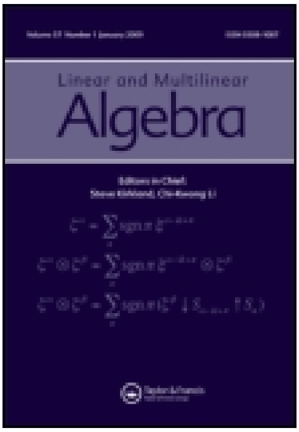

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Algebraic connectivity of directed graphs

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Algebraic connectivity of directed graphs

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We consider a generalization of Fiedler's notion of algebraic connectivity to directed graphs. We show that several properties of Fiedler's definition remain valid for directed graphs and present properties peculiar to directed graphs. We prove inequalities relating the algebraic connectivity to quantities such as the bisection width, maximum directed cut and the isoperimetric number. Finally, we illustrate an application to the synchronization in networks of coupled chaotic systems.

Keywords: Connectivity; Directed graphs; Dynamical systems; Eigenvalues; Laplacian matrix; Synchronization

AMS Mathematics Subject Classifications: 05C20; 05C50; 15A18; 37C75

1. Introduction

In [1], Fiedler defined the algebraic connectivity of an undirected graph as the second smallest eigenvalue of its Laplacian matrix. In this article, we generalize this concept to directed graphs. There are several choices for the generalization possible; one generalization is to define it as the second smallest eigenvalue of the Laplacian matrix of the directed graph. We choose our generalization differently based on two reasons. First, for the chosen definition of algebraic connectivity several properties of Fiedler's definition remain valid for directed graphs. Second, this definition has applications to deriving criteria for synchronization in networks of chaotic systems.

We consider finite weighted directed graphs (V, E) with no loops and with adjacency matrix A, where $A_{ij} \neq 0$ if there is a directed edge from vertex i into vertex j, and 0 otherwise. Without loss of generality, we assume that $A_{ij} \leq 1$. An exception to this assumption is when a graph is *unweighted*, defined as the case when A_{ij} are natural numbers, with $A_{ij} = k$ denoting k edges from vertex i to vertex j. The number of vertices and edges will be denoted as $n \geq 2$ and m respectively. We will mainly be interested

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in graphs with nonnegative weights. However, since some of the results are valid for arbitrary weights, the condition of nonnegative weights will only be imposed when necessary. The indegree and outdegree of vertex k are given by $d_i(k) = \sum_j A_{jk}$ and $d_o(k) = \sum_j A_{kj}$ respectively. If $A_{jk} \neq 0$, then vertex j is the *parent* of vertex k and vertex k is the *child* of vertex j. The complement of a graph G without multiple edges is defined as the graph $\overline{G}$ with the same vertex set as G and adjacency matrix $\overline{A}$ where $\overline{A}_{ij} = 1 - A_{ij}$ for $i \neq j$. The Laplacian matrix of a directed graph is defined as $L = D - A$, where D is the diagonal matrix of vertex outdegrees. For unweighted undirected graphs, this definition coincides with the usual definition of the combinatorial Laplacian matrix or Kirchhoff matrix [2]. There are other ways to define the Laplacian matrix of directed graphs (see for instance [3]). Let $e = (1, \dots, 1)^T$. It is clear that L is a zero row sums matrix and thus e is a right eigenvector of L, i.e. $Le = 0$. From Perron–Frobenius theory [4], if all weights are nonnegative, then all eigenvalues of L are nonnegative, with 0 being a simple eigenvalue if the graph is strongly connected.

An undirected graph is equivalent to a directed graph by considering each undirected edge with weight w as two directed edges with weight w and opposite orientation. Similarly, a mixed graph (a graph with directed and undirected edges) can be represented as a directed graph. We define the symmetric part of a graph by replacing each directed edge with an undirected edge of half the weight. This means that the symmetric part of an undirected graph is itself. If A is the adjacency matrix of G then $1/2(A + A^T)$ is the adjacency matrix of the symmetric part of G. We define the *reversal* of a directed graph as the directed graph obtained by reversing the orientation of all the edges. The adjacency matrix of the reversal of G is A^T. The Laplacian matrix of the complete graph will be denoted as $L_K = nI - ee^T$. Note that for $x \perp e$, $x^T L_K x = nx^T x$.

2. Algebraic connectivity

Let P be the set $\{x \in \mathbb{R}^V, x \perp e, \|x\| = 1\}$, i.e. the set of real vectors of unit norm in $e^\perp$, the orthogonal complement of e.

Definition 1 For a directed graph G with Laplacian matrix L, the algebraic connectivity is the real number defined as:

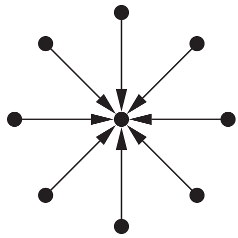
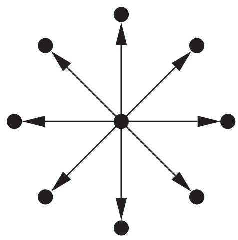
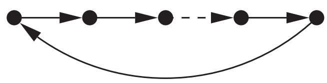
$$a(G) = \min_{x \in P} x^T L x = \min_{x \in \mathbb{R}^V, x \neq 0, x \perp e} \frac{x^T L x}{x^T x}$$

For a real symmetric matrix A of order n, let the eigenvalues of A be arranged as:

$$\lambda_1(A) \leq \lambda_2(A) \cdots \leq \lambda_n(A)$$

We will also write $\lambda_1(A)$ and $\lambda_n(A)$ as $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively. For a graph with n vertices, $a(G)$ can be efficiently computed as $a(G) = \min_{x \in \mathbb{R}^{n-1}, \|Qx\|=1} x^T Q^T L Q x = \lambda_{\min}((1/2)Q^T(L + L^T)Q)$, where Q is an n by $n-1$ matrix whose columns form an orthonormal basis of $e^\perp$. The algebraic connectivity is generally not invariant under reversal of the graph, unless the graph is balanced (section 3.1). The algebraic connectivity of some directed graphs are shown in table 1.

Table 1. $a(G)$ and $b(G)$ of some directed graphs.

	G	$a(G)$	$b(G)$
Imploding star		1	1
Exploding star ($n \geq 3$)		0	$n-1$
Directed cycle		$2\sin^2(\pi/n)$	2, n even $2\sin^2(\pi(n-1)/(2n))$, n odd

By the Courant–Fischer min–max theorem [5], Definition 1 coincides with Fiedler's definition of algebraic connectivity when restricted to undirected graphs. The following related quantity will also be of interest.

Definition 2 For a graph G with Laplacian matrix L, $b(G)$ is defined as

$$b(G) = \max_{x \in P} x^T L x = \lambda_{\max}\left(\frac{1}{2} Q^T (L + L^T) Q\right) \geq a(G)$$

It is clear that $a(G)$ and $b(G)$ are independent of the ordering of the vertices as P is invariant under permutation of coordinates.

3. Properties of $a(G)$ and $b(G)$

As alluded to before, the choice of Definition 1 is partly due to the fact that several properties of Fiedler's definition for undirected graphs remain valid. In fact, some of the proofs are similar to the corresponding proofs for undirected graphs. See [6,7] for excellent surveys of the properties of the eigenvalues of L for undirected graphs.

LEMMA 1 (super- and sub-additivity) *If the graphs G and H have the same vertex set, then*

$$a(G) + a(H) \leq a(G \cup H) \leq b(G \cup H) \leq b(G) + b(H)$$

Proof Since $L(G \cup H) = L(G) + L(H)$,

$$a(G \cup H) = \min_{x \in P} x^T L(G \cup H)x \geq \min_{x \in P} x^T L(G)x + \min_{x \in P} x^T L(H)x = a(G) + a(H)$$

The proofs of the other inequalities are similar. ■

LEMMA 2 *Let $G \times H$ be the Cartesian product of the graphs G and H, then*

$$a(G \times H) \leq \min(a(G), a(H)) \leq \max(b(G), b(H)) \leq b(G \times H)$$

Proof The Laplacian matrix L of $G \times H$ is $L(G) \otimes I + I \otimes L(H)$ [8]. Let $x^T L(G)x = a(G)$ and $y^T L(H)y = a(H)$, $x \in P, y \in P$. Since $(1/\sqrt{n})x \otimes e \in P$, $a(G \times H) \leq ((1/\sqrt{n})x \otimes e)^T L((1/\sqrt{n})x \otimes e) = (1/n)x^T L(G)x \otimes e^T e = a(G)$. Similarly using the vector $(1/\sqrt{n})e \otimes y$, we get $a(G \times H) \leq a(H)$. The proof for $b(G \times H)$ is similar. ■

LEMMA 3 *For a graph G with Laplacian matrix L*:

$$\lambda_1\left(\frac{1}{2}(L + L^T)\right) \leq a(G) \leq \lambda_2\left(\frac{1}{2}(L + L^T)\right)$$
$$\lambda_{n-1}\left(\frac{1}{2}(L + L^T)\right) \leq b(G) \leq \lambda_n\left(\frac{1}{2}(L + L^T)\right)$$

Proof Follows from the Courant–Fischer min–max theorem. ■

LEMMA 4

$$a(G) + b(\overline{G}) = n$$

Proof As in [1], the proof follows from the facts that $L(G) + L(\overline{G}) = L_K$ and $x^T L_K x = n$ for all $x \in P$. ■

LEMMA 5 *Consider a graph G with Laplacian matrix L. If λ is an eigenvalue of L not corresponding to the eigenvector e, then $a(G) \leq \mathrm{Re}(\lambda)$. In particular, if G has nonnegative weights, then $a(G) \leq \mathrm{Re}(\lambda)$ for all nonzero eigenvalues λ of L.*

Proof See [9].

Definition 3 Let S_1 and S_2 be subsets of vertices. Define

$$e(S_1, S_2) = \sum_{v_1 \in S_1, v_2 \in S_2} A_{v_1, v_2}$$

which is the sum of the weights of edges which start in S_1 and ends in S_2. In general, $e(S_1, S_2) \neq e(S_2, S_1)$.

Since $A_{ij} \leq 1$, $e(S_1, S_2) \leq |S_1||S_2|$. The following result is quite useful and will be used throughout this article.

Lemma 6 *Let S_1 and S_2 be two nontrivial disjoint subsets of vertices of a graph G (i.e., $0 < |S_1|$, $0 < |S_2|$ and $S_1 \cap S_2 = \emptyset$) and $\overline{S_i} = V \backslash S_i$. Then*

$$a(G) \leq \frac{|S_2|^2 e(S_1, \overline{S}_1) + |S_1||S_2|(e(S_1, S_2) + e(S_2, S_1)) + |S_1|^2 e(S_2, \overline{S}_2)}{|S_1||S_2|^2 + |S_1|^2|S_2|} \leq b(G)$$

If all weights are nonnegative, then

$$\begin{aligned} a(G) &\leq \frac{e(S_1, \overline{S}_1)}{|S_1|} + \frac{e(S_2, \overline{S}_2)}{|S_2|} \\ b(G) &\geq \frac{e(S_1, S_2)}{|S_1|} + \frac{e(S_2, S_1)}{|S_2|} \end{aligned}$$

Proof Let x be a vector such that $x_v = |S_2|$ if $v \in S_1$, $x_v = -|S_1|$ if $v \in S_2$ and $x_v = 0$ otherwise. Then $x \perp e$ and $x^T x = |S_1||S_2|^2 + |S_1|^2|S_2|$.

$$\begin{aligned} x^T D x &= |S_2|^2 \sum_{v \in S_1} d_o(v) + |S_1|^2 \sum_{v \in S_2} d_o(v) \\ x^T A x &= |S_2|^2 e(S_1, S_1) - |S_1||S_2| e(S_1, S_2) - |S_1||S_2| e(S_2, S_1) + |S_1|^2 e(S_2, S_2) \end{aligned}$$

Since $e(S, S) + e(S, \overline{S}) = e(S, V) = \sum_{v \in S} d_o(v)$, this implies that

$$\begin{aligned} x^T L x &= x^T D x - x^T A x \\ &= |S_2|^2 e(S_1, \overline{S}_1) + |S_1||S_2|(e(S_1, S_2) + e(S_2, S_1)) + |S_1|^2 e(S_2, \overline{S}_2) \end{aligned}$$

Since $a(G) \leq (x^T L x)/(x^T x) \leq b(G)$, the first set of inequalities follows. If all weights are nonnegative, then the last two inequalities follow from the fact that $e(S_1, \overline{S}_1) \geq e(S_1, S_2)$ and $e(S_2, \overline{S}_2) \geq e(S_2, S_1)$. ∎

Corollary 1 *Let S be a nontrivial subset of vertices of a graph G (i.e., $0 < |S| < n$) and $\overline{S} = V \backslash S$. Then*

$$\begin{aligned} a(G) &\leq \frac{e(S, \overline{S})}{|S|} + \frac{e(\overline{S}, S)}{n - |S|} \leq b(G) \\ a(G) &\leq \frac{e(S, \overline{S})}{|S|} + |S| \end{aligned}$$

Proof Follows from Lemma 6 and choosing $S_1 = S$, $S_2 = \overline{S}$. The last inequality follows from $e(S, \overline{S}) \leq |S||\overline{S}|$. ∎

Lemma 7 *Let v, w be nonadjacent vertices of a graph G, i.e. $A_{vw} = A_{wv} = 0$. Then*

$$a(G) \leq \frac{1}{2}(d_o(v) + d_o(w)) \leq b(G)$$

In particular, if G has two vertices with zero outdegrees, then $a(G) \leq 0$.

Proof Follows from Lemma 6 and choosing $S_1 = \{v\}$, $S_2 = \{w\}$. ■

Let $\Delta_o = \max_{v\in V} d_o(v)$, $\delta_o = \min_{v\in V} d_o(v)$, $\Delta_i = \max_{v\in V} d_i(v)$ and $\delta_i = \min_{v\in V} d_i(v)$.

LEMMA 8

$$a(G) \le \min_{v\in V}\left\{d_o(v) + \frac{1}{n-1}d_i(v)\right\} \le \max_{v\in V}\left\{d_o(v) + \frac{1}{n-1}d_i(v)\right\} \le b(G)$$
$$a(G) \le \min\left\{\delta_o + \frac{1}{n-1}\Delta_i, \Delta_o + \frac{1}{n-1}\delta_i\right\} \le \frac{n}{n-1}\min\{\Delta_o, \Delta_i\}$$
$$b(G) \ge \max\left\{\delta_o + \frac{1}{n-1}\Delta_i, \Delta_o + \frac{1}{n-1}\delta_i\right\} \ge \frac{n}{n-1}\max\{\delta_o, \delta_i\}$$

Proof The first set of inequalities follows from Corollary 1 and choosing $S = \{v\}$. We define $|E| = \sum_k d_i(k) = \sum_k d_o(k) \le n(n-1)$. Since

$$\begin{aligned}\min_{v\in V}\left\{d_o(v) + \frac{1}{n-1}d_i(v)\right\} &\le \min\left\{\delta_o + \frac{1}{n-1}\Delta_i, \Delta_o + \frac{1}{n-1}\delta_i\right\}\\ &\le \max\left\{\delta_o + \frac{1}{n-1}\Delta_i, \Delta_o + \frac{1}{n-1}\delta_i\right\}\\ &\le \max_{v\in V}\left\{d_o(v) + \frac{1}{n-1}d_i(v)\right\}\end{aligned}$$

and $\delta_o \le |E|/n \le \Delta_o$, $\delta_i \le |E|/n \le \Delta_i$, we have

$$a(G) \le \frac{|E|}{n} + \frac{\Delta_i}{n-1} \le \frac{n}{n-1}\Delta_i$$
$$a(G) \le \frac{|E|}{n(n-1)} + \Delta_o \le \frac{n}{n-1}\Delta_o$$
$$b(G) \ge \frac{|E|}{n} + \frac{\delta_i}{n-1} \ge \frac{n}{n-1}\delta_i$$
$$b(G) \ge \frac{|E|}{n(n-1)} + \delta_o \ge \frac{n}{n-1}\delta_o$$

which prove the last two sets of inequalities. ■

LEMMA 9 *Let G be a graph with nonnegative weights. Then*

$$\frac{1}{2}\min_{v\in V}\{d_o(v) - d_i(v)\} \le a(G) \le \min_{v\ne w}\{d_o(v) + d_o(w)\}$$
$$b(G) \le \max_{v\in V}\left\{\frac{3}{2}d_o(v) + \frac{1}{2}d_i(v)\right\}$$

Proof The upper bound on $a(G)$ follows from Lemma 6. Let $B = (1/2)(L + L^T)$. Then for each v, $B_{vv} + \sum_{v\ne w}|B_{vw}| = (3/2)d_o(v) + (1/2)d_i(v)$ and

$B_{vv} - \sum_{v \neq w} |B_{vw}| = (1/2)d_o(v) - (1/2)d_i(v)$. By Gershgorin's circle criterion, $\lambda_{\max}(B) \leq \max_{v \in V}\{(3/2)d_o(v) + (1/2)d_i(v)\}$ and $\lambda_{\min}(B) \geq \min_{v \in V}\{(1/2)d_o(v) - (1/2)d_i(v)\}$. The result then follows from Lemma 3. ■

LEMMA 10 *Let H be constructed from a graph G by removing a subset of vertices with zero indegree from G and all adjacent edges. Then $a(H) \geq a(G)$.*

Proof It is clear we only need to prove the case where one vertex of zero indegree is removed. Let $y \in P$ be such that $a(H) = y^T L(H)y$. The Laplacian matrix of G is of the form:

$$L(G) = \begin{pmatrix} L(H) & 0 \\ w^T & z \end{pmatrix}$$

Since $(y^T \quad 0)^T \in P$ and $(y^T \quad 0)L(G)(y^T \quad 0)^T = a(H)$, it follows that $a(G) \leq a(H)$. ■

LEMMA 11 *For a graph G with nonnegative weights, let H be constructed from G by removing k vertices from G and all adjacent edges. Then $a(H) \geq a(G) - k$.*

Proof It is clear that we only need to consider the case $k = 1$. The Laplacian of graph G can be written as:

$$L(G) = \begin{pmatrix} L(H) + D & -v \\ w^T & z \end{pmatrix}$$

where D is a diagonal matrix with $D_{ii} = v_i \leq 1$. Define

$$F = \begin{pmatrix} L(H) + I & -e \\ w^T - (e^T - v^T) & z + n - \sum_i v_i \end{pmatrix}$$

Let $y \in P$ be such that $a(H) = y^T L(H)y$. Then

$$\begin{aligned} (y^T \quad 0)F(y^T \quad 0)^T &= a(H) + 1 \geq \min_{x \in P} x^T F x \\ &\geq \min_{x \in P} x^T L(G)x + \min_{x \in P} x^T (F - L(G))x \\ &= a(G) + \min_{x \in P} x^T (F - L(G))x \end{aligned}$$

Now

$$F - L(G) = \begin{pmatrix} I - D & -(e - v) \\ -(e^T - v^T) & n - \sum_i v_i \end{pmatrix}$$

which is a symmetric zero row sums matrix with nonpositive off-diagonal elements. Therefore $\min_{x \in P} x^T (F - L(G))x = \lambda_2(F - L(G)) \geq 0$ where the last inequality follows by Gershgorin circle criterion. ■

COROLLARY 2 *For a graph G, let (V_1, V_2) be a partition of V and let G_i be the subgraph generated from V_i. Then*

$$a(G) \leq \min(a(G_1) + |V_2|, a(G_2) + |V_1|)$$

Proof Follows from Lemma 11. ■

3.1. *Balanced graphs*

Definition 4 A vertex is balanced if its indegree is equal to its outdegree. A directed graph is balanced if every vertex is balanced.

In particular, undirected graphs are balanced. An unweighted directed graph is balanced if and only if each strongly connected component is a directed Eulerian graph [10].

LEMMA 12 *Consider a graph G with nonnegative weights and Laplacian matrix L. If the graph is balanced, then* $a(G) = \lambda_2((1/2)(L + L^T)) \geq 0$ *and* $b(G) = \lambda_{\max}((1/2)(L + L^T))$.

Proof For a balanced graph, the Laplacian matrix has zero row sums and zero column sums, i.e. $Le = L^T e = 0$. By the Courant–Fischer min–max theorem,

$$a(G) = \min_{x \in P} x^T L x = \min_{x \in P} \frac{1}{2} x^T (L + L^T) x = \lambda_2\left(\frac{1}{2}(L + L^T)\right)$$

Similarly, $b(G) = \lambda_{\max}\big((1/2)(L + L^T)\big)$. ■

It is easy to see that the algebraic connectivity of a balanced graph is equal to the algebraic connectivity of its symmetric part. Furthermore, for a balanced graph, the algebraic connectivity remains the same if the graph is reversed.

LEMMA 13 *Let* $T = \{x \in \mathbb{R}^V, x \notin \mathrm{span}(e)\}$. *If G is balanced,*

$$a(G) = n \min_{x \in T} \frac{x^T L x}{x^T L_K x} \leq n \max_{x \in T} \frac{x^T L x}{x^T L_K x} = b(G)$$

Proof Decompose $x \in T$ as $x = \alpha e + y$, where $y \perp e$. Since $e^T L = Le = e^T L_K = L_K e = 0$, the proof is then complete by noting that

$$\frac{x^T L x}{x^T L_K x} = \frac{y^T L y}{y^T L_K y} = \frac{y^T L y}{n y^T y}$$ ■

LEMMA 14 *If G is a balanced graph, then*

$$a(G) \leq \frac{n}{n-1} \delta_o \leq \frac{n}{n-1} \Delta_o \leq b(G)$$

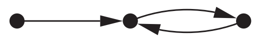
If in addition all weights are nonnegative, then

$$0 \leq a(G) \leq b(G) \leq 2\Delta_o$$

Proof Follows from Lemmas 8 and 9. ■

The following graphs show that Lemma 14 is in general not true when the graph is not balanced, although Lemma 8 indicates that the first set of inequalities in Lemma 14 is almost true as $n \to \infty$ when the indegrees are bounded by some number for all n. In the following graph, $a(G) = b(G) = 1$, whereas $\delta_o = \delta_i = 0$ and $\Delta_o = \Delta_i = 1$.

As for the second set of inequalities in Lemma 14, the following graph satisfies $b(G) > 2\Delta_o$ ($b(G) = 2.0774$, $\Delta_o = 1$).



The reversal of this graph satisfies $b(G) > 2\Delta_i$ ($b(G) = 2.6547$, $\Delta_i = 1$).

Corollary 3 *Let S be a subset of vertices of a balanced graph G and $\overline{S} = V \backslash S$. Then*

$$a(G)\frac{|S|(n-|S|)}{n} \leq e(S,\overline{S}) \leq b(G)\frac{|S|(n-|S|)}{n}$$

Proof Follows from Corollary 1 and the fact that $e(S,\overline{S}) = e(\overline{S},S)$ is a balanced graph. ■

3.2. *Directed bipartite graphs and trees*

Definition 5 ([10]) A directed graph G is a *directed tree* or *arborescence* if the symmetric part of G is a tree and there exists a vertex of G, called the root of G, which has directed paths to all remaining vertices of G. A subgraph of G is a spanning directed tree if it is a directed tree with the same vertex set as G.

Lemma 15 *If the reversal of G does not contain a spanning directed tree, then $a(G) \leq 0$.*

Proof There exists a spanning directed tree in the reversal of G if and only if for any pair of vertices v and w, there exists a vertex z such that there is a directed path from v to z and a directed path from w to z [10]. If the reversal of G does not have a spanning directed tree, then there exist a pair of vertices v and w such that for all vertices z, there is either no directed paths from v to z or no directed paths from w to z. Let $R(v)$ and $R(w)$ be the set of vertices reachable from v and w respectively. Let $H(v)$ and $H(w)$ be the subgraphs of G corresponding to $R(v)$ and $R(w)$ respectively. Express the Laplacian matrix of $H(v)$ in Frobenius normal form [11]:

$$H(v) = M\begin{pmatrix} B_1 & B_{12} & \dots & B_{1k} \\ & B_2 & \dots & B_{2k} \\ & & \ddots & \vdots \\ & & & B_k \end{pmatrix}M^T \tag{1}$$

where M is a permutation matrix and B_i are square irreducible matrices. Let $B(v) = B_k$ be the square irreducible matrix in the lower right corner. We define $B(w)$ similarly.

Note that $B(w)$ and $B(v)$ are zero row sums singular matrices. By the construction, it is easy to see that $B(v) = B_i$ and $B(w) = B_j$ in the Frobenius normal form (equation (1)) of the Laplacian matrix of G for some i, j. This means that $B_{ik} = 0$ for $k > i$ and $B_{jk} = 0$ for $k > j$.

To B_i and B_j correspond two nontrivial disjoint subsets of vertices S_1, S_2 of G such that the edges starting in S_i do not point outside of S_i, i.e. $A_{vw} \neq 0$ and $v \in S_i \Rightarrow w \in S_i$. The proof then follows from Lemma 6 as $e(S_i, \overline{S}_i) = 0$ and $e(S_i, S_j) = 0$. ∎

Definition 6 A directed graph is bipartite if its vertices can be partitioned into two sets V and W such that each edge starts from a vertex in V and ends in a vertex in W. If $|V| = p$ and $|W| = q$, then we use $G_{p,q}$ to denote such a graph.

Corollary 4 *If $q \geq 2$ for a bipartite directed graph $G_{p,q}$, then $a(G) \leq 0$.*

Proof Follows from Lemma 15. ∎

Theorem 1 *If G is a directed tree and some vertex is the parent of at least two vertices, then $a(G) \leq 0$. If the reversal of G is a directed tree then $a(G) \leq (d_i(r))/(n-1)$, where r is the root of the tree.*

Proof If G is a directed tree and some vertex is the parent of at least two vertices, consider the subtrees rooted at these two children with vertices S_1 and S_2. These two sets satisfy the condition in the proof of Lemma 15 and thus $a(G) \leq 0$. If the reversal of G is a directed tree, then Lemma 8 applied to the root results in the upper bound $(d_i(r))/(n-1)$. ∎

3.3. *Random directed graphs*

Juhász [12] shows that the algebraic connectivity of random undirected graphs grows as pn when $n \to \infty$ where p is the density of the edges. This bound can be extended to directed graphs.

Theorem 2 *Let $G_d(n,p)$ be a random directed graph with n vertices and adjacency matrix A where for $i \neq j$ A_{ij} are independent random variables such that $P(A_{ij} = 1) = p$ and $P(A_{ij} = 0) = q = 1 - p$. Then for any $\epsilon > 0$, the algebraic connectivity of G_d satisfies*

$$a(G_d) = pn + o(n^{1/2+\epsilon}) \text{ in probability.}$$

Proof Consider the symmetric matrix $B = 1/2(A + A^T)$. Since $P(B_{ij} = 0) = q^2$, $P(B_{ij} = 1/2) = 2pq$, $P(B_{ij} = 1) = p^2$, by [13],

$$\max_{i \leq n-1} |\lambda_i(B)| = o(n^{1/2+\epsilon}) \text{ in probability.}$$

Let $C = 1/2(L + L^T) - (D_B - B)$ where D_B is the diagonal matrix with the row sums of B on the diagonal. Note that e is an eigenvector of $D_B - B$ and thus $\min_{x \in P} x^T(D_B - B)x = \lambda_2(D_B - B)$. Consider the diagonal matrix $F = (D_B - p(n-1)I)$. As in [12] the interlacing properties of eigenvalues of symmetric matrices imply that $|\lambda_2(D_B - B) - \lambda_2(p(n-1)I - B)| \leq \rho(F) \leq \|F\|_\infty$. An application of a generalization of Chernoff's inequality [14] (also known as Hoeffding's inequality) shows that $P(\|F\|_\infty \geq Kn^{1/2+\epsilon}) \leq \sum_i P(|F_{ii}| \geq Kn^{1/2+\epsilon}) \leq ne^{-\beta n^{2\epsilon}}$ and thus $\|F\|_\infty = o(n^{1/2+\epsilon})$ in probability. Therefore $|\lambda_2(D_B - B) - p(n-1) + \lambda_{n-1}(B)| = o(n^{1/2+\epsilon})$, i.e. $|\lambda_2(D_B - B)| = pn + o(n^{1/2+\epsilon})$. Next note that $C = D - D_B$ is a diagonal matrix and

$$a(G_d) = \min_{x \in P} x^T L x \leq \lambda_2(D_B - B) + \max_{x \in P} x^T C x \leq \lambda_2(D_B - B) + \max_i C_{ii}$$

Similarly,

$$a(G_d) \geq \lambda_2(D_B - B) + \min_{x \in P} x^T C x \geq \lambda_2(D_B - B) + \min_i C_{ii}$$

i.e., $|a(G_d) - \lambda_2(D_B - B)| \leq \|C\|_\infty$. Similar applications of Hoeffding's inequality show that $\|C\|_\infty = o(n^{1/2+\epsilon})$ in probability which implies that $\|F\|_\infty + \|C\|_\infty = o(n^{1/2+\epsilon})$ in probability and thus the theorem is proved. ■

4. Graph connectivity

For undirected graphs with nonnegative weights, $a(G) \geq 0$ with the inequality being strict if and only if G is connected. However, the situation is different for general directed graphs. In particular, $a(G)$ can be negative as illustrated by the following disconnected graph with algebraic connectivity equal to -0.0774:

Lemmas 1 and 12 show that adding undirected edges with positive weights cannot decrease the algebraic connectivity of a graph. The above graph shows that adding directed edges can decrease the algebraic connectivity by noting that the graph with 3 vertices and no edges has algebraic connectivity equal to zero.

Definition 7 ([10]) A directed graph is *strongly connected* if for any pair of distinct vertices v and w, there is a directed path from v to w. A directed graph is *quasi-strongly connected* if for any pair of distinct vertices v and w, there exists a vertex z, such that there is a directed path from z to v and a directed path from z to w. A directed graph with nonnegative weights is *connected*[1] if its symmetric part is strongly connected.

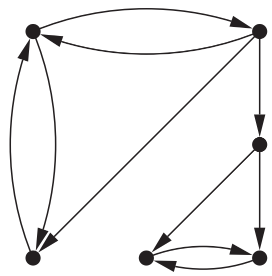
Strongly connected graphs correspond to Laplacian matrices which are irreducible.

LEMMA 16 *If G is not connected, $a(G) \leq 0$.*

Proof Follows from Lemma 15. ■

[1]This is also referred to as *weakly* connected.

Even if G is quasi-strongly connected, which is equivalent to G containing a spanning directed tree, $a(G)$ can still be nonpositive as the exploding star and Theorem 1 indicate. In fact, it can be negative, as illustrated in the following quasi-strongly connected graph ($a(G) = -0.0427$):



On the other hand, the imploding star, which is not quasi-strongly connected, has a positive algebraic connectivity. Other conditions for which $a(G) \leq 0$ are given by Lemmas 7 and 15. The continuity of eigenvalues can be used to show that there exists strongly connected graphs with nonnegative weights such that $a(G) < 0$. Let L be the Laplacian matrix of a graph such that $a(G) < 0$. Then for sufficiently small $\epsilon > 0$, the algebraic connectivity of the strongly connected graph with Laplacian matrix $L + \epsilon L_K$ is also negative.

However, when the graph is balanced, positivity of $a(G)$ can be used to determine connectedness.

Lemma 17 *For a balanced graph G with nonnegative weights, $a(G) > 0 \Leftrightarrow G$ is connected $\Leftrightarrow G$ is strongly connected.*

Proof One direction follows from Lemma 16. If G is connected, then $L + L^T$ is irreducible. By Perron–Frobenious theory and Lemma 12, $a(G) = (1/2)\lambda_2(L + L^T) > 0$. If G is balanced, then $e^T L = 0$ and Lemma 1 in [15] can be used to show that connected implies strongly connected. ■

Lemma 18 *If the Laplacian matrix L of a graph is normal, then L is irreducible if $L + L^T$ is irreducible.*

Proof See Lemma 4 in [16]. ■

Example of normal matrices include circulant matrices such as the Laplacian matrix of the directed cycle in table 1.

Lemma 19 *If the Laplacian matrix L of a graph G is normal, then G is connected if and only if G is strongly connected.*

Proof Clearly strongly connectedness imply connectedness. If G is connected, then $L + L^T$ is irreducible. By Lemma 18 L is irreducible and thus G is strongly connected. ■

Lemma 20 *If the Laplacian matrix $L(G)$ of a graph G is a normal matrix then the graph is balanced and $a(G) = \min_{\lambda \in M} \mathrm{Re}(\lambda)$ where M is the set of eigenvalues of $L(G)$ not corresponding to the eigenvector e.*

Proof See [9].

4.1. *Vertex and edge connectivity*

Definition 8 The vertex connectivity $v(G)$ of a graph is defined as the size of the smallest subset of vertices such that its removal along with all adjacent edges results in a disconnected graph. If no such vertex set exists, $v(G) = n$.

COROLLARY 5 $a(G) \leq v(G)$.

Proof This is a direct consequence of Lemma 11. ■

Definition 9 The edge connectivity $e(G)$ of a graph is defined as the smallest weighted sum among all subsets of edges such that its removal results in a disconnected graph.

THEOREM 3 *For a graph with nonnegative weights,* $a(G) \leq e(G)$.

Proof Let S be a connected component of the disconnected graph resulting from removal of a minimal set of edges. Then $e(S, \overline{S}) + e(\overline{S}, S) = e(G)$. By Corollary 1, $a(G) \leq ((e(S, \overline{S}))/|S|) + ((e(\overline{S}, S))/|\overline{S}|) \leq e(S, \overline{S}) + e(\overline{S}, S)$. ■

5. Graph partitions

5.1. *Maximum directed cut*

Definition 10 The maximum directed cut md(G) is defined as:

$$\mathrm{md}(G) = \max_{0<|S|<n} \{e(S, \overline{S})\}$$

THEOREM 4 *For a graph with nonnegative weights,*

$$\mathrm{md}(G) \leq (n-1)\min(b(G), b(G^R))$$
$$\mathrm{md}(G) \leq (n-2)b(G) + \max(0, b(G) - (n-1)\delta_o)$$

where G^R is the reversal of G.

Proof Since $|S| \leq n-1$, by Corollary 1 we have

$$\frac{e(S, \overline{S})}{n-1} \leq \frac{e(S, \overline{S}) + e(\overline{S}, S)}{n-1} \leq \frac{e(S, \overline{S})}{|S|} + \frac{e(\overline{S}, S)}{|\overline{S}|} \leq b(G)$$

Similarly, $(e(S, \overline{S}))/(n-1) \leq b(G^R)$. If md$(G)$ is achieved with $|S| \leq n-2$, then md$(G) \leq (n-2)b(G)$. If md(G) is achieved with $|S| \leq n-1$, then $b(G) \geq (e(S, \overline{S}))/(n-1) + \delta_o$, so in either case md$(G) \leq (n-2)b(G) + \max(0, b(G) - (n-1)\delta_o)$. ■

Note that for the imploding star the bound in Theorem 4 is tight.

5.2. *Edge-forwarding index*

The definition of edge-forwarding index in [17] can also be applied to directed graphs.

Definition 11 Given a strongly connected unweighted directed graph, a routing is defined as a set of $n(n-1)$ paths $R(u,v)$ between any pair of distinct vertices v, w of G. The load of an edge e, $\pi(G,R,e)$, is defined as the number of paths in the routing R which traverse it. The edge-forwarding index of (G,R) is defined as $\pi(G,R) = \max_{e\in E}(G,R,e)$. The edge-forwarding index of the graph G is defined as $\pi(G) = \min_R \pi(G,R)$.

THEOREM 5 *Let G be a strongly connected unweighted directed graph. For $S \subset V$,*

$$\pi(G) \geq \max\left(\frac{|S|(n-|S|)}{e(S,\overline{S})}, \frac{|S|(n-|S|)}{e(\overline{S},S)}\right) \geq \frac{n}{b(G)}$$

Proof The proof is similar to [7]. Let R be a routing. Each path in R from vertex v in S to vertex w in $\overline{S}$ contains at least one edge in the edge cut of S. Since there are $|S|(n-|S|)$ such paths, $\pi(G) \geq (|S|(n-|S|))/(e(S,\overline{S}))$. Similarly, $\pi(G) \geq (|S|(n-|S|))/(e(\overline{S},S))$. Let $t = \min(e(S,\overline{S}), e(\overline{S},S))$. By Corollary 1,

$$b(G) \geq \frac{t}{|S|} + \frac{t}{n-|S|} = t\frac{n}{|S|(n-|S|)}$$

which implies the second inequality. ■

5.3. *Bisection width*

Definition 12 The bisection width is defined as:

$$\mathrm{bw}(G) = \min_{|S|=\lfloor n/2\rfloor}\left\{e(S,\overline{S})\right\}$$

A related quantity is

$$\overline{\mathrm{bw}}(G) = \max_{|S|=\lfloor n/2\rfloor}\left\{e(S,\overline{S})\right\}$$

It is easy to see that

$$\mathrm{bw}(G) + \overline{\mathrm{bw}}(\overline{G}) = \left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil \tag{2}$$

For the exploding and imploding stars with more than 2 vertices, $\mathrm{bw}(G) = 0$.

THEOREM 6

$$\mathrm{bw}(G) \geq \left\lfloor\frac{n}{2}\right\rfloor\left(a(G) - \left\lfloor\frac{n}{2}\right\rfloor\right)$$
$$\overline{\mathrm{bw}}(G) \leq \left\lfloor\frac{n}{2}\right\rfloor b(G)$$

Proof The first inequality follows from Corollary 1 by setting $|S| = \lfloor n/2 \rfloor$. The second inequality follows from equation (2) and Lemma 4. ■

As in [18] (see also [7]), the bound on the bisection width can be improved by the use of correction functions c.

THEOREM 7 *Let n be even. Then*

$$\mathrm{bw}(G) \geq \frac{n}{2}\left(a^*(G) - \frac{n}{2}\right)$$

where

$$a^*(G) = \max_{c \perp e} \min_{x \in P} x^T(\mathrm{diag}(c) + L(G))x$$

Proof In the proof of Lemma 6, for $|S_1| = |S_2| = n/2$, the vector x as defined satisfies $x_v^2 = x_w^2$ and thus $x^T \mathrm{diag}(c) x = \sum_{v \in V} c_v x_v^2 = 0$ implying $a^*(G) \leq (x^T L x)/(x^T x)$. The rest of the proof is similar to that of Lemma 6. ■

5.4. *Isoperimetric number*

Definition 13 The isoperimetric number $i(G)$ is defined as:

$$i(G) = \min_{0 < |S| \leq (n/2)} \left\{ \frac{e(S, \overline{S})}{|S|} \right\}$$

THEOREM 8 *The isoperimetric number of a graph satisfies*:

$$i(G) \geq a(G) - \left\lfloor \frac{n}{2} \right\rfloor$$

Proof Follows from Corollary 1. ■

5.5. *Minimum ratio cut [19]*

Definition 14 The minimum ratio cut $\mathrm{rc}(G)$ is defined as:

$$\mathrm{rc}(G) = \min_S \left\{ \frac{e(S, \overline{S})}{|S||\overline{S}|} \right\}$$

THEOREM 9 *The minimum ratio cut of a graph satisfies*:

$$\mathrm{rc}(G) \geq \frac{a(G) - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor}$$

Proof Follows from Corollary 1. ■

The lower bounds on the bisection width, isoperimetric number and minimum ratio cut in Theorems 6–9 are nontrivial only when $a(G) \geq \lfloor n/2 \rfloor$, and they are tight for the following graph with n vertices: let V_1 be a subset of vertices with $\lfloor n/2 \rfloor$ vertices. The edges of the graph are all the edges which starts at a vertex in V and ends in a vertex in V_1. It is easy to see that the Laplacian matrix can be written as $\lfloor n/2 \rfloor I - [J \quad 0]$, where J is the n by $\lfloor n/2 \rfloor$ matrix of all 1's. If $x \perp e$, then $x^T J = 0$, and thus $x^T L x = \lfloor n/2 \rfloor x^T x$. Therefore $a(G) = b(G) = \lfloor n/2 \rfloor$. Since there are no edges out of V_1, $\mathrm{bw}(G) = i(G) = \mathrm{rc}(G) = 0$.

5.6. *Independence number*

Definition 15 An independent set of vertices is a set of vertices such that no two distinct vertices in the set are adjacent. The independence number of a graph $\alpha(G)$ is the size of the largest independent set of vertices.

THEOREM 10 *Let the indegrees and outdegrees be ordered as* $d_o(1) \leq d_o(2) \leq \cdots \leq d_o(n)$, $d_i(1) \leq d_i(2) \leq \cdots \leq d_i(n)$ *respectively and define* $e_o(r) = (1/r)\sum_{j=1}^{r} d_o(j)$ *and* $e_i(r) = (1/r)\sum_{j=1}^{r} d_i(j)$. *If* r_0 *is the smallest integer of* r *such that*

$$r(b(G) + e_i(r) - e_o(r)) > n(b(G) - e_o(r))$$

then $\alpha(G) \leq r_0 - 1$.

Proof Let S be an independent set such that $|S| = r$. Since S is independent, $e(S, \overline{S}) = \sum_{v \in S} d_o(v) \geq r e_o(r)$ and $e(\overline{S}, S) = \sum_{v \in S} d_i(v) \geq r e_i(r)$. Then by Corollary 1

$$b(G) \geq e_o(r) + \frac{r e_i(r)}{n - r}$$

which implies that

$$r(b(G) + e_i(r) - e_o(r)) \leq n(b(G) - e_o(r)) \qquad \blacksquare$$

For a graph G, construct the unweighted undirected graph by ignoring the weight, multiplicity and orientation of each edge. It is clear that the independence number of these two graphs are the same. Thus Theorem 10 suggests an algorithm for improving the upper bound of the independence number considered in [6]. Given an undirected unweighted graph G with adjacency matrix A, consider the class of graphs U whose edges are at the same place as G, i.e. graphs in U has adjacency matrices $\tilde{A}$ such that $(\tilde{A}_{vw} \neq 0$ or $\tilde{A}_{wv} \neq 0) \Leftrightarrow A_{vw} \neq 0$. For a given r, find $\tilde{G} \in U$ such that $(b(\tilde{G}) - e_o(r))/(b(\tilde{G}) + e_i(r) - e_o(r))$ is minimized[2] and see if this reduces the value of r_0. If so, set r equal to r_0 and repeat. It is not clear what the best strategy is to find a good graph in U which reduces r_0.

Since vertex connectivity is also independent of the orientations and weights of the edges, similar statements can be made about the relationship between $a(G)$ and $v(G)$.

[2] Where $e_o(r)$ and $e_i(r)$ are calculated using the in- and outdegrees of $\tilde{G}$.

6. Semibalanced graphs

The bounds on the bisection width, maximum directed cut, the isoperimetric number and minimum ratio cut can be improved if the difference between $e(S,\overline{S})$ and $e(\overline{S},S)$ is small.

Definition 16 A graph is (α,β)-semibalanced if $-\beta \le d_i(v) - d_o(v) \le \alpha$ for all $v \in V$.

LEMMA 21 *If G is (α,β)-semibalanced, then*

$$a(G) \le \frac{n}{n-1}\delta_o + \frac{\alpha}{n-1}$$
$$b(G) \ge \frac{n}{n-1}\Delta_o - \frac{\beta}{n-1}$$

If in addition all weights are nonnegative,

$$a(G) \ge -\frac{\alpha}{2}, \qquad b(G) \le 2\Delta_o + \frac{\alpha}{2}$$

Proof Follows from Lemmas 8 and 9. ■

Since balanced graphs are $(0,0)$-semibalanced, Lemma 21 is a generalization of Lemma 14.

LEMMA 22 *If G is (α,α)-semibalanced, then*

$$|e(S,\overline{S}) - e(\overline{S},S)| \le \alpha \min(|S|, n-|S|)$$

Proof Follows from the fact that $e(S,\overline{S}) - e(\overline{S},S) = \sum_{v\in S} d_o(v) - d_i(v)$. ■

THEOREM 11 *If G is (α,α)-semibalanced with nonnegative weights, then*

$$\mathrm{md}(G) \le \left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil\frac{b(G)+\alpha}{n}$$

Proof From Corollary 1,

$$\begin{aligned} b(G) &\ge \frac{e(S,\overline{S})}{|S|} + \frac{e(\overline{S},S)}{|\overline{S}|} \\ &= \left(\frac{1}{|S|} + \frac{1}{|\overline{S}|}\right)e(S,\overline{S}) + \frac{e(\overline{S},S) - e(S,\overline{S})}{|\overline{S}|} \\ &\ge \frac{n}{|S||\overline{S}|}e(S,\overline{S}) - \alpha\frac{\min(|S|,|\overline{S}|)}{|\overline{S}|} \ge \frac{n}{|S||\overline{S}|}e(S,\overline{S}) - \alpha \end{aligned}$$

Since

$$\frac{n}{|S|(n-|S|)} \ge \frac{n}{\lfloor n/2\rfloor\lceil n/2\rceil}$$

the result follows. ■

THEOREM 12 *If G is* (α, α)-*semibalanced with nonnegative weights*, *then*

$$\overline{\mathrm{bw}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{b(G) + \alpha}{n}$$

$$\mathrm{bw}(G) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{a(G) - \alpha}{n}$$

Proof The first inequality follows from a similar proof as Theorem 11. The second inequality is due to equation (2) and Lemma 4. ■

THEOREM 13 *If G is* (α, α)-*semibalanced with nonnegative weights*, *then*

$$i(G) \geq \frac{(a(G) - \alpha)\lceil n/2 \rceil}{n}$$

Proof Similar to the proof of Theorem 11,

$$a(G) \leq \frac{n}{|S||\overline{S}|} e(S, \overline{S}) + \alpha$$

Since $|\overline{S}| \geq \lceil n/2 \rceil$, the result follows. ■

THEOREM 14 *If G is* (α, α)-*semibalanced with nonnegative weights*, *then*

$$\mathrm{rc}(G) \geq \frac{a(G) - \alpha}{n}$$

Proof The proof is similar to the proof of Theorem 13. ■

7. Synchronization in networks of coupled chaotic systems

Another reason for defining the algebraic connectivity as in Definition 1 is its usefulness in deriving synchronization criteria for networks of coupled chaotic systems.

Definition 17 Given a square matrix V, a function $f(y, t): \mathbb{R}^{n+1} \to \mathbb{R}^n$ is V-uniformly decreasing if $(y - z)^T V(f(y, t) - f(z, t)) \leq -c\|y - z\|^2$ for some $c > 0$ and for all y, z, t.

Consider the following synchronization result [16,20–22] for the coupled network of identical dynamical systems with state equations

$$\dot{x} = (f(x_1, t), \ldots, f(x_n, t))^T + (C(t) \otimes D(t))x + u(t) \tag{3}$$

where $x = (x_1, \ldots, x_n)^T$ and $C(t)$ is a zero row sums matrix for all t.

THEOREM 15 *Let* $W(t)$ *be some time-varying matrix and* V *be a symmetric positive definite matrix such that* $f(x, t) + W(t)x$ *is* V-*uniformly decreasing*. *Then the array in* *equation* (3) *synchronizes in the sense that* $\|x_i - x_j\| \to 0$ *as* $t \to \infty$ *if*

(1) $\lim_{t\to\infty} \|u_i - u_j\| = 0$ *for all* i, j,
(2) *There exists a symmetric irreducible zero row sums matrix* U *with nonpositive off-diagonal elements such that* $(U \otimes V)(C(t) \otimes D(t) - I \otimes W(t))$ *is negative semidefinite for all* t.

7.1. *Constant coupling*

Definition 18 Let $\mu(C)$ be the supremum of the set of real numbers μ such that $U(C - \mu I) + (C^T - \mu I)U$ is positive semidefinite for some symmetric zero row sums matrix U with nonpositive off-diagonal elements.

Using Theorem 15 it is easy to show the following [23]:

THEOREM 16 *The coupled network*

$$\dot{x} = (f(x_1, t), \ldots, f(x_n, t))^T + (C \otimes D)x + u(t) \tag{4}$$

synchronizes if

(1) $\lim_{t\to\infty} \|u_i - u_j\| = 0$ *for all* i, j,
(2) $f(y, t) + \alpha D y$ *is* V*-uniformly decreasing for some symmetric positive definite* V,
(3) VD *is symmetric negative semidefinite and*
(4) $\mu(C) \geq \alpha$.

The matrix C describes the coupling topology between systems whereas the matrix D describes the coupling term between two systems. The term Dy is the amount of feedback needed to stabilize $\dot{y} = f(y, t)$. The array can be considered as coupled via a graph where for $i \neq j$, $C_{ij} \neq 0$ means that there is a term $C_{ij} D x_j$ in $\dot{x}_i$, i.e. system i is influenced by system j. If we assign a directed edge of weight $-C_{ij}$ from system i to system j, then C is exactly the Laplacian matrix of the underlying graph.[3]

In [9] it was shown that $\mu(L) \geq a(G)$ where L is the Laplacian matrix of G. In particular, $L_K(C - a(C)I) + (C^T - a(C)I)L_K$ is positive semidefinite. Thus the algebraic connectivity of the underlying graph provides a lower bound on the amount of coupling needed to synchronize the array. In [9] it was also shown that $a(G) = \mu(L)$ when L is normal. The reader is referred to [15] for a better lower bound for $\mu(L)$.

7.2. *Time-varying coupling*

Since $L_K(C - a(C)I) + (C^T - a(C)I)L_K$ is positive semidefinite, by choosing $U = L_K$ in Theorem 15 the following can be proved:

THEOREM 17 *Let* $f(x, t) + \alpha D x$ *be* V*-uniformly decreasing for some symmetric positive definite matrix* V. *Suppose that* $VD(t)$ *is symmetric negative semidefinite. Then the*

[3] From a dynamics point of view, it is probably more appropriate to define the edge to go from system j into system i, but the above definition is consistent with the definition of adjacency matrix.

array in equation (3) *synchronizes if* $\lim_{t\to\infty}\|u_i - u_j\| = 0$ *for all* i, j *and* $a(G(t)) \geq \alpha$ *for all* t *where* $G(t)$ *is the graph with Laplacian matrix* $C(t)$.

Thus the array synchronizes if the algebraic connectivity of the underlying graph is large enough at each time t. Extensions to coupled networks with coupling between delayed state variables can be found in [24].

8. Conclusions

We propose a generalization of Fiedler's algebraic connectivity to directed graphs and prove several relationships between the algebraic connectivity and several graph–theoretical properties. An application to the synchronization in networks of coupled dynamical systems is shown. In particular, the algebraic connectivity of the underlying directed graph provides a lower bound on the coupling needed to synchronize the network.

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