Spectral Rankability Update

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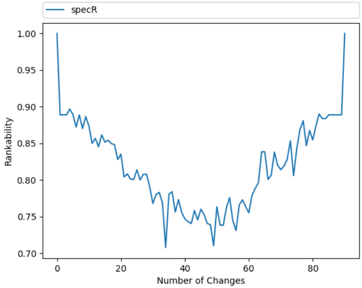
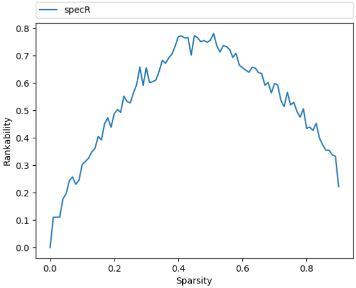
Updated Algorithm

Algorithm 1 Spectral Rankability of Graph Data Γ.

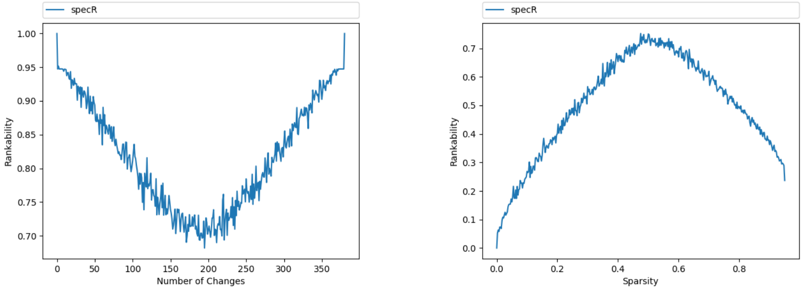
function $[r] = \mathrm{SpecR}\,(\Gamma)$:
- $n \leftarrow$ the number of vertices in Γ
- $D \leftarrow$ the out-degree matrix of Γ
- $L \leftarrow$ graph Laplacian of Γ
- $S = \mathrm{diag}\,(n-1, n-2, \ldots, 0)$
- $r = 1 - \frac{\mathrm{hd}(D,S)+\mathrm{hd}(L,S)}{2(n-1)}$
- **return**

Size 10

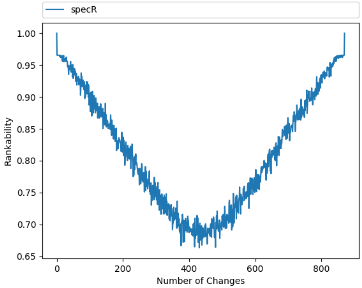

specR
1.00
0.95
0.90
0.85
0.80
0.75
0.70
Rankability
0
20
40
60
80
Number of Changes

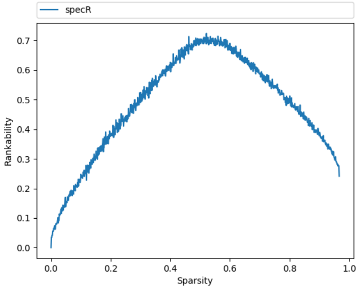

specR
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Rankability
0.0
0.2
0.4
0.6
0.8
Sparsity

Size 20

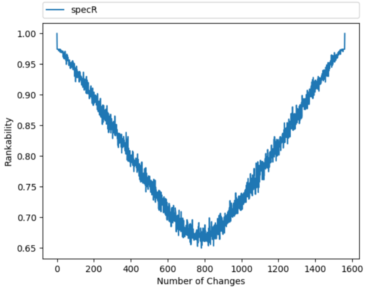

specR
Rankability
Number of Changes
specR
Rankability
Sparsity

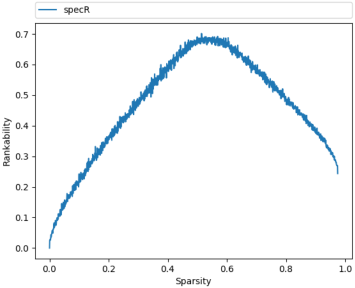
Size 30


specR
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
Rankability
0
200
400
600
800
Number of Changes

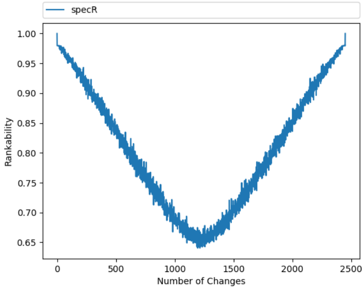

specR
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Rankability
0.0
0.2
0.4
0.6
0.8
1.0
Sparsity

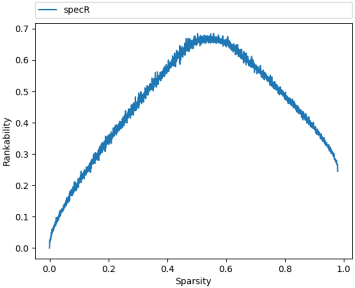
Size 40


specR
Rankability
Number of Changes
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0
200
400
600
800
1000
1200
1400
1600


specR
Rankability
Sparsity
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Size 50


specR
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
Rankability
0
500
1000
1500
2000
2500
Number of Changes


specR
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Rankability
0.0
0.2
0.4
0.6
0.8
1.0
Sparsity

Tournament Graphs

A Tournament graph is a directed graph obtained by assigning a direction to each edge in a complete undirected graph.

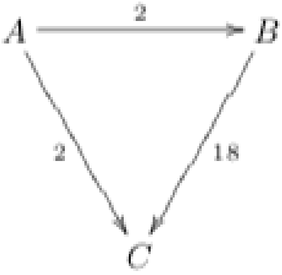
Our rankability measures make sense for data that can be modeled by a tournament (or near-tournament) graph.

Applications

- Sports where each pair of distinct teams play at least one game.
- Social networks that display dominance relations [Lan53].
- Preference list voting systems.

On Election Voting Systems

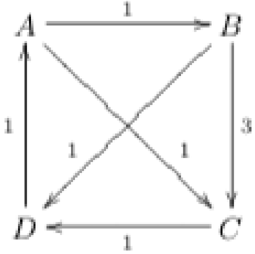
# Voters	Ranking
10	$A > B > C$
1	$A > C > B$
5	$C > A > B$
0	$C > B > A$
9	$B > C > A$
5	$B > A > C$



Courtesy of [Rat10]

On Election Voting Systems

# Voters	Ranking
3	$A > B > C > D^*$
1	$D > B > A > C^\dagger$
1	$D > C > A > B$
1	$B > D > C > A$
1	$C > D > B > A$



Courtesy of [Rat10]

A Different Measure of Rankability

Model voting preference with binary graph.

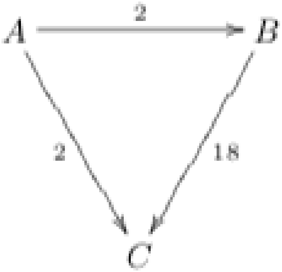
Theorem

There exists a Condorcet Winner if and only if the graph Laplacian has an eigenvalue of $(n-1)$ and there exists a vertex with out-degree $(n-1)$.

Results

Rankability Measure = 1.00

# Voters	Ranking
10	$A > B > C$
1	$A > C > B$
5	$C > A > B$
0	$C > B > A$
9	$B > C > A$
5	$B > A > C$

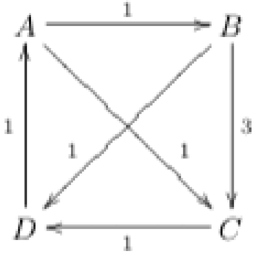


Courtesy of [Rat10]

Results

Rankability Measure = 0.67

# Voters	Ranking
3	$A > B > C > D^*$
1	$D > B > A > C^\dagger$
1	$D > C > A > B$
1	$B > D > C > A$
1	$C > D > B > A$



Courtesy of [Rat10]

References I

H. G. Landau, *On dominance relations and the structure of animal societies: III*, Bull. Math. Biophys. **15** (1953), 143–148.

T. C. Ratliff, *Lewis Carroll, voting, and the taxicab metric*, College Math. J. **41** (2010), 303–311.