3

Matrices and Digraphs

3.1 Basic Concepts

A *digraph* (*directed graph*) D consists of a finite set V of elements called *vertices* (*points*) together with a prescribed set E of *ordered* pairs of *not necessarily distinct* vertices of V. Every ordered pair α of vertices a and b in E is called an *arc* (*directed edge, directed line*) of the digraph D, written

$$\alpha = (a, b).$$

Notice that a digraph may contain both the arcs (a, b) and (b, a) as well as *loops* of the form (a, a). The generalization of a digraph by allowing multiple arcs results in a *directed general graph* (*general digraph*). Here the arcs sets are required to be finite. Most of the terminology in section 2.1 carries over without ambiguity to the directed case. The vertices a and b of an arc $\alpha = (a, b)$ are the *endpoints* of α, but now a is called the *initial vertex* and b is called the *terminal vertex* of α. The number of arcs issuing from a vertex is the *outdegree* of the vertex. The number of entering arcs is the *indegree* of the vertex. We agree that a loop at a vertex contributes one to the outdegree and also one to the indegree. If the outdegrees and indegrees equal a fixed integer k for every vertex of D, then D is *regular of degree* k.

Let D be a general digraph of order n whose set of vertices is $V = \{a_1, a_2, \ldots, a_n\}$. We let a_{ij} equal the multiplicity $m(a_i, a_j)$ of the arcs of the form (a_i, a_j). The resulting matrix

$$A = [a_{ij}], (i, j, = 1, 2, \ldots, n)$$

of order n is called the *adjacency matrix* of D. The entries of A are nonnegative integers. But A need no longer be symmetric. In the event that A is symmetric, then the general digraph D is said to be *symmetric*.

The sum of row i of the adjacency matrix A is the outdegree of vertex a_i. The sum of column j of A is the indegree of vertex a_j. Notice how nicely loops behave in that they contribute one to both the outdegree and indegree. The assertion that D is regular of degree k is equivalent to the assertion that A has all of its line sums equal to k.

In the general digraph we now deal with *directed walks* of the form

$$(a_0, a_1), (a_1, a_2), \ldots, (a_{m-1}, a_m), (m > 0)$$

which is also denoted by

$$a_0 \to a_1 \to a_2 \to \cdots \to a_{m-1} \to a_m.$$

Most of the related concepts already discussed for general graphs carry over without ambiguity to the directed case; in particular we now refer to *directed chains* (*paths*), *directed trails*, and *directed cycles* (*circuits*). Notice that a directed cycle can have any of the lengths $1, 2, \ldots, n$.

Two vertices a and b are called *strongly connected* provided there are directed walks from a to b and from b to a. A vertex is regarded as trivially strongly connected to itself. Strong connectivity between vertices is reflexive, symmetric, and transitive. Hence strong connectivity defines an equivalence relation on the vertices of D and yields a partition

$$V_1 \cup V_2 \cup \cdots \cup V_t$$

of the vertices of D. The subdigraphs $D(V_1), D(V_2), \ldots, D(V_t)$ formed by taking the vertices in an equivalence class and the arcs incident to them are called the *strong components* of D. The general digraph D is *strongly connected* (*strong*) if it has exactly one strong component. Thus D is strongly connected if and only if each pair of vertices is strongly connected.

A *tournament* of order n is a digraph which can be obtained from the complete graph K_n by assigning a direction to each of its edges. Let A be the adjacency matrix of a tournament. Then A is a (0,1)-matrix satisfying the equation

$$A + A^T = J - I$$

and is called a *tournament matrix*.

Exercises

1. Let A be the adjacency matrix of a general digraph D. Show that there is a directed walk of length m from vertex a_i to vertex a_j if and only if the element in position (i, j) of A^m is positive.
2. Let D be a general digraph each of whose vertices has a positive indegree. Prove that D contains a directed cycle.
3. Prove that a digraph is strongly connected if and only if there is a closed directed walk which contains each vertex (at least once).

4. Prove that a tournament of order n contains a path of length $n-1$. Conclude that the term rank of a tournament matrix of order n equals $n-1$ or n.
5. Let A be the adjacency matrix of a digraph of order n. Prove that the term rank of A equals the maximal size of a set of vertices which can be partitioned into parts each of which is the set of vertices of a directed chain (that is, a path or directed cycle).

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3.2 Irreducible Matrices

Let $A = [a_{ij}], (i, j, = 1, 2, \ldots, n)$ be a matrix of order n consisting of real or complex numbers. To A there corresponds a digraph $D = D(A)$ of order n as follows. The vertex set is the n-set $V = \{a_1, a_2, \ldots, a_n\}$. There is an arc $\alpha = (a_i, a_j)$ from a_i to a_j if and only if $a_{ij} \neq 0, (i, j = 1, 2, \ldots, n)$. We may think of a_{ij} as being a nonzero weight attached to the arc α. In the event that A is a matrix of nonnegative integers, the weight a_{ij} of α can be regarded as the multiplicity $m(\alpha)$ of α. Then D is a general digraph and A is its adjacency matrix. However, unless specified to the contrary, D is the unweighted digraph as defined above.

The matrix A of order n is called *reducible* if by simultaneous permutations of its lines we can obtain a matrix of the form

$$\begin{bmatrix} A_1 & O \\ A_{21} & A_2 \end{bmatrix}$$

where A_1 and A_2 are square matrices of order at least one. If A is not reducible, then A is called *irreducible*. Notice that a matrix of order 1 is irreducible.

Irreducibility has a direct interpretation in terms of the digraph D of A.

Theorem 3.2.1. *Let A be a matrix of order n. Then A is irreducible if and only if its digraph D is strongly connected.*

Proof. First assume that A is reducible. Then the vertex set V of D can be partitioned into two nonempty sets V_1 and V_2 in such a way that there is no arc from a vertex in V_1 to a vertex in V_2. If a is a vertex in V_1 and b is a vertex in V_2 there is no directed walk from a to b. Hence D is not strongly connected.

Now assume that D is not strongly connected. Then there are distinct vertices a and b of D for which there is no directed walk from a to b. Let W_1 consist of b and all vertices of D from which there is a directed walk to b, and let W_2 consist of a and all vertices to which there is a directed walk from a. The sets W_1 and W_2 are nonempty and disjoint. Let W_3 be the set consisting of those vertices which belong to neither W_1 nor W_2. We now simultaneously permute the lines of A so that the lines corresponding to the vertices in W_2 come first followed by those corresponding to the vertices in W_3:

$$\begin{array}{c c} & \begin{array}{ccc} W_2 & W_3 & W_1 \end{array} \\ \begin{array}{c} W_2 \\ W_3 \\ W_1 \end{array} & \left[\begin{array}{ccc} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{array}\right]. \end{array}$$

Since there is no directed walk from a to b there is no arc from a vertex in W_2 to a vertex in W_1. Also there is no arc from a vertex c in W_3 to a vertex in W_1, because such an arc implies that c belongs to W_1. Hence $X_{13} = O$ and $X_{23} = O$, and A is reducible. □

If A is irreducible, then for each pair of distinct vertices a_i and a_j there is a walk from a_i to a_j with length at most equal to $n - 1$. Hence if the elements of A are nonnegative numbers, then it follows from Theorem 3.2.1 that A is irreducible if and only if the elements of the matrix $(I + A)^{n-1}$ are all positive.

We return to the general case of a matrix A of order n. Let D be the digraph of A and let $D(V_1), D(V_2), \ldots, D(V_t)$ be the strong components of D. Let D^* be the digraph of order t whose vertices are the sets $V_1, V_2, \ldots, V_t$ in which there is an arc from V_i to V_j if and only if $i \neq j$ and there is an arc in D from some vertex in V_i to some vertex in V_j. The digraph D^* is the *condensation digraph* of D. The digraph D^* has no loops.

Lemma 3.2.2. *The condensation digraph D^* of the digraph D has no closed directed walks.*

Proof. Suppose that D^* has a closed directed walk. Since D^* has no loops, its length is at least two. If V_k and V_l $(k \neq l)$ are two vertices of D^* of this walk, and $a \in V_k$ and $b \in V_l$, then a and b are strongly connected vertices of D in different strong components. This contradiction implies that D^* has no closed directed walks. □

Lemma 3.2.3. *Let D^* be a digraph of order t which has no closed directed walks. Then the vertices of D^* can be ordered $b_1, b_2, \ldots, b_t$ so that each arc of D^* is of the form (b_i, b_j) for some i and j with $1 \leq i < j \leq t$.*

Proof. The proof proceeds by induction on t. If $t = 1$ the digraph has no arcs. Assume that $t > 1$. The assumption that there are no closed directed walks implies that there is a vertex b_1 with indegree equal to zero. We now delete the vertex b_1 and all incident arcs and apply the induction hypothesis to the resulting digraph of order $t - 1$. □

We now show that a square matrix can be brought to a very special form by simultaneous permutations of its lines.

Theorem 3.2.4. *Let A be a matrix of order n. Then there exists a permutation matrix P of order n and an integer $t \geq 1$ such that*

$$PAP^T = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1t} \\ O & A_2 & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_t \end{bmatrix}, \tag{3.1}$$

where $A_1, A_2, \ldots, A_t$ are square irreducible matrices. The matrices $A_1, A_2, \ldots, A_t$ that occur as diagonal blocks in (3.1) *are uniquely determined to within simultaneous permutation of their lines, but their ordering in* (3.1) *is not necessarily unique.*

Proof. Let $D(V_1), D(V_2), \ldots, D(V_t)$ be the strong components of the digraph D of A. By Lemma 3.2.2 the condensation graph D^* has no closed directed walks. We apply Lemma 3.2.3 to D^* and obtain an ordering $W_1, W_2, \ldots, W_t$ of $V_1, V_2, \ldots, V_t$ with the property that each arc of D^* is of the form (W_i, W_j) for some i and j with $1 \leq i < j \leq t$. We now simultaneously permute the lines of A so that the lines corresponding to the vertices in W_1 come first, followed in order by those corresponding to the vertices in $W_2, \ldots, W_t$, and then partition the resulting matrix to obtain

$$\begin{bmatrix} A_1 & A_{12} & \cdots & A_{1t} \\ A_{21} & A_2 & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \cdots & A_t \end{bmatrix}. \tag{3.2}$$

In (3.2) $D(W_i)$ is the digraph corresponding to the matrix A_i, $(i = 1, 2, \ldots, t)$. Since there is no arc of the form (W_i, W_j) with $i > j$, each of the matrices A_{ij} with $i > j$ is a zero matrix. Hence (3.2) has the form given in (3.1). Because $D(W_i)$ is strongly connected, each A_i is an irreducible matrix.

We now establish the uniqueness assertion in the theorem. Let Q be a permutation matrix of order n such that

$$QAQ^T = \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1s} \\ O & B_2 & \cdots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_s \end{bmatrix}$$

where each B_i is an irreducible square matrix. The subdigraphs $D'_1, D'_2, \ldots, D'_s$ of D corresponding to the diagonal blocks $B_1, B_2, \ldots, B_s$, respectively, are strongly connected. Moreover, since for $i > j$ there is no directed walk in D from a vertex of D'_i to a vertex of D'_j, the digraphs $D'_1, D'_2, \ldots, D'_s$ are the strong components of D. Since the strong components of a digraph are uniquely determined, The digraphs $D'_1, D'_2, \ldots, D'_s$ are $D(V_1), D(V_2), \ldots, D(V_t)$ in some order. Thus $s = t$ and to within simultaneous permutations of lines, the matrices $B_1, B_2, \ldots, B_t$ are the matrices $A_1, A_2, \ldots, A_t$ in some order. □

The form in (3.1) appears in the works of Frobenius[1912] and is called the *Frobenius normal form* of the square matrix A. The irreducible matrices $A_1, A_2, \ldots, A_t$ that occur along the diagonal are the *irreducible components* of A. By Theorem 3.2.4 the irreducible components are uniquely determined only to within simultaneous permutations of their lines. This slight ambiguity causes no difficulty. Notice that the matrix A is irreducible if and only if it has exactly one irreducible component. Whether the ordering of the irreducible components along the diagonal in the Frobenius normal form is uniquely determined depends on the matrices A_{ij} $(1 \le i < j \le t)$. For example, let

$$\left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{X}} \\ 1 & 0 & & \\ \multicolumn{2}{c}{\multirow{2}{*}{O}} & 1 & 1 \\ & & 1 & 1 \end{array}\right].$$

Then A is in Frobenius normal form and the irreducible components of A are

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If X is a zero matrix, then we may change the order of the diagonal blocks A_1 and A_2 and obtain a different Frobenius normal form for A. If, however, X is not a zero matrix, then the Frobenius normal form of A is unique. In general, the irreducible components of the square matrix A that correspond to vertices of the condensation graph D^* of $D(A)$ which are incident with no arc can always be put in the first positions along the diagonal of the Frobenius normal form (3.1).

Let A be an irreducible matrix of order n. If B is obtained from A by simultaneous line permutations, then B is also irreducible. However, a matrix B obtained from A by arbitrary line permutations may be reducible. For example, interchanging rows 1 and 2 of the irreducible matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

we obtain the reducible matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus it may be possible to permute the lines of a reducible matrix and obtain an irreducible matrix. The following theorem of Brualdi[1979] characterizes those square matrices which can be obtained from irreducible matrices by arbitrary line permutations. Since for permutation matrices P and Q, $PAQ = (PAP^T)PQ$, it suffices to consider only column permutations.

Theorem 3.2.5. *Let A be a matrix of order n. There exists a permutation matrix Q of order n such that AQ is an irreducible matrix if and only if A has at least one nonzero element in each line.*

Proof. If A has a zero line, then for each permutation matrix Q of order n, AQ has a zero line and hence is reducible. Conversely, assume that A has no zero line. Without loss of generality we assume that A is in Frobenius normal form

$$\begin{bmatrix} A_1 & A_{12} & \cdots & A_{1t} \\ O & A_2 & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_t \end{bmatrix}.$$

If $t = 1$, then A is irreducible and we may take $Q = I$. Now assume that $t > 1$. Let $D(V_1), D(V_2), \ldots, D(V_t)$ be the strong components of the digraph D of A corresponding, respectively, to the irreducible matrices $A_1, A_2, \ldots, A_t$. Any arc that leaves the vertex set V_i enters the vertex set $V_{i+1} \cup \cdots \cup V_t$, $(1 \leq i \leq t-1)$. For each $i = 1, 2, \ldots, t$ we choose a vertex a_i in V_i. Let B be the matrix obtained from A by cyclically permuting the columns corresponding to the vertices $a_1, a_2, \ldots, a_t$. The digraph D' of B is obtained from the digraph D of A by changing the arcs that entered

a_i into arcs that enter $a_{i+1}, (i = 1, \ldots, t-1)$, and changing the arcs that entered a_t into arcs that enter a_1. We show that B is irreducible by proving that the digraph D' is strongly connected. For convenience of notation we define a_{t+1} to be a_1.

If each of the strong components of D either has order greater than one or has order one and contains a loop, the proof is simple to express. In this case each strong component $D(V_i)$ has a sequence $\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{ik_i}$ of $k_i \geq 1$ closed walks of nonzero length which begin and end at $a_i, (i = 1, 2, \ldots, t)$. Each vertex in V_i belongs to at least one of these walks and they may be chosen so that a_i occurs only as the first and the last vertex. By repeating walks if necessary we may assume that the k_i have a common value k. In D' the last arcs of these walks enter a_{i+1}. Let γ'_{ij} be the directed walk in D' obtained from γ_{ij} by replacing the last vertex a_i of γ_{ij} with a_{i+1} and the last arc with an arc entering a_{i+1} $(1 \leq i \leq t)$. Then

$$\gamma'_{11}, \gamma'_{12} \cdots, \gamma'_{1k}, \gamma'_{21}, \gamma'_{22}, \cdots, \gamma'_{2k}, \cdots, \gamma'_{t1}, \gamma'_{t2}, \cdots, \gamma'_{tk}$$

is a closed directed walk in D' which contains each vertex of D' at least once. It follows that D' is strongly connected in this case.

In the general case some of the strong components of D may consist of a single isolated vertex without a loop. However since A has no zero lines, neither the first component $D(V_1)$ nor the last component $D(V_t)$ can be of this form. We prove that D' is strongly connected by showing that for each vertex a there is a directed walk from a_1 to a and a directed walk from a to a_1. Suppose that a is a vertex of V_i. We may assume that $i > 1$. In order to obtain a directed walk from a_1 to a it suffices to obtain a directed walk from a_1 to a_i. Since the column of A corresponding to vertex a_{i-1} contains a 1, there is an arc in D' from some vertex b in $V_1 \cup \cdots \cup V_{i-1}$ to a_i. Arguing inductively, there is a directed walk in D' from a_1 in V_1 to b. Hence there is a directed walk in D' from a_1 to a.

An argument similar to the above, but using the assumption that each row of A has a 1, allows us to conclude that there is a directed walk in D' from a to a vertex c in V_t. It then follows that there is a directed walk in D' from a to a_1. Hence D' is strongly connected. □

For some historical remarks on the origins of the property of irreducibility of matrices, we refer the reader to Schneider[1977].

Exercises

1. Determine the special nature of the Frobenius normal form of a tournament matrix.
2. What is the Frobenius normal form of a permutation matrix of order n?
3. Determine the smallest number of nonzero elements of an irreducible matrix of order n.

4. Show by example that the product of two irreducible matrices may be reducible (even if the matrices have nonnegative elements).
5. Let A be an irreducible matrix of order n with nonnegative elements. Assume that each element on the main diagonal of A is positive. Let x be a column vector with nonnegative elements. Prove that if x contains at least one 0, then Ax has fewer 0's than x.
6. Let D be a strongly connected digraph of order n and assume that each directed cycle of D has length 2. Prove that D can be obtained from a tree of order n by replacing each edge $\{a, b\}$ with the two oppositely directed arcs (a, b) and (b, a).

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3.3 Nearly Reducible Matrices

Let D be a strong digraph of order n with vertex set $V = \{a_1, a_2, \ldots, a_n\}$, and let $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ be its (0,1)-adjacency matrix of order n. By Theorem 3.2.1 A is an irreducible matrix. The digraph D is called *minimally strong* provided each digraph obtained from D by the removal of an arc is not strongly connnected. Evidently, a minimally strong digraph has no loops. Each arc of the digraph D corresponds to a 1 in the adjacency matrix A. Thus the removal of an arc in D corresponds in the adjacency matrix A to the replacement of a 1 with a 0. The irreducible matrix A is called *nearly reducible* (Hedrick and Sinkhorn[1970]) provided each matrix obtained from A by the replacement of a 1 with a 0 is a reducible matrix. Thus the digraph D is minimally strong if and only if its adjacency matrix A is nearly reducible. A nearly reducible matrix has zero trace. The matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is an example of a nearly reducible matrix.

More generally we say that an arbitrary matrix A of order n is *nearly reducible* if its digraph D is minimally strong. However, in discussing the combinatorial structure of nearly reducible matrices, it suffices to consider (0,1)-matrices.

We now investigate the structure of minimally strong digraphs as determined by Luce[1952] (see also Berge[1973]). Let D be a strong digraph of order n with vertex set V. Each vertex has indegree and outdegree at least equal to 1. A vertex whose indegree and outdegree both equal 1 is called *simple*. If all the vertices of D are simple, then D consists of the vertices and arcs of a directed cycle of length n, and D is minimally strong. A directed walk

$$a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_{m-1} \rightarrow a_m, (m \geq 1)$$

is called a *branch* provided the following three conditions hold:

(i) a_0 and a_m are not simple vertices;
(ii) the set $W = \{a_1, \ldots, a_{m-1}\}$ contains only simple vertices;
(iii) the subdigraph $D(V - W)$ is strongly connected.

Notice that a branch may be closed and the set W may be empty. If the removal of an arc (a, b) from D results in a strongly connected digraph, then $a \rightarrow b$ is a branch. Let U be a nonempty subset of m vertices of V. The *contraction of D by U* is the general digraph $D(\otimes U)$ of order $n-m+1$ defined as follows: The vertex set of $D(\otimes U)$ is $V - U$ with an additional vertex labeled (U). The arcs of D which have both of their endpoints in $V - U$ are arcs of $D(\otimes U)$; in addition, for each vertex a in $V - U$ there is an arc in $D(\otimes U)$ from a to (U) [respectively, from (U) to a] of multiplicity k if there are k arcs in D from a to vertices in U [respectively, from vertices in U to a].

Theorem 3.3.1. *Let D be a minimally strong digraph with vertex set V. Let U be a nonempty subset of V for which the subdigraph $D(U)$ is strongly connected. Then both $D(U)$ and $D(\otimes U)$ are minimally strong digraphs.*

Proof. If the removal of an arc of $D(U)$ leaves a strong digraph, then the removal of that arc from D leaves a strong digraph. Hence $D(U)$ is a minimally strong digraph.

It follows in an elementary way that $D(\otimes U)$ is strongly connected. Now let α be an arc of $D(\otimes U)$. First assume that α is also an arc of D. If the removal of α from $D(\otimes U)$ leaves a strong digraph, then the removal of α from D also leaves a strong digraph. Now assume that α is an arc joining the vertex (U) and some vertex a in $V-U$. If the multiplicity of α is greater than one, then since $D(U)$ is strongly connected all but one of the arcs of D that contribute to the multiplicity of α can be removed from D to leave a strong digraph. It follows that α has multiplicity one and $D(\otimes U)$ is a digraph. Let α' be the arc of D joining a and some vertex in U which corresponds to the arc α of $D(\otimes U)$. Since $D(U)$ is strongly connected, the removal of α' from D leaves a strong digraph if the removal of α from $D(\otimes U)$ leaves a strong digraph. Since D is minimally strong, we deduce that no arc can be removed from $D(\otimes U)$ to leave a strong digraph. Hence $D(\otimes)$ is a minimally strong digraph. □

A special case of Theorem 3.3.1 asserts that in a minimally strong digraph D the only arcs joining the vertices of a directed cycle in D are the arcs of the directed cycle. We now show that minimally strong digraphs contain simple vertices.

Lemma 3.3.2. *Let D be a minimally strong digraph of order $n \geq 2$. Then D has at least two simple vertices.*

Proof. Since D is strongly connected, there must be a directed cycle in D. If D is a directed cycle, then all its vertices are simple. This happens, in particular, when $n = 2$. We now assume that $n > 2$ and proceed by induction on n. First assume that all directed cycles in D have length 2. Then D can be obtained from a tree by replacing each of its edges $\{a, b\}$ by the arcs (a, b) and (b, a) in opposite directions. A tree with $n > 2$ vertices has at least two pendant vertices and each of these pendant vertices is a simple vertex of D.

Now assume that D has a directed cycle μ of length $m \geq 3$. Since D is not itself a directed cycle, we have $m \leq n - 1$. Let U be the set of vertices on the arcs of μ. Then the subdigraph $D(U)$ contains no arcs other than those of μ. The contraction $D(\otimes U)$ has order $n - m + 1 \geq 2$ and, by the induction hypothesis, has (at least) two simple vertices. A simple vertex of $D(\otimes U)$ different from the vertex (U) is a simple vertex of D. Suppose that one of the two simple vertices of $D(\otimes U)$ is (U). Then in D there is exactly one arc (a, c) with a in U and c in $V - U$, and exactly one arc (d, b) with d in $V - U$ and b in U. Since $m \geq 3$ there is a vertex e in U different from a and b. But then e is a simple vertex of D. It follows that D contains at least two simple vertices. □

In a minimally strong digraph a branch cannot have length one and hence a branch contains at least one simple vertex. A digraph D which is a directed cycle is minimally strong, but since all its vertices are simple, D contains no branches.

Lemma 3.3.3. *Let D be a minimally strong digraph of order $n \geq 3$ which is not a directed cycle. Then D has a branch of length $k \geq 2$.*

Proof. Since D is not a directed cycle, it has at least one vertex which is not simple. We define a general digraph D^* whose vertex set V^* is the set of nonsimple vertices of D. Let a and b be vertices of D^*. Each directed walk α of D from a to b all of whose vertices other than a and b are simple determines an arc $\alpha^* = (a, b)$ on D^*. The general digraph D^* is strongly connected and has no simple vertices. If D^* has a loop or a multiple arc α^*, then α is a branch of D. Otherwise D^* is a strong digraph of order at least two with no simple vertices. By Theorem 3.3.2 D^* is not minimally strong and hence there is an arc α^* whose removal from D^* leaves a strong digraph. The directed walk α is a branch of D. □

It follows from Lemma 3.3.3 that any minimally strong digraph can be constructed by beginning with a directed cycle and sequentially adding branches. However, while every digraph constructed in this way is strongly connected, it need not be minimally strong.

We now apply Lemma 3.3.3 to determine an inductive structure for nearly reducible matrices (Hartfiel[1970]).

Theorem 3.3.4. *Let A be a nearly reducible $(0,1)$-matrix of order $n \geq 2$. Then there exists a permutation matrix P of order n and an integer m with $1 \leq m \leq n-1$ such that*

$$PAP^T = \left[\begin{array}{cccccc|c} 0 & 0 & 0 & \cdots & 0 & 0 & \\ 1 & 0 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 0 & \cdots & 0 & 0 & \\ 0 & 0 & 1 & \cdots & 0 & 0 & F_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ & & & F_2 & & & A_1 \end{array}\right] \qquad (3.3)$$

where A_1 is a nearly reducible matrix of order m. The matrix F_1 contains a single 1 and this 1 belongs to the first row and column j of F_1 for some j with $1 \leq j \leq m$. The matrix F_2 also contains a single 1 and this 1 belongs to the last column and row i of F_2 for some i with $1 \leq i \leq m$. The element in the position (i, j) of A_1 is 0.

Proof. If the digraph $D(A)$ is a directed cycle, then there is a permutation matrix P such that (3.3) holds where A_1 is a zero matrix of order 1. Assume that $D(A)$ is not a directed cycle. Then $n \geq 3$ and by Lemma 3.3.3 $D(A)$ has a branch. A direct translation of the defining properties of a branch implies the existence of a permutation matrix P such that (3.3) and the ensuing properties hold. □

A matrix A satisfying the conclusions of Theorem 3.3.4 is not necessarily nearly reducible. An example with $n = 6$ and $m = 5$ is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix obtained from A by replacing the 1 in row 2 and column 3 with a 0 is irreducible and hence A is not nearly reducible. However, given a nearly reducible (0,1)-matrix A_1 of order $m \geq 1$ it is possible to choose the matrices F_1 and F_2 so that the matrix in (3.3) is nearly reducible. Indeed by choosing F_1 to have a 1 in position (1,1) and by choosing F_2 to have a 1 in position $(1, n-m)$, we obtain a nearly reducible matrix of order n for each integer $n > m$.

The inductive structure of minimally strong digraphs provided by Lemma 3.3.3 can be used to bound the number of arcs in a minimally strong digraph and hence the number of 1's in a nearly reducible matrix. Let T be a tree of order n. We denote by $\overleftrightarrow{T}$ the digraph of order n obtained from T by replacing each edge $\{a, b\}$ with the two oppositely directed arcs (a, b) and (b, a). The digraph $\overleftrightarrow{T}$ is called a *directed tree*. The directed tree $\overleftrightarrow{T}$ is minimally strong and has $2(n-1)$ arcs. The following theorem is due to Gupta[1967] (see also Brualdi and Hedrick[1979]).

Theorem 3.3.5. *Let D be a minimally strong digraph of order $n \geq 2$. Then the number of arcs of D is between n and $2(n-1)$. The number of arcs of D is n if and only if D is a directed cycle. The number of arcs of D is $2(n-1)$ if and only if D is a directed tree.*

Proof. The outdegree of each vertex of a strongly connected graph is at least one and hence D has at least n arcs. There are exactly n arcs in D if and only if D is a directed cycle.

The upper bound on the number of arcs of D and the characterization of equality is proved by induction on the order n. If $n = 2$, then D is a directed cycle with two arcs. Assume that $n \geq 3$. If D is a directed cycle then D has $n < 2(n-1)$ arcs. Now suppose that D is not a directed cycle. By Lemma 3.3.3 D has a branch

$$\alpha : a_0 \to a_1 \to \cdots \to a_{m-1} \to a_m$$

of length $m \geq 2$. The subdigraph D_1 of D obtained by removing the vertices $a_1, \ldots, a_{m-1}$ is a minimally strong digraph of order $n - m + 1$. By the induction hypothesis D_1 has at most $2(n-m)$ arcs, and hence the number of arcs of D is at most

$$2(n-m) + m = 2n - m \leq 2n - 2.$$

Assume that D has $2n-2$ arcs. Then $m = 2$, α is the branch $a_0 \to a_1 \to a_2$ and D_1 is a minimally strong digraph of order $n-1$ with $2(n-2)$ arcs. By the induction hypothesis there is a tree T_1 of order $n-1$ such that $D_1 = \overleftrightarrow{T_1}$. Suppose that the vertex a_0 is different from the vertex a_2. Then in D_1 there is a directed chain $a_2 \to x_1 \to x_2 \to \cdots \to x_s \to a_0$ from a_2 to a_0. The arc (x_1, a_2) is an arc of D_1. Moreover,

$$x_1 \to x_2 \to \cdots \to a_0 \to a_1 \to a_2$$

is a directed chain in D from x_1 to a_2 which does not use the arc (x_1, a_2). It follows that the removal of the arc (x_1, a_2) of D leaves a strong digraph which contradicts the assumption that D is a minimally strong digraph. Hence $a_0 = a_2$. Let T be the tree obtained from T_1 by including the vertex a_1 and the edge $\{a_0, a_1\}$. Then $D = \overleftrightarrow{T}$. □

A direct translation of the preceding theorem yields the following.

Theorem 3.3.6. *Let A be a nearly reducible $(0,1)$-matrix of order $n \geq 2$. Then the number of 1's of A is between n and $2(n-1)$. The number of 1's of A equals n if and only if there is a permutation matrix P of order n such that*

$$PAP^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

The number of 1's of A equals $2(n-1)$ if and only if there is a tree T of order n such that A is the adjacency matrix of $\overleftrightarrow{T}$. □

It follows from Theorem 3.3.1 that an irreducible principal submatrix of order m of a nearly reducible (0,1)-matrix is itself nearly reducible and hence by Theorem 3.3.5 has at most $2(m-1)$ 1's. But a principal submatrix of a nearly reducible matrix need not be irreducible. A simple example is furnished by the leading principal submatrix of order 2 of the nearly reducible matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Nonetheless a principal submatrix of order m of a nearly reducible (0,1)-matrix has at most $2(m-1)$ 1's. This and other properties of nearly reducible matrices can be found in Brualdi and Hedrick[1979].

Lemma 3.3.3 can also be used to determine an inductive structure for strongly connected digraphs (or strongly connected general digraphs), and hence for irreducible matrices.

Theorem 3.3.7. *Let D be a strong digraph of order $n \geq 2$ with vertex set V. Then there exists a partition of V into $m \geq 2$ nonempty sets $W_1, W_2, \ldots, W_m$ such that the subdigraphs $D(W_1), D(W_2), \ldots, D(W_m)$ are strongly connected. Let $W_{m+1} = W_1$. Each arc of D that does not belong to one of these sub digraphs issues from W_i and enters W_{i+1} for some $i = 1, 2, \ldots, m$.*

Proof. We remove arcs from D in order to obtain a minimally strong digraph D'. It follows from Lemma 3.3.3 that D' has the cyclical structure in the theorem. Indeed the partition can be chosen so that $|W_i| = 1$ $(1 \leq i \leq m-1)$, $D(W_m)$ is minimally strong ($|W_m| = 1$ is possible) and there is exactly one arc issuing from W_i and entering W_{i+1} $(1 \leq i \leq m)$.

Consider a digraph with the cyclical structure of the theorem. If we add a new arc, then either this cyclical structure is retained or else the arc issues from some W_i and enters some W_j where $j \neq i, i+1$. In the latter case we obtain the cyclical structure of the theorem with vertex partition

$$W_{i+1}, \ldots, W_{j-1}, W_j \cup W_{j+1} \cup \cdots \cup W_i.$$

Since the minimally strong digraph D' has the cyclical structure in the theorem and since D is obtained from D' by adding arcs, it now follows that D has the desired cyclical structure. □

A direct translation of the previous theorem yields the following inductive structure for irreducible matrices.

Theorem 3.3.8. *Let A be an irreducible matrix of order $n \geq 2$. Then there exists a permutation matrix P of order n and an integer $m \geq 2$ such that*

$$PAP^T = \begin{bmatrix} A_1 & O & \cdots & O & E_1 \\ E_2 & A_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{m-1} & O \\ O & O & \cdots & E_m & A_m \end{bmatrix}$$

where $A_1, A_2, \ldots, A_m$ are irreducible matrices and $E_1, E_2, \ldots, E_m$ are matrices having at least one nonzero entry.

Exercises

1. Show that a minimally strong regular digraph is a directed cycle.
2. Prove that the permanent of a nearly reducible (0,1)-matrix of order n equals 0 or 1. For each integer $n \geq 2$ construct an example of a nearly reducible (0,1)-matrix with permanent equal to 0 and one with permanent equal to 1 (Hedrick and Sinkhorn[1970]).
3. Let $n \geq 2$ be an integer. Show that there exists a nearly reducible (0,1)-matrix of order n with exactly k 1's for each integer k with $n \leq k \leq 2(n-1)$ (Brualdi and Hedrick[1979]).
4. Let $n \geq 3$ be an integer and let D be a minimally strong digraph of order n with exactly $2n-3$ arcs. Prove that D has a directed cycle of length 3 and does not have a directed cycle of length greater than 3 (Brualdi and Hedrick[1979]).
5. Let A be a nearly reducible (0,1)-matrix of order n. Deduce from Theorem 3.3.1 that if a principal submatrix B of A is irreducible, then in fact B is nearly reducible. Give an example of a nearly reducible matrix which has a reducible principal submatrix.
6. Let A be a nearly reducible (0,1)-matrix and let B be a principal submatrix of A of order k. Prove that the number of 1's in B is at most equal to $2(k-1)$ (Brualdi and Hedrick[1979]).

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3.4 Index of Imprimitivity and Matrix Powers

Let D denote a strongly connected digraph of order n whose set of vertices is $V = \{a_1, a_2, \ldots, a_n\}$. Let $k = k(D)$ be the greatest common divisor of the lengths of the closed directed walks of D. (If $n = 1$ and D does not contain a loop, k is undefined.) The integer k is called the *index of imprimitivity* of D. The digraph D is *primitive* if $k = 1$ and *imprimitive* if $k > 1$. The length of a closed directed walk is the sum of the lengths of one or more directed cycles, and hence the index of imprimitivity k is also the greatest common divisor of the lengths of the directed cycles of D. The integer k does not exceed the length of any directed cycle of D. There are a number of elementary facts concerning the index of imprimitivity which we collect in the following lemma.

Lemma 3.4.1. *Let D be a strongly connected digraph of order n with index of imprimitivity equal to k.*

(i) For each vertex a of D, k equals the greatest common divisor of the lengths of the closed directed walks containing a.
(ii) For each pair of vertices a and b, the lengths of the directed walks from a to b are congruent modulo k.
(iii) The set V of vertices of D can be partitioned into k nonempty sets $V_1, V_2, \ldots, V_k$ where, with $V_{k+1} = V_1$, each arc of D issues from V_i and enters V_{i+1} for some i with $1 \leq i \leq k$.
(iv) For $x_i \in V_i$ and $x_j \in V_j$ the length of a directed walk from x_i to x_j is congruent to $j - i$ modulo $k, (1 \leq i, j \leq k)$.

Proof. Let a and b be two vertices of D and let k_a and k_b denote the greatest common divisors of the lengths of the closed directed walks containing a and b, respectively. Let α be a closed directed walk containing a and suppose that α has length r. Since D is strongly connected there is a directed walk β from a to b of some length s and a directed walk γ from b to a of some length t. We may combine α, β and γ and obtain closed directed walks containing b with lengths $s + t$ and $r + s + t$, respectively. Thus k_b is a divisor of r, and since α was an arbitrary closed directed walk containing a, k_b is a divisor of k_a. In a similar way one proves that k_a is a divisor of k_b. Therefore $k_a = k_b$. But a and b are arbitrary vertices of D and (i) follows. (We note that (i) does not hold in general if we consider only the directed cycles containing the vertex a.)

Now let β' be another directed walk from a to b, and let s' be the length of β'. We may combine β and γ and also β' and γ to obtain closed directed walks containing a with lengths $s + t$ and $s' + t$, respectively. Since k is a divisor of $s + t$ and $s' + t$, k is a divisor of $s - s'$. Hence (ii) holds.

Let V_i denote the set of vertices x_i for which there is a directed walk from vertex a to x_i with length congruent to i modulo $k, (i = 1, 2, \ldots, k)$. By (ii) the sets $V_1, V_2, \ldots, V_k$ are mutually disjoint. Notice that the vertex a belongs to V_k. Since D is strongly connected each vertex belongs to one of the sets $V_1, V_2, \ldots, V_k$, and none of these sets can be empty. Let (x_i, x_j) be any arc of D where $x_i \in V_i$ and $x_j \in V_j$. There is a directed walk from a to x_i whose length is congruent to i modulo k and thus a directed walk from a to x_j whose length is congruent to $i + 1$ modulo k. Hence $i + 1$ is congruent to j modulo k, and (iii) follows. The proof of (iv) is quite similar to that of (iii). $\square$

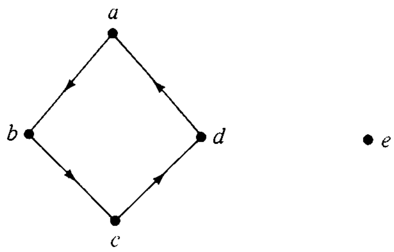

Figure 3.1

The sets $V_1, V_2, \ldots, V_k$ in (iii) of Lemma 3.4.1 are called the *sets of imprimitivity* of D. Although their construction depended on a choice of vertex a, they are uniquely determined. Indeed much more is true. Call a digraph D *cyclically r-partite with ordered partition* $U_1, U_2, \ldots, U_r$ provided $U_1, U_2, \ldots, U_r$ is a partition of the vertex set V of D into r nonempty sets where, with $U_{r+1} = U_1$, each arc of D issues from U_i and enters U_{i+1} for some $i = 1, 2, \ldots, r$. If D is cyclically r-partite then r is a divisor of the length of each directed cycle of D, and in addition, D is cyclically s-partite for each positive integer s which is a divisor of r. Thus *if D is strongly connected, then D is cyclically r-partite if and only if r is a divisor of the index of imprimitivity k of D*. Let D be a strongly connected digraph with sets of imprimitivity $V_1, V_2, \ldots, V_k$. Suppose that D is cyclically r-partite with ordered partition $U_1, U_2, \ldots, U_r$. Then except for a possible cyclic rearrangement,

$$\begin{aligned} U_1 &= V_1 \cup V_{r+1} \cup \cdots, \\ U_2 &= V_2 \cup V_{r+2} \cup \cdots, \\ &\cdots \\ U_r &= V_r \cup V_{2r} \cup \cdots. \end{aligned}$$

If D is not strongly connected, it may be cyclically r-partite with respect to two ordered partitions which are not cyclic rearrangements. For example, the digraph in Figure 3.1 is cyclically 4-partite with ordered partition $\{a\}, \{b\}, \{c\}, \{d, e\}$. It is also cyclically 4-partite with respect to the ordered partition $\{a\}, \{b\}, \{c, e\}, \{d\}$.

Now let A be an irreducible matrix of order n. By Theorem 3.2.1 the digraph $D(A)$ is strongly connected. We define the *index of imprimitivity* of A to be the index of imprimitivity k of $D(A)$. In addition we call A *primitive* (respectively, *imprimitive*) if $D(A)$ is primitive (respectively, imprimitive). We also say that A is *r-cyclic* if $D(A)$ is cyclically r-partite. Imprimitive matrices, more generally, r-cyclic matrices with $r > 1$ have a restrictive structure which is a consequence of the structure of cyclically r-partite digraphs discussed above.

Suppose the digraph $D(A)$ is cyclically r-partite with ordered partition $U_1, U_2, \ldots, U_r$. Let U_i contain n_i vertices $(i = 1, 2, \ldots, r)$. Then $n = n_1 + n_2 + \cdots + n_r$, and by simultaneously permuting the lines of A so that the rows corresponding to the vertices in U_1 come first, followed in order by those corresponding to the vertices in $U_2, \ldots, U_r$, we may determine a permutation matrix P of order n so that

$$PAP^T = \begin{bmatrix} O & A_{12} & O & \cdots & O \\ O & O & A_{23} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_{r-1,r} \\ A_{r1} & O & O & \cdots & O \end{bmatrix}. \tag{3.4}$$

In (3.4) the zero matrices on the diagonal are square matrices of orders $n_1, n_2, \ldots, n_r$, respectively. The matrices $A_{i,i+1}$ display the adjacencies between vertices in U_i and U_{i+1}. If $r = 1$ then (3.4) reduces to $PAP^T = A$ with P equal to the identity matrix of order n. If r is the index of imprimitivity k of A, then (3.4) holds with $r = k$. The matrices $A_{12}, A_{23}, \ldots, A_{r-1,r}, A_{r1}$ in (3.4) are called the *r-cyclic components* of the matrix A. The r-cyclic components may be cyclically permuted in (3.4). In addition, since the elements in the sets U_i can be given in any specified order,

$$P_1 A_{12} P_2^T, P_2 A_{23} P_3^T, \ldots, P_{r-1} A_{r-1,r} P_r^T, P_r A_{r1} P_1^T$$

can be taken as the r-cyclic components of A for any choice of permutation matrices $P_1, P_2, \ldots, P_r$ of orders $n_1, n_2, \ldots, n_r$, respectively. If A is irreducible, the r-cyclic components are uniquely determined apart from these transformations.

Suppose that A is r-cyclic and a permutation matrix P has been determined so that (3.4) holds. Then

$$PA^rP^T = \begin{bmatrix} B_1 & O & \cdots & O \\ O & B_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_r \end{bmatrix} \tag{3.5}$$

where

$$\begin{aligned} B_1 &= A_{12}A_{23} \cdots A_{r-1,r}A_{r1} \\ B_2 &= A_{23}A_{34} \cdots A_{r1}A_{12} \\ &\cdots \\ B_r &= A_{r1}A_{12} \cdots A_{r-2,r-1}A_{r-1,r}. \end{aligned}$$

In particular, if $r > 1$ then A^r is reducible and has at least r irreducible components. If A is a matrix whose entries are nonnegative real numbers, further information can be obtained.

We now assume that $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ is a nonnegative matrix of order n. Let t be a positive integer and let the element in the (i, j) position of A^t be denoted by $a_{ij}^{(t)}, (i, j = 1, 2, \ldots, n)$. Then $a_{ij}^{(t)}$ is positive if and only if there is a directed walk of length t from vertex a_i to vertex a_j in the digraph $D(A)$. In particular, the locations of the zeros and nonzeros in A^t are wholly determined by the digraph $D(A)$.

The following lemma is usually attributed to Schur (see Kemeny and Snell[1960]).

Lemma 3.4.2. *Let S be a nonempty set of positive integers which is closed under addition. Let d be the greatest common divisor of the integers in S. Then there exists a positive integer N such that td is in S for every integer $t \geq N$.*

Proof. We may divide each integer in S by d and this allows us to assume that $d = 1$. There exist integers $r_1, r_2, \ldots, r_m$ in S which are relatively prime. Each integer k can be expressed as a linear combination of $r_1, r_2, \ldots, r_m$ with integral, but not necessarily nonnegative, coefficients. Let $q = r_1 + r_2 + \cdots + r_m$. Then we may determine integers c_{ij} such that

$$i = c_{i1}r_1 + c_{i2}r_2 + \cdots + c_{im}r_m, (i = 0, 1, \ldots, q - 1).$$

Let M be the maximum of the integers $|c_{ij}|$, let $N = Mq$ and let t be any integer with $t \geq N$. There exist integers p and l such that $t = pq + l$ where $p \geq M$ and $0 \leq l \leq q - 1$. Then

$$\begin{aligned} t &= pq + l = p(r_1 + r_2 + \cdots + r_m) + (c_{l1}r_1 + c_{l2}r_2 + \cdots + c_{lm}r_m) \\ &= (p + c_{l1})r_1 + (p + c_{l2})r_2 + \cdots + (p + c_{lm})r_m. \end{aligned}$$

Since $p \geq M$, each of the integers $p + c_{lj}$ is nonnegative. Because S is closed under addition, we conclude that t is in S whenever $t \geq N$. □

Let S and d satisfy the hypotheses of Lemma 3.4.2. Then there exists a smallest positive integer $\phi(S)$ such that nd is in S for every integer $n \geq \phi(S)$. The integer $\phi(S)$ is called the *Frobenius–Schur index* of S. (Frobenius was one of the first to consider the evaluation of this number.) If S consists of all nonnegative linear combinations of the positive integers $r_1, r_2, \ldots, r_m$, then we write the Frobenius–Schur index of S as $\phi(r_1, r_2, \ldots, r_m)$.

Lemma 3.4.3. *Let D be a strongly connected digraph of order n with vertex set V. Let k be the index of imprimitivity of D and let $V_1, V_2, \ldots, V_k$ be the sets of imprimitivity of D. Then there exists a positive integer N for which the following holds: If x_i and x_j are vertices belonging respectively* *to V_i and V_j, then there are directed walks from x_i to x_j of every length $j - i + tk$ with $t \geq N, (1 \leq i, j \leq k)$.*

Proof. Let a and b be vertices in V and suppose that $a \in V_i$ and $b \in V_j$. By (iv) of Lemma 3.4.1 each directed walk from a to b has length $j - i + tk$ for some nonnegative integer k. Let t_{ab} be an integer such that $j - i + t_{ab}k$ is the length of some directed walk from a to b. The lengths of the closed directed walks containing b form a nonempty set S_b of positive integers which is closed under addition. By (i) of Lemma 3.4.1 k is the greatest common divisor of the integers in S_b. We apply Lemma 3.4.2 to S_b and obtain a positive integer N_b such that $tk \in S_b$ for every integer $t \geq N_b$. There exists a directed walk from a to b with length $j - i + tk$ for every integer $t \geq t_{ab} + N_b$. We now let N be the maximum integer in the set $\{t_{ab} + N_b | a, b \in V\}$. □

We return to the irreducible, nonnegative matrix A of order n with index of imprimitivity equal to k. There exists a permutation matrix P of order n such that (3.4) and (3.5) hold with $r = k$. The matrices $A_{12}, A_{23}, \ldots$, $A_{k-1,k}, A_{k1}$ are the k-cyclic components of A, and these arise from the sets of imprimitivity of the digraph $D(A)$. It follows from (3.5) that for each positive integer t we have

$$PA^{tk}P^T = \begin{bmatrix} B_1^t & O & \cdots & O \\ O & B_2^t & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_k^t \end{bmatrix}, \tag{3.6}$$

where

$$\begin{aligned} B_1 &= A_{12}A_{23} \cdots A_{k-1,k}A_{k1} \\ B_2 &= A_{23}A_{34} \cdots A_{k1}A_{12} \\ &\cdots \\ B_k &= A_{k1}A_{12} \cdots A_{k-2,k-1}A_{k-1,k}. \end{aligned}$$

We apply Lemma 3.4.3 and conclude that there exists a positive integer N such that $B_1^t, B_2^t, \ldots, B_k^t$ are positive matrices for all $t \geq N$. In the special case that $k = 1$, A is primitive and A^t is a positive matrix for each integer $t \geq N$. If $k > 1$ and A is imprimitive we may apply (iv) of Lemma 3.4.1 and conclude that no positive integral power of A is a positive matrix. Hence we obtain the following characterization of primitive matrices.

Theorem 3.4.4. *Let A be a nonnegative matrix of order n. Then A is primitive if and only if some positive integral power of A is a positive matrix. If A is primitive then there exists a positive integer N such that A^t is a positive matrix for each integer $t \geq N$.*

Proof. The fact that A is primitive if and only if some positive integral power of A is a positive matrix has been proved in the above paragraph under the additional assumption that A is irreducible. Since a positive integral power of a reducible matrix can never be positive, the theorem follows. □

Let A be a primitive nonnegative matrix. By Theorem 3.4.4 there exists a smallest positive integer $\exp(A)$ such that A^t is a positive matrix for all integers $t \geq \exp(A)$. The integer $\exp(A)$ is called the *exponent of the primitive matrix* A. The exponent is the subject of the next section. We continue now with the general development of irreducible matrices.

A matrix which is a positive integral power of a reducible matrix is reducible. However, positive integral powers of irreducible matrices may be either reducible or irreducible. In the case of a nonnegative matrix A those positive integral powers of A which are irreducible can be characterized (Dulmage and Mendelsohn[1967] and Brualdi and Lewin[1982]).

Theorem 3.4.5. *Let A be an irreducible nonnegative matrix of order n with index of imprimitivity equal to k. Let m be a positive integer. Then A^m is irreducible if and only if k and m are relatively prime. In general there is a permutation matrix P of order n (independent of m) such that*

$$PA^mP^T = \begin{bmatrix} C_1 & O & \cdots & O \\ O & C_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & C_r \end{bmatrix} \tag{3.7}$$

where r is the greatest common divisor of k and m. The matrices $C_1, C_2, \ldots,$ C_r in (3.7) *are irreducible matrices and each has index of imprimitivity equal to k/r.*

Proof. The digraph $D = D(A)$ is strongly connected with index of imprimitivity equal to k. Let $V_1, V_2 \ldots, V_k$ be the sets of imprimitivity of D. Then D is cyclically k-partite with ordered partition $V_1, V_2, \ldots, V_k$. Since r is a divisor of k, D is also cyclically r-partite with ordered partition $U_1, U_2, \ldots, U_r$ where

$$\begin{aligned} U_1 &= V_1 \cup V_{r+1} \cup \cdots, \\ U_2 &= V_2 \cup V_{r+2} \cup \cdots, \\ &\cdots \\ U_r &= V_r \cup V_{2r} \cup \cdots. \end{aligned}$$

We may choose a permutation matrix P of order n such that (3.4) holds.

For this permutation matrix P, PA^rP^T has the form given in (3.5). Since r is also a divisor of m, we may write

$$PA^mP^T = (PA^rP^T)^{m/r} = \begin{bmatrix} C_1 & O & \cdots & O \\ O & C_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & C_r \end{bmatrix}$$

where $C_1 = B_1^{m/r}, C_2 = B_2^{m/r}, \ldots, C_r = B_r^{m/r}$. If $r > 1$ then A^m is reducible. We first show that the matrices $C_1, C_2, \ldots, C_r$ are irreducible.

Let a and b be vertices in U_i where $1 \le i \le r$. There exist integers u and v such that $a \in V_{ur+i}$ and $b \in V_{vr+i}$. By Lemma 3.4.3 there exists a positive integer N such that there are directed walks in D from a to b of every length $(v-u)r + tk$ with $t \ge N$. Since r is the greatest common divisor of k and m, it follows from Lemma 3.4.2 that there is an integer t' such that

$$(v-u)r + t'k = ek + fm$$

for some nonnegative integers e and f. For each nonnegative integer s we have

$$(v-u)r + (t' - e + sm)k = (f + sk)m.$$

We now choose s large enough so that $t' - e + sm \ge N$. Then there is a directed walk in D from a to b with length $(f+sk)m$ and thus a directed walk in $D(A^m)$ with length $f + sk$. Since a and b are arbitrary vertices in U_i, we conclude that $D(C_i)$ is strongly connected and hence that C_i is irreducible, $(1 \le i \le r)$.

Let l be the length of a closed directed walk of $D(A^m)$. Then there is a closed directed walk in D with length lm. Because the index of imprimitivity of D is k, k is a divisor of lm. Since the greatest common divisor of k and m is r, k/r is a divisor of l. Therefore the index of imprimitivity of each digraph $D(C_i)$ is a multiple of k/r. We now show that the index of imprimitivity equals k/r.

We now take a and b to be the same vertex of U_i. Then $v = u$ and there are closed directed walks containing a in D of every length tk with $t \ge N$, and hence of every length

$$t\frac{m}{r}k = (tm)\frac{k}{r}$$

with $t \ge N$. It follows that in $D(C_i)$ there are closed directed walks containing a of every length $t(k/r)$ with $t \ge N$. We now take $t = N$ and $t = N+1$ and obtain closed directed walks in $D(C_i)$ whose lengths have a greatest common divisor equal to k/r. We now conclude that the index of imprimitivity of $D(C_i)$, and thus of C_i equals k/r for each integer $i = 1, 2, \ldots, r$. □

A matrix A of order n is *completely reducible* provided there exists a permutation matrix P of order n such that

$$PAP^T = \begin{bmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_t \end{bmatrix}$$

where $t \geq 2$ and $A_1, A_2, \ldots, A_t$ are square irreducible matrices. Thus A is completely reducible if and only if A is reducible and the matrices $A_{ij}, (1 \leq i < j \leq t)$ that occur in the Frobenius normal form (2.1) are all zero matrices. The following corollary is an immediate consequence of Theorem 3.4.5.

Corollary 3.4.6. *Let A be an irreducible nonnegative matrix. Let m be a positive integer. If A^m is reducible, then A^m is completely reducible.*

Let $A = [a_{ij}], (1 \leq i, j \leq n)$ be a matrix of order n which is r-cyclic. We conclude this section by showing how the r-cyclicity of A implies a special structure for the characteristic polynomial of A. First we recall the definition of the determinant. The determinant of A is given by

$$\det(A) = \sum_{\pi} (\operatorname{sign} \pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$$

where the summation extends over all permutations π of $\{1, 2, \ldots, n\}$ and $(\operatorname{sign} \pi) = \pm 1$ is the *sign* of the permutation π. We let the set V of vertices of $D(A)$ be $\{1, 2, \ldots, n\}$ where there is an arc (i, j) from vertex i to vertex j if and only if $a_{ij} \neq 0$. Suppose that $a_{1i_1} a_{2i_2} \cdots a_{ni_n} \neq 0$. Then $U = \{(1, i_1), (2, i_2), \ldots, (n, i_n)\}$ is a set of n arcs of $D(A)$. For each vertex there is exactly one arc in U leaving the vertex and exactly one arc entering it. Thus the set U of arcs can be partitioned into nonempty sets each of which is the set of arcs of a directed cycle of $D(A)$.

Now let

$$\varphi(\lambda) = \det(\lambda I - A) = c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_{n-1} \lambda + c_n, (c_0 = 1)$$

be the *characteristic polynomial* of A. The n roots of $\varphi(\lambda)$ are the *characteristic roots* (*eigenvalues*) of A. The coefficient c_i of λ^{n-i} equals $(-1)^i$ times the sum of the determinants of the principal submatrices of A of order i. It follows from the above discussion that $c_i \neq 0$ only if there exists in $D(A)$ a collection of directed cycles the sum of whose lengths is i which have no vertex in common. If the digraph $D(A)$ is cyclically r-partite, each directed cycle has a length which is divisible by r. Hence if $D(A)$ is cyclically r-partite, $c_i \neq 0$ only if r is a divisor of i. Hence the only powers of λ which can appear with a nonzero coefficient in $\varphi(\lambda)$ are $\lambda^n, \lambda^{n-r}, \lambda^{n-2r}, \ldots$. The following theorem is due to Dulmage and Mendelsohn[1963,1967].

Theorem 3.4.7. *Let A be an r-cyclic matrix of order n. Let P be a permutation matrix of order n such that* (3.4) *and* (3.5) *hold. Then there exists a monic polynomial $f(\lambda)$ and nonnegative integers $p_1, p_2, \ldots, p_r$ such that the following hold:*

(i) $f(0) \neq 0$;
(ii) The characteristic polynomial of B_i is $f(\lambda)\lambda^{p_i}$, $(i = 1, 2, \ldots, r)$. For each root μ of $f(\lambda)$ the elementary divisors corresponding to μ are the same for each of $B_1, B_2, \ldots, B_r$;
(iii) The characteristic polynomial of A is $f(\lambda^r)\lambda^{p_1+p_2+\cdots+p_r}$;
(iv) The characteristic polynomial of A^r is $f(\lambda)^r\lambda^{p_1+p_2+\cdots+p_r}$.

Proof. Since A is r-cyclic its characteristic polynomial $\varphi(\lambda)$ can be written as $\varphi(\lambda) = f(\lambda^r)\lambda^p$ where $f(\lambda)$ is a monic polynomial with $f(0) \neq 0$ and p is a nonnegative integer. Since the eigenvalues of A^r are the rth powers of the eigenvalues of A, the characteristic polynomial of A^r is $(f(\lambda))^r\lambda^p$. Let $\varphi_i(\lambda)$ be the characteristic polynomial of B_i $(i = 1, 2, \ldots, r)$. We have

$$\varphi_1(\lambda)\varphi_2(\lambda)\cdots\varphi_r(\lambda) = (f(\lambda))^r\lambda^p, \tag{3.8}$$

and the nonzero eigenvalues of $B_1, B_2, \ldots, B_r$ are all roots of $f(\lambda)$. Next we observe that

$$B_1 = A_{12}(A_{23}\cdots A_{r1}), \qquad \text{and} \qquad B_2 = (A_{23}\cdots A_{r1})A_{12}.$$

Standard results in matrix theory now allow us to conclude that the nonzero eigenvalues of B_1 are the same as those of B_2 and the elementary divisors of B_1 corresponding to its nonzero eigenvalues are the same as those corresponding to the nonzero eigenvalues of B_2. The same conclusions hold for B_2 and B_3, B_3 and B_4, $\ldots$, B_r and B_1. Hence $B_1, B_2, \ldots, B_r$ all have the same nonzero eigenvalues and the same elementary divisors corresponding to each of their nonzero eigenvalues. We are now able to assert that there exists a monic polynomial $g(\lambda)$ with $g(0) \neq 0$ and nonnegative integers $p_1, p_2, \ldots, p_r$ such that $\varphi_i(\lambda) = g(\lambda)\lambda^{p_i}$ $(i = 1, 2, \ldots, r)$. Substituting these last equations into (3.8) we obtain

$$(g(\lambda))^r\lambda^{p_1+p_2+\cdots+p_r} = (f(\lambda))^r\lambda^p. \tag{3.9}$$

Since $g(0) \neq 0$ and $f(0) \neq 0$ we conclude from (3.9) that $f(\lambda) = g(\lambda)$ and $p = p_1 + p_2 + \cdots + p_p$. Now each of (i)-(iv) holds. □

Most of the results in this section can be found in Dulmage and Mendelsohn[1967]. Other early papers which treat some of the topics, in some cases from a different viewpoint or with a different starting point, include Gantmacher[1959], Pták[1958], Romanovsky[1936], Pták and Sedláček[1958] and Varga[1962]. Spectral properties of irreducible nonnegative matrices relating to the index of imprimitivity will be studied in the book *Combinatorial Matrix Classes.*

Exercises

1. Show that the index of imprimitivity of a strongly connected digraph is not always equal to the greatest common divisor of the lengths of the directed cycles containing a specified vertex.
2. Prove or disprove that the product of two primitive matrices is primitive.
3. Prove that a primitive (0,1)-matrix of order $n \geq 2$ contains at least $n+1$ 1's and construct an example with exactly $n+1$ 1's.
4. Prove that the index of imprimitivity of an irreducible imprimitive symmetric matrix of order $n \geq 2$ equals 2.
5. Let A be a nonnegative matrix of order n and assume that A has no zero lines. Suppose that A is cyclically r-partite and has the form given in (3.4). Prove that A is irreducible if and only if $A_{12} \cdots A_{r-1,r}A_{r1}$ is irreducible (Dulmage and Mendelsohn[1967]; see also Minc[1974]).
6. (Continuation of Exercise 5) Prove that the number of irreducible components of A equals the number of irreducible components of $A_{12} \cdots A_{r-1,r}A_{r1}$ (Brualdi and Lewin[1982]).

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3.5 Exponents of Primitive Matrices

The exponent $\exp(A)$ of a primitive nonnegative matrix A has been defined to be the smallest positive integer k such that A^t is a positive matrix for all integers $t \geq k$. The exponent of A depends only on the digraph $D(A)$ (and not on the magnitude of the elements of A) and equals the smallest positive integer k such that for each ordered pair a, b of vertices there is a directed walk from a to b of length k, and thus a directed walk from a to b of every length greater than or equal to k. In investigating the exponent there is therefore no loss in generality in considering only

(0,1)-matrices. As a result we assume throughout this section that A is a primitive (0,1)-matrix of order n. The vertex set of the digraph $D(A)$ is denoted by $V = \{a_1, a_2, \ldots, a_n\}$.

The exponent of the matrix A can be evaluated in terms of other more basic quantities. Let $\exp(A : i, j)$ equal the smallest integer k such that the element in position (i, j) of A^t is nonzero for all integers $t \geq k$, $(1 \leq i, j \leq n)$. Let $\exp(A : i)$ equal the smallest positive integer p such that all the elements in row i of A^p are nonzero, $(1 \leq i \leq n)$. Thus $\exp(A : i, j)$ equals the smallest positive integer k such that there is a directed walk of length t from a_i to a_j in $D(A)$ for all $t \geq k$, and $\exp(A : i)$ equals the smallest positive integer p such that there are directed walks of length p from a_i to each vertex of $D(A)$.

Lemma 3.5.1. *The exponent of A equals the maximum of the integers*

$$\exp(A : i, j), \quad (i, j = 1, 2, \ldots, n).$$

It also equals the maximum of the integers

$$\exp(A : i), \quad (i = 1, 2, \ldots, n).$$

Proof. The first conclusion is an immediate consequence of the definitions involved. Suppose that there is a directed walk of length p in $D(A)$ from vertex a_i to vertex a_j for each j with $1 \leq j \leq n$. There is an arc $\alpha = (a_k, a_j)$ for some choice of vertex a_k. A directed walk from a_i to a_k of length p combined with the arc α determines a directed walk of length $p + 1$ from a_i to a_j. It follows that there are directed walks from a_i to each vertex a_j of every length $t \geq p$, and the second conclusion also holds. □

Lemma 3.5.1 is useful for obtaining upper bounds for the exponent of the primitive matrix A. If $f_1, f_2, \ldots, f_n$ are integers such that there are directed walks of length f_i from a_i to every vertex of $D(A)$, $(i = 1, 2, \ldots, n)$, then $\exp(A) \leq \max\{f_1, f_2, \ldots, f_n\}$.

An irreducible matrix with at least one nonzero element on its main diagonal is primitive, since its digraph has a directed cycle of length 1. The following theorem of Holladay and Varga[1958] gives a bound for the exponent of such a matrix.

Theorem 3.5.2. *Let A be an irreducible matrix of order n having $p \geq 1$ nonzero elements on its main diagonal. Then A is a primitive matrix and* $\exp(A) \leq 2n - p - 1$.

Proof. The digraph $D(A)$ has p loops, and we let W be the set of p vertices which are incident with a loop. Let a_i and a_j be two vertices. There is a directed path from a_i to a vertex a_k in W whose length is at most $n - p$ and a directed path from a_k to a_j whose length is at most $n - 1$. Combining these two directed paths we obtain a directed path from a_i to

a_j of length at most equal to $2n - p - 1$. Taking advantage of the loop at vertex a_k we obtain a directed walk from a_i to a_j whose length is exactly $2n - p - 1$. □

If the irreducible matrix A in Theorem 3.5.2 has no zeros on its main diagonal, then the exponent of A is at most $n - 1$. This special case of Theorem 3.5.2 is equivalent to the property noted in Section 2 that for an irreducible nonnegative matrix A of order n, $(I + A)^{n-1}$ is a positive matrix.

The characterization of those matrices achieving the bound in the following theorem is due to Shao[1987].

Theorem 3.5.3. *Let A be a symmetric irreducible $(0,1)$-matrix of order $n \geq 2$. Then A is primitive if and only if its associated digraph $D(A)$ has a directed cycle of odd length. If the symmetric matrix A is primitive, then* $\exp(A) \leq 2n-2$ *and equality occurs if and only if there exists a permutation matrix P of order n such that*

$$PAP^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

Proof. The digraph $D(A)$ is a symmetric digraph and has a directed cycle of length 2. Hence A is primitive if and only if $D(A)$ has a directed cycle of odd length. Assume that A is primitive. The matrix A^2 is primitive and has no zeros on its main diagonal. By Theorem 3.5.2 $(A^2)^{n-1}$ is a positive matrix and hence $\exp(A) \leq 2n - 2$. First suppose that A is the matrix displayed in the theorem. The smallest *odd* integer k for which there is a directed walk from vertex a_1 to itself is $2n - 1$. Hence $\exp(A) \geq \exp(A : 1, 1) \geq 2n - 2$. Now suppose that $\exp(A) = 2n - 2$. Examining the proof of Theorem 3.5.2 as applied to the matrix A^2 we see that there are two vertices whose distance in $D(A^2)$ is $n-1$. Thus $D(A^2)$ is a directed chain of length $n-1$ with a loop incident at each vertex. If there were three vertices a, b and c each two of which were adjacent in the symmetric digraph $D(A)$, then $a \to b \to c \to a$ would be a directed cycle in $D(A^2)$. It follows that $D(A)$ is a directed chain of length $n - 1$ with at least one vertex incident with a loop. Since $\exp(A) = 2n - 2$ there is exactly one loop in $D(A)$ and it is incident with one of the end vertices of the directed chain. □

Theorem 3.5.2 gives the upper bound $2n-2$ for the exponent of a primitive matrix whose digraph has a directed cycle of length 1. The following theorem of Sedláček[1959] and Dulmage and Mendelsohn[1964] furnishes a bound for the exponent in terms of the lengths of directed cycles.

Theorem 3.5.4. *Let A be a primitive $(0,1)$-matrix of order n. Let s be the smallest length of a directed cycle in the digraph $D(A)$. Then*

$$\exp(A) \leq n + s(n-2).$$

Proof. The matrix A^s has at least s positive elements on its main diagonal. Let W be the set of vertices of $D(A^s)$ which are incident with a loop. Then $|W| \geq s$ and each vertex can be reached by a directed walk in $D(A^s)$ of length $n-1$ starting from any vertex in W. Let the set of vertices of $D(A)$ be $V = \{a_1, a_2, \ldots, a_n\}$. In $D(A)$ each vertex a_j can be reached by a directed walk of length $s(n-1)$ starting from any vertex in W. For each vertex a_i there is a directed walk of length l_i to some vertex in W where $l_i \leq n-s$. It follows that

$$\exp(D(A) : i) \leq l_i + s(n-1) \leq n + s(n-2), \quad (i = 1, 2, \ldots, n).$$

We now apply Lemma 3.5.1 and obtain the conclusion of the theorem. □

Shao[1985] has characterized the (0,1)-matrices A whose exponent $\exp(A)$ achieves the upper bound $n + s(n-2)$ in the theorem.

Theorem 3.5.4 can be used to determine the largest exponent possible for a primitive matrix of order n. First we determine the Frobenius–Schur index of two relatively prime integers.

Lemma 3.5.5. *Let p and q be relatively prime positive integers. Then* $\phi(p,q) = (p-1)(q-1) = pq - p - q + 1$.

Proof. We first show that $\phi(p,q) \geq pq - p - q + 1$. Suppose that there are nonnegative integers a and b such that $pq - p - q = ap + bq$. The relative primeness of p and q implies that p is a divisor of $b+1$ and q is a divisor of $a+1$. Hence

$$pq - p - q = ap + bq \geq (q-1)p + (p-1)q = pq - p - q + pq,$$

a contradiction.

We next show that every integer $m > pq$ can be expressed as a *positive* integral linear combination of p and q. There exists an integer a with $1 \leq a \leq q$ such that $m \equiv ap \pmod{q}$. Let $b = (m - ap)/q$. Then b is a positive integer and $m = ap + bq$. It now follows that every integer $m > pq - p - q$ can be expressed as a nonnegative linear combination of p and q. □

The following inequality was stated by Wielandt[1950]; the first published proofs appeared in Rosenblatt[1957], Holladay and Varga[1958] and Pták[1958].

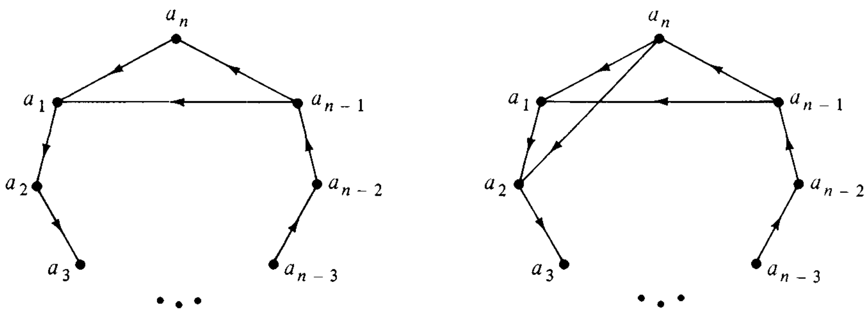

Figure 3.2

Theorem 3.5.6. *Let A be a primitive $(0,1)$-matrix of order $n \geq 2$. Then*

$$\exp(A) \leq (n-1)^2 + 1. \tag{3.10}$$

Equality holds in (3.10) if and only if there exists a permutation matrix P of order n such that

$$PAP^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.11}$$

Proof. Let s denote the smallest length of a directed cycle in the digraph $D(A)$. Since A is primitive we have $s \leq n-1$. Equation (3.10) now follows from Theorem 3.5.4. Assume that $\exp(A) = (n-1)^2 + 1$. Then $s = n-1$ and the primitivity of A implies that $D(A)$ also has a directed cycle of length n. [If $n = 2$ we use the fact that $D(A)$ is strongly connected.] Since $D(A)$ does not have a directed cycle with length smaller than $n-1$, it follows readily that apart from the labeling of the vertices $D(A)$ is one of the two digraphs D_1 and D_2 shown in Figure 3.2.

The digraph D_1 is the digraph of the matrix displayed in (3.11).

First assume that the digraph $D(A)$ equals D_1. Every closed directed walk from a_n to a_n has length $n + a(n-1) + bn$ for some nonnegative integers a and b. It follows from Lemma 3.5.5 with $p = n-1$ and $q = n$ that the integer $(n-2)(n-1) - 1$ cannot be expressed as $a(n-1) + bn$ for any choice of nonnegative integers a and b. Hence there is no directed walk from a_n to a_n of length $(n-2)(n-1) + n - 1 = (n-1)^2$. Using Lemma 3.5.1 we now see that $\exp(A) \geq \exp(A : n) \geq (n-1)^2 + 1$. Thus $\exp(A) = (n-1)^2 + 1$.

Now assume that the digraph $D(A)$ equals D_2. Every directed walk from a_1 to a_n has length $(n-1) + a(n-1) + bn$ for some nonnegative integers a

and b. Lemma 3.5.5 implies that there is no directed walk from a_1 to a_n with length $(n-1)+(n-2)(n-1)-1=(n-1)^2-1$. Hence $\exp(A)\geq(n-1)^2$. We now show that $\exp(A)=(n-1)^2$. Since each vertex is on a directed cycle of length n and is also on a directed cycle of length $n-1$, we apply Lemma 3.5.5 again and conclude that each vertex belongs to a closed directed walk of length t for each integer $t\geq(n-2)(n-1)$. Let a_i and a_j be any two vertices. There is a directed walk from a_i to a_j with length $l_{ij}\leq n-1$ and hence a directed walk from a_i to a_j of length $l_{ij}+t$ for each integer $t\geq(n-2)(n-1)$. Thus

$$\exp(A:i,j)\leq l_{ij}+(n-2)(n-1)\leq(n-1)+(n-2)(n-1)=(n-1)^2.$$

We now apply Lemma 3.5.1 to obtain $\exp(A)\leq(n-1)^2$. Hence $\exp(A)=(n-1)^2$. Thus the primitive matrix of order n has exponent equal to $(n-1)^2+1$ if and only if there is a permutation matrix P of order n such that (3.11) holds. □

In the proof of Theorem 3.5.6 we have also established the fact that a primitive (0,1)-matrix A of order n has exponent equal to $(n-1)^2$ if and only if the digraph $D(A)$ is, apart from the labeling of its vertices, the digraph D_2 of Figure 3.2. Thus A has exponent $(n-1)^2$ if and only if there is a permutation matrix P of order n such that

$$PAP^T=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&0&0&\cdots&1\\1&1&0&\cdots&0\end{bmatrix}.$$

Let n be a positive integer and let E_n denote the set of integers t for which there is a primitive matrix of order n with exponent equal to t. By Theorem 3.5.6, $E_n\subseteq\{1,2,\ldots,w_n\}$ where $w_n=(n-1)^2+1$. The exponent of a positive matrix is 1, and thus $1\in E_n$. By Theorem 3.5.6 and the discussion immediately following its proof, w_n and $w_n-1=(n-1)^2$ are in E_n. The following theorem of Shao[1985] shows that the exponent sets E_n form a nondecreasing chain of sets.

Theorem 3.5.7. *For all $n\geq1$,*

$$E_n\subseteq E_{n+1}.$$

Proof. Let A be a primitive (0,1)-matrix of order n with exponent equal to t. Let B be the matrix of order $n+1$ whose leading principal submatrix of order n equals A and whose last two rows and last two columns, respectively, are equal. The digraph $D(B)$ is obtained from the digraph $D(A)$ by adjoining a new vertex a_{n+1} to the vertex set $V=\{a_1,a_2,\ldots,a_n\}$ of

$D(A)$ and then adding arcs (a_i, a_{n+1}) and (a_{n+1}, a_j) whenever (a_i, a_n) and (a_n, a_j) are, respectively, arcs of $D(A)$. In addition, (a_{n+1}, a_{n+1}) is an arc of $D(B)$ if and only if (a_n, a_n) is an arc of $D(A)$.

An elementary argument based on these digraphs reveals that B is a primitive matrix and that B has exponent equal to t. □

Dulmage and Mendelsohn[1964] showed that if $n \geq 4$, E_n is a proper subset of $\{1, 2, \ldots, w_n\}$. Specifically they showed that there is no primitive matrix A of odd order n such that

$$n^2 - 3n + 5 \leq \exp(A) \leq (n-1)^2 - 1$$

or

$$n^2 - 4n + 7 \leq \exp(A) \leq n^2 - 3n + 1.$$

If n is even, there is no primitive matrix A with

$$n^2 - 4n + 7 \leq \exp(A) \leq (n-1)^2 - 1.$$

Intervals in the set $\{1, 2, \ldots, w_n\}$ containing no integer which is the exponent of a primitive matrix of order n have been called *gaps* in E_n. Lewin and Vitek[1981] obtained a family of gaps in E_n and in doing so obtained a test for deciding whether or not an integer m satisfying

$$\lfloor \frac{w_n}{2} \rfloor + 2 \leq m \leq w_n$$

belongs to E_n. The following theorem plays an important role in this test.

Theorem 3.5.8. *Let the exponent of a primitive* $(0,1)$*-matrix* A *of order* n *satisfy*

$$\exp(A) \geq \lfloor \frac{w_n}{2} \rfloor + 2.$$

Then the digraph $D(A)$ *has directed cycles of exactly two different lengths.*

Theorem 3.5.8 puts severe restrictions on a primitive matrix of order n whose exponent is at least $\lfloor w_n/2 \rfloor + 2$. Such a matrix must contain a large number of zeros. Lewin and Vitek[1981] conjectured that every positive integer m with $m \leq \lfloor w_n/2 \rfloor + 1$ is the exponent of at least one primitive matrix of order n. Shao[1985] disproved this conjecture by showing that the integer 48 which satisfies $48 < \lfloor w_{11}/2 \rfloor + 1 = 51$ is not the exponent of a primitive matrix of order 11. In addition he proved that

$$\{1, 2, \ldots, \lfloor \frac{w_n}{4} \rfloor + 1\} \subseteq E_n$$

for all n and that

$$\{1, 2, \ldots, \lfloor \frac{w_n}{2} \rfloor + 1\} \subseteq E_n$$

for all sufficiently large n. In doing so he showed that the conjecture of Lewin and Vitek would be true if one could establish the validity of a certain number theoretical question. Zhang[1987] proved the validity of the number theoretical question thereby settling the question which arose from the conjecture of Lewin and Vitek.

Theorem 3.5.9. *Let n be an integer with $n \geq 2$. Then for each positive integer m with $m \leq \lfloor w_n/2 \rfloor + 1$ there is a primitive matrix of order n with exponent equal to m with the exception of the integer $m = 48$ when $n = 11$.*

Let E_n^s denote the set of integers t for which there exists a symmetric primitive (0,1)-matrix of order n with exponent equal to t. By Theorem 3.5.3, $E_n^s \subseteq \{1, 2, \ldots, 2n-2\}$. Shao[1987] proved that $E_n^s = \{1, 2, \ldots, 2n-2\} - S$ where S is the set of odd integers m with $n \leq m \leq 2n-2$. Liu, McKay, Wormald and Zhang[1990] proved that the set of exponents of primitive (0,1)-matrices with zero trace is $\{2, 3, \ldots, 2n-4\} - S'$ where S' is the set of odd integers m with $n-2 \leq m \leq 2n-5$. Liu[1990] proved that for $n \geq 4$ every integer $m > 1$ which is the exponent of a primitive (0,1)-matrix of order n is the exponent of a primitive (0,1)-matrix of order n with zero trace.

Let A be a primitive matrix of order n. A matrix obtained from A by simultaneous line permutations is also primitive. But a matrix obtained from A by arbitrary line permutations need not be primitive even if it is irreducible. For example, the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

is a primitive matrix. Suppose we move column 1 so that it is between columns 3 and 4. We obtain the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is irreducible but not primitive.

The following theorem of Shao[1985] characterizes those matrices which can be obtained from primitive matrices by arbitrary line permutations. As in the case of irreducible matrices in Theorem 3.2.5, it suffices to consider only column permutations.

Theorem 3.5.10. *Let A be a $(0,1)$-matrix of order n. There exists a permutation matrix Q of order n such that AQ is a primitive matrix if and only if the following three conditions hold:*

(i) A has at least one 1 in each row and column;
(ii) A is not a permutation matrix;
(iii) A is not the matrix

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & O & \\ 1 & & & \end{bmatrix}$$

or any matrix obtained from it by line permutations.

We now discuss a theorem of Moon and Moser[1966]. Let $\mathcal{A}_n$ denote the set of all (0,1)-matrices of order n. Let $\mathcal{P}_n$ denote the subset of $\mathcal{A}_n$ consisting of the primitive matrices. The proportion of primitive matrices among all the (0,1)-matrices of order n is

$$\frac{|\mathcal{P}_n|}{|\mathcal{A}_n|} = \frac{|\mathcal{P}_n|}{2^{n^2}}.$$

Theorem 3.5.11. *Almost all $(0,1)$-matrices of order n are primitive, that is*

$$\lim_{n\to\infty} \frac{|\mathcal{P}_n|}{2^{n^2}} = 1.$$

Indeed almost all $(0,1)$-matrices of order n are primitive and have exponent equal to 2.

Since a primitive matrix is irreducible, almost all (0,1)-matrices of order n are irreducible.

The study of the exponent of primitive, nearly reducible matrices was initiated by Brualdi and Ross[1980] and further investigated by Ross[1982], Yang and Barker[1988] and Li[1990]. Other bounds for the exponent are in Heap and Lynn [1964], Lewin[1971] and Lewin[1974]. Some generalizations of the exponent are considered in Brualdi and Li[1990], Chao[1977], Chao and Zhang[1983] and Schwarz[1973].

Exercises

1. Let A be primitive symmetric $(0,1)$-matrix of order $n \geq 2$ having $p \geq 1$ 1's on its main diagonal. Prove that

$$\exp(A) \leq \max\{n-1, 2(n-p)\}$$

(Lewin[1971]).

2. Let A be an irreducible $(0,1)$-matrix of order $n > 2$. Consider the digraph $D(A)$ and assume that there are vertices a_i and a_j (possibly the same vertex) such that there are directed walks from a_i to a_j of each of the lengths $1, 2, \ldots, n-1$. Prove that A is primitive and that the exponent of A does not exceed $2d+1$ where d is the diameter of $D(A)$ (Lewin[1971]).
3. Prove that there does not exist a primitive matrix of order $n \geq 5$ whose exponent equals $n^2 - 2n$ (Dulmage and Mendelsohn[1964]).
4. Let A be a $(0,1)$-matrix of order n. Prove that the conditions (i), (ii) and (iii) in Theorem 3.5.10 must be satisfied if there is to be a permutation matrix Q such that AQ is a primitive matrix.
5. Let A be the matrix (3.11) displayed in Theorem 3.5.6. Determine the numbers $\exp(A; i), (i = 1, 2, \ldots, n)$ (Brualdi and Li[1990]).
6. Let n be a positive integer and let E_n^s denote the set of integers t for which there exists a primitive symmetric matrix of order n and exponent t. Prove that $E_n^s \subseteq E_{n+1}^s, (n \geq 1)$. Use this fact and the displayed matrix in Theorem 3.5.3 to show that $k \in E_n^s$ for each even positive integer $k \leq 2n-2$ (Shao[1987]).
7. Prove that the exponent of a primitive, nearly reducible matrix of order n is at least 4. [In fact, it is at least 6, and for each $n \geq 4$ there exists a primitive, nearly reducible matrix of order n whose exponent equals 6 (Brualdi and Ross[1980]).]
8. Let A be a tournament matrix of order n. If $n \geq 4$, prove that A is primitive if and only if A is irreducible. If $n \geq 5$ and A is primitive, prove that the exponent of A is at most equal to $n+2$ (Moon and Pullman[1967]).

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3.6 Eigenvalues of Digraphs

Let D be a general digraph of order n and let its vertex set be the n-set $V = \{a_1, a_2, \ldots, a_n\}$. Let A be the adjacency matrix of D. The characteristic polynomial of A is called the *characteristic polynomial* of D and the collection of the n eigenvalues of A is called the *spectrum* of D. If D is a symmetric digraph, then the spectrum of D consists of n real numbers. In general, the spectrum of D consists of n complex numbers. Information about the eigenvalues in the spectrum of the digraph D is not in general easy to obtain. In addition, the combinatorial significance of the eigenvalues is less evident than it is for the eigenvalues of a graph. The directed cycles of D can be used to obtain regions in the complex plane which contain the spectrum of D. Such *eigenvalue inclusion regions* can be obtained more generally for complex matrices and their associated digraphs.

Let $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ be a complex matrix of order n. Inclusion regions for the eigenvalues of A are closely connected with conditions which guarantee that A is nonsingular. The most classical results of this type are the Gerŝgorin theorem and the Lévy–Desplanques theorem.

Let

$$R_i = |a_{i1}| + \cdots + |a_{i,i-1}| + |a_{i,i+1}| + \cdots + |a_{in}|, \quad (i = 1, 2, \ldots, n) \quad (3.12)$$

denote the sum of the moduli of the off-diagonal elements in row i of A. The matrix A is called *diagonally dominant* provided

$$|a_{ii}| > R_i, \quad (i = 1, 2, \ldots, n).$$

Notice that a diagonally dominant matrix can have no zeros on its main diagonal. The theorem of Lévy[1881] and Desplanques[1887] (see Marcus and Minc[1964]) gives a sufficient condition for A to be nonsingular.

Theorem 3.6.1. *If the matrix A is diagonally dominant, then* $\det(A) \neq 0$.

The theorem of Gerŝgorin[1931] (see also Taussky[1949]) determines an inclusion region for the eigenvalues of A.

Theorem 3.6.2. *The eigenvalues of the matrix A of order n lie in the region of the complex plane determined by the union of the n closed discs*

$$Z_i = \{z : |z - a_{ii}| \leq R_i\}, \quad (i = 1, 2, \ldots, n).$$

It is straightforward to derive either of these theorems from the other. Let λ be an eigenvalue of A. Then $\det(\lambda I - A) = 0$ and hence by Theorem 3.6.1, $\lambda I - A$ is not diagonally dominant. Thus for at least one integer i with $1 \leq i \leq n$, $|\lambda - a_{ii}| \leq R_i$. Thus Theorem 3.6.2 follows from Theorem 3.6.1. If A is diagonally dominant, then none of the discs Z_i contains the number 0. Therefore Theorem 3.6.2 implies that a diagonally dominant matrix A has no eigenvalue equal to 0 and so $\det(A) \neq 0$.

In order to obtain generalizations of Theorems 3.6.1 and 3.6.2 which utilize the digraph of a matrix, we need one elementary result about digraphs. Let D be a digraph of order n with vertices $\{a_1, a_2, \ldots, a_n\}$. Corresponding to each vertex a_i let there be given a real number $w_i, (i = 1, 2, \ldots, n)$. Under these circumstances we call D a *vertex-weighted digraph.* An arc (a_i, a_j) issuing from vertex a_i is a *dominant arc from vertex a_i* provided $w_j \geq w_k$ for all integers k for which (a_i, a_k) is an arc of D. There may be more than one dominant arc from a given vertex. A directed cycle

$$a_{i_1} \to a_{i_2} \to \cdots \to a_{i_p} \to a_{i_{p+1}} = a_{i_1}$$

in which $(a_{i_j}, a_{i_{j+1}})$ is a dominant arc from vertex a_{i_j} for each $j = 1, 2, \ldots, p$ is called a *dominant directed cycle* (with respect to the given vertex-weighting).

Lemma 3.6.3. *Assume that each vertex of the digraph D has an arc issuing from it. Then D has a dominant directed cycle.*

Proof. Let a_{k_1} be any vertex of D. Choose a dominant arc (a_{k_1}, a_{k_2}) issuing from a_{k_1}, then a dominant arc (a_{k_2}, a_{k_3}) issuing from a_{k_2} and so on. In this way we obtain a directed walk

$$a_{k_1} \to a_{k_2} \to a_{k_3} \to \cdots.$$

Let s be the smallest integer for which there is an integer r with $1 \le r < s$ such that $k_r = k_s$. Then

$$a_{k_r} \to a_{k_{r+1}} \to \cdots \to a_{k_s}$$

is a directed cycle each of whose arcs is dominant. □

We return to the complex matrix $A = [a_{ij}]$ of order n and denote by $D_0(A)$ the digraph obtained from $D(A)$ by removing all loops. If a digraph is vertex-weighted with weights $w_1, w_2, \ldots, w_n$ and γ is a directed cycle, then $\prod_\gamma w_i$ denotes the product of the weights of the vertices that are on γ. We may regard the digraphs $D(A)$ and $D_0(A)$ as vertex-weighted by the numbers $|a_{11}|, |a_{22}|, \ldots, |a_{nn}|$ as well as by the numbers $R_1, R_2, \ldots, R_n$.

Theorem 3.6.4. *Let all the elements on the main diagonal of the complex matrix $A = [a_{ij}]$ of order n be different from zero. If*

$$\prod_\gamma |a_{ii}| > \prod_\gamma R_i \tag{3.13}$$

for all directed cycles of $D(A)$ with length at least 2, then

$$\det(A) \neq 0.$$

Proof. Since the determinant of A is the product of the determinants of its irreducible components and since no main diagonal element of A equals zero, we assume that A is an irreducible matrix of order at least two. Suppose that $\det(A) = 0$. There exists a nonzero vector $x = (x_1, x_2, \ldots, x_n)^T$ such that

$$Ax = 0. \tag{3.14}$$

Let W consist of those vertices a_i for which $x_i \neq 0$. Let D_0' be the subdigraph of $D_0(A)$ obtained by deleting the vertices not in W and all arcs incident with at least one of the deleted vertices. The equation (3.14) implies that each vertex of D_0' has an arc issuing from it. We weight the vertices of W by assigning to the vertex a_i the weight $|x_i|$ and apply Lemma 3.6.2 to obtain a dominant directed cycle

$$\gamma' : a_{i_1} \to a_{i_2} \to \cdots \to a_{i_p} \to a_{i_{p+1}} = a_{i_1}$$

in D'_0 of length $p \geq 2$. Let j be any integer with $1 \leq j \leq p$. By (3.14) and the definition of W we obtain

$$a_{i_j i_j} x_{i_j} = -\sum_{k \neq i_j} a_{i_j k} x_k = -\sum_{\{k : a_k \in W - \{a_{i_j}\}\}} a_{i_j k} x_k.$$

Since γ' is a dominant directed cycle, we obtain

$$\begin{aligned} |a_{i_j i_j}||x_{i_j}| &\leq \sum_{\{k : a_k \in W - \{a_{i_j}\}\}} |a_{i_j k}||x_k| \\ &\leq \left(\sum_{\{k : a_k \in W - \{a_{i_j}\}\}} |a_{i_j k}| \right) |x_{i_{j+1}}|. \end{aligned}$$

Hence

$$|a_{i_j i_j}||x_{i_j}| \leq \left(\sum_{k \neq i_j} |a_{i_j k}| \right) |x_{i_{j+1}}|, (1 \leq j \leq p). \tag{3.15}$$

By (3.12)

$$|a_{i_j i_j}||x_{i_j}| \leq R_{i_j} |x_{i_{j+1}}|, (1 \leq j \leq p). \tag{3.16}$$

We multiply the p inequalities in (3.16) and use the fact that $i_{p+1} = i_1$ and obtain

$$\prod_{\gamma'} |a_{ii}| \prod_{\gamma'} |x_i| \leq \prod_{\gamma'} R_i \prod_{\gamma'} |x_i|.$$

Since $x_i \neq 0$ if $a_i \in W$, we obtain

$$\prod_{\gamma'} |a_{ii}| \leq \prod_{\gamma'} R_i. \tag{3.17}$$

Inequality (3.17) is in contradiction to our assumption (3.13), and hence $\det(A) \neq 0$. □

If the matrix $A = [a_{ij}]$ of order n does not have an irreducible component of order 1, then each vertex of $D(A)$ belongs to at least one directed cycle of length at least 2 and hence (3.13) implies that $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$. It follows that the assumption in Theorem 3.6.3 that the elements on the main diagonal of A are different from zero is not needed if A is an irreducible matrix of order at least 2. A zero matrix shows that the assumption cannot be removed in general.

We may apply Theorem 3.6.4 to obtain an eigenvalue inclusion region.

Theorem 3.6.5. *Let $A = [a_{ij}]$ be a complex matrix of order n. Then the eigenvalues of A lie in that part of the complex plane determined by the union of the regions*

$$Z_\gamma = \left\{ z : \prod_\gamma |z - a_{ii}| \leq \prod_\gamma R_i \right\}$$

over all directed cycles γ of $D(A)$ having length at least 2.

Proof. Since the vertices of a directed cycle all belong to the same strong component, it suffices to prove the theorem for an irreducible matrix A. We assume that A is irreducible matrix of order $n \geq 2$ and proceed as in the derivation of Theorem 3.6.2 from Theorem 3.6.1. Let λ be an eigenvalue of A. Then $\det(\lambda I - A) = 0$. The digraphs $D(A)$ and $D(\lambda I - A)$ have the same set of directed cycles of length at least 2. Moreover, R_i is also the sum of the moduli of the off-diagonal elements in row i of $\lambda I - A, (1 \leq i \leq n)$. We apply Theorem 3.6.4 to $\lambda I - A$, and noting the discussion following its proof, we conclude that there is a directed cycle γ of $D(A)$ with length at least 2 such that

$$\prod_\gamma |\lambda - a_{ii}| \leq \prod_\gamma R_i.$$

Hence the eigenvalue λ is in Z_γ. □

Notice that Theorem 3.6.1 and hence Theorem 3.6.2 are direct consequences of Theorem 3.6.4. Another consequence of Theorem 3.6.4 is the following theorem of Ostrowski[1937] and Brauer[1947].

Theorem 3.6.6. *Let $A = [a_{ij}]$ be a complex matrix of order $n \geq 2$. If*

$$|a_{ii}||a_{jj}| > R_i R_j, \quad (i, j = 1, 2, \ldots, n; i \neq j), \tag{3.18}$$

then $\det(A) \neq 0$. The eigenvalues of the matrix A lie in the region of the complex plane determined by the union of the ovals

$$Z_{ij} = \{z : |z - a_{ii}||z - a_{jj}| \leq R_i R_j\}, \quad (i, j = 1, 2, \ldots, n; i \neq j).$$

Proof. Assume that (3.18) holds. Then $a_{ii} \neq 0, (i = 1, 2, \ldots, n)$ and (3.13) holds for all directed cycles γ of $D(A)$ of length 2. We show that (3.13) holds also for all directed cycles of length at least 3. If

$$|a_{ii}| > R_i \tag{3.19}$$

holds for all i, then (3.13) is satisfied. It follows from (3.18) that the only other possibility is for (3.19) to fail for exactly one integer i. Let γ be a directed cycle of length $p \geq 3$. We choose distinct integers i and j such that a_i and a_j are vertices of γ and $|a_{kk}| > R_k$ whenever a_k is a vertex of γ with $k \neq i, j$. It now follows from (3.18) that (3.13) holds for γ. We now apply Theorem 3.6.4 and obtain $\det(A) \neq 0$. The second conclusion of the theorem follows by applying the first conclusion to the matrix $\lambda I - A$ for each eigenvalue λ of A. □

The *girth* of a digraph D is defined to be the smallest integer $k \geq 2$ such that D has a directed cycle with length k. Notice that we exclude loops in the calculation of the girth. If D has no directed cycle with length at least 2, the girth of D is undefined.

If the square matrix A has only zeros on its main diagonal, then Theorem 3.6.5 simplifies considerably. The numbers $R_1, R_2, \ldots, R_n$ defined in (3.12) are then the sum of the moduli of the elements in the rows of A.

Corollary 3.6.7. *Let $A = [a_{ij}]$ be a complex matrix of order n with only 0's on its main diagonal. Suppose that the numbers $R_1, R_2, \ldots, R_n$ defined in* (3.12) *satisfy $R_1 \leq R_2 \leq \cdots \leq R_n$. Then each eigenvalue λ of A satisfies*

$$|\lambda| \leq \sqrt[g]{R_{n-g+1} \cdots R_n}$$

where g is the girth of $D(A)$.

Proof. Let λ be an eigenvalue of A. By Theorem 3.6.5 there exists a directed cycle γ of $D(A)$ of length $p \geq g$ such that

$$|\lambda|^p \leq \prod_{\gamma} R_i \leq R_{n-p+1} \cdots R_n.$$

Hence

$$|\lambda| \leq \sqrt[p]{R_{n-p+1} \cdots R_n} \leq \sqrt[g]{R_{n-g+1} \cdots R_n}.$$ □

Applying Corollary 3.6.7 to the adjacency matrix of a digraph D of girth g with no loops, we conclude that the absolute value of each eigenvalue of D does not exceed the gth root of the product of the g largest outdegrees of its vertices.

If the matrix A is irreducible, the sufficient conditions obtained for the nonvanishing of the determinant and the corresponding eigenvalue inclusion regions can be improved. The improvement of Theorems 3.6.1 and 3.6.2 as given in the next theorem is due to Taussky[1949].

Theorem 3.6.8. *Let $A = [a_{ij}]$ be an irreducible complex matrix of order n. If*

$$|a_{ii}| \geq R_i, \quad (i = 1, 2, \ldots, n)$$

with strict inequality for at least one i, then $\det(A) \neq 0$. *A boundary point w of the union of the n closed discs*

$$Z_i = \{z : |z - a_{ii}| \leq R_i\}, \quad (i = 1, 2, \ldots, n)$$

can be an eigenvalue of A only if w is a boundary point of each of the n discs.

The more general Theorems 3.6.4 and 3.6.5 admit a similar improvement.

Theorem 3.6.9. *Let $A = [a_{ij}]$ be an irreducible complex matrix of order $n \geq 2$. If*

$$\prod_{\gamma} |a_{ii}| \geq \prod_{\gamma} R_i \tag{3.20}$$

for all directed cycles γ of $D(A)$ with length at least 2, with strict inequality for at least one such directed cycle, then $\det(A) \neq 0$. *A boundary point w of the union of the regions*

$$Z_\gamma = \left\{ z : \prod_{\gamma} |z - a_{ii}| \leq \prod_{\gamma} R_i \right\}$$

can be an eigenvalue of A only if w is a boundary point of each Z_γ.

Proof. Assume that (3.20) holds for all γ. Since A is irreducible, $R_i \neq 0, (i = 1, 2, \ldots, n)$ and (3.20) implies that $a_{ii} \neq 0, (i = 1, 2, \ldots, n)$. We suppose that $\det(A) = 0$ and proceed as in the proof of Theorem 3.6.4. However, since we only assume the weaker inequality (3.20), when we reach (3.17) we can only conclude that equality holds, that is

$$\prod_{\gamma'} |a_{ii}| = \prod_{\gamma'} R_i. \tag{3.21}$$

It follows that equality holds in (3.16). We conclude from the derivation of (3.16) that for each integer $j = 1, 2, \ldots, p$

$$a_{i_j k} \neq 0 \text{ implies } |x_k| = |x_{i_{j+1}}|, \quad (k = 1, 2, \ldots, n; k \neq i_j).$$

Thus for each vertex a_{i_j} of γ', the weights of the vertices to which there is an arc from a_{i_j} are constant. Suppose there is a vertex of $D(A)$ which is not a vertex of γ. Since A is irreducible, $D(A)$ is strongly connected and hence there is an arc (a_{i_j}, a_k) from some vertex a_{i_j} of γ to some vertex a_k not belonging to γ. We may argue as in Lemma 3.6.3 and obtain a dominant directed cycle γ'' which has at least one vertex different from a vertex of γ'. Replacing γ' with γ'' we conclude as above that the weights are constant over those vertices to which there is an arc from a specified vertex of γ''. Continuing like this, we conclude that each vertex a_i has the property that the weights of the vertices to which there is an arc from a_i are constant. This implies that every directed cycle of $D(A)$ of length at least 2 is a dominant directed cycle. Hence the proof of Theorem 3.6.4 applies to all directed cycles of length at least 2. This means that (3.21) holds for all directed cycles γ' of length at least 2, contradicting the assumption in the theorem. □

Consider once again the complex matrix $A = [a_{ij}]$ of order n and let

$$S_i = |a_{1i}| + \cdots + |a_{i-1,i}| + |a_{i+1,i}| + \cdots + |a_{ni}|, \quad (i = 1, 2, \ldots n),$$

the sum of the moduli of the off-diagonal elements in column i of A. The theorems proved in this section remain true if the numbers $R_1, R_2, \ldots, R_n$ are replaced, respectively, by the numbers $S_1, S_2, \ldots, S_n$. This is because we may apply the theorems to the transposed matrix A^T. The eigenvalues of A^T are identical to those of A. The digraph $D(A^T)$ is obtained from the digraph $D(A)$ by reversing the directions of all arcs. As a result the directed cycles of $D(A^T)$ are obtained from those of $D(A)$ by reversing the cyclical order of the vertices.

Ostrowski[1951] combined the sequences $R_1, R_2, \ldots, R_n$ and $S_1, S_2, \ldots,$ S_n to obtain extensions of Theorems 3.6.1 and 3.6.2.

Theorem 3.6.10. *Let $A = [a_{ij}]$ be a complex matrix of order n, and let p be a real number with $0 \le p \le 1$. If*

$$|a_{ii}| > R_i^p S_i^{1-p}, \quad (1 \le i \le n)$$

then $\det(A) \ne 0$. *The eigenvalues of the matrix A lie in the region of the complex plane determined by the union of the n discs*

$$\{z : |z - a_{ii}| \le R_i^p S_i^{1-p}\}, \quad (1 \le i \le n).$$

Theorems 3.6.4 and 3.6.5 can be similarly extended (Brualdi[1982]).

Exercises

1. Let $A = [a_{ij}]$ be a real matrix of order n such that $a_{ii} \ge 0, (i = 1, 2, \ldots, n)$ and $a_{ij} \le 0, (i, j = 1, 2, \ldots, n; i \ne j)$. Assume that each row sum of A is positive. Prove that the determinant of A is positive and that the real part of each eigenvalue of A is positive.
2. For each integer $n \ge 2$ determine a matrix A of order n whose eigenvalues are not contained in the union of the regions

$$\{z : |z - a_{ii}||z - a_{jj}||z - a_{kk}| \le R_i R_j R_k\}.$$

   Here i, j and k are distinct integers between 1 and n and R_i denotes the sum of the moduli of the off-diagonal elements in row i of A.
3. Let D be a digraph of order n and let D' be the digraph obtained from D by reversing the direction of each arc. The digraphs D and D' have the same spectrum. Construct a digraph D such that D' is not isomorphic to D and thereby obtain a pair of cospectral digraphs.
4. Let A be a tournament matrix of order n, that is a (0,1)-matrix satisfying $A + A^T = J - I$. Prove that the real part of each eigenvalue of A lies between $-1/2$ and $(n-1)/2$ (Brauer and Gentry[1968] and de Oliveira[1974]).

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3.7 Computational Considerations

Let $A = [a_{ij}], (i, j = 1, 2, \ldots, n)$ be a matrix of order n. In Theorem 3.2.4 we have established the existence of a permutation matrix P of order n such that PAP^T is in the Frobenius normal form given in (3.1). The diagonal blocks $A_1, A_2, \ldots, A_t$ in (3.1) are the irreducible components of A. By Theorem 3.2.4 they are uniquely determined apart from simultaneous permutations of their lines. In this section we discuss two algorithms. The first algorithm, due to Tarjan[1972] (see also Aho, Hopcroft and Ullman[1975]), obtains the irreducible components $A_1, A_2, \ldots, A_t$ of A including their ordering in (3.1). The second algorithm, due to Denardo[1977] and Atallah[1982], determines the index of imprimitivity k of an irreducible matrix A and the k-cyclic components $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ as given in (3.4) with $r = k$. These algorithms are best discussed in the language of digraphs. The equivalent formulation of these considerations in terms of the digraph $D(A)$ has been discussed in sections 3.2 and 3.4.

We begin by recalling some definitions from the theory of digraphs. A *directed tree with root* r is a digraph with a distinguished vertex r having the property that for each vertex a different from r there is a unique directed chain from r to a. It follows that a directed tree with root r can be obtained from a tree T by labeling one vertex of T with r, thereby obtaining a tree rooted at r, and directing all edges of T away from r. In particular, a directed tree of order n has exactly $n-1$ arcs. A *directed forest* is a digraph consisting of one or more directed trees no two of which have a vertex in common.

Throughout we let D be a digraph of order n with vertex set V. Let F be a subset of the arcs of D. The subdigraph of D consisting of the arcs in F and all vertices of D which are incident with at least one arc in F is denoted by $\langle F \rangle$. If the vertex set of $\langle F \rangle$ is V, then $\langle F \rangle$ is a *spanning subdigraph* of D. The spanning subdigraph of D whose set of vertices is V and whose set of arcs is F is denoted by $\langle F \rangle^*$. A spanning subdigraph of D which is a directed tree or a directed forest is called, respectively, a *spanning directed tree* or a *spanning directed forest* of D. For each vertex $a \in V$ we define the *out-list* of a to be the set $L(a)$ of vertices b for which there is an arc (a, b) from a to b.

We first discuss an algorithm for obtaining a spanning directed forest of D. The algorithm is based on a technique called *depth-first search* for visiting all the vertices of D, and as a result the spanning directed forest that it determines is called a *depth-first spanning directed forest.* As the name suggests, in the search for new vertices preference is given to the *deepest* (or *forward*) direction. In the algorithm each vertex $a \in V$ is assigned a positive integer between 1 and n which is called its *depth-first number* and is denoted by $df(a)$. The depth-first numbers give the order in which the vertices of D are visited in the search.

Initially, all vertices of D are labeled *new*, F is empty and a function *COUNT* has the value 0. We choose a vertex a and apply the following procedure *Search*(a) described below.

Search(a)

1. $COUNT \leftarrow COUNT + 1$.
2. $df(a) \leftarrow$ *Count.*
3. Change the label of a to *old.*
4. For each vertex b in $L(a)$, do
    (i) If b is *new*, then
        (a) Put the arc (a, b) in F.
        (b) Do *Search*(b).

If upon the completion of *Search*(a), all vertices of D are now labeled *old*, then $\langle F \rangle$ is a spanning directed tree of D rooted at a. If vertices labeled *new* remain, we choose any *new* vertex c and proceed with *Search*(c). We continue in this way until every vertex has the label *old.* Then $\langle F \rangle^*$ is a spanning directed forest of D consisting of l directed trees $T_1, T_2, \ldots, T_l$ for some integer $l \geq 1$. We assume that these directed trees have been obtained in the order specified by their subscripts, and we speak of T_i as being *to the left* of T_j if $i < j$.

We illustrate the depth-first procedure with the digraph D in Figure 3.3. A depth-first spanning directed forest $\langle F \rangle^*$ produced by the algorithm is illustrated in Figure 3.4.

The unbroken lines designate arcs of F; the broken lines designate the remaining arcs of D. The numbers in the parentheses designate the depth-first numbers of the vertices.

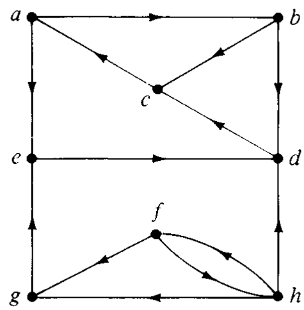

Figure 3.3

Application of the depth-first algorithm to a digraph D determines a partition of the arcs of D into four (possibly empty) sets specified below:

forest arcs: these are the arcs in F produced by the algorithm;

forward arcs: these are the arcs which go from a vertex to a proper descendant of that vertex in one of the directed trees of $\langle F \rangle^*$, but which are not forest arcs. (For the determination of the strong components of D, these arcs are of no importance and can be ignored.)

back arcs: these are the arcs which go from a vertex to an ancestor of that vertex in one of the directed trees of $\langle F \rangle^*$. (Here we can include the loops of D.)

cross arcs: these are the arcs which join two vertices neither of which is an ancestor of the other. The vertices may belong to the same directed tree or different directed trees of $\langle F \rangle^*$.

Suppose that (c, d) is an arc of D. If (c, d) is either a forest arc or a forward arc then $df(c) < df(d)$. If (c, d) is a back arc then $df(c) \geq df(d)$ (equality can hold only if $c = d$). For cross arcs we have the following.

Lemma 3.7.1. *If (c, d) is a cross arc of D then $df(c) > df(d)$.*

Proof. Let (c, d) be an arc of D satisfying $df(c) < df(d)$. When c is changed from a *new* vertex to an *old* vertex, d is still *new*. Since d is in $L(c)$, *Search*(c) cannot end until d is reached. It follows that (c, d) is either a forest arc or a forward arc. □

Let the strong components of D be $D(V_1), D(V_2), \ldots, D(V_k)$. The next lemma is the first step in the identification of $V_1, V_2, \ldots, V_k$.

Lemma 3.7.2. *The vertex set of a strong component of D is the set of vertices of some directed subtree of one of the directed trees $T_1, T_2, \ldots, T_l$ of the depth-first spanning directed forest $\langle F \rangle^*$ of D.*

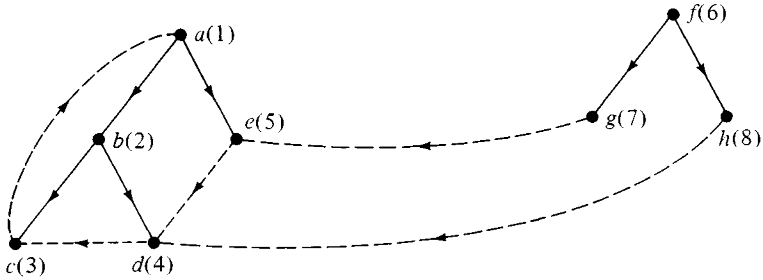

Figure 3.4

Proof. Let c and d be two vertices of D which belong to the same strong component $D(V_i)$. We first show that there is a vertex in V_i which is a common ancestor of c and d. Assume that $df(c) < df(d)$. Since c and d are in the same strong component of D, there is a directed chain γ from c to d all of whose vertices belong to V_i. Let x be the vertex of γ with the smallest depth-first number. The vertices which come after x in γ are each descendants of x in one of the directed trees $T_1, T_2, \ldots, T_l$. This is true because by Lemma 3.7.1 each of the arcs of γ beginning with the one leaving x is either a forest arc or a forward arc. In particular, d is a descendant of x. Since $df(x) \le df(c) < df(d)$, it follows from the way that depth-first search is carried out, that c is also a descendant of x. Therefore each pair of vertices in V_i have a common ancestor which also belongs to V_i. We conclude that there is a vertex s_i in V_i such that s_i is a common ancestor of all vertices in V_i. In particular, V_i is a subset of the vertex set of one of the directed trees of $\langle F\rangle^*$.

Now let c be any vertex in V_i and let d be a vertex on the directed chain in $\langle F\rangle^*$ from s_i to c. Since c and s_i belong to the strong component $D(V_i)$ of D, there is a directed chain in D from c to s_i. Hence there are directed chains from s_i to d and from d to s_i, and we conclude that d is also in V_i. It follows that V_i is the set of vertices of a directed subtree of one of the directed trees of $\langle F\rangle^*$. □

By Lemma 3.7.2 the vertex sets $V_1, V_2, \ldots, V_k$ of the strong components of D are the vertex sets of directed subtrees of the depth-first spanning forest $\langle F\rangle^*$. Let the roots of these directed subtrees be $s_1, s_2, \ldots, s_k$, respectively, where we have chosen the ordering of $V_1, V_2, \ldots, V_k$ in which depth-first search of their roots has terminated. Thus *Search*(s_i) terminates before *Search*(s_{i+1}) for $i = 1, 2, \ldots, k-1$.

The strong components of D can be determined from the roots as follows.

Lemma 3.7.3. *For each integer i with $1 \le i \le k$, the vertex set V_i of the strong component $D(V_i)$ of D consists of those vertices which are descendants of s_i but which are not descendants of any of $s_1, s_2, \ldots, s_{i-1}$.*

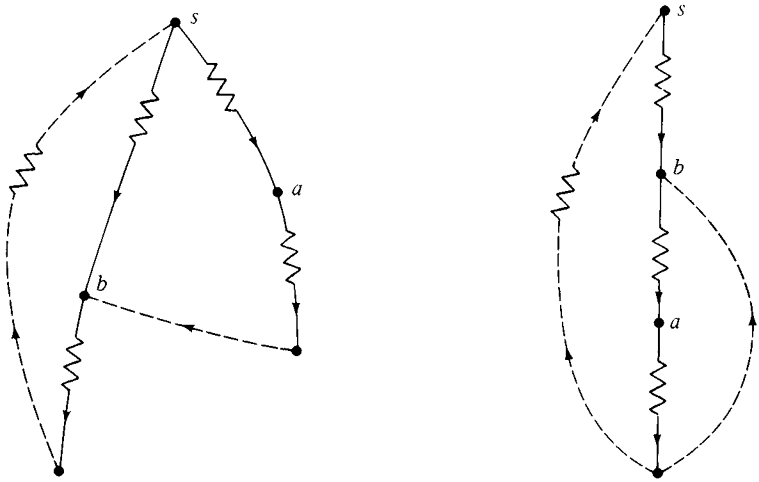

Figure 3.5

Proof. We first observe that the vertex set V_1 consists of all descendants of s_1. This is because *Search*(s_1) terminates before *Search*(s_i) for $i = 2, 3, \ldots, k$ and hence by definition of depth-first search, no s_i with $i > 1$ can be a descendant of s_1. Similarly, for $j > 1$, s_j cannot be a descendant of $s_1, \ldots, s_{j-1}$ and the lemma follows. □

By Lemma 3.7.3 the vertex sets of the strong components of D can be determined once the roots of the strong components are known. To find the roots, a new function g, called *LOWLINK*, defined on the vertex set V of D is introduced. If a is a vertex in V, $g(a)$ is the smallest depth-first number of the vertices in the set consisting of a and those vertices b satisfying the property: there is a cross arc or back arc from a descendant of a (possibly a itself) to b where the root s of the strong component containing b is an ancestor of a. The two possibilities, namely those of a cross arc and a back arc, are illustrated in Figure 3.5. In that figure s and b are in the same strong component and hence there must be a directed chain from a descendant of b to s. Notice that for all vertices a we have $g(a) \leq df(a)$.

The function *LOWLINK* provides a characterization of the strong components.

Lemma 3.7.4. *A vertex a of the digraph D is the root of one of its strong components if and only if $g(a) = df(a)$.*

Proof. We first assume that $g(a) \neq df(a)$. Then there is a vertex b satisfying

(i) there is a back arc or cross arc from a descendant of a to b;

(ii) the root s of the strong component containing b is an ancestor of a;
(iii) $df(b) < df(a)$.

We have $df(s) \leq df(b) < df(a)$ and hence $s \neq a$. Since there is a directed chain from s to a and a directed chain from a to s (through b), a, b and s are in the same strong component of D. Thus a is not the root of a strong component of D.

We now assume that $g(a) = df(a)$. Let r be the root of the strong component containing a and suppose that $r \neq a$. There is a directed chain γ from a to r. Since r is an ancestor of a, there is a first arc α of γ which goes from a descendant of a to a vertex b which is not a descendant of a. The arc α is either a cross arc or a back arc. In either case $df(b) < df(a)$. The directed chain γ implies the existence of a directed chain from b to r. Since there is a directed chain from r to a, there is also a directed chain from r to b. Hence r and b are in the same strong component. By definition of *LOWLINK* we have $g(a) \leq df(b)$, contradicting $g(a) = df(a) > df(b)$. □

The computation of *LOWLINK* can be incorporated into the depth-first search algorithm by replacing *Search* with *SearchComp*. This enhancement allows us to obtain the vertex sets of the strong components.

SearchComp(a)

1. *Count* $\leftarrow$ *Count* $+1$.
2. $df(a) \leftarrow$ *Count*.
3. Change the label of a to *old*.
4. $g(a) \leftarrow df(a)$.
5. Push a on a *Stack*.
6. For each vertex b in $L(a)$, do
    (i) if b is *new*, then
        (a) Do *SearchComp*(b).
        (b) $g(a) \leftarrow \min\{g(a), g(b)\}$.
    (ii) if b is *old* (hence $df(b) < df(a)$), then
        (a) if b is on the *Stack*, $g(a) \leftarrow \min\{g(a), df(b)\}$.
7. If $g(a) = df(a)$, then *pop* x from the *Stack* until $x = a$. The vertices popped are declared the set of vertices of a strong component of D and a is declared its root.

With this enhancement we obtain the following.

Theorem 3.7.5. *The enhanced depth-first search algorithm correctly determines the vertex sets of the strong components of the digraph D.*

Proof. We first observe that a vertex a is declared a root of a strong component if and only if $g(a) = df(a)$. Hence by Lemma 3.7.4 the roots of the strong components are computed correctly provided the function *LOWLINK* is. Moreover, when a is declared a root, the vertices put into the strong component with a are precisely the vertices above a on the *Stack*, that is the descendants of a which have not yet been put into a strong component. This is in agreement with Lemma 3.7.3. It thus remains to prove that *LOWLINK* is correctly computed. We accomplish this by using induction on the number of calls to *SearchComp* that have terminated.

We first show that the computed value for $g(a)$ is at least equal to the correct value. There are two places in *SearchComp* where the computed value of $g(a)$ could be less than $df(a)$. In $6(i)$ it can happen if b is a child of a and $g(b) < df(a)$. In this case, since $df(a) < df(b)$, there is a vertex x with $df(x) = g(b)$ such that x can be reached from a descendant y of b by either a cross arc or a back arc. The vertex x has the additional property that the root r of the strong component containing x is an ancestor of b and hence of a. Thus the correct value of $g(a)$ should be at least as low as $g(b) = df(x)$. In $6(ii)$ the computed value of $g(a)$ could be less than $df(a)$ if there is a cross arc or back arc from a to b and the strong component C containing b has not yet been found. In this case the call of *SearchComp* on the root r of C has not yet terminated, so that r is an ancestor of a. As in the previous case $g(a)$ should be at least as low as $df(b)$.

Now we show that the computed value for $g(a)$ is at most equal to the correct value. Suppose that x is a descendant of a for which there is a cross arc or a back arc from x to a vertex y where the root r of the strong component containing y is an ancestor of a. We need to show that the computed value of $g(a)$ is at least as small as $df(y)$. We distinguish two cases. In the first case $x = a$. By the inductive assumption, all strong components found thus far are correct. Since *SearchComp*(a) has not yet terminated, neither has *SearchComp*(r). Hence y is still on the *Stack*. Thus $6(ii)$ sets $g(a)$ to $df(y)$ or lower. In the second case, $x \neq a$. Then there exists a child z of a of which x is a descendant. By the inductive assumption, when *SearchComp*(z) terminates, $g(z)$ has been set to $df(y)$ or lower. In $6(ii)$, $g(a)$ is set this low unless it is already lower. Thus it follows by induction that the computed values of *LOWLINK* are correct.

□

The number of steps used in the preceding algorithm for determining the strong components of a digraph D of order n is bounded by $c\max\{n, e\}$ where c is a constant independent of the number n of vertices and e is the number of arcs of D.

We return now to a matrix A of order n. Let the vertex sets of the strong components of the digraph $D(A)$ be determined by the algorithm in the order $V_1, V_2, \ldots, V_k$. We simultaneously permute the lines of A so that the lines corresponding to the vertices in V_i come before the lines corresponding to the vertices in V_{i-1} $(2 \leq i \leq k)$. The Frobenius normal form is then

$$\begin{bmatrix} A_1 & A_{12} & \cdots & A_{1k} \\ O & A_2 & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_k \end{bmatrix}$$

where A_i is the adjacency matrix of the strong component $D(V_{k+1-i})$, $(i = 1, 2, \ldots, k)$.

We now discuss an algorithm for determining the index of imprimitivity k of an *irreducible* matrix A and for determining the k-cyclic components $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ of A. As in the previous algorithm we frame our discussion in the language of digraphs and determine the index of imprimitivity k and the sets of imprimitivity of a strongly connected digraph.

Let D be a strongly connected digraph of order n with vertex set V. We recall that for a vertex a of V, $L(a)$ denotes the set of vertices b for which (a, b) is an arc of D. Since D is strongly connected, a depth-first spanning directed forest of D is a directed tree. In the algorithm we assume that a spanning directed tree T with root r has been determined. We also assume that the length $d(a)$ of the unique directed chain in T from r to a has also been computed for each vertex a [we define $d(r) = 0$]. The algorithm *INDEX* computes the index of imprimitivity of D and its sets of imprimitivity.

INDEX

1. $\delta \leftarrow 0$.
2. For each vertex a in V, do
   (i) For each b in $L(a)$, do
      (a) $\delta \leftarrow \gcd\{\delta, d(a) - d(b) + 1\}$.
3. $W_1 \leftarrow \{a : d(a) \equiv 0 \pmod{\delta}\}$,
   $W_2 \leftarrow \{a : d(a) \equiv 1 \pmod{\delta}\}$,
   $\vdots$
   $W_\delta \leftarrow \{a : d(a) \equiv \delta - 1 \pmod{\delta}\}$.

The greatest common divisor gcd in the algorithm is always taken to be a nonnegative integer. We use the convention that $\gcd\{0, 0\} = 0$.

We show that upon termination of *INDEX*, the value of δ is the index of imprimitivity of D, and D is cyclic with respect to the ordered partition $W_1, W_2, \ldots, W_\delta$.

Lemma 3.7.6. *The strong digraph D is cyclically r-partite if and only if for each arc (a,b) of D, r is a divisor of $d(a)-d(b)+1$.*

Proof. First we assume that D is cyclically r-partite with ordered partition $U_1, U_2, \ldots, U_r$. Let (a,b) be an arc of D. In T there are directed chains α_a and α_b from r to a and b with lengths $d(a)$ and $d(b)$, respectively. Because D is strongly connected there is a directed chain β in D from b to r. Let the length of β be p. These directed chains along with the arc (a,b) determine closed directed walks of lengths $d(a)+1+p$ and $d(b)+p$, respectively. Since D is cyclically r-partite, we have

$$d(a)+1+p \equiv 0 \pmod{r},$$

and

$$d(b)+p \equiv 0 \pmod{r}.$$

Hence

$$d(a)-d(b)+1 \equiv 0 \pmod{r}.$$

Conversely, suppose that r is a divisor of $d(a)-d(b)+1$ for each arc (a,b) of D. Let $W_1, W_2, \ldots, W_r$ be defined as in Step 3 of *INDEX* with r replacing δ. Let (a,b) be any arc of D and suppose that a is in W_i and b is in W_j. We then have

$$d(a)-d(b)+1 \equiv 0 \pmod{r},$$

$$d(a) \equiv i-1 \pmod{r},$$

and

$$d(b) \equiv j-1 \pmod{r}.$$

From these three relations it follows that $j \equiv i+1 \pmod{r}$. Hence D is cyclically r-partite with respect to the ordered partition $W_1, W_2, \ldots, W_r$. □

Theorem 3.7.7. *Let D be a strongly connected digraph of order n. The number δ computed by the algorithm INDEX is the index of imprimitivity of D. Moreover, D is cyclically δ-partite with respect to the ordered partition $W_1, W_2, \ldots, W_\delta$.*

Proof. As shown in section 3.4, the index of imprimitivity of D equals the largest integer k such that D is cyclically k-partite. It thus follows from Lemma 3.7.6 that

$$k = \gcd\{d(a)-d(b)+1 : (a,b) \text{ is an arc of } D\}.$$

Hence when algorithm *INDEX* terminates, δ has the value k. The proof that $\dot{D}$ is cyclically δ-partite with respect to the ordered partition $W_1, W_2, \ldots,$ W_δ is the same as the one used in the proof of Lemma 3.7.6. □

The algorithm *INDEX* can be implemented so that the number of steps taken is bounded by $c\max\{n, e\}$ where e is the number of arcs of the digraph D. For a strongly connected digraph, $e \geq n$ and hence this bound is ce.

Let A be an irreducible matrix of order n with index of imprimitivity equal to k. We apply the algorithm *INDEX* to the strong digraph $D(A)$. The computed value of δ is k. Let $W_1, W_2, \ldots, W_k$ be the partition of the vertex set of $D(A)$ produced by *INDEX*. If we simultaneously permute the lines of A so that the lines corresponding to the vertices in W_i come before those corresponding to $W_{i+1}, (i = 1, 2, \ldots, k-1)$, we obtain

$$\begin{array}{c} \\ W_1 \\ W_2 \\ \vdots \\ W_{k-1} \\ W_k \end{array} \begin{array}{c} \begin{array}{ccccc} W_1 & W_2 & W_3 & \cdots & W_k \end{array} \\ \left[\begin{array}{ccccc} O & A_{12} & O & \cdots & O \\ O & O & A_{23} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_{k-1,k} \\ A_{k1} & O & O & \cdots & O \end{array}\right] \end{array}.$$

The matrices $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ are the k-cyclic components of A.

Exercise

1. Use the algorithms in this section to show that the matrix below is irreducible and to determine its index k of imprimitivity and its k-cyclic components:

$$\left[\begin{array}{cccccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

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