5

Some Special Graphs

5.1 Regular Graphs

We begin our study of special graphs with two lemmas on nonnegative matrices. We again let e_n denote the column vector of n 1's.

Lemma 5.1.1. *Let A be a nonnegative real matrix of order n and let all of the line sums of A equal k. Then k is an eigenvalue of A corresponding to the eigenvector e_n and the modulus of every other eigenvalue of A does not exceed k. Furthermore, if $n > 1$ then the eigenvalue k is of multiplicity one if and only if A is irreducible.*

Proof. The equation $Ae_n = ke_n$ implies at once that k is an eigenvalue of A corresponding to the eigenvector e_n. By Theorem 3.6.2 no other eigenvalue can have larger modulus. If A is reducible, then all of the line sums of each irreducible component of A also equal k and it follows that the multiplicity of the eigenvalue k is at least two. If A is irreducible, then it follows from the Perron–Frobenius theory (see, e.g., Horn and Johnson[1985]) of nonnegative matrices that the multiplicity of k as an eigenvalue of A equals one. □

Lemma 5.1.2. *Let A be a nonnegative real matrix of order n. Then there exists a polynomial $p(x)$ such that*

$$J = p(A) \tag{5.1}$$

if and only if A is irreducible and all of the line sums of A are equal.

Proof. Suppose that (5.1) is valid. Then $AJ = JA$ and it follows that all of the line sums of A are equal. If A is reducible, then all of the positive integral powers of A have certain fixed positions occupied by zeros and this contradicts (5.1).

Conversely, suppose that A is irreducible and that all of the line sums of A are equal to k. Then by Lemma 5.1.1 we know that k is a simple eigenvalue of A. We may write the minimum polynomial of A in the form

$$m(\lambda) = (\lambda - k)q(\lambda)$$

and this implies that

$$Aq(A) = kq(A).$$

Thus each nonzero column of $q(A)$ is an eigenvector of A corresponding to the eigenvalue k. But the eigenspace associated with the eigenvalue k has dimension one and hence each column of $q(A)$ is a suitable multiple of e_n. The same argument applies to the transposed situation

$$A^T q(A)^T = kq(A)^T,$$

and we may conclude that each column of $q(A)^T$ is also a multiple of e_n. Hence each row of $q(A)$ is a multiple of e_n. But this means that $q(A)$ is a multiple of J. We cannot have $q(A) = O$ because $m(\lambda)$ is the minimum polynomial of A. Thus J is a polynomial in A. □

We may apply the preceding lemma directly to the adjacency matrix of a graph and obtain the following theorem of Hoffman[1963].

Theorem 5.1.3. *Let A be the adjacency matrix of a graph G of order $n > 1$. Then there exists a polynomial $p(x)$ such that*

$$J = p(A) \tag{5.2}$$

if and only if G is a regular connected graph.

Corollary 5.1.4. *Let G be a regular connected graph of order $n > 1$ and let the distinct eigenvalues of G be denoted by $k > \lambda_1 > \cdots > \lambda_{t-1}$. Then if*

$$q(\lambda) = \prod_{i=1}^{t-1} (\lambda - \lambda_i),$$

we have

$$J = \left(\frac{n}{q(k)}\right) q(A).$$

The polynomial

$$p(\lambda) = \left(\frac{n}{q(k)}\right) q(\lambda)$$

is the unique polynomial of lowest degree such that $p(A) = J$.

Proof. Since A is symmetric we know that the zeros of the minimum polynomial of A are distinct. Then by the proof of Lemma 5.1.2 we have that $q(A) = cJ$ for some nonzero constant c. The eigenvalues of $q(A)$ are $q(k)$ and $q(\lambda_i)$ $(i = 1, 2, \ldots, t-1)$ and all of these are zero with the exception of $q(k)$. But the only nonzero eigenvalue of cJ is cn and hence $c = q(k)/n$.

Let $p(\lambda)$ be a polynomial such that $p(A) = J$. The eigenvalues of $p(A)$ are $p(k)$ and $p(\lambda_i)$ $(i = 1, 2, \ldots, t-1)$. Since e_n is an eigenvector of $p(A)$ and of J corresponding to the eigenvalues $p(k)$ and n, respectively, we have $p(\lambda_i) = 0$ for $i = 1, 2, \ldots, t-1$. □

The polynomial

$$p(\lambda) = \left(\frac{n}{q(k)}\right) q(\lambda)$$

in Corollary 5.1.4 is called the *Hoffman polynomial* of the regular connected graph G.

We illustrate the preceding discussion by showing that the only connected graph G of order n with exactly two distinct eigenvalues is the complete graph K_n. Let A be the adjacency matrix of such a graph with eigenvalues $\lambda_1 > \lambda_2$. Then we have

$$A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I = O.$$

Since A is symmetric and of trace zero, it follows that G is regular of degree $-\lambda_1\lambda_2$. Thus the Hoffman polynomial of G is of degree 1 and this implies that $J = A + I$.

In the following section we study in some detail regular connected graphs with exactly three distinct eigenvalues, that is, graphs whose Hoffman polynomial is of degree 2.

Exercises

1. Let G be a graph of order n which is regular of degree k. Prove that the sum of the squares of its eigenvalues equals kn.
2. Determine the spectrum and Hoffman polynomial of the complete bipartite graph $K_{m,m}$.
3. Determine the spectrum and Hoffman polynomial of the complete multipartite graph $K_{m,m,\ldots,m}(k\ m's)$. (This graph has km vertices partitioned into k parts of size m and there is an edge joining two vertices if and only if they belong to different parts.)

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5.2 Strongly Regular Graphs

Throughout this section G denotes a graph of order $n, (n \geq 3)$ with vertices $a_1, a_2, \ldots, a_n$ and we let A denote the adjacency matrix of G.

A *strongly regular graph* on the parameters (n, k, λ, μ) is a graph G of order $n, (n \geq 3)$ which is regular of degree k and satisfies the following additional requirements:

(i) If a and b are any two distinct vertices of G which are joined by an edge, then there are exactly λ further vertices of G which are joined to both a and b.

(ii) If a and b are any two distinct vertices of G which are not joined by an edge, then there are exactly μ further vertices of G which are joined to both a and b.

We exclude from consideration the complete graph K_n and its complement, the void graph, so that neither property (i) nor (ii) is vacuous. Strongly regular graphs were introduced by Bose[1963] and have subsequently been investigated by many authors. We mention, in particular, the studies of Seidel[1968,1969,1974,1976] and the book by Brouwer, Cohen and Neumaier[1989].

We begin with some simple examples of strongly regular graphs.

The 4-cycle and the 5-cycle are strongly regular graphs on the parameters

$$(4, 2, 0, 2) \qquad \text{and} \qquad (5, 2, 0, 1),$$

respectively. No other n-cycle qualifies as a strongly regular graph.

The *Petersen graph* in Figure 5.1 is a strongly regular graph on the parameters

$$(10, 3, 0, 1).$$

The graph with two connected components each of which is a 3-cycle is a strongly regular graph on the parameters

$$(6, 2, 1, 0).$$

The complete bipartite graph $K_{m,m}, (m \geq 2)$ is a strongly regular graph on the parameters

$$(2m, m, 0, m).$$

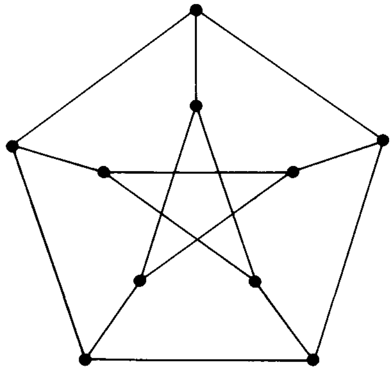

Figure 5.1

Let G be a strongly regular graph on the parameters (n, k, λ, μ) and let A be its adjacency matrix. We know that the entry in the (i, j) position of A^2 equals the number of walks of length 2 with a_i and a_j as endpoints. This number equals k, λ or μ according as these vertices are equal, adjacent or nonadjacent. Hence we have

$$A^2 = kI + \lambda A + \mu(J - I - A), \tag{5.3}$$

or, equivalently,

$$A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J. \tag{5.4}$$

We introduce another parameter, namely,

$$l = n - k - 1. \tag{5.5}$$

The integer l is the degree of the complement $\bar{G}$ of G. An elementary calculation involving (5.3) tells us that

$$(J - I - A)^2 = lI + (l - k + \mu - 1)(J - I - A) + (l - k + \lambda + 1)A.$$

Hence it follows that if G is a strongly regular graph, then its complement $\bar{G}$ is also a strongly regular graph on the parameters

$$(\bar{n} = n, \bar{k} = l, \bar{\lambda} = l - k + \mu - 1, \bar{\mu} = l - k + \lambda + 1).$$

If we multiply the equation (5.3) by the column vector e_n then we obtain

$$k^2 = k + \lambda k + \mu(n - 1 - k),$$

and we write this relation in the form

$$l\mu = k(k - \lambda - 1). \tag{5.6}$$

In case $\mu = 0$, then by (5.6) we have $\lambda = k - 1$. This means that every vertex of G belongs to a complete graph K_{k+1}, and thus G is a disconnected graph whose connected components are all of the form K_{k+1}. The requirement $\mu \geq 1$ is equivalent to the assertion that the strongly regular graph G is connected. For this reason we frequently require strongly regular graphs to have $\mu \geq 1$.

Theorem 5.2.1. *Let G be a strongly regular connected graph on the parameters (n, k, λ, μ). Let the parameters d and δ be defined by*

$$d = (\lambda - \mu)^2 + 4(k - \mu), \qquad \delta = (k + l)(\lambda - \mu) + 2k. \tag{5.7}$$

Then the adjacency matrix A of G has the maximal eigenvalue k of multiplicity 1, and A has exactly two additional eigenvalues

$$\rho = \frac{1}{2}(\lambda - \mu + \sqrt{d}) \geq 0, \qquad \sigma = \frac{1}{2}(\lambda - \mu - \sqrt{d}) \leq -1 \tag{5.8}$$

of multiplicities

$$r = \frac{1}{2}\left(k + l - \frac{\delta}{\sqrt{d}}\right), \qquad s = \frac{1}{2}\left(k + l + \frac{\delta}{\sqrt{d}}\right), \tag{5.9}$$

respectively.

Proof. Since G is a connected graph and is not a complete graph, A has at least three distinct eigenvalues. The first assertion in the theorem follows from Lemma 5.1.1. We next multiply (5.4) by $A - kI$ and this implies

$$(A - kI)(A^2 - (\lambda - \mu)A - (k - \mu)I) = O.$$

Thus the quantities ρ and σ displayed in (5.8) are eigenvalues of A.

If $d = 0$ then $\lambda = \mu = k$. But since G is regular of degree k we must have $\lambda \leq k - 1$ so that $d \neq 0$ and $\rho > \sigma$. Notice that the parameters λ and μ are expressible in terms of the quantities $k > \rho > \sigma$:

$$\lambda = k + \rho + \sigma + \rho\sigma, \qquad \mu = k + \rho\sigma.$$

We know that $\mu \leq k$ so that $\rho \geq 0$ and $\sigma \leq 0$. But $\sigma = 0$ implies that $\lambda = k + \rho$ and this contradicts $\lambda \leq k - 1$. Hence we have $\rho \geq 0$ and $\sigma < 0$.

We now turn to the complement $\bar{G}$ of G. An elementary calculation tells us that for $\bar{G}$ we have

$$\bar{d} = d, \qquad \bar{\rho} = -\sigma - 1, \qquad \bar{\sigma} = -\rho - 1.$$

But again for $\bar{G}$ we have $\bar{\rho} \geq 0$ so that we may conclude that $\rho \geq 0$ and $\sigma \leq -1$, as required.

Let r and s denote the multiplicities of ρ and σ, respectively, as eigenvalues of A. Then we have

$$r + s = n - 1,$$

and since A has trace zero, we have

$$k + r\rho + s\sigma = 0.$$

We solve these equations for r and s and this gives (5.9). □

The eigenvalue multiplicities r and s are nonnegative integers, and this fact in conjunction with (5.9) places severe restrictions on the parameter sets for strongly regular graphs.

Theorem 5.2.2. *Let G be a strongly regular connected graph on the parameters (n, k, λ, μ).*

(i) If $\delta = 0$, then

$$\lambda = \mu - 1, \qquad k = l = 2\mu = r = s = (n-1)/2.$$

(ii) If $\delta \neq 0$, then $\sqrt{d}$ is an integer and the eigenvalues ρ and σ are also integers. Furthermore if n is even, then $\sqrt{d} \mid \delta$ whereas $2\sqrt{d} \not\mid \delta$, and if n is odd, then $2\sqrt{d} \mid \delta$.

Proof. If $\delta = 0$, then $k + l = 2k/(\mu - \lambda) > k$ and thus $0 < \mu - \lambda < 2$. Therefore we have $\lambda = \mu - 1$. The remaining equations of (1) now follow from (5.6) and (5.9).

If $\delta \neq 0$, then the conclusion (2) follows directly from (5.8) and (5.9). □

Strongly regular graphs of the form (1) in Theorem 5.2.2 are called *conference graphs.* They arise in a wide variety of mathematical investigations (see Cameron and van Lint[1975], Goethals and Seidel[1967, 1970 and van Lint and Seidel[1966]). They have the same parameter sets as their complements and have been constructed for orders n equal to a prime power congruent to 1 (modulo 4). Let F be a finite field on n elements, where n is a prime power congruent to 1 (modulo 4). Then we may construct a graph G of order n whose vertices are the elements of F. Two vertices a and b are adjacent in G if and only if $a - b$ is a nonzero square in F. Notice that -1 is a square in F so that G is undirected. The resulting graph is a strongly regular graph on the parameters

$$(n, k = (n-1)/2, \lambda = (n-5)/4, \mu = (n-1)/4).$$

These special conference graphs are called *Paley graphs.*

We now apply the preceding theory to a proof of the *friendship theorem* of Erdös, Rényi and Sós[1966]. In other terms the theorem says that in a finite society in which each pair of members has exactly one common friend, there is someone who is a friend to everyone else. Our account follows Cameron[1978].

Theorem 5.2.3. *Let G be a graph of order n and suppose that for any two distinct vertices a and b there is a unique vertex c which is joined to* *both a and b. Then n is odd and G consists of a number of triangles with a common vertex.*

Proof. Let G be a graph fulfilling the hypothesis of the theorem. Let a and b be nonadjacent vertices of G. Then there is a unique vertex c which is adjacent to both a and b. There are also unique vertices $d \neq b$ adjacent to both a and c and $e \neq a$ adjacent to both b and c. If x is any vertex different from c and d which is adjacent to a then there exists a unique vertex y different from c and e which is adjacent to both x and b. A similar statement holds with a and b interchanged. Hence the degrees of the vertices a and b are equal.

Now suppose that G is not a regular graph. Let a and b be vertices of unequal degrees, and let c be the unique vertex which is adjacent to both a and b. The preceding paragraph implies that a and b are adjacent.

We may suppose by interchanging a and b if necessary that the degrees of a and c are unequal. Let d be any further vertex. Then d is adjacent to at least one of a and b because a and b are of unequal degrees. Similarly, d is adjacent to at least one of a and c. But d is not adjacent to both b and c because a is already adjacent to both b and c. Hence d is adjacent to a. It follows that G consists of a number of triangles with a common vertex a.

Hence we may assume that G is regular of degree k. By the hypothesis of the theorem we then have a strongly regular graph with $\lambda = \mu = 1$. By Theorem 5.2.1 it follows that $s - r = \delta/\sqrt{d} = k/\sqrt{k-1}$ is an integer. But then $(k-1)|k^2$ and it follows easily that the only possibilities are $k = 0$ and $k = 2$. These yield the cases of a single vertex and a triangle. □

We look next at some further examples of strongly regular graphs. The *triangular graph* $T(m)$ is defined as the line graph of the complete graph K_m, $(m \geq 4)$. Thus the vertices of $T(m)$ may be identified as the 2-subsets of $\{1, 2, \ldots, m\}$, and two vertices are adjacent in $T(m)$ provided the corresponding 2-subsets have a nonempty intersection. An inspection of the structure of $T(m)$ reveals that $T(m)$ is a strongly regular graph on the parameters

$$(n = m(m-1)/2, k = 2(m-2), \lambda = m-2, \mu = 4).$$

The following classification theorem is due to Chang[1959, 1960] and Hoffman[1960].

Theorem 5.2.4. *Let G be a strongly regular graph on the parameters $(m(m-1)/2, 2(m-2), m-2, 4)$, $(m \geq 4)$. If $m \neq 8$, then G is isomorphic to the triangular graph $T(m)$. If $m = 8$, then G is isomorphic to one of four graphs, one of which is $T(8)$.*

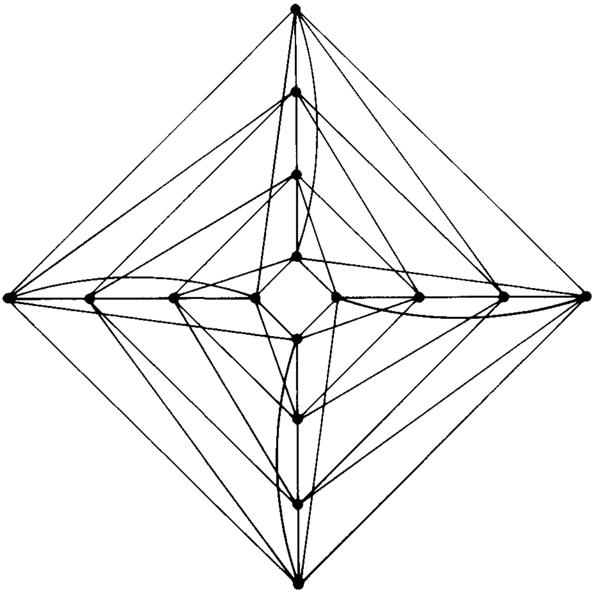

Figure 5.2

The *lattice graph* $L_2(m)$ is defined as the line graph of the complete bipartite graph $K_{m,m}$, $(m \geq 2)$. These are strongly regular graphs on the parameters

$$(n = m^2, k = 2(m-2), \lambda = m-2, \mu = 2).$$

The following classification theorem is due to Shrikhande[1959].

Theorem 5.2.5. *Let G be a strongly regular graph on the parameters $(m^2, 2(m-2), m-2, 2)$, $(m \geq 2)$. If $m \neq 4$, then G is isomorphic to the lattice graph $L_2(m)$. If $m = 4$, then G is isomorphic to $L_2(4)$ or to the graph in Figure* 5.2.

A *Moore graph* (of diameter 2) is a strongly regular graph with $\lambda = 0$ and $\mu = 1$. These graphs contain no triangles and for any two nonadjacent vertices there is a unique vertex adjacent to both. Hoffman and Singleton[1960] showed that the parameter sets of Moore graphs are severely restricted.

Theorem 5.2.6. *The only possible parameter sets (n, k, λ, μ) of a Moore graph are*

$$(5, 2, 0, 1), (10, 3, 0, 1), (50, 7, 0, 1) \textit{ and } (3250, 57, 0, 1).$$

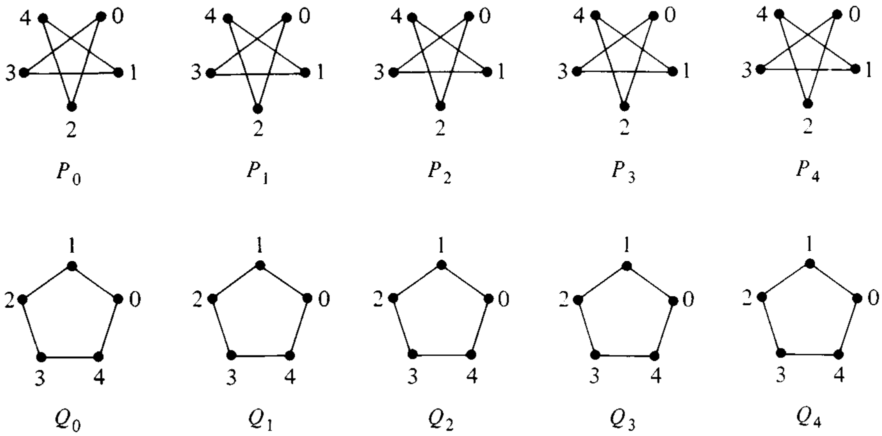

Figure 5.3

Proof. Condition (1) of Theorem 5.2.2 occurs precisely for the parameters (5,2,0,1). We next apply condition (2) of Theorem 5.2.2. We have that $d = 4k - 3$ is equal to a square. Equation (5.6) asserts that $k + l = k^2$ and hence $\delta = k(2-k)$. Thus we have $k(2-k) \equiv 0 \pmod{\sqrt{d}}$. We also have $4k - 3 \equiv 0 \pmod{\sqrt{d}}$. Multiplying the first of these congruences by 4 and the second by k and then adding we obtain $5k \equiv 0 \pmod{\sqrt{d}}$. This and $4k - 3 \equiv 0 \pmod{\sqrt{d}}$ now imply that $15 \equiv 0 \pmod{\sqrt{d}}$. Thus the only possibilities for $\sqrt{d}$ are 1,3,5 and 15. The first case is an excluded degeneracy, and the other three values yield the last three parameter sets displayed in the theorem. □

The first of the parameter sets in Theorem 5.2.6 is satisfied by the pentagon, the second by the Petersen graph and the third by the *Hoffman–Singleton graph.* They are the unique strongly regular graphs on these parameter sets. The existence of a strongly regular graph corresponding to the last of the parameter sets is unknown. Aschbacher[1971] has shown that its automorphism group cannot be too large.

The Hoffman–Singleton graph may be represented by the ten cycles of order 5 labeled as shown in Figure 5.3, where vertex i of P_j is joined to vertex $i + jk \pmod 5$ of Q_k (Bondy and Murty[1976]).

We remark that Moore graphs may be defined under certain more general conditions so that their diameter is allowed to exceed 2 (see Cameron[1978] and Cameron and van Lint[1975]). But in this case Bannai and Ito[1973] and Damerell[1973] have shown that the only additional graphs introduced consist of a single cycle.

A *generalized Moore graph* is a strongly regular graph with $\mu = 1$. The parameter λ is allowed to take on any value in such a graph, but none has yet been found with $\lambda \geq 1$.

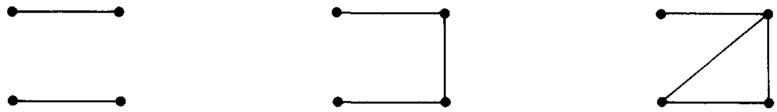

Figure 5.4

Exercises

1. Prove that a regular connected graph with three distinct eigenvalues is strongly regular.
2. Let G be a connected graph of order n which is regular of degree k. Assume that G satisfies requirements (i) and (ii) for a strongly regular graph but with the words *exactly* λ and *exactly* μ replaced by *at most* λ and *at most* μ, respectively. Prove that

$$n \leq k + 1 + k(k - 1 - \lambda)/\mu$$

with equality if and only if G is strongly regular on the parameters (n, k, λ, μ) (Seidel[1979]).
3. A $(0, 1, -1)$-matrix C of order $n+1$ all of whose main diagonal elements equal 0 is a *conference matrix* provided $CC^T = nI$. Prove that there exists a symmetric conference matrix of order $n + 1$ if and only if there exists a conference graph of order n.
4. Construct the conference matrices of orders 6 and 10 corresponding to the Paley graphs of orders 5 and 9.
5. Prove Theorem 5.2.4 when $m > 8$.
6. Let G be a regular connected graph of order n with at most 4 distinct eigenvalues. Prove that a graph H of order n is cospectral with G if and only if H is a connected regular graph having the same set of distinct eigenvalues as G (Cvetković, Doob and Sachs[1982]).
7. Let G be a graph with no vertex of degree 0 which is not a complete multipartite graph. Prove that G contains one of the three graphs in Figure 5.4 as an induced subgraph.
8. Let G be a graph with no vertex of degree 0. Assume that G has exactly one positive eigenvalue. Use Exercise 7 and the interlacing inequalities for the eigenvalues of symmetric matrices to prove that G is a complete multipartite graph (Smith[1970]).

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5.3 Polynomial Digraphs

We may directly generalize the proofs of Theorem 5.1.3 and Corollary 5.1.4 and obtain the following theorem of Hoffman and McAndrew[1965].

Theorem 5.3.1. *Let A be the adjacency matrix of a digraph D of order $n > 1$. Then there exists a polynomial $p(x)$ such that*

$$J = p(A) \tag{5.10}$$

if and only if D is a regular strongly connected digraph. Let D be a strongly connected digraph which is regular of degree k and let $m(\lambda)$ be the minimum polynomial of A. If

$$q(\lambda) = \frac{m(\lambda)}{\lambda - k}$$

then the polynomial

$$p(\lambda) = \left(\frac{n}{q(k)}\right) q(\lambda)$$

is the unique polynomial of lowest degree such that $p(A) = J$.

By Lemma 5.1.1 the modulus of each eigenvalue of A is at most equal to k. The roots of the polynomial $p(\lambda)$ are eigenvalues of A and it follows that $|p(\lambda)|$ is a monotone increasing function if λ is real and $\lambda \geq k$. We have $p(k) = n$ and we therefore conclude that the degree k of regularity of D equals the greatest real root of the equation $p(\lambda) = n$. Extending our definition in section 5.1 to digraphs, we call the polynomial $p(\lambda)$ in Theorem 5.3.1 the *Hoffman polynomial* of the regular strongly connected digraph D.

Let A be the adjacency matrix of a digraph D. We say that A is *regular of degree k* provided D is regular of degree k. Similarly, two adjacency matrices are called *isomorphic* provided their corresponding digraphs are isomorphic.

We now consider the special polynomials $p(\lambda) = (\lambda^m + d)/c$, where c is a positive integer and d is a nonnegative integer.

Theorem 5.3.2. *Let m and c be positive integers and let d be a nonnegative integer. Let A be a $(0,1)$-matrix of order n satisfying the equation*

$$A^m = -dI + cJ. \tag{5.11}$$

Then there exists a positive integer k such that A is regular of degree k and $k^m = -d + cn$. If $d = 0$ then the trace of A is also equal to k.

Proof. The regularity of A is a consequence of Theorem 5.3.1. We now multiply (5.11) by J and this gives $k^m = -d + cn$. Now assume that $d = 0$.

Let the characteristic roots of A be $\lambda_1, \lambda_2, \ldots, \lambda_n$. The characteristic roots of cJ are cn of multiplicity one and 0 of multiplicity $n-1$. Hence we may write $\lambda_1 = k, \lambda_2 = 0, \ldots, \lambda_n = 0$ and the trace of A is k. □

We note that if $d = 0$ in Theorem 5.3.2 it is essential that the digraph D associated with A have loops because otherwise the configuration is impossible. In terms of D equation (5.11) asserts that for each pair of distinct vertices a and b of D there are exactly c walks of length m from a to b and each vertex is on exactly $c-d$ closed walks of length m. In the case $c = 1$ and $d = 0$, a (0,1)-matrix A satisfying $A^m = J$ is a primitive matrix of exponent m, and the polynomial $p(\lambda) = \lambda^m$ is the Hoffman polynomial of D.

We next turn to showing that if $d = 0$ then the condition $k^m = cn$ in Theorem 5.3.2 is sufficient for there to exist a (0,1)-matrix A of order n satisfying $A^m = cJ$. A *g-circulant matrix* is a matrix of order n in which each row other than the first is obtained from the preceding row by shifting the elements cyclically g columns to the right. Let $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ be a g-circulant. Then

$$a_{ij} = a_{i+1,j+g}$$

in which the subscripts are computed modulo n. A 1-circulant matrix is more commonly called a *circulant matrix.* A detailed study of circulant matrices can be found in Ablow and Brenner[1963] and in the book by Davis[1979].

Let $a_0, a_1, \ldots, a_{n-1}$ be the first row of the g-circulant matrix A of order n. The *Hall polynomial* of A is defined to be the polynomial

$$\theta_A(x) = \sum_{i=0}^{n-1} a_i x^i.$$

The following lemma is a direct consequence of the definitions involved.

Lemma 5.3.3. *Let A be a g-circulant matrix of order n and let B be an h-circulant matrix of order n. Then the product AB is a gh-circulant matrix of order n and we have*

$$\theta_{AB}(x) \equiv \theta_A(x^h)\theta_B(x) \pmod{x^n - 1}.$$

Corollary 5.3.4. *Let A be a g-circulant matrix of order n. Then for each positive integer m A^m is a g^m-circulant matrix and we have*

$$\theta_{A^m}(x) \equiv \theta_A(x)\theta_A(x^g)\cdots\theta_A(x^{g^{m-1}}) \pmod{x^n - 1}. \tag{5.12}$$

Proof. We apply Lemma 5.3.3 and use induction on m. □

The g-circulant solutions of the equation $A^m = -dI + cJ$ are characterized by their Hall polynomials in the following result of Lam[1977].

Lemma 5.3.5. *Let A be a g-circulant matrix. Then $A^m = cJ$ if and only if*

$$\theta_A(x)\theta_A(x^g)\cdots\theta_A(x^{g^{m-1}}) \equiv c(1+x+\cdots+x^{n-1}) \pmod{x^n-1}. \quad (5.13)$$

If $d \neq 0$, then $A^m = dI + cJ$ if and only if

$$\theta_A(x)\theta_A(x^g)\cdots\theta_A(x^{g^{m-1}}) \equiv d+c(1+x+\cdots+x^{n-1}) \pmod{x^n-1} \quad (5.14)$$

and

$$g^m \equiv 1 \pmod{n}. \quad (5.15)$$

Proof. By Corollary 5.3.4 A^m is a g^m-circulant and its Hall polynomial is given by equation (5.12). If $A^m = cJ$, then the first row of A^m is $(c, c, \ldots, c)$ and this implies (5.13). Conversely, if (5.13) is satisfied, then the first row of A^m is $(c, c, \ldots, c)$, and it follows that $A^m = cJ$.

Suppose that $d \neq 0$ and $A^m = dI + cJ$. It follows as above that (5.14) holds. Moreover, since $dI+cJ$ is a circulant, we have $g^m \equiv 1 \pmod{n}$ and (5.15) holds. Next suppose that $d \neq 0$ and (5.14) and (5.15) are satisfied. Then the first row of A^m equals $(d+c, c, \ldots, c)$. By (5.15) A^m is a 1-circulant and hence we have $A^m = dI + cJ$. □

The following theorem is from Lam[1977].

Theorem 5.3.6. *Suppose that $k^m = cn$. Then the k-circulant matrix A of order n whose first row consists of k 1's followed by $n-k$ 0's satisfies*

$$A^m = cJ.$$

Proof. The Hall polynomial of the matrix A defined in the theorem satisfies

$$\theta_A(x) = 1 + x + \cdots + x^{k-1}.$$

It follows by induction on k that

$$\theta_A(x)\theta_A(x^k)\cdots\theta_A(x^{k^m-1}) = 1 + x + x^2 + \cdots + x^{k^m-1}.$$

Since $k^m = cn$, we have

$$\begin{aligned}\theta_A(x)\theta_A(x^k)\cdots\theta_A(x^{k^{m-1}}) &= 1 + x + x^2 + \cdots + x^{cn-1}\\ &\equiv c(1 + x + \cdots + x^{n-1}) \pmod{x^n - 1}.\end{aligned}$$

We now apply Lemma 5.3.5 and obtain the desired conclusion. □

If $d \neq 0$, then the condition $k^m = -d + cn$ is not in general sufficient to guarantee the existence of a (0,1)-matrix A of order n satisfying $A^m = -dI + cJ$. Let $d = -1$. If $k^m = 1 + cn$, then Lemma 5.3.5 also implies that the k-circulant matrix A constructed in Theorem 5.3.6 satisfies $A^m = I + cJ$.

The following theorem of Lam and van Lint[1978] completely settles the existence question in the case $c = d = 1$.

Theorem 5.3.7. *Let m be a positive integer. There exists a $(0,1)$-matrix A of order n satisfying the equation*

$$A^m = -I + J$$

if and only if m is odd and $n = k^m + 1$ for some positive integer k.

Proof. Suppose that A is a (0,1)-matrix satisfying $A^m = -I + J$. Then A has trace equal to zero and by Theorem 5.3.2 A is a regular matrix of degree k with $n = k^m + 1$. First assume that $m = 2$. The eigenvalues of $J - I$ are $n - 1 = k^2$ with multiplicity 1 and -1 with multiplicity $n - 1$. Hence the eigenvalues of A are k with multiplicity 1 and $\pm i$ with equal multiplicities. This contradicts the fact that A has zero trace. If m is even, then $(A^{m/2})^2 = -I + J$ where $A^{m/2}$ is also a (0,1)-matrix. We conclude that m is an odd integer.

We now suppose that m is an odd integer and $n = k^m + 1$. Let $g = -k$ and let A be the g-circulant matrix of order n whose first row consists of 0 followed by k 1's and $(n - 1 - k)$ 0's. The Hall polynomial of A satisfies

$$\theta_A(x) = x + x^2 + \cdots + x^k.$$

We have

$$\theta_A(x)\theta_A(x^g)\cdots\theta_A(x^{g^{m-1}}) \equiv -1 + (1 + x + \cdots + x^{n-1}) \pmod{x^n - 1},$$

and

$$g^m = (-k)^m = -k^m = -n + 1 \equiv 1 \pmod{n}.$$

We now deduce from Lemma 5.3.5 that $A^m = -I + J$. □

Although the matrix equation $A^m = -dI + cJ$ has a simple form some difficult questions emerge. A complete characterization of those integers m,d and c for which there exists a (0,1)-matrix solution is very much unsettled. In those instances where a solution is known to exist, virtually nothing is known about the number of nonisomorphic solutions for general n. If $n = k^2$ the number of regular (0,1)-matrices of order n satisfying $A^2 = J$ is unknown (Hoffman[1967]).

If in the equation $A^m = dI + cJ$, we do not regard d and c as prescribed, then different questions emerge. In these circumstances we seek (0,1)-matrices A of order n for which A^m has all elements on its main diagonal equal and all off-diagonal elements equal. A trivial solution is the matrix $A = J$, and in this case $d = 0$ and $c = n^{m-1}$. Ma and Waterhouse[1987] have completely settled the existence of nontrivial solutions in the case $d = 0$.

Theorem 5.3.8. *Let $m \geq 2$ and $n \geq 2$ be integers. There exists a $(0,1)$-matrix A of order n with $A \neq J$ such that A^m has all of its elements equal if and only if n is divisible by the square of some prime number. Let g be the product of the distinct prime divisors of n. If n is divisible by the square of some prime number, then there exists a g-circulant matrix $A \neq J$ for which all elements of A^m are equal.*

Let A be a (0,1)-matrix of order n. We now consider the more general matrix equation

$$A^2 = E + cJ, \tag{5.16}$$

where E is a diagonal matrix and c is a positive integer. This corresponds to the case of a digraph D of order n with exactly c walks of length 2 between every pair of *distinct* vertices.

It is at once evident that the matrix A of (5.16) need no longer be regular. The following matrices with $c = 1$ provide counterexamples:

$$\left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & O & \\ 1 & & & \end{array}\right] (n \geq 2), \tag{5.17}$$

where O is the zero matrix of order $n-1$, and

$$\left[\begin{array}{cccc} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & Q & \\ 1 & & & \end{array}\right] (n \geq 4), \tag{5.18}$$

where Q is a symmetric permutation matrix of order $n-1$.

In this connection Ryser[1970] has established the following.

Theorem 5.3.9. *Let A be a $(0,1)$-matrix of order $n > 1$ that satisfies the matrix equation*

$$A^2 = E + cJ,$$

where E is a diagonal matrix and c is a positive integer. Then there exists an integer k such that A is regular of degree k except for the $(0,1)$-matrices of order n with $c = 1$ isomorphic to (5.17) or (5.18) and the $(0,1)$-matrix of order 5 with $c = 2$ isomorphic to

$$\left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right].$$

Furthermore, if A is regular of degree k, then

$$A^2 = dI + cJ,$$

where

$$k^2 = d + cn$$

and

$$-c < d \leq k - c.$$

Bridges[1971] has extended the preceding result and found all of the non-regular solutions of

$$A^2 - aA = E + cJ.$$

In addition Bridges[1972] and Bridges and Mena[1981] have completely settled the regularity question for matrix equations of the form $A^r = E+cJ$.

Theorem 5.3.10. *Let A be a $(0,1)$-matrix of order n that satisfies the equation*

$$A^r = E + cJ,$$

where E is a diagonal matrix and r and c are positive integers. Then A is regular provided $n > 3$ and $r > 3$.

Additional information on regular solutions of the various types of matrix equations described here can be found in Chao and Wang[1987], King and Wang[1985], Knuth[1970], Lam[1975] and Wang[1980, 1981, 1982].

Exercises

1. Determine the Hoffman polynomial of the strongly connected digraph of order n each of whose vertices has indegree and outdegree equal to 1 (a directed cycle of length n).
2. Determine the Hoffman polynomial of the digraph obtained from the complete bipartite graph $K_{m,m}$ by replacing each edge by two oppositely directed arcs.
3. Prove Lemma 5.3.3.
4. Suppose that $k^m = 1 + cn$. Prove that the k-circulant matrix A in Theorem 5.3.6 satisfies the equation $A^m = I + cJ$.
5. Construct the digraph of the solution A of order $n = 9$ of the equation $A^3 = -I + J$ given in the proof of Theorem 5.3.7.

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