7

The Permanent

7.1 Basic Properties

Let $A = [a_{ij}], (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ be a matrix of size m by n with elements in a field F and assume that $m \leq n$. As defined in Chapter 1 the *permanent* of A is

$$\operatorname{per}(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{mi_m}, \tag{7.1}$$

where the summation extends over all the m-permutations $(i_1, i_2, \ldots, i_m)$ of the integers $1, 2, \ldots, n$. There is a nonzero product in the summation (7.1) if and only if there exist m nonzero elements of A no two of which are on the same line. By Theorem 1.2.1 all of the products in the summation (7.1) are zero if and only if A has a p by q zero submatrix with $p+q = n+1$.

The permanent of A is unchanged under arbitrary permutations of the lines of A and also under transposition of A if $m = n$. The permanent of the matrix obtained from A by multiplying the elements of some row by the scalar c equals $c\operatorname{per}(A)$. In addition, if the kth row α of A is the sum of two row vectors α' and α'', then $\operatorname{per}(A) = \operatorname{per}(A') + \operatorname{per}(A'')$ where A' is obtained from A by replacing α with α' and A'' is obtained from A by replacing α with α''. Thus the permanent is a linear function of each of the rows of the matrix A.

If A has the block form

$$\begin{bmatrix} A_1 & O \\ A_3 & A_2 \end{bmatrix}$$

where A_1 is a square matrix, then

$$\operatorname{per}(A) = \operatorname{per}(A_1)\operatorname{per}(A_2).$$

For a square matrix the definition of the permanent is the same as that of the determinant apart from a factor of ± 1 preceding each of the products in the summation (7.1). However, this similarity of the permanent to the determinant does not extend to an effective computational procedure for the permanent which is analogous to that for the determinant. This is because the permanent is in general greatly altered by the addition of a multiple of one row to another.

The Laplace expansion for the determinant has a simple counterpart for the permanent. Let $A(i,j)$ denote the matrix of size $m-1$ by $n-1$ obtained from A by deleting row i and column j,$(i=1,2,\ldots,m; j=1,2,\ldots,n)$. It follows as a direct consequence of the definition of the permanent that

$$\operatorname{per}(A)=\sum_{j=1}^{n} a_{ij}\operatorname{per}(A(i,j)), \quad (i=1,2,\ldots,m). \tag{7.2}$$

The expansion of the permanent in (7.2) is called the *Laplace expansion by row* i. If $m=n$ there is also a Laplace expansion by column i for the permanent.

We now describe a procedure of Ryser[1963] for the evaluation of the permanent. It is the best known general computational procedure for the permanent (see Nijenhuis and Wilf[1978]).

Theorem 7.1.1. *Let $A=[a_{ij}]$ be a matrix of size m by n with $m \leq n$. For each integer r between $n-m$ and $n-1$ let A_r denote an m by $n-r$ submatrix of A obtained by deleting r columns of A. Let $\prod(A_r)$ denote the product of the row sums of A_r. Let $\sum\prod(A_r)$ denote the sum of the products $\prod(A_r)$ taken over all choices for A_r. Then*

$$\operatorname{per}(A)=\sum\prod(A_{n-m})-\binom{n-m+1}{1}\sum\prod(A_{n-m+1})+$$
$$\binom{n-m+2}{2}\sum\prod(A_{n-m+2})-\cdots+(-1)^{m-1}\binom{n-1}{m-1}\sum\prod(A_{n-1}). \tag{7.3}$$

Proof. Let U denote the set of all sequences

$$j_1,j_2,\ldots,j_m \text{ where } 1\leq j_i\leq n, \quad (i=1,2,\ldots,m). \tag{7.4}$$

For each sequence (7.4) let

$$f(j_1,j_2,\ldots,j_m)=a_{1j_1}a_{2j_2}\cdots a_{mj_m}.$$

Let P_i denote the set of all sequences (7.4) that do *not* contain the integer i, $(i=1,2,\ldots,n)$. Then

$$\operatorname{per}(A)=\sum f(j_1,j_2,\ldots,j_m)$$

where the summation extends over all those sequences (7.4) which belong to exactly $n-m$ of the sets $P_1, P_2, \ldots, P_n$. For each choice of $i_1, i_2, \ldots, i_r$ with

$$1 \leq i_1 < i_2 < \cdots < i_r \leq n, \tag{7.5}$$

let

$$f(P_{i_1}, P_{i_2}, \ldots, P_{i_r}) = \sum f(j_1, j_2, \ldots, j_m)$$

where the summation extends over all those sequences (7.4) which are in the intersection $P_{i_1} \cap P_{i_2} \cap \cdots \cap P_{i_r}$. Let A_r denote the submatrix of A obtained by deleting columns $i_1, i_2, \ldots, i_r$. Then

$$\prod(A_r) = f(P_{i_1}, P_{i_2}, \ldots, P_{i_r})$$

and the sum

$$\sum f(P_{i_1}, P_{i_2}, \ldots, P_{i_r})$$

taken over all $i_1, i_2, \ldots, i_r$ satisfying (7.5) equals

$$\sum \prod(A_r).$$

We now apply the inclusion-exclusion principle (Ryser[1963]) and use the fact that

$$\mathrm{per}(A) = \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-m} \leq n} f(P_{i_1}, P_{i_2}, \ldots, P_{i_{n-m}})$$

to obtain the equation (7.3). □

Corollary 7.1.2. *Let A be a square matrix of order n. Then the permanent of A equals*

$$\prod(A) - \sum \prod(A_1) + \sum \prod(A_2) - \cdots + (-1)^{n-1} \sum \prod(A_{n-1}). \tag{7.6}$$

If A is the 2 by 2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then (7.6) gives the following formula for the permanent:

$$\mathrm{per}(A) = (a_{11} + a_{12})(a_{21} + a_{22}) - (a_{11}a_{21} + a_{12}a_{22}).$$

Exercises

1. Use Corollary 7.1.2 in order to evaluate the permanent of the matrix J_n of all 1's thereby obtaining an identity for $n!$.
2. Use Corollary 7.1.2 in order to evaluate the permanent of the (0,1)-matrix of order n all of whose elements equal 1 except for p elements in the first row.
3. Compare the number of multiplications required to evaluate the permanent of a matrix using the definition (7.1) of the permanent and the formula (7.3).
4. Let A be a (0,1)-matrix of order n and let α and β denote, respectively, the first row and second row of A. Let A_1 denote the (0,1)-matrix obtained from A by replacing α by the row which has 1's in those positions in which at least one of α and β has a 1 and replacing β by the row which has 1's in those positions in which both α and β have 1's. Let A_2 be the (0,1)-matrix obtained from A by replacing α by the row which has 1's in those positions in which α has a 1 and β has a zero and replacing β by the row which has 1's in those positions in which β has a 1 and α has a 0. Prove that

$$\operatorname{per}(A) = \operatorname{per}(A_1) + \operatorname{per}(A_2)$$

(Kallman[1982]).

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7.2 Permutations with Restricted Positions

We now let A be the incidence matrix for m subsets $X_1, X_2, \ldots, X_m$ of the n-set $X = \{1, 2, \ldots, n\}$. As noted in Theorem 1.2.3, per(A) equals the number of distinct SDR's for this configuration of subsets, that is, per(A) equals the number of m-permutations $i_1, i_2, \ldots, i_m$ of the n-set X in which i_j is restricted by the condition

$$i_j \in X_j, \quad (j = 1, 2, \ldots, m).$$

This simple observation accounts for the importance of the permanent as a counting function in combinatorics.

A classical combinatorial enumeration problem known as "le problème des recontres" asks for the number of permutations $i_1, i_2, \ldots, i_n$ of the n-set $\{1, 2, \ldots, n\}$ such that $i_j \neq j, (j = 1, 2, \ldots, n)$. Such permutations have no element in their natural position and are called *derangements* of order

n. Let D_n denote the number of derangements of order $n, (n \geq 1)$. If we choose

$$X_i = \{1, 2, \ldots, n\} - \{i\}, \quad (i = 1, 2, \ldots, n),$$

then D_n equals the number of SDR's of this configuration of sets and hence

$$D_n = \text{per}(J_n - I_n),$$

the permanent of the n by n matrix

$$J_n - I_n = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}.$$

Applying (7.6) we obtain the formula

$$D_n = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r)^r (n-r-1)^{n-r}, \quad (n \geq 1). \tag{7.7}$$

A different formula for D_n can be obtained by applying the inclusion-exclusion principle to the set of all permutations of $\{1, 2, \ldots, n\}$ and the sets $P_1, P_2, \ldots, P_n$ where P_i consists of those permutations which have i in position i, $(i = 1, 2, \ldots, n)$. This yields

$$D_n = n! \sum_{r=0}^{n} (-1)^r \frac{1}{r!}. \tag{7.8}$$

Another classical combinatorial enumeration problem that can be formulated as a permanent of a (0,1)-matrix is the "problème des ménages." This problem asks for the number M_n of ways to seat n married couples at a round table with men and women in alternate positions and with no husband next to his wife. The wives may be seated first and this may be done in $2(n!)$ ways. Each husband is then excluded from the two seats next to his wife. The number of ways the husbands may be seated is the same for each seating arrangement of the wives. Let

$$A_i = \{1, 2, \ldots, n\} - \{i, i+1\}, \quad (i = 1, 2, \ldots, n)$$

where $n + 1$ is interpreted as 1. Then

$$M_n = 2(n!)U_n \tag{7.9}$$

where U_n is the number of permutations $i_1, i_2, \ldots, i_n$ of $\{1, 2, \ldots, n\}$ satisfying $i_j \in A_i, (i = 1, 2, \ldots, n)$. The numbers U_n are called the *ménage* *numbers.* Let C_n denote the permutation matrix of order n with 1's in positions $(1,2),(2,3),\ldots,(n-1,n),(n,1)$. We have

$$U_n = \text{per}(J_n - I_n - C_n).$$

Before obtaining a formula for the ménage numbers U_n by evaluating the permanent of the matrix $J_n - I_n - C_n$, we derive a general formula for the evaluation of permanents of matrices of 0's and 1's.

Let k be a nonnegative integer and let n be a positive integer. The collection of all k-subsets of the n-set $\{1,2,\ldots,n\}$ is denoted by $\mathcal{P}_{k,n}$. Let A be an m by n matrix over a field F. As usual for α in $\mathcal{P}_{k,m}$ and β in $\mathcal{P}_{k,n}$, $A[\alpha,\beta]$ is the k by k submatrix of A determined by rows i with $i \in \alpha$ and columns j with $j \in \beta$. The permanent $\text{per}(A[\alpha,\beta])$ is called a *permanental k-minor* of A, or sometimes a *permanental minor* of A. The sum of the permanental k-minors of A is

$$p_k(A) = \sum_{\beta\in\mathcal{P}_{k,n}} \sum_{\alpha\in\mathcal{P}_{k,m}} \text{per}(A[\alpha,\beta]). \tag{7.10}$$

We define $p_0(A) = 0$, and note that if $m \le n$ then $p_m(A) = \text{per}(A)$ and $p_k(A) = 0$ for $k > m$. If the matrix A is a (0,1)-matrix, then $p_k(A)$ counts the number of different selections of k 1's of A with no two of the 1's on the same line.

The next theorem allows us to evaluate the permanent of an m by n (0,1)-matrix A in terms of the permanental minors of the *complementary matrix* $J_{m,n} - A$.

Theorem 7.2.1. *Let A be an m by n $(0,1)$-matrix satisfying $m \le n$. Then*

$$\text{per}\,(A) = \sum_{k=0}^{m} (-1)^k p_k(J_{m,n} - A)\frac{(n-k)!}{(n-m)!}. \tag{7.11}$$

Proof. Let A be the incidence matrix of the configuration $X_1, X_2, \ldots, X_m$ of subsets of the n-set $X = \{1,2,\ldots,n\}$. We shall apply the inclusion-exclusion principle to the set $S_{m,n}$ of all m-permutations of $\{1,2,\ldots,n\}$. Let P_j denote the subset of $S_{m,n}$ consisting of all those m-permutations $i_1, i_2, \ldots, i_m$ such that

$$i_j \in \overline{X}_j = \{1,2,\ldots,n\} - X_j, \quad (j = 1,2,\ldots,m).$$

Then $\text{per}(A)$ equals the number of m-permutations in $S_{m,n}$ which are in none of the sets $P_1, P_2, \ldots, P_m$. Let α be a k-subset of $\{1, 2, \ldots, m\}$ and let $g(\alpha)$ denote the number of m-permutations in $\cap_{i\in\alpha} P_i$. Then

$$g(\alpha) = \frac{(n-k)!}{(n-m)!}\operatorname{per}((J_{m,n} - A)[\alpha, \{1, 2, \ldots, n\}])$$

$$= \frac{(n-k)!}{(n-m)!}\sum_{\beta\in\mathcal{P}_{k,n}} \operatorname{per}((J_{m,n} - A)[\alpha, \beta]).$$

Hence

$$\sum_{\alpha\in\mathcal{P}_{k,m}} g(\alpha) =$$

$$\frac{(n-k)!}{(n-m)!}\sum_{\alpha\in\mathcal{P}_{k,m}}\sum_{\beta\in\mathcal{P}_{k,n}} \operatorname{per}((J_{m,n} - A)[\alpha, \beta]) = \frac{(n-k)!}{(n-m)!} p_k(J_{m,n} - A).$$

The formula (7.11) now follows from the inclusion-exclusion principle. □

The formula (7.8) for $D_n = \operatorname{per}(J_n - I_n)$ follows from (7.11) by taking $m = n$ and $A = J_n - I_n$. The matrix $J_{m,n} - A$ in the right-hand side of (7.11) is then I_n and $p_k(I_n) = \binom{n}{k}$.

We now return to the ménage numbers U_n. In obtaining a formula for the numbers U_n we shall make use of the results of Kaplansky[1943] given in the next lemma.

Lemma 7.2.2. *Let $f(n,k)$ denote the number of ways to select k objects, no two adjacent, from n objects arranged in a line and let $g(n,k)$ denote the number of ways when the objects are arranged in a circle. Then*

$$f(n,k) = \binom{n-k+1}{k}, \quad (k = 1, 2, \ldots, n) \tag{7.12}$$

and

$$g(n,k) = \frac{n}{n-k}\binom{n-k}{k}, \quad (k = 1, 2, \ldots, n). \tag{7.13}$$

Proof. First suppose that the n objects are arranged in a line. We have $f(n,1) = n = \binom{n}{1}, (n \geq 1)$, and $f(n,n) = 0 = \binom{1}{n}, (n \geq 2)$. Now assume that $1 \leq k \leq n$. The selections may be partitioned into those that contain the first object [there are $f(n-2, k-1)$ of these] and those that do not [there are $f(n-1, k)$]. Hence

$$f(n,k) = f(n-1,k) + f(n-2,k-1).$$

Using this recurrence relation and arguing inductively we obtain

$$f(n,k) = \binom{n-k}{k} + \binom{n-k}{k-1} = \binom{n-k+1}{k}.$$

Now suppose that the objects are arranged in a circle. Partitioning the selections as above, we obtain

$$\begin{aligned} g(n,k) &= f(n-3,k-1) + f(n-1,k) \\ &= \binom{n-k-1}{k-1} + \binom{n-k}{k} \\ &= \tfrac{n}{n-k}\binom{n-k}{k}. \end{aligned}$$

□

The following formula is due to Touchard[1934].

Theorem 7.2.3. *The ménage numbers are given by the formula*

$$U_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!. \tag{7.14}$$

Proof. We apply Theorem 7.2.1 and obtain

$$U_n = \operatorname{per}(J_n - I_n - C_n) = \sum_{k=0}^{n} (-1)^k p_k(I_n + C_n)(n-k)!.$$

But $p_k(I_n + C_n) = g(2n,k)$ where the function g was defined in Lemma 7.2.2. By (7.13)

$$g(2n,k) = \frac{2n}{2n-k}\binom{2n-k}{k},$$

and (7.14) follows. □

We return to a general m by n (0,1)-matrix A satisfying $m \le n$. We may identify A with an m by n chessboard in which the square in row i and column j has been removed from play for all those i and j satisfying $a_{ij} = 0$. The number $r_k = p_k(A)$ counts the number of ways to place k identical rooks on this board so that no rook can attack another (that is, no two rooks lie on the same line), and is called the kth *rook number* of $A, (k = 0, 1, \ldots, m)$. The $(m+1)$-tuple $(r_0, r_1, \ldots, r_m)$ is the *rook vector* of A and

$$r(x) = r_0 + r_1 x + \cdots + r_m x^m$$

is the *rook polynomial* of A. The rook polynomial of $A = I_n$ is

$$\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n$$

and that of $A = I_n + C_n$ is

$$g(2n, 0) + g(2n, 1)x + \cdots + g(2n, n)x^n.$$

Theorem 7.2.1 shows how to compute the mth rook number of A from the rook vector of the complementary matrix $J_{mn} - A$. The entire rook vector of the matrix A may be obtained in this way.

Theorem 7.2.4. *Let A be an m by n $(0, 1)$-matrix with $m \leq n$. Then*

$$p_t(A) = \sum_{k=0}^{t} (-1)^k p_k(J_{m,n} - A) \binom{m-k}{t-k} \frac{(n-k)!}{(n-t)!}, \quad (t = 0, 1, \ldots, m). \tag{7.15}$$

Proof. Using Theorem 7.2.1 we calculate that

$$\begin{aligned} p_t(A) &= \sum_{\alpha \in \mathcal{P}_{t,m}} \operatorname{per}(A[\alpha, \{1, 2, \ldots, n\}]) \\ &= \sum_{\alpha \in \mathcal{P}_{t,m}} \sum_{k=0}^{t} (-1)^k p_k(J_{t,n} - A[\alpha, \{1, 2, \ldots, n\}]) \frac{(n-k)!}{(n-t)!} \\ &= \sum_{k=0}^{t} \sum_{\alpha \in \mathcal{P}_{t,m}} (-1)^k p_k(J_{t,n} - A[\alpha, \{1, 2, \ldots, n\}]) \frac{(n-k)!}{(n-t)!} \\ &= \sum_{k=0}^{t} (-1)^k p_k(J_{mn} - A) \binom{m-k}{t-k} \frac{(n-k)!}{(n-t)!}. \end{aligned}$$

□

The computation of the rook vector of an m by n (0,1)-matrix is in general a difficult problem. There is, however, a special class of (0,1)-matrices whose rook polynomials have an exceedingly simple form.

Let $b_1, b_2, \ldots, b_m$ be integers with $0 \leq b_1 \leq b_2 \leq \cdots \leq b_m$. The m by b_m (0,1)-matrix $A = [a_{ij}]$ defined by

$$a_{ij} = 1 \text{ if and only if } 1 \leq j \leq b_i, \quad (i = 1, 2, \ldots, m)$$

is a *Ferrers matrix* and is denoted by $F(b_1, b_2, \ldots, b_m)$. In what follows the number of columns of the Ferrers matrix can be taken to be any integer $n \geq b_m$ with no change in the conclusions. We define

$$[x]_k = x(x-1)\cdots(x-k+1), \quad (k \geq 1) \qquad \text{and} \qquad [x]_0 = 1.$$

Theorem 7.2.5. *Let $F(b_1, b_2, \ldots, b_m)$ be a Ferrers matrix with rook vector $(r_0, r_1, \ldots, r_m)$. Then*

$$\sum_{k=0}^{m} r_k [x]_{m-k} = \prod_{i=1}^{m} (x + q_i) \tag{7.16}$$

where $q_i = b_i - i + 1, (i = 1, 2, \ldots, m)$.

Proof. Let x be a nonnegative integer and consider the Ferrers matrix

$$F(x + b_1, x + b_2, \ldots, x + b_m) = \begin{bmatrix} J_{m,x} & A \end{bmatrix}.$$

We evaluate the mth rook number r'_m of $F(x + b_1, x + b_2, \ldots, x + b_m)$ in two different ways. Counting the number of ways to select m 1's of $F(x + b_1, x + b_2, \ldots, x + b_m)$, no two from the same line, according to the number of 1's selected from $J_{m,x}$, we obtain

$$r'_m = \sum_{k=0}^{m} r_k [x]_{m-k}.$$

We may also evaluate r'_m by first choosing a 1 in row 1, then choosing a 1 in row 2, ..., then a 1 in row m in such a way that no two 1's chosen belong to the same column. Because

$$x + b_1 \leq x + b_2 \leq \cdots \leq x + b_m$$

we obtain

$$\begin{aligned} r'_m &= (x + b_1)(x + b_2 - 1) \cdots (x + b_m - (m-1)) \\ &= (x + q_1)(x + q_2) \cdots (x + q_m). \end{aligned}$$

Equating these two counts we obtain (7.16). □

The following two corollaries are direct consequences of Theorem 7.2.5.

Corollary 7.2.6.

$$\operatorname{per}(F(b_1, b_2, \ldots, b_m)) = \prod_{i=1}^{m} (b_i - i + 1).$$

Corollary 7.2.7. *Let $F(b_1, b_2, \ldots, b_m)$ and $F(c_1, c_2, \ldots, c_m)$ be two Ferrers matrices, and let $q_i = b_i - i + 1$ and $q'_i = c_i - i + 1$ for $i = 1, 2, \ldots, m$. Then $F(b_1, b_2, \ldots, b_m)$ and $F(c_1, c_2, \ldots, c_m)$ have the same rook polynomial if and only if the numbers $q_1, q_2 \ldots, q_m$ are identical with the numbers $q'_1, q'_2, \ldots, q'_m$ including multiplicities.*

One may also define the *rook polynomial* of an m by n matrix A over a field F by

$$r(x) = \sum_{k=0}^{m} p_k(A)x^k.$$

We conclude this section with the statement of the following theorem of Nijenhuis[1976].

Theorem 7.2.8. *If A is a nonnegative matrix then all the roots of its rook polynomial are real numbers.*

Exercises

1. Obtain formula (7.8) directly from the inclusion-exclusion principle.
2. Determine the rook vector of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

3. Let A be the (0,1)-matrix of order $n = 2m$ given by

$$A = J_n - (J_2 \oplus J_2 \oplus \cdots \oplus J_2).$$

   Use Theorem 7.2.1 to obtain a formula for per(A).
4. Let $a_i, (i = 1, 2, \ldots, r)$ and $b_i, (i = 1, 2, \ldots, r)$ be positive integers such that $a_1 + a_2 + \cdots + a_r = n$ and $b_1 + b_2 + \cdots + b_r \leq n$. Let $A = A_{(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r)}$ denote the (0,1)-matrix of order n obtained from the matrix J_n of all 1's of order n by choosing for each $i = 1, 2, \ldots, r$ "line disjoint" submatrices of sizes a_i by b_i and replacing their 1's by 0's. Prove that

$$\text{per}(A) = \sum (-1)^{i_1+i_2+\cdots+i_r} (n - i_1 - i_2 - \cdots - i_r) \prod_{k=1}^{r} \binom{a_k}{i_k} \binom{b_k}{i_k} i_k!$$

   where the summation extends over all $i_1, i_2, \ldots, i_r$ satisfying $i_k \leq a_k, (k = 1, 2, \ldots, r)$ (Kaplansky[1939,1944] and Chung, Diaconis, Graham and Mallows[1981]).
5. Prove that the rook vector of the Ferrers matrix $F(0, 1, 2, \ldots, m-1)$ of order m equals $(S(m, m), S(m, m-1), \ldots, S(m, 1), S(m, 0))$ where $S(m, k)$ denotes a Stirling number of the second kind (see Stanley[1986] for the definition of the Stirling numbers of the second kind).

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7.3 Matrix Factorization of the Permanent and the Determinant

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a matrix of order 3. We observe that the result of the following matrix multiplication

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} a_{22} & a_{23} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{33} \\ a_{32} \\ a_{31} \end{bmatrix} \tag{7.17}$$

is a matrix of order 1 whose unique element equals the permanent of A. Each of the three matrices in this factorization of $\operatorname{per}(A)$ depends only on one row of the matrix A. A matrix factorization for the determinant of A is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} a_{22} & a_{23} & 0 \\ -a_{21} & 0 & a_{23} \\ 0 & -a_{21} & -a_{22} \end{bmatrix} \begin{bmatrix} a_{33} \\ -a_{32} \\ a_{31} \end{bmatrix}. \tag{7.18}$$

The factorization (7.18) is obtained from the factorization (7.17) by affixing minus signs to some of the elements of the matrix factors. These two factorizations for the permanent and the determinant of a matrix of order 3 are instances of factorizations of the permanent and determinant that exist for all square matrices. These factorizations were discovered by Jurkat and Ryser[1966], but we shall follow the simplified account of Brualdi and Shader[1990] (see also Brualdi[1990]).

Let n be a positive integer. We partially order the collection $\mathcal{P}_n = \mathcal{P}_{n,n}$ of all subsets of $\{1, 2, \ldots, n\}$ by inclusion and thereby obtain a partially ordered set. A *chain* in this partially ordered set is a sequence of distinct subsets

$$X_0 \subset X_1 \subset \cdots \subset X_k \tag{7.19}$$

which are related by inclusion as shown. The *length of the chain* (7.19) is k. The maximal length of a chain is n. Let

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n \tag{7.20}$$

be a chain of length n. Then Y_j is in $\mathcal{P}_{j,n}, (j = 0, 1, \ldots, n)$, and there is a unique element i_j such that

$$Y_j = Y_{j-1} \cup \{i_j\}, \quad (j = 1, 2, \ldots, n). \tag{7.21}$$

Moreover,

$$i_1, i_2, \ldots, i_n \tag{7.22}$$

is a permutation of $\{1, 2, \ldots, n\}$. Conversely, given a permutation (7.22) of $\{1, 2, \ldots, n\}$, if we define $Y_0 = \emptyset$ and recursively define Y_j using (7.21) then we obtain a chain (7.20) of length n. Thus the chains (7.20) of length n are in one-to-one correspondence with the permutations (7.22) of $\{1, 2, \ldots, n\}$.

Now assume that the collection $\mathcal{P}_{j,n}$ of subsets of $\{1, 2, \ldots, n\}$ of size j have been linearly ordered in some way for each $j = 0, 1, \ldots, n$. To fix the notation, let us choose for each j the lexicographic order and let the elements of $\mathcal{P}_{j,n}$ in the lexicographic order be

$$Z_1^{(j)}, Z_2^{(j)}, \ldots, Z_{q_{j,n}}^{j}, \quad (j = 0, 1, \ldots, n)$$

where $q_{j,n}$ denotes $\binom{n}{j}$.

Let $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ be a matrix of order n over a field F (a ring will do). We use the elements of the matrix A in order to define a *weighted incidence matrix*

$$W^{(j)} = \left[w_{k,l}^{(j)} \right], \quad (k = 1, 2, \ldots, q_{j-1,n}; l = 1, 2, \ldots, q_{j,n})$$

of size $q_{j-1,n}$ by $q_{j,n}$ for the inclusions that hold between the sets in $\mathcal{P}_{j-1,n}$ and the sets in $\mathcal{P}_{j,n}$, $(j = 1, 2, \ldots, n)$. We define

$$w_{kl}^{(j)} = \begin{cases} 0 & \text{if } Z_k^{(j-1)} \not\subset Z_l^{(j)} \\ a_{jt} & \text{if } Z_k^{(j-1)} \cup \{t\} = Z_l^{(j)}. \end{cases} \tag{7.23}$$

We remark that only the elements of row j of A are used to define the matrix $W^{(j)}$.

We now define the *weight of the chain* (7.20) corresponding to the permutation (7.22) to be

$$a_{1i_1}a_{2i_2}\cdots a_{ni_n}.$$

Thus the sum of the weights of all the chains (7.20) of length n in $\mathcal{P}_n$ equals the permanent of the matrix A.

We now compute the matrix product

$$W^{(1)}W^{(2)}\ldots W^{(n)}. \tag{7.24}$$

The matrix (7.24) is a matrix of size 1 by 1 whose unique element equals

$$\sum_{i_1=1}^{q_{1,n}}\sum_{i_2=1}^{q_{2,n}}\cdots\sum_{i_{n-1}=1}^{q_{n-1,n}} w_{1i_1}^{(1)}w_{i_1i_2}^{(2)}\cdots w_{i_{n-1}1}^{(n)}. \tag{7.25}$$

The product

$$w_{1i_1}^{(1)}w_{i_1i_2}^{(2)}\cdots w_{i_{n-1}1}^{(n)} \tag{7.26}$$

equals zero if

$$Z_1^{(0)} = \emptyset, Z_{i_1}^{(1)}, Z_{i_2}^{(2)}, \ldots, Z_1^{(n)} = \{1, 2, \ldots, n\} \tag{7.27}$$

is not a chain. If, however, (7.27) is a chain, then the product (7.26) equals its weight. Hence the matrix (7.24) of order 1 has unique element equal to $\operatorname{per}(A)$ and

$$\operatorname{per}(A) = W^{(1)}W^{(2)}\ldots W^{(n)} \tag{7.28}$$

is a matrix factorization of the permanent of A.

We now alter the definition of the weighted incidence matrices $W^{(j)}$ as given in (7.23) by defining

$$w_{kl}^{(j)} = \begin{cases} 0 & \text{if } Z_k^{(j-1)} \not\subset Z_l^{(j)} \\ (-1)^{c_j}a_{jt} & \text{if } Z_k^{(j-1)} \cup \{t\} = Z_l^{(j)} \end{cases} \tag{7.29}$$

where c_j elements of $Z_k^{(j-1)}$ are greater than t. The *weight of the chain* (7.20) corresponding to the permutation (7.22) is now

$$(-1)^{\mathrm{inv}(i_1,i_2,\ldots,i_n)} a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where $\mathrm{inv}(i_1, i_2, \ldots, i_n)$ equals the number of inversions in the permutation $(i_1, i_2, \ldots, i_n)$. Thus the sum of all the weights of the chains (7.20) of length n in $\mathcal{P}_n$ equals the determinant of A. Arguing as above we now obtain the matrix factorization

$$\det(A) = W^{(1)} W^{(2)} \ldots W^{(n)} \tag{7.30}$$

of the determinant of A.

The matrices $W^{(j)}$ which occur in the factorizations (7.28) and (7.30) of the permanent and determinant have an inductive structure which facilitates their computation. We first treat the $W^{(j)}$'s that arise in the factorization of the permanent. The matrix $W^{(j)}$ depends only on row j of A, and we now denote row j of A by

$$x = (x_1, x_2, \ldots, x_n)$$

and write

$$W^{(j)} = W^{(j)}(x).$$

We have

$$W^{(1)}(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

and

$$W^{(n)}(x) = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix}.$$

Let j be an integer with $1 < j < n$. Let

$$U_1^{(j-1)}, U_2^{(j-1)}, \ldots, U_{q_{j-2,n-1}}^{(j-1)}$$

be the sets of $\mathcal{P}_{j-1,n}$ containing 1, arranged in lexicographic order, and let

$$V_1^{(j-1)}, V_2^{(j-1)}, \ldots, V_{q_{j-1,n-1}}^{(j-1)}$$

be the sets in $\mathcal{P}_{j-1,n}$ not containing 1, arranged in lexicographic order. Then

$$U_1^{(j-1)}, U_2^{(j-1)}, \ldots, U_{q_{j-2,n-1}}^{(j-1)}, V_1^{(j-1)}, V_2^{(j-1)}, \ldots, V_{q_{j-1,n-1}}^{(j-1)}$$

are the sets in $\mathcal{P}_{j-1,n}$ arranged in lexicographic order. In a similar way we obtain the sets

$$U_1^{(j)}, U_2^{(j)}, \ldots, U_{q_{j-1,n-1}}^{(j)}, V_1^{(j)}, V_2^{(j)}, \ldots, V_{q_{j,n-1}}^{(j)}$$

of $\mathcal{P}_{j,n}$ arranged in lexicographic order. Let

$$x^* = (x_2, x_3, \ldots, x_n).$$

It now follows from (7.23) that

$$W^{(j)}(x) = \begin{bmatrix} W^{(j-1)}(x^*) & O \\ x_1 I_{q_{j-1,n-1}} & W^{(j)}(x^*) \end{bmatrix}, \quad (j = 1, 2, \ldots, n-1) \qquad (7.31)$$

where O denotes a zero matrix of size $q_{j-2,n-1}$ by $q_{j,n-1}$. The inductive structure of $W^{(j)}(x)$ given in (7.31) implies that exactly $n-(j-1)$ components of x occur in each row of $W^{(j)}(x)$ and they occur in the same order as their occurrence in x. Also exactly j components of x occur in each column of $W^{(j)}(x)$ and they occur in the reverse order to their occurrence in x.

There is a similar inductive structure for the matrices $W^{(j)}(x)$ that occur in the factorization of the determinant. In this case we have

$$W^{(1)}(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

and

$$W^{(n)}(x) = \begin{bmatrix} x_n \\ -x_{n-1} \\ \cdots \\ (-1)^{n-1}x_1 \end{bmatrix}.$$

The j components of x that occur in each column of $W^{(j)}(x)$ occur in the reverse order to their occurrence in x. It follows from (7.29) that the signs in front of these j components of x alternate and that the sign of the topmost component is $+1$ whereas the sign of the bottommost component is $(-1)^{j-1}$. These observations allow us to obtain the inductive structure

$$W^{(j)}(x) = \begin{bmatrix} W^{(j-1)}(x^*) & O \\ (-1)^{j-1} x_1 I_{q_{j-1,n-1}} & W^{(j)}(x^*) \end{bmatrix}, \quad (j = 1, 2, \ldots, n-1)$$

in this case.

Let C be a rectangular complex matrix. The l_2-*norm* of C is the square root of the largest eigenvalue of the positive semidefinite matrix CC^* and is denoted by $||C||$. Here C^* denotes the conjugate transpose of the matrix C. The l_2-norm is a matrix norm, that is, $||C_1C_2|| \leq ||C_1||||C_2||$ whenever the product C_1C_2 is defined. Jurkat and Ryser[1966] used the factorization (7.28) of the permanent and an evaluation of $||W^{(j)}(x)||$ in the case that x is a (0,1)-vector to obtain an upper bound for the permanent of a

(0,1)-matrix of order n in terms of its row sum vector $(r_1, r_2, \ldots, r_n)$. This bound is not as good as the one given in Theorem 7.4.5 in the next section. They also used the factorization (7.30) of the determinant and the evaluation

$$||W^{(j)}(x)|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \quad (j = 1, 2, \ldots, n)$$

for x a complex vector in order to obtain Hadamard's determinant inequality

$$|\det(A)| \leq \prod_{i=1}^{n} \sqrt{a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2}.$$

Exercises

1. Compute the matrix factorizations of the permanent and determinant of a matrix of order 4.
2. In the matrix factorizations $W^{(1)}W^{(2)} \cdots W^{(n)}$ of the permanent and the determinant, determine the elements of the matrix $W^{(r)} \cdots W^{(s)}$ where r and s are integers with $0 \leq r < s \leq n$.

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7.4 Inequalities

Let A be a (0,1)-matrix of size m by n with $m \leq n$. The permanent of A satisfies the inequality

$$0 \leq \operatorname{per}(A) \leq n(n-1)\cdots(n-m+1) \tag{7.32}$$

Equality holds on the right in (7.32) if and only if A is the matrix $J_{m,n}$ of all 1's. Equality holds on the left if and only if the term rank of A is strictly less than m. In this section we shall improve the inequalities (7.32) by taking into account the number of 1's in each row of A, the number of 1's in each column of A, or the total number of 1's of A.

The following theorem of Ostrand[1970] is a strengthening of earlier results of Hall[1948], Jurkat and Ryser[1966] and Rado[1967].

Theorem 7.4.1. *Let $A = [a_{ij}]$ be a $(0,1)$-matrix of size m by n with* $\operatorname{per}(A) > 0$. *Let the row sums of A be $r_1, r_2, \ldots, r_m$ and assume that the rows of A have been arranged so that $r_1 \le r_2 \le \cdots \le r_m$. Then*

$$\operatorname{per}(A) \ge \prod_{i=1}^{m} \max\{1, r_i - i + 1\}. \tag{7.33}$$

Proof. We prove (7.33) by induction on m. If $m = 1$ then (7.33) clearly holds. Now assume that $m > 1$. Because $\operatorname{per}(A) > 0$, if P and Q are permutation matrices such that

$$PAQ = \begin{bmatrix} A_1 & O \\ A_2 & A_3 \end{bmatrix}, \tag{7.34}$$

the number k of rows and the number l of columns of A_1 satisfies $l \ge k$. We consider two cases.

Case 1. Whenever (7.34) holds with $k < m$ we have $l > k$.

Without loss of generality we assume that the r_1 1's in the first row of A occur in the first r_1 columns. The Laplace expansion by row 1 yields

$$\operatorname{per}(A) = \sum_{j=1}^{r_1} a_{1j} \operatorname{per}(A(1,j)) \tag{7.35}$$

where $A(1,j)$ is the submatrix of A obtained by deleting row 1 and column j. The assumption in this case and Theorem 1.2.3 imply that

$$\operatorname{per}(A(1,j)) > 0, \quad (j = 1, 2, \ldots, r_1).$$

The number $r_i(j)$ of 1's in row i of $A(1,j)$ satisfies

$$r_i(j) \ge r_{i+1} - 1, \quad (i = 1, 2, \ldots, m-1; j = 1, 2, \ldots, r_1). \tag{7.36}$$

Let

$$r'_1(j), r'_2(j), \ldots, r'_{m-1}(j)$$

denote a rearrangement of $r_1(j), r_2(j), \ldots, r_{m-1}(j)$ satisfying

$$r'_1(j) \le r'_2(j) \le \cdots \le r'_{m-1}(j), \quad (j = 1, 2, \ldots, r_1). \tag{7.37}$$

Applying the inductive hypothesis to each of the matrices $A(1,j), (j = 1, 2, \ldots, r_1)$, we obtain

$$\operatorname{per}(A(1,j)) \ge \prod_{i=1}^{m-1} \max\{1, r'_i(j) - i + 1\}$$

$$\ge \prod_{i=1}^{m-1} \{1, r_{i+1} - 1 - i + 1\} = \prod_{i=2}^{m} \max\{1, r_i - i + 1\}. \tag{7.38}$$

The second inequality in (7.38) follows from (7.36) and the fact that the rearrangement of the rows that achieves (7.37) can be accomplished by rearrangements only among those rows of $A(1, j)$ that correspond to rows of A with the same number of 1's. Combining (7.35) and (7.38) we obtain

$$\operatorname{per}(A) \geq r_1 \prod_{i=2}^{m} \max\{1, r_i - i + 1\}$$

and (7.33) holds in this case.

Case 2. There exist permutation matrices P and Q and an integer k with $1 \leq k \leq n-1$ such that (7.34) holds where A_1 has order k.

There exists a k-subset α of the rows of A and a k-subset β of the columns such that the submatrix $A[\alpha, \overline{\beta}]$ of A determined by the rows in α and the columns not in β is a zero matrix. We have

$$\operatorname{per}(A) = \operatorname{per}(A[\alpha, \beta])\operatorname{per}(A[\overline{\alpha}, \overline{\beta}]). \tag{7.39}$$

Let $\alpha = \{i_1, i_2, \ldots, i_k\}$ and let $\overline{\alpha} = \{i_{k+1}, i_{k+2}, \ldots, i_m\}$ where $i_1 \leq i_2 \leq \cdots \leq i_k$ and $i_{k+1} \leq i_{k+2} \leq \cdots \leq i_m$. Because $A[\alpha, \overline{\beta}] = O$ and $r_1 \leq r_2 \leq \cdots \leq r_m$ we have $r_i \leq k, (i = 1, 2, \ldots, i_k)$ and hence

$$\max\{1, r_i - i + 1\} = 1, \quad (i = k, k+1, \ldots, i_k). \tag{7.40}$$

Applying the inductive hypothesis and using (7.40) we obtain

$$\operatorname{per}(A[\alpha, \beta]) \geq \prod_{j=1}^{k} \max\{1, r_{i_j} - j + 1\}$$

$$\geq \prod_{j=1}^{k} \max\{1, r_j - j + 1\} = \prod_{j=1}^{i_k} \max\{1, r_j - j + 1\}. \tag{7.41}$$

Let the row sums of $A[\overline{\alpha}, \overline{\beta}]$ be $r'_{k+1}, r'_{k+2}, \ldots, r'_m$ arranged so that $r'_{k+1} \leq r'_{k+2} \leq \cdots \leq r'_m$. Applying the inductive hypothesis again and using (7.40) we obtain

$$\operatorname{per}(A[\overline{\alpha}, \overline{\beta}]) \geq \prod_{i=1}^{m-k} \max\{1, r'_{k+i} - i + 1\}$$

$$\geq \prod_{j=i_k+1}^{m} \max\{1, r'_j - (j-k) + 1\} \geq \prod_{j=i_k+1}^{m} \max\{1, r_j - j + 1\}. \tag{7.42}$$

Using (7.39), (7.41) and (7.42) we obtain (7.33). Hence the theorem holds by induction. □

Let A be a Ferrers matrix $F(b_1, b_2, \ldots, b_m)$ as defined in section 7.2. The row sums of A satisfy $b_1 \leq b_2 \leq \cdots \leq b_m$ and $\operatorname{per}(A) > 0$ if and only if $b_i \geq i, (i = 1, 2, \ldots, m)$. By Corollary (7.2.6), $\operatorname{per}(A) = \prod_{i=1}^{m}(b_i - i + 1)$ and hence equality holds in (7.33). Equality also holds in (7.33) for all permutation matrices.

The following result is due to Hall[1948].

Corollary 7.4.2. *Let A be a $(0,1)$-matrix of size m by n with $m \leq n$, and let t be a positive integer such that each row of A contains at least t 1's. If $t < m$ and* $\operatorname{per}(A) > 0$, *then* $\operatorname{per}(A) \geq t!$. *If $t \geq m$, then* $\operatorname{per}(A) \geq t!/(t-m)!$.

Proof. If $t < m$ and $\operatorname{per}(A) > 0$, then by Theorem 7.4.1

$$\operatorname{per}(A) \geq \prod_{i=1}^{m} \max\{1, t - i + 1\} = t!.$$

Now suppose that $t \geq m$. It follows from Theorem 1.2.3 that $\operatorname{per}(A) > 0$. By Theorem 7.4.1

$$\operatorname{per}(A) \geq \prod_{i=1}^{m} \max\{1, t - i + 1\} = t!/(t-m)!. \qquad \square$$

Inequality (7.33) gives the best known general lower bound for the permanent, if not zero, of a (0,1)-matrix in terms of the numbers r_i of 1's in its rows. We now obtain the best known general upper bound $\prod_{i=1}^{n}(r_i!)^{1/r_i}$ for the permanent of a (0,1)-matrix A of order n in terms of the r_i's. This bound was conjectured by Minc[1963] as a generalization of a conjecture of Ryser[1960]. Minc[1974] proved the conjecture under the assumption that no row of A has more than eight 1's. Weaker upper bounds were obtained by Jurkat and Ryser[1966], Minc[1963,1967], Wilf[1968] and Nijenhuis and Wilf[1970]. The conjecture was first proved by Brégman [1973]. A simpler proof was obtained by Schrijver[1978] and it is this proof that we present. We first prove two lemmas. We adopt the convention that $0^0 = 1$.

Lemma 7.4.3. *Let $t_1, t_2, \ldots, t_m$ be nonnegative numbers. Then*

$$\left(\frac{1}{m}\sum_{i=1}^{m} t_i\right)^{\sum_{i=1}^{m} t_i} \leq \prod_{i=1}^{m} t_i^{t_i}. \tag{7.43}$$

Proof. The function $f(x) = x \log x$ is a convex function on $(0, \infty)$ and hence satisfies

$$f\left(\frac{1}{m}\sum_{i=1}^{m} t_i\right) \leq \frac{1}{m}\sum_{i=1}^{m} f(t_i).$$

This yields

$$\frac{1}{m}\left(\sum_{i=1}^{m} t_i\right)\log\left(\frac{1}{m}\sum_{i=1}^{m} t_i\right) \le \frac{1}{m}\sum_{i=1}^{m} t_i \log t_i,$$

from which (7.43) for positive t_i's follows. By taking limits we see that (7.43) holds for nonnegative t_i's. □

Lemma 7.4.4. *Let $A = [a_{ij}]$ be a $(0,1)$-matrix of order n. Let T be the set of all permutations σ of $\{1, 2, \ldots, n\}$ satisfying $\prod_{i=1}^{n} a_{i\sigma(i)} = 1$. Then*

$$\prod_{\{(i,k):a_{ik}=1\}} \operatorname{per}(A(i,k))^{\operatorname{per}(A(i,k))} = \prod_{\sigma\in T}\prod_{i=1}^{n} \operatorname{per}(A(i,\sigma(i))). \qquad (7.44)$$

Proof. The only factors $\operatorname{per}(A(i,k))$ that occur on either side of (7.44) are those for which $a_{ik} = 1$. If $a_{ik} = 1$, then the number of times that $\operatorname{per}(A(i,k))$ occurs as a factor on the right of (7.44) equals the number of permutations σ of $\{1, 2, \ldots, n\}$ satisfying $\sigma(i) = k$ and $\prod_{i=1}^{n} a_{i\sigma(i)} = 1$; this number equals $\operatorname{per}(A(i,k))$. □

Theorem 7.4.5. *Let $A = [a_{ij}]$ be a $(0,1)$-matrix with row sums $r_1, r_2, \ldots, r_n$. Then*

$$\operatorname{per}(A) \le \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}. \qquad (7.45)$$

Proof. The inequality holds if $n = 1$. We suppose that $n > 1$ and proceed by induction on n. Let T be defined as in Lemma 7.4.4. The Laplace expansion of the permanent by rows and the inequality (7.43) and the identity (7.44) yield

$$\begin{aligned}
(\operatorname{per}(A))^{n\operatorname{per}(A)} &= \prod_{i=1}^{n} (\operatorname{per}(A))^{\operatorname{per}(A)} \\
&= \prod_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}\operatorname{per}(A(i,k))\right)^{\sum_{k=1}^{n} a_{ik}\operatorname{per}(A(i,k))} \\
&\le \prod_{i=1}^{n}\left(r_i^{\operatorname{per}(A)} \prod_{\{k:a_{ik}=1\}} \operatorname{per}(A(i,k))^{\operatorname{per}(A(i,k))}\right) \\
&= \prod_{i=1}^{n} r_i^{\operatorname{per}(A)} \prod_{\{(i,k):a_{ik}=1\}} \operatorname{per}(A(i,k))^{\operatorname{per}(A(i,k))} \\
&= \prod_{i=1}^{n} r_i^{\operatorname{per}(A)} \prod_{\sigma\in T}\prod_{i=1}^{n} \operatorname{per}(A(i,\sigma(i))). \qquad (7.46)
\end{aligned}$$

Let σ be a permutation in T. Applying the inductive hypothesis to $\mathrm{per}(A(i,\sigma(i)),(i=1,2,\ldots,n)$ we obtain

$$\prod_{i=1}^{n} \mathrm{per}(A(i,\sigma(i)))$$

$$\leq \prod_{i=1}^{n}\left(\prod_{\{j:j\neq i,a_{j\sigma(i)}=0\}}(r_j!)^{1/r_j}\right)\left(\prod_{\{j:j\neq i,a_{j\sigma(i)}=1\}}((r_j-1)!)^{1/(r_j-1)}\right)$$

$$=\prod_{j=1}^{n}\left(\prod_{\{i:j\neq i,a_{j\sigma(i)}=0\}}(r_j!)^{1/r_j}\right)\left(\prod_{\{i:j\neq i,a_{j\sigma(i)}=1\}}((r_j-1)!)^{1/(r_j-1)}\right)$$

$$=\prod_{j=1}^{n}(r_j!)^{(n-r_j)/r_j}((r_j-1)!)^{(r_j-1)/(r_j-1)}$$

$$=\prod_{j=1}^{n}(r_j!)^{(n-r_j)/r_j}(r_j-1)!. \tag{7.47}$$

Using (7.47) in (7.46) we obtain

$$(\mathrm{per}(A)^{n\mathrm{per}(A)} \leq \prod_{i=1}^{n} r_i^{\mathrm{per}(A)}\left(\prod_{j=1}^{n}(r_j!)^{(n-r_j)/r_j}(r_j-1)!\right)^{\mathrm{per}(A)}$$

$$=\left(\prod_{j=1}^{n}(r_j!)^{(n-r_j)/r_j}r_j!\right)^{\mathrm{per}(A)}=\left(\prod_{j=1}^{n}(r_j!)^{1/r_j}\right)^{n\mathrm{per}(A)}$$

Hence $\mathrm{per}(A)\leq\prod_{j=1}^{n}(r_j!)^{1/r_j}$, and the theorem follows by induction. □

The inequality (7.45) is an improvement for (0,1)-matrices of the upper bound

$$\mathrm{per}(A)\leq r_1r_2\cdots r_n$$

which is valid more generally for nonnegative matrices of order n. We remark that the inequalities obtained from these inequalities by replacing the row sums $r_1,r_2,\ldots,r_n$ of A with its column sums $s_1,s_2,\ldots,s_n$ also hold. Thus for A a (0,1)-matrix we have

$$\mathrm{per}(A)\leq\min\left\{\prod_{i=1}^{n}(r_i!)^{1/r_i},\prod_{i=1}^{n}(s_i!)^{1/s_i}\right\}.$$

If A is a nonnegative matrix, then

$$\operatorname{per}(A) \leq \min\left\{\prod_{i=1}^{n} r_i, \prod_{i=1}^{n} s_j\right\}.$$

The following theorem of Jurkat and Ryser[1967] improves this last inequality.

Theorem 7.4.6. *Let $A = [a_{ij}]$ be a nonnegative matrix of order n with row sums $r_1, r_2, \ldots, r_n$ and column sums $s_1, s_2, \ldots, s_n$. Assume that the rows and columns of A have been arranged so that $r_1 \leq r_2 \leq \cdots \leq r_n$ and $s_1 \leq s_2 \leq \cdots \leq s_n$. Then*

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} \min\{r_i, s_i\}.$$

Proof. We first observe that an easy induction shows that

$$\prod_{i=1}^{n} \min\{r_i, s_{j_i}\} \leq \prod_{i=1}^{n} \min\{r_i, s_i\}$$

holds for each permutation $j_1, j_2, \ldots, j_n$ of $\{1, 2, \ldots, n\}$.

The theorem holds if $n = 1$. We assume that $n > 1$ and proceed by induction on n. Without loss of generality we assume that $r_1 \leq s_1$. From the Laplace expansion by row 1 we obtain

$$\operatorname{per}(A) = \sum_{j=1}^{n} a_{1j} \operatorname{per}(A(1, j)).$$

Let the row sums and the column sums of $A(1, j)$ be, respectively,

$$r_2(j), r_3(j), \ldots, r_n(j), \quad (j = 1, 2, \ldots, n)$$

and

$$s_1(j), \ldots, s_{j-1}(j), s_{j+1}(j), \ldots, s_n(j), \quad (j = 1, 2, \ldots, n).$$

Let $i_1, i_2, \ldots, i_{n-1}$ be a permutation of $\{2, 3, \ldots, n\}$ such that

$$r_{i_1}(j) \leq r_{i_2}(j) \leq \cdots \leq r_{i_{n-1}}(j).$$

Let $k_1, k_2, \ldots, k_{n-1}$ be a permutation of $\{1, \ldots, j-1, j+1, \ldots, n\}$ such that

$$s_{k_1}(j) \leq s_{k_2}(j) \leq \cdots \leq s_{k_{n-1}}(j).$$

We have $r_{i_t}(j) \leq r_{i_t}$ and $s_{k_t}(j) \leq s_{k_t}$ for $t = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, n$. Hence by the inductive hypothesis

$$\operatorname{per}(A(1, j)) \leq \prod_{t=1}^{n-1} \min\{r_{i_t}, s_{k_t}\}, \quad (j = 1, 2, \ldots, n).$$

Therefore

$$\begin{aligned}
\operatorname{per}(A) = \sum_{j=1}^{n} a_{1j}\operatorname{per}(A(1,j)) &\le \left(\sum_{j=1}^{n} a_{1j}\right)\prod_{t=1}^{n-1}\min\{r_{i_t}, s_{k_t}\} \\
&= r_1 \prod_{t=1}^{n-1}\min\{r_{i_t}, s_{k_t}\} \\
&\le r_1 \prod_{i=2}^{j}\min\{r_i, s_{i-1}\}\prod_{i=j+1}^{n}\min\{r_i, s_i\} \\
&\le r_1 \prod_{i=2}^{j}\min\{r_i, s_i\}\prod_{i=j+1}^{n}\min\{r_i, s_i\} \\
&= r_1 \prod_{i=2}^{n}\min\{r_i, s_i\}.
\end{aligned}$$

Because $r_1 \le s_1$ the theorem now follows. □

Theorem 7.4.5 implies the validity of a conjecture of Ryser[1960]. Let n and k be integers with $1 \le k \le n$, and let $\mathcal{A}_{n,k}$ denote the set of all (0,1)-matrices of order n with exactly k 1's in each line. By Theorem 7.4.2 the permanent of each matrix A in $\mathcal{A}_{n,k}$ satisfies $\operatorname{per}(A) \ge k!$.

Theorem 7.4.7. *If k is a divisor of n, then the maximum permanent of a matrix in $\mathcal{A}_{n,k}$ equals $(k!)^{n/k}$.*

Proof. Suppose that k is a divisor of n and that A is a matrix in $\mathcal{A}_{n,k}$. By (7.45), $\operatorname{per}(A) \le (k!)^{n/k}$. The matrix $A = J_k \oplus \cdots \oplus J_k$, which is the direct sum of n/k matrices each of which equals the all 1's matrix J_k of order k, satisfies $\operatorname{per}(A) = (k!)^{n/k}$. □

Brualdi, Goldwasser and Michael[1988] generalized Theorem 7.4.7 by showing that $\operatorname{per}(A) \le (k!)^{n/k}$ provided that k is a divisor of n and A is a (0,1)-matrix of order n and the *average* number of 1's in each row equals k.

Now let

$$\beta(n,k) = \max\{\operatorname{per}(A) : A \in \mathcal{A}_{n,k}\}$$

denote the largest permanent achieved by a matrix in $\mathcal{A}_{n,k}$. If k is a divisor of n, then by Theorem 7.4.7, $\beta(n,k) = (k!)^{n/k}$. If $k = 2$ then Merriell[1980] proved that

$$\beta(n,2) = 2^{\lfloor n/2 \rfloor}$$

holds for all n. Brualdi, Goldwasser and Michael[1988] showed that if $A \in \mathcal{A}_{n,2}$ satisfies $per(A) = 2^{\lfloor n/2 \rfloor}$, then there are permutation matrices P and Q of order n such that either

$$PAQ = J_2 \oplus J_2 \oplus \cdots \oplus J_2, \quad (n \text{ even})$$

or

$$PAQ = (J_3 - I_3) \oplus J_2 \oplus \cdots \oplus J_2, \quad (n \text{ odd}).$$

If $k = 3$ and 3 is not a divisor of n, then Merriell determined $\beta(n, 3)$ as follows:

$$\beta(3t + 1, 3) = 6^{t-1}9, \quad (t \geq 1),$$

$$\beta(5, 3) = 13, \beta(3t + 2) = 6^{t-2}9^2, \quad (t \geq 2).$$

The matrix

$$(J_4 - I_4) \oplus J_3 \oplus \cdots \oplus J_3$$

in $\mathcal{A}_{3t+1,3}$ has permanent equal to $6^{t-1}9, (t \geq 1)$. The matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

in $\mathcal{A}_{5,3}$ has permanent equal to 13. The matrix

$$(J_4 - I_4) \oplus (J_4 - I_4) \oplus J_3 \oplus \cdots \oplus J_3$$

in $\mathcal{A}_{3t+2,3}$ has permanent equal to $6^{t-2}9^2, (t \geq 2)$.

If $k = 4$ then Bol'shakov[1986] determined that

$$\beta(4t + 1, 4) = 24^{t-1}44, \quad (t \geq 1).$$

The matrix

$$(J_5 - I_5) \oplus J_4 \oplus \cdots \oplus J_4$$

in $\mathcal{A}_{4t+1}$ has permanent equal to $24^{t-1}44$.

If $k = n - 1$ then $\beta(n, n - 1) = D_n$, the nth derangement number, and every matrix in $\mathcal{A}_{n.k}$ has permanent equal to D_n. Now let $k = n - 2$. If $n \geq 8$, then Brualdi, Goldwasser and Michael[1988] showed that if A is a matrix in $\mathcal{A}_{n,n-2}$ satisfying per$(A) = \beta(n, n - 2)$, then there exist permutation matrices P and Q of order n such that

$$PAQ = J_n - (J_2 \oplus J_2 \oplus \cdots \oplus J_2), \quad (n \text{ even}), \tag{7.48}$$

or

$$PAQ = J_n - ((J_3 - I_3) \oplus J_2 \oplus \cdots \oplus J_2), \quad (n \text{ odd}). \tag{7.49}$$

A simple expression for $\beta(n, n-2)$ is not known, but recurrence relations can be obtained from (7.48) and (7.49). If $n < 8$ the matrices A in $\mathcal{A}_{n,n-2}$ with $\text{per}(A) = \beta(n, n-2)$ are given in Brualdi, Goldwasser and Michael[1988].

Now let

$$\lambda(n, k) = \min\{\text{per}(A) : A \in \mathcal{A}_{n,k}\}, \quad (k = 1, 2, \ldots, n)$$

denote the smallest permanent achieved by a matrix in $\mathcal{A}_{n,k}$. We have $\lambda(n, 1) = 1$ and because the matrix $I_n + C_n$[1] in $\mathcal{A}_{n,2}$ has permanent equal to 2, $\lambda(n, 2) = 2$. We also have $\lambda(n, n) = n!$ and $\lambda(n, n-1) = D_n$, the nth derangement number. Henderson[1975] has shown that $\lambda(n, n-2) = U_n$, the nth ménage number, if n is even and $\lambda(n, n-2) = -1 + U_n$ if n is odd. The matrix $J_n - I_n - C_n$ in $\mathcal{A}_{n,n-2}$ has permanent equal to U_n. If n is odd and $n = 2k+1$, the matrix $J_n - ((I_k + C_k) \oplus (I_{k+1} + C_{k+1}))$ has permanent equal to $-1 + U_n$.

The exact value of $\lambda(n, 3)$ is not known in general. However the following bound of Voorhoeve[1979] gives the exponential lower bound

$$\lambda(n, 3) \geq 6 \left(\frac{4}{3}\right)^{n-3}, \quad (n \geq 3).$$

This bound holds for a wider class of matrices whose introduction facilitates its proof.

Let $\mathcal{B}_{n,k}$ denote the class of all nonnegative integral matrices of order n each of whose line sums equals k. The class $\mathcal{A}_{n,k}$ consists of all (0,1)-matrices in $\mathcal{B}_{n,k}$. Let

$$\lambda^*(n, k) = \min\{\text{per}(A) : A \in \mathcal{B}_{n,k}\}, \quad (k = 1, 2, \ldots, n)$$

denote the smallest permanent achieved by a matrix in $\mathcal{B}_{n,k}$. We have $\lambda(n, k) \geq \lambda^*(n, k)$.

Theorem 7.4.8. *For all $n \geq 3$,*

$$\lambda^*(n, 3) \geq 6 \left(\frac{4}{3}\right)^{n-3}. \tag{7.50}$$

Proof. Let $\mathcal{B}'_{n,3}$ denote the class of matrices obtained by subtracting 1 from a positive entry of matrices in $\mathcal{B}_{n,3}$. Thus a nonnegative integral matrix of order n belongs to $\mathcal{B}'_{n,3}$ if and only if its row sums and its

[1] Recall that C_n is the permutation matrix of order n with 1's in positions $(1, 2), (2, 3), \ldots, (n-1, n), (n, 1)$.

column sums are $3, \ldots, 3, 2$ in some order. We denote the smallest permanent achieved by a matrix in $\mathcal{B}'_{n,3}$ by $\lambda'(n,3)$. We prove (7.50) by showing that

$$\lambda^*(n,3) \geq \frac{3}{2}\lambda'(n,3), \quad (n \geq 3), \tag{7.51}$$

and

$$\lambda'(n,3) \geq \frac{4}{3}\lambda'(n-1,3), \quad (n \geq 4). \tag{7.52}$$

Let A be a matrix in $\mathcal{B}_{n,3}$. Without loss of generality we assume that the first row of A equals

$$a_1, a_2, a_3, 0, \ldots, 0$$

where a_1, a_2 and a_3 are nonnegative integers with $a_1 + a_2 + a_3 = 3$. We have

$$\begin{aligned}&(2a_1, 2a_2, 2a_3)\\ &= a_1(a_1 - 1, a_2, a_3) + a_2(a_1, a_2 - 1, a_3) + a_3(a_1, a_2, a_3 - 1).\end{aligned} \tag{7.53}$$

Because the permanent is a linear function of each of the rows of a matrix, it follows from (7.53) that

$$2\mathrm{per}(A) = a_1\mathrm{per}(A_1) + a_2\mathrm{per}(A_2) + a_3\mathrm{per}(A_3) \tag{7.54}$$

where A_1, A_2 and A_3 are obtained from A by subtracting one, respectively, from the elements of A in positions (1,1), (1,2) and (1,3). We note that one or two of the a_i's may be zero, but then the term $a_i\mathrm{per}(A_i) = 0$. If $a_i \neq 0$ then A_i belongs to the class $\mathcal{B}'_{n,3}$, and it follows from (7.54) that

$$2\mathrm{per}(A) \geq (a_1 + a_2 + a_3)\lambda'(n,3) = 3\lambda'(n,3).$$

Hence (7.51) holds.

Now assume that $n \geq 4$, and let A denote a matrix in $\mathcal{B}'_{n,3}$ satisfying $\mathrm{per}(A) = \lambda'(n,3)$. Without loss of generality we assume that the first row sum of A equals 2 and that the first row of A is either $1, 1, 0, \ldots, 0$ or $2, 0, \ldots, 0$.

First assume that the first row of A is $1, 1, 0, \ldots, 0$. We write

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ u & v & & B & \end{bmatrix}.$$

Because the permanent is a linear function of each of the columns of a square matrix, we have $\mathrm{per}(A) = \mathrm{per}(C)$ where

$$C = \begin{bmatrix} u+v & B \end{bmatrix},$$

a matrix of order $n-1$. The sum s of the entries of $u+v$ equals 3 or 4. If $s = 3$, then C is in the class $\mathcal{B}_{n-1,3}$ and by (7.51)

$$\begin{aligned}\lambda'(n,3) &= \mathrm{per}(A) = \mathrm{per}(C) \geq \lambda^*(n-1,3)\\ &\geq \frac{3}{2}\lambda'(n-1,3) \geq \frac{4}{3}\lambda'(n-1,3).\end{aligned}$$

Now suppose that $s = 4$. Without loss of generality we assume that

$$u + v = [\ d_1 \quad d_2 \quad d_3 \quad d_4 \quad 0 \quad \cdots \quad 0\ ]^{\mathrm{T}},$$

where d_1, d_2, d_3, d_4 are nonnegative integers satisfying $d_1+d_2+d_3+d_4 = 4$. We have

$$\begin{gathered}(3d_1, 3d_2, 3_3, 3d_4)\\ = d_1(d_1-1, d_2, d_3, d_4) + d_2(d_1, d_2-1, d_3, d_4) + d_3(d_1, d_2, d_3-1, d_4)\\ +d_4(d_1, d_2, d_3, d_4-1).\end{gathered}$$

Using the linearity of the permanent again we obtain

$$3\mathrm{per}(C) = \sum_{i=1}^{4} d_i \mathrm{per}(C_i)$$

where C_i is the matrix obtained by subtracting one from the element d_i in the first column of C,$(i = 1, 2, 3, 4)$. If $d_i \neq 0$ then C_i is a matrix in the class $\mathcal{B}'_{n-1,3}$ and hence

$$3\lambda'(n,3) = 3\mathrm{per}(A) = 3\mathrm{per}(C) \geq \sum_{i=1}^{4} d_i\lambda'(n-1,3) = 4\lambda'(n-1,3).$$

Therefore $\lambda'(n,3) \geq (4/3)\lambda'(n-1,3)$ if $s = 4$.

Now assume that the first row of A is $(2, 0, \ldots, 0)$. We then write

$$A = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ u & & B & \end{bmatrix}.$$

We have $\text{per}(A) = 2\text{per}(B)$ where either B is in the class $\mathcal{B}_{n-1,3}$ or B is in the class $\mathcal{B}'_{n-1,3}$. Hence

$$\lambda'(n,3) = \text{per}(A) \geq 2\min\{\lambda(n-1,3), \lambda'(n-1,3)\} = 2\lambda'(n-1,3)$$

and hence

$$\lambda'(n,3) \geq \frac{4}{3}\lambda'(n-1,3).$$

Thus (7.52) holds for all $n \geq 4$. The proof of the theorem is completed by noting that $\lambda'(3,3) = 4$. □

Let A be a matrix in the class $\mathcal{B}_{n,k}$ where $n \geq 3$ and $k \geq 3$. Then there exist matrices A_1 in $\mathcal{B}_{n,3}$ and A_2 in $\mathcal{B}_{n,k-3}$ such that $A = A_1 + A_2$. Hence

$$\text{per}(A) \geq \text{per}(A_1) \geq \lambda^*(n,3) \geq 6\left(\frac{4}{3}\right)^{n-3},$$

and it follows that

$$\lambda^*(n,k) \geq 6\left(\frac{4}{3}\right)^{n-3} \geq \left(\frac{4}{3}\right)^n, \quad (n \geq 3, k \geq 3). \tag{7.55}$$

Let

$$\theta_k = \liminf_{n\to\infty} \lambda^*(n,k)^{1/n}, \quad (k \geq 3). \tag{7.56}$$

The number θ_k gives the best exponential lower bound

$$\text{per}(A) \geq \theta_k^n$$

for matrices A in $\mathcal{B}_{n,k}$ with n sufficently large. It follows from (7.55) that

$$\theta_k \geq \frac{4}{3}, \quad (k \geq 3). \tag{7.57}$$

The following upper bound for θ_k is due to Schrijver and Valiant[1980].

Theorem 7.4.9. *For all $k \geq 3$,*

$$\theta_k \leq \frac{(k-1)^{k-1}}{k^{k-2}}.$$

Proof. Let $X = \{1,2,3,\ldots,nk\}$ and let $\mathcal{X}_{n,k}$ denote the collection of all ordered partitions $\mathcal{U} = (U_1, U_2, \ldots, U_n)$ of X into n sets of size k. An elementary combinatorial count shows that the number $c(n,k)$ of ordered partitions in $\mathcal{X}_{n,k}$ satisfies

$$c(n,k) = \frac{(nk)!}{(k!)^n}. \tag{7.58}$$

Let $\mathcal{V} = (V_1, V_2, \ldots, V_n)$ denote another ordered partition in $\mathcal{X}_{n,k}$. A set $R = \{x_1, x_2, \ldots, x_n\}$ of size n which consists of one element from each of the sets $U_1, U_2, \ldots, U_n$ is called a *transversal* of $\mathcal{U}$. The collection of transversals of $\mathcal{U}$ is denoted by $\mathcal{T}_{\mathcal{U}}$. A transversal of $\mathcal{U}$ which is also a transversal of $\mathcal{V}$ is called a *common transversal* of $\mathcal{U}$ and $\mathcal{V}$. (Thus the elements of a common transversal can be ordered to give an SDR of $\mathcal{U}$, and they can also be ordered to give an SDR of $\mathcal{V}$.) We denote the collection of all common transversals of $\mathcal{U}$ and $\mathcal{V}$ by $\mathcal{T}_{\mathcal{U},\mathcal{V}}$, and denote their number by $t_{\mathcal{U},\mathcal{V}}$. The number $t_{\mathcal{U},\mathcal{V}}$ can be computed as a permanent. Let $E = [e_{ij}]$ be the nonnegative integral matrix of order n defined by

$$e_{ij} = |U_i \cap V_j|, \quad (i, j, = 1, 2, \ldots, n).$$

The matrix E belongs to the class $\mathcal{B}_{n,k}$. If $j_1, j_2, \ldots, j_n$ is a permutation of $\{1, 2, \ldots, n\}$, then $e_{1j_1} e_{2j_2} \cdots e_{nj_n}$ counts the number of common transversals $\{x_1, x_2, \ldots, x_n\}$ of $\mathcal{U}$ and $\mathcal{V}$ in which $x_i \in U_i \cap V_{j_i}, (i = 1, 2, \ldots, n)$. Hence

$$t_{\mathcal{U},\mathcal{V}} = \operatorname{per}(E) \geq \theta_k^n. \tag{7.59}$$

We now fix the ordered partition $\mathcal{U}$ and allow the ordered partition $\mathcal{V}$ to vary over the set $\mathcal{X}_{n,k}$. It follows from (7.58) and (7.59) that

$$\sum_{\mathcal{V} \in \mathcal{X}_{n,k}} t_{\mathcal{U},\mathcal{V}} \geq \frac{(nk)!}{(k!)^n} \theta_k^n. \tag{7.60}$$

We also have

$$\begin{aligned} \sum_{\mathcal{V} \in \mathcal{X}_{n,k}} t_{\mathcal{U},\mathcal{V}} &= \sum_{\mathcal{V} \in \mathcal{X}_{n,k}} \sum_{R \in \mathcal{T}_{\mathcal{U},\mathcal{V}}} 1 = \sum_{R \in \mathcal{T}_{\mathcal{U}}} \sum_{\{\mathcal{V}: \mathcal{V} \in \mathcal{X}_{n,k}, R \in \mathcal{T}_{\mathcal{V}}\}} 1 \\ &= \sum_{R \in \mathcal{T}_{\mathcal{U}}} n! c(n, k-1) = k^n n! c(n, k-1). \end{aligned} \tag{7.61}$$

In the previous calculation we have used the facts that a transversal of $\mathcal{U}$ is a common transversal of $\mathcal{U}$ and $\mathcal{V}$ for exactly $n!c(n, k-1)$ ordered partitions $\mathcal{V}$ in $\mathcal{X}_{n,k}$ and that $\mathcal{U}$ has exactly k^n transversals. We now use (7.58) (with k replaced with $k-1$), (7.60) and (7.61) and obtain

$$k^n n! \frac{(nk-n)!}{(k-1)!^n} \geq \frac{(nk)!}{k!^n} \theta_k^n$$

and hence

$$\theta_k^n \leq \frac{k^{2n}}{\binom{nk}{n}}. \tag{7.62}$$

Applying Stirling's formula in (7.62) we obtain $\theta_k \leq (k-1)^{k-1}/k^{k-2}$. □

Corollary 7.4.10.

$$\theta_3 = \frac{4}{3}.$$

Proof. By (7.57), $\theta_3 \geq 4/3$. By Theorem 7.4.9, $\theta_3 \leq 4/3$. □

It has been *conjectured* by Schrijver and Valiant[1980] that

$$\theta_k = (k-1)^{k-1}/k^{k-2}, k \geq 3.$$

The solution of the van der Waerden conjecture for the minimum permanent of a doubly stochastic matrix of order n (Egoryčev[1981] and Falikman[1981]) yields the bound $\theta_k \geq k/e$ which is better than the bound in (7.57) for all $k \geq 4$. In addition, Schrijver[1983] has shown that $\theta_4 \geq 3/2$ and $\theta_6 \geq 20/9$.

We now consider lower and upper bounds for the permanent of a (0,1)-matrix which depend on the total number of 1's in the matrix (and not on how these 1's are distributed in the rows and columns of the matrix).

We recall from Chapter 4 the definitions of a fully indecomposable and nearly decomposable matrix. A (0,1)-matrix A of order n is fully indecomposable provided A does not have an r by $n-r$ zero submatrix for any integer r with $1 \leq r \leq n-1$. The fully indecomposable matrix A is nearly decomposable provided the replacement of a 1 of A with a 0 always results in a matrix which is not fully indecomposable. By Theorem 1.2.1 (cf. Theorem 4.2.2) the (0,1)-matrix A of order n is fully indecomposable if and only if

$$\operatorname{per}(A(i,j)) > 0, \text{ for all } i,j = 1,2,\ldots,n. \tag{7.63}$$

A fully indecomposable (0,1)-matrix has a nonzero permanent. Moreover, it follows from (7.63) that if the fully indecomposable matrix A has row sums $r_1, r_2, \ldots, r_n$ and column sums $s_1, s_2, \ldots, s_n$, then

$$\operatorname{per}(A) \geq \max\{r_1, r_2, \ldots, r_n, s_1, s_2, \ldots, s_n\}. \tag{7.64}$$

Now let A be a (0,1)-matrix of order n with a nonzero permanent, and let $A_1, A_2, \ldots, A_t, (t \geq 1)$, be the fully indecomposable components of A (cf. Theorem 4.2.6). Then

$$\operatorname{per}(A) = \prod_{i=1}^{t} \operatorname{per}(A_i). \tag{7.65}$$

Hence bounds for the permanent of a fully indecomposable matrix will give bounds for the permanent of any matrix with a nonzero permanent. Minc[1969] obtained the following lower bound for the permanent of a fully indecomposable (0,1)-matrix. We follow the proof of Hartfiel[1970].

Theorem 7.4.11. *Let A be a fully indecomposable $(0,1)$-matrix of order n with exactly $\sigma(A)$ 1's. Then*

$$\operatorname{per}(A) \geq \sigma(A) - 2n + 2. \tag{7.66}$$

Proof. We first establish (7.66) for nearly decomposable matrices A by induction on n. If $n = 1$ then (7.66) clearly holds. Now assume that $n \geq 2$. It follows from Theorem 4.3.4 that we may assume that

$$A = \left[\begin{array}{cccccccc} 1 & 0 & 0 & \cdots & 0 & 0 & & \\ 1 & 1 & 0 & \cdots & 0 & 0 & & \\ 0 & 1 & 1 & \cdots & 0 & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & F_1 & \\ 0 & 0 & 0 & \cdots & 1 & 0 & & \\ 0 & 0 & 0 & \cdots & 1 & 1 & & \\ & & & F_2 & & & B & \end{array}\right]$$

where B is a nearly decomposable matrix of order m with $1 \leq m \leq n-1$, and F_1 and F_2 each contains exactly one 1. Applying the inductive hypothesis and (7.63) to B we obtain

$$\operatorname{per}(A) \geq \operatorname{per}(B) + 1 \geq \sigma(B) - 2m + 2 + 1 = \sigma(A) - 2n + 2.$$

Hence (7.66) holds if A is nearly decomposable.

Now assume that A is not nearly decomposable. Because A is fully indecomposable there exists a (0,1)-matrix C of order n such that $A - C$ is a nearly decomposable (0,1)-matrix. Applying (7.63) to A and (7.66) to $A - C$, we obtain

$$\operatorname{per}(A) \geq \operatorname{per}(A-C) + \sigma(C) \geq \sigma(A-C) - 2n + 2 + \sigma(C) = \sigma(A) - 2n + 2.$$

□

Corollary 7.4.12. *Let A be a fully indecomposable, nonnegative integral matrix of order n the sum of whose elements equals $\sigma(A)$. Then*

$$\operatorname{per}(A) \geq \sigma(A) - 2n + 2. \tag{7.67}$$

Proof. If A is a (0,1)-matrix we apply Theorem 7.4.11. Suppose that some entry a_{rs} of A is greater than 1. Let B be the matrix obtained from A by subtracting 1 from a_{rs}. Then $\sigma(B) < \sigma(A)$ and B is a fully indecomposable, nonnegative integral matrix. Arguing by induction on the sum of elements we obtain

$$\operatorname{per}(A) = \operatorname{per}(B) + \operatorname{per}(A(r,s)) \geq \sigma(B) - 2n + 2 + 1 = \sigma(A) - 2n + 2.$$

□

Brualdi and Gibson[1977] characterized the fully indecomposable, nonnegative integral matrices A of order n for which equality holds in (7.67) as follows. Assume that $n \geq 2$. Then $\text{per}(A) = \sigma(A) - 2n + 2$ if and only if there exists an integer p with $0 \leq p \leq n-1$ and permutation matrices P and Q of order n such that

$$PAQ = \begin{bmatrix} A_3 & A_1 \\ A_2 & O \end{bmatrix} \tag{7.68}$$

where A_3 is a nonnegative integral matrix of size $n-p$ by $p+1$, and A_1^T and A_2 are (0,1)-matrices with exactly two 1's in each row. The full indecomposability assumption on A implies that the matrices A_1^T and A_2 are incidence matrices of graphs which are trees.

We now turn to an upper bound of Foregger[1975] for the permanent of a fully indecomposable, nonnegative integral matrix. The following lemma is a step in its proof.

Lemma 7.4.13. *Let $A = [a_{ij}]$ be a fully indecomposable, nonnegative integral matrix of order $n \geq 2$. Then there exists an integer $j \geq 0$ and a fully indecomposable $(0,1)$-matrix B of order n with $B \leq A$ such that*

$$\text{per}(A) \leq 2^j \text{per}(B) - (2^j - 1) \text{ where } \sigma(A) - \sigma(B) = j. \tag{7.69}$$

Proof. If A is a (0,1)-matrix, then $B = A$ and $j = 0$ satisfy (7.69). Now suppose that there exist an element a_{rs} of A with $a_{rs} \geq 2$. Let A' be the matrix obtained from A by subtracting 1 from a_{rs}. Because A is fully indecomposable and $n \geq 2$, there exists an integer $t \neq s$ such that $a_{rt} \geq 1$. From the Laplace expansion by row r we obtain

$$\begin{aligned} \text{per}(A) &= \sum_{k=1}^{n} a_{rk}\text{per}(A(r,k)) \\ &\geq a_{rs}\text{per}(A(r,s)) + a_{rt}\text{per}(A(r,t)) \geq 2\text{per}(A(r,s)) + 1. \end{aligned}$$

Hence

$$\text{per}(A(r,s)) \leq \frac{(\text{per}(A) - 1)}{2}. \tag{7.70}$$

We also have

$$\text{per}(A) = \text{per}(A') + \text{per}(A(r,s)). \tag{7.71}$$

From (7.70) and (7.71) we get

$$\text{per}(A) \leq 2\text{per}(A') - 1. \tag{7.72}$$

The proof of the lemma is now completed by induction on the sum

$$\sum (a_{ij} - 1 : a_{ij} \geq 2, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n). \qquad \square$$

Theorem 7.4.14. *Let A be a fully indecomposable, nonnegative integral matrix of order n, the sum of whose elements equals $\sigma(A)$. Then*

$$\operatorname{per}(A) \leq 2^{\sigma(A)-2n} + 1. \tag{7.73}$$

Proof. If $n = 1$ the inequality holds. We now assume that $n \geq 2$, and proceed by induction on n. If A is a (0,1)-matrix and each row sum of A is at least equal to three, then (7.45) implies that we have strict inequality in (7.73).

First suppose that A is a (0,1)-matrix with at least one row sum equal to 2. Without loss of generality we assume that row 1 of A equals $1, 1, 0, \ldots, 0$. The linearity of the permanent implies that

$$\operatorname{per}(A) = \operatorname{per}(A(1,1)) + \operatorname{per}(A(1,2)) = \operatorname{per}(A')$$

where A' is the matrix of order $n-1$ obtained from A by adding column 1 to column 2 and then deleting row 1 and column 1. The matrix A' is fully indecomposable and $\sigma(A') = \sigma(A) - 2$. Using the inductive assumption we obtain

$$\operatorname{per}(A) = \operatorname{per}(A') \leq 2^{\sigma(A')-2(n-1)} + 1 = 2^{\sigma(A)-2n} + 1.$$

Now suppose that A has at least one element which is strictly larger than 1. Let B be a (0,1)-matrix and j an integer satisfying the conclusions of Lemma 7.4.13. Applying what we have just shown to the fully indecomposable (0,1)-matrix B, we obtain

$$\begin{aligned} \operatorname{per}(A) &\leq 2^j \operatorname{per}(B) - (2^j - 1) \\ &\leq 2^j(2^{\sigma(B)-2n} + 1) - (2^j - 1) = 2^{\sigma(A)-2n} + 1. \end{aligned}$$

Hence the theorem follows by induction. □

The fully indecomposable, nonnegative integral matrices A for which equality holds in (7.73) have been characterized by Foregger[1975]. Up to row and column permutations such matrices are equal to a matrix of the form

$$\begin{bmatrix} I_{k_1} + C_{k_1} & O & \cdots & O & E_1 \\ E_2 & I_{k_2} + C_{k_2} & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & I_{k_{p-1}} + C_{k_{p-1}} & O \\ O & O & \cdots & E_p & I_{k_p} + C_{k_p} \end{bmatrix}$$

where p is a positive integer, E_i is a (0,1)-matrix containing exactly one 1, and C_{k_i} as usual denotes the permutation matrix of order k_i with 1's in positions $(1,2),(2,3),\ldots,(k_i-1,k_i),(k_i,1)$. (If $k_i=1$, then $C_{k_i}=[1]$ and $I_{k_i}+C_{k_i}=[2]$.)

As noted in the proof of Theorem 7.4.14, if A is a fully indecomposable (0,1)-matrix with all row sums at least equal to three, then (7.73) is implied by (7.45). If some row sum equals two, then (7.73) may be better than (7.45). For example, let

$$A=\begin{bmatrix}1&0&1&0\\1&1&0&1\\0&1&1&0\\0&0&1&1\end{bmatrix}.$$

Then $\operatorname{per}(A)=3$ and (7.73) gives $\operatorname{per}(A)\le 3$. However, (7.45) gives $\operatorname{per}(A)\le 2^{3/2}3^{1/3}=4.079\ldots$. The inequality (7.73) holds for integral matrices A, but (7.45) need not hold if A is not a (0,1)-matrix.

Donald et al.[1984] have improved (7.7.3) by showing that a fully indecomposable, nonnegative integral matrix A with row sums $r_1,\ldots,r_n$ and column sums $s_1,\ldots,s_n$ satisfies

$$\operatorname{per}(A)\le 1+\min\left\{\prod_i(r_i-1),\prod_i(s_i-1)\right\}.$$

To conclude this section we derive the following theorem of Brualdi and Gibson[1977] in which the full indecomposability assumption in Theorem 7.4.14 is replaced by the assumption of total support. We recall from section 4.2 that a matrix of total support is, up to permutation of rows and columns, a direct sum of $t\ge 1$ fully indecomposable matrices, and these t matrices are the fully indecomposable components of A.

Theorem 7.4.15. *Let A be a nonnegative integral matrix of order n with total support, and let t be the number of fully indecomposable components of A. Then*

$$\operatorname{per}(A)\le 2^{\sigma(A)-2n+t}. \tag{7.74}$$

Proof. Without loss of generality we assume that $A=A_1\oplus A_2\oplus\cdots\oplus A_t$ where A_i is a fully indecomposable matrix of order $n_i\ge 1,(i=1,2,\ldots,t)$. We have

$$\operatorname{per}(A)=\prod_{i=1}^{t}\operatorname{per}(A_i).$$

If for some i we have $n_i=1$ and $A_i=[1]$, then $2^{\sigma(A_i)-2n_i+1}=1$ and $\operatorname{per}(A_i)=1$. It follows that we may assume that $\sigma(A_i)-2n_i\ge 0,(i=1,2,\ldots,t)$. If for some i we have $n_i=1$ and $A_i=[2]$, then $2^{\sigma(A_i)-2n_i+1}=2$

and $\operatorname{per}(A_i) = 2$. Hence we may now assume that $\sigma(A_i) - 2n_i \geq 1$, $(i = 1, 2, \ldots, t)$. Using Theorem 7.4.14 we obtain

$$\operatorname{per}(A) = \prod_{i=1}^{t} \operatorname{per}(A_i) \leq \prod_{i=1}^{t} (2^{\sigma(A_i)-2n_i} + 1)$$

$$\leq 2^{(\sum_{i=1}^{t} \sigma(A_i)-2n_i)+t-1} + 1 = 2^{\sigma(A)-2n+t-1} + 1 < 2^{\sigma(A)-2n+t}. \qquad \square$$

It follows from the proof of Theorem 7.4.15 that equality holds in (7.74) if and only if each fully indecomposable component of A is either a matrix of order 1 whose unique element is 1 or 2, or $I_k + C_k$ for some $k \geq 2$.

Exercises

1. Let A be a (0,1)-matrix of size m by n with $m \leq n$. Prove that
$$\operatorname{per}(A) \leq n(n-1)\cdots(n-m+1)$$
with equality if and only if $A = J_{m,n}$.
2. Let A be a fully indecomposable (0,1)-matrix of order n and let r be the maximum row sum of A. Prove that $\operatorname{per}(A) \geq r$ with equality if and only if at least $n-1$ of the row sums of A equal 2 (Minc[1973]).
3. Let A be a fully indecomposable (0,1)-matrix of order n and let r be the minimal row sum of A. Prove that
$$\operatorname{per}(A) \geq \sigma(A) - 2n + 2 + \sum_{k=1}^{r-1} (k! - 1)$$
(Gibson[1972]).
4. Use Theorem 7.4.5 to show that if A is a (0,1)-matrix of order n with row sums $r_1, r_2, \ldots, r_n$, then
$$\operatorname{per}(A) \leq \prod_{i=1}^{n} \frac{r_i + 1}{2}$$
(Minc[1963]).
5. Determine all fully indecomposable (0,1)-matrices with permanent equal to 3.
6. Determine all fully indecomposable (0,1)-matrices with permanent equal to 4.
7. Let k be an integer with $0 \leq k \leq 2^{n-1}$. Show that there is a (0,1)-matrix of order n with permanent equal to k (Brualdi and Newman[1965]).
8. Let p be a prime number and let A be a circulant, nonnegative integral matrix of order p. Let r be the common value of the row sums of A. Prove that
$$\operatorname{per}(A) \equiv r \pmod{p}$$
(Brualdi and Newman[1965]).
9. Let A be a (0,1)-matrix of order n with exactly n 0's. Prove that $\operatorname{per}(A) \leq D_n$, the nth derangement number, with equality if and only if there are permutation matrices P and Q such that $PAQ = J - I$ (Brualdi, Goldwasser and Michael[1988]).

10. Let A be a (0,1)-matrix of order n of the form

$$\left[\begin{array}{cc} A_3 & A_1 \\ A_2 & O \end{array}\right]$$

where O denotes a zero matrix of size p by $n-p-1$, and A_1^T and A_2 are incidence matrices of trees. Prove that $\text{per}(A) = \sigma(A_3)$, the number of 1's of A_3 (Brualdi and Gibson[1977]).
11. Let $n = 2k+1$. Show that the permanent of the matrix $J_n - ((I_k + C_k) \oplus (I_{k+1} + C_{k+1}))$ of order n with exactly two 0's in each row and column equals $-1 + U_n$, where U_n denotes the nth ménage number (Henderson[1975]).
12. Let B be a nonnegative matrix of order n each of whose line sums is at most 1. Use Corollary 7.1.2 to prove that $\text{per}(I - B) \geq 0$ (Gibson[1966]).

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7.5 Evaluation of Permanents

Let $X = [x_{ij}], (i, j = 1, 2, \ldots, n)$ be a real matrix of order n. The definition

$$\text{per}(X) = \sum x_{1\pi(1)} x_{2\pi(2)} \cdots x_{n\pi(n)}$$

of the permanent of X is very much like the definition

$$\det(X) = \sum (\text{sign}\pi) x_{1\pi(1)} x_{2\pi(2)} \cdots x_{n\pi(n)}$$

of the determinant. Both summations are over all permutations π of $\{1, 2, \ldots, n\}$. In the case of the determinant a negative sign is affixed to those summands which correspond to even permutations. The similarity of the two definitions and the existence of efficient computational procedures for determinants suggest the possibility of directly affixing negative signs to some of the elements of the matrix X in order to obtain a matrix $X' = [x'_{ij}]$ with $x'_{ij} = \pm x_{ij}, (i, j = 1, 2, \ldots, n)$ which satisfies

$$\det(X') = \mathrm{per}(X).$$

Let $E = [e_{ij}]$ be a matrix of order n each of whose elements equals 1 or -1, and let

$$E * X = [e_{ij}x_{ij}], \quad (i, j = 1, 2, \ldots, n),$$

be the *elementwise* product of E and X. We say that the matrix E *converts the permanent of n by n matrices into the determinant* provided

$$\mathrm{per}(X) = \det(E * X) \tag{7.75}$$

for all matrices X of order n. If (7.75) holds then we have an effective procedure to calculate the permanent of any real matrix X of order n. Since (7.75) is to hold for all matrices X of order n, (7.75) is a polynomial identity in the n^2 elements of the matrix X. Hence the $(1, -1)$-matrix E converts the permanent of n by n matrices into the determinant if and only if

$$e_{1\pi(1)}e_{2\pi(2)} \cdots e_{n\pi(n)} = \mathrm{sign}(\pi(1), \pi(2), \ldots, \pi(n)) \tag{7.76}$$

for each permutation $(\pi(1), \pi(2), \ldots, \pi(n))$ of $\{1, 2, \ldots, n\}$. Equivalently, the $(1, -1)$-matrix E converts the permanent of n by n matrices into the determinant if and only if

$$\det(E) = n!.$$

Let $n = 2$ and let

$$E = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then $\det(E) = 2$ and it follows that (7.75) holds. Pólya[1913] (see also Szegö[1913]) observed that for $n \geq 3$ there is no matrix E which converts the permanent of n by n matrices into the determinant.

Theorem 7.5.1. *Let $n \geq 3$. Then there does not exist a $(1, -1)$-matrix E which converts the permanent of n by n matrices into the determinant.*

Proof. First suppose that $n = 3$ and that $E = [e_{ij}]$ is a $(1, -1)$-matrix of order 3. There are three even permutations of $\{1, 2, 3\}$ and also three odd permutations, and both the even permutations and the odd permutations partition the elements of E. Since the even permutations partition the elements of E we have

$$\prod_{i,j=1}^{3} e_{ij} = (e_{11}e_{22}e_{33})(e_{12}e_{23}e_{31})(e_{13}e_{21}e_{31}) = (1)(1)(1) = 1.$$

Since the odd permutations partition the elements of E we also have

$$\prod_{i,j=1}^{3} e_{ij} = (e_{11}e_{23}e_{32})(e_{13}e_{22}e_{31})(e_{12}e_{21}e_{33}) = (-1)(-1)(-1) = -1.$$

Hence E does not convert the permanent of 3 by 3 matrices into the determinant. If $n > 3$, then by considering matrices of the form $X = I_{n-3} \oplus X'$, where X' is a matrix of order 3, we also conclude that there is no $(1, -1)$-matrix E which converts the permanent of n by n matrices into the determinant. □

Let $\mathcal{M}_n$ denote the linear space of n by n matrices over the real field. Let E be a $(1, -1)$-matrix of order n. The mapping $T : \mathcal{M}_n \to \mathcal{M}_n$ defined by $T(X) = E * X$ for all X in $\mathcal{M}_n$ is an instance of a linear transformation. Marcus and Minc[1961] generalized Theorem 7.5.1 by showing that for $n \geq 3$ no linear transformation $T : \mathcal{M}_n \to \mathcal{M}_n$ satisfies

$$\operatorname{per}(X) = \det(T(X)), X \in \mathcal{M}_n. \tag{7.77}$$

Theorem 7.5.1 was further generalized by von zur Gathen[1987a] who showed that there is no affine transformation T on $\mathcal{M}_n$ for which (7.77) holds. These results all hold for matrices over an arbitrary infinite field of characteristic different from 2.

We now consider the possibility of converting the permanent into the determinant on "coordinate" subspaces of the linear space $\mathcal{M}_n$. Let $A = [a_{ij}]$ be a (0,1)-matrix of order n. Then

$$\mathcal{M}_n(A) = \{A * X : X \in \mathcal{M}_n\}$$

is the linear subspace of $\mathcal{M}_n$ consisting of the matrices which have 0's in those positions in which A has 0's (and possibly other 0's as well). If $\operatorname{per}(A) = 0$, then $\operatorname{per}(A * X) = \det(A * X) = 0$ for all X in $\mathcal{M}_n$. As a result *we henceforth assume that* $\operatorname{per}(A) \neq 0$. Let E be an n by n $(0, 1, -1)$-matrix, and assume that an element of E equals 0 if and only if the corresponding element of A equals 0. Thus if $|E|$ denotes the matrix obtained from E by replacing each element with its absolute value, then $|E| = A$. We say that the matrix E *converts the permanent of matrices in* $\mathcal{M}_n(A)$ *into the determinant* provided

$$\operatorname{per}(Y) = \det(E * Y) \tag{7.78}$$

for all matrices Y in $\mathcal{M}_n(A)$. Since A and E have 0's in exactly the same positions, (7.78) is equivalent to

$$\text{per}(A * X) = \det(E * X) \tag{7.79}$$

for all matrices X in $\mathcal{M}_n$. Equation (7.79) is a polynomial identity in the elements of X corresponding to the 1's of A. Hence (7.79) holds if and only if

$$\text{per}(A) = \det(E).$$

Since $|E| = A$, we conclude that E converts the permanent of matrices in $\mathcal{M}_n(A)$ into the determinant if and only if

$$\text{per}(|E|) = \det(E). \tag{7.80}$$

Thus the problem of finding coordinate subspaces of $\mathcal{M}_n$ on which the permanent can be evaluated as a determinant is equivalent to the problem of finding $(0, 1, -1)$-matrices E of order n satisfying (7.80).

For $n = 3$ the matrix

$$E = \left[\begin{array}{rrr} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{array}\right]$$

satisfies (7.80) and hence converts the permanent of 3 by 3 matrices with a 0 in position (1,3) into the determinant.

For a real number x, the *sign* of x is defined by

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

The *sign pattern* of the real matrix $X = [x_{ij}]$ of order n is the $(0, 1, -1)$-matrix

$$\text{sign}(X) = [\text{sign}(x_{ij})], (i, j = 1, 2, \ldots, n)$$

obtained by replacing each of the elements of X with its sign. The matrix X is *sign-nonsingular* provided that each matrix with the same sign pattern as X is nonsingular. A matrix obtained from a sign-nonsingular matrix by arbitrary line permutations or by transposition is also sign-nonsingular. In discussing sign-nonsingular matrices there is no loss in generality in restricting our attention to $(0, 1, -1)$-matrices X, that is matrices X for which $X = \text{sign}(X)$. We now show the equivalence of sign-nonsingular matrices and matrices which convert the permanent into the determinant (Brualdi[1988] and Brualdi and Shader[1991]).

Theorem 7.5.2. *The $(0, 1, -1)$-matrix X of order n is sign-nonsingular* *if and only if X converts the permanent of matrices in* $\mathcal{M}_n(|X|)$ *into the determinant or the negative of the determinant, that is if and only if*

$$\operatorname{per}(|X|) = \pm \det(X) \neq 0. \tag{7.81}$$

Proof. If (7.81) holds then there is a nonzero term in the determinant expansion of X and all nonzero terms have the same sign, and it follows that X is sign-nonsingular. Now suppose that $X = [x_{ij}]$ is sign-nonsingular. Let $(i_1, i_2, \ldots, i_n)$ be a permutation of $\{1, 2, \ldots, n\}$ for which $x_{1i_1}x_{2i_2}\cdots x_{ni_n} \neq 0$. Let $Y = [y_{ij}]$ be a nonnegative matrix for which y_{ij_i} equals 1 for $i = 1, 2, \ldots, n$ and y_{rs} equals a positive number ϵ otherwise. Then $Z = X * Y$ has the same sign pattern as X and for ϵ sufficiently small

$$\operatorname{sign}(\det(Z)) = \operatorname{sign}(i_1, i_2, \ldots, i_n)x_{1i_1}x_{2i_2}\cdots x_{ni_n}.$$

It now follows by continuity that each nonzero term in the determinant expansion of X has the same sign, and hence (7.81) holds. □

It follows from Theorem 7.5.2 that a sign-nonsingular matrix gives both a coordinate subspace $\mathcal{M}_n(A)$ of $\mathcal{M}_n$ on which the permanent can be evaluated as a determinant and a prescription for doing so. We now turn to demonstrating that under the assumption of full indecomposability this prescription is unique up to diagonal equivalence. First we prove the following lemma which contains a result first proved by Sinkhorn and Knopp[1969] (see also Ryser[1973] and Engel and Schneider[1973]).

Lemma 7.5.3. *Let* $A = [a_{ij}]$ *be a fully indecomposable matrix for which there exists a nonzero number d such that for each permutation* $(i_1, i_2, \ldots, i_n)$ *of* $\{1, 2, \ldots, n\}$

$$a_{1i_1}a_{2i_2}\cdots a_{ni_n} = 0 \text{ or } d.$$

Then there exist nonsingular diagonal matrices D_1 *and* D_2 *such that* D_1AD_2 *is a* $(0,1)$*-matrix. Moreover, the matrices* D_1 *and* D_2 *are unique up to multiplication of* D_1 *by a nonzero scalar* θ *and multiplication of* D_2 *by* θ^{-1}.

Proof. We prove the lemma by induction on n. If $n = 1$ the lemma clearly holds. Now assume that $n > 1$. By Theorem 4.2.8 we may assume that

$$A = \begin{bmatrix} A_1 & O & \cdots & O & E_1 \\ E_2 & A_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{m-1} & O \\ O & O & \cdots & E_m & A_m \end{bmatrix},$$

where $m \geq 2$, A_i is fully indecomposable of order $n_i, (i = 1, 2, \ldots, m)$, and E_i contains at least one nonzero element $e_i, (i = 1, 2, \ldots, m)$. By the induction hypothesis we may assume that each of the matrices A_i is a (0,1)-matrix and that $d = 1$. Since each A_i is fully indecomposable it follows from Theorem 4.2.2 that the nonzero elements of E_i have a common value e_i and that $e_1 e_2 \cdots e_m = 1$. Letting

$$D_2 = e_1 I_{n_1} \oplus (e_1 e_2) I_{n_2} \oplus \cdots \oplus (e_1 e_2 \cdots e_m) I_{n_m} \qquad \text{and} \qquad D_1 = D_2^{-1},$$

we see that $D_1 A D_2$ is a (0,1)-matrix. The uniqueness statement in the lemma follows easily from the inductive hypothesis. □

The following three theorems are from Brualdi and Shader[1991].

Theorem 7.5.4. *Let A be a $(0,1)$-matrix with total support and let X and Y be sign-nonsingular matrices with $|X| = |Y| = A$. Then there exist diagonal matrices D_1 and D_2 of order n whose diagonal elements equal ± 1 such that $Y = D_1 X D_2$. If A is a fully indecomposable matrix then D_1 and D_2 are unique up to a scalar factor of -1.*

Proof. Assume that A is fully indecomposable. It follows from Theorem 7.5.2 that both X and Y satisfy the hypotheses of Lemma 7.5.3 with $d = 1$. Hence there exist nonsingular diagonal matrices D_1 and D_2, unique up to multiplication by θ and θ^{-1}, respectively, such that $X = D_1 Y D_2$. Let

$$D_1 = \mathrm{diag}(d_1, d_2, \ldots, d_n) \qquad \text{and} \qquad D_2 = \mathrm{diag}(d'_1, d'_2, \ldots, d'_n)$$

be diagonal matrices, and let $K = \{i : |d_i| = |d_1|, 1 \le i \le n\}$ and $L = \{j : |d'_j| = |d_1|^{-1}, 1 \le j \le n\}$. Since X and Y are $(0,1,-1)$-matrices, it follows that the submatrices $A[K, \overline{L}]$ and $A[\overline{K}, L]$ of A in rows K and columns $\overline{L}$ and rows $\overline{K}$ and columns L, respectively, are zero matrices. Because A is fully indecomposable, we have $K = L = \{1, 2, \ldots, n\}$. Let $D_1^* = d_1^{-1} D_1$ and $D_2^* = d_1 D_2$. Then D_1^* and D_2^* are diagonal matrices whose main diagonal elements equal ± 1 such that $Y = D_1^* X D_2^*$. The theorem now follows. □

Theorem 7.5.5. *Let A be a fully indecomposable $(0,1)$-matrix of order n, and let $X = [x_{ij}]$ be a sign-nonsingular matrix with $|X| = A$. Let B be a $(0,1)$-matrix obtained from A by replacing a 0 in position (u, v) with a 1. Then the following are equivalent:*

(i) There exists a sign-nonsingular matrix $Z = [z_{ij}]$ with $|Z| = B$.
(ii) There exists a sign-nonsingular matrix $\hat{Z} = [\hat{z}_{ij}]$ which can be obtained from X by changing x_{uv} to 1 or -1.
(iii) The matrix obtained from X by deleting row u and column v is a sign-nonsingular matrix.

Proof. Statements (ii) and (iii) are clearly equivalent and (ii) implies (i). Now suppose that (i) holds. Let $\tilde{Z}$ be the matrix obtained from Z by replacing z_{uv} with 0. Then $|\tilde{Z}| = A$ and $\tilde{Z}$ is a sign-nonsingular matrix.

By Theorem 7.5.4 there exist diagonal matrices D_1 and D_2 with diagonal elements equal to ± 1 such that $D_1 \tilde{Z} D_2 = X$. The matrix $D_1 Z D_2$ now satisfies (ii). □

A similar proof allows one to extend Theorem 7.5.5 to matrices with total support.

Theorem 7.5.6. *Let $A = [a_{ij}]$ be a $(0,1)$-matrix with total support and assume that $A = A_1 \oplus \cdots \oplus A_k \oplus A_{k+1}$ where the matrices $E_1, \ldots, E_k$ are fully indecomposable. Let $X = [x_{ij}] = X_1 \oplus \cdots \oplus X_k \oplus X_{k+1}$ be a sign-nonsingular matrix with $|X| = A$. Let*

$$B = \begin{bmatrix} A_1 & O & O & \cdots & O & F_k \\ F_1 & A_2 & O & \cdots & O & O \\ O & F_2 & A_3 & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & A_{k-1} & O \\ O & O & O & \cdots & F_{k-1} & A_k \end{bmatrix} \oplus A_{k+1}$$

where the matrix F_i is a $(0,1)$-matrix with exactly one 1 and this 1 is in position (u_i, v_i) of $F_i, (i = 1, 2, \ldots, k)$. Then the following are equivalent:

(i) There exists a sign-nonsingular matrix Z with $|Z| = B$.
(ii) There exists a sign-nonsingular matrix $\hat{Z} = [\hat{z}_{ij}]$ with $|\hat{Z}| = B$ such that $\hat{z}_{ij} = x_{ij}$ for all (i, j) for which $a_{ij} \neq 0$.
(iii) For $i = 1, 2, \ldots, k$ the matrix obtained from X_i by deleting row u_{i-1} and column v_i is a sign-nonsingular matrix (here we interpret u_0 as u_k).

Let A be a (0,1)-matrix of order n with total support. It follows from Theorem 4.3.4 that there exist permutation matrices P and Q of order n such that starting with the identity matrix I_n, we can obtain PAQ by applying the constructions for the matrices B in Theorems 7.5.5 and 7.5.6. Let $B_0 = I_n, B_1, \ldots, B_l = PAQ$ be one such way. Theorems 7.5.5 and 7.5.6 imply that given any conversion $\hat{B}_i$ of B_i, there is a conversion of B_{i+1} if and only if there is a conversion $\hat{B}_{i+1}$ of B_{i+1} which extends the conversion $\hat{B}_i$. Starting with the conversion $\hat{B}_0 = I_n$ of A_0, we attempt to extend a conversion of A_i to a conversion of A_{i+1}. If at some point we are unable to do so, then Theorems 7.5.5 and 7.5.6 imply that no conversion of A exists. This provides us with an algorithm which avoids backtracking to determine whether or not there exists a sign-nonsingular matrix X with $|X| = A$, equivalently to determine whether or not the permanent of the matrices in $\mathcal{M}_n(A)$ can be converted into the determinant and a prescription for doing so.

Little[1975] characterized the coordinate subspaces $\mathcal{M}_n(A)$ of $\mathcal{M}_n$ on which the permanent can be converted into the determinant, and thus the

(0,1)-matrices $|E|$ which result from sign-nonsingular $(0,1,-1)$-matrices E. We now describe without proof his characterization.

Let $Y = [y_{ij}]$ be a matrix of order n, and suppose that there exist integers p, q, r, and s with $1 \leq p, q, r, s \leq n$ and $p \neq q$ and $r \neq s$ such that $y_{pj} = 0, (j \neq r, s)$, $y_{ir} = 0, (i \neq p, q)$, $y_{pr} = 1$ and $y_{qs} = 0$. Let Y' be the matrix of order $n-1$ obtained from Y by replacing y_{qs} by $y_{ps}y_{qr}$ and deleting both row p and column q. The matrix Y' is said to be obtained from Y by *contraction.* It follows from the Laplace expansion for the permanent that

$$\operatorname{per}(Y) = \operatorname{per}(Y').$$

If Y is a (0,1)-matrix then Y' is also a (0,1)-matrix.

Theorem 7.5.7. *Let A be a $(0,1)$-matrix of order n. There exists a $(0,1,-1)$-matrix E of order n such that $|E| = A$ and $\operatorname{per}(A) = \det(E)$ if and only if there do not exist permutation matrices P and Q of order n and a $(0,1)$-matrix B with $B \leq A$ such that for some integer $k \leq n-3$,*

$$PBQ = \begin{bmatrix} I_k & O \\ O & B' \end{bmatrix} \tag{7.82}$$

where the all 1's matrix J_3 of order 3 can be obtained by a sequence of contractions starting with the matrix B'.

For example, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{7.83}$$

Then with $B = A$ in Theorem 7.5.7, $P = Q = I_5$ and $k = 0$ [the identity matrix I_k in (7.82) is vacuous], we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

after one contraction and J_3 after two contractions. Thus there is no matrix which converts the permanent into the determinant on the coordinate subspace $\mathcal{M}_5(A)$.

It follows from Theorem 7.5.7 that the permanent can be converted into the determinant on a coordinate subspace $\mathcal{M}_n(A)$ if and only if it can be converted on each coordinate subspace $\mathcal{M}_n(B)$ where B is a (0,1)-matrix with $B \leq A$ and $\text{per}(B) = 6$.

We now discuss some connections between sign-nonsingular matrices and digraphs. Let $E = [e_{ij}]$ be a $(0, 1, -1)$-matrix of order n. A necessary condition for E to be sign-nonsingular is that $\text{per}(|E|) \neq 0$. Since neither line permutations nor the multiplication of certain lines by -1 affect the sign-nonsingularity of E, we henceforth assume that the elements on the main diagonal of E are all equal to -1. Let $D(E)$ be the digraph of E with vertices $1, 2, \ldots, n$. We use the elements of E in order to assign weights to the arcs of $D(E)$. The resulting *weighted digraph* $D_w(E)$ has an arc $i \to j$ from vertex i to vertex j if and only if $e_{ij} \neq 0$ and the *weight* of this arc equals $e_{ij}, (i, j = 1, 2, \ldots, n)$. The *weight of a directed cycle* is defined to be the product of the weights of the arcs of the cycle. The following characterization of sign-nonsingular matrices is due to Bassett, Maybee and Quirk[1968].

Theorem 7.5.8. *Let $E = [e_{ij}]$ be a $(0, 1, -1)$-matrix of order n with $e_{ii} = -1, (i = 1, 2, \ldots, n)$. Then E is sign-nonsingular if and only if the weight of each directed cycle of $D_w(E)$ equals -1.*

Proof. The matrix E is sign-nonsingular if and only if for each permutation $(\pi(1), \pi(2), \ldots, \pi(n))$ of $\{1, 2, \ldots, n\}$

$$e_{1\pi(1)}e_{2\pi(2)} \cdots e_{n\pi(n)} \neq 0 \text{ implies } \text{sign}(\pi)e_{1\pi(1)}e_{2\pi(2)} \cdots e_{n\pi(n)} = (-1)^n.$$

Let $\gamma : i_1 \to i_2 \to \cdots \to i_k \to i_1$ be a directed cycle of $D_w(E)$. Then $\pi(i_j) = i_{j+1}$ for $j = 1, 2, \ldots, k$ and $\pi(i) = i$, otherwise defines a permutation of $\{1, 2, \ldots, n\}$ with $\text{sign}(\pi)e_{1\pi(1)}e_{2\pi(2)} \cdots e_{n\pi(n)}$ equal to the $(-1)^{n-1}$ times the weight of γ. The theorem readily follows. $\square$

If in Theorem 7.5.8 we assume that all main diagonal elements of E equal 1, then the criterion for sign-nonsingularity is that the weight of each cycle have the opposite parity of its length.

Another characterization of (0,1)-matrices A for which there exists a (0,1,-1)-matrix E such that $|E| = A$ and $\text{per}(A) = \det(E)$ is contained in the work of Seymour and Thomassen[1987] and we now describe this characterization.

Let D be a digraph which has no loops. A *splitting of an arc* (x, y) of D is the result of adjoining a new vertex z to D and replacing the arc (x, y) by the two arcs (x, z) and (z, y). A *subdivision* of D is a digraph obtained from D by successively splitting arcs (perhaps none). The digraph D is *even* provided every subdivision of D (including D itself) has a directed cycle of even length. The digraph is even if and only if for every weighting of the arcs of D with weights 1 and -1 there exists a directed cycle whose weight equals 1. A *splitting of a vertex* u of D is the result of adjoining a new vertex v and a new arc (u, v) and replacing each arc of the form (u, w) with an arc of the form (v, w). A *splitting* of the digraph D is a subdivision of a digraph obtained from D by splitting some (perhaps none) of its vertices.

Let k be an integer with $k \geq 3$ and let D_n^* denote the digraph which is obtained from an (undirected) cycle of length k by replacing each of its edges $\{x, y\}$ with two oppositely directed arcs (x, y) and (y, x). The following characterization of even digraphs is due to Seymour and Thomassen[1987].

Theorem 7.5.9. *The digraph D is even if and only if it contains a splitting of D_k^* for some odd integer $k \geq 3$.*

Now let A be a (0,1)-matrix of order n with a nonzero permanent. Permuting lines if necessary, we may assume that all of the main diagonal elements of A equal 1. If there exists a (0,1,−1)-matrix E such that $|E| = A$ and $|\det(E)| = \operatorname{per}(A)$ then we may assume that all of the main diagonal elements of E equal −1. It follows from Theorem 7.5.8 that $|\det(E)| = \operatorname{per}(A)$ if and only if the digraph $D'(A)$ obtained from $D(A)$ by removing the loops at each of its vertices is even. We thus have the following characterization of coordinate subspaces on which the permanent can be converted into the determinant.

Theorem 7.5.10. *Let A be a $(0,1)$-matrix of order n with all of its main diagonal elements equal to 1. There exists a $(0,1,-1)$-matrix E such that $|E| = A$ and $\operatorname{per}(A) = \det(E)$ if and only if the digraph $D'(A)$ does not contain a splitting of D_k^* for any odd integer $k \geq 3$.*

The direct equivalence of the characterizations for converting the permanent into the determinant contained in Theorems 7.5.7 and 7.5.10 is discussed in Brualdi and Shader[1991].

The largest dimension of a coordinate subspace of $\mathcal{M}_n$ on which the permanent can be converted into the determinant is determined in the following theorem of Gibson[1971]. This result was independently obtained by Thomassen[1986] in the context of sign-nonsingular matrices. The description of the case of equality is due to Gibson.

Theorem 7.5.11. *Let A be a $(0,1)$-matrix of order n such that $\operatorname{per}(A) \neq 0$ and suppose that there exists a $(0,1,-1)$-matrix E such that $|E| = A$ and $\det(E) = \operatorname{per}(A)$. Then the number of 1's of A is at most equal to $(n^2+3n-2)/2$, with equality if and only if there exist permutation matrices P and Q such that*

$$PAQ = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Thomassen[1986] has observed that an algorithm which decides whether or not a square (0,1)-matrix A is sign-nonsingular, that is which satisfies $\text{per}(A) = \det(A)$, can be used to determine whether or not a square $(0, 1, -1)$-matrix is sign-nonsingular. The reason is as follows. Let B be a $(0, 1, -1)$-matrix and asssume without loss of generality that B has all 1's on its main diagonal. Let $D_w(B)$ be the weighted digraph associated with B. We construct an (unweighted) digraph $\hat{D}(B)$ by replacing each weighted arc $i \overset{-1}{\to} j$ with two arcs forming a path $i \to k_{ij} \to j$ of length 2, where k_{ij} is a new vertex. The number of vertices of $\hat{D}(B)$ is n plus the number of arcs of weight -1 of $D_w(B)$ (the number of -1's of B). Let $\hat{B}$ be the (0,1)-matrix with all 1's on its main diagonal which satisfies $D(\hat{B}) = \hat{D}(B)$. By Theorem 7.5.8 B is sign-nonsingular if and only if $\hat{D}(B) = D(\hat{B})$ has no directed cycle of even length. By Theorem 7.5.8 again, $D(\hat{B})$ has no directed cycle of even length if and only if the (0,1)-matrix $\hat{B}$ is sign-nonsingular.

We conclude this section by discussing the work of Valiant[1979] concerning the algorithmic complexity of computing the permanent of a (0,1)-matrix. We begin with a brief and informal discussion of the classes of decision problems known as **P** and **NP**.

A decision problem P has one of the two answers "yes" or "no." Assume that there is some "natural" way to measure the *size* of the problem P. For example, consider the decision problem, known as the *Hamilton cycle problem: Does a graph have a cycle whose length equals the order of the graph?* An instance of this problem is a specification of a graph, and we measure the size by the order n of the graph, that is by the number of its vertices. An algorithm to solve a decision problem P is a *nondeterministic polynomial algorithm* provided there exists a polynomial $p(n)$ such that if the solution to an instance of the problem of size n is "yes," then there is some guess which when input to the algorithm answers "yes" in a number of steps which is bounded by $p(n)$, while if the solution is "no," then for every guess either the algorithm answers "no" or does not halt. The set of decision problems which can be solved by a nondeterministic polynomial algorithm is denoted by **NP**. A nondeterministic algorithm can be converted to a deterministic algorithm by inputting all possible guesses. However, such an algorithm requires a number of steps which are not bounded by a polynomial in the size n.

The problems in **NP** have the property that they can be "certified" in a polynomial number of steps. The Hamilton cycle problem is in **NP**, since specifying a graph of order n and guessing a sequence of vertices γ, one can check in a number of steps bounded by a polynomial in n whether γ is a cycle of length n, that is whether γ is a cycle through all the vertices of the graph. Another problem in **NP** is the problem: *Is an integer N not a prime number?* A nondeterministic algorithm to solve this problem in a polynomial number of steps in the size n, the number of decimal digits in

N, can be described as follows. Specifying a factor F we divide N by F and obtain a remainder R. If $R = 0$, the answer is "yes"; if $R \neq 0$, the answer is "no." If N is not a prime, then a "certification" is a factor of N and such a factor can be checked in a polynomial number of steps.

Let **P** be the set of decision problems which can be solved deterministically in a number of steps bounded by a polynomial in the size of the problem. Thus for a problem P in **P** there exists an algorithm to determine (not just to check the correctness of a guess) whether the answer is "yes" or "no." The set **P** is contained in the set **NP** because given any guess, we set it aside and apply the polynomial deterministic algorithm. This results in a nondeterministic polynomial algorithm for the problem P. The question of whether **P**=**NP** has not yet been resolved, although the general belief is that $\mathbf{P} \neq \mathbf{NP}$.

A decision problem P_1 is said to be *polynomially reducible* to another decision problem P_2 provided there is a function f from the set of inputs of P_1 to the set of inputs of P_2 such that the answer to the input I_1 for P_1 is "yes" if and only if the answer to the corresponding input $f(I_1)$ of P_2 is "yes," and there is a polynomially bounded algorithm to compute $f(I_1)$. If P_1 is polynomially reducible to P_2 and if P_2 is in **P**, then it follows that P_1 is also in **P**. A decision problem in **NP** is *NP-complete* provided every problem in **NP** can be polynomially reduced to it. It follows that a problem in **NP** to which an NP-complete problem can be polynomially reduced is also NP-complete. Also if some NP-complete problem belongs to **P** then $\mathbf{P} = \mathbf{NP}$.

It was a fundamental contribution of Cook[1971] that there exist NP-complete problems. The problem that Cook established as NP-complete is the satisfiability problem SAT of conjunctive normal forms:

SAT

Given a finite set U of variables and a finite set C of clauses each of which is a disjunction of variables in U and their negations, is there a truth assignment for the variables in U for which all clauses in C are true?

There are many fundamental problems which are now known to belong to the class of NP-complete problems. These include the combinatorial problems of the existence of a coloring of the vertices of a graph with k colors, the existence of a complete subgraph of order k of a graph, and the existence of a Hamilton cycle in a graph. For more examples of NP-complete problems and a more formal description of the class **NP** we refer the reader to Karp[1972] and the books by Garey and Johnson[1979], Even[1979], and Wilf[1986].

Associated with a decision problem is the counting question: *How many solutions does the problem have?* The answer to the counting question is to be a number; a complete listing of all the solutions is not required. An NP-problem P is called *sharp P complete*, written *$\#P$-complete*, provided the counting question for every other NP-complete problem is polynomially reducible to the counting question for P. Most "natural" NP-complete problems are also $\#P$-complete (although it is not known whether this is always the case). The reason is that the reductions used to establish polynomial reducibility "preserve" the number of solutions. Thus, for instance, counting the number of satisfying assignments in SAT is $\#P$-complete. The set of all problems in **NP** which are $\#P$-complete is denoted by $\#\mathbf{P}$.

It was a fundamental contribution of Valiant[1979a,b] that there exist $\#P$-complete problems for which the decision question can be decided in a number of steps which is bounded by a polynomial in the size of the problem, that is, there exist problems in **P** which are in $\#\mathbf{P}$. Thus unless $\mathbf{P}=\mathbf{NP}$, and perhaps even if $\mathbf{P}\neq\mathbf{NP}$, there are NP-problems for which the decision question can be answered in a number of steps bounded by the size of the problem but the counting question cannot. The problem identified by Valiant is the problem of finding a system of distinct representatives:

SDR

Given a family $(X_1, X_2, \ldots, X_n)$ of subsets of the set $\{1, 2, \ldots, n\}$, does the family have a system of distinct representatives?

There are algorithms which show that *SDR* is in **P** (see, e.g., Even[1979]). A system of distinct representatives of $(A_1, A_2, \ldots, A_n)$ corresponds to a perfect matching in the associated bipartite graph of order $2n$.

Let A be the incidence matrix of order n of the family $(A_1, A_2, \ldots, A_n)$ of subsets of $\{1, 2, \ldots, n\}$. The problem *SDR* is equivalent to the problem

$$\text{Is per}(A) \neq 0?$$

The counting question for *SDR* is: *How many systems of distinct representatives does $(A_1, A_2, \ldots, A_n)$ have?* In terms of the incidence matrix A, the counting question is

What is the value of per(A)*?*

Thus determining the permanent of a square (0,1)-matrix is a $\#P$-complete problem. It follows that the computation of the permanent of square matrices of 0's and 1's is a fundamental counting problem. The existence of an algorithm to compute the permanent of a (0,1)-matrix of order n in a number of steps bounded by a polynomial in n would imply the existence of polynomially bounded algorithms to compute the number of satisfying truth assignments in SAT, to compute the number of ways to color the vertices of a graph with k colors, to compute the number of Hamilton cycles in a graph and so on. However, *Valiant's hypothesis is that* *no polynomially bounded algorithm exists for computing the permanent of a* $(0,1)$*-matrix* (see also von zur Gathen[1987b].

Valiant showed that permanents of matrices $X = [x_{ij}]$ of order n can be calculated by determinants if one is allowed to increase the size of the matrix. Specifically he showed that it is possible to find an m by m matrix Y each of whose entries is either a constant or one of the elements $x_{11}, \ldots, x_{1n}, \ldots, x_{nn}$ of X such that $\operatorname{per}(X) = \det(Y)$. The size m of the matrix Y is, however, exponential in n, roughly a constant times $n^2 2^n$. Thus although the determinant of Y can be computed in a number of steps which is bounded by a polynomial in m, this number is exponential in n. It is shown in von zur Gathen[1987a] that every Y of the above type has order $m \geq \sqrt{2}n - 6\sqrt{n}$, even if one allows the elements of Y equal to negatives of elements of X. A matrix Y of order m where m is bounded by a polynomial in n would imply a polynomially bounded algorithm to compute the permanent of a matrix.

In closing we note that Everett and Stein[1973] have shown that the number of (0,1)-matrices of order n with zero permanent is asymptotic to $n \cdot 2^{n^2-2n+1}$. Hence it follows that almost all (0,1)-matrices of order n have a nonzero permanent. Komlós[1967] has shown that almost all (0,1)-matrices of order n have a nonzero determinant.

Exercises

1. Determine whether the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

is sign-nonsingular.

2. Show how to affix minus signs to some of the 1's of the matrix in Theorem 7.5.11 so that the determinant of the new matrix is the permanent of the original matrix.
3. Show that the incidence matrix A of the projective plane of order 2 given in (1.16) of Chapter 1 is a sign-nonsingular matrix (with no -1's). Show that each matrix obtained from A by replacing a 0 with ± 1 is not sign-nonsingular (thus A is a *maximal* sign-nonsingular matrix). (The matrix A, after line permutations, is the circulant matrix $I_7 + C_7 + C_7^3$.) (Brualdi and Shader[1991]).
4. Prove Theorem 7.5.6.
5. Prove that a digraph which contains a splitting of D_k^* for some odd integer $k \geq 3$ is even.

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