices given in Theorem 4.3.4 can be any nearly decomposable matrix except for the 2 by 2 matrix of all 1's (Hartfiel[1971]). The nearly decomposable matrix of order 1, whose unique entry is a 1, occurs when the matrix of all 1's of order 2 is written in the form (4.13). Now let $B$ be any nearly decomposable matrix of order $n \geq 3$. Without loss of generality we may assume that $B$ has the form (4.13) and the conclusions of Theorem 4.3.4 are satisfied. Let $A$ be the matrix

$$\begin{bmatrix} 1 & E_1 \\ E_2 & B \end{bmatrix}$$

where $E_1$ is a 1 by $n$ (0,1)-matrix with a single 1 and this 1 belongs to the same column in which $F_1$ has its 1, and $E_2$ is an $n$ by 1 (0,1)-matrix with a single 1 and this 1 belongs to the same row in which $F_1$ has its 1. By Lemma

4.3.2 the matrix $A$ is fully indecomposable. The near decomposability of $B$ implies the near decomposability of $A$.

It was pointed out in section 3.3 that an irreducible principal submatrix of a nearly reducible matrix is nearly reducible. Since a nearly decomposable matrix remains nearly decomposable under arbitrary line permutations, one might suspect that a fully indecomposable submatrix $B$ of a nearly decomposable matrix $A$ is nearly decomposable. This turns out to be false. However, if the submatrix of $A$ which is complementary to $B$ has a nonzero diagonal, then $B$ is nearly decomposable. These two properties of nearly decomposable matrices can be found in Brualdi and Hedrick[1979].

A nearly decomposable (0,1)-matrix of order 1 has exactly one 1. A nearly decomposable (0,1)-matrix of order 2 has exactly four. Minc[1972] determined the largest number of 1's that a nearly decomposable (0,1)-matrix of order $n$ can have. Combining Theorem 4.3.1 with Theorem 3.3.6 we see that $3n-2$ is an upper bound for the number of 1's in a nearly decomposable matrix of order $n$. This bound cannot be attained for $n \geq 3$.

**Theorem 4.3.5.** *Let $A$ be a nearly decomposable $(0,1)$-matrix of order $n \geq 3$. Then the number of 1's in $A$ is between $2n$ and $3(n-1)$. The number of 1's in $A$ is $2n$ if and only if there are permutation matrices $P$ and $Q$ of order $n$ such that $PAQ$ equals*

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{4.14}$$

*The number of 1's in $A$ equals $3(n-1)$ if and only if there are permutation matrices $P$ and $Q$ of order $n$ such that $PAQ$ equals the matrix*

$$S_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}. \tag{4.15}$$

*Proof.* The matrices in (4.14) and (4.15) are nearly decomposable, since each of its 1's is contained in a line which has exactly two 1's. Since $n \geq 3$, each line of the nearly decomposable matrix $A$ has at least two 1's. Hence $A$ has at least $2n$ 1's. Assume that $A$ has exactly $2n$ 1's. We permute the lines of $A$ so that there are only 1's on the main diagonal. The permuted $A$ contains exactly one off-diagonal 1 in each line and hence equals $I+R$ for some permutation matrix $R$ of order $n$. Since $A$ is fully indecomposable, the permutation of $\{1,2,\ldots,n\}$ derived from $R$ is a cycle of length $n$. It follows that (4.14) holds for some permutation matrices $P$ and $Q$ of order $n$.

We next investigate the largest number of 1's in a nearly decomposable matrix $A$ of order $n\geq 3$. We verify the conclusions of the theorem by induction on $n$. Each nearly decomposable matrix of order 3 is a permuted form of the matrix $S_n$ in (4.15) and thus has exactly 6 1's. [We remark that if $n=3$, the matrix in (4.14) is a permuted form of the matrix in (4.15).] Now assume that $n>3$. We use the notation $\sigma(X)$ for the number of 1's in a (0,1)-matrix $X$. Let $P$ and $Q$ be permutation matrices such that $PAQ$ satisfies the conclusions of Theorem 4.3.4. Then

$$\sigma(A)\leq 2(n-m)+1+\sigma(B_1) \tag{4.16}$$

where $B_1$ is a nearly decomposable matrix of order $m$. If $m=1$, then $\sigma(B_1)=1$ and by (4.16), $\sigma(A)=2n$ which is strictly less than $3(n-1)$ for $n>3$. The case $m=2$ cannot occur in Theorem 4.3.4, so we now assume that $m\geq 3$. By the induction hypothesis, $\sigma(B_1)\leq 3(m-1)$. Using this inequality in (4.16) we obtain

$$\sigma(A)\leq 2n+m-2\leq 2n+(n-1)-2=3(n-1).$$

Suppose that $\sigma(A)=3(n-1)$. Then we must have $m=n-1$ and $\sigma(B_1)=3(n-2)$. By the inductive assumption, the lines of $B_1$ can be permuted to obtain the matrix $S_{n-1}$. Thus the lines of $A$ can be permuted to obtain

$$\left[\begin{array}{cc} 1 & F_1 \\ F_2 & S_{n-1} \end{array}\right] \tag{4.17}$$

where the matrices $F_1$ and $F_2$ each contain exactly one 1. Let the 1 of $F_2$ occur in row $i$ of $F_2$, and let the 1 of $F_1$ occur in column $j$ of $F_1$. If $i=j=n-1$, then (4.17) is the same as (4.15). We now assume that it is not the case that $i=j=n-1$. It now follows from Theorem 4.3.4 that $i\neq j$ and that $i\neq n-1$ and $j\neq n-1$. We apply additional line permutations to $A$ and assume that $i=1$ and $j=2$. The matrix in (4.17) can be repartitioned to give

$$\left[\begin{array}{cc|ccccc} 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & & & & \\ 0 & 0 & & & & & \\ \vdots & \vdots & & & S_{n-2} & & \\ 0 & 0 & & & & & \\ 0 & 1 & & & & & \end{array}\right]. \tag{4.18}$$

If $n > 4$, the matrix obtained from (4.18) by replacing the 1 in position $(2, n)$ with a 0 is fully indecomposable by Lemma 4.3.2. Hence we must have $n = 4$. But in this case (4.18) is a permuted form of (4.15). $\square$

Lovász and Plummer[1977] call a bipartite graph *elementary* provided it is connected and each edge is contained in a perfect matching. A *minimal elementary bipartite graph* is one such that the removal of any edge results in a bipartite graph which is not elementary. By Theorem 4.2.7 a bipartite graph is elementary if and only if its reduced adjacency matrix is fully indecomposable. The reduced adjacency matrix of a minimal elementary bipartite graph is a nearly decomposable matrix. Estimates for the number of lines in a nearly decomposable matrix which have exactly two 1's follow from their investigations.

## Exercises

1. For each integer $n \geq 3$ give an example of a nearly reducible matrix $A$ of order $n$ such that $A + I$ is not nearly decomposable.
2. Let $n \geq 3$ and $k$ be integers with $2n \leq k \leq 3(n-1)$. Prove that there exists a nearly decomposable (0,1)-matrix of order $n$ with exactly $k$ 1's (Brualdi and Hedrick[1979].
3. Give an example to show that a fully indecomposable submatrix of a nearly decomposable matrix need not be nearly decomposable (Brualdi and Hedrick [1979]).
4. Let $A$ be a nearly decomposable (0,1)-matrix which is partitioned as

$$A = \begin{bmatrix} A_1 & B_1 \\ B_2 & A_2 \end{bmatrix}$$

where $A_1$ is a square matrix with 1's everywhere on its main diagonal and $A_2$ is a fully indecomposable matrix. Prove that $A_2$ is a nearly decomposable matrix (Brualdi and Hedrick[1979]).

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## 4.4 Decomposition Theorems

Let $A$ be an $m$ by $n$ matrix, and let $\mathcal{P}$ denote a class of matrices. By a *decomposition theorem* we mean a theorem which asserts that there is an expression for $A$ of the form

$$A = P_1 + P_2 + \cdots + P_k + X \tag{4.19}$$

where the matrices $P_1, P_2, \ldots, P_k$ are in the class $\mathcal{P}$. We may require $X$ to be restricted in some way, perhaps equal to a zero matrix. The purpose of the theorem may be to maximize $k$ in (4.19) or to minimize $k$ in the event that $X$ is required to be a zero matrix, or the purpose may be to maximize or to minimize some other quantity that can be associated with the decomposition (4.19).

The König theorem (Theorem 1.2.1) can be viewed as a decomposition theorem. Recall that an $m$ by $n$ (0,1)-matrix $P$ is a *subpermutation matrix of rank* $r$ (an $r$*-subpermutation matrix*) provided $P$ has exactly $r$ 1's and no two 1's of $P$ are on the same line. Let $A$ be an $m$ by $n$ (0,1)-matrix. Then the König theorem asserts that $A$ *can be expressed in the form*

$$A = P + X$$

*where $P$ is an $r$-subpermutation matrix and $X$ is a $(0,1)$-matrix if and only if $A$ does not have a line cover consisting of fewer than $r$ lines.* The theorem of Vizing (Theorem 2.6.2) is a decomposition theorem for symmetric (0,1)-matrices in which the $P_i$ are required to be symmetric permutation matrices and $X$ is required to be a zero matrix.

Let $A$ be an $m$ by $n$ (0,1)-matrix having no zero lines. We define the *co-term rank* of $A$ to be the minimal number of 1's in $A$ with the property that each line of $A$ contains at least one of these 1's. We denote this basic invariant by $\rho^*(A)$ and derive the following basic relationship.

**Theorem 4.4.1.** *Let $A$ be an $m$ by $n$ $(0,1)$-matrix having no zero lines. Then the co-term rank $\rho^*(A)$ equals*

$$\max\{r + s\} \tag{4.20}$$

*where the maximum is taken over all $r$ by $s$ (possibly vacuous) zero submatrices of $A$ with $0 \le r \le m$ and $0 \le s \le n$.*

*Proof.* Suppose that $r$ and $s$ are nonnegative integers for which $A$ has an $r$ by $s$ zero submatrix. We permute the lines of $A$ to bring $A$ to the form

$$\begin{array}{cc} & \begin{array}{cc} n-s & s \end{array} \\ \begin{array}{c} m-r \\ r \end{array} & \left[\begin{array}{cc} A_3 & A_1 \\ A_2 & O \end{array}\right] \end{array}.$$

Clearly, $\rho^*(A) \geq r + s$. We now assume that $r$ and $s$ are integers for which the maximum occurs in (4.20) and verify the reverse inequality. The first $m - r$ rows and the first $n - s$ columns of the permuted $A$ form a line cover with the fewest number of lines. Hence by Theorem 1.2.1

$$\rho(A) = m - r + n - s = m + n - (r + s). \tag{4.21}$$

In addition it follows from Theorem 1.2.1 that $\rho(A_1) = m - r$ and $\rho(A_2) = n - s$. We select $m - r$ 1's of $A_1$ with no two of the 1's in the same line and then $s - (m - r)$ additional 1's one from each of the remaining columns of $A_1$. We also select $n - s$ 1's of $A_2$ with no two of the 1's from the same line and then $r - (n - s)$ additional 1's one from each of the remaining rows of $A_2$. We obtain in this way a total $r + s$ 1's of $A$ with the property that each line of $A$ contains at least one of these 1's. Therefore $\rho^*(A) \leq r + s$ and hence we have equality. □

We remark that for an $m$ by $n$ (0,1)-matrix $A$ with no zero lines, it follows from equation (4.21) above that the two basic invariants are related by the equation

$$\rho(A) + \rho^*(A) = m + n. \tag{4.22}$$

Stated as a decomposition theorem, Theorem 4.4.1 asserts: The $m$ by $n$ (0,1)-matrix $A$ with no zero lines can be expressed in the form

$$A = Q + Y$$

where $Q$ is a (0,1)-matrix with no zero lines and with at most $t$ 1's if and only if $A$ does not have an $r$ by $s$ zero submatrix with $r + s > t$.

Every (0,1)-matrix $A$ has a decomposition (4.19) in which $P_1, P_2, \ldots, P_k$ are subpermutation matrices and $X = O$. We may, for instance, choose the $P_i$ to have rank 1 and the integer $k$ to be the number of 1's of $A$. It is natural to ask for the smallest $k$ for which a decomposition of $A$ into subpermutation matrices exists. Now assume that $A$ has no zero lines. Then $A$ has a decomposition (4.19) in which the matrices $P_1, P_2, \ldots, P_k$ have no zero lines and $X = O$. We may, for instance, choose $k = 1$ and $P_1 = A$. It is also natural to ask for the largest $k$ for which $A$ admits a decomposition into matrices with no zero lines. Both of these questions can be answered by appealing to the following theorem of Gupta[1967, 1974, 1978]. This theorem is stated in terms of nonnegative integral matrices, but its proof is more conveniently expressed in terms of bipartite multigraphs.

**Theorem 4.4.2.** *Let $A$ be an $m$ by $n$ nonnegative integral matrix with row sums $r_1, r_2, \ldots, r_m$ and column sums $s_1, s_2, \ldots, s_n$. Let $k$ be a positive integer. Then $A$ has a decomposition of the form*

$$A = P_1 + P_2 + \cdots + P_k \tag{4.23}$$

*where* $P_1, P_2, \ldots, P_k$ *are nonnegative integral matrices satisfying the following two properties:*

*(i) The number of* $P_t$*'s which have a positive element in row* $i$ *equals* $\min\{k, r_i\}$, $(i = 1, 2, \ldots, m)$.

*(ii) The number of* $P_t$*'s which have a positive element in column* $j$ *equals* $\min\{k, s_j\}, (j = 1, 2, \ldots, n)$.

*Proof.* The matrix $A = [a_{ij}], (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ is the reduced adjacency matrix of a bipartite multigraph $G$ with bipartition $\{X, Y\}$ where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The edge $\{x_i, y_j\}$ has multiplicity $a_{ij}$. The degree of the vertex $x_i$ equals $r_i$ and the degree of the vertex $y_j$ equals $s_j$. Let $\sigma$ be a function which assigns to each edge of $G$ an integer $k$ from the set $\{1, 2, \ldots, k\}$. As in the proof of Theorem 2.6.2 we think of $\sigma$ as assigning a *color* to each edge of $G$ from a set $\{1, 2, \ldots, k\}$ of $k$ colors. Adjacent edges of $G$ need not be assigned different colors by $\sigma$, nor are the $a_{ij}$ edges of the form $\{x_i, y_j\}$ required to have identical colors.

For each integer $r = 1, 2, \ldots, k$, we define a bipartite multigraph $G_r(\sigma)$ with bipartition $\{X, Y\}$. The multiplicity of the edge $\{x_i, y_j\}$ in $G_r(\sigma)$ equals the number of edges of $G$ of the form $\{x_i, y_j\}$ which are assigned color $r$ by $\sigma$,$(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Let $P_r = A_r(\sigma)$ be the reduced adjacency matrix of $G_r(\sigma)$. Then $A = P_1 + P_2 + \cdots + P_r$. The properties $(i)$ and $(ii)$ in the theorem hold for this decomposition of $A$ if and only if $\sigma$ assigns $\min\{k, r_i\}$ distinct colors to the edges of $G$ incident to vertex $x_i, (i = 1, 2, \ldots, m)$ and $\min\{k, s_j\}$ distinct colors to the edges incident to vertex $y_j, (j = 1, 2, \ldots, n)$.

For each vertex $z$ of $G$ let $f(z, \sigma)$ be the number of distinct colors assigned by $\sigma$ to the edges of $G$ which are incident to $z$. We have

$$f(z, \sigma) \leq \min\{k, \text{degree of } z\}, (z \in X \cup Y). \tag{4.24}$$

We now assume that $\sigma$ has been chosen so that

$$\sum_{z \in X \cup Y} f(z, \sigma)$$

is as large as possible. We show that for this choice of $\sigma$ equality holds throughout (4.24). Assume that this were not the case. Without loss of generality we may assume that for a vertex $z = x_{i_0}$ we have

$$f(x_{i_0}, \sigma) < \min\{k, r_{i_0}\}. \tag{4.25}$$

It follows that some color $p$ is assigned to two or more edges incident to $x_{i_0}$ while some color $q$ is assigned to no edge incident to $x_{i_0}$. Starting at vertex $x_{i_0}$ we determine a walk

$$x_{i_0} \to y_{i_0} \to x_{i_1} \to y_{i_1} \to x_{i_2} \to \cdots \tag{4.26}$$

in which the edges $\{x_{i_0}, y_{i_0}\}, \{x_{i_1}, y_{i_1}\}, \cdots$ are assigned color $p$ and the edges $\{y_{i_0}, x_{i_1}\}, \{y_{i_1}, x_{i_2}\}, \cdots$ are assigned color $q$. The walk (4.26) continues until one of the following occurs:

(a) a vertex is reached which is incident to another edge of the same assigned color as the incoming edge.

(b) a vertex is reached which is not incident with any edge of one of the two colors $p$ and $q$.

Since $G$ is a bipartite multigraph and since there is no edge incident to $x_{i_0}$ which is assigned color $q$, the vertex $x_{i_0}$ occurs exactly once on the walk (4.26). We now reassign colors to the edges that occur on the walk (4.26) by interchanging the two colors $p$ and $q$. Let $\tau$ be the coloring of the edges of $G$ thus obtained. It follows that

$$f(x_{i_0}, \tau) = f(x_{i_0}, \sigma) + 1,$$

and

$$f(z, \tau) \geq f(z, \sigma)$$

for all vertices $z$ of $G$. But then

$$\sum_{z \in X \cup Y} f(z, \tau) > \sum_{z \in X \cup Y} f(z, \sigma),$$

contradicting our choice of $\sigma$. Hence equality holds throughout (4.24). □

There are two special cases of Theorem 4.4.2 which are of particular interest. The first of these is another theorem of König[1936].

**Theorem 4.4.3.** *Let $A$ be an $m$ by $n$ nonnegative integral matrix with maximal line sum equal to $k$. Then $A$ has a decomposition of the form*

$$A = P_1 + P_2 + \cdots + P_k \tag{4.27}$$

*where $P_1, P_2, \ldots, P_k$ are $m$ by $n$ subpermutation matrices.*

*Proof.* Let $r_1, r_2, \ldots, r_m$ be the row sums of $A$ and let $s_1, s_2, \ldots, s_n$ be the column sums. We apply Theorem 4.4.2 to $A$ with $k$ equal to the maximum line sum of $A$. We obtain a decomposition (4.23) in which the $P_i$ are nonnegative integral matrices satisfying $(i)$ and $(ii)$ in Theorem 4.4.2. For the chosen $k$, we have $\min\{k, r_i\} = r_i, (i = 1, 2, \ldots, m)$ and $\min\{k, s_j\} = s_j, (j = 1, 2, \ldots, n)$. It follows that $P_1, P_2, \ldots, P_k$ are subpermutation matrices. □

**Corollary 4.4.4.** *Let $A$ be an $m$ by $n$ nonnegative integral matrix with maximum line sum equal to $k$ and let $r$ be an integer with $0 < r < k$. Then $A$ has a decomposition $A = A_1 + A_2$ where $A_1$ is a nonnegative integral matrix* *with maximum line sum equal to $r$, and $A_2$ is a nonnegative integral matrix with maximum line sum equal to $k - r$.*

*Proof.* By Theorem 4.4.3 there is a decomposition (4.27) of $A$ into subpermutation matrices. We now choose $A_1 = P_1 + P_2 + \cdots + P_r$ and $A_2 = A - A_1$. □

**Corollary 4.4.5.** *Let $A$ be an $m$ by $n$ nonnegative integral matrix each of whose line sums equals $k$ or $k-1$. Let $r$ be an integer with $1 \leq r < k$. Then $A$ has a decomposition $A = A_1 + A_2$ into nonnegative integral matrices $A_1$ and $A_2$ where each line sum of $A_1$ equals $r$ or $r - 1$ and each line sum of $A_2$ equals $k - r$ or $k - r + 1$.*

*Proof.* We again use the decomposition (4.27). Since the line sums of $A$ equal $k$ or $k - 1$, at most one of $P_1, P_2, \ldots, P_k$ does not have a 1 in any specified line. The result now follows as in the preceding corollary. □

There are analogues of the preceding two corollaries for symmetric matrices. The following theorem of Thomassen[1981] is a slight extension of a theorem of Tutte[1978].

**Theorem 4.4.6.** *Let $A$ be a symmetric nonnegative integral matrix of order $n$ each of whose line sums equals $k$ or $k - 1$. Let $r$ be an integer with $1 \leq r < k$. Then $A$ has a decomposition $A = A_1 + A_2$ into symmetric nonnegative integral matrices $A_1$ and $A_2$ where each line sum of $A_1$ equals $r$ or $r - 1$.*

*Proof.* It suffices to prove the theorem in the case that $r = k - 1$. Let $p$ be the number of rows of $A$ which sum to $k$. We assume that $p > 0$, for otherwise we may choose $A_1 = A$. We simultaneously permute the lines of $A = [a_{ij}]$ and assume that $A$ has the form

$$\begin{bmatrix} C & B \\ B^T & D \end{bmatrix},$$

where $C$ is a matrix of order $p$ and $D$ is a matrix of order $n - p$, and the first $p$ rows of $A$ are the rows which sum to $k$. Suppose that $a_{ij} \neq 0$ for some $i$ and $j$ with $1 \leq i, j \leq p$. Then we may subtract 1 from $a_{ij}$ and, in the case $i \neq j$, 1 from $a_{ji}$ and argue on the resulting matrix. Thus we may assume that $C = O$. Then the matrix $B$ has all row sums equal to $k$ and all column sums at most equal to $k$. It follows from Theorem 1.2.1 (or from Theorem 4.4.3) that $B$ has a decomposition $B = P + B_1$ where $P$ is a subpermutation matrix of rank $p$. We now let

$$A_2 = \begin{bmatrix} O & P \\ P^T & D \end{bmatrix},$$

and $A_1 = A - A_2$. □

The following corollary is due to Lovász[1970].

**Corollary 4.4.7.** *Let $B$ be a symmetric nonnegative integral matrix of order $n$ with maximum line sum at most equal to $r+s-1$ where $r$ and $s$ are positive integers. Then there is a decomposition $A = A_1 + A_2$ where $A_1$ and $A_2$ are symmetric nonnegative integral matrices with maximum line sum at most equal to $r$ and $s$, respectively.*

*Proof.* We increase some of the diagonal elements of $B$ and obtain a symmetric nonnegative integral matrix $A$ each of whose line sums equals $r+s-1$ and apply Theorem 4.4.6 with $k = r+s-1$. □

The following theorem was discovered by Gupta[1967, 1974, 1978] and Fulkerson[1968].

**Theorem 4.4.8.** *Let $A$ be a nonnegative integral matrix with minimum line sum equal to a positive integer $k$. Then $A$ has a decomposition of the form*

$$A = P_1 + P_2 + \cdots + P_k$$

*where $P_1, P_2, \ldots, P_k$ are $m$ by $n$ nonnegative integral matrices each of whose line sums is positive.*

*Proof.* As in the proof of Theorem 4.4.3 we apply Theorem 4.4.2 but this time with $k$ equal to the minimum line sum of $A$. In order for $(i)$ and $(ii)$ of Theorem 4.4.2 to be satisfied each of the line sums of the matrices $P_1, P_2 \ldots, P_k$ in (4.23) must be positive. □

We now turn to a decomposition theorem of a different type. Theorem 4.1.3 asserts that in a decomposition of the complete graph $K_n$ of order $n$ into complete bipartite subgraphs, the number of complete bipartite graphs is at least $n-1$. This theorem, as was done in its proof, can be viewed as a decomposition theorem for the matrix $J - I$ of order $n$. We now prove a general theorem of Graham and Pollak[1971, 1973] which gives a lower bound for the number of bipartite graphs in a decomposition of a multigraph.

**Theorem 4.4.9.** *Let $G$ be a multigraph of order $n$ and let $G_1, G_2, \ldots G_r$ be a decomposition of $G$ into complete bipartite subgraphs. Let $A = [a_{ij}]$, $(i, j = 1, 2, \ldots, n)$ be the adjacency matrix of $G$ and let $n_+$ be the number of positive eigenvalues of $A$ and let $n_-$ be the number of negative eigenvalues. Then $r \geq \max\{n_+, n_-\}$.*

*Proof.* Let the set of vertices of $G$ be $V = \{a_1, a_2, \ldots, a_n\}$ . A complete bipartite subgraph of $G$ is obtained by specifying two nonempty subsets $X$ and $Y$ of $V$ for which $\{x, y\}$ is an edge of $G$ for each $x$ in $X$ and each $y$ in $Y$.

The pair $\{X, Y\}$ is the bipartition of the subgraph and since $G$ has no loops, the sets $X$ and $Y$ are disjoint. Let $\{X_i, Y_i\}$ be the bipartition of the complete bipartite graph $G_i$ in the decomposition of $G$ $(i = 1, 2, \ldots, r)$.

Let $z_1, z_2, \ldots, z_n$ be $n$ indeterminates and let $z = (z_1, z_2, \ldots, z_n)^T$. We consider the quadratic form

$$q(z) = z^T Az = 2 \sum_{1 \le i < j \le n} a_{ij} z_i z_j.$$

With each of the bipartite graphs $G_i$ in the decomposition of $G$ we associate the quadratic form

$$q_i(z) = q_i(z_1, z_2, \ldots, z_n) = \left( \sum_{\{k : a_k \in X_i\}} z_k \right) \left( \sum_{\{l : a_l \in Y_i\}} z_l \right).$$

Since $G_1, G_2, \ldots, G_r$ is a decomposition of $G$ we have

$$q(z) = z^T Az = 2 \sum_{i=1}^{r} q_i(z). \tag{4.28}$$

We apply the elementary algebraic identity

$$ab = \frac{1}{4}((a+b)^2 - (a-b)^2)$$

to $q_i(z)$ and obtain from (4.28)

$$q(z) = z^T Az = \frac{1}{2} \left( \sum_{i=1}^{r} l_i'(z)^2 - \sum_{i=1}^{r} l_i''(z)^2 \right), \tag{4.29}$$

where the $l_i'(z)$ and the $l_i''(z)$ are linear forms in $z_1, z_2, \ldots, z_n$. The linear forms $l_1'(z), l_2'(z), \ldots, l_r'(z)$ vanish on a subspace $W$ of dimension at least $n - r$ of real $n$-space. Hence the quadratic form $q(z)$ is negative semi-definite on $W$. Let $E^+$ be the linear space of dimension $n_+$ spanned by the eigenvectors of $A$ corresponding to its positive eigenvalues. Then $q(z)$ is positive definite on $E^+$. It follows that

$$(n - r) + n_+ = \dim W + \dim E^+ \le n$$

and hence $r \ge n_+$. One concludes in a similar way that $r \ge n_-$. □

Alon, Brualdi and Shader[1991] have extended Theorem 4.4.9 by proving that every bipartite decomposition of the graph $G$ has an additional special property. We state this theorem without proof below. Given a coloring of a graph $G$, we say that a subgraph of $G$ is *multicolored* provided that no two of its edges have the same color. A bipartite decomposition of $G$ corresponds to an edge coloring in which two edges receive the same color if and only if they belong to the same bipartite graph of the decomposition.

**Theorem 4.4.10.** *Let $G$ be a graph of order $n$ and suppose the adjacency matrix of $G$ has $n_+$ positive eigenvalues and $n_-$ negative eigenvalues. Then in any decomposition of $G$ into complete bipartite graphs there is a multicolored forest with at least* $\max\{n_+, n_-\}$ *edges. In any decomposition of $K_n$ into complete bipartite graphs there is a multicolored spanning tree.*

Let $G$ be the complete graph $K_n$ of order $n$. The adjacency matrix $A = J - I$ has $n-1$ negative eigenvalues. Hence Theorem 4.1.3, which asserts that the complete graph of order $n$ cannot be decomposed into fewer than $n-1$ complete bipartite subgraphs, is a special case of Theorem 4.4.9.

If $n$ is even, the complete graph $K_n$ can be decomposed into $n-1$ complete bipartite subgraphs, each of which is isomorphic to $K_{1,n/2}$. If $n$ is odd, $K_n$ cannot have a decomposition into $n-1$ isomorphic complete bipartite subgraphs $K_{1,m}$ for any positive integer $m$. The following theorem of de Caen and Hoffman[1989], stated without proof, asserts that other decompositions of $K_n$ into $n-1$ isomorphic complete bipartite subgraphs are impossible.

**Theorem 4.4.11.** *Let $n$ be a positive integer. If $r$ and $s$ are integers with $r, s \geq 2$, then there does not exist a decomposition of $K_n$ into complete bipartite subgraphs each of which is isomorphic to the complete bipartite graph $K_{r,s}$.*

Theorem 4.4.9 has implications for an addressing problem in graphs. We refer to the papers of Graham and Pollak cited above and to Winkler[1983] and van Lint[1985].

Let $K^*_{n,n}$ be the bipartite graph of order $2n$ which is obtained from the complete bipartite graph $K_{n,n}$ by removing the edges of a perfect matching. The reduced adjacency matrix of $K^*_{n,n}$ is the matrix $J - I$ of order $n$. The graph $K^*_{n,n}$ can be decomposed into $n$ complete bipartite subgraphs each of which is isomorphic to $K_{1,n-1}$. The following theorem of de Caen and Gregory [1987] asserts that there are no decompositions with fewer than $n$ complete bipartite graphs.

**Theorem 4.4.12.** *Let $n \geq 2$. Let the bipartite graph $K^*_{n,n}$ of order $2n$ have a decomposition $G_1, G_2, \ldots, G_r$ into complete bipartite subgraphs. Then $r \geq n$. If $r = n$, then there exist positive integers $p$ and $q$ such that $pq = n - 1$ and each $G_i$ is isomorphic to $K_{p,q}$.*

*Proof.* Let $\{X, Y\}$ be a bipartition of $K^*_{n,n}$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Each of the bipartite subgraphs $G_i$ has a bipartition $\{X_i, Y_i\}$. Let $G'_i$ be the spanning subgraph of $K^*_{n,n}$ with the same set of edges as $G_i$ and let $A_i$ be the reduced adjacency matrix of $G_i'$, $(i = 1, 2, \ldots, r)$. The hypothesis of the theorem implies that

$$J - I = A_1 + A_2 + \cdots + A_r. \tag{4.30}$$

Let $\widehat{X}_i$ be the (0,1)-matrix of size $n$ by 1 whose $k$th component equals 1 if and only if $x_k$ is in $X_i$, $(k = 1, 2, \ldots, n)$. Let $\widehat{Y}_i$ be the (0,1)-matrix of size 1 by $n$ whose $k$th component is 1 if and only if $y_k$ is in $Y_i$, $(k = 1, 2, \ldots, n)$. We have $A_i = \widehat{X}_i \widehat{Y}_i$, $(i = 1, 2, \ldots, r)$. We define an $n$ by $r$ (0,1)-matrix by

$$\widehat{X} = \begin{bmatrix} \widehat{X}_1 & \widehat{X}_2 & \cdots & \widehat{X}_r \end{bmatrix}$$

and we define an $r$ by $n$ (0,1)-matrix by

$$\widehat{Y} = \begin{bmatrix} \widehat{Y}_1 \\ \widehat{Y}_2 \\ \vdots \\ \widehat{Y}_n \end{bmatrix}.$$

By (4.30) we have

$$J - I = \widehat{X}\widehat{Y}. \tag{4.31}$$

From equation (4.31) we conclude that a decomposition of $K_{n,n}^*$ into $r$ complete bipartite subgraphs is equivalent to a factorization of $J - I$ into two (0,1)-matrices of sizes $n$ by $r$ and $r$ by $n$, respectively. The matrix $J - I$ has rank equal to $n$, and the ranks of $\widehat{X}$ and $\widehat{Y}$ cannot exceed $r$. Hence it follows from (4.31) that

$$n = \operatorname{rank}(J - I) \le r.$$

We now assume that $r = n$. Since the elements on the main diagonal of $J - I$ equal 0, we have $\widehat{Y}_i^T \widehat{X}_i = 0, (i = 1, 2, \ldots, n)$. Let $i$ and $j$ be distinct integers between 1 and $n$. Let $U$ be the $n$ by $n - 1$ matrix obtained from $\widehat{X}$ by deleting columns $i$ and $j$ and appending a column of 1's as a new first column. Let $V$ be the $n - 1$ by $n$ matrix obtained from $\widehat{Y}$ by deleting rows $i$ and $j$ and appending a row of 1's as a new first row. Then

$$UV = I + \widehat{X}_i\widehat{Y}_i + \widehat{X}_j\widehat{Y}_j$$

is a singular matrix. Let

$$U' = \begin{bmatrix} \widehat{X}_i & \widehat{X}_j \end{bmatrix}$$

and

$$V' = \begin{bmatrix} \widehat{Y}_i \\ \widehat{Y}_j \end{bmatrix}.$$

Using these equations and taking determinants we obtain

$$0 = \det(UV) = \det(I + U'V') = \det(I_2 + V'U') = 1 - (\widehat{Y}_i\widehat{X}_j)(\widehat{Y}_j\widehat{X}_i),$$

where $I_2$ denotes the identity matrix of order 2. Hence $\widehat{Y}_i\widehat{X}_j = \widehat{Y}_j\widehat{X}_i = 1$ for all $i$ and $j$ with $i \neq j$. Thus in the case $r = n$ the equation (4.31) implies that

$$\widehat{Y}\widehat{X} = J - I. \tag{4.32}$$

The two equations (4.31) and (4.32) imply that both $X$ and $Y$ commute with $J$, and hence $\widehat{X}$ and $\widehat{Y}$ each have constant line sums. There exist integers $p$ and $q$ such that

$$XJ = JX = pJ \qquad \text{and} \qquad YJ = JY = qJ.$$

Now

$$(n-1)J = (J - I)J = (\widehat{X}\widehat{Y})J = \widehat{X}(\widehat{Y}J) = \widehat{X}(qJ) = pqJ.$$

Thus $pq = n - 1$ and the theorem now follows. □

In Chapter 6 we shall obtain additional decomposition theorems.

## Exercises

1. Let $A$ be an $m$ by $n$ (0,1)-matrix with no zero lines. Prove that $\rho(A) \leq \rho^*(A)$. Investigate the case of equality.
2. Determine the largest co-term rank possible for an $m$ by $n$ (0,1)-matrix with no zero lines and characterize those matrices for which equality holds.
3. Let $D$ be a digraph with no isolated vertices. A *matching* of $D$ is a collection of pairwise vertex-disjoint directed chains and cycles of $D$. A *cover* of $D$ is a collection of arcs which meet all vertices of $D$. Let $\chi(D)$ denote the minimum number of matchings into which the arcs of $D$ can be partitioned, and let $\kappa(D)$ denote the maximum number of covers of $D$ into which the arcs of $D$ can be partitioned. Prove that $\chi(D)$ equals the smallest number $s$ such that both the indegree and outdegree of each vertex of $D$ are at most $s$, and that $\kappa(D)$ equals the largest number $t$ such that both the indegree and outdegree of each vertex of $D$ are at least $t$ (Gupta[1978]).
4. Let $A$ be a nonnegative integral matrix of order $n$ each of whose line sums equals $k$. Theorem 4.4.4 asserts that $A$ can be written as a sum of permutation matrices of order $n$. Prove this special case of Theorem 4.4.4 using Theorem 1.2.1. In Exercises 5 and 6 we refer to this special case of Theorem 4.4.4 as the *regular* case.

5. Prove Theorem 4.4.4 from the regular case as follows: Let $A$ be an $m$ by $n$ nonnegative integral matrix with maximal line sum equal to $k$. Assume that $m \geq n$. Extend $A$ to a matrix $B$ of order $m$ by including $m - n$ additional columns of 0's. Now increase the elements of $B$ by integer values in order to obtain a nonnegative integral matrix $B'$ of order $m$ each of whose line sums equals $k$. Apply the regular case of Theorem 4.4.4 to $B'$ and deduce that $A$ is the sum of $k$ subpermutation matrices. Give an example to show that for a given choice of $B'$ not every decomposition of $A$ as a sum of $k$ subpermutation matrices arises in this way (Brualdi and Csima[1991]).
6. Prove Theorem 4.4.4 from the regular case as follows: Let $A$ be an $m$ by $n$ nonnegative integral matrix with maximal line sum equal to $k$. Let the row sums of $A$ be $r_1, r_2, \ldots, r_m$ and let the column sums of $A$ be $s_1, s_2, \ldots, s_n$. Let $A'$ be the matrix of order $m + n$ defined by

$$\begin{bmatrix} D_1 & A \\ A^T & D_2 \end{bmatrix},$$

where $D_1$ is the diagonal matrix of order $m$ with diagonal elements $k - r_1, k - r_2, \ldots, k - r_m$ and $D_2$ is the diagonal matrix of order $n$ with diagonal elements $k - s_1, k - s_2, \ldots, k - s_n$. Apply the regular case of Theorem 4.4.4 to $A'$ and deduce that $A$ is the sum of $k$ subpermutation matrices. Show that every decomposition of $A$ as a sum of $k$ subpermutation matrices arises in this way (Brualdi and Csima[1991]).
7. Let $G$ be the graph of order $n$ which is obtained from the complete graph $K_n$ by removing an edge. Determine the smallest number of complete bipartite graphs into which the edges of $G$ can be partitioned.
8. Let $m$ and $n$ be positive integers with $m \leq n$. Let $G$ be the graph of order $n$ obtained from the complete graph $K_n$ by removing the edges of a complete graph $K_m$. Prove that the smallest number of complete bipartite graphs into which the edges of $G$ can be partitioned equals $n - m$ (Jones, Lundgren, Pullman and Rees[1988]).

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## 4.5 Diagonal Structure of a Matrix

Let $A = [a_{ij}], (i, j = 1, 2, \dots, n)$ be a (0,1)-matrix of order $n$. A nonzero diagonal of $A$, as defined in section 4.2, is a collection of $n$ 1's of $A$ with no two of the 1's on a line. More formally, a *nonzero diagonal* of $A$ is a set

$$D = \{(1, j_1), (2, j_2), \dots, (n, j_n)\} \tag{4.33}$$

of $n$ positions of $A$ for which $(j_1, j_2, \dots, j_n)$ is a permutation of the set $\{1, 2, \dots, n\}$ and $a_{1j_1} = a_{2j_2} = \cdots = a_{n,j_n} = 1$. Let $\mathcal{D} = \mathcal{D}(A)$ be the set of all nonzero diagonals of $A$. The cardinality of the set $\mathcal{D}$ equals the permanent of $A$. In this section we are concerned with some basic properties of the *diagonal structure* $\mathcal{D}$ of $A$. Let $X$ be the set of positions of $A$ which contain 1's. The pair $(X, \mathcal{D})$ is called the *diagonal hypergraph* of $A$.[2]

Those positions of $A$ containing 1's which do not belong to any nonzero diagonal are of no importance for the diagonal structure of $A$. Thus throughout this section we assume that each position in $X$ is contained in a nonzero diagonal, that is, the matrix $A$ has total support. Moreover, we implicitly assume that $A$ is not a zero matrix.

Let $B$ be another (0,1)-matrix of order $n$ with total support, and let $Y$ be the set of positions of $B$ which contain 1's. An *isomorphism of the diagonal hypergraphs* $(X, \mathcal{D}(A))$ and $(Y, \mathcal{D}(B))$ is a bijection $\phi : X \to Y$

[2] In the terminology of hypergraphs (see Berge[1973]), the elements of $X$ are *vertices* and the elements of $\mathcal{D}$ are *hyperedges*. Thus a graph is a hypergraph in which all hyperedges have cardinality equal to two. All the hyperedges of $\mathcal{D}$ have cardinality equal to $n$. In general, a hypergraph may have edges of different cardinalities.

with the property that for each $D \subseteq X$, $D$ is a nonzero diagonal of $A$ if and only if $\phi(D)$ is a nonzero diagonal of $B$.

A collection of 1's of the matrix $A$ with the property that no two of the 1's belong to the same nonzero diagonal of $A$ is called strongly stable.[3] More formally, a set

$$S = \{(i_1, k_1), (i_2, k_2), \ldots, (i_t, k_t)\}$$

is a *strongly stable set* of $A$ provided $a_{i_1k_1} = a_{i_2k_2} = \cdots = a_{i_tk_t} = 1$ and each nonzero diagonal $D$ of $A$ has at most one element in common with $S$. The collection of strongly stable sets of $A$ is denoted by $\mathcal{S} = \mathcal{S}(A)$, and the pair $(X, \mathcal{S})$ is the *strongly stable hypergraph* of $A$. An isomorphism of the strongly stable hypergraphs of two matrices is defined very much like an isomorphism of diagonal hypergraphs.

By its very definition the strongly stable hypergraph of a matrix is determined by its diagonal hypergraph. It follows from the following theorem of Brualdi and Ross[1981] that the diagonal hypergraph is determined by the strongly stable hypergraph.

**Theorem 4.5.1.** *Let $A$ and $B$ be $(0,1)$-matrices of order $n$ with total support. Let $X$ be the set of positions of $A$ which contain 1's and let $Y$ be the set of positions of $B$ that contain 1's. Let $\phi : X \rightarrow Y$ be a bijection. Then $\phi$ is an isomorphism of the diagonal hypergraphs $(X, \mathcal{D}(A))$ and $(Y, \mathcal{D}(B))$ if and only if $\phi$ is an isomorphism of the strongly stable hypergraphs $(X, \mathcal{S}(A))$ and $(Y, \mathcal{S}(B))$.*

*Proof.* First assume that $\phi$ is an isomorphism of $(X, \mathcal{D}(A))$ and $(Y, \mathcal{D}(B))$. Let $F$ be a subset of $X$ and let $D$ be a nonzero diagonal of $A$. Then $|D \cap F| \leq 1$ if and only if $|\phi(D) \cap \phi(F)| \leq 1$. It follows that $F$ is a strongly stable set of $A$ if and only if $\phi(A)$ is a strongly stable set of $B$. Hence $\phi$ is an isomorphism of $(X, \mathcal{S}(A))$ and $(Y, \mathcal{S}(B))$.

Now assume that $\phi$ is an isomorphism of $(X, \mathcal{S}(A))$ and $(Y, \mathcal{S}(B))$. Let $D$ be a nonzero diagonal of $A$. Suppose that $\phi(D)$ is not a nonzero diagonal of $B$. Since $\phi(D)$ is a set of $n$ positions of $B$ containing 1's, there are distinct positions $p_1$ and $p_2$ in $\phi(D)$ which belong to the same line of $B$. The set $\{p_1, p_2\}$ is a strongly stable set of $B$, but since $\{\phi^{-1}(p_1), \phi^{-1}(p_2)\}$ is a subset of $D$, $\{\phi^{-1}(p_1), \phi^{-1}(p_2)\}$ is not a strongly stable set of $A$. This contradicts the assumption that $\phi$ is an isomorphism of the strongly stable hypergraphs of $A$ and $B$. Hence $\phi(D)$ is a nonzero diagonal of $B$. In a similar way one proves that if $D'$ is a nonzero diagonal of $B$ then $\phi^{-1}(D')$ is a nonzero diagonal of $A$. Thus $\phi$ is an isomorphism of the diagonal hypergraphs of $A$ and $B$. $\square$

[3] This terminology comes from the theory of hypergraphs. A *strongly stable set* of a hypergraph is a subset of its vertices containing at most one vertex of each hyperedge.

The strongly stable sets of the (0,1)-matrix $A$ have been defined in terms of the nonzero diagonals of $A$. The next theorem of Brualdi[1979] contains a simple instrinsic characterization of strongly stable sets.

**Theorem 4.5.2.** *Let $A$ be a $(0,1)$-matrix of order $n$ with total support. Let $X$ be the set of positions of $A$ which contain 1's, and let $S$ be a subset of $X$. Then $S$ is a strongly stable set of $A$ if and only if there exist nonnegative integers $p$ and $q$ with $p+q=n-1$ and a $p$ by $q$ zero submatrix $B$ of $A$ such that $S$ is a subset of the positions of the submatrix of $A$ which is complementary to $B$.*

*Proof.* First assume that there exists a zero submatrix $B$ satisfying the properties stated in the theorem. If $p=0$ or $q=0$, then the submatrix complementary to $B$ is a line and $S$ is a strongly stable set. Now suppose that $p>0$ and $q>0$. Let $(i_1,j_1)$ and $(i_2,j_2)$ be two positions of $S$ which belong to different lines of the complementary submatrix of $B$. The submatrix of $A$ of order $n-2$ obtained by deleting rows $i_1$ and $i_2$ and columns $j_1$ and $j_2$ contains the $p$ by $q$ zero submatrix $B$. Since $p+q=n-1$, it follows from Theorem 1.2.1 that there does not exist a nonzero diagonal of $A$ containing both of the positions $(i_1,j_1)$ and $(i_2,j_2)$. Therefore $S$ is a strongly stable set of $A$.

Now assume that $S$ is a strongly stable set of positions of $A$, and let $|S|=m$. We first assume that $A$ is fully indecomposable, and prove by induction on $m$ that there exists a $p$ by $q$ zero submatrix $B$ of $A$ with $p+q=n-1$ such that $S$ is a subset of the positions of the submatrix of $A$ which is complementary to $B$. If the positions of $S$ all belong to one line, we may find $B$ with $p=0$ or $q=0$. If $m=2$ the existence of $B$ is a consequence of Theorem 1.2.1. We now proceed under the added assumption that $m>2$ and that $S$ contains positions from at least two different rows and at least two different columns. If there is a position $(k,l)$ in $S$ which is in the same row as another position in $S$ and in the same column as a third position in $S$, then the conclusions hold by applying the induction hypothesis to $S-\{(k,l)\}$. Hence we further assume that $S$ satisfies the condition:

*(*) For each position $(k,l)$ in $S$, $(k,l)$ is either the only position of $S$ in row $k$ or the only position of $S$ in column $l$.*

We now distinguish two cases.

*Case 1. There exist distinct positions $s_1$ and $s_2$ in $S$ which belong to the same line of $A$.* Without loss of generality we assume that the line containing both $s_1$ and $s_2$ is a row. Let $S_1=S-\{s_1\}$ and let $S_2=S-\{s_2\}$. We apply the induction hypothesis to $S_1$ and $S_2$ and obtain for $k=1$ and $k=2$ a $p_k$ by $q_k$ zero submatrix $B_k$ with $p_k+q_k=n-1$ such that $S_k$ is a subset of the positions in the submatrix of $A$ complementary to $B_k$. If the column containing $s_1$ does not meet $B_1$ or the column containing $s_2$ does not meet $B_2$, then the conclusion holds. Hence we assume that the column containing $s_1$ meets $B_1$ and the column containing $s_2$ meets $B_2$. It now follows from (*) that the column containing $s_1$ contains no other position in $S, (k = 1, 2)$ and that each of the integers $p_1, q_1, p_2$ and $q_2$ is positive. Let $B_1$ and $B_2$ lie in exactly $v$ common rows of $A$ and exactly $u$ common columns of $A$. Then $A$ has zero submatrices $B_3$ and $B_4$ of sizes $v$ by $q_1 + q_2 - u$ and $p_1 + p_2 - v$ by $u$, respectively. We have

$$(v + (q_1 + q_2 - u)) + ((p_1 + p_2 - v) + u) = 2(n - 1). \tag{4.34}$$

Hence either $v + (q_1 + q_2 - u) \geq n - 1$ or $u + (p_1 + p_2 - v) \geq n - 1$. Suppose that $v + (q_1 + q_2 - u) \geq n$. Since $q_1 > 0$ and $q_2 > 0$, we have $q_1 + q_2 - u > 0$. Since all positions in $S - \{s_1, s_2\}$ lie in the $n - (q_1 + q_2 - u)$ columns of $A$ complementary to those of $B_3$ and since $|S| > 2$, we have $q_1 + q_2 - u < n$. Thus $B_3$ is a nonvacuous zero submatrix of $A$ and we contradict the full indecomposable assumption of $A$. Hence we have $v + (q_1 + q_2 - u) \leq n - 1$ and it follows from (4.34) that $(p_1 + p_2 - v) + u \geq n - 1$. Since $S$ is contained in the set of positions of the submatrix complementary to the $(p_1 + p_2 - v)$ by $u$ zero submatrix $B_4$ of $A$, the conclusion holds in this case.

*Case 2. Each line of $A$ contains at most one position in $S$.* We permute the lines of $A$ and assume that $S = \{(1, 1), (2, 2), \ldots, (m, m)\}$. For $i = 1, 2$ and 3, let $S_i = S - \{(i, i)\}$. By the induction assumption there exists a $p_i$ by $q_i$ zero submatrix $B_i$ of $A$ with $p_i + q_i = n - 1$ such that the positions of $S_i$ are positions of the submatrix of $A$ complementary to $B_i$. Since $|S_i| \geq 2$, $p_i$ and $q_i$ are both positive, $(i = 1, 2, 3)$. If for some $i$, neither row $i$ nor column $i$ meets $B_i$, then $S$ is a subset of the positions of the submatrix complementary to $B_i$. Hence we assume that row $i$ or column $i$ meets $B_i$, $(i = 1, 2, 3)$. If for some $i$ and $j$ with $i \neq j$ row $i$ does not meet $B_i$ and row $j$ does not meet $B_j$, or column $i$ does not meet $B_i$ and column $j$ does not meet $B_j$, then an argument like that in Case 1 completes the proof. Without loss of generality, we assume that row 1 does not meet $B_1$ and column 2 does not meet $B_2$. But now if row 3 does not meet $B_3$ we apply the argument of Case 1 to $B_1$ and $B_3$, and if column 3 does not meet $B_3$ we apply the argument of Case 1 to $B_2$ and $B_3$. Hence the conclusion follows by induction if $A$ is fully indecomposable. A strongly stable set of $A$ can contain positions from only one fully indecomposable component of $A$, and the conclusion now holds in general. □

Let $A$ be a (0,1)-matrix of order $n$ with total support, and let $X$ be the set of positions of $A$ that contain 1's. A *linear set* of $A$ [or of the diagonal hypergraph $(X, \mathcal{D}(A))$] is the set of positions occupied by 1's in a line of $A$. According as the line is a row or column, we speak of a *row-linear set* and a *column-linear set*. Let $P = [p_{ij}], (i, j = 1, 2, \ldots, n)$ and $Q = [q_{ij}], (i, j = 1, 2, \ldots, n)$ be permutation matrices of order $n$, and let $\sigma$

and $\tau$ be the permutations of $\{1, 2, \ldots, n\}$ defined by $\sigma(i) = j$ if $p_{ij} = 1$ and $\tau(i) = j$ if $q_{ij} = 1$ $(i, j = 1, 2, \ldots, n)$. Let $Y$ be the set of positions occupied by 1's in $PAQ$. The bijection $\psi$ from the set $X$ to the set $Y$ defined by $\psi((i,j)) = (\sigma(i), \tau(j))$ is an isomorphism of the diagonal hypergraphs $(X, \mathcal{D}(A))$ and $(Y, \mathcal{D}(PAQ))$. The isomorphism $\psi$ is the *isomorphism induced by the permutation matrices $P$ and $Q$*. The bijection $\theta$ from the set $X$ to the set $Z$ of positions occupied by 1's in the transposed matrix $A^T$ defined by $\theta((i,j)) = (j,i)$ is an isomorphism of the diagonal hypergraphs $(X, \mathcal{D}(A))$ and $(Z, \mathcal{D}(A^T))$. The isomorphism $\theta$ is the *isomorphism induced by transposition.* Isomorphisms induced by permutation matrices or by transposition map the linear sets of one diagonal hypergraph onto the linear sets of another diagonal hypergraph.

The following theorem is from Brualdi and Ross[1981].

**Theorem 4.5.3.** *Let $A$ and $B$ be $(0,1)$-matrices of order $n$ with $B$ fully indecomposable, and let $\theta$ be an isomorphism of the diagonal hypergraphs $(X, \mathcal{D}(A))$ and $(Y, \mathcal{D}(B))$. Then each linear set of $A$ is mapped by $\theta$ onto a linear set of $B$ if and only if there are permutation matrices $P$ and $Q$ of order $n$ such that one of the following holds:*

*(i) $B = PAQ$ and $\theta$ is induced by $P$ and $Q$;*
*(ii) $B = PA^TQ$ and $\theta = \phi\rho$ where $\phi$ is an isomorphism induced by transposition and $\rho$ is an isomorphism induced by $P$ and $Q$.*

*Proof.* If $(i)$ or $(ii)$ holds, then it is evident that each linear set of $A$ is mapped by $\theta$ onto a linear set of $B$. We now prove the converse statement. Suppose that $e$ row-linear sets of $A$ are mapped onto row-linear sets of $B$ and $(n-e)$ row-linear sets are mapped onto column-linear sets. Since the row-linear sets of $A$ are pairwise disjoint, $B$ has an $e$ by $n-e$ zero submatrix. Since $B$ is fully indecomposable, we have $e = 0$ or $e = n$. If $e = n$ the column-linear sets of $A$ sets are mapped by $\theta$ onto the column-linear sets of $B$, and $(i)$ holds. If $e = 0$ the column-linear sets of $A$ are mapped onto the row linear sets of $B$, and $(ii)$ holds. □

If the assumptions of Theorem 4.5.3 hold and each linear set of $A$ is mapped onto a linear set of $B$, then Theorem 4.5.3 implies that $A$ is fully indecomposable. More generally, the number of fully indecomposable components is invariant under a diagonal hypergraph isomorphism (Brualdi and Ross[1979]).

**Theorem 4.5.4.** *Let $A$ and $B$ be $(0,1)$-matrices of order $n$ with total support, and let $A_1, A_2, \ldots, A_r$ and $B_1, B_2, \ldots, B_s$ be the fully indecomposable components of $A$ and $B$, respectively. Then the diagonal hypergraph $(X, \mathcal{D}(A))$ of $A$ is isomorphic to the diagonal hypergraph of $B$ if and only if $r = s$ and there is a permutation $\sigma$ of $\{1, 2, \ldots, r\}$ such that the diagonal hypergraph of $A_i$ is isomorphic to the diagonal hypergraph of $B_{\sigma(i)}$ for each $i = 1, 2, \ldots, r$.*

*Proof.* Without loss of generality we assume that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$ and that $B = B_1 \oplus B_2 \oplus \cdots \oplus B_s$. First assume that $\psi$ is an isomorphism of $(X, \mathcal{D}(\mathcal{A}))$ and $(Y, \mathcal{D}(\mathcal{B}))$. If $r = s = 1$, then the conclusions hold. Without loss of generality we now assume that $r > 1$. Then $A = A_1 \oplus A'$ where $A' = A_2 \oplus \cdots \oplus A_r$ is a matrix of order $k < n$. The set $X$ can be partitioned into sets $X_1$ and $X'$ where $X_1$ is the set of positions of $A_1$ that contain 1's. Each nonzero diagonal $D$ of $A$ can be partitioned into sets $D_1$ and $D'$ where $D_1 \subseteq X_1$ and $D' \subseteq X'$. Let $E = E_1 \cup E'$ be any nonzero diagonal of $A$. Then for each nonzero diagonal $D$ of $A$, $D_1 \cup E'$ and $E_1 \cup D'$ are nonzero diagonals of $A$. Since $A$ has total support, each position in $X_1$ belongs to a nonzero diagonal $D_1$ of $A_1$. Since $D_1 \cup E'$ is a nonzero diagonal of $A$, $\psi(D_1) \cup \psi(E')$ is a nonzero diagonal of $B$. It now follows that $\psi(X_1)$ is contained in rows and columns of $B$ which are complementary to those of $\psi(E')$. In a similar way we conclude that $\psi(X')$ is contained in the rows and columns complementary to those of $\psi(E_1)$. Thus $\psi$ induces an isomorphism of the diagonal hypergraphs of $A_1$ and $B_{i_1} \oplus \cdots \oplus B_{i_m}$ for some positive integer $m$ and some $i_1, \ldots, i_m$ with $1 \leq i_1 < \cdots < i_m \leq s$. It follows that $r \leq s$, and in a similar way that $s \leq r$. Hence $r = s$ and there exists a permutation $\sigma$ of $\{1, 2, \ldots, r\}$ such that the diagonal hypergraph of $A_i$ is isomorphic to the diagonal hypergraph of $B_{\sigma(i)}$ for each $i = 1, 2, \ldots, r$.

The converse is immediate. □

Let $A$ and $B$ be (0,1)-matrices of order $n$ with total support. The isomorphisms of the diagonal hypergraphs of $A$ and $B$ which map the linear sets of $A$ onto the linear sets of $B$ can be characterized by using Theorems 4.5.3 and 4.5.4. Informally described, such isomorphisms are obtained by replacing some of the fully indecomposable components of $A$ with their transposes and permuting the lines of $A$. We note that in general Theorem 4.5.5 implies that for the further investigation of isomorphisms of diagonal hypergraphs, it suffices to consider only fully indecomposable matrices.

In order to have an isomorphism of diagonal hypergraphs which is different from those described in $(i)$ and $(ii)$ of Theorem 4.5.4, there must be a linear set of one of the matrices which is mapped onto a nonlinear set of the other. That such isomorphisms exist is demonstrated in the following example.

Let $A$ and $B$ be the fully indecomposable matrices of order 5 defined by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{4.35}$$

Notice that $B$ has a line sum equal to five while no line sum of $A$ equals five. Thus there do not exist permutation matrices $P$ and $Q$ such that

$PAQ = B$ or $PA^TQ = B$. Yet the diagonal hypergraphs of $A$ and $B$ are isomorphic. If we label the positions of $A$ and $B$ with the elements of the set $\{a, b, \ldots, l, m\}$, then an isomorphism of the diagonal hypergraphs of $A$ and $B$ is defined schematically by

$$\begin{bmatrix} 0 & j & a & 0 & 0 \\ k & l & b & 0 & 0 \\ m & 0 & c & d & e \\ 0 & 0 & g & 0 & f \\ 0 & 0 & 0 & h & i \end{bmatrix}, \quad \begin{bmatrix} 0 & j & a & 0 & 0 \\ k & l & b & 0 & 0 \\ m & 0 & c & g & 0 \\ 0 & 0 & d & 0 & h \\ 0 & 0 & e & f & i \end{bmatrix}. \tag{4.36}$$

The nonzero diagonals of $A$ and of $B$ are:

$$\{a, f, h, l, m\}, \{b, f, h, j, m\}, \{c, f, h, j, k\}, \{e, g, h, j, k\}, \{d, g, i, j, k\}.$$

The set of positions $\{a, b, c, d, e\}$ is a linear set of $B$ but not of $A$.

We now generalize the idea used in the construction of the previous example.

Let $C = [c_{ij}], (i, j = 1, 2, \ldots, m)$ and $D = [d_{kl}], (k, l = 1, 2, \ldots, n)$ be matrices of orders $m$ and $n$, respectively, such that $c_{mm} = d_{11}$. The matrix $C \star D$ of order $m + n - 1$ is defined schematically by

$$A = C \star D = \begin{bmatrix} C & & O_{m-1,n-1} \\ & x & \\ O_{n-1,m-1} & & D \end{bmatrix} (x = c_{mm} = d_{11}).$$

In the matrix $C \star D$ the matrices $C$ and $D$ "overlap" in one position with common value $x$ and there is an $m-1$ by $n-1$ zero submatrix in the upper right corner and an $n-1$ by $m-1$ zero submatrix in the lower left corner. The matrices $C^T \star D$ and $C \star D^T$ are said to be obtained from $A = C \star D$ by a *partial transposition* on $C$ and $D$, respectively. We remark that if the order of $C$ is 1, then $A = D$ and $A^T = C \star D^T$. Thus transposition of a matrix is a special instance of partial transposition.

Suppose that the matrix $A = C \star D$ has total support. Then the matrix $B = C^T \star D$ also has total support. Let the diagonal hypergraphs of $A$ and $B$ be $(X, \mathcal{D}(A))$ and $(Y, \mathcal{D}(B))$, respectively. Then the mapping $\theta : X \to Y$ defined by

$$\theta(i, j) = \begin{cases} (j, i) & \text{if } (i, j) \text{ is a position in } C, \\ (i, j) & \text{otherwise}, \end{cases}$$

is an isomorphism of the diagonal hypergraphs of $A$ and $B$. The mapping $\theta$ is called the *isomorphism of the diagonal hypergraphs of $A$ and $B$ induced by partial transposition on $C$*. In an analogous way we define the isomorphism induced by partial transposition on $D$.

The binary operation $\star$ is an associative operation and we may write without ambiguity $A = A_1 \star A_2 \star \cdots \star A_k$ whenever $A_1, A_2, \ldots, A_k$ are matrices such that the element in the lower right corner of $A_i$ equals the entry in the upper left corner of $A_{i+1}, (i = 1, 2, \ldots, k-1)$. Let $B = A_1' \star A_2' \star \cdots \star A_k'$ where for each $i = 1, 2, \ldots, k$ we have $A_i' = A_i$ or $A_i^T$. Then it follows by an inductive argument that the diagonal hypergraph of $A$ is isomorphic to the diagonal hypergraph of $B$ under a composition of isomorphisms induced by partial transposition. For instance, if $k = 3$ and $B = A_1 \star A_2^T \star A_3$, then a partial transposition of $A = A_1 \star A_2 \star A_3$ on $A_1 \star A_2$ followed by a partial transposition on $A_1^T$ results in $B$. It has been *conjectured* by Brualdi and Ross[1981] that if $A$ and $B$ are matrices with total support and $\theta$ is an isomorphism of the diagonal hypergraphs of $A$ and $B$, then $\theta$ is a composition of isomorphisms induced by permutation matrices and partial transpositions. If each linear set of $A$ is mapped by $\theta$ onto a linear set of $B$, then the validity of the conjecture is a consequence of Theorem 4.5.4. The conjecture has also been verified in Brualdi and Ross[1981] in another circumstance which we briefly describe.

Let $A$ be a (0,1)-matrix of order $n$ with total support, and let $S$ be a subset of the positions of $A$ occupied by 1's. Then $S$ is called a *linearizable set* of $A$ provided there is a (0,1)-matrix $B$ of order $n$ with total support and an isomorphism $\theta$ of the diagonal hypergraphs of $A$ and $B$ such that $\theta(S)$ is a subset of a linear set of $B$. A linearizable set $S$ of $A$ contains at most $n$ elements, and it is a consequence of Theorem 4.5.1 that $S$ is a strongly stable set of $A$. For the matrix $A$ of order $n = 5$ in (4.35), the set $S = \{a, b, c, d, e\}$ of positions indicated in (4.36) is a linearizable set of $A$ with 5 elements. Linearizable sets are characterized in Brualdi and Ross[1981] where the following theorem is also proved.

**Theorem 4.5.5.** *Let $A$ be a $(0,1)$-matrix of order $n$ with total support such that $A$ has a linearizable set $S$ of $n$ elements. Let $B$ be a $(0,1)$-matrix of order $n$ with total support such that there is an isomorphism of the diagonal hypergraphs of $A$ and $B$ for which $\theta(S)$ is a linear set of $B$. Then $\theta$ is a composition of isomorphisms induced by permutation matrices and partial transpositions.*

Ross[1980] has obtained the same conclusion under a weakening of the hypothesis of the theorem.

### Exercises

1. Let $A$ be a fully indecomposable (0,1)-matrix of order $n$. Prove that the size of a strongly stable set of the diagonal hypergraph of $A$ does not exceed $\sigma(A) - 2n + 2$, where $\sigma(A)$ denotes the number of 1's of $A$ (Brualdi[1979]).
2. Prove that the mapping induced by partial transposition is an isomorphism of diagonal hypergraphs.

3. Let $A$ be a (0,1)-matrix of order $n$ and let $X$ be the set of positions of $A$ which contain 1's. Let $\mathcal{C} = \mathcal{C}(A)$ denote the collection $C$ of subsets of $X$ for which $C$ is the set of edges of a cycle of the bipartite graph whose reduced adjacency matrix is $A$. We call $(X, \mathcal{C})$ the *cycle hypergraph* of $A$. Now assume that $A$ and $B$ are (0,1)-matrices of order $n$ with total support. Let $\theta$ be an isomorphism of the diagonal hypergraphs of $A$ and $B$. Prove that $\theta$ is an isomorphism of the cycle hypergraphs of $A$ and $B$. Conclude that a linearizable set of $A$ does not contain a member of $\mathcal{C}(A)$ (Brualdi and Ross[1981]).
4. (Continuation of Exercise 3) Show that the converse of Exercise 3 is false by exhibiting an isomorphism $\theta$ of the cycle hypergraphs of

$$A = B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

which is not a diagonal hypergraph isomorphism (Brualdi and Ross[1981]).
5. (Continuation of Exercises 3 and 4) Prove that the converse of Exercise 3 is true if $\theta$ maps some nonzero diagonal of $A$ to a nonzero diagonal of $B$ (Brualdi and Ross[1981]).

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