THE THEORY OF ROUND ROBIN TOURNAMENTS

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In this review paper, we make a detailed study of a class of directed graphs, known as tournaments. The reason they are called tournaments is that they represent the structure of round robin tournaments, in which players or teams engage in a game that cannot end in a tie and in which every player plays each other exactly once.

Although tournaments are quite restricted structurally, they are realized by a great many empirical phenomena in addition to round robin competitions. For example, it is known that many species of birds and mammals develop dominance relations so that for every pair of individuals, one dominates the other. Thus, the digraph of the "pecking structure" of a flock of hens is asymmetric and complete, and hence a tournament.

Still another realization of tournaments arises in the method of scaling, known as "paired comparisons." Suppose, for example, that one wants to know the structure of a person's preferences among a collection of competing brands of a product. He can be asked to indicate for each pair of brands which one he prefers. If he is not allowed to indicate indifference, the structure of his stated preferences can be represented by a tournament.

Tournaments appear similarly in the theory of committees and elections. Suppose that a committee is considering four alternative policies. It has been argued that the best decision will be reached by a series of votes in which each policy is paired against each other. The outcome of these votes can be represented by a digraph whose points are policies and whose lines indicate that one policy defeated the other. Such a digraph is clearly a tournament.

After giving some essential definitions, we develop properties that all tournaments display. We then turn our attention to transitive tournaments, namely those that are complete orders. It is well known that not all preference structures are transitive. There is considerable interest, therefore, in knowing how transitive any given tournament is. Such an index is presented toward the end of the second section. In the final section, we consider some properties of strongly connected tournaments.

1. Definitions and preliminary concepts. It is necessary, alas, to include quite a few definitions, to make the treatment precise and self-contained.

A *directed graph*, or *digraph* D for short, consists of a finite set $V=\{v_1, v_2, \cdots, v_p\}$ of *points* together with a subset of $V\times V$, whose elements are called *lines*. Each line $(v_i, v_j)=v_iv_j$ is directed and goes from its first point v_i to a different second point v_j, so that a digraph is *irreflexive*. A digraph is *asymmetric* if whenever line v_iv_j is in it, then v_jv_i is not. It is *transitive* if for every three distinct points v_i, v_j, v_k, the existence of lines v_iv_j and v_jv_k implies the existence of line v_iv_k. In a *complete* digraph, for any two distinct points v_i and v_j, line v_iv_j or line v_jv_i exists.

A *subgraph* of a digraph D is a subset of the points and lines of D which themselves form a digraph. The *removal of a point* v from D results in the maximal subgraph, D-v, not containing v. That is, D-v has all points of D except v and all lines except those to and from v. If U is a proper subset of the set V of points of D, then D-U is the digraph obtained by removing the points of U in succession. The *subgraph* $\langle U\rangle$ *generated by a set* U *of points* of D contains the points of U and those lines of D from one point of U to another.

If uv is a line of a digraph, then u is said to be *adjacent to* v and v is *adjacent from* u. The *outdegree*, denoted od v, of a point v is the number of points adjacent from v; its *indegree*, id v, is the number of points adjacent to v. A *transmitter* is a point with positive outdegree and zero indegree; *a receiver* has positive indegree and zero outdegree.

A *walk from* u *to* v is an alternating sequence of points and lines of the form $u_1, u_1u_2, u_2, u_2u_3, \cdots, u_{n-1}u_n, u_n$, in which $u=u_1$ and $v=u_n$. For brevity this is written $u_1u_2 \cdots u_n$ since then the lines are clear from context. A walk in which all points (and hence all lines) are distinct is called a *path*. A walk of positive length in which only the first and last points are the same is called a *cycle*. The *length* of a path or a cycle is the number of lines in it. A *complete path* or *cycle* contains all the points of the given digraph. If there is a path from u to v, then v is *reachable from* u. The *distance* from u to v, denoted $d(u, v)$, is the length of a shortest such path.

A digraph is called *strongly connected*, or *strong*, if every pair of points are mutually reachable. A *strong component* of a digraph is a maximal strongly connected subgraph. The *condensation* D^* of digraph D has as its points the strong components of D; there is a line in D^* from one strong component S_i to another S_j if there is a line in D from some point of S_i to a point of S_j.

The *converse* D' of a digraph D has the same points as D and contains line v_iv_j if and only if v_jv_i is a line of D. In other words, D' is obtained from D by reversing the direction of every line. Every concept in directed graph theory has a converse concept. For example, the outdegree and indegree of a point are converse concepts of each other, as are "adjacent to" and "adjacent from," etc. A point and a line are their own converses. A valuable principle in the theory of directed graphs is the following, borrowed from the theory of relations.

Directional Duality Principle. For each theorem about digraphs, there is a corresponding theorem which is obtained by replacing every concept by its converse.

2. Some properties of tournaments. A *tournament* is a complete asymmetric digraph. A brief discussion of tournaments is given by Ore in [16]. The smallest tournaments are shown in Figure 1. Clearly, if U is a proper subset of the points in a tournament T, then T-U is also a tournament.

It follows directly from the definitions that if v is any point in a tournament

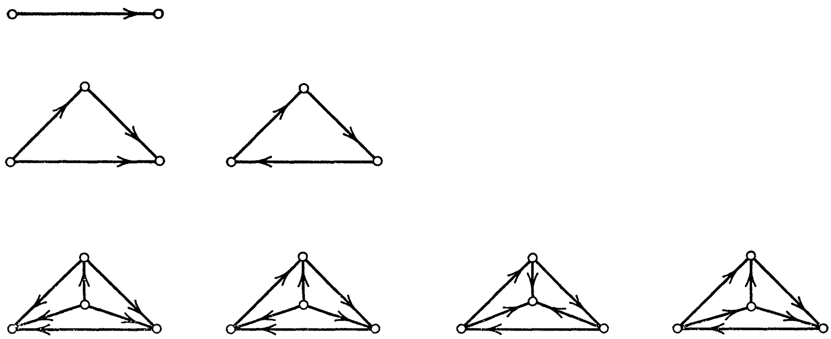

FIG. 1

T with p points, then od v+id $v=p-1$. Also, the total number of lines in T is $\frac{1}{2}p(p-1)$. In the tournament of a round robin competition, the outdegree of a point is the number of victories won by that player. For this reason, we shall call the outdegree of a point v_i of a tournament its *score*, denoted s_i.

The *score sequence* of a tournament T is the ordered sequence of integers $(s_1, s_2, \cdots, s_p)$. We assume without loss of generality that the points v_i have been ordered in such a way that $s_1 \leqq s_2 \leqq \cdots \leqq s_p$. The following theorem by Landau [**13**] gives a necessary and sufficient condition for a sequence of nonnegative integers to be the scores of some tournament.

THEOREM 1. *A sequence of nonnegative integers* $s_1 \leqq s_2 \leqq \cdots \leqq s_p$ *is a score sequence if and only if their sum satisfies the equation*:

$$\text{(I)} \qquad \sum_{i=1}^{p} s_i = \tfrac{1}{2}p(p-1),$$

and the following inequalities hold for every positive integer $k<p$:

$$\text{(II)} \qquad \sum_{i=1}^{k} s_i \geqq \tfrac{1}{2}k(k-1).$$

We prove the necessity of conditions (I) and (II) by taking $s_1 \leqq s_2 \leqq \cdots \leqq s_p$ as the score sequence of a tournament T. Since the sum of the scores of T is the number q of lines, and since $q=\frac{1}{2}p(p-1)$, equation (I) is verified. To establish the inequalities (II), we note that, for any integer $k<p$, the subtournament of T whose points are $v_1, v_2, \cdots, v_k$ contains exactly $\frac{1}{2}k(k-1)$ lines. Hence in the entire tournament T, $\sum_{i=1}^{k} s_i \geqq \frac{1}{2}k(k-1)$, since there may occur in T a line to one of the other $p-k$ points. The proof of the converse is considerably more involved, and is omitted.

Moon has proved in [**15**] that Theorem 1 can be generalized to tournaments with non-integral scores, with conditions (I) and (II) still serving as a criterion for a score sequence.

Theorem 1 may be illustrated by the tournament with five points shown in Figure 2. Clearly, the score sequence of this tournament is (1, 1, 2, 3, 3,). It is immediately apparent that equation (I) and the inequalities of (II) are satisfied.

Consider a basketball league consisting of ten teams in which each team plays every other team once. Since no game can end in a tie, the digraph of the outcomes of all games at the end of the season is a tournament. What are the possible distributions of the number of victories among the teams? Clearly, each distribution must satisfy conditions (I) and (II) of Theorem 1. This fact provides information concerning certain questions that may be asked about the final standings of the team. For example, what is the largest number of teams that can have a winning season? The answer is nine, since the sequence of integers (0, 5, 5, 5, 5, 5, 5, 5, 5, 5) satisfies the conditions of Theorem 1 and no sequence containing ten integers, all greater than 4, does. Can the season end in a complete tie? Clearly not, since the average of the scores is not an integer.

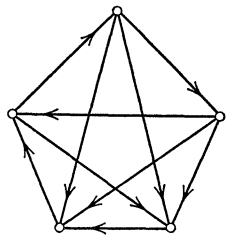

FIG. 2

If a tournament T has a transmitter v, then v is adjacent to every point of T. Thus, its distance to every point is 1, and its score is $p-1$. Clearly a tournament has at most one transmitter. If T does not have a transmitter, there will, of course, be at least one point with maximum score, which is less than $p-1$. The next theorem provides information concerning the location of such points in T; it is a "folk-theorem" cited in [10].

THEOREM 2. *In any tournament, the distance from a point with maximum score to any other point is* 1 *or* 2.

To prove this theorem, let v be any point whose score s is maximum. Without loss of generality, we denote the points to which v is adjacent by $v_1, v_2, \cdots, v_s$. Since T is a tournament, v is adjacent from the remaining $p-1-s$ points $u_1, u_2, \cdots, u_{p-1-s}$, as in Figure 3. The proof will be complete if we show that each point u_k is adjacent from at least one point v_j, for then each distance $d(v, v_j)=1$ and $d(v, u_k)=2$. Assume that this is not the case for the point u_1. Then u_1 is adjacent to every point $v_1, v_2, \cdots, v_s$ as well as to v itself. Hence, its score is $\text{od}(u_1) \geqq s+1$. This contradicts the hypothesis that s is the largest score of any point.

An interesting consequence of Theorem 2 follows when it is applied to a round robin competition. Let v be a player with maximum score in such a competition consisting of at least three players. Then any player who defeats v is himself defeated by another player defeated by v. The directional dual of this conclusion says that if v is a player with minimum score, then any player defeated by v defeats another player who defeats v.

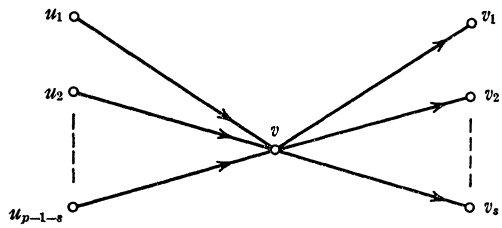

FIG. 3

The directional dual of the next theorem was stated by Silverman [18] in the following picturesque terminology: Consider a club in which among any two members, one is a creditor of the other. A "bum" is defined as a member who is in debt to everyone else. A "deadbeat" is not a "bum," but for each member he does not owe, he owes someone who owes this member. Then if the club has no bums, it has at least three deadbeats. In the future, the directional duals of theorems will not be given, but the reader is urged to develop them for himself.

THEOREM 3. *If a tournament has no transmitter, then it contains at least three points each of which can reach every point in at most two steps.*

Let u be a point of T with maximum score. By Theorem 2, u can reach every point in at most 2 steps. Since T has no transmitter, there is at least one point adjacent to u. Among all such points, let v have maximum score. Suppose there is a point v_0 not reachable from v within 2 steps. It follows that v_0 is necessarily adjacent to v and to every point which is adjacent from v, in particular, the point u. But then $s(v_0) \geqq s(v)+1$, contradicting the choice of v. Therefore every point is reachable from v in at most 2 steps. By the same argument, if w has greatest score among the points adjacent to v, then w can reach every point with 2 steps. Since T is asymmetric and lines wv and vu are in T, necessarily u, v, and w are distinct points. Since every point is within distance 2 from each of these, the theorem is proved.

The next theorem due to Rédei [17] is certainly the best known result concerning tournaments. It also holds for any complete digraph, as stated by König in [12]. Actually, Rédei showed that every tournament has an odd number of complete paths.

THEOREM 4. *Every tournament has a complete path.*

The proof is given by induction on the number p of points. Referring to Figure 2, we see that every tournament with 2, 3, or 4 points has a complete path. As the inductive hypothesis, let the theorem hold for all tournaments with n points. Let T be any tournament with $n+1$ points. To complete the proof of the theorem, it is necessary to show that T has a complete path.

Let v_0 be any point of T. Then $T-v_0$ is a tournament with n points. Since the inductive hypothesis applies to $T-v_0$, it has a complete path which may be denoted by $P=v_1v_2v_3 \cdots v_n$. Let us return to T and see how the point v_0 can be added to P in order to obtain a complete path of T. Consider the two points v_0 and v_1 of T. There are two possibilities: either line v_0v_1 or v_1v_0 is in T. If v_0v_1 is a line of T, then $v_0v_1v_2v_3 \cdots v_n$ is a complete path of T. On the other hand, if v_1v_0 is in T, then let v_i be the first point of P, if any, for which the line v_0v_i is in T. Then necessarily line $v_{i-1}v_0$ is in T. Therefore $v_1v_2 \cdots v_{i-1}v_0v_i \cdots v_n$ is a complete path of T (as shown in Figure 4). But there may not be any such first point v_i, since v_0 might be a receiver of T. In that case, $v_1v_2v_3 \cdots v_nv_0$ is a complete path of T, completing the proof.

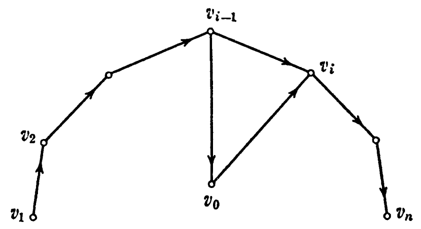

FIG. 4

Since every tournament has a complete path, it is possible to order all the players in a round robin competition so that each defeats the succeeding one. Thus the integers $1, 2, \cdots, p$ can be assigned to the players to indicate their rank in this order. There are, however, two serious difficulties in such a procedure. First, there is no necessary relation, in general, between such a ranking of players and their scores. Second, a tournament may have more than one complete path, so that several different rankings may be possible. Figure 2 illustrates these observations. For example, there is a complete path from one of the two points with lowest score to the other one. In fact, each point has every possible rank in some complete path of this tournament.

Another way of stating these difficulties is to say that a tournament need not be a complete order. For if there is a complete order on a set of p points, then there exists a one-to-one correspondence between these points and the integers $1, 2, \cdots, p$, in their natural order. Thus, whenever there is a complete order on a set of points, each point can be assigned a distinct rank. A complete order is a relation: irreflexive, asymmetric, complete, and transitive. Since every tournament has the first three of these properties, a transitive tournament is a complete order. There is considerable interest, therefore, in knowing the properties of transitive tournaments and, for any particular tournament, how much transitivity it displays.

How Transitive is a Tournament?

In analyzing the degree of transitivity of a tournament, it is useful to refer to the subtournament generated by any three of its points. A *triple* $\langle uvw \rangle$ of a tournament is the subtournament generated by the points u, v, and w. We saw in Figure 1 that there are only two tournaments with three points; one of these is transitive and the other is cyclic. Obviously every triple in a transitive tournament is transitive. Also, if a tournament is not transitive, it must contain at least one cyclic triple. It is possible therefore to quantify its degree of transitivity by using the number of triples of each kind.

Before dealing directly with the question of the degree of transitivity of a tournament, we first state the well-known structural characterization of a complete order. If v is any point of a complete order with at least three points, then $T-v$ is also a complete order. Also, every complete order has a unique transmitter and a unique receiver.

THEOREM 5. *If T is a complete order with p points, then T is isomorphic with the tournament T_p whose points are $v_1, v_2, \cdots, v_p$ in which v_i is adjacent to v_j if and only if $i<j$.*

The first corollary of this theorem lists without proof the major equivalent characterizations of a complete order.

COROLLARY 5a. *The following statements are equivalent for any tournament T with p points.*

(1) *T is transitive.*
(2) *T is acyclic.*
(3) *T has a unique complete path.*
(4) *The score sequence of T is $(0, 1, 2, \cdots, p-1)$.*
(5) *T has $p(p-1)(p-2)/6$ transitive triples.*

COROLLARY 5b. *The condensation of a tournament is a complete order.*

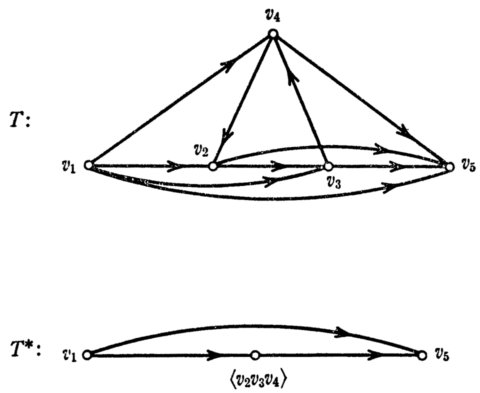

FIG. 5

If T is strong, then T^* is a single point. Therefore, take T as not strong. Obviously the condensation of a tournament is itself a tournament. And since T^* is acyclic, it is transitive.

Figure 5 shows a tournament T and its condensation T^*. Clearly, T is not transitive but T^* is.

If a tournament is not transitive, then it is often useful to know how many transitive triples it has. The next theorem shows that the number of transitive triples in any tournament may be easily calculated from the score sequence of the tournament. Theorem 6 and its first two corollaries are implicitly contained in Kendall and Babington Smith [**11**]; Corollary 6c appears there explicitly.

THEOREM 6. *The number b of transitive triples in a tournament T with score sequence $(s_1, s_2, \cdots, s_p)$ is*

$$b = \sum_{i=1}^{p} \tfrac{1}{2}s_i(s_i - 1).$$

To prove this formula, let $b_i = \frac{1}{2}s_i(s_i - 1)$. Then b_i is the number of combinations of s_i objects taken two at a time. But s_i is the number of points adjacent from v_i. Therefore b_i is the number of pairs of points adjacent from v_i. But any transitive triple in the tournament T has a unique transmitter within it. Hence b_i is the number of transitive triples whose transmitter is v_i. Clearly, the number b of transitive triples in T is obtained by adding these number b_i for all points, proving the theorem.

A little algebraic manipulation transforms the equation of Theorem 6 into the following equivalent form.

COROLLARY 6a. *The number b of transitive triples in T is*

$$b = \tfrac{1}{2} \sum_{i=1}^{p} s_i^2 - \tfrac{1}{4} p(p - 1).$$

Since the total number of triples in any tournament with p points is $\binom{p}{3}$, and since each triple is either transitive or cyclic, we obtain the following formula for the number of cyclic triples.

COROLLARY 6b. *The number c of cyclic triples in a tournament satisfies the equation*

$$c = \tfrac{1}{6} p(p - 1)(p - 2) - \tfrac{1}{2} \sum_{i=1}^{p} s_i(s_i - 1).$$

The next corollary gives the maximum number of cyclic triples that can occur in any tournament with a given number of points. Its proof is omitted.

COROLLARY 6c. *Among all the tournaments with p points, the maximum number of cyclic triples is*

$$c_{\max}(p) = \begin{cases} \dfrac{p^3 - p}{24} & \textit{if } p \textit{ is odd, and} \\[2ex] \dfrac{p^3 - 4p}{24} & \textit{if } p \textit{ is even.} \end{cases}$$

Very recently, we (Beineke and Harary [2]) have generalized the result of the preceding corollary by obtaining a formula for the maximum number of subsets of n points which generate a strongly connected subtournament that can occur in any tournament with a given number p of points. Using the methods of flows in networks as developed in Ford and Fulkerson [6], a recent result of Fulkerson [8] determines the maximum possible number of upsets (defeat of a player by another player of lower score) that can occur in a round robin tournament with a given score sequence.

Coefficient of Consistency. In research employing paired comparisons, it is usually assumed that if a judge is entirely "consistent" in his decisions, the result will be a complete order, which therefore has no cyclic triples. Since in actual practice judges are seldom completely consistent, it is useful to have a coefficient indicative of the degree of consistency among the comparisons. Such a coefficient has been proposed by Kendall and Babington Smith [11]. They wanted their coefficient of consistency to be normalized in such a way that its value is 1 when the tournament of comparisons is transitive and 0 when the tournament contains as many cyclic triples as possible, i.e., when it is as inconsistent as possible. Making use of the results in Corollary 6c, they define a coefficient of consistency ξ by the following equation in which c is the number of cyclic triples of a given tournament T with p points resulting from paired comparisons,

$$\xi = 1 - \frac{c}{c_{\max}(p)} \cdot$$

Thus we see that $0 \leqq \xi \leqq 1$ for any tournament T. On substituting the result of Corollary 6c, this equation becomes:

$$\xi = \begin{cases} 1 - \dfrac{24c}{p^3 - p} & \text{when } p \text{ is odd, and} \\[2ex] 1 - \dfrac{24c}{p^3 - 4p} & \text{when } p \text{ is even.} \end{cases}$$

Another coefficient of consistency has been proposed by Berge [3]. His ratio for measuring consistency is $b/\binom{p}{3}$, where b is the number of transitive triples of a tournament T, and the denominator $\binom{p}{3}$ is the total number of triples. This ratio will take on the value 1 if and only if T is transitive. But it will never have the value 0 when $p>3$ since every tournament with more than 3 points must contain at least one transitive triple.

Unsolved Problem. To each tournament T, one can assign a positive integer trans (T) which gives the order of a largest transitive subtournament contained in T. What is the minimum value $f(p)$ of trans (T) as T varies over all tournaments with p points, expressed as an explicit function of p? In an unpublished work, P. Erdös and L. Moser have shown that this value has the order of magnitude $c \log_2 p$ for large p, but the constant c is undetermined. From Figure **1**, we see that $f(2)=f(3)=2$ and $f(4)=3$.

The Voting Paradox. Consider an electorate which, by majority vote, is to choose among a set of motions or candidates. Assume that the chairman casts a deciding vote in case of a tie. The possible outcomes resulting from the pairing of each motion against each other can be represented by a tournament in which each motion corresponds to a point and the fact that motion v_i can defeat motion v_j corresponds to a line v_iv_j. It has long been known that such a tournament may contain a cycle. In fact, the term "cyclical majorities" was employed by the Rev. C. L. Dodgson (Lewis Carroll), cf. Black [**4**], to describe this kind of situation.

This voting paradox has stimulated a considerable literature concerning which procedure is "best," but no completely satisfactory method has been devised. Arrow [**1**] has considered the more general problem of finding a "social welfare function," whereby the preferences of individuals are equitably combined into a preference ordering by society. He has shown that it is impossible to satisfy one set of five plausible conditions for an equitable welfare function. A discussion of Arrow's Impossibility Theorem may be found in Luce and Raiffa [**14**].

Strong Tournaments

In Corollary 5b, we saw that the condensation of any tournament is a complete order. Thus if a tournament is not transitive, one may wish to study its strong components. For this reason, there is considerable interest in investigating the properties of strong tournaments. In this section we present two criteria for a tournament to be strong as well as several conditions which are sufficient but not necessary.

THEOREM 7. *If a tournament T is strong, then it contains a cycle of each length $k=3, 4, \cdots, p$.*

This theorem is proved by induction. First we observe that since T is not transitive, it has a cyclic triple. We now take as our inductive hypothesis the statement that T has a cycle Z of length $k<p$. We will show that T must have a cycle of length $k+1$. Let us label the cycle Z as $v_1v_2v_3 \cdots v_kv_1$. There are just two possibilities. Either there is a point u not in Z such that u is adjacent to some point of Z and adjacent from another point of Z, or there is no such point u.

Case 1. There is a point u not in Z such that for some points v and w in Z, lines uv and wu occur in T. The proof for this case is illustrated in Figure 6. Let us assume that the line v_1u is in T. Let v_i be the first point, going around the cycle from v_1, such that the line uv_i is in T. Then line $v_{i-1}u$ must be in T. Thus we see that T contains a cycle of length $k+1$, namely $v_1v_2v_3 \cdots v_{i-1}uv_i \cdots v_kv_1$.

Case 2. There is no point u as in Case 1. In this case, all points of T which are not in Z can be partitioned into two subsets U_1 and U_2, where U_1 is the set of all points not in Z adjacent to (every point of) Z, and U_2 is the set of all points not in Z adjacent from (every point of) Z. The sets U_1 and U_2 are not empty since T is strong by hypothesis, and they are disjoint by the hypothesis of Case 2. Since T is strong, there exist points u_1 in U_1 and u_2 in U_2 such that line u_2u_1 is in T. Then we may again construct a cycle of length $k+1$ in T, namely: $u_1v_1v_2 \cdots v_{k-1}u_2u_1$.

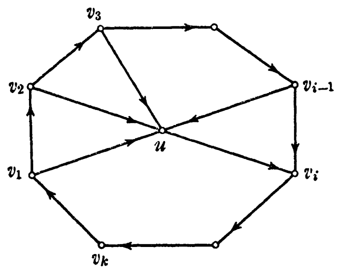

FIG. 6

By proving for each of these two cases that if T has a cycle of length k, it has a cycle of length $k+1$, we have completed the proof of the theorem.

By Corollary 5a, one criterion for a tournament to be transitive is that it have a unique complete path. The first corollary of Theorem 7 provides an analogous criterion for a tournament to be strong, cf. Camion [5] and Foulkes [7].

COROLLARY 7a. *A tournament is strong if and only if it has a complete cycle.*

By Theorem 7, every strong tournament with p points has a cycle of length p, which is a complete cycle. If a digraph has a complete cycle, it is obviously strong.

COROLLARY 7b. *If T is a strong tournament, it has at least $p-2$ cyclic triples.*

We prove the result by induction. If a strong tournament has exactly 3 points, it clearly has $p-2$ cyclic triples. As the inductive hypothesis, we take the result as true when $p=n$. Let T be strong with $n+1$ points. By Theorem 7, T has a cycle of length $p-1$. Hence, there is a point v_0 such that $T-v_0$ has a complete cycle $Z=v_1v_2 \cdots v_nv_1$. Because T is strong, v_0 has positive score. Without loss of generality we take the line v_0v_1 in T. Since v_0 also has positive indegree, we let v_i be the first point, going around the cycle Z from v_1, such that the line v_iv_0 is in T. Then $v_iv_0v_{i-1}v_i$ is a cycle so that T has at least one more cyclic triple than does $T-v_0$. But by the inductive hypothesis, $T-v_0$, having n

points and being strong, has at least $n-2$ cyclic triples. Hence T has at least $n-1$ cyclic triples, completing the proof.

COROLLARY 7c. *There exists a strong tournament of p points with $p-2$ cyclic triples.*

That no strong tournament can have fewer cyclic triples is shown in the preceding corollary. We now show that a tournament with this few cyclic triples can be constructed. Starting with a transitive tournament with p points, we replace the line from its transmitter u to its receiver v by vu forming a strong tournament T. Since $T-u$ and $T-v$ are transitive and therefore contain no cyclic triples, every cyclic triple of T contains both u and v and hence also vu. Every point of T other than u and v forms a cycle with vu. There are $p-2$ such points and hence $p-2$ cyclic triples in T.

The preceding corollaries give the number of cyclic triples necessary for a tournament to be strong. The next theorem, due to L. W. Beineke, gives a sufficient condition, in terms of the maximum number of cyclic triples in tournaments with p points, as in Corollary 6c.

THEOREM 8. *If T is a tournament with p points in which there are more cyclic triples than can occur in any tournament with $p-1$ points, then T is strong.*

Let T be a tournament with p points having more than $c_{\max}(p-1)$ cyclic triples. Suppose T is not strong, thus having at least two strong components. Clearly, all points of a cyclic triple lie in the same strong component. Hence if one strong component of T has k points, the number c of cyclic triples in T is no greater than the sum of the maximum numbers in tournaments with k and $p-k$ points: $c \leqq c_{\max}(k)+c_{\max}(p-k)$. However, it can be verified that if $0<k<p$, then $c_{\max}(p-1) \geqq c_{\max}(k)+c_{\max}(p-k)$. But this contradicts the assumption that $c>c_{\max}(p-1)$, thereby proving the theorem.

COROLLARY 8a. *If a tournament has p points and $c_{\max}(p)$ cyclic triples, then it is strong.*

It will be recalled that formulas (I) and (II) of Theorem 1 provide a necessary and sufficient condition for a sequence of nonnegative integers to be the scores of some tournament with p points. Formulas (I) and (II′) of the next theorem give the corresponding criterion for a strong tournament.

THEOREM 9. *Let T be a tournament with score sequence $s_1 \leqq s_2 \leqq \cdots \leqq s_p$. Then T is strong if and only if their sum satisfies the equation:*

$$\text{(I)} \qquad \sum_{i=1}^{p} s_i = \tfrac{1}{2}p(p-1),$$

and the following inequalities hold for every positive integer $k<p$:

$$\text{(II}'\text{)} \qquad \sum_{i=1}^{k} s_i > \tfrac{1}{2}k(k-1).$$

We first show that if T is a strong tournament, then conditions (I) and (II′) hold. We already know that condition (I) holds, since Theorem 1 has established it for any tournament. To verify the inequalities (II′), we note that for any integer $k<p$, the subtournament generated by $\{v_1, v_2, \cdots, v_k\}$ contains exactly $\frac{1}{2}k(k-1)$ lines. But since T is strong, there must be a line from one of these points to one of the $p-k$ points. Hence, in the entire tournament T, $\sum_1^k s_i > \frac{1}{2}k(k-1)$.

To prove the converse, consider conditions (I) and (II′) as given. We know by Theorem 1 that there exists a tournament T with these scores. Assume that such a tournament T is not strong. Then it has exactly one strong component S which is a receiver of the condensation T^*. Obviously the points in S have the smallest scores among all the points of T. If m is the number of points in S, then $m<p$ and $\sum_1^m s_i = \frac{1}{2}m(m-1)$, since there are no lines in T from a point in S to a point not in S. But one of the inequalities of the given condition (II′) is $\sum_1^m s_i > \frac{1}{2}m(m-1)$. This contradiction establishes the converse.

We next give some bounds for the scores of a tournament.

THEOREM 10. *Let T be a tournament with score sequence $s_1 \leqq s_2 \leqq \cdots \leqq s_p$. Then every score satisfies the inequalities:* $\frac{1}{2}(k-1) \leqq s_k \leqq \frac{1}{2}(p+k-2)$.

First, we suppose that $s_k < \frac{1}{2}(k-1)$. Then, for every $i<k$, $s_i \leqq s_k < \frac{1}{2}(k-1)$, so that $\sum_1^k s_i < \frac{1}{2}k(k-1)$. But by Theorem 1, $\sum_1^k s_i \geqq \frac{1}{2}k(k-1)$, which is a contradiction. Hence, $\frac{1}{2}(k-1) \leqq s_k$.

The second inequality is dual to the first. In the converse tournament T' with score sequence $t_1 \leqq t_2 \leqq \cdots \leqq t_p$ in which

$$t_i = (p-1) - s_{p-i+1}, \quad t_{p-k+1} \geqq \tfrac{1}{2}[(p-k+1)-1] = \tfrac{1}{2}(p-k),$$

by the first inequality of the theorem. But $s_k = (p-1) - t_{p-k+1}$, so

$$s_k \leqq (p-1) - \tfrac{1}{2}(p-k) = \tfrac{1}{2}(p+k-2),$$

proving the result.

The next theorems and their corollaries provide information regarding the scores of points and strong components of tournaments.

THEOREM 11. *Let v_i and v_j be points in different strong components of a tournament T. Then the line v_iv_j is in T if and only if $s_i > s_j$.*

Let v_i and v_j be points in different strong components of T. Let the line v_iv_j be in T. There is no cycle containing them. Every point adjacent from v_j is also adjacent from v_i, for otherwise we have a cycle containing v_i and v_j. In addition, v_j is adjacent from v_i. Therefore, $s_i > s_j$. Conversely, if $s_i > s_j$, the line v_jv_i cannot occur in T by this same argument. Hence, since T is complete, the line v_iv_j is in T.

COROLLARY 11a. *In a tournament, any two points with the same score are in the same strong component.*

The next theorem gives a sufficient condition for points with unequal scores to be in the same strong component. Although it is not a necessary condition, it may often be used in determining the strong components of a tournament.

THEOREM 12. *Let T be a tournament with score sequence $s_1 \leqq s_2 \leqq \cdots \leqq s_p$. If $0 \leqq s_n - s_m < \frac{1}{2}(n-m+1)$, then v_m and v_n are in the same strong component.*

Suppose that there are points v_m and v_n with $0 \leqq s_n - s_m < \frac{1}{2}(n-m+1)$, but they are not in the same strong component. Let j be the greatest integer less than n such that v_j is not in $S(v_n)$, the strong component containing v_n. By Theorem 10 we know that the score of v_n in the subtournament $S(v_n)$is not less than $\frac{1}{2}(n-j-1)$. Also by Theorem 11, there is a line from v_n to each of the j points v_i with $s_i \leqq s_j$. Therefore,

$$s_n \leqq j + \tfrac{1}{2}(n - j - 1) = \tfrac{1}{2}(n + j - 1).$$

Using Theorem 11 again, we know that every point adjacent from v_m is in the set $\{v_1, v_2, \cdots, v_j\}$. Applying Theorem 10 to the subtournament whose points are $v_1, v_2, \cdots, v_j$, we see that $s_m \leqq \frac{1}{2}(j+m-2)$. Combining these results with the assumption, we have $\frac{1}{2}(n-m+1) > s_n - s_m \geqq \frac{1}{2}(n+j-1) - \frac{1}{2}(j+m-2) = \frac{1}{2}(n-m+1)$, which is a contradiction.

T:

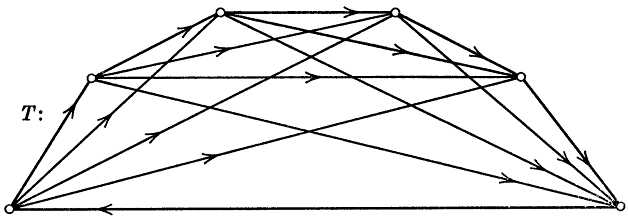

FIG. 7

We may illustrate Theorem 12 by referring again to Figure 7. From Corollary 11a we were able to conclude that v_1, v_2, v_3 all lie in the same strong component since the scores of these points are all 1. We see that $s_7 = 5$ and $s_4 = 4$. Substituting into the inequality of Theorem 12 we obtain $0 \leqq 5 - 4 = 1 < \frac{1}{2}(7-4+1) = 2$. Therefore, we conclude that v_4 and v_7 are in the strong component $S(v_4)$ and, *a fortiori*, so are v_5 and v_6.

The next two corollaries give sufficient conditions for a tournament to be strong.

COROLLARY 12a. *If the difference between every two scores in a tournament T is less than $\frac{1}{2}p$, then T is strong.*

Since the scores are $s_1 \leqq s_2 \leqq \cdots \leqq s_p$, the greatest difference between any two scores is $s_p - s_1$. By hypothesis, $s_p - s_1 < \frac{1}{2}p$ so that s_p and s_1 are in the same strong component by Theorem 12. Similarly for any other point s_i, it follows from the theorem that s_i and s_1 are in the same strong component. Thus the entire tournament is strong.

COROLLARY 12b. *If both the outdegree and indegree of each point of a tournament T is at least $\frac{1}{4}(p-1)$, then T is strong.*

If id $v \geqq \frac{1}{4}(p-1)$, then

$$\text{od } v = p - 1 - \text{id } v \leqq p - 1 - \tfrac{1}{4}(p - 1) = \tfrac{3}{4}(p - 1).$$

Thus $s_1 \geqq \frac{1}{4}(p-1)$ and $s_p \leqq \frac{3}{4}(p-1)$. Hence

$$s_p - s_1 \leqq \tfrac{3}{4}(p - 1) - \tfrac{1}{4}(p - 1) = \tfrac{1}{4}(2p - 2) < \tfrac{1}{2}p.$$

Therefore, by the preceding corollary, T is strong.

The strong tournament T displayed in Figure 7 shows that the conditions of these corollaries are not necessary for a tournament to be strong. In T, the greatest and least scores are 4 and 1, so that $s_p - s_1 = 3 = \frac{1}{2}p$. Also, the least score is less than $\frac{1}{4}(p-1)$. Nevertheless, the criteria are so simple that they sometimes are of considerable value in determining strong tournaments.

Let T_p be the number of distinct (non-isomorphic) tournaments with p points and let S_p be the number of strong tournaments among these. An explicit formula has been found for T_p by R. L. Davis [**19**]. The same formula is obtained in [**20**] from the number of oriented graphs by taking those oriented graphs with p points having $\frac{1}{2}p(p-1)$ lines. Thus the generating function

$$T(x) = \sum_{p=1}^{\infty} T_p \, x^p$$

may be regarded as known. J. W. Moon has pointed out that it is very easy to determine the corresponding generating function $S(x)$ in terms of $T(x)$ by using Corollary 5b. First write $T(x) = \sum_{n=1}^{\infty} T_n(x)$, where $T_n(x)$ counts those tournaments having exactly n strong components. Then substitute $T_n(x) = [S(x)]^n$ to obtain

$$S(x) = \frac{T(x)}{1 + T(x)} \cdot$$

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TRANSLATIONS ON AN ABSTRACT SPACE

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Introduction. Let $\mathcal{E}$ be a nonempty set, and let $\mathfrak{I}$ be such a partition of $\mathcal{E}\times\mathcal{E}$ that each member of $\mathfrak{I}$ is a function of $\mathcal{E}$ into $\mathcal{E}$. For any $(A, B)\in\mathcal{E}\times\mathcal{E}$, (A, B) is contained in only one member of the partition of $\mathcal{E}\times\mathcal{E}$, and so each member of $\mathfrak{I}$ is determined by a point of $\mathcal{E}$ and its image. The function in $\mathfrak{I}$ which contains (A, B) will be denoted by $B-A$. The image of $X\in\mathcal{E}$ under the mapping $B-A$ will be denoted by $(B-A)+X$. We shall also use '+' for function composition, and we define:

$$(A - B) + (C - D) = (A - B) \circ (C - D).$$

We also assume that the members of $\mathfrak{I}$ satisfy:

$$A - A = B - C \Leftrightarrow B = C, \tag{1}$$

$$(A - B) + (B - C) = A - C. \tag{2}$$