Socially Desirable Approximations for Dodgson's Voting Rule

IOANNIS CARAGIANNIS, CHRISTOS KAKLAMANIS, and NIKOS KARANIKOLAS,
CTI and University of Patras
ARIEL D. PROCACCIA, Carnegie Mellon University

In 1876, Charles Lutwidge Dodgson suggested the intriguing voting rule that today bears his name. Although Dodgson's rule is one of the most well-studied voting rules, it suffers from serious deficiencies, both from the computational point of view—it is $\mathcal{NP}$-hard even to approximate the Dodgson score within sublogarithmic factors—and from the social choice point of view—it fails basic social choice desiderata such as monotonicity and homogeneity. However, this does not preclude the existence of approximation algorithms for Dodgson that are monotonic or homogeneous, and indeed it is natural to ask whether such algorithms exist.

In this article, we give definitive answers to these questions. We design a monotonic exponential-time algorithm that yields a 2-approximation to the Dodgson score, while matching this result with a tight lower bound. We also present a monotonic polynomial-time $\mathcal{O}(\log m)$-approximation algorithm (where m is the number of alternatives); this result is tight as well due to a complexity-theoretic lower bound. Furthermore, we show that a slight variation on a known voting rule yields a monotonic, homogeneous, polynomial-time $\mathcal{O}(m \log m)$-approximation algorithm and establish that it is impossible to achieve a better approximation ratio even if one just asks for homogeneity. We complete the picture by studying several additional social choice properties; for these properties, we prove that algorithms with an approximation ratio that depends only on m do not exist.

1. INTRODUCTION

Social choice theory is concerned with aggregating the preferences of a set of *n agents* over a set of *m alternatives*. It is often assumed that each agent holds a private ranking of the alternatives; the collection of agents' rankings is known as a *preference profile*.

The preference profile is reported to a *voting rule*, which then singles out the winning alternative.

When there are two alternatives (and an odd number of agents), majority voting is unanimously considered a perfect method of selecting the winner. However, when there are at least three alternatives, it is sometimes unclear which alternative is best. In the 18th century, marquis de Condorcet, perhaps the founding father of the mathematical theory of voting, suggested a solution by extending majority voting to multiple alternatives [Condorcet 1785]. An alternative x is said to beat alternative y in a *pairwise election* if a majority of agents prefer x to y (i.e., rank x above y). An alternative that beats every other alternative in a pairwise election is easy to accept as the winner of the entire election; in the modern literature, such an alternative is known as a *Condorcet winner*. Unfortunately, there are preference profiles for which no alternative is a Condorcet winner.

Almost a century after Condorcet, a refinement of Condorcet's ideas was proposed by Charles Lutwidge Dodgson (today better known by his pen name Lewis Carroll), despite apparently being unfamiliar with Condorcet's work [Black 1958]. Dodgson proposed selecting the alternative "closest" to being a Condorcet winner, in the following sense. The *Dodgson score* of an alternative x is the number of exchanges between adjacent alternatives in the agents' rankings that must be introduced in order for x to become a Condorcet winner (see Section 2 for an example). A *Dodgson winner* is an alternative with minimum Dodgson score.

Although Dodgson's rule is intuitively appealing, it has been heavily criticized over the years for failing to satisfy desirable properties that are considered by social choice theorists to be extremely basic. Most prominent among these properties are *monotonicity* and *homogeneity*; a voting rule is said to be *monotonic* if it is indifferent to pushing a winning alternative upward in the preferences of the agents and is said to be *homogeneous* if it is invariant under duplication of the electorate. In fact, several authors have commented that it is somewhat unfair to attribute the previously mentioned rule to Dodgson, since Dodgson himself seems to have questioned it due to its serious defects (see, e.g., the papers by Tideman [1987, p. 194] and Fishburn [1977, p. 474]).

To make matters worse, the rise of computational complexity theory, a century after the conception of Dodgson's rule, has made it clear that it suffers from yet another serious deficiency: it is intractable to single out the winner of the election. Indeed, it is the first voting rule where winner determination was known to be $\mathcal{NP}$-hard [Bartholdi et al. 1989]; as a consequence, the computation of the Dodgson score of a given alternative is also $\mathcal{NP}$-hard. The question of the exact complexity of winner determination under Dodgson's rule was resolved by Hemaspaandra et al. [1997]: it is complete for the class Θ_2^p. The parameterized complexity of the computation of the Dodgson score of a given alternative is studied in Betzler et al. [2010]. These results have sparked great interest in Dodgson's rule among computer scientists, making it "one of the most studied voting rules in computational social choice" [Brandt 2009].

In previous work with numerous colleagues [Caragiannis et al. 2009], we have largely taken the computational complexity point of view by considering the computation of the Dodgson score as an optimization problem. Among other results, we have given two polynomial-time algorithms that guarantee an approximation ratio of $\mathcal{O}(\log m)$ to the Dodgson score (where m is the number of alternatives); this bound is asymptotically tight with respect to polynomial-time algorithms (unless $\mathcal{P} = \mathcal{NP}$). Approximating Dodgson's rule (using slightly different notions of approximation) has also been considered by Homan and Hemaspaandra [2009], McCabe-Dansted et al. [2008], and Tideman [2006, pp. 199–201].

Taking the social choice point of view, we suggest that an algorithm that approximates the Dodgson score is a voting rule in its own right in the sense that it naturally

induces a voting rule that selects an alternative with minimum score according to the algorithm. Hence, such algorithms should be evaluated not only by their computational properties (e.g., approximation ratio and complexity) but also by their social choice properties (e.g., monotonicity and homogeneity). In other words, they should be "socially desirable." This issue was briefly explored in Caragiannis et al. [2009]: we observed that one of our two approximation algorithms satisfies a weak flavor of monotonicity, whereas the other does not. Both algorithms, as well as Dodgson's rule itself, are neither monotonic (in the usual sense) nor homogeneous, but this does not preclude the existence of monotonic or homogeneous approximation algorithms for the Dodgson score. A natural question is therefore: Are there such approximation algorithms that yield a good approximation ratio?

In the following, we refer to algorithms approximating the Dodgson score (as well as to the voting rules they induce) using the term *Dodgson approximations*. A nice property that Dodgson approximations enjoy is that a finite approximation ratio implies *Condorcet consistency*—that is, a Condorcet winner (if one exists) is elected as the unique winner. One might wish for approximations of the Dodgson ranking (i.e., the ranking of the alternatives with respect to their Dodgson scores) directly instead of approximating the Dodgson score. Unfortunately, it is known that distinguishing whether an alternative is the Dodgson winner or in the last $\mathcal{O}(\sqrt{m})$ positions in the Dodgson ranking is $\mathcal{NP}$-hard [Caragiannis et al. 2009]. This extreme inapproximability result provides a complexity-theoretic explanation of the discrepancies that have been observed in the social choice literature when comparing Dodgson's rule to simpler polynomial-time voting rules (see the discussion in Caragiannis et al. [2009]) and implies that, as long as we care about efficient algorithms, reasonable approximations of the Dodgson ranking are impossible. However, the cases where the ranking is hard to approximate are cases where the alternatives have very similar Dodgson scores. We would argue that in those cases, it is not crucial, from Dodgson's point of view, which alternative is elected, since they are all almost equally close to being Condorcet winners. Put another way, if the Dodgson score is a measure of an alternative's quality, the goal is simply to elect a good alternative according to this measure.

1.1. Our Results and Techniques

In this article, we give definitive (and mostly positive) answers to the questions raised earlier; our results are tight.

In Section 3, we study monotonic Dodgson approximations. We first design an algorithm that we denote by M. Roughly speaking, this algorithm "monotonizes" Dodgson's rule by explicitly defining a winner set for each given preference profile and assigning an alternative to the winner set if it is a Condorcet winner in some preference profile such that the former profile is obtained from the latter by pushing the alternative upwards. We prove the following result:

THEOREM 3.2. *M is a monotonic Dodgson approximation with an approximation ratio of 2.*

We furthermore show that there is no monotonic Dodgson approximation with a ratio smaller than 2 (Theorem 3.4); hence, M is optimal among monotonic Dodgson approximations. Note that the lower bound is independent of computational assumptions, and, crucially, computing an alternative's score under M requires exponential time. This is to be expected because the Dodgson score is computationally hard to approximate within a factor better than $\Omega(\log m)$ [Caragiannis et al. 2009].

It is now natural to ask whether there is a monotonic *polynomial-time* Dodgson approximation with an approximation ratio of $\mathcal{O}(\log m)$. We give a positive answer to

this question as well. Indeed, we design a Dodgson approximation denoted by Q and establish the following result:

THEOREM 3.10. *Q is a monotonic polynomial-time Dodgson approximation with an approximation ratio of $\mathcal{O}(\log m)$.*

The result relies on monotonizing an existing Dodgson approximation that is based on linear programming. The main obstacle is to perform the monotonization in polynomial time rather than looking at an exponential number of profiles, as described earlier. Our main tool is the notion of *pessimistic estimator*, which allows the algorithm to restrict its attention to a single preference profile. Pessimistic estimators are obtained by solving a linear program that is a variation on the one that approximates the Dodgson score.

In Section 4, we turn to homogeneity. We consider Tideman's *simplified Dodgson rule* [Tideman 2006, pp. 199–201], which was designed to overcome the deficiencies of Dodgson's rule. The former rule is computable in polynomial time and is moreover known to be monotonic and homogeneous. By scaling the score given by the simplified Dodgson rule, we obtain a rule, denoted Td$'$, that is identical as a voting rule and moreover has the following properties:

THEOREM 4.1. *Td$'$ is a monotonic, homogeneous, polynomial-time Dodgson approximation with an approximation ratio of $\mathcal{O}(m \log m)$.*

Note that the Dodgson score can be between 0 and $\Theta(nm)$, so this bound is far from trivial. The analysis is tight when there is an alternative that is tied against many other alternatives in pairwise elections (and hence has relatively high Dodgson score), whereas another alternative strictly loses in pairwise elections to few alternatives (so it has a relatively low Dodgson score). By homogeneity, the former alternative must be elected since its score does not scale when the electorate is replicated (we elaborate in Section 4). This intuition leads to the following result, which applies to any (even exponential-time) homogeneous Dodgson approximation:

THEOREM 4.2. *Any homogeneous Dodgson approximation has approximation ratio at least $\Omega(m \log m)$.*

In particular, the upper bound given in Theorem 4.1 (which is achieved by an algorithm that is moreover monotonic and efficient) is asymptotically tight. The heart of our construction is the design of a preference profile such that an alternative is tied against $\Omega(m)$ other alternatives; this is equivalent to a construction of a family of subsets of a set U, $|U| = m$, such that each element of U appears in roughly half the subsets but the minimum cover is of size $\Omega(\log m)$.

In order to complete the picture, in Section 5, we discuss some other, less prominent, social choice properties not satisfied by Dodgson's rule [Tideman 2006, Chapter 13]: combinativity, Smith consistency, mutual majority, invariant loss consistency, and independence of clones. We show that any Dodgson approximation that satisfies one of these properties has an approximation ratio of $\Omega(nm)$ (in the case of the former two properties) or $\Omega(n)$ (in the case of the latter three). An $\Omega(nm)$ ratio is a completely trivial one, but we also consider an approximation ratio of $\Omega(n)$ to be impractical, as the number of agents n is very large in many settings of interest.

1.2. Discussion

Our results with respect to monotonicity are positive across the board. In particular, we find Theorem 3.2 surprising, as it indicates that Dodgson's lack of monotonicity can be circumvented by slightly modifying the definition of the Dodgson score; in a sense, this suggests that Dodgson's rule is not fundamentally far from being monotonic. Theorem 3.10 strengthens the result of Caragiannis et al. [2009]. Indeed, if one is interested in computationally tractable algorithms, then an approximation ratio of $\mathcal{O}(\log m)$

is optimal; the theorem implies that we can additionally satisfy monotonicity without (asymptotically) increasing the approximation ratio. Our monotonization techniques may be of independent interest.

Our results regarding homogeneity, Theorem 4.1 and Theorem 4.2, can be interpreted both positively and negatively. Consider first the case where the number of alternatives m is small (e.g., in political elections). A major advantage of Theorem 4.1 is that it concerns Tideman's simplified Dodgson rule, which is already recognized as a desirable voting rule, as it is homogeneous, monotonic, Condorcet consistent, and resolvable in polynomial time. The theorem lends further justification to this rule by establishing that it always elects an alternative relatively close (according to Dodgson's notion of distance) to being a Condorcet winner—that is, the spirit of Dodgson's ideas is indeed preserved by the "simplification," and (due to Theorem 4.2) this is accomplished in the best possible way.

Viewed negatively, when the number of alternatives is large (an extreme case is a multiagent system where the agents are voting over joint plans), Theorem 4.2 strengthens the criticism against Dodgson's rule: not only is the rule itself nonhomogeneous, but any (even exponential-time computable) conceivable variation that tries to roughly preserve Dodgson's notion of proximity to Condorcet is also nonhomogeneous. We believe that both interpretations of the homogeneity results are of interest to social choice theorists as well as computer scientists.

A constructive versus descriptive point of view. Our interpretation of the results implicitly incorporates two points of view. The first is *constructive*: we would like to design new polynomial-time computable voting rules that preserve the qualities of Dodgson's rule (e.g., the proximity of alternatives to becoming Condorcet winners) while satisfying additional properties. The second is *descriptive*: the results provide us with a novel way to *quantify* how "close" Dodgson's rule is to satisfying certain properties. It is the descriptive point of view that, we believe, social choice theorists would find appealing.

The constructive point of view gives rise to the following conceptual question: does it make sense to use a Dodgson approximation when several voting rules that are monotonic, homogeneous, and polynomial time are available? However, our results regarding Tideman's simplified Dodgson rule offer the best of both worlds: it is both an optimal Dodgson approximation and an established socially desirable voting rule. Moreover, note that this question is a nonissue when taking the descriptive point of view.

An important advantage of the descriptive point of view is that our ideas can also be applied to voting rules where winner determination is not computationally hard, thereby significantly broadening the intellectual scope of the results. For example, the Plurality rule is not Condorcet consistent, but how far is it from being Condorcet consistent?

Both points of view rely on the underlying assumption that the score of an alternative according to a voting rule is proportional (or inversely proportional, in the case of Dodgson) to a measure of that alternative's social desirability. One can argue that a similar implicit assumption is in fact usually made in the context of many standard optimization problems (e.g., a vertex cover that is twice the size of the optimal cover is twice as bad). This approach treats a voting rule as being coupled with a specific notion of score. Hence, although scaling the score by an additive term would yield the same function from preference profiles to alternatives, it would alter the problem in a fundamental way (in the same way that scaling the vertex cover objective by an additive term would make the problem easier to approximate but render its solution useless).

Truthfulness. Note that almost all work in *algorithmic mechanism design* [Nisan and Ronen 2001] seeks *truthful* approximation algorithms—that is, algorithms such that the agents cannot benefit by lying. However, it is well known that in the standard social choice setting, truthfulness cannot be achieved. Indeed, the Gibbard–Satterthwaite Theorem [Gibbard 1973; Satterthwaite 1975] (see also Nisan [2007]) implies that any minimally reasonable voting rule is not truthful. Therefore, social choice theorists strive for other socially desirable properties, and in particular, the ones discussed earlier. To avoid confusion, we remark that although notions of monotonicity are often studied in mechanism design as ways of obtaining truthfulness (see, e.g., Archer and Tardos [2001]), in social choice theory monotonicity is a very basic desirable property in its own right (and has been so long before mechanism design was conceived).

Future work. In the future, we envision the extension of our agenda of socially desirable approximation algorithms to other important voting rules. Positive results in this direction would provide us with tools to circumvent the deficiencies of known voting rules without sacrificing their core principles; negative results would further enhance our understanding of such deficiencies. As noted earlier, these questions are relevant even with respect to tractable voting rules that do not satisfy certain properties but from a computational perspective may be especially challenging in the context of voting and rank aggregation rules that are hard to compute (e.g., Kemeny's and Slater's rules [Ailon et al. 2005; Coppersmith et al. 2006; Kenyon-Mathieu and Schudy 2007]). The work in this direction might involve well-known tractable, Condorcet-consistent, monotonic, and homogeneous rules such as Copeland and Maximin (see, e.g., Tideman [2006]) in the same way that we use Tideman's simplified Dodgson rule in the current article. It might also use different notions of approximation (e.g., additive or differential approximations) besides the standard definition of the approximation ratio as a multiplicative factor used in this article.

2. PRELIMINARIES

We consider a set of *agents* $N = \{0, 1, \ldots, n-1\}$ and a set of *alternatives* A, $|A| = m$. Each agent has linear preferences over the alternatives—that is, a ranking over the alternatives. Formally, the preferences of agent i are a binary relation $\succ_i$ over A that satisfies irreflexivity, asymmetry, transitivity, and totality; given $x, y \in A$, $x \succ_i y$ means that i prefers x to y. We let $\mathcal{L} = \mathcal{L}(A)$ be the set of linear preferences over A. A *preference profile* $\succ = \langle \succ_0, \ldots, \succ_{n-1} \rangle \in \mathcal{L}^n$ is a collection of preferences for all of the agents. A *voting rule* (also known as a *social choice correspondence*) is a function $f : \mathcal{L}^n \to 2^A \setminus \{\emptyset\}$ from preference profiles to nonempty subsets of alternatives, which designates the winner(s) of the election.

Let $x, y \in A$ and $\succ \in \mathcal{L}^n$. We say that x beats y in a *pairwise election* if $|\{i \in N : x \succ_i y\}| > n/2$—that is, if a (strict) majority of agents prefer x to y. A *Condorcet winner* is an alternative that beats every other alternative in a pairwise election. The *Dodgson score* of an alternative $x \in A$ with respect to a preference profile $\succ \in \mathcal{L}^n$, denoted $\mathrm{sc}_D(x, \succ)$, is the number of swaps between adjacent alternatives in the individual rankings that are required in order to make x a Condorcet winner. A *Dodgson winner* is an alternative with a minimum Dodgson score.

Consider, for example, the profile $\succ$ in Table I; in this example, $N = \{0, \ldots, 4\}$, $A = \{a, b, c, d, e\}$, and the ith column is the ranking reported by agent i. Alternative a loses in pairwise elections to b and e (two agents prefer a to b, one agent prefers a to e). In order to become a Condorcet winner, four swaps suffice: swapping a and e, and then a and b, in the ranking of agent 1 (after the swaps, the ranking becomes

Table I. An Example of a Profile

0	1	2	3	4
a	b	e	e	b
b	e	b	c	e
c	a	c	d	d
d	c	a	a	a
e	d	d	b	c

For this profile $\succ$, it holds that
$\mathrm{sc}_D(b, \succ) = 0$, $\mathrm{sc}_D(a, \succ) = 4$.

$a \succ_1 b \succ_1 e \succ_1 c \succ_1 d$), and swapping a and d, and then a and e, in the ranking of agent 4. Alternative a cannot be made a Condorcet winner with fewer swaps; hence, we have $\mathrm{sc}_D(a, \succ) = 4$ in this profile. However, in the profile of Table I, there is a Condorcet winner, namely alternative b, and hence b is the Dodgson winner with $\mathrm{sc}_D(b, \succ) = 0$.

Given a preference profile $\succ \in \mathcal{L}^n$ and $x, y \in A$, the *deficit* of x against y, denoted $\mathrm{defc}(x, y, \succ)$, is the number of additional agents that must rank x above y in order for x to beat y in a pairwise election. Formally,

$$\mathrm{defc}(x, y, \succ) = \max\left\{ 0, \left\lceil \frac{n+1}{2} \right\rceil - |\{i \in N : x \succ_i y\}| \right\}.$$

In particular, if x beats y in a pairwise election, then it holds that $\mathrm{defc}(x, y, \succ) = 0$. Note that if n is even and x and y are tied—that is, $|\{i \in N : x \succ_i y\}| = n/2$—then $\mathrm{defc}(x, y, \succ) = 1$. For example, in the profile of Table I, we have that $\mathrm{defc}(a, b, \succ) = 1$, $\mathrm{defc}(a, c, \succ) = 0$, $\mathrm{defc}(a, d, \succ) = 0$, and $\mathrm{defc}(a, e, \succ) = 2$.

We consider algorithms that receive as input an alternative $x \in A$ and a preference profile $\succ \in \mathcal{L}^n$, and return a score for x. We denote the score returned by an algorithm V on the input, which consists of an alternative $x \in A$ and a profile $\succ \in \mathcal{L}^n$ by $\mathrm{sc}_V(x, \succ)$. We call such an algorithm V a *Dodgson approximation* if $\mathrm{sc}_V(x, \succ) \geq \mathrm{sc}_D(x, \succ)$ for every alternative $x \in A$ and every profile $\succ \in \mathcal{L}^n$. We also say that V has an *approximation ratio* of ρ if

$$\mathrm{sc}_D(x, \succ) \leq \mathrm{sc}_V(x, \succ) \leq \rho \cdot \mathrm{sc}_D(x, \succ),$$

for every $x \in A$ and every $\succ \in \mathcal{L}^n$. A Dodgson approximation naturally induces a voting rule by electing the alternative(s) with minimum score. Hence, when we say that a Dodgson approximation satisfies a social choice property, we are referring to the voting rule induced by the algorithm. Observe that the voting rule induced by a Dodgson approximation with finite approximation ratio is *Condorcet consistent*. Indeed, by the definition of the approximation ratio ρ of a Dodgson approximation V, if ρ is finite and x is a Condorcet winner (i.e., $\mathrm{sc}_D(x, \succ) = 0$), then it should also hold that $\mathrm{sc}_V(x, \succ) = 0$ and x is the (sole) winner under V.

Let us give an example. Consider the algorithm V that, given an alternative $x \in A$ and a preference profile $\succ \in \mathcal{L}^n$, returns a score of $\mathrm{sc}_V(x, \succ) = m \cdot \sum_{y \in A \setminus \{x\}} \mathrm{defc}(x, y, \succ)$. It is easy to show that this algorithm is a Dodgson approximation and, furthermore, has approximation ratio at most m. Indeed, it is possible to make x beat y in a pairwise election by pushing x to the top of the preferences of $\mathrm{defc}(x, y, \succ)$ agents, and this requires at most $m \cdot \mathrm{defc}(x, y, \succ)$ swaps. By summing over all $y \in A \setminus \{x\}$, we obtain an upper bound of $\mathrm{sc}_V(x, \succ)$ on the Dodgson score of x. On the other hand, given $x \in A$, for every $y \in A \setminus \{x\}$, we require $\mathrm{defc}(x, y, \succ)$ swaps that push x above y in the preferences of some agent in order for x to beat y in a pairwise election. Moreover, these swaps do not

decrease the deficit against any other alternative. Therefore, $\sum_{y \in A \setminus \{x\}} \mathrm{defc}(x, y, \succ) \leq \mathrm{sc}_D(x, \succ)$, and by multiplying by m, we get that $\mathrm{sc}_V(x, \succ) \leq m \cdot \mathrm{sc}_D(x, \succ)$.[1]

3. MONOTONICITY

In this section, we present our results on monotonic Dodgson approximations. A voting rule is *monotonic* if a winning alternative remains winning after it is pushed upward in the preferences of some of the agents. Dodgson's rule is known to be nonmonotonic (see, e.g., Brandt [2009]). The intuition is that if an agent ranks x directly above y and y above z, swapping x and y may not help y if it already beats x, but it may help z defeat x.

As a warm-up, we observe that the Dodgson approximation mentioned at the end of the previous section is monotonic as a voting rule. Indeed, consider a preference profile $\succ$ and a winning alternative x. Pushing x upward in the preferences of some of the agents can neither increase its score (since its deficit against any other alternative does not increase) nor decrease the score of any other alternative $y \in A \setminus \{x\}$ (since the deficit of y against any alternative in $A \setminus \{x, y\}$ remains unchanged and its deficit against x does not decrease). In the following, we present a much stronger result.

3.1. Monotonizing Dodgson's Voting Rule

Using a natural monotonization of Dodgson's voting rule, we obtain a monotonic Dodgson approximation with approximation ratio at most 2. The main idea is to define the winning set of alternatives for a given profile first and then assign the same score to the alternatives in the winning set and a higher score to the nonwinning alternatives. Roughly speaking, the winning set is defined so that it contains the Dodgson winners for the given profile as well as the Dodgson winners of other profiles that are necessary so that monotonicity is satisfied.

More formally, we say that a preference profile $\succ' \in \mathcal{L}^n$ is a *y-improvement* of $\succ$ for some alternative $y \in A$ if $\succ'$ is obtained by starting from $\succ$ and pushing y upward in the preferences of some of the agents. In particular, a profile is a y-improvement of itself for any alternative $y \in A$. The next statement is obvious.

Observation 3.1. Let $y \in A$, and let $\succ, \succ' \in \mathcal{L}^n$ be profiles such that $\succ'$ is a y-improvement of $\succ$. Then,

$$\mathrm{sc}_D(y, \succ') \leq \mathrm{sc}_D(y, \succ).$$

We monotonize Dodgson's voting rule as follows. Let M denote the new voting rule that we are constructing. We denote by $W(\succ)$ the set of winners of M (to be defined shortly) for profile $\succ \in \mathcal{L}^n$. Let $\Delta = \max_{y \in W(\succ)} \mathrm{sc}_D(y, \succ)$. The voting rule M assigns a score of $\mathrm{sc}_M(y, \succ) = \Delta$ to each alternative $y \in W(\succ)$ and a score of

$$\mathrm{sc}_M(y, \succ) = \max\{\Delta + 1, \mathrm{sc}_D(y, \succ)\}$$

to each alternative $y \notin W(\succ)$. All that remains is to define the set of winners $W(\succ)$ for profile $\succ$. This is done as follows: for each preference profile $\succ^* \in \mathcal{L}^n$ and each Dodgson winner y^* at $\succ^*$, include y^* in the winner set $W(\succ')$ of each preference profile $\succ' \in \mathcal{L}^n$ that is a y^*-improvement of $\succ^*$.

THEOREM 3.2. *M is a monotonic Dodgson approximation with an approximation ratio of* 2.

---

[1]Recently, a similar reasoning was used by Faliszewski et al. [2011] to prove that a voting rule known as Maximin (which is homogeneous and monotonic) is a Dodgson approximation with approximation ratio at most m^2; see Faliszewski et al. [2011] for details.

PROOF. Clearly, the voting rule M is a Dodgson approximation (i.e., $sc_M(y, \succ) \geq sc_D(y, \succ)$ for each alternative $y \in A$ and profile $\succ \in \mathcal{L}^n$). Furthermore, it is monotonic by definition; if y is a winner for a profile $\succ$, y stays a winner for each y-improvement of $\succ$. In the following, we show that it has an approximation ratio of 2 as well. We will need the following lemma; informally, it states that by pushing an alternative upward, we cannot significantly decrease—that is, improve—the Dodgson score of another alternative. □

LEMMA 3.3. *Let $y \in A$, and let $\succ, \succ' \in \mathcal{L}^n$ be profiles such that $\succ'$ is a y-improvement of $\succ$. For any alternative $z \in A \setminus \{y\}$, it holds that $sc_D(z, \succ) \leq 2 \cdot sc_D(z, \succ')$.*

PROOF. The fact that the Dodgson score of alternative z at profile $\succ'$ is $sc_D(z, \succ')$ means that z can become a Condorcet winner by pushing it $sc_D(z, \succ')$ positions upward in the preferences of some agents; let $\succ''$ be the resulting profile (where z is a Condorcet winner). For $i \in N$, denote

$$S_i = \{x \in A: \ x \succ_i' z \wedge z \succ_i'' x\}.$$

Clearly, $\sum_{i \in N} |S_i| = sc_D(z, \succ')$.

Next consider the profile $\succ$ and observe that for all alternatives $x \in A \setminus \{z\}$, $defc(z, x, \succ) \leq defc(z, x, \succ')$. Hence, z becomes a Condorcet winner when it is pushed upward in each agent i so that it bypasses all of the alternatives in S_i. This involves no swaps at agent i if $S_i = \emptyset$ while pushing z upward for $|S_i| + 1$ positions at agent i is sufficient to bypass the alternatives in S_i and, possibly, alternative y that may lie in between them in $\succ$ (and not in $\succ'$). Hence, the Dodgson score of z at profile $\succ$ is

$$sc_D(z, \succ) \leq \sum_{i \in N: S_i \neq \emptyset} (|S_i| + 1) \leq 2 \sum_{i \in N} |S_i| = 2 \cdot sc_D(z, \succ'),$$

and the lemma follows. □

Now, consider a profile $\succ \in \mathcal{L}^n$. Let y^* be an alternative in $W(\succ)$ with highest Dodgson score (equal to Δ). If y^* is a Dodgson winner at $\succ$, then $sc_M(z, \succ) = sc_D(z, \succ)$ for each alternative $z \in A$, and we are done. Hence, we can assume that y^* is not a Dodgson winner but belongs to $W(\succ)$.

By definition, there must be a profile $\succ^* \in \mathcal{L}^n$ such that y^* is a Dodgson winner of $\succ^*$ and $\succ$ is a y^*-improvement of $\succ^*$. By Observation 3.1, since $\succ$ is a y^*-improvement of $\succ^*$, we have

$$sc_D(y^*, \succ) \leq sc_D(y^*, \succ^*). \tag{1}$$

Since y^* is a Dodgson winner of $\succ^*$, we have

$$sc_D(y^*, \succ^*) \leq sc_D(z, \succ^*) \tag{2}$$

for each alternative $z \in W(\succ)$. In addition, by applying Lemma 3.3, we have

$$sc_D(z, \succ^*) \leq 2 \cdot sc_D(z, \succ) \tag{3}$$

for each alternative $z \in W(\succ)$. Now, using the definition of M and inequalities (1), (2), and (3), for any alternative $z \in W(\succ)$, we have

$$sc_M(z, \succ) = \Delta = sc_D(y^*, \succ) \leq 2 \cdot sc_D(z, \succ).$$

It remains to establish the approximation ratio with respect to the alternatives in $A \setminus W(\succ)$. Let $y' \in A \setminus \{y^*\}$ be a Dodgson winner of $\succ$. Since $y' \in W(\succ)$, the earlier inequality implies that

$$\Delta \leq 2 \cdot sc_D(y', \succ). \tag{4}$$

Table II. The Preference Profile $\succ$ Used in the Proof of Theorem 3.4

$\times k$	$\times k$	$\times k$	$\times k$	$\times k$	$\times k$	$\times 2k$	$\times 2k$	$\times 2k$	$\times 2k$	$\times 2k$	$\times 2k$
c	c	a	a	b	b	a	a	b	b	c	c
b	b	c	c	a	a	d	e	d	e	d	e
a	a	b	b	c	c	b	b	c	c	a	a
d	e	d	e	d	e	d	e	d	e	e	d
e	d	e	d	e	d	c	c	a	a	b	b

Let $z' \in A \setminus W(\succ)$. By definition, $\mathrm{sc}_M(z', \succ) = \mathrm{sc}_D(z', \succ)$ when $\mathrm{sc}_D(z', \succ) > \Delta + 1$, and we are done. Otherwise, it holds that $\mathrm{sc}_M(z', \succ) = \Delta + 1$. Since z' is not a Dodgson winner for $\succ$, we have $\mathrm{sc}_D(z', \succ) \geq \mathrm{sc}_D(y', \succ) + 1$ in this case; using (4), we obtain

$$\mathrm{sc}_M(z', \succ) = \Delta + 1 \leq 2 \cdot \mathrm{sc}_D(y', \succ) + 1 \leq 2 \cdot \mathrm{sc}_D(z', \succ).$$

To conclude, the score of each alternative under M is at most twice its Dodgson score.

In general, the Dodgson approximation M is computable in exponential time. However, it can be implemented to run in polynomial time when m is a constant; in this special case, the number of different profiles with n agents is polynomial and the Dodgson score can be computed exactly in polynomial time [Bartholdi et al. 1989].

The next statement shows that the voting rule M is the best possible monotonic Dodgson approximation. Note that it is not based on any complexity assumptions and, hence, it holds for exponential-time Dodgson approximations as well.

THEOREM 3.4. *A monotonic Dodgson approximation cannot have an approximation ratio smaller than 2.*

PROOF. Let k be a positive integer. We use the preference profile $\succ$ (see Table II) with $18k$ agents and five alternatives a, b, c, d, and e.

The deficit of d and e against any of the alternatives a, b, and c is $3k + 1$, and d is tied against e—that is,

$$\mathrm{defc}(d, e, \succ) = \mathrm{defc}(e, d, \succ) = 1.$$

Moreover,

$$\mathrm{defc}(a, c, \succ) = \mathrm{defc}(b, a, \succ) = \mathrm{defc}(c, b, \succ) = k + 1,$$

whereas the remaining deficits are zero.

Hence, the Dodgson score of d and e is at least $9k + 4$. In addition, observe that c is ranked either below a or two positions above a by each agent. It follows that a must be pushed two positions upward in $k + 1$ agents in order to become a Condorcet winner. This is also sufficient since a becomes a Condorcet winner by pushing it two positions upward in $k + 1$ agents among those in the first and second columns of profile $\succ$. Hence, $\mathrm{sc}_D(a, \succ) = 2(k + 1)$. With a similar argument, we obtain that

$$\mathrm{sc}_D(b, \succ) = \mathrm{sc}_D(c, \succ) = 2(k + 1)$$

as well.

Now, consider any monotonic approximation algorithm M' for Dodgson. Given $\succ$, if it returns alternative d or e as a winner, then

$$\begin{aligned}
\mathrm{sc}_{M'}(a, \succ) &\geq \min\{\mathrm{sc}_{M'}(d, \succ), \mathrm{sc}_{M'}(e, \succ)\} \\
&\geq \min\{\mathrm{sc}_D(d, \succ), \mathrm{sc}_D(e, \succ)\} \\
&\geq 9k + 4 > 4(k + 1) = 2 \cdot \mathrm{sc}_D(a, \succ)
\end{aligned}$$

(i.e., the approximation ratio is more than 2).

If the algorithm returns alternative a as the winner, then consider the profile $\succ^a$ in which a is pushed one position upward in the $2k$ agents in the fifth and sixth columns

of profile $\succ$. Due to monotonicity, a should be the winner in the profile $\succ^a$ as well. Additionally, note that now c can become a Condorcet winner by pushing it just one position upward in $k+1$ among the $2k$ agents in the fifth and sixth columns of profile $\succ^a$ and, hence, $\mathrm{sc}_D(c, \succ^a) = k+1$, whereas the Dodgson score of a in $\succ^a$ is the same as in $\succ$. We have

$$\mathrm{sc}_{M'}(c, \succ^a) \geq \mathrm{sc}_{M'}(a, \succ^a) \geq \mathrm{sc}_D(a, \succ^a)$$
$$= 2(k+1) = 2 \cdot \mathrm{sc}_D(c, \succ^a)$$

(i.e., the approximation ratio is at least 2).

If the algorithm returns alternative b as the winner, then consider the profile $\succ^b$ in which b is pushed one position upward in the $2k$ agents in the first and second columns of profile $\succ$. Due to monotonicity, we have that b is a winner in the profile $\succ^b$ as well. In addition, note that now a can become a Condorcet winner by pushing it just one position upward in $k+1$ among the $2k$ agents in the first and second columns of profile $\succ^b$ and, hence, $\mathrm{sc}_D(a, \succ^b) = k+1$. Using a similar calculation, we get that the approximation ratio is at least 2.

Finally, if the algorithm returns alternative c as the winner, then consider the profile $\succ^c$ in which c is pushed one position upward in the $2k$ agents in the third and fourth columns of profile $\succ$. This time, we have that $\mathrm{sc}_D(b, \succ^c) = k+1$, and we obtain the lower bound of 2 as before. We have thus covered all possible cases, and the theorem follows.   □

We remark that in the earlier proof, it is possible to replace each column of k or $2k$ agents with a single agent or two agents, respectively. However, the current proof has the advantage of demonstrating that the approximation ratio cannot improve if the number of agents is assumed to be large. Clearly, it is also possible to increase the number of alternatives by adding more alternatives at the bottom of the agents' preferences.

3.2. A Monotonic Polynomial-Time $\mathcal{O}(\log m)$-Approximation Algorithm

We now present a monotonic polynomial-time Dodgson approximation that achieves an approximation ratio of $\mathcal{O}(\log m)$. Given the $\Omega(\log m)$ inapproximability bound for the Dodgson score [Caragiannis et al. 2009], this rule is asymptotically optimal with respect to polynomial-time algorithms. To be precise, it is optimal within a factor of 4, assuming that problems in $\mathcal{NP}$ do not have quasi-polynomial-time algorithms.

We remark that there are two main obstacles that we have to overcome in order to implement the monotonization in polynomial time. First, the computation of the Dodgson score and the decision problem of detecting whether a given alternative is a Dodgson winner on a particular profile are $\mathcal{NP}$-hard problems [Bartholdi et al. 1989]. We overcome this obstacle by using a polynomial-time Dodgson approximation R from Caragiannis et al. [2009] instead of the Dodgson score itself. Even in this case, given a profile, we still need to be able to detect whether an alternative $y \in A$ is the winner according to R at some profile of which the current profile is a y-improvement; if this is the case, y should be included in the winning set. Note that, in general, this requires checking an exponential number of profiles in order to determine the winning set of the current one. We tackle this second obstacle using the notion of *pessimistic estimators*; these are quantities defined in terms of the current profile only and are used to identify its winning alternatives.

In order to define the algorithm R that we will monotonize, we consider an alternative definition of the Dodgson score for an alternative $z^* \in A$ and a profile $\succ \in \mathcal{L}^n$. Define the set $S_k^{\succ_i}(z^*)$ to be the set of alternatives that z^* bypasses as it is pushed k positions upward in the preference of agent i. Denote by $\mathcal{S}^{\succ_i}(z^*)$ the collection of all possible such

sets for agent i—that is,

$$\mathcal{S}^{\succ_i}(z^*) = \left\{ S_k^{\succ_i}(z^*) : k = 1, \ldots, r_i(z^*, \succ) - 1 \right\},$$

where $r_i(z^*, \succ)$ denotes the rank of alternative z^* in the preferences of agent $i \in N$ (e.g., the most and least preferred alternatives have rank 1 and m, respectively). Let $\mathcal{S}(z^*) = \bigcup_{i \in N} \mathcal{S}^{\succ_i}(z^*)$. Then, the problem of computing the Dodgson score of alternative z^* on the profile $\succ$ is equivalent to selecting sets from $\mathcal{S}(z^*)$ of minimum total size so that at most one set is selected among the ones in $\mathcal{S}^{\succ_i}(z^*)$ for each agent $i \in N$ and each alternative $z \in A \setminus \{z^*\}$ appears in at least $\mathrm{defc}(z^*, z, \succ)$ selected sets. This can be expressed by the following integer linear program:

$$\text{minimize} \sum_{i \in N} \sum_{k=1}^{r_i(z^*, \succ)-1} k \cdot \mathbf{x}\left(S_k^{\succ_i}(z^*)\right)$$

$$\text{subject to } \forall z \in A \setminus \{z^*\},$$

$$\sum_{i \in N} \sum_{S \in \mathcal{S}^{\succ_i}(z^*) : z \in S} \mathbf{x}(S) \geq \mathrm{defc}(z^*, z, \succ) \qquad (5)$$

$$\forall i \in N, \quad \sum_{S \in \mathcal{S}^{\succ_i}(z^*)} \mathbf{x}(S) \leq 1$$

$$\forall S \in \mathcal{S}(z^*), \quad \mathbf{x}(S) \in \{0, 1\}.$$

The binary variable $\mathbf{x}(S)$ indicates whether the set $S \in \mathcal{S}^{\succ_i}(z^*)$ is selected ($\mathbf{x}(S) = 1$) or not ($\mathbf{x}(S) = 0$). Now, consider the linear programming relaxation of integer linear program (5), in which the last constraint is relaxed to $\mathbf{x}(S) \geq 0$. We define the voting rule R that sets $\mathrm{sc}_R(z^*, \succ)$ equal to the optimal value of the linear programming relaxation multiplied by H_{m-1}, where H_k is the kth harmonic number. In Caragiannis et al. [2009], it is shown that

$$\mathrm{sc}_D(y, \succ) \leq \mathrm{sc}_R(y, \succ) \leq H_{m-1} \cdot \mathrm{sc}_D(y, \succ)$$

for every alternative $y \in A$—that is, R is a Dodgson approximation with an approximation ratio of H_{m-1}. The following observation is analogous to Observation 3.1.

Observation 3.5. Let $y \in A$, and let $\succ, \succ' \in \mathcal{L}^n$ be profiles such that $\succ'$ is a y-improvement of $\succ$. Then,

$$\mathrm{sc}_R(y, \succ') \leq \mathrm{sc}_R(y, \succ).$$

We now present a new voting rule Q by monotonizing R. The voting rule Q defines a set of alternatives $W(\succ)$ that is the set of winners on a particular profile $\succ$. Then, it sets $\mathrm{sc}_Q(y, \succ) = 2 \cdot \mathrm{sc}_R(y^*, \succ)$ for each alternative $y \in W(\succ)$, where y^* is the winner according to the voting rule R. In addition, it sets $\mathrm{sc}_Q(y, \succ) = 2 \cdot \mathrm{sc}_R(y, \succ)$ for each alternative $y \notin W(\succ)$.

In order to define the set $W(\succ)$, we will use another (slightly different) linear program defined for two alternatives $y, z^* \in A$ and a profile $\succ \in \mathcal{L}^n$. The new linear program has the same set of constraints as the relaxation of (5) used in the definition of $\mathrm{sc}_R(z^*, \succ)$ and the following objective function:

$$\text{minimize} \sum_{i \in N} \sum_{k=1}^{r_i(z^*, \succ)-1} k \cdot \mathbf{x}\left(S_k^{\succ_i}(z^*)\right)$$

$$+ \sum_{i \in N : y \succ_i z^*} \sum_{k=1}^{r_i(z^*, \succ)-r_i(y, \succ)-1} \mathbf{x}\left(S_k^{\succ_i}(z^*)\right). \qquad (6)$$

We define the *pessimistic estimator* $pe(z^*, y, \succ)$ for alternative $z^* \in A$ with respect to another alternative $y \in A \setminus \{z^*\}$ and a profile $\succ \in \mathcal{L}^n$ to be equal to the objective value of linear program (6) multiplied by H_{m-1}. As it will become apparent shortly, the pessimistic estimator $pe(z^*, y, \succ')$ upper bounds the score of alternative z^* under R on every profile $\succ$ such that $\succ'$ is a y-improvement of $\succ$, hence the pessimism with respect to estimating the score of z^*. These pessimistic estimators will be our main tool in order to monotonize R.

We are now ready to complete the definition of the voting rule Q. The set $W(\succ)$ is defined as follows. First, it contains all winners according to voting rule R. An alternative y that is not a winning alternative according to R is included in the set $W(\succ)$ if $pe(z, y, \succ) \geq sc_R(y, \succ)$ for every alternative $z \in A \setminus \{y\}$.

Clearly, Q is polynomial time; computing the scores of all alternatives involves solving m^2 linear programs of polynomial size. We next show that it is monotonic as well. This is done in Lemma 3.8 after establishing some properties of pessimistic estimators. The first property (stated in Lemma 3.6) has a long and technically involved proof that can be skipped at first reading. The second one (in Lemma 3.7) follows easily by the definition of pessimistic estimators.

LEMMA 3.6. *Let $y, z^* \in A$ be different alternatives, and let $\succ, \succ' \in \mathcal{L}^n$ be two profiles such that $\succ'$ is a y-improvement of $\succ$. Then, $pe(z^*, y, \succ') \geq pe(z^*, y, \succ)$.*

PROOF. In order to prove the lemma, it suffices to consider only the case in which $\succ'$ is obtained from $\succ$ by pushing alternative y one position upward in the preference of a single agent $j \in N$ (i.e., $r_j(y, \succ) = r_j(y, \succ') + 1$).

Consider linear program (6) used in the definition of $pe(z^*, y, \succ')$, and let $\mathbf{x}$ be an optimal solution. We will show how to transform this solution to a feasible solution $\bar{\mathbf{x}}$ for linear program (6) used in the definition of $pe(z^*, y, \succ)$ in such a way that the objective of the latter is not larger than that of the former linear program. Next, we use a simplified notation that omits z^* whenever it is clear from context. In particular, we use $\mathcal{S}^{\succ i}$ instead of $\mathcal{S}^{\succ i}(z^*)$ and $S_k^{\succ i}$ instead of $S_k^{\succ i}(z^*)$.

We define the solution $\bar{\mathbf{x}}$ as follows. First, we set $\bar{\mathbf{x}}(S_k^{\succ i}) = \mathbf{x}(S_k^{\succ' i})$ for each agent $i \in N \setminus \{j\}$ and $k = 1, 2, \ldots, r_j(z^*, \succ) - 1$. Then, we distinguish between three cases depending on the type of agent j.

We say that agent j is of type BB if y is ranked below z^* in both profiles $\succ$ and $\succ'$. In this case, observe that $r_j(z^*, \succ) = r_j(z^*, \succ')$ and $S_k^{\succ j} = S_k^{\succ' j}$ for $k = 1, \ldots, r_j(z^*, \succ) - 1$. We set $\bar{\mathbf{x}}(S_k^{\succ j}) = \mathbf{x}(S_k^{\succ' j})$, for $k = 1, 2, \ldots, r_j(z^*, \succ) - 1$.

We say that agent j is of type AA if y is ranked above z^* in both profiles $\succ$ and $\succ'$. Again, we have $r_j(z^*, \succ) = r_j(z^*, \succ')$. In this case, we set

$$-\bar{\mathbf{x}}(S_k^{\succ j}) = \mathbf{x}(S_k^{\succ' j}) \text{ for } k = 1, \ldots, r_j(z^*, \succ) - r_j(y, \succ) - 1;$$
$$-\bar{\mathbf{x}}(S_{r_j(z^*, \succ) - r_j(y, \succ)}^{\succ j}) = 0;$$
$$-\bar{\mathbf{x}}(S_{r_j(z^*, \succ) - r_j(y, \succ) + 1}^{\succ j}) = \mathbf{x}(S_{r_j(z^*, \succ) - r_j(y, \succ)}^{\succ' j}) + \mathbf{x}(S_{r_j(z^*, \succ) - r_j(y, \succ) + 1}^{\succ' j});$$
$$-\bar{\mathbf{x}}(S_k^{\succ j}) = \mathbf{x}(S_k^{\succ' j}) \text{ for } k = r_j(z^*, \succ) - r_j(y, \succ) + 2, \ldots, r_j(z^*, \succ) - 1.$$

In addition, we say that agent j is of type AB if y is ranked above z^* in $\succ'$ but below z^* in $\succ$. Note that $r_j(z^*, \succ') = r_j(z^*, \succ) + 1$ in this case. We set $\bar{\mathbf{x}}(S_k^{\succ j}) = \mathbf{x}(S_{k+1}^{\succ' j})$ for $k = 1, \ldots, r_j(z^*, \succ) - 1$.

Now consider linear program (6) used in the definition of $pe(z^*, y, \succ)$. Clearly, since the solution $\mathbf{x}$ is nonnegative, the solution $\bar{\mathbf{x}}$ is nonnegative as well. The previous

definitions guarantee that

$$\sum_{S \in \mathcal{S}^{\succ i}} \bar{\mathbf{x}}(S) \leq \sum_{S \in \mathcal{S}^{\succ'_i}} \mathbf{x}(S)$$

for each agent $i \in N$ (actually, the two sums are equal when $i \in N \setminus \{j\}$, or $i = j$ and j is of type BB or AA). Hence, the second set of constraints is satisfied since $\mathbf{x}$ satisfies the second set of constraints in linear program (6) used in the definition of pe($z^*, y, \succ'$).

Furthermore, observe that defc($z^*, z, \succ$) = defc($z^*, z, \succ'$) for each alternative $z \in A \setminus \{y, z^*\}$ since the relative ranking of z and z^* in the preference of each agent is the same in both profiles $\succ$ and $\succ'$. In addition, the definition of solution $\bar{\mathbf{x}}$ guarantees that

$$\sum_{S \in \mathcal{S}^{\succ i} : z \in S} \bar{\mathbf{x}}(S) = \sum_{S \in \mathcal{S}^{\succ'_i} : z \in S} \mathbf{x}(S)$$

for each agent $i \in N$ and each alternative $z \in A \setminus \{y, z^*\}$. Hence, the first set of constraints is satisfied for each alternative $z \in A \setminus \{y, z^*\}$ since the solution $\mathbf{x}$ satisfies the first set of constraints of linear program (6) in the definition of pe($z^*, y, \succ'$).

Concerning alternative y, we first consider the cases where agent j is of type BB or AA. In both cases, the relative ranking of y and z^* in each agent is the same in both profiles $\succ$ and $\succ'$; hence, defc($z^*, y, \succ$) = defc($z^*, y, \succ'$). Furthermore, in both cases, the definition of the solution $\bar{\mathbf{x}}$ guarantees that

$$\sum_{S \in \mathcal{S}^{\succ i} : y \in S} \bar{\mathbf{x}}(S) \geq \sum_{S \in \mathcal{S}^{\succ'_i} : y \in S} \mathbf{x}(S)$$

for each agent $i \in N$ (actually, the two sums are equal when $i \in N \setminus \{j\}$, or $i = j$ and j is of type BB). Hence, in both cases, the first constraint for alternative y is satisfied since $\mathbf{x}$ satisfies the corresponding constraint of linear program (6) in the definition of pe($z^*, y, \succ'$).

If agent j is of type AB, we further distinguish between two cases. If defc($z^*, y, \succ'$) = 0, then, clearly,

$$\sum_{i \in N} \sum_{S \in \mathcal{S}^{\succ i} : y \in S} \bar{\mathbf{x}}(S) \geq 0 = \text{defc}(z^*, y, \succ).$$

Otherwise (if defc($z^*, y, \succ'$) ≥ 1), observe that defc($z^*, y, \succ$) = defc($z^*, y, \succ'$) $- 1$. Since no set in $\mathcal{S}^{\succ i}$ contains alternative y, we have

$$\sum_{i \in N} \sum_{S \in \mathcal{S}^{\succ i} : y \in S} \bar{\mathbf{x}}(S) = \sum_{i \in N \setminus \{j\}} \sum_{S \in \mathcal{S}^{\succ i} : y \in S} \bar{\mathbf{x}}(S)$$

$$= \sum_{i \in N \setminus \{j\}} \sum_{S \in \mathcal{S}^{\succ'_i} : y \in S} \mathbf{x}(S)$$

$$= \sum_{i \in N} \sum_{S \in \mathcal{S}^{\succ'_i} : y \in S} \mathbf{x}(S) - \sum_{S \in \mathcal{S}^{\succ'_j} : y \in S} \mathbf{x}(S)$$

$$\geq \sum_{i \in N} \sum_{S \in \mathcal{S}^{\succ'_i} : y \in S} \mathbf{x}(S) - 1$$

$$\geq \text{defc}(z^*, y, \succ') - 1$$

$$= \text{defc}(z^*, y \succ),$$

which implies that the solution $\bar{\mathbf{x}}$ satisfies the first constraint of linear program (6) used in the definition of pe($z^*, y, \succ$) for alternative y. Note that the two last inequalities

follow since the solution $\mathbf{x}$ satisfies the constraints in linear program (6) used in the definition of $\text{pe}(z^*, y, \succ')$.

So far, we have shown that $\bar{\mathbf{x}}$ is a feasible solution for linear program (6) used in the definition of $\text{pe}(z^*, y, \succ)$. It remains to upper bound its objective value by the optimal objective value of linear program (6) used in the definition of $\text{pe}(z^*, y, \succ')$. In order to simplify the calculations, for each agent $i \in N$, we define

$$\mathbf{O}_i = \sum_{k=1}^{r_i(z^*,\succ)-1} k \cdot \bar{\mathbf{x}}\big(S_k^{\succ_i}\big) + \sum_{k=1}^{r_i(z^*,\succ)-r_i(y,\succ)-1} \bar{\mathbf{x}}\big(S_k^{\succ_i}\big)$$

if $y \succ_i z^*$ and

$$\mathbf{O}_i = \sum_{k=1}^{r_i(z^*,\succ)-1} k \cdot \bar{\mathbf{x}}\big(S_k^{\succ_i}\big)$$

otherwise. The quantity $\mathbf{O}'_i$ is defined analogously by replacing $\succ$ with $\succ'$ and $\bar{\mathbf{x}}$ with $\mathbf{x}$. Observe that the objective values of linear program (6) used in the definition of $\text{pe}(z^*, y, \succ)$ and $\text{pe}(z^*, y, \succ')$ for the solutions $\bar{\mathbf{x}}$ and $\mathbf{x}$ are $\sum_{i \in N} \mathbf{O}_i$ and $\sum_{i \in N} \mathbf{O}'_i$, respectively. Additionally, the definition of solution $\bar{\mathbf{x}}$ yields that $\mathbf{O}_i = \mathbf{O}'_i$ when $i \in N \setminus \{j\}$, or $i = j$ and j is of type BB. In order to complete the proof, it suffices to show that $\mathbf{O}_j \leq \mathbf{O}'_j$ when j is of type AA or AB.

First, consider the case where agent j is of type AA. By the definition of $\mathbf{O}_j$ and by just rearranging the sums, we have

$$\mathbf{O}_j = \sum_{k=1}^{r_j(z^*,\succ)-1} k \cdot \bar{\mathbf{x}}\big(S_k^{\succ_j}\big) + \sum_{k=1}^{r_j(z^*,\succ)-r_j(y,\succ)-1} \bar{\mathbf{x}}\big(S_k^{\succ_j}\big)$$

$$= \sum_{k=1}^{r_j(z^*,\succ)-r_j(y,\succ)-1} (k+1) \cdot \bar{\mathbf{x}}\big(S_k^{\succ_j}\big) + (r_j(z^*,\succ)-r_j(y,\succ)) \cdot \bar{\mathbf{x}}\big(S_{r_j(z^*,\succ)-r_j(y,\succ)}^{\succ_j}\big)$$

$$+ (r_j(z^*,\succ)-r_j(y,\succ)+1) \cdot \bar{\mathbf{x}}\big(S_{r_j(z^*,\succ)-r_j(y,\succ)+1}^{\succ_j}\big) + \sum_{k=r_j(z^*,\succ)-r_j(y,\succ)+2}^{r_j(z^*,\succ)-1} k \cdot \bar{\mathbf{x}}\big(S_k^{\succ_j}\big)$$

Using this equality, the definition of the variables $\bar{\mathbf{x}}$ on the sets of agent j, equalities $r_j(z^*, \succ) = r_j(z^*, \succ')$ and $r_j(y, \succ) = r_j(y, \succ') + 1$, and the definition of $\mathbf{O}'_j$, we have

$$\mathbf{O}_j = \sum_{k=1}^{r_j(z^*,\succ)-r_j(y,\succ)-1} (k+1) \cdot \mathbf{x}\big(S_k^{\succ'_j}\big) + (r_j(z^*,\succ)-r_j(y,\succ)) \cdot 0$$

$$+ (r_j(z^*,\succ)-r_j(y,\succ)+1) \cdot \Big(\mathbf{x}\big(S_{r_j(z^*,\succ)-r_j(y,\succ)}^{\succ'_j}\big) + \mathbf{x}\big(S_{r_j(z^*,\succ)-r_j(y,\succ)+1}^{\succ'_j}\big)\Big)$$

$$+ \sum_{k=r_j(z^*,\succ)-r_j(y,\succ)+2}^{r_j(z^*,\succ)-1} k \cdot \mathbf{x}\big(S_k^{\succ'_j}\big)$$

$$= \sum_{k=1}^{r_j(z^*,\succ')-r_j(y,\succ')-2} (k+1) \cdot \mathbf{x}\big(S_k^{\succ'_j}\big)$$

$$+ \left(r_j(z^*, \succ') - r_j(y, \succ')\right) \cdot \left(\mathbf{x}\big(S^{\succ'_j}_{r_j(z^*,\succ')-r_j(y,\succ')-1}\big) + \mathbf{x}\big(S^{\succ'_j}_{r_j(z^*,\succ')-r_j(y,\succ')}\big)\right)$$

$$+ \sum_{k=r_j(z^*,\succ')-r_j(y,\succ')+1}^{r_j(z^*,\succ')-1} k \cdot \mathbf{x}\big(S^{\succ'_j}_k\big)$$

$$= \sum_{k=1}^{r_j(z^*,\succ')-1} k \cdot \mathbf{x}\big(S^{\succ'_j}_k\big) + \sum_{k=1}^{r_j(z^*,\succ')-r_j(y,\succ')-1} \mathbf{x}\big(S^{\succ'_j}_k\big)$$

$$= \mathbf{O}'_j.$$

Now, consider the case where agent j is of type AB, and recall that $z^* \succ_j y$ and $y \succ'_j z^*$. Using the definition of $\mathbf{O}_j$, the definition of the variables $\bar{\mathbf{x}}$ on the sets of agent j, equality $r_j(z^*, \succ) = r_j(z^*, \succ') - 1$, and the definition of $\mathbf{O}'_j$, we have

$$\mathbf{O}_j = \sum_{k=1}^{r_j(z^*,\succ)-1} k \cdot \bar{\mathbf{x}}\big(S^{\succ_j}_k\big)$$

$$= \sum_{k=1}^{r_j(z^*,\succ)-1} k \cdot \mathbf{x}\big(S^{\succ'_j}_{k+1}\big)$$

$$= \sum_{k=2}^{r_j(z^*,\succ')-1} (k-1) \cdot \mathbf{x}\big(S^{\succ'_j}_k\big)$$

$$\le \sum_{k=1}^{r_j(z^*,\succ')-1} k \cdot \mathbf{x}\big(S^{\succ'_j}_k\big) + \sum_{k=1}^{r_j(z^*,\succ')-r_j(y,\succ')-1} \mathbf{x}\big(S^{\succ'_j}_k\big)$$

$$= \mathbf{O}'_j.$$

This completes the proof of the lemma.   □

LEMMA 3.7. *Let $y, z^* \in A$ be different alternatives, and let $\succ \in \mathcal{L}^n$ be a profile. Then, $sc_R(z^*, \succ) \le pe(z^*, y, \succ) \le 2 \cdot sc_R(z^*, \succ)$.*

PROOF. The lemma follows directly from the observations that the objective of linear program (6) is lower bounded by the objective of the linear programming relaxation of (5) and also upper bounded by the latter multiplied by 2.   □

LEMMA 3.8. *The voting rule Q is monotonic.*

PROOF. Let $y \in A$, and consider a profile $\succ \in \mathcal{L}^n$ such that $y \in W(\succ)$. We will show that $y \in W(\succ')$ for each profile $\succ'$, which is a y-improvement of $\succ$. This is clearly true if y is a winning alternative according to R at $\succ'$. If this is not the case, we distinguish between two cases:

Case 1. y is a winning alternative according to R at $\succ$. Then, for every alternative $z \in A\backslash\{y\}$, we have

$$\mathrm{pe}(z, y, \succ') \ge \mathrm{pe}(z, y, \succ) \ge \mathrm{sc}_R(z, \succ) \ge \mathrm{sc}_R(y, \succ)$$
$$\ge \mathrm{sc}_R(y, \succ');$$

therefore, $y \in W(\succ')$. The first inequality follows by Lemma 3.6, the second follows by Lemma 3.7, the third is true since y is the winner under R in profile $\succ$, and the fourth follows from Observation 3.5.

Case 2. y is not a winning alternative according to R at $\succ$. Since $y \in W(\succ)$, it must hold that $\mathrm{pe}(z, y, \succ) \geq \mathrm{sc}_R(y, \succ)$ for every alternative $z \in A \backslash \{y\}$. We therefore have that

$$\mathrm{pe}(z, y, \succ') \geq \mathrm{pe}(z, y, \succ) \geq \mathrm{sc}_R(y, \succ) \geq \mathrm{sc}_R(y, \succ')$$

for every alternative $z \in A \backslash \{y\}$ and, hence, $y \in W(\succ')$. The first inequality follows by Lemma 3.6, and the third is implied by Observation 3.5.   □

The following lemma provides the desired bound on the approximation ratio.

LEMMA 3.9.   *Q is a Dodgson approximation with an approximation ratio of $2H_{m-1}$.*

PROOF.   We have to show that

$$\mathrm{sc}_D(y, \succ) \leq \mathrm{sc}_Q(y, \succ) \leq 2H_{m-1} \cdot \mathrm{sc}_D(y, \succ)$$

for any alternative $y \in A$ and profile $\succ \in \mathcal{L}^n$. This is clearly the case if y is a winning alternative according to R or $y \notin W(\succ)$, since $\mathrm{sc}_Q(y, \succ) = 2 \cdot \mathrm{sc}_R(y, \succ)$ (by the definition of Q), and

$$\mathrm{sc}_D(y, \succ) \leq \mathrm{sc}_R(y, \succ) \leq H_{m-1} \cdot \mathrm{sc}_D(y, \succ),$$

since R is a Dodgson approximation with approximation ratio H_{m-1}.

Now assume that y is not a winning alternative according to R but that it belongs to $W(\succ)$. Let z be a winning alternative according to R. Since $y \in W(\succ)$, it must be the case that $\mathrm{pe}(z, y, \succ) \geq \mathrm{sc}_R(y, \succ)$. So, using in addition the definition of Q, Lemma 3.7, and the fact that R is a Dodgson approximation, we have

$$\mathrm{sc}_Q(y, \succ) = 2 \cdot \mathrm{sc}_R(z, \succ) \geq \mathrm{pe}(z, y, \succ) \geq \mathrm{sc}_R(y, \succ)$$
$$\geq \mathrm{sc}_D(y, \succ).$$

Furthermore, using the definition of Q, the fact that z is a winning alternative under R, and the approximation bound of R, we have

$$\mathrm{sc}_Q(y, \succ) = 2 \cdot \mathrm{sc}_R(z, \succ) \leq 2 \cdot \mathrm{sc}_R(y, \succ)$$
$$\leq 2H_{m-1} \cdot \mathrm{sc}_D(y, \succ).   □$$

We summarize the earlier discussion with the following statement.

THEOREM 3.10.   *Q is a monotonic polynomial-time Dodgson approximation with an approximation ratio of $2H_{m-1}$.*

4. HOMOGENEITY

In this section, we present our results on homogeneous Dodgson approximations. A voting rule is *homogeneous* if duplicating the electorate—that is, duplicating the preference profile—does not change the outcome of the election. An example (due to Brandt [2009]) that demonstrates that Dodgson's rule fails homogeneity can be found in Table III. The intuition is that if alternatives x and y are tied in a pairwise election, the deficit of x against y does not increase by duplicating the profile, whereas if x strictly loses to y in a pairwise election, then the deficit scales with the number of copies.

4.1. The Simplified Dodgson Rule

Tideman [2006, pp. 199–201] defines the following simplified Dodgson rule and proves that it is monotonic and homogeneous. Consider a profile $\succ \in \mathcal{L}^n$. If an alternative is a Condorcet winner, then this alternative is the sole winner. Otherwise, the simplified Dodgson rule assigns a score to each alternative, and the alternative with the minimum

Table III. An Example Demonstrating That Dodgson's Rule Does Not
Satisfy Homogeneity

×2	×2	×2	×2	×2	×1	×1
d	b	c	d	a	a	d
c	c	a	b	b	d	a
a	a	b	c	c	b	b
b	d	d	a	d	c	c

A column headed by $\times k$ represents k identical agents. In the pro-
file, a is the Dodgson winner with a score of 3. By duplicating the
electorate three times, we obtain a profile in which the winner is
d with a score of 6.

score wins. According to the simplified Dodgson rule, the score of an alternative x is

$$\mathrm{sc}_{\mathrm{Td}}(x, \succ) = \sum_{y \in A \setminus \{x\}} \max\{0, n - 2 \cdot |\{i \in N : x \succ_i y\}|\}.$$

Observe that $\mathrm{sc}_{\mathrm{Td}}(x, \succ)$ can be smaller than the Dodgson score of x, and hence this definition does not correspond to a Dodgson approximation. For example, in profiles with an even number of agents, $\mathrm{sc}_{\mathrm{Td}}(x, \succ)$ is 0 when x is tied against some alternatives and beats the rest. Therefore, we present an alternative definition of the simplified Dodgson rule as a Dodgson approximation by scaling the original definition. If an alternative x is a Condorcet winner, then it has score $\mathrm{sc}_{\mathrm{Td}'}(x, \succ) = 0$. Otherwise,

$$\mathrm{sc}_{\mathrm{Td}'}(x, \succ) = m \cdot \mathrm{sc}_{\mathrm{Td}}(x, \succ) + m(\log m + 1).$$

It is clear that this alternative definition is equivalent to the original one of the simplified Dodgson rule in the sense that it elects the same set of alternatives. It is also clear that $\mathrm{sc}_{\mathrm{Td}}(x, \succ)$ can be computed in polynomial time, and, as mentioned earlier, Td is known to be monotonic and homogeneous. Hence, in order to prove the following theorem, it is sufficient to prove that Td$'$ is a Dodgson approximation and to bound its approximation ratio.

THEOREM 4.1. *Td$'$ is a monotonic, homogeneous, polynomial-time Dodgson approximation with an approximation ratio of $\mathcal{O}(m \log m)$.*

PROOF. We will show that given any profile $\succ \in \mathcal{L}^N$ and alternative $x \in A$, it holds that $\mathrm{sc}_D(x, \succ) \leq \mathrm{sc}_{\mathrm{Td}'}(x, \succ) \leq m(\log m + 3) \cdot \mathrm{sc}_D(x, \succ)$. We will consider the case in which x is not a Condorcet winner, since otherwise the inequalities clearly hold.

In order to show that Td$'$ is a Dodgson approximation, we distinguish between two cases.

If the number of agents is odd, then $\mathrm{sc}_{\mathrm{Td}}(x, \succ)$ can be expressed in terms of the deficits of alternative x against the other alternatives as follows:

$$\mathrm{sc}_{\mathrm{Td}}(x, \succ) = \sum_{y \in A \setminus \{x\}} \max\{0, 2 \cdot \mathrm{defc}(x, y, \succ) - 1\}.$$

Observe that each term of the previous sum is nonzero only when $\mathrm{defc}(x, y, \succ) > 0$. Since $2 \cdot \mathrm{defc}(x, y, \succ) - 1 \geq \mathrm{defc}(x, y, \succ)$ in this case, we have that

$$\begin{aligned} \mathrm{sc}_{\mathrm{Td}'}(x, \succ) &= m \cdot \mathrm{sc}_{\mathrm{Td}}(x, \succ) + m(\log m + 1) \\ &> m \cdot \mathrm{sc}_{\mathrm{Td}}(x, \succ) \\ &\geq m \sum_{y \in A \setminus \{x\}} \mathrm{defc}(x, y, \succ) \\ &\geq \mathrm{sc}_D(x, \succ). \end{aligned}$$

If the number of agents is even, then $\mathrm{sc}_{\mathrm{Td}}(x, \succ)$ can be expressed in terms of the deficits of alternative x against the other alternatives as follows:

$$\mathrm{sc}_{\mathrm{Td}}(x, \succ) = \sum_{y \in A \setminus \{x\}} \max\{0, 2 \cdot \mathrm{defc}(x, y, \succ) - 2\}. \tag{7}$$

Let

$$S_x = \{y \in A \setminus \{x\} : \mathrm{defc}(x, y, \succ) \geq 2\}$$

and

$$T_x = \{y \in A \setminus \{x\} : \mathrm{defc}(x, y, \succ) = 1\}.$$

We will now prove that it is sufficient to push x to the top of the preferences of at most $\log m + 1$ agents in order to cover the deficits against the alternatives in T_x. Since the number of agents n is even, the fact that $\mathrm{defc}(x, y, \succ) = 1$ means that exactly $n/2$ agents rank x above y. For every $i \in N$, let $A_i = \{y \in T_x : y \succ_i x\}$. By the pigeonhole principle, there exists an agent i_1 such that $|A_{i_1}| \geq |T_x|/2$; we add i_1 to our cover and denote $X_1 = T_x \setminus A_{i_1}$. Next, there must exist an agent i_2 such that $|A_{i_2} \cap X_1| \geq \frac{|X_1|}{2}$. We add i_2 to our cover and define $X_2 = X_1 \setminus A_{i_2}$. Continuing inductively in this way, we cover all of the alternatives in T_x after $\log |T_x| + 1 \leq \log m + 1$ steps.

Moreover, in order to make x beat the alternatives in S_x, it suffices to push it to the top of the preferences of at most $\sum_{y \in S_x} \mathrm{defc}(x, y, \succ)$ agents. Hence, the Dodgson score of x is

$$\mathrm{sc}_D(x, \succ) \leq m \sum_{y \in S_x} \mathrm{defc}(x, y, \succ) + m(\log m + 1).$$

Now observe that each term of the sum in the equivalent definition of $\mathrm{sc}_{\mathrm{Td}}(x, \succ)$ in equality (7) is nonzero only when $\mathrm{defc}(x, y, \succ) \geq 2$ (i.e., when $y \in S_x$). Since $2 \cdot \mathrm{defc}(x, y, \succ) - 2 \geq \mathrm{defc}(x, y, \succ)$ in this case, we have that

$$\mathrm{sc}_{\mathrm{Td}'}(x, \succ) = m \cdot \mathrm{sc}_{\mathrm{Td}}(x, \succ) + m(\log m + 1)$$
$$\geq m \sum_{y \in S_x} \mathrm{defc}(x, y, \succ) + m(\log m + 1)$$
$$\geq \mathrm{sc}_D(x, \succ).$$

We have completed the proof that Td$'$ is a Dodgson approximation. In order to prove the bound on the approximation ratio, in both cases we have

$$\mathrm{sc}_{\mathrm{Td}'}(x, \succ) = m \cdot \mathrm{sc}_{\mathrm{Td}}(x, \succ) + m(\log m + 1)$$
$$\leq 2m \sum_{y \in A \setminus \{x\}} \mathrm{defc}(x, y, \succ) + m(\log m + 1)$$
$$\leq m(\log m + 3) \cdot \mathrm{sc}_D(x, \succ).$$

The last inequality holds since $\mathrm{sc}_D(x, \succ)$ is lower bounded by both $\sum_{y \in A \setminus \{x\}} \mathrm{defc}(x, y, \succ)$ and 1. □

4.2. Lower Bound

We next show that Td$'$ is the asymptotically optimal homogeneous Dodgson approximation by proving a matching lower bound on the approximation ratio of homogeneous Dodgson approximations. The lower bound is not based on any complexity assumptions and holds for exponential-time Dodgson approximations as well. This is quite striking, since, as stated in Theorem 4.1, Td$'$ is also monotonic and polynomial time.

THEOREM 4.2. *Any homogeneous Dodgson approximation has an approximation ratio of at least $\Omega(m \log m)$.*

The proof is based on the construction of a preference profile with an alternative $b \in A$ that defeats some of the alternatives in pairwise elections and is tied against many others. Hence, it has a high Dodgson score. On the other hand, there is a second alternative that has a Dodgson score of two, simply because it has a deficit of two against another alternative. In order to obtain a good approximation ratio, the algorithm must not select b in this profile. However, when the profile is replicated, the Dodgson score of b does not increase: it is still tied against the same alternatives. In contrast, the Dodgson score of the other alternatives scales with the number of copies. By homogeneity, we cannot select b in the replicated profile, which yields the lower bound.

We can think of an agent as the subset of alternatives that are ranked above b. If b is tied against an alternative, then that alternative is a member of exactly half the subsets. The argument used in the proof of Theorem 4.1 implies that there is always a cover of logarithmic size; the proof of Theorem 4.2 establishes that this bound is tight. Indeed, the combinatorial core of the theorem's proof is the construction of a set cover instance with the following properties: each element of the ground set appears in roughly half the subsets, but every cover requires a logarithmic number of subsets (see Claim 4.3). This construction is based on a lower bound for the worst-case integrality gap of a natural linear programming relaxation for Set Cover (see, e.g., Vazirani [2001, pp. 111–112]).

PROOF. (*of Theorem 4.2*) Given an integer $r \geq 3$, we construct a preference profile $\succ$ with $n = 2^r$ agents and $m = 2^{r+1} + 1$ alternatives. There is a set $X = \{x_1, x_2, \ldots, x_{2^r - 1}\}$ with $2^r - 1$ alternatives, two sets Y and Z with 2^{r-1} alternatives each, and two additional alternatives a and b.

For $i = 1, \ldots, 2^r - 1$, denote by X_i the set of alternatives x_j such that the inner product of the binary vectors corresponding to the binary representations of i and j equals 1 modulo 2. Denote by $\mathcal{X}$ the collection of all sets X_i for $i = 1, \ldots, 2^r - 1$. □

CLAIM 4.3. *The sets of $\mathcal{X}$ have the following properties:*

(1) *Each alternative $x \in X$ belongs to 2^{r-1} different sets of $\mathcal{X}$.*
(2) *Each set of $\mathcal{X}$ contains exactly 2^{r-1} alternatives.*
(3) *There are r different sets in $\mathcal{X}$ whose union contains all alternatives in X.*
(4) *For each subcollection of at most $r - 1$ sets in $\mathcal{X}$, there exists an alternative of X that does not belong to their union.*

PROOF. Properties 1 and 2 follow easily by the definition of the sets in $\mathcal{X}$.

In order to establish property 3, it suffices to consider the r sets X_{2^i} for $i = 0, \ldots, r - 1$—that is, the ones whose binary representation has just one 1 in the $(i + 1)$-th bit position.

Turning to property 4, we consider a binary r-vector $\mathbf{z} = \langle z_1, z_2, \ldots, z_r \rangle \in \{0, 1\}^r$. Now, consider the set X_k, and let $\langle b_1(k), b_2(k), \ldots, b_r(k) \rangle$ be the r-vector corresponding to the binary representation of k. It holds that the equation $\sum_{j=1}^{r} b_j(k) \cdot z_j = 0 \mod 2$ is true if and only if the alternative such that $\mathbf{z}$ is the binary representation of its index is not contained in set X_k. Since any homogeneous system of less than r linear equations modulo 2 with r unknowns has a nontrivial (i.e., nonzero) solution, it follows that for any subcollection of less than r sets in $\mathcal{X}$, there exists an alternative in X that is not contained in their union. □

Table IV. The Preference Profile $\succ$ Used in the Proof of Theorem 4.2

0	1	2	...	2^{r-1}	$2^{r-1}+1$	...	$2^r - 1$
b	b	a	...	a	$X_{2^{r-1}+1}$	...	X_{2^r-1}
a	a	X_2	...	$X_{2^{r-1}}$	Z	...	Z
Y	X_1	Y	...	Y	b	...	b
Z	Z	b	...	b	Y	...	Y
X	Y	Z	...	Z	$X \setminus X_{2^{r-1}+1}$	...	$X \setminus X_{2^r-1}$
	$X \setminus X_1$	$X \setminus X_2$	...	$X \setminus X_{2^{r-1}}$	a	...	a

We construct the preference profile $\succ$ as follows (see Table IV):

—Agent 0 ranks b first, then a, then the alternatives in Y (in arbitrary order), then the alternatives in Z (also in arbitrary order), and then the alternatives of X (in arbitrary order).
—Agent 1 ranks b first, then a, then the alternatives in X_1 (in arbitrary order), then the alternatives in Z, then the alternatives in Y, and then the alternatives in $X \setminus X_1$ (in arbitrary order).
—For $i = 2, \ldots, 2^{r-1}$, agent i ranks a first, then the alternatives of X_i (in arbitrary order), then the alternatives in Y, then b, then the alternatives in Z, and then the alternatives in $X \setminus X_i$ (in arbitrary order).
—For $i = 2^{r-1}+1, \ldots, 2^r-1$, agent i ranks the alternatives in X_i (in arbitrary order) first, then the alternatives of Z, then b, then the alternatives in Y, then the alternatives in $X \setminus X_i$ (in arbitrary order), and then a.

The next four claims state important properties of the profile $\succ$.

CLAIM 4.4. *The Dodgson score of a is at most 2.*

PROOF. After swapping a and b in the rankings of agents 0 and 1, alternative a is ranked first by a majority of agents; hence, it clearly becomes the Condorcet winner. □

CLAIM 4.5. *Alternative b has a deficit of at most 1 against any other alternative.*

PROOF. By property 1 of Claim 4.3 and the construction of the profile, we have that b is ranked below any alternative x_i of $X \setminus X_1$ by 2^{r-1} agents—that is, b is tied with these alternatives in pairwise elections. It follows that $\mathrm{defc}(b, x_i, \succ) = 1$. In addition, b is ranked above any alternative in $X_1 \cup Y \cup Z \cup \{a\}$ by $2^{r-1} + 1$ agents—that is, $\mathrm{defc}(b, x, \succ) = 0$ for any alternative $x \in X_1 \cup Y \cup Z \cup \{a\}$. □

CLAIM 4.6. $r2^{r-2} \le sc_D(b, \succ) \le (r-1)2^r$.

PROOF. By property 4 of Claim 4.3, alternative b has to be pushed upward in the rankings of at least $r-1$ among the agents $2, \ldots, 2^r-1$ in order to eliminate its deficit against the $2^{r-1} - 1$ alternatives of $X \setminus X_1$. This requires at least $(r-1)2^{r-1}$ swaps in order to push above the alternatives of Y (in the case of agents $2, \ldots, 2^{r-1}$) or Z (in the case of agents $2^{r-1} + 1, \ldots, 2^r-1$) in the rankings of $r-1$ agents, plus at least $2^{r-1} - 1$ additional swaps in order to defeat each of the alternatives in $X \setminus X_1$; the total is $r2^{r-1} - 1 \ge r2^{r-2}$ swaps.
The upper bound follows by Properties 2 and 3 of Claim 4.3, since b becomes a Condorcet winner by pushing it above the alternatives of X in the rankings of at most $r-1$ additional agents, and using at most $|X_i| + |Y| = 2^r$ or $|X_i| + |Z| = 2^r$ swaps per agent. □

CLAIM 4.7. *Any alternative besides b has a deficit of at least 2 against some other alternative.*

PROOF. Alternative a is ranked higher than alternative b by $2^{r-1} - 1$ agents; therefore, it holds that $\mathrm{defc}(a, b, \succ) = 2$. Moreover, alternative a is ranked higher than the alternatives in X, Y, and Z by $2^{r-1} + 1$ agents. So, $\mathrm{defc}(x, a, \succ) = 2$ for any alternative $x \in X \cup Y \cup Z$. □

Now, consider a homogeneous Dodgson approximation H. If it selects b as a winner of profile $\succ$, then, using Claims 4.4 and 4.6, and since H is a Dodgson approximation, we have

$$\mathrm{sc}_H(a, \succ) \geq \mathrm{sc}_H(b, \succ) \geq \mathrm{sc}_D(b, \succ) \geq r2^{r-2} \geq r2^{r-3}\mathrm{sc}_D(a, \succ).$$

Hence, H has an approximation ratio of at least

$$r2^{r-3} = \frac{m-1}{16} \cdot \log \frac{m-1}{2} = \Omega(m \log m).$$

Assume otherwise that the winner under H is some alternative $x \in A \setminus \{b\}$. Consider the preference profile $\succ'$ obtained by making $r(r-1)2^{2r-3}$ copies of the profile $\succ$. By Claim 4.5, we have that b has a deficit of at most 1 against any other alternative in the new profile as well; its Dodgson score in the new profile is in the range defined in Claim 4.6—that is, $\mathrm{sc}_D(b, \succ') \leq (r-1)2^r$. By the definition of the deficit and Claim 4.7, we have that alternative x has a deficit of at least $r(r-1)2^{2r-3}$ against some other alternative, and hence its Dodgson score in the new profile is $\mathrm{sc}_H(x, \succ') \geq r(r-1)2^{2r-3}$.

By the homogeneity property, x should be a winner under H in the profile $\succ'$. Then,

$$\begin{aligned}\mathrm{sc}_H(b, \succ') &\geq \mathrm{sc}_H(x, \succ') \geq \mathrm{sc}_D(x, \succ') \\ &\geq r(r-1)2^{2r-3} \geq r2^{r-3}\mathrm{sc}_D(b, \succ').\end{aligned}$$

Therefore, H has approximation ratio $r2^{r-3} = \Omega(m \log m)$ in this case as well.

5. ADDITIONAL PROPERTIES

In this section, we present our results with respect to several additional social choice properties that are not satisfied by Dodgson's rule. In general, our lower bounds with respect to these properties are at least linear in n, the number of agents. Since n is almost always large, these results should strictly be interpreted as impossibility results—that is, normally an upper bound of $\mathcal{O}(n)$ is not useful. We now (informally) formulate the five properties in question; for more formal definitions, the reader is referred to Tideman [2006].

We say that a voting rule satisfies *combinativity* if, given two preference profiles where the rule elects the same winning set, the rule would also elect this winning set under the profile obtained from appending one of the original preference profiles to the other. Note that combinativity implies homogeneity.

A *dominating set* is a nonempty set of alternatives such that each alternative in the set beats every alternative outside the set in pairwise elections. The *Smith set* is the unique inclusion-minimal dominating set. A voting rule satisfies *Smith consistency* if winners under the rule are always contained in the Smith set.

We say that a voting rule satisfies *mutual majority consistency* if, given a preference profile where more than half the agents rank a subset of alternatives $X \subseteq A$ above $A \setminus X$, only alternatives from X can be elected. A voting rule satisfies *invariant loss consistency* (or *Condorcet-loser consistency*) if an alternative that loses to every other alternative in pairwise elections cannot be elected.

Independence of clones was introduced by Tideman [1987]; see also the paper by Schulze [2003]. For ease of exposition, we use a slightly weaker definition previously employed by Brandt [2009]; since we are proving a lower bound, a weaker definition only strengthens the bound. Given a preference profile, two alternatives $x, y \in A$ are

Table V. The Preference Profiles $\succ$ and $\succ'$ Used in the Proof of Theorem 5.2

0	...	$\lambda-3$	$\lambda-2$	...	$2\lambda-3$	$2\lambda-2$	...	$3\lambda-3$	$3\lambda-2$	...	$4\lambda-3$
a	...	a	b	...	b	$\Psi_{2\lambda-2}$	...	$\Psi_{3\lambda-3}$	a	...	a
Ψ_0	...	$\Psi_{\lambda-3}$	a	...	a	b	...	b	$\Psi_{3\lambda-2}$	...	$\Psi_{4\lambda-3}$
b	...	b	$\Psi_{\lambda-2}$	...	$\Psi_{2\lambda-3}$	a	...	a	b	...	b

0	1	2	...	$\lambda-1$	λ	...	$2\lambda-2$
a	a	a	...	a	b	...	b
b	b	b	...	b	Ψ_λ	...	$\Psi_{2\lambda-2}$
Ψ_0	Ψ_1	Ψ_2	...	$\Psi_{\lambda-1}$	a	...	a

The subscripts of the sets Ψ_i are modulo k.

considered *clones* if they are adjacent in the rankings of all agents—that is, their order with respect to every alternative in $A \setminus \{x, y\}$ is identical everywhere. A voting rule is *independent of clones* if a losing alternative cannot be made a winning alternative by introducing clones.

We have the following theorem.

THEOREM 5.1. *Let V be a Dodgson approximation. If V satisfies combinativity or Smith consistency, then its approximation ratio is at least $\Omega(nm)$. If V satisfies mutual majority consistency, invariant loss consistency, or independence of clones, then its approximation ratio is at least $\Omega(n)$.*

Each of the following subsections deals with a property (or two). Our main purpose is to prove Theorem 5.1; however, in some cases, we elaborate a bit regarding (simple) upper bounds.

5.1. Combinativity

An upper bound of $\mathcal{O}(nm)$ that satisfies combinativity can be obtained by selecting a Condorcet winner if one exists, and otherwise selecting some fixed alternative (by setting its score equal to $nm - 1$ and setting the scores of all other alternatives equal to nm).

THEOREM 5.2. *Let V be a Dodgson approximation. If V satisfies the combinativity property, then its approximation ratio is at least $\Omega(nm)$.*

PROOF. Let k, λ be positive integers such that k is even and divides $4\lambda - 2$. Consider the following preference profile $\succ$ with $n = 4\lambda - 2$ agents and $m = k + 2$ alternatives. There is a set Ψ of k alternatives $\psi_0, \ldots, \psi_{k-1}$ and two additional alternatives a and b. For $i = 0, \ldots, k - 1$, denote by Ψ_i the ordered set that contains the alternatives in Ψ ordered as $\psi_i, \psi_{i+1 \bmod k}, \ldots, \psi_{i+k-1 \bmod k}$.

The preference profile $\succ$ is as follows (see Table V):

—For $i = 0, \ldots, \lambda - 3$, agent i ranks a first, then the alternatives of $\Psi_{i \bmod k}$, and then b.
—For $i = \lambda - 2, \ldots, 2\lambda - 3$, agent i ranks b first, then a, and then the alternatives of $\Psi_{i \bmod k}$.
—For $i = 2\lambda - 2, \ldots, 3\lambda - 3$, agent i ranks the alternatives of $\Psi_{i \bmod k}$ first, then b, and then a.
—For $i = 3\lambda - 2, \ldots, 4\lambda - 3$, agent i ranks a first, then the alternatives of $\Psi_{i \bmod k}$, and then b.

We have the following deficits with respect to $\succ$: $\mathrm{defc}(a, b, \succ) = 2$, $\mathrm{defc}(b, \psi_i, \succ) = \lambda$ for any $\psi_i \in \Psi$, and $\mathrm{defc}(\psi_i, a, \succ) = \lambda$ for any $\psi_i \in \Psi$. Alternative a has a Dodgson score of 2 since it suffices to push it one position upward in the rankings of two agents in order to defeat b twice more. Alternative b has to defeat all of the k alternatives in Ψ λ

additional times and hence has a Dodgson score of at least $k\lambda$. Each alternative ψ_i of Ψ has to beat a λ times. By the definition of the sets Ψ_i, alternative ψ_i is ranked higher than alternative $\psi_{i+j \bmod k}$ by $(k-j)\frac{4\lambda-2}{k}$ agents, for $j = 1, 2, \ldots, k-1$, whereas it needs to defeat $\psi_{i+j \bmod k}$ 2λ times in total in order to beat it in their pairwise elections. Hence, for $j = k/2, \ldots, k-1$, we have

$$\mathrm{defc}(\psi_i, \psi_{i+j \bmod k}, \succ) = 2\lambda - (k-j)\frac{4\lambda-2}{k}$$

$$= j\frac{4\lambda-2}{k} - 2\lambda + 2.$$

and the Dodgson score of ψ_i is

$$\mathrm{sc}_D(\psi_i, \succ) \geq \mathrm{defc}(\psi_i, a, \succ) + \sum_{j=k/2}^{k-1} \mathrm{defc}(\psi_i, \psi_{i+j \bmod k}, \succ)$$

$$= \lambda + \sum_{j=k/2}^{k-1} \left(j\frac{4\lambda-2}{k} - 2\lambda + 2 \right)$$

$$= \lambda + \frac{4\lambda-2}{k} \sum_{j=k/2}^{k-1} j - k(\lambda - 1)$$

$$= \frac{k\lambda}{2} + \frac{k}{4} + \frac{1}{2} > \frac{k\lambda}{2}.$$

Now consider the following preference profile $\succ'$ with $n' = 2\lambda - 1$ agents and the same m alternatives (see Table V).

—For $i = 0, \ldots, \lambda - 1$, agent i ranks a first, then b, and then the alternatives of $\Psi_{i \bmod k}$.
—For $i = \lambda, \ldots, 2\lambda - 2$, agent i ranks b first, then the alternatives of $\Psi_{i \bmod k}$, and then a.

In this preference profile, we have that alternative a is the Condorcet winner. If we combine the two preference profiles, we get a new one $\succ''$ in which alternative b is the Condorcet winner. We can assume that V is Condorcet consistent, since otherwise it would have an infinite approximation ratio. Hence, a should be the winner under V in $\succ'$ and b the winner under V in $\succ''$. Since V has the combinativity property, a is not the winner under V in $\succ$—that is, its score $\mathrm{sc}_V(a, \succ)$ is either not smaller than $\mathrm{sc}_V(b, \succ)$ or not smaller than $\mathrm{sc}_V(\psi_i, \succ)$ for some alternative $\psi_i \in \Psi$. The minimum Dodgson score among these alternatives is at least $k\lambda/2$, whereas a has Dodgson score of 2. Using the previous information, the fact that V is a Dodgson approximation, and the definition of n and m, we obtain that

$$\mathrm{sc}_V(a, \succ) \geq \min_{y \in \Psi \cup \{b\}} \mathrm{sc}_V(y, \succ) \geq \min_{y \in \Psi \cup \{b\}} \mathrm{sc}_D(y, \succ) \geq \frac{k\lambda}{2}$$

$$= \frac{k\lambda}{4} \mathrm{sc}_D(a, \succ) = \frac{(n+2)(m-2)}{16} \cdot \mathrm{sc}_D(a, \succ)$$

(i.e., V has an approximation ratio of $\Omega(nm)$). □

5.2. Smith Consistency

It is not difficult to define a trivial $\mathcal{O}(nm)$-approximation algorithm for the Dodgson score that satisfies Smith consistency. The algorithm selects a Condorcet winner if one exists. Otherwise, for each alternative in the Smith set, we set its score equal to $nm - 1$,

Table VI. The Preference Profile $\succ$ Used in the Proof
of Theorem 5.3

$\times k$	$\times k$	$\times k$	$\times k$	$\times k$	$\times k$	$\times 1$
d	a	d	b	d	c	a
a	X_1	b	X_3	c	X_3	b
X_1	b	X_1	c	X_3	a	c
b	X_2	c	X_2	a	X_2	d
X_2	c	X_2	a	X_2	b	X_1
c	X_3	a	X_1	b	X_1	X_2
X_3	d	X_3	d	X_1	d	X_3

and we set the score of any other alternative equal to nm. Note that the Smith set can be computed in polynomial time [Brandt et al. 2009]. The following statement shows that no asymptotic improvement is possible.

THEOREM 5.3. *Let V be a Dodgson approximation. If V satisfies the Smith consistency property, then its approximation ratio is at least $\Omega(nm)$.*

PROOF. Let $k, t \geq 1$ be integers. We construct a preference profile $\succ$ with three alternatives a, b, and c that belong to the Smith set, an alternative d, and a set of alternatives $X = \{x_1, \ldots, x_{3t}\}$ that do not belong to the Smith set, in a way that the Dodgson score of d is at most 3 and the Dodgson score of a, b, and c is at least $\Omega(nm)$.

The set X is partitioned into three sets $X_1 = \{x_1, \ldots, x_t\}$, $X_2 = \{x_{t+1}, \ldots, x_{2t}\}$, and $X_3 = \{x_{2t+1}, \ldots, x_{3t}\}$. We have $n = 6k + 1$ agents, $m = 3t + 4$ alternatives, and the preference profile $\succ$ of Table VI; note that the order of alternatives in the sets X_1, X_2, and X_3 is arbitrary.

Alternative a beats all alternatives besides c in a pairwise election and, furthermore, $\text{defc}(a, c, \succ) = k$. Alternative b beats all alternatives besides a in a pairwise election and, furthermore, $\text{defc}(b, a, \succ) = k + 1$. Alternative c beats all alternatives besides b in a pairwise election and, furthermore, $\text{defc}(c, b, \succ) = k + 1$. Clearly, the Smith set is $\{a, b, c\}$.

Alternative d beats all of the alternatives in X and has $\text{defc}(d, a, \succ) = \text{defc}(d, b, \succ) = \text{defc}(d, c, \succ) = 1$. Clearly, the Dodgson score of d is (at most) 3 since it can be made a Condorcet winner by pushing it three positions in the ranking of the last agent.

Now, observe that in order to defeat c in the preference of any agent where it is ranked lower, a has to be pushed at least $t + 1$ positions upward. This means that its Dodgson score is at least $k(t + 1)$. Similarly, b (respectively, c) can defeat a (respectively, b) in the ranking of the last agent by rising one position upward but needs at least $t + 1$ pushes in the preference of any other agent where it is ranked below a (respectively, b). So, we have that the Dodgson score of a is at least $k(t + 1)$, and the Dodgson score of b and c is at least $1 + k(t + 1)$.

Since V has the Smith consistency property, some alternative among a, b, and c must be a winner. Using the bounds on the Dodgson scores and the definition of k and t, we have that

$$
\begin{aligned}
\text{sc}_V(d, \succ) &\geq \min\{\text{sc}_V(a, \succ), \text{sc}_V(b, \succ), \text{sc}_V(c, \succ)\} \\
&\geq \min\{\text{sc}_D(a, \succ), \text{sc}_D(b, \succ), \text{sc}_D(c, \succ)\} \\
&\geq \frac{k(t + 1)}{3} \text{sc}_D(d, \succ) \\
&= \frac{(n - 1)(m - 1)}{54} \text{sc}_D(d, \succ)
\end{aligned}
$$

(i.e., V has an approximation ratio of $\Omega(nm)$). $\square$

Table VII. The Preference Profile $\succ$ Used in the Proof
of Theorem 5.4

0	...	$\lfloor \frac{\lambda(m-1)}{2} \rfloor - 1$	$\lfloor \frac{\lambda(m-1)}{2} \rfloor$	...	$\lambda(m-1) - 1$
a	...	a	Ψ_0	...	Ψ_{m-2}
Ψ_0	...	Ψ_{m-2}	a	...	a

5.3. Mutual Majority Consistency and Invariant Loss Consistency

It is not difficult to see that the following trivial (super-polynomial-time) algorithm satisfies invariant loss consistency and has approximation ratio $\mathcal{O}(n)$ for the Dodgson score. For each alternative, set its score equal to nm if it loses to every other alternative and equal to its Dodgson score otherwise. The following statement shows that no asymptotic improvement is possible.

THEOREM 5.4. *Let V be a Dodgson approximation. If V is mutual majority consistent or invariant loss consistent, then its approximation ratio is at least $\Omega(n)$.*

PROOF. Consider the following preference profile $\succ$ with $m \geq 4$ alternatives and $n = \lambda(m-1)$ agents where λ is odd and m is even (hence n is odd). There is a set Ψ of $m-1$ alternatives $\psi_0, \ldots, \psi_{m-2}$ and one additional alternative a. For $i = 0, \ldots, m-2$, denote by Ψ_i the ordered set that contains the alternatives in Ψ ordered as $\psi_i, \psi_{i+1 \bmod (m-1)}, \ldots, \psi_{i+m-2 \bmod (m-1)}$.

The preference profile is as follows (see Table VII):

—For $i = 0, \ldots, \lfloor \frac{\lambda(m-1)}{2} \rfloor - 1$, agent i ranks a first, and then the alternatives of $\Psi_{i \bmod (m-1)}$.
—For $i = \lfloor \frac{\lambda(m-1)}{2} \rfloor, \ldots, \lambda(m-1) - 1$, agent i ranks the alternatives of $\Psi_{i \bmod (m-1)}$ first, and then a.

The Dodgson score of a is $sc_D(a, \succ) \leq m-1$ since it suffices to push a upward for $m-1$ positions in the ranking of one of the last $\lfloor n/2 \rfloor$ agents in order to make it a Condorcet winner.

CLAIM 5.5. *For every $\psi_i \in \Psi$, it holds that $sc_D(\psi_i, \succ) > \frac{\lambda(m-1)^2}{18}$.*

PROOF. Observe that ψ_i is ranked higher than $\psi_{i+j \bmod (m-1)}$ by $\lambda(m-1-j)$ agents. Hence, in order to beat $\psi_{i+j \bmod (m-1)}$ in the preferences of $\lceil n/2 \rceil = \frac{\lambda(m-1)+1}{2}$ agents, the number of additional agents in whose preferences ψ_i must defeat $\psi_{i+j \bmod (m-1)}$ is

$$\text{defc}(\psi_i, \psi_{i+j \bmod (m-1)}, \succ) = \frac{\lambda(m-1)+1}{2} - \lambda(m-1-j)$$
$$> \lambda\left(j - \frac{m}{2} + \frac{1}{2}\right).$$

Therefore,

$$sc_D(\psi_i, \succ) \geq \sum_{j=m/2}^{m-2} \text{defc}(\psi_i, \psi_{i+j \bmod (m-1)}, \succ)$$
$$> \lambda \sum_{j=m/2}^{m-2} \left(j - \frac{m}{2} + \frac{1}{2}\right)$$
$$= \lambda \sum_{j=1}^{m/2-1} \left(j - \frac{1}{2}\right)$$
$$= \frac{\lambda}{2}\left(\frac{m}{2} - 1\right)^2 \geq \frac{\lambda(m-1)^2}{18}.$$

The last inequality holds since $m \geq 4$. □

Clearly, every alternative in Ψ beats a in their pairwise election, and in particular, the alternatives in Ψ are ranked higher than a by more than half of the agents. Since V is mutual majority consistent or invariant loss consistent, a is not the winner under V in $\succ$—that is, $\mathrm{sc}_V(a, \succ) \geq \min_{\psi_i \in \Psi} \mathrm{sc}_V(\psi_i, \succ)$. Using the previous claim, we have that

$$
\begin{aligned}
\mathrm{sc}_V(a, \succ) &\geq \min_{\psi_i \in \Psi} \mathrm{sc}_V(\psi_i, \succ) \geq \min_{\psi_i \in \Psi} \mathrm{sc}_D(\psi_i, \succ) \\
&> \frac{\lambda(m-1)^2}{18} \geq \frac{\lambda(m-1)}{18} \mathrm{sc}_D(a, \succ) \\
&= \frac{n}{18} \mathrm{sc}_D(a, \succ),
\end{aligned}
$$

which means that V has an approximation ratio of $\Omega(n)$.

5.4. Independence of Clones

THEOREM 5.6. *Let V be a Dodgson approximation. If V is independent of clones, then its approximation ratio is at least $\Omega(n)$.*

PROOF. Consider the preference profile $\succ$ with two alternatives a and b and n agents (n is a multiple of 4). The preferences are such that $\mathrm{defc}(a, b, \succ) = 0$ (i.e., a is a Condorcet winner) and $\mathrm{defc}(b, a, \succ) = 2$. Now, consider a Dodgson approximation V. If it selects b as a winner, then, clearly, its approximation ratio is infinite. So, b should be a losing alternative.

Next, consider the profile $\succ'$ obtained by cloning alternative a four times so that the Dodgson score of the clones a_0, a_1, a_2, and a_3 of a is at least $n/4$. In order to do this, it suffices to replace a with the ranking of its clones $a_i \succ_i' a_{i+1 \bmod 4} \succ_i' a_{i+2 \bmod 4} \succ_i' a_{i+3 \bmod 4}$ in the ranking of agent i. The Dodgson score of b is at most 8 since it can become a Condorcet winner by pushing it to the top of the rankings of two agents. By the independence of clones property, b should be a losing alternative in this new profile— that is, it should have $\mathrm{sc}_V(b, \succ') \geq \mathrm{sc}_V(a', \succ')$ for some clone a' of a. We obtain that

$$
\begin{aligned}
\mathrm{sc}_V(b, \succ') &\geq \mathrm{sc}_V(a', \succ') \geq \mathrm{sc}_D(a', \succ') \\
&\geq n/4 \geq n \cdot \mathrm{sc}_D(b, \succ')/32,
\end{aligned}
$$

which means that V has an approximation ratio of $\Omega(n)$. $\square$

REFERENCES

N. Ailon, M. Charikar, and A. Newman. 2005. Aggregating inconsistent information: Ranking and clustering. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*. 684–693.

A. Archer and É. Tardos. 2001. Truthful mechanisms for one-parameter agents. In *Proceedings of the 42nd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*. 482–491.

J. Bartholdi, C. A. Tovey, and M. A. Trick. 1989. Voting schemes for which it can be difficult to tell who won the election. *Social Choice and Welfare* 6, 157–165.

N. Betzler, J. Guo, and R. Niedermeier. 2010. Parameterized computational complexity of Dodgson and Young elections. *Information and Computation* 208, 2, 165–177.

D. Black. 1958. *Theory of Committees and Elections*. Cambridge University Press.

F. Brandt. 2009. Some remarks on Dodgson's voting rule. *Mathematical Logic Quarterly* 55, 4, 460–463.

F. Brandt, F. Fischer, and P. Harrenstein. 2009. The computational complexity of choice sets. *Mathematical Logic Quarterly* 55, 4, 444–459.

I. Caragiannis, J. A. Covey, M. Feldman, C. M. Homan, C. Kaklamanis, N. Karanikolas, A. D. Procaccia, and J. S. Rosenschein. 2009. On the approximability of Dodgson and Young elections. In *Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 1058–1067.

D. Coppersmith, L. Fleischer, and A. Rudra. 2006. Ordering by weighted number of wins gives a good rank-
    ing for weighted tournaments. In *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete
    Algorithms (SODA)*. 776–782.

M. D. Condorcet, 1785. Essai sur l'application de l'analyse à la probabilité de décisions rendues à la pluralité
    de voix. Imprimerie Royal. Facsimile published in 1972 by Chelsea Publishing Company, New York.

P. Faliszewski, E. Hemaspaandra, and L. A. Hemaspaandra. 2011. Multimode control attacks on elections.
    Journal of Artificial Intelligence Research 40, 305–351.

P. C. Fishburn, 1977. Condorcet social choice functions. *SIAM Journal on Applied Mathematics* 33, 3, 469–
    487.

A. Gibbard. 1973. Manipulation of voting schemes. *Econometrica* 41, 587–602.

E. Hemaspaandra, and L. A. Hemaspaandra, and J. Rothe. 1997. Exact analysis of Dodgson elections: Lewis
    Carroll's 1876 voting system is complete for parallel access to NP. *Journal of the ACM* 44, 6, 806–825.

C. Homan, and L. A. Hemaspaandra. 2009. Guarantees for the success frequency of an algorithm for finding
    Dodgson election winners. *Journal of Heuristics* 15, 4, 403–423.

C. Kenyon-Mathieu, and C. Schudy. 2007. How to rank with few errors. In *Proceedings of the 39th Annual
    ACM Symposium on Theory of Computing (STOC)*. 95–103.

J. C. McCabe-Dansted, G. Pritchard, and A. M. Slinko. 2008. Approximability of Dodgson's rule. *Social Choice
    and Welfare* 31, 2, 311–330.

N. Nisan, 2007. Introduction to mechanism design (for computer scientists). In *Algorithmic Game Theory*,
    N. Nisan, T. Roughgarden, É. Tardos, and V. V. Vazirani, Eds., Cambridge University Press, Chapter 9.

N. Nisan, and A. Ronen. 2001. Algorithmic mechanism design. *Games and Economic Behavior* 35(1–2),
    166–196.

M. Satterthwaite. 1975. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems
    for voting procedures and social welfare functions. *Journal of Economic Theory* 10, 187–217.

M. Schulze. 2003. A new monotonic and clone-independent single-winner election method. *Voting Matters*
    17, 9–19.

N. Tideman. 1987. Independence of clones as a criterion for voting rules. *Social Choice and Welfare* 4, 3,
    185–206.

N. Tideman. 2006. *Collective Decisions and Voting*. Ashgate.

V. V. Vazirani. 2001. *Approximation Algorithms*. Springer.