6

Existence Theorems

6.1 Network Flows

Let D be a digraph of order n whose set of vertices is the n-set V. Let E be the set of arcs of D and let $c : E \to Z^+$ be a function which assigns to each arc $\alpha = (x, y)$ a nonnegative integer

$$c(\alpha) = c(x, y).$$

The integer $c(x, y)$ is called the *capacity* of the arc (x, y) and the function c is a *capacity function* for the digraph D. In this chapter, loops (arcs joining a vertex to itself) are of no significance and thus we implicitly assume throughout that D has no loops.

Let s and t be two distinguished vertices of D, which we call the *source* and *sink*, respectively, of D. The quadruple

$$N = \langle D, c, s, t \rangle$$

is called a *capacity-constrained network*. We could replace D with a general digraph in which the arc (x, y) of D has multiplicity $c(x, y)$. However, it is more convenient and suggestive to continue with a digraph in which $c(x, y)$ represents the capacity of the arc (x, y).

A *flow from s to t* in the network N is a function $f : E \to Z^+$ from the set of arcs of D to the nonnegative integers which satisfies the constraints

$$0 \leq f(x, y) \leq c(x, y) \text{ for each arc } (x, y) \text{ of } D, \tag{6.1}$$

and

$$\sum_y f(x, y) - \sum_z f(z, x) = 0 \text{ for each vertex } x \neq s, t. \tag{6.2}$$

 https://doi.org/10.1017/CBO9781107325708.007

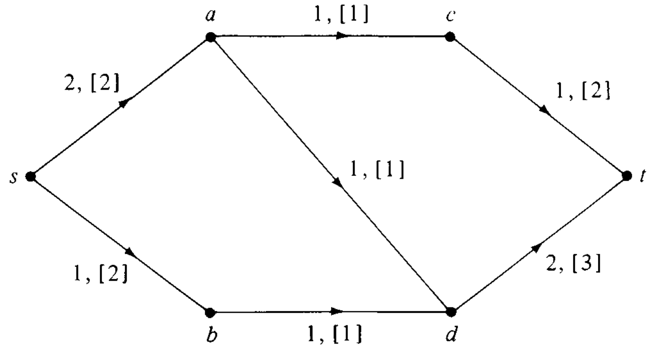

Figure 6.1

In (6.2) the first summation is over all vertices y such that (x, y) is an arc of D and the second summation is over all vertices z such that (z, x) is an arc of D. Since

$$\sum_{x \in V}\left(\sum_{y} f(x,y) - \sum_{z} f(z,x)\right) = \sum_{x \in V}\sum_{y} f(x,y) - \sum_{x \in V}\sum_{z} f(z,x) = 0,$$

it follows from (6.2) that

$$\sum_{y} f(s,y) - \sum_{z} f(z,s) = \sum_{z} f(z,t) - \sum_{y} f(t,y). \tag{6.3}$$

Equations (6.2) are interpreted to mean that the net flow out of a vertex different from the source s and sink t is zero, while equation (6.3) means that the net flow out of the source s equals the net flow into the sink t. We let $v = v(f)$ be the common value of the two quantities in (6.3) and call v the *value of the flow* f. In a capacity-constrained network it is usual to allow the capacity and flow functions (and other imposed constraints) to take on any nonnegative real values. For combinatorial applications integer values are required. The theorems to follow remain valid if real values are permitted for both the capacity and flow values. With our restriction to integer values, there are only finitely many flows.

For an example, let N be the capacity-constrained network illustrated in Figure 6.1 where the numbers in brackets denote capacities of arcs and the other numbers denote the function values of a flow f. The value of this flow f is $v = 3$. No flow from s to t in N can have value greater than 3, since the value of a flow cannot exceed the amount of flow from the set $X = \{s, a, b\}$ of vertices to the set $Y = \{t, c, d\}$ of vertices and this latter quantity is bounded by the sum of the capacities of those arcs whose initial vertices lie in X and whose terminal vertices lie in Y. We now investigate in general the maximum value of a flow in a capacity-constrained network.

Let X and Y be subsets of the vertex set V of the digraph D. The set of arcs (x, y) of D for which x is in X and y is in Y is denoted by (X, Y). If X is a set consisting of only one vertex x then we write (x, Y) instead of $(\{x\}, Y)$; a similar remark applies to Y. If g is any real-valued function defined on the set E of arcs of D, then we define

$$g(X, Y) = \sum_{(x,y)\in(X,Y)} g(x, y).$$

Now consider the network $N = \langle D, c, s, t\rangle$. Let X be a set of vertices such that s is in X and t is in the complement $\overline{X}$. Then $(X, \overline{X})$ is called a *cut in N separating s and t*. The *capacity of the cut* $(X, \overline{X})$ is $c(X, \overline{X})$. We first show that the value of a flow is bounded by the capacity of each cut.

Lemma 6.1.1. *Let f be a flow with value v in the network $\langle D, c, s, t\rangle$, and let $(X, \overline{X})$ be a cut separating s and t. Then*

$$v = f(X, \overline{X}) - f(\overline{X}, X) \leq c(X, \overline{X}).$$

Proof. It follows from (6.2) and the definition of v as given by (6.3) that

$$f(x, V) - f(V, x) = \begin{cases} 0 & \text{if } x \neq s, t \\ v & \text{if } x = s \\ -v & \text{if } x = t \end{cases}.$$

Since s is in X and t is in $\overline{X}$, we have

$$\begin{aligned} v &= \textstyle\sum_{x\in X}(f(x, V) - f(V, x)) = f(X, V) - f(V, X) \\ &= f(X, X \cup \overline{X}) - f(X \cup \overline{X}, X) \\ &= f(X, X) + f(X, \overline{X}) - f(X, X) - f(\overline{X}, X) \\ &= f(X, \overline{X}) - f(\overline{X}, X). \end{aligned}$$

From (6.1) we conclude that $f(X, \overline{X}) \leq c(X, \overline{X})$ and $f(\overline{X}, X) \geq 0$, and the conclusion follows. □

We now state and prove the fundamental *maxflow-mincut theorem* of Ford and Fulkerson[1956,1962] which asserts that there is a cut for which equality holds in Lemma 6.1.1.

Theorem 6.1.2. *In a capacity-constrained network $N = \langle D, c, s, t\rangle$ the maximum value of a flow from the source s to the sink t equals the minimum capacity of a cut separating s and t.*

Proof. Let f be a flow from s to t in N whose value v is largest. By Lemma 6.1.1 it suffices to define a cut $(X, \overline{X})$ separating s and t whose capacity equals v. The set X is defined recursively as follows:

(i) $s \in X$;
(ii) if $x \in X$ and y is a vertex for which (x, y) is an arc and $f(x, y) < c(x, y)$, then $y \in X$;
(iii) if $x \in X$ and y is a vertex for which (y, x) is an arc and $f(y, x) > 0$, then $y \in X$.

We first show that $(X, \overline{X})$ is a cut separating s and t by verifying that $t \notin X$. Assume, to the contrary, that $t \in X$. It follows from the definition of X that there is a sequence $x_0 = s, x_1, \ldots, x_m = t$ of vertices such that for each $i = 0, 1, \ldots, m-1$ either (x_i, x_{i+1}) or (x_{i+1}, x_i) is an arc, and

$$(x_i, x_{i+1}) \text{ is an arc and } c(x_i, x_{i+1}) - f(x_i, x_{i+1}) > 0 \tag{6.4}$$

$$\text{or } (x_{i+1}, x_i) \text{ is an arc and } f(x_{i+1}, x_i) > 0. \tag{6.5}$$

Let a be the positive integer equal to the minimum of the numbers occurring in (6.4) and (6.5). Let g be the function defined on the arcs of the digraph D which has the same values as f except that

$$g(x_i, x_{i+1}) = f(x_i, x_{i+1}) + a \text{ if } (x_i, x_{i+1}) \text{ is an arc,}$$

and

$$g(x_{i+1}, x_i) = f(x_{i+1}, x_i) - a \text{ if } (x_{i+1}, x_i) \text{ is an arc.}$$

Then g is a flow in N from s to t with value $v + a > v$ contradicting the choice of f. Hence $t \in \overline{X}$ and $(X, \overline{X})$ is a cut separating s and t.

From the definition of X it follows that

$$f(x, y) = c(x, y) \text{ if } (x, y) \in (X, \overline{X})$$

and

$$f(y, x) = 0 \text{ if } (y, x) \in (\overline{X}, X).$$

Hence $f(X, \overline{X}) = c(X, \overline{X})$ and $f(\overline{X}, X) = 0$. Applying Lemma 6.1.1 we conclude that $v = c(X, \overline{X})$. ☐

Let $l : E \to Z^+$ be an integer-valued function defined on the set E of arcs of the digraph D such that for each arc (x, y),

$$0 \leq l(x, y) \leq c(x, y).$$

Suppose that in the definition of a flow f we replace (6.1) with

$$l(x, y) \leq f(x, y) \leq c(x, y) \text{ for each arc } (x, y) \text{ of } D. \tag{6.6}$$

Thus l determines lower bounds on the flows of arcs. The proof of Theorem 6.1.2 can be adapted to yield: *the maximum value of a flow equals the minimum value of* $c(X, \overline{X}) - l(\overline{X}, X)$ *taken over all cuts separating* s *and* t *provided there is a flow satisfying* (6.2) *and* (6.6).

We now discuss the existence of flows satisfying (6.6) which satisfy (6.2) for *all* vertices x. More general results can be found in Ford and Fulkerson[1962].

A function $f : E \to Z^+$ defined on the set E of arcs of the digraph D with vertex set V which satisfies (6.6) and

$$f(x, V) - f(V, x) = 0 \text{ for all } x \in V \tag{6.7}$$

is called a *circulation on* D *with constraints* (6.6). The following fundamental theorem of Hoffman[1960] establishes necessary and sufficient conditions for the existence of circulations.

Theorem 6.1.3. *There exists a circulation on the digraph* D *with constraints* (6.6) *if and only if for every subset* X *of the vertex set* V,

$$c(X, \overline{X}) \geq l(\overline{X}, X).$$

Proof. We define a network $N^* = (D^*, c^*, s, t)$ as follows. The digraph D^* is obtained from D by adjoining two new vertices s and t and all the arcs (s, x) and (x, t) with $x \in V$. If (x, y) is an arc of D, then $c^*(x, y) = c(x, y) - l(x, y)$. If $x \in V$, then $c^*(s, x) = l(V, x)$ and $c^*(x, t) = l(x, V)$.

From the rules

$$\begin{aligned} f^*(x, y) &= f(x, y) - l(x, y) \text{ if } (x, y) \text{ is an arc of } D, \\ f^*(s, x) &= l(V, x) \text{ if } x \in V, \\ f^*(x, t) &= l(x, V) \text{ if } x \in V, \end{aligned}$$

we see that there is a circulation f on D with constraints (6.6) if and only if there is a flow f^* in N^* with value equal to $l(V, V)$. The subsets X of V and the cuts $(X^*, \overline{X}^*)$ separating s and t are in one-to-one correspondence by the rules

$$X^* = X \cup \{s\}, \overline{X}^* = \overline{X} \cup \{t\}.$$

Moreover,

$$\begin{aligned} c^*(X^*, \overline{X}^*) &= c^*(X \cup \{s\}, \overline{X} \cup \{t\}) \\ &= c^*(X, \overline{X}) + c^*(s, \overline{X}) + c^*(X, t) \\ &= c(X, \overline{X}) - l(X, \overline{X}) + l(V, \overline{X}) + l(X, V) \\ &= c(X, \overline{X}) + l(\overline{X}, \overline{X}) + l(X, V) \\ &= c(X, \overline{X}) + l(V, V) - l(\overline{X}, X). \end{aligned}$$

Hence by Theorem 6.1.2 there is a flow f^* in N^* with value $l(V,V)$ if and only if $c(X,\overline{X}) \geq l(\overline{X},X)$ for all $X \subseteq V$. □

Theorem 6.1.3 can be used to obtain conditions for the existence of a flow f in a network N satisfying the lower and upper bound constraints (6.6). We add new arcs (s,t) and (t,s) with infinite capacity (any capacity larger than the capacity of each cut suffices) and apply Theorem 6.1.3. The resulting necessary and sufficient condition is that $c(X,\overline{X}) \geq l(\overline{X},X)$ for all subsets X of vertices for which $\{s,t\} \subseteq X$ or $\{s,t\} \subseteq \overline{X}$.

The final flow theorem that we present concerns a network with multiple sources and multiple sinks in which an upper bound is placed on the net flow out of each of the source vertices and a lower bound is placed on the net flow into each of the sink vertices.

Let D be a digraph of order n with vertex set V. Let c be a nonnegative integer-valued capacity function defined on the set E of arcs of D. Suppose that S and T are disjoint subsets of the vertex set V, and let $W = V-(S \cup T)$. Let $a : S \to Z^+$ be a nonnegative integer-valued function defined on the vertices in S, and let $b : T \to Z^+$ be a nonnegative integer-valued function defined on the vertices in T. For $s \in S$, a(s) can be regarded as the *supply* at the source vertex s. For $t \in T$, b(t) is the *demand* at the sink vertex t. We call

$$N = \langle D, c, S, a, T, b \rangle$$

a *capacity-constrained, supply-demand network*, and we are interested in when a flow exists that satisfies the demands at the vertices in T without exceeding the supplies at the vertices in S. Let $f : E \to Z^+$ be a function which assigns to each arc in E a nonnegative integer. Then f is a *supply-demand flow* in N provided

$$f(s,V) - f(V,s) \leq a(s), (s \in S), \tag{6.8}$$

$$f(V,t) - f(t,V) \geq b(t), (t \in T), \tag{6.9}$$

$$f(x,V) - f(V,x) = 0, (x \in W), \tag{6.10}$$

and

$$0 \leq f(x,y) \leq c(x,y), ((x,y) \in E). \tag{6.11}$$

The following theorem is from Gale[1957].

Theorem 6.1.4. *In the capacity-constrained, supply-demand network $N = \langle D, c, S, a, T, b \rangle$ there exists a supply-demand flow if and only if*

$$b(T \cap \overline{X}) - a(S \cap \overline{X}) \leq c(X, \overline{X}) \tag{6.12}$$

for each subset X of the vertex set V.

Proof. We remark that if $X = \emptyset$, then (6.12) asserts that $\sum_{s\in S} a(s) \geq \sum_{t\in T} b(t)$, that is, the total demand does not exceed the total supply.

First suppose that there is a flow f satisfying (6.8)–(6.11). Let $X \subseteq V$. Then summing the constraints (6.8)–(6.10) over all vertices in $\overline{X}$ and using (6.11) we obtain

$$\begin{aligned} b(T\cap\overline{X}) - a(S\cap\overline{X}) &\leq f(V,\overline{X}) - f(\overline{X},V) \\ &= f(X,\overline{X}) - f(\overline{X},X) \leq c(X,\overline{X}). \end{aligned}$$

Hence (6.12) holds for all $X \subseteq V$.

Now suppose that (6.12) holds for all $X \subseteq V$. We define a capacity-constrained network $N^* = \langle D^*, c^*, s^*, t^* \rangle$ by adjoining to D a new source vertex s^* and a new sink vertex t^* and all arcs of the form (s^*, s) with $s \in S$ and all arcs of the form (t, t^*) with $t \in T$. If (x, y) is an arc of D we define $c^*(x,y) = c(x,y)$. If $s \in S$, then $c^*(s^*, s) = a(s)$. If $t \in T$, then $c^*(t,t^*) = b(t)$. Let $(X^*, \overline{X}^*)$ be any cut of N^* which separates s^* and t^*, and let $X = X^* - \{s^*\}$ and $\overline{X} = \overline{X}^* - \{t^*\}$. Then

$$\begin{aligned} & c^*(X^*,\overline{X}^*) - c^*(T,t^*) \\ &= c^*(X,t^*) + c^*(s^*,\overline{X}) + c^*(X,\overline{X}) - c^*(T,t^*) \\ &= (T\cap X) + a(S\cap\overline{X}) + c(X,\overline{X}) - b(T) \\ &= -b(T\cap\overline{X}) + a(S\cap\overline{X}) + c(X,\overline{X}). \end{aligned}$$

Hence by (6.12)

$$c^*(X^*,\overline{X}^*) \geq c^*(T,t^*)$$

for all cuts $(X^*, \overline{X}^*)$ of N^* which separate s^* and t^*. It follows that the minimum capacity of a cut separating s^* and t^* equals $c^*(T,t^*)$. By Theorem 6.1.2 there is a flow f^* in N^* with value equal to $c^*(T,t^*)$. Since $c^*(t,t^*) = b(t)$ for all t in T we have $f^*(t,t^*) = b(t), (t \in T)$. Let f be the restriction of f^* to the arcs of D. Then f satisfies (6.10) and (6.11). Moreover, for s in S

$$a(s) \geq f^*(s^*,s) = f^*(s,V) - f^*(V,s) = f(s,V) - f(V,s),$$

and for t in T

$$b(t) = f^*(t,t^*) = f^*(V,t) - f^*(t,V) = f(V,t) - f(t,V).$$

Thus f also satisfies (6.8) and (6.9). □

A reformulation of Theorem 6.1.4 asserts that a supply-demand flow exists in N if and only if for each set of sink vertices there is a flow that satisfies the combined demand of those sink vertices without exceeding the supply at each source vertex. A precise statement is given below.

Corollary 6.1.5. *There exists a supply-demand flow f in $N=\langle D,c,S,a,T,b\rangle$ if and only if for each $U \subseteq T$, there is a flow f_U satisfying* (6.8), (6.10) *and* (6.11) *and*

$$f_U(V,U) - f_U(U,V) \geq b(U). \tag{6.13}$$

Proof. If f is a supply-demand flow in N, then for each $U \subseteq T$ we may choose $f_U = f$ to satisfy (6.8), (6.10), (6.11) and (6.13).

Conversely, suppose that for each $U \subseteq V$ there exists an f_U satisfying (6.8), (6.10), (6.11) and (6.13). Let X be a subset of V. By Theorem 6.1.4 it suffices to show that (6.12) is satisfied. Let $S' = S \cap \overline{X}$, $W' = W \cap \overline{X}$ and $U = T \cap \overline{X}$. Since f_U satisfies (6.8), (6.10) and (6.13) we have

$$-a(S') \leq f_U(V,S') - f_U(S',V),$$

$$0 = f_U(V,W') - f_U(W',V),$$

$$b(U) \leq f_U(V,U) - f_U(U,V).$$

Adding and using (6.11), we obtain

$$\begin{aligned} b(U) - a(S') &\leq f_U(V,\overline{X}) - f_U(\overline{X},V) \\ &= f_U(X,\overline{X}) - f_U(\overline{X},X) \leq c(X,\overline{X}). \end{aligned}$$ □

A corollary very similar to Corollary 6.1.5 holds if the set T is replaced by the set S.

In the next sections we shall use the flow theorems presented here in order to obtain existence theorems for matrices, graphs and digraphs.

Exercises

1. Let $N = \langle D,c,s,t\rangle$ be a capacity-constrained network and let $l : E \rightarrow Z^+$ be an integer-valued function defined on the set E of arcs of D. Prove that the maximum value of a flow f satisfying $l(x,y) \leq f(x,y) \leq c(x,y)$ for each arc (x,y) of D equals the minimum value of $c(X,\overline{X}) - l(\overline{X},X)$ taken over all cuts separating s and t, *provided at least one such flow f exists.*
2. Use Theorem 6.1.3 to show that there is a flow f in the network N satisfying $l(x,y) \leq f(x,y) \leq c(x,y)$ for each arc (x,y) if and only if $c(X,\overline{X}) \geq l(\overline{X},X)$ for all subsets X of vertices for which $\{s,t\} \subseteq X$ or $\{s,t\} \subseteq \overline{X}$.
3. Suppose we drop the assumptions that the capacity function and flow function are integer valued. Prove that Theorem 6.1.2 remains valid.
4. Construct an example of a capacity-constrained network N whose capacity function is integer valued for which there is a flow f of maximum value, such that $f(x,y)$ is not an integer for at least one arc (x,y). (By Theorem 6.1.2 there is also an integer-valued flow of maximum value.)

References

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6.2 Existence Theorems for Matrices

Let $A = [a_{ij}], (i = 1,2,\ldots,m; j = 1,2,\ldots,n)$ be an m by n matrix whose entries are nonnegative integers. Let

$$r_i = a_{i1} + a_{i2} + \cdots + a_{in}, \quad (i = 1,2,\ldots,m)$$

be the sum of the elements in row i of A, and let

$$s_j = a_{1j} + a_{2j} + \cdots + a_{mj}, \quad (j = 1,2,\ldots,m)$$

be the sum of the elements in column j of A. Then

$$R = (r_1, r_2, \ldots, r_m)$$

is the *row sum vector* of A and

$$S = (s_1, s_2, \ldots, s_n)$$

is the *column sum vector* of A. The vectors R and S consist of nonnegative integers and satisfy the fundamental equation

$$r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n. \tag{6.14}$$

The matrix A can be regarded as the reduced adjacency matrix of a bipartite multigraph G with bipartition $\{X, Y\}$ where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The multiplicity of the edge $\{x_i, y_j\}$ equals $a_{ij}, (i = 1,2,\ldots,m; j = 1,2,\ldots,n)$. The vector R records the degrees of the vertices in X and the vector S records the degrees of the vertices in Y. Without loss of generality we choose the ordering of the vertices in X and in Y so that

$$r_1 \geq r_2 \geq \cdots \geq r_m \qquad \text{and} \qquad s_1 \geq s_2 \geq \cdots \geq s_n.$$

The vectors R and S are then said to be *monotone*.

Theorem 6.2.1. *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors. There exists an m by n nonnegative integral matrix with row sum vector R and column sum vector S if and only if* (6.14) *is satisfied.*

Proof. If there exists an m by n nonnegative integral matrix with row sum vector R and column sum vector S then (6.14) holds. Conversely, suppose that (6.14) is satisfied. We inductively construct an m by n nonnegative integral matrix $A = [a_{ij}]$ with row sum vector R and column sum vector S. If $m = 1$ we let

$$A = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}.$$

If $n = 1$ we let

$$A = \begin{bmatrix} r_1 & r_2 & \cdots & r_m \end{bmatrix}^T.$$

Now we assume that $m > 1$ and $n > 1$ and proceed by induction on $m+n$. Let

$$a_{11} = \min\{r_1, s_1\}.$$

First suppose that $a_{11} = r_1$. We then let $a_{12} = \cdots = a_{1n} = 0$, and define $R' = (r_2, \ldots, r_m)$ and $S' = (s_1 - r_1, s_2, \ldots, s_n)$. We have

$$r_2 + \cdots + r_m = (s_1 - r_1) + s_2 + \cdots + s_n,$$

and by the induction assumption there exists a nonnegative integral matrix A' with row sum vector R' and column sum vector S'. The matrix

$$\begin{bmatrix} r_1 & 0 & \cdots & 0 \\ & A' & & \end{bmatrix}$$

has row sum vector R and column sum vector S. If $a_{11} = s_1$, a similar construction works. $\square$

If $m = n$, the nonnegative integral matrix A of order n with row sum vector $R = (r_1, r_2, \ldots, r_n)$ and column sum vector $S = (s_1, s_2, \ldots, s_n)$ can also be regarded as the adjacency matrix of a general digraph of order n. The set of vertices of D is $V = \{a_1, a_2, \ldots, a_n\}$ and a_{ij} equals the multiplicity of the arc (a_i, a_j), $(i, j = 1, 2, \ldots, n)$. The vector R now records the outdegrees of the vertices and is the *outdegree sequence* of D. The vector S records the indegrees of the vertices and is the *indegree sequence* of D. When dealing with digraphs we may assume without loss of generality that R or S is monotone, but we cannot in general assume that both R and S are monotone. Theorem 6.2.1 provides a necessary and sufficient condition that nonnegative integral vectors R and S of the same length be the indegree sequence and outdegree sequence, respectively, of a general digraph.

We now investigate existence questions similar to the above in which a uniform bound is placed on the elements of the matrix. Although our formulations are in terms of matrices, there are equivalent formulations in terms of bipartite multigraphs (with a uniform bound on the multiplicities of the edges) and, in the case of square matrices, in terms of general digraphs (with a uniform bound on the mutliplicities of arcs). More general results with nonuniform bounds can be derived in a very similar way.

Theorem 6.2.2. *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors, and let p be a positive integer. There exists an m by n nonnegative integral matrix $A = [a_{ij}]$ such that*

$$\begin{array}{ll} a_{ij} \leq p, & (1 \leq i \leq m, 1 \leq j \leq n) \\ \sum_{j=1}^{n} a_{ij} \leq r_i, & (1 \leq i \leq m) \\ \sum_{i=1}^{m} a_{ij} \geq s_j, & (1 \leq j \leq n) \end{array} \tag{6.15}$$

if and only if

$$p|I||J| \geq \sum_{j \in J} s_j - \sum_{i \in \bar{I}} r_i, \quad (I \subseteq \{1,2,\ldots,m\}; J \subseteq \{1,2,\ldots,n\}). \tag{6.16}$$

Proof. We define a capacity-constrained, supply-demand network $N = \langle D, c, S, a, T, b \rangle$ as follows. The digraph D has order $m + n$ and its set of vertices is $V = S \cup T$ where $S = \{x_1, x_2, \ldots, x_m\}$ is an m-set and $T = \{y_1, y_2, \ldots, y_n\}$ is an n-set. There is an arc of D from x_i to y_j with capacity equal to p for each $i = 1, 2, \ldots, m$ and each $j = 1, 2, \ldots, n$. There are no other arcs in D. We define $a(x_i) = r_i, (i = 1, 2, \ldots, m)$ and $b(y_j) = s_j, (j = 1, 2, \ldots, n)$. If f is a supply-demand flow in N, then defining

$$a_{ij} = f(x_i, y_j), \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

we obtain a nonnegative integral matrix A satisfying (6.15). It follows from Theorem 6.1.4 that there is a supply-demand flow f in N if and only if (6.16) is satisfied. □

If in Theorem 6.2.2 both R and S are monotone, then (6.16) is equivalent to

$$pkl \geq \sum_{j=1}^{l} s_j - \sum_{i=k+1}^{m} r_i, \quad (0 \leq k \leq m, 0 \leq l \leq n). \tag{6.17}$$

The special case of Theorem 6.2.2 obtained by choosing $p = 1$ and by assuming (6.14) is recorded in the following corollary.

Corollary 6.2.3. *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors satisfying* (6.14). *There exists an m by n $(0,1)$-matrix with row sum vector R and column sum vector S if and only if*

$$|I||J| \geq \sum_{j \in J} s_j - \sum_{i \in \bar{I}} r_i, (I \subseteq \{1,2,\ldots,m\}; J \subseteq \{1,2,\ldots,n\}). \tag{6.18}$$

The conditions given in Corollary 6.2.3 for the existence of an m by n (0,1)-matrix with row sum vector R and column sum vector S can be formulated in terms of the concepts of conjugation and majorization of vectors. Let $R = (r_1, r_2, \ldots, r_m)$ be a nonnegative integral vector of length m, and suppose that $r_k \leq n, (k = 1, 2, \ldots, m)$. The *conjugate* of R is the nonnegative integral vector $R^* = (r_1^*, r_2^*, \ldots, r_n^*)$ where

$$r_k^* = |\{i : r_i \geq k, i = 1, 2, \ldots, m\}|.$$

(There is a certain arbitrariness in the length of the conjugate R^* of R in that its length n can be any integer which is not smaller than any component of R.) The conjugate of R is monotone even if R is not. We also have the elementary relationship

$$\sum_{i=1}^{k} r_i^* = \sum_{j=1}^{m} \min\{r_j, k\}. \tag{6.19}$$

There is a geometric way to view the conjugate vector R^*. Consider an array of m rows which has r_i 1's in the first positions of row $i, (i = 1, 2, \ldots, m)$. Then $R^* = (r_1^*, r_2^*, \ldots, r_n^*)$ is the vector of column sums of the array. For example, if $R = (5, 3, 3, 2, 1, 1)$ and $n = 5$, then using the array

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & & \\ 1 & 1 & & & \\ 1 & & & & \\ 1 & & & & \end{array}$$

we see that $R^* = (6, 4, 3, 1, 1)$.

Now let $E = (e_1, e_2, \ldots, e_n)$ and $F = (f_1, f_2, \ldots, f_n)$ be two monotone, nonnegative integral vectors. Then we write $E \preceq F$ and say that E is *majorized* by F provided the partial sums of E and F satisfy

$$e_1 + e_2 + \cdots + e_k \leq f_1 + f_2 + \cdots + f_k, \quad (k = 1, 2, \ldots, n)$$

with equality for $k = n$.

Theorem 6.2.4. *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors, and let p be a positive integer. Assume also that $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$ and that S is monotone. There exists an m by n nonnegative integral matrix $A = [a_{ij}]$ with row sum vector R and column sum vector S satisfying $a_{ij} \leq p, (1 \leq i \leq m, 1 \leq j \leq n)$ if and only if*

$$\sum_{j=1}^{k} s_j \leq \sum_{i=1}^{m} \min\{r_i, pk\}, \quad (k = 1, 2, \ldots, n) \tag{6.20}$$

Proof. We consider the capacity-constrained supply-demand network $N = \langle D, c, S, a, T, b\rangle$ defined in the proof of Theorem 6.2.2. It follows from Corollary 6.1.5 that the matrix A exists if and only if $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$ and for each set $U \subseteq T$ there is a flow f_U satisfying (6.8) and (6.13) each of whose values lies between 0 and p. The set of arcs (U, V) is empty and $f_U(U, V) = 0$, and a flow f_U that maximizes $f_U(V, U) = f_U(S, U)$ is obtained by defining f_U so that

$$f_U(x_i, U) = \min\{r_i, p|U|\}, \quad (i = 1, 2, \ldots, m).$$

Thus the desired matrix A exists if and only if

$$\sum_{i=1}^{m} \min\{r_i, p|U|\} \geq \sum_{\{j:y_j \in U\}} s_j \text{ for all } U \subseteq T. \tag{6.21}$$

The left side of (6.21) is the same for all subsets U of a fixed cardinality k. Since S is monotone, for fixed k the largest value of the right side of (6.21) is $\sum_{j=1}^{k} s_j$. Hence (6.21) holds if and only if

$$\sum_{i=1}^{m} \min\{r_i, pk\} \geq \sum_{j=1}^{k} s_j, \quad (k = 1, 2, \ldots, n). \tag{6.22}$$

□

The special case of Theorem 6.2.4 obtained by taking $p = 1$ is known as the *Gale–Ryser theorem* (see Gale[1957] and Ryser[1957]).

Corollary 6.2.5. *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors. Assume that S is monotone and that $r_i \leq n$, $(i = 1, 2, \ldots, m)$. There exists an m by n $(0, 1)$-matrix with row sum vector R and column sum vector S if and only if $S \preceq R^*$.*

Proof. The corollary follows from Theorem 6.2.4 by choosing $p = 1$ and using the relationship (6.19). □

If $m = n$, Corollary 6.2.3 provides necessary and sufficient conditions for the existence of a digraph with prescribed indegree sequence R and outdegree sequence S. Alternative conditions are given by Corollary 6.2.5 because we may assume without loss of generality that the outdegree sequence S is monotone. We now determine conditions for existence with the added restriction that the digraph has no loops. This is equivalent to determining conditions for the existence of a square (0,1)-matrix with zero trace having prescribed row and column sum vectors. Following the procedure used in the proof of Theorem 6.2.4 we obtain the following theorem of Fulkerson[1960].

Theorem 6.2.6. *Let $R = (r_1, r_2, \ldots, r_n)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors and let p be a positive integer. There exists a nonnegative integral matrix $A = [a_{ij}]$ of order n such that*

$$a_{ij} \le p, \quad (i, j = 1, 2, \ldots, n),$$

$$a_{ii} = 0, \quad (i = 1, 2, \ldots, n),$$

$$\sum_{j=1}^{n} a_{ij} \le r_i, \quad (i = 1, 2, \ldots, n),$$

$$\sum_{i=1}^{n} a_{ij} \ge s_j, \quad (j = 1, 2, \ldots, n)$$

if and only if

$$\sum_{j \in J} s_j \le \sum_{j \in J} \min\{r_j, p(|J| - 1)\} + \sum_{j \in \overline{J}} \min\{r_j, p|J|\}, \quad (J \subseteq \{1, 2, \ldots, n\}). \tag{6.23}$$

Now assume that $p = 1$, and that both R and S are monotone. (This assumption entails some loss of generality because of the requirement that the trace is to be zero.) Then conditions (6.23) simplify considerably. Let k be an integer with $1 \le k \le n$. For $J \subseteq \{1, 2, \ldots, n\}$ with $|J| = k$, the left side of (6.23) is maximal if $J = \{1, 2, \ldots, k\}$ and the right side is minimal if $J = \{1, 2, \ldots, k\}$. Thus (6.23) holds if and only if

$$\sum_{j=1}^{k} s_j \le \sum_{j=1}^{k} \min\{r_j, k-1\} + \sum_{j=k+1}^{n} \min\{r_j, k\}, \quad (k = 1, 2, \ldots, n). \tag{6.24}$$

Consider an array of n rows which has r_i 1's in the first positions of row i with the exception that there is a 0 in position i of row i if $r_i \ge i, (1 \le i \le n)$. For example, if $R = (5, 3, 3, 2, 1, 1)$ the array is

$$\begin{array}{cccccc}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & & \\
1 & 1 & 0 & 1 & & \\
1 & 1 & & & & \\
1 & & & & & \\
1 & & & & &
\end{array}.$$

Let $R^{**} = (r_1^{**}, r_2^{**}, \ldots)$ be the vector of column sums of the array. Then R^{**} is called the *diagonally restricted conjugate* of R, and it follows that

$$\sum_{i=1}^{k} r_i^{**} = \sum_{j=1}^{k} \min\{r_j, k-1\} + \sum_{j=k+1}^{n} \min\{r_j, k\}.$$

We thus obtain from Theorem 6.2.6 the following result of Fulkerson[1960].

Theorem 6.2.7. *Let R and S be monotone, nonnegative integral vectors of length n. There exists a $(0,1)$-matrix of order n with row sum vector R and column sum vector S and trace zero if and only if $S \preceq R^{**}$.*

Theorems 6.2.6 and 6.2.7 have been generalized in Anstee[1982] by replacing the requirement that the matrix have zero trace with the requirement that there be a prescribed zero in at most one position in each column. Existence theorems more general than Theorem 6.2.2 can be derived from more general flow theorems than those presented in section 6.1.

Algorithms for the construction of the matrices considered in this section can be found in the references. In addition they will be discussed in the book *Combinatorial Matrix Classes.*

Exercises

1. Prove that the matrix A constructed inductively in the proof of Theorem 6.2.1 contains at most $m+n-1$ positive elements and is the reduced adjacency matrix of a bipartite graph which is a forest.
2. Let A be an m by n (0,1)-matrix with row sum vector R and column sum vector S. Interpret the quantity

$$|I||J| - \sum_{j\in J} s_j + \sum_{i\in \overline{I}} r_i$$

   appearing in Corollary 6.2.3 as a counting function.
3. Prove that there exists an m by n (0,1)-matrix with all row sums equal to the positive integer p and column sum vector equal to the nonnegative integral vector $S = (s_1, s_2, \ldots, s_n)$ if and only if $p \le n$, $s_j \le m, (j = 1, 2, \ldots, n)$ and $pm = \sum_{j=1}^{n} s_j$.
4. Generalize Theorem 6.2.2 by replacing the requirement $a_{ij} \le p$ with $a_{ij} \le c_{ij}$ where $c_{ij}, (1 \le i \le m, 1 \le j \le n)$ are nonnegative integers.
5. Let $0 \le r_i' \le r_i, 0 \le s_j' \le s_j, c_{ij} \ge 0, (1 \le i \le m, \le j \le n)$ be integers. Prove that there exists an m by n nonnegative integral matrix $A = [a_{ij}]$ such that

$$a_{ij} \le c_{ij}, \quad (1 \le i \le m, 1 \le j \le n),$$

$$r_i' \le \sum_{j=1}^{n} a_{ij} \le r_i, \quad (1 \le i \le m),$$

$$s_j' \le \sum_{i=1}^{m} a_{ij} \le s_j, \quad (1 \le j \le n)$$

if and only if

$$\sum_{i\in I, j\in J} c_{ij} \ge \max\left\{ \sum_{i\in I} r_i' - \sum_{j\in \overline{J}} s_j, \sum_{j\in J} s_j' - \sum_{i\in \overline{I}} r_i \right\}$$

(Mirsky[1968]).

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6.3 Existence Theorems for Symmetric Matrices

Let $A = [a_{ij}]$ be a symmetric matrix of order n whose entries are nonnegative integers, and let $R = (r_1, r_2, \ldots, r_n)$ be the row sum vector of A. Since A is symmetric R is also the column sum vector of A. The matrix A is the adjacency matrix of a general graph G of order n. The vertex set of G is an n-set $V = \{a_1, a_2, \ldots, a_n\}$, and a_{ij} equals the multiplicity of the edge $\{a_i, a_j\}, (i, j = 1, 2, \ldots, n)$. The vector R records the degrees of the vertices and is called the *degree sequence* of G. A reordering of the vertices of G replaces A by the symmetric matrix $P^T AP$ for some permutation matrix P of order n. Thus without loss of generality we may assume that R is monotone.

Let $R = (r_1, r_2, \ldots, r_n)$ be an arbitrary nonnegative integral vector of length n. The diagonal matrix

$$\begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}$$

is a symmetric, nonnegative integral matrix of order n with row sum vector R. We obtain necessary and sufficient conditions for the existence of a symmetric, nonnegative integral matrix with a uniform bound on its elements and with row sum vector equal to R from Theorem 6.2.2. We first prove a lemma which allows us to dispense with the symmetry requirement.

For a real matrix $X = [x_{ij}]$ of order n we define

$$q(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} |x_{ij} - x_{ji}|.$$

We note that $q(X) = 0$ if and only if X is a symmetric matrix.

Lemma 6.3.1. *Let $R = (r_1, r_2, \ldots, r_n)$ be a nonnegative integral vector and let p be a positive integer. Assume that there is a nonnegative integral matrix $A = [a_{ij}]$ of order n whose row and column sum vectors equal R and whose elements satisfy $a_{ij} \leq p, (i, j = 1, 2, \ldots, n)$. Then there exists a symmetric, nonnegative integral matrix $B = [b_{ij}]$ of order n whose row and column sum vectors equal R and whose elements satisfy $b_{ij} \leq p, (i, j = 1, 2, \ldots, n)$.*

Proof. Let $B = [b_{ij}]$ be a nonnegative integral matrix of order n whose row and column sum vectors equal R and whose elements satisfy $b_{ij} \leq p$, $(i, j = 1, 2, \ldots, n)$ such that $q(B)$ is minimal. Suppose that $q(B) > 0$. Let D be the digraph of order n whose vertex set is $V = \{a_1, a_2, \ldots, a_n\}$ in which there is an arc (a_i, a_j) if and only if $b_{ij} > b_{ji}, (i, j = 1, 2, \ldots, n)$. The digraph D has no loops and since $q(B) > 0$, D has at least one arc. Let a_i be a vertex whose outdegree is positive. Since r_i equals the sum of the elements in row i of B and also equals the sum of the elements in column i of B, the indegree of a_i is also positive. Conversely, if the indegree of a_i is positive, then the outdegree of a_i is also positive. It follows that there exists in D a directed cycle

$$a_{i_1} \to a_{i_2} \to \cdots \to a_{i_k} \to a_{i_1}$$

of length k for some integer k with $2 \leq k \leq n$. We now define a new nonnegative integral matrix $B' = [b'_{ij}]$ of order n for which $b'_{ij} \leq p, (i, j = 1, 2, \ldots n)$ and for which R is both the row and column sum vector. If k is even then B' is obtained from B by decreasing the elements $b_{i_1 i_2}, b_{i_3 i_4}, \ldots$, $b_{i_{k-1} i_k}$ by 1 and increasing the elements $b_{i_3 i_2}, b_{i_5 i_4}, \ldots, b_{i_{k-1} i_{k-2}}, b_{i_1 i_k}$ by 1. Now suppose that k is odd. If $b_{i_1 i_1} = p$ then B' is obtained from B by decreasing $b_{i_1 i_1}, b_{i_2 i_3}, b_{i_4 i_5}, \ldots, b_{i_{k-1} i_k}$ by 1 and increasing $b_{i_2 i_1}, b_{i_4 i_3}, \ldots$, $b_{i_{k-1} i_{k-2}}, b_{i_1 i_k}$ by 1. If $b_{i_1 i_1} < p$, then B' is obtained from B by increasing $b_{i_1 i_1}, b_{i_3 i_2}, b_{i_5 i_4}, \ldots, b_{i_k i_{k-1}}$ by 1 and decreasing $b_{i_1 i_2}, b_{i_3 i_4}, \ldots, b_{i_{k-2} i_{k-1}}, b_{i_k i_1}$ by 1. The matrix B' satisfies $q(B') < q(B)$ and this contradicts our choice of B. Hence $q(B) = 0$ and the matrix B is symmetric. □

Theorem 6.3.2. *Let $R = (r_1, r_2, \ldots, r_n)$ be a nonnegative integral vector and let p be a positive integer. There exists a symmetric, nonnegative integral matrix $B = [b_{ij}]$ whose row sum vector equals R and whose elements satisfy $b_{ij} \leq p, (i, j = 1, 2, \ldots, n)$ if and only if*

$$p|I||J| \geq \sum_{j \in J} r_j - \sum_{i \in \overline{I}} r_i, \quad (I, J \subseteq \{1, 2, \ldots, n\}). \tag{6.25}$$

Proof. The theorem is an immediate consequence of Lemma 6.3.1 and Theorem 6.2.2. □

If R is a monotone vector then (6.25) is equivalent to

$$pkl \geq \sum_{j=1}^{l} r_j - \sum_{i=k+1}^{n} r_i, \quad (k, l = 1, 2, \ldots, n). \tag{6.26}$$

Corollary 6.3.3. *Let $R = (r_1, r_2, \ldots, r_n)$ be a monotone, nonnegative integral vector. The following are equivalent:*

(i) There exists a symmetric $(0,1)$-matrix with row sum vector equal to R.
(ii) $kl \geq \sum_{j=1}^{l} r_j - \sum_{i=k+1}^{n} r_i, \quad (k, l = 1, 2, \ldots, n)$.
(iii) $R \preceq R^*$.

Proof. The equivalence of (i) and (ii) is a consequence of Theorem 6.3.2. The equivalence of (i) and (iii) is a consequence of Lemma 6.3.1 and Corollary 6.2.5 and the observation that $R \preceq R^*$ implies that $r_1 \leq n$. □

We now consider criteria for the existence of a symmetric, nonnegative integral matrix with zero trace having a prescribed row sum vector $R = (r_1, r_2, \ldots, r_n)$ and a uniform bound on its elements. Since the sum of the entries of a symmetric integral matrix with zero trace is even, a necessary condition is that $r_1 + r_2 + \cdots + r_n$ is an even integer. The following lemma is a special case of a more general theorem of Fulkerson, Hoffman and McAndrew[1965].

Lemma 6.3.4. *Let $R = (r_1, r_2, \ldots, r_n)$ be a nonnegative integral vector such that $r_1 + r_2 + \cdots + r_n$ is an even integer, and let p be a positive integer. Assume that there exists a nonnegative integral matrix $A = [a_{ij}]$ of zero trace whose row and column sum vectors equal R and whose elements satisfy $a_{ij} \leq p, (i, j = 1, 2, \ldots, n)$. Then there exists a symmetric, nonnegative integral matrix $B = [b_{ij}]$ of zero trace whose row and column sum vectors equal R and whose elements satisfy $b_{ij} \leq p$.*

Proof. Let $B = [b_{ij}]$ be a nonnegative integral matrix of zero trace such that R equals the row and column sum vectors of B, $b_{ij} \leq p, (i, j = 1, 2, \ldots, n)$ and $q(B)$ is minimal. Suppose that $q(B) > 0$. Let D be the general digraph of order n whose vertex set is $V = \{a_1, a_2, \ldots, a_n\}$ in which there is an arc (a_i, a_j) of multiplicity $a_{ij} - a_{ji}$ if $a_{ij} > a_{ji}, (i, j = 1, 2, \ldots, n)$. The digraph D has no loops and has exactly $q(B)$ arcs. Since R is both the row and column sum vector of B, the indegree of each vertex equals its outdegree. It follows that the arcs of D can be partitioned into sets each of which is the set of arcs of a directed cycle.

First suppose that there exists in D a closed directed walk γ: $a_{i_1} \to a_{i_2} \to \cdots \to a_{i_k} \to a_{i_1}$ of even length k in which the multiplicity of each arc on the walk does not exceed its multiplicity in D. We obtain a matrix B' from B as follows. We increase the elements $b_{i_2 i_1}, b_{i_4 i_3}, \ldots, b_{i_k i_{k-1}}$ by 1 and decrease the elements $b_{i_2 i_3}, \ldots, b_{i_{k-2} i_{k-1}}, b_{i_k i_1}$ by 1. The resulting matrix B' is a nonnegative integral matrix of zero trace whose row and column sum vectors equal R and whose elements do not exceed p. Moreover, $q(B') < q(B)$ contradicting our choice of B. It follows no such directed walk γ exists in D. We now conclude that the arcs of D can be partitioned into directed cycles of odd length and no two of these directed cycles have a vertex in common.

Let $a_{i_1} \to a_{i_2} \to \cdots \to a_{i_k} \to a_{i_1}$ be a directed cycle of D of odd length k. Since $r_1 + r_2 + \cdots + r_n$ is an even integer, D has another directed cycle $a_{j_1} \to a_{j_2} \to \cdots \to a_{j_l} \to a_{j_1}$ of odd length l. Neither of the arcs (a_{i_1}, a_{j_1}) and (a_{j_1}, a_{i_1}) belongs to D and hence $b_{i_1 j_1} = b_{j_1 i_1}$.

First suppose that $b_{i_1 j_1} = b_{j_1 i_1} \geq 1$. Let B' be the matrix obtained from B by decreasing each of $b_{i_1 j_1}, b_{j_1 i_1}, b_{i_2 i_3}, \ldots, b_{i_{k-1}, i_k}, b_{j_2 j_3}, \ldots, b_{j_{l-1} j_l}$ by 1 and increasing $b_{i_2 i_1}, \ldots, b_{i_{k-1} i_{k-2}}, b_{i_1 i_k}, b_{j_2 j_1}, \ldots, b_{j_{l-1} j_{l-2}}, b_{j_1 j_l}$ by 1. Then B' is a nonnegative integral matrix of zero trace whose row and column sum vectors equal R and whose elements do not exceed p, and B satisfies $q(B') < q(B)$ contradicting our choice of B. A similar construction results in a contradiction if $b_{i_1 j_1} = b_{j_1 i_1} = 0$. We conclude that $q(B) = 0$ and hence that B is symmetric. □

We now obtain the conditions of Chungphaisan[1974] for the existence of a symmetric nonnegative integral matrix with a uniform bound on its elements and with prescribed row sum vector.

Theorem 6.3.5. *Let $R = (r_1, r_2, \ldots, r_n)$ be a monotone, nonnegative integral vector such that $r_1 + r_2 + \cdots + r_n$ is an even integer, and let p be a positive integer. There exists a symmetric, nonnegative integral matrix $A = [a_{ij}]$ of order n with zero trace whose row sum vector equals R and whose elements satisfy $a_{ij} \leq p, (i, j = 1, 2, \ldots, n)$ if and only if*

$$\sum_{i=1}^{k} r_i \leq pk(k-1) + \sum_{i=k+1}^{n} \min\{r_i, pk\}, \quad (k = 1, 2, \ldots, n). \tag{6.27}$$

Proof. It follows from Theorem 6.2.6 and Lemma 6.3.4 that a matrix satisfying the properties of the theorem exists if and only if

$$\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k} \min\{r_i, p(k-1)\} + \sum_{i=k+1}^{n} \min\{r_i, pk\}, \quad (k = 1, 2, \ldots, n). \tag{6.28}$$

Suppose that (6.27) holds but (6.28) does not hold for some integer k. Clearly $k > 1$. Let q be the largest integer such that $r_q \geq p(k-1)$. Then $q < k$ and hence

$$\sum_{i=1}^{k} r_i > pq(k-1) + \sum_{i=q+1}^{k} r_i + \sum_{i=k+1}^{n} r_i.$$

Hence

$$\sum_{i=1}^{q} r_i > pq(k-1) + \sum_{i=k+1}^{n} r_i.$$

On the other hand by (6.27) we have

$$\begin{aligned}\sum_{i=1}^{q} r_i &\le pq(q-1) + \sum_{i=q+1}^{n} \min\{r_i, pq\}\\ &\le pq(q-1) + \sum_{i=q+1}^{k} \min\{r_i, pq\} + \sum_{i=k+1}^{n} r_i\\ &\le pq(q-1) + (k-q)pq + \sum_{i=k+1}^{n} r_i\\ &\le pq(k-1) + \sum_{i=k+1}^{n} r_i.\end{aligned}$$

This contradiction shows that (6.27) implies (6.28). The converse clearly holds and the theorem follows. □

We now deduce the theorem of Erdös and Gallai[1960] for the existence of a symmetric (0,1)-matrix with zero trace and prescribed row sum vector (a graph with prescribed degree sequence).

Theorem 6.3.6. *Let $R = (r_1, r_2, \ldots, r_n)$ be a monotone, nonnegative integral vector such that $r_1 + r_2 + \cdots + r_n$ is an even integer. Then the following statements are equivalent:*

(i) There exists a symmetric $(0,1)$-matrix with zero trace whose row sum vector equals R.

(ii) $R \preceq R^{**}$.

(iii) $\sum_{i=1}^{k} r_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{k, r_i\}, \quad (k = 1, 2, \ldots, n)$.

Proof. The theorem is a direct consequence of Theorem 6.3.5, Theorem 6.2.7 and Lemma 6.3.4. □

Algorithms for the construction of the matrices considered in this section can be found in Havel[1955], Hakimi[1962], Fulkerson[1960], Chungphaisan[1974] and Brualdi and Michael[1989].

Exercises

1. Prove the following generalization of Lemma 6.3.4: Let C be a symmetric, nonnegative integral matrix with zero trace. Let D be the general digraph of order n whose adjacency matrix is C. Assume that for each pair of cycles

γ and γ' of D of odd length, either γ and γ' have a vertex in common or there is an edge joining a vertex of γ and a vertex of γ'. If there exists a nonnegative integral matrix B of order n with row and column sum vector R such that $B \le C$ (entrywise), then there exists a symmetric, nonnegative integral matrix A of order n with row and column sum vector R such that $A \le C$ (Fulkerson, Hoffman and McAndrew[1965]).

2. Prove that there exists a symmetric, nonnegative integral matrix of order n with row sum vector $R = (r_1, r_2, \ldots, r_n)$ which is the adjacency matrix of a tree of order n if and only if $r_i \ge 1, (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} r_i = 2(n-1)$.

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6.4 More Decomposition Theorems

Several decomposition theorems for matrices have already been established in section 4.4. In this section we obtain additional decomposition theorems by applying the network flow theorems of this chapter. We recall that Theorem 4.4.3 asserts that the m by n nonnegative integral matrix A with maximum line sum equal to k can be decomposed into k (and no fewer) subpermutation matrices $P_1, P_2, \ldots, P_k$ of size m by n. The ranks of these subpermutation matrices are unspecified.

Theorem 6.4.1. *Let A be an m by n nonnegative integral matrix, and let k and l be positive integers. Then A has a decomposition of the form*

$$A = P_1 + P_2 + \cdots + P_l$$

where $P_1, P_2, \ldots, P_l$ are subpermutation matrices of rank k if and only if each line sum of A is at most equal to l and the sum of the elements of A equals lk.

Proof. It is clear that the conditions given are necessary for there to exist a decomposition of A into l subpermutation matrices of rank k. Now suppose that no line sum of A exceeds l and that the sum of the entries of A equals lk. If necessary we augment A by including additional lines of zeros and assume that $m = n$. Let the row and column sum vectors of A be $(r_1, r_2, \ldots, r_n)$ and $(s_1, s_2, \ldots, s_n)$, respectively. By Theorem 6.2.1 there exists an n by $n - k$ nonnegative integral matrix A_1 with row sum vector $(l - r_1, l - r_2, \ldots, l - r_n)$ and column sum vector $(l, l, \ldots, l)$. There also exists an $n - k$ by n nonnegative integral matrix A_2 with row sum vector $(l, l, \ldots, l)$ and column sum vector $(l - s_1, l - s_2, \ldots, l - s_n)$. Let

$$B = \begin{bmatrix} A & A_1 \\ A_2 & O \end{bmatrix}$$

where O denotes a zero matrix of order $n - k$. Then B is a nonnegative integral matrix of order $2n - k$ with all line sums equal to l. By Theorem 4.4.3 B has a decomposition

$$B = Q_1 + Q_2 + \cdots + Q_l$$

into l permutation matrices of order $2n-k$. Each of the permutation matrices Q_i has $n-k$ 1's in positions corresponding to those of A_1 and $n-k$ 1's in positions corresponding to those of A_2. Let P_i be the leading principal submatrix of Q_i of order n, $(i = 1, 2, \ldots, l)$. Then each P_i is a subpermutation matrix of rank k and $A = P_1 + P_2 + \cdots + P_l$. □

For each positive integer k a nonnegative integral matrix has a decomposition into subpermutation matrices of rank at most equal to k. However, not every nonnegative integral matrix can be expressed as a sum of subpermutation matrices of rank at least equal to k.

Recall that the sum of the elements of a matrix A is denoted by $\sigma(A)$.

Corollary 6.4.2. *Let A be an m by n nonnegative integral matrix and let l be the maximum line sum of A. Let k be a positive integer. Then A has a decomposition into subpermutation matrices each of whose ranks is at least equal to k if and only if $\sigma(A) \geq lk$. Moreover, if $\sigma(A) \geq lk$, then there exists an integer $k' \geq k$ such that A has a decomposition into subpermutation matrices where the rank of each subpermutation matrix equals k' or $k' + 1$.*

Proof. If A has a decomposition of the type described in the theorem, then $\sigma(A) \geq lk$. Now suppose that $\sigma(A) \geq lk$. Let $l' = \lfloor \sigma(A)/k \rfloor$. Then $l' \geq l$ and

$$l'k \leq \sigma(A) < (l' + 1)k,$$

and there exists an integer $k' \geq k$ such that

$$l'k' \leq \sigma(A) < l'(k' + 1).$$

Let $\sigma(A) = l'k' + p$ where $0 \le p < l'$ and define

$$A' = \begin{bmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & l' - p \end{bmatrix}.$$

Then A' is a nonnegative integral matrix with all line sums at most equal to l' and with sum of elements $\sigma(A') = l'(k'+1)$. By Theorem 6.4.1 A' has a decomposition into l' subpermutation matrices of rank $k'+1$. Hence A has a decomposition into l' subpermutation matrices of rank k' or $k'+1$.

□

Let A be an arbitrary nonnegative integral matrix. We now consider decompositions of A of the form

$$A = P_1 + P_2 + \cdots + P_l + X$$

where $P_1, P_2, \ldots, P_l$ are subpermutation matrices of a prescribed rank k and X is a nonnegative integral matrix. We define $\pi_k(A)$ to be the maximum integer l for which such a decomposition exists and we seek to determine $\pi_k(A)$. It follows from Theorem 1.2.1 that $\pi_k(A) \ge 1$ if and only if A does not have a line cover consisting of fewer than k lines. By Theorem 6.4.1 $\pi_k(A)$ equals the maximum integer l such that A has a decomposition of the form

$$A = B + X,$$

where B is a nonnegative integral matrix with sum of elements $\sigma(B) = kl$ and with line sums not exceeding l, and X is a nonnegative integral matrix. The following theorem is from Folkman and Fulkerson[1969] (see also Fulkerson[1964]).

Theorem 6.4.3. *Let $A = [a_{ij}]$ be an m by n nonnegative integral matrix and let k be a positive integer. Then*

$$\pi_k(A) = \min \left\lfloor \frac{\sigma(A')}{e + f + k - m - n} \right\rfloor$$

where the minimum is taken over all e by f submatrices A' of A with $e + f > m + n - k$.

Proof. Let l be a nonnegative integer. We define a network $N = \langle D, c, s, t\rangle$ as follows. The digraph D has order $m+n$ and its set of vertices is $V = X \cup Y$ where $X = \{x_1, x_2, \ldots, x_m\}$ is an m-set and $Y = \{y_1, y_2, \ldots, y_n\}$ is an n-set. There is an arc from x_i to y_j of capacity $a_{ij}, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. There are also arcs from s to $x_i, (i = 1, 2, \ldots, m)$ and from y_j to $t, (j = 1, 2, \ldots, n)$ each of capacity l. By Theorem 6.4.1 $\pi_k(A) \ge l$ if and only if there exists a flow in N with value lk. Applying Theorem 6.1.2 we see that a flow with value lk exists if and only if

$$\sigma(A') + l(m-e) + l(n-f) \geq lk$$

for every e by f submatrix A' of $A, (e = 0, 1, \ldots, m; f = 0, 1, \ldots, n)$. The theorem now follows. □

In Theorem 6.4.1 we determined when a nonnegative integral matrix has a decomposition into subpermutation matrices of a specified rank k. A more general question was considered by Folkman and Fulkerson[1969]. Let $K = (k_1, k_2, \ldots, k_l)$ be a monotone vector of positive integers, and let A be an m by n nonnegative integral matrix. A K-*decomposition* of A is a decomposition

$$A = P_1 + P_2 + \cdots + P_l$$

where P_i is an m by n subpermutation matrix of rank $k_i, (i = 1, 2, \ldots, l)$. Theorem 6.4.1 determines when the matrix A has a $(k, k, \ldots, k)$-decomposition. Since every nonnegative integral matrix is the sum of subpermutation matrices of rank 1, we can obtain from Theorem 6.4.3 necessary and sufficient conditions for A to have a K-decomposition if the vector K has the form $(k, k, \ldots, k, 1, 1, \ldots, 1)$, that is, if K has at most two different components one of which equals 1. The general case in which K has only two different components is settled in the following theorem of Folkman and Fulkerson[1969], which we state without proof.

Theorem 6.4.4. *Let $K = (k_1, k_2, \ldots, k_l)$ be a monotone vector of positive integers, and let $K^* = (k_1^*, k_2^*, \ldots)$ be the conjugate of K. Let A be an m by n nonnegative integral matrix. If A has a K-decomposition, then*

$$\sigma(A') \geq \sum_{j \geq m-e+n-f+1} k_j^* \tag{6.29}$$

for all e by f submatrices A' of $A, (e = 0, 1, \ldots, m; f = 0, 1, \ldots, n)$ with equality for $A' = A$. If the components of K take on at most two different values and (6.29) *holds, with equality for $A' = A$, then A has a K-decomposition.*

We conclude this section by stating without proof another decomposition theorem of Folkman and Fulkerson[1969] which is used in their proof of Theorem 6.4.4. It can be proved using the network flow theorems of Section 6.1.

Theorem 6.4.5. *Let A be an m by n nonnegative integral matrix with row sum vector $R = (r_1, r_2, \ldots, r_m)$ and column sum vector $S = (s_1, s_2, \ldots, s_n)$. Let $R' = (r_1', r_2', \ldots, r_m')$, $R'' = (r_1'', r_2'', \ldots, r_m'')$, $S' = (s_1', s_2', \ldots, s_n')$, and $S'' = (s_1'', s_2'', \ldots, s_n'')$ be nonnegative integral vectors such that $r_i \leq r_i' + r_i'', (i = 1, 2, \ldots, m)$ and $s_j \leq s_j' + s_j'', (j = 1, 2, \ldots, n)$. Let σ' and*

σ'' be nonnegative integers such that $\sigma(A) = \sigma' + \sigma''$. Then there exist m by n nonnegative integral matrices A' and A'' such that $A = A' + A''$ where $\sigma(A') = \sigma'$, $\sigma(A'') = \sigma''$, the components of the row and column sum vectors of A' do not exceed the corresponding components of R' and S', respectively, and the components of the row and column sum vectors of A'' do not exceed the corresponding components of R'' and S'', respectively, if and only if

$$\sigma' - \sum_{i\in\overline{I}} r'_i - \sum_{j\in\overline{J}} s'_j \leq \sigma(A[I,J]) \leq \sum_{i\in I} r'_i + \sum_{j\in J} s''_j.$$

and

$$\sigma'' - \sum_{i\in\overline{I}} r''_i - \sum_{j\in\overline{J}} s''_j \leq \sigma(A[I,J]) \leq \sum_{i\in I} r''_i + \sum_{j\in J} s'_j$$

for all $I \subseteq \{1,2,\ldots,m\}$ and all subsets $J \subseteq \{1,2,\ldots,n\}$. Here $A[I,J]$ denotes the submatrix of A with row indices in I and column indices in J.

The proof of the Theorem 6.4.5 proceeds by defining a digraph with certain lower and upper bounds on arc flows and then applying the circulation theorem, Theorem 6.1.3.

Exercises

1. Let A be an m by n nonnegative integral matrix and let p be the maximal line sum of A. Let k be a nonnegative integer. Prove that A has a decomposition into subpermutation matrices of ranks k or $k+1$ if and only if

$$pk \leq \sigma(A) \leq \lfloor \sigma(A)/k \rfloor (k+1).$$

2. Prove Theorem 6.4.5.

References

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6.5 A Combinatorial Duality Theorem

Let $A = [a_{ij}], (1 \leq i \leq m; 1 \leq j \leq n)$ be a nonnegative integral matrix and let k be a positive integer. We define

$$\sigma_k(A) = \max\{\sigma(P_1 + P_2 + \cdots + P_k)\}$$

where the maximum is taken over all m by n subpermutation matrices $P_1, P_2, \ldots, P_k$ such that

$$P_1 + P_2 + \cdots + P_k \leq A \text{ (entrywise)}.$$

Thus $\sigma_k(A)$ equals the maximum sum of the ranks of k subpermutation matrices whose sum does not exceed A. If A is a (0,1)-matrix, then $\sigma_k(A)$ equals the maximum sum of the ranks of k "disjoint" subpermutation matrices contained in A. By Theorem 4.4.3

$$\sigma_k(A) = \max\{\sigma(X)\}$$

where the maximum is taken over all m by n nonnegative integral matrices X such that $X \leq A$ and each line sum of X is at most equal to k. In terms of the bipartite graph G whose reduced adjacency matrix is A, $\sigma_k(A)$ equals the maximum number of edges of G which can be covered by k matchings. The integer $\sigma_1(A)$ equals the term rank $\rho(A)$ of A. If k equals the maximum line sum of A, then $\sigma_k(A)$ equals $\sigma(A)$, the sum of the entries of A. We define $\sigma_0(A)$ to be equal to 0, and let

$$\underline{\sigma}(A) = (\sigma_0(A), \sigma_1(A), \sigma_2(A), \ldots).$$

The sequence $\underline{\sigma}(A)$ is an infinite nondecreasing sequence with maximal element equal to $\sigma(A)$ and is called the *matching sequence* of A. In this section we discuss some elegant results of Saks[1986] concerning the matching sequence of a nonnegative integral matrix.

The following theorem of Vogel[1963] contains as a special case an evaluation of the terms of the matching sequence.

Theorem 6.5.1. *Let $A = [a_{ij}]$ be an m by n nonnegative integral matrix, and let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors. Then the maximum value of $\sigma(X)$ taken over all m by n nonnegative integral matrices $X = [x_{ij}]$ such that*

$$X \leq A; \quad \sum_{j=1}^{n} x_{ij} \leq r_i, \ \ (i = 1, 2, \ldots, m); \quad \sum_{i=1}^{m} x_{ij} \leq s_j, \ \ (j = 1, 2, \ldots, n)$$

equals

$$\min_{I \subseteq \{1,2,\ldots,m\}, J \subseteq \{1,2,\ldots,n\}} \left\{ \sum_{i \in I} r_i + \sum_{j \in J} s_j + \sum_{i \in \bar{I}, j \in \bar{J}} a_{ij} \right\}.$$

Proof. The proof is a straightforward consequence of the maxflow-mincut theorem, Theorem 6.1.2. We define a capacity-constrained network $N = \langle D, c, s, t \rangle$ as follows. The digraph D has order $m + n + 2$ and its set of vertices is $V = \{s, t\} \cup X \cup Y$ where $X = \{x_1, x_2, \ldots, x_m\}$ and

$Y = \{y_1, y_2, \ldots, y_n\}$. There is an arc from s to x_i of capacity $c(s, x_i) = r_i, (i = 1, 2, \ldots, m)$, an arc from y_j to t of capacity $c(y_j, t) = s_j$, $(j = 1, 2, \ldots, n)$ and an arc from x_i to y_j of capacity $c(x_i, y_j) = a_{ij}$, $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. There are no other arcs in D. If f is a flow in N of value v, then defining

$$x_{ij} = f(x_i, y_j), \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

we obtain an m by n nonnegative integral matrix $X = [x_{ij}]$ such that $x_{ij} \leq a_{ij}$ for all i and j, the row sum vector R' and column sum vector S' of X satisfy $R' \leq R$ and $S' \leq S$, and $\sigma(X) = v$. Conversely, given such an X there is a flow with value $\sigma(X)$. Let $(Z, \overline{Z})$ be a cut separating s and t where

$$\overline{Z} \cap X = \{x_i : i \in I\} \qquad \text{and} \qquad Z \cap Y = \{y_j : j \in J\}.$$

Then the capacity of $(Z, \overline{Z})$ is

$$c(Z, \overline{Z}) = \sum_{i \in I} r_i + \sum_{j \in J} s_j + \sum_{i \in \overline{I}, j \in \overline{J}} a_{ij}.$$

The theorem now follows. □

Let $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$. Then we define

$$\sigma_{I,J}(A) = \sigma(A) - \sigma(A[\overline{I}, \overline{J}]),$$

the sum of the elements of A which belong to the union of the rows indexed by I and the columns indexed by J. Recall that $A[\overline{I}, \overline{J}]$ denotes the submatrix of A with row indices in $\overline{I}$ and column indices in $\overline{J}$. For $p \geq 0$, we let

$$\tau_p(A) = \max\{\sigma_{I,J}(A) : I \subseteq \{1, 2, \ldots, m\}, J \subseteq \{1, 2, \ldots, n\}, |I| + |J| = p\}, \tag{6.30}$$

the maximum sum of the elements of A lying in the union of p lines. We note that $\tau_p(A) = \sigma(A)$ for all $p \geq m + n$. The infinite nondecreasing sequence

$$\underline{\tau}(A) = (\tau_0(A), \tau_1(A), \tau_2(A), \ldots)$$

is called the *covering sequence* of A. The matching sequence of a nonnegative integral matrix can be obtained from its covering sequence.

Corollary 6.5.2. *Let A be an m by n nonnegative integral matrix and let k be a nonnegative integer. Then*

$$\sigma_k(A) = \min\{\sigma(A) + kp - \tau_p(A) : p \geq 0\}, \quad (k \geq 0).$$

Proof. By Theorem 6.5.1 with $r_i = k, (i = 1, 2, \ldots, m)$, and $s_j = k, (j = 1, 2, \ldots, n)$, $\sigma_k(A)$ equals the minimum value of the quantities

$$k(|I| + |J|) + \sum_{i \in \overline{I}, j \in \overline{J}} a_{ij} = \sigma(A) + k(|I| + |J|) - \sigma_{I,J}(A),$$

over all subsets I of $\{1, 2, \ldots, m\}$ and J of $\{1, 2, \ldots, n\}$. Setting $p = |I| + |J|$ we see that $\sigma_k(A)$ equals

$$\begin{aligned} \min_{p \geq 0} \min\{\sigma(A) + kp - \sigma_{I,J}(A) : I &\subseteq \{1, 2, \ldots, m\}; \\ J &\subseteq \{1, 2, \ldots, n\}, |I| + |J| = p\}, \end{aligned}$$

and the result follows from (6.30). □

We now digress and study some properties of integer sequences. Let n be a nonnegative integer. We denote by T_n the set of all infinite integer sequences

$$\underline{t} = (t_0, t_1, t_2, \ldots.)$$

satisfying

(i) $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq n$, and
(ii) $t_l = n$ for some integer l (and hence for all integers greater than l).

We partially order the set T_n by defining $\underline{s} \leq_T \underline{t}$ provided $s_k \leq t_k$ for all $k \geq 0$. This partial order determines a lattice with meet and join given as follows:

$$s \wedge_T t = (\min\{s_0, t_0\}, \min\{s_1, t_1\}, \min\{s_2, t_2\}, \ldots.),$$

$$s \vee_T t = (\max\{s_0, t_0\}, \max\{s_1, t_1\}, \max\{s_2, t_2\}, \ldots.).$$

We note that $(0, n, n, n, \ldots.)$ is the unique maximal element of T_n.

A *composition* of a positive integer n is an infinite sequence

$$\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots.)$$

of nonnegative integers with sum $\sum_{k \geq 1} \lambda_k = n$. Notice that there is no monotonicity assumption on the terms of a composition. The set of all compositions of the integer n is denoted by C_n. We partially order the set C_n by defining $\underline{\mu} \leq_C \underline{\lambda}$ provided

$$\sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i, \quad (k \geq 1).$$

Let $\underline{t} = (t_0, t_1, t_2, \ldots.)$ be a sequence of integers. The *difference sequence*

$$\delta \underline{t} = ((\delta \underline{t})_1, (\delta \underline{t})_2, (\delta \underline{t})_3, \ldots.)$$

is defined by $(\delta\underline{t})_k = t_k - t_{k-1}, (k \geq 1)$. If $\underline{t}$ is in T_n then $\delta\underline{t}$ is in C_n. Conversely, if $\underline{\lambda}$ is in C_n and we define $\underline{t} = (0, \lambda_1, \lambda_1 + \lambda_2, \ldots)$, then $\delta\underline{t} = \underline{\lambda}$. Moreover it follows that $\underline{s} \leq_T \underline{t}$ if and only if $\delta\underline{s} \leq_C \delta\underline{t}$. Hence $\delta : T_n \to C_n$ defines an isomorphism of the partially ordered sets T_n and C_n. Since T_n is a lattice, C_n is also a lattice and we denote its meet and join by $\wedge_C$ and $\vee_C$, respectively.

A *partition* of the integer n is a composition $\underline{\lambda}$ of n which satisfies the monotonicity assumption $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. The partitions of n form a finite subset P_n of the set C_n of compositions of n. A sequence $\underline{t}$ in T_n is called *convex* provided that

$$2t_k \geq t_{k+1} + t_{k-1}, (k = 1, 2, \ldots).$$

The set of convex sequences in T_n is denoted by X_n. A sequence $\underline{t}$ in T_n belongs to X_n if and only if the difference sequence $\delta\underline{t}$ belongs to P_n. Thus $\delta : X_n \to P_n$ defines an isomorphism of the partially ordered sets X_n and P_n. In particular, X_n is a finite set.

The join $\vee_T$ of two convex sequences in X_n need not be a convex sequence. For instance, the join of the convex sequences

$$\underline{s} = (0, 4, 8, 12, 12, 12, \ldots), \qquad \underline{t} = (0, 5, 7, 9, 11, 12, 12, \ldots)$$

in X_{12} is

$$\underline{s} \vee_T \underline{t} = (0, 5, 8, 12, 12, 12, \ldots)$$

and is not convex. Thus X_n is not in general a sublattice of the lattice T_n, and P_n is not in general a sublattice of C_n. However, X_n is closed under the meet operation $\wedge_T$ of T_n, and hence P_n is closed under the meet operation $\wedge_C$ of C_n.

Lemma 6.5.3. *If $\underline{s}$ and $\underline{t}$ are in X_n then their meet $\underline{s} \wedge_T \underline{t}$ is also in X_n. If $\underline{\lambda}$ and $\underline{\mu}$ are in P_n then their meet $\underline{\lambda} \wedge_C \underline{\mu}$ is also in P_n.*

Proof. We have

$$\begin{aligned} 2(\underline{s} \wedge_T \underline{t})_k &= 2\min\{s_k, t_k\} \\ &= \min\{2s_k, 2t_k\} \geq \min\{s_{k+1} + s_{k-1}, t_{k+1} + t_{k-1}\} \\ &\geq \min\{s_{k+1}, t_{k+1}\} + \min\{s_{k-1}, t_{k-1}\} \\ &= (\underline{s} \wedge_T \underline{t})_{k+1} + (\underline{s} \wedge_T \underline{t})_{k-1} \end{aligned}$$

and the result follows. □

Let $\underline{t}$ belong to T_n and consider the nonempty set

$$\{\underline{s} : \underline{s} \in X_n, \underline{t} \leq_T \underline{s}\} \tag{6.31}$$

of convex sequences above $\underline{t}$. It follows from Lemma 6.5.3 that the set (6.31) has a unique minimal element, namely

$$\underline{t}^X = \wedge_T \{\underline{s} : \underline{s} \in X_n, \underline{t} \leq_T \underline{s}\},$$

and this minimal element is convex. The convex sequence $\underline{t}^X$ is called the *convex closure* of $\underline{t}$. Similarly it follows that for a composition $\underline{\lambda}$ in C_n there is a unique minimal partition $\underline{\lambda}^P \geq_C \underline{\lambda}$, called the *partition closure* of $\underline{\lambda}$. We have

$$(\delta \underline{t})^P = \delta \underline{t}^X, \quad (\underline{t} \in T_n). \tag{6.32}$$

We next determine the relationship between two convex sequences whose difference sequences are conjugate partitions.

Lemma 6.5.4. *Let $\underline{s}$ and $\underline{t}$ be convex sequences in X_n. The following are equivalent:*

(i) $\delta\underline{s}$ and $\delta\underline{t}$ are conjugate partitions of n;
(ii) $s_k = \min\{n + kp - t_p : p \geq 0\}, \quad (k = 0, 1, 2, \ldots)$;
(iii) $t_k = \min\{n + kp - s_p : p \geq 0\}, \quad (k = 0, 1, 2, \ldots)$.

Proof. For $p \geq 0$ we have $t_p = \sum_{i=0}^{p} (\delta \underline{t})_i$ and hence

$$n + kp - t_p = \sum_{i=1}^{p} k + \sum_{i \geq p+1} (\delta \underline{t})_i.$$

It follows from this equation that

$$\min\{n + kp - t_p : p \geq 0\} = \sum_{l \geq 1} \min\{(\delta \underline{t})_l, k\} = \sum_{i=1}^{k} \max\{j : (\delta \underline{t})_j \geq i\}.$$

Thus (ii) is equivalent to

$$(\delta \underline{s})_k = \max\{j : (\delta \underline{t})_j \geq k\} = ((\delta \underline{t})^*)_k, \quad (k = 0, 1, 2, \ldots).$$

This proves the equivalence of (i) and (ii) and hence also the equivalence of (i) and (iii). □

By Lemma 6.5.4 the function $\Phi : X_n \to X_n$ defined by

$$(\Phi \underline{t})_k = \min\{n + kp - t_p : p \geq 0\}, \quad (k \geq 0); \quad \underline{t} \in X_n$$

is the bijection on X_n corresponding to the bijection $\Psi : P_n \to P_n$ determined by conjugation on P_n, that is,

$$\Phi = \delta \Psi \delta^{-1},$$

where $\Psi \underline{\lambda} = \underline{\lambda}^*, (\underline{\lambda} \in P_n)$.

The function Φ can be extended to all of T_n by defining

$$(\Phi\underline{t})_k = \min\{n + kp - t_p : p \geq 0\}, \quad (k \geq 0),$$

for each $\underline{t}$ in T_n. We now show that $\Phi\underline{t}$ is always convex.

Lemma 6.5.5. *For all $\underline{t}$ in T_n, $\Phi\underline{t}$ is in X_n.*

Proof. We have

$$(\Phi\underline{t})_0 = \min\{n - t_p : p \geq 0\} = 0,$$

since $t_p = n$ for some $p \geq 0$;

$$(\Phi\underline{t})_k = \min\{n + kp - t_p\} \leq n + k0 - t_0 = n,$$

with equality holding for $k = n$;

$$\begin{aligned}(\Phi\underline{t})_{k+1} &= \min\{n + (k+1)p - t_p : p \geq 0\} \\ &\geq \min\{n + kp - t_p : p \geq 0\} = (\Phi\underline{t})_R, \quad (k \geq 0).\end{aligned}$$

Hence $\Phi\underline{t}$ is in T_n. We also have

$$\begin{aligned}2(\Phi\underline{t})_k &= 2\min\{n + kp - t_p : p \geq 0\} \\ &= \min\{n + (k+1)p - t_p + n + (k-1)p + t_p : p \geq 0\} \\ &\geq \min\{n + (k+1)p - t_p; p \geq 0\} + \min\{n + (k-1)p - t_p : p \geq 0\} \\ &= (\Phi\underline{t})_{k+1} + (\Phi\underline{t})_{k-1}, (k \geq 1).\end{aligned}$$

Hence $\Phi\underline{t}$ is in X_n. □

In summary, $\Phi : T_n \to X_n$ is a function which is a bijection on X_n. Moreover, it is a consequence of the definition of Φ that Φ is order-reversing, that is, $\underline{s} \leq_T \underline{t}$ implies that $\Phi\underline{t} \leq_T \Phi\underline{s}$.

Lemma 6.5.6. *For all $\underline{t}$ in T_n,*

$$\underline{t} \leq_T \Phi^2\underline{t}.$$

Proof. We have

$$\begin{aligned}(\Phi^2\underline{t})_k &= \min\{n + kp - (\Phi\underline{t})_p : p \geq 0\} \\ &= \min\{n + kp - \min\{n + pq - t_q : q \geq 0\} : p \geq 0\} \\ &= \min\{\max\{p(k-q) + t_q : q \geq 0\}; p \geq 0\} \\ &\geq \min\{t_k : p \geq 0\} = t_k, \quad (k \geq 0).\end{aligned}$$

□

We now show that $\Phi^2\underline{t}$ is the convex closure of a sequence $\underline{t}$ in T_n.

Theorem 6.5.7. *For all $\underline{t}$ in T_n,*

$$\Phi^2\underline{t} = \underline{t}^X.$$

Proof. Using Lemma 6.5.5 and Lemma 6.5.6 we see that $\Phi^2\underline{t}$ is a convex sequence and

$$\underline{t} \leq_T \underline{t}^X \leq_T \Phi^2\underline{t}.$$

Since Φ is order reversing we have

$$\Phi\underline{t} \leq_T \Phi^3\underline{t} \leq_T \Phi\underline{t}^X \leq_T \Phi\underline{t}.$$

Thus $\Phi^3\underline{t} = \Phi\underline{t}^X$. Since Φ is a bijection on X_n, we conclude that $\Phi^2\underline{t} = \underline{t}^X$. □

The isomorphism δ of T_n and C_n now yields the following.

Theorem 6.5.8. *Let $\underline{t}$ be a sequence in T_n. Then*

$$\delta\Phi\underline{t} = ((\delta\underline{t})^P)^*.$$

Proof. By Theorem 6.5.7 $\Phi^2\underline{t} = \underline{t}^X$ and hence $\delta\Phi^2\underline{t} = \delta\underline{t}^X$. By Lemmas 6.5.4 and 6.5.5, $\delta\Phi^2\underline{t} = (\delta\Phi\underline{t})^*$ and hence

$$(\delta\Phi\underline{t})^* = \delta\underline{t}^X.$$

By (6.32) $\delta\underline{t}^X = (\delta\underline{t})^P$ and thus $(\delta\Phi\underline{t})^* = (\delta\underline{t})^P$. □

We now return to the m by n nonnegative integral matrix A and its matching sequence $\underline{\sigma}(A) = (\sigma_0(A), \sigma_1(A), \sigma_2(A), \ldots.)$ and covering sequence $\underline{\tau}(A) = (\tau_0(A), \tau_1(A), \tau_2(A), \ldots.)$. The sequences $\underline{\sigma}(A)$ and $\underline{\tau}(A)$ belong to $T_{\sigma(A)}$.

Theorem 6.5.9. *Let A be an m by n nonnegative integral matrix. Then*

$$\delta\underline{\sigma}(A) = ((\delta\underline{\tau}(A))^P)^*.$$

In particular, the sequence of differences of the matching sequence $\underline{\sigma}(A)$ is convex.

Proof. By Corollary 6.5.2, $\underline{\sigma}(A) = \Phi\underline{\tau}(A)$, and hence by Lemma 6.5.5, $\underline{\sigma}(A)$ is a convex sequence. The theorem now follows by applying Theorem 6.5.8. □

In words, Theorem 6.5.9 says that the sequence of differences of the matching sequence is the conjugate of the partition closure of the sequence of differences of the covering sequence. The sequence of differences of the covering sequences is not in general convex. For instance, let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\underline{\tau}(A) = (0, 4, 8, 9, 12, 12, \ldots.)$$

and

$$\delta\underline{\tau}(A) = (4, 4, 1, 3, 0, 0, \ldots.).$$

The partition closure of $\delta\underline{\tau}(A)$ is

$$(4, 4, 2, 2, 0, 0, \ldots.),$$

which is self-conjugate and equals the sequence of differences of the matching sequence $\underline{\sigma}(A)$.

More general results can be found in Saks[1986].

Exercises

1. Show how König's theorem (Theorem 1.2.1) follows from Theorem 6.5.1.
2. Let $\underline{\lambda}$ be a composition of n. Prove that the following procedure always results in the partition closure $\underline{\lambda}^P$ of $\underline{\lambda}$: Choose any j such that $\lambda_j < \lambda_{j+1}$. Replace λ_j by $\lambda_j + 1$ and λ_{j+1} by $\lambda_{j+1} - 1$. Repeat until a partition of n is obtained (Saks[1986]).
3. Determine the partition closure of the composition $(2, 3, 6, 2, 4, 4, 5, 0, 0, \ldots)$ of 26.
4. Determine the matching sequence and covering sequence of the m by n matrix of all 1's. Verify Theorem 6.5.9 in this case.
5. Determine the matching sequence and covering sequence of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and verify Theorem 6.5.9 in this case.

References

M. Saks[1986], Some sequences associated with combinatorial structures, *Discrete Math.*, 59, pp. 135–166.

W. Vogel[1963], Bemerkungen zur Theorie der Matrizen aus Nullen und Einsen, *Archiv der Math.*, 14, pp. 139–144.