Introduction to the Graph Laplacian and Rankability

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Weighted Directed Graphs

We are interested in finite simple loopless weighted (non-negative) directed graphs $\mathbb{G}^+$.

For $\Gamma = (V, E, w) \in \mathbb{G}^+$, we denote by V the vertex set, by E the edge set, and by $w\colon V \times V \to \mathbb{R}_{\geq 0}$ the associated weight function.

If $(i, j) \in E$, then there is an edge from i to j; the weight of (i, j) is given by w_{ij}.

We use the convention that $w_{ij} = 0$ if and only if $(i, j) \notin E$.

The Laplacian

The in-degree and out-degree of the vertex $k \in V$ is defined by

$$d_i(k) = \sum_j w_{jk} \quad \text{and} \quad d_o(k) = \sum_j w_{kj},$$

respectively.

Let $D = \operatorname{diag}(d_o(1), \ldots, d_o(n))$ denote the out degree matrix and $A = [w_{ij}]_{i,j=1}^n$ denote the adjacency matrix.

Then, the graph Laplacian is defined by

$$L = D - A.$$

Irreducible Matrix

The matrix A is reducible if there exists a permutation matrix P such that

$$P^T AP = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}.$$

We say that the matrix A is irreducible if it is not reducible.

Frobenius Normal Form

The Frobenius normal form of the Laplacian is

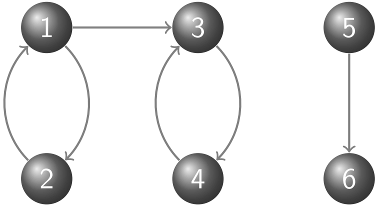
$$L = \begin{bmatrix} L_1 & L_{12} & \cdots & L_{1k} \\ & L_2 & \cdots & L_{2k} \\ & & \ddots & \vdots \\ & & & L_k \end{bmatrix}.$$

The blocks L_i are irreducible matrices that correspond to the strongly connected components of Γ, denoted Γ_i.

We call $(k-1)$ the degree of reducibility of L.

Example

The graph Γ:



The graph Laplacian:

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Isolated Vertices

The vertex $i \in V$ is isolated if $w_{ij} = 0$ for all $j \in V$.

Let $V_R \subset V$ denote the subset of vertices that are not isolated

Let L_R denote the principle submatrix of L whose columns and rows correspond to the vertices in V_R.

Note that $\sigma(L) = \sigma(L_R) \cup \{|V \setminus V_R| \text{ times the eigenvalue } 0\}$

Isolated Components

Let $\Gamma = (V, E, w) \in \mathbb{G}^+$ and let $\Gamma' = (V', E', w')$ be an induced subgraph of Γ, i.e., $V' \subseteq V$, $E' = E \cap (V' \times V')$, and $w' = w \restriction_{E'}$.

We say that Γ' is isolated if $w_{ij} = 0$ for all $i \in V'$ and $j \notin V'$.

Let L be in its Frobenius normal form, then Γ_i is isolated if and only if $L_{ij} = 0$ for $j = i + 1, \ldots, k$.

Lemma 1

Let $\Gamma \in \mathbb{G}^+$ and let Γ' be an induced subgraph of Γ. If Γ' is isolated, then

$$\sigma(L(\Gamma')) \subseteq \sigma(L(\Gamma)).$$

Theorem 1 (Taussky)

A complex $n \times n$ matrix A is non-singular if A is irreducible and

$$|A_{ii}| \geq \sum_{j \neq i} |A_{ij}|,$$

with equality in at most $(n-1)$ cases.

Lemma 2

Let $\Gamma \in \mathbb{G}^+$ and let L be the graph Laplacian of Γ in Frobenius normal form.

Then, zero is an eigenvalue (in fact a simple eigenvalue) of L_i if and only if Γ_i is isolated.

Theorem 2

Let $\Gamma \in \mathbb{G}^+$. Then, the following statements are equivalent:

i. The algebraic multiplicity of the zero eigenvalue of L is equal to $k \in \mathbb{N}$.

ii. There exists k isolated strongly connected components of Γ.

iii. The minimum number of directed trees needed to span the whole graph is equal to $k \in \mathbb{N}$.

Directed Cycles

Let $\Gamma \in \mathbb{G}^+$. A directed cycle in Γ is a cycle with all edges oriented in the same direction.

A vertex of Γ that is contained in at least one directed cycle is called a cyclic vertex.

We say that Γ is acyclic if none of its vertices are cyclic.

Lemma 3

The following statements are equivalent:

i. The graph $\Gamma \in \mathbb{G}^+$ is acyclic.

ii. Every strongly connected component of Γ consists of exactly one vertex.

iii. The graph Laplacian of Γ in its Frobenius normal form is upper triangular.

Theorem 3

Let $\Gamma \in \mathbb{G}^+$ and let L be the graph Laplacian of Γ. Then,

$$\sigma(L) = \{d_o(1), \ldots, d_o(n)\}$$

if and only if Γ is acyclic.

Corollary 1

Let $\Gamma \in \mathbb{G}^+$ have binary weights and let L be the graph Laplacian of Γ.

Then, Γ is a perfect dominance graph if and only if

$$\sigma(L) = \{d_o(1), \ldots, d_o(n)\}$$

and there exists a re-ordering of the vertices such that $d_o(i) = n - i$ for $i = 1, \ldots, n$.

Corollary 2

Let $\Gamma \in \mathbb{G}^+$ have binary weights and let L be the graph Laplacian of Γ.

Then, Γ is a perfect dominance graph if and only if there exists a permutation matrix P such that the eigenpairs (λ_i, v_i) of L satisfy

$$\lambda_i = n - i$$

and

$$Pv_i = \sum_{k=1}^{i} e_i,$$

for $i = 1, \ldots, n$.

Rankability Measure

Let $\Gamma \in \mathbb{G}^+$ have binary weights and let L be the graph Laplacian of Γ.

Denote the eigenvalues of L and the out-degrees of Γ by

$$\sigma = \{\lambda_1, \ldots, \lambda_n\},$$
$$\delta = \{d_o(1), \ldots, d_o(n)\},$$

respectively.

The rankability measure of Γ is defined by

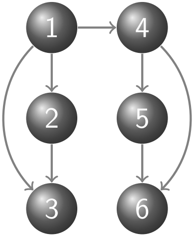
$$\frac{d(\sigma, s) + d(\delta, s)}{2(n-1)},$$

where $s = \{n-1, \ldots, 1, 0\}$ and d is the Hausdorff distance between two sets.

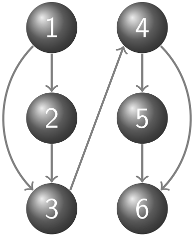
Example

Graph	Rankability Measure
Dominance Graph	0.000
Perturbed Dominance Graph	0.062
Perturbed Random Graph	0.223
Nearly Disconnected	0.400
Random	0.565
Cyclic	0.700
Completely Connected	0.800
Empty Graph	1.000

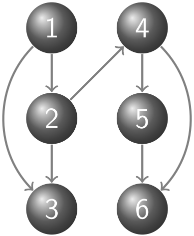
Example



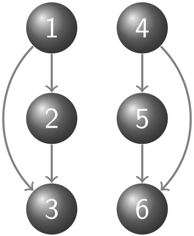
rankability $= 0.4$



rankability $= 0.6$



rankability $= 0.6$



rankability $= 0.6$

Algebraic Connectivity

We define the algebraic connectivity of $\Gamma \in \mathbb{G}^+$ by

$$\alpha(\Gamma) = \min_{x \in S} x^T L x,$$

where L is the graph Laplacian of Γ and

$$S = \{x \in \mathbb{R}^n : \ \ \|x\| = 1, \ x \perp e\}.$$

Another useful quantity is

$$\beta(\Gamma) = \max_{x \in S} x^T L x.$$

Properties

- $\alpha(\Gamma)$ and $\beta(\Gamma)$ are independent of the ordering of vertices since S is an invariant subspace of permutation matrices.

- Let Q be an orthonormal matrix whose columns span S, then

$$\alpha(\Gamma) = \lambda_{\min}\left(\frac{1}{2}Q^T\left(L + L^T\right)Q\right)$$

and

$$\beta(\Gamma) = \lambda_{\max}\left(\frac{1}{2}Q^T\left(L + L^T\right)Q\right)$$

Properties

- If $\Gamma_1, \Gamma_2 \in \mathbb{G}^+$ have the same vertex set, then

$$\alpha(\Gamma_1) + \alpha(\Gamma_2) \leq \alpha(\Gamma_1 \cup \Gamma_2) \leq \beta(\Gamma_1 \cup \Gamma_2) \leq \beta(\Gamma_1) + \beta(\Gamma_2).$$

- Let $\Gamma_1 \times \Gamma_2$ be the cartesian product of two graphs Γ_1 and Γ_2, then

$$\alpha(\Gamma_1 \times \Gamma_2) \leq \min(\alpha(\Gamma_1), \alpha(\Gamma_2)) \leq \max(\beta(\Gamma_1), \beta(\Gamma_2)) \leq \beta(\Gamma_1 \times \Gamma_2).$$

Properties

- Let $\Gamma \in \mathbb{G}^+$ with graph Laplacian L. Then,

$$\lambda_1\left(\frac{1}{2}(L+L^T)\right) \leq \alpha(\Gamma) \leq \lambda_2\left(\frac{1}{2}(L+L^T)\right)$$

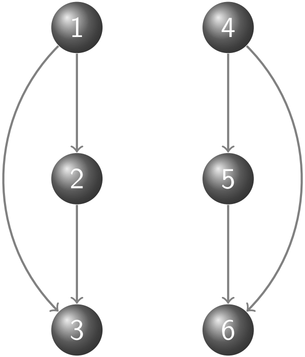
  and

$$\lambda_{n-1}\left(\frac{1}{2}(L+L^T)\right) \leq \beta(\Gamma) \leq \lambda_n\left(\frac{1}{2}(L+L^T)\right)$$

- Let i, j be non-adjacent vertices of $\Gamma \in \mathbb{G}^+$. Then,

$$\alpha(\Gamma) \leq \frac{1}{2}\left(d_o(i) + d_o(j)\right) \leq \beta(\Gamma).$$

Example



Eigenvalues of the Laplacian:

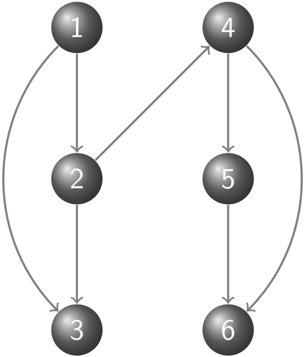
$$\sigma(L) = \{2, 1, 0, 2, 1, 0\}$$

Spectral Rankability: 0.6

Algebraic Connectivity: -0.389

AC Rankability: 1.0

Example



Eigenvalues of the Laplacian:

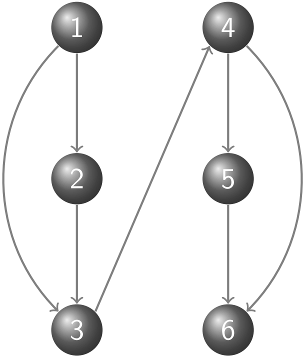
$$\sigma(L) = \{2, 2, 0, 2, 1, 0\}$$

Spectral Rankability: 0.6

Algebraic Connectivity: -0.303

AC Rankability: 0.929

Example



Eigenvalues of the Laplacian:

$$\sigma(L) = \{2, 1, 1, 2, 1, 0\}$$

Spectral Rankability: 0.6

Algebraic Connectivity: -0.068

AC Rankability: 0.738