On the Algebraic Connectivity of Directed Graphs

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Definition

In [Wu05a], the algebraic connectivity of a directed graph G on n vertices is defined by [1].

$$\alpha(G) = \min_{x \in P} x^T L x,$$

where L is the graph Laplacian and

$$P = \{x \in \mathbb{R}^n : \ \ \|x\| = 1, \ x \perp e\}$$

[1] Another definition of algebraic connectivity is given in [Wu05b]

Properties

- $\alpha(G)$ is independent of the ordering of vertices since P is an invariant subspace of permutation matrices.

- Let Q be a unitary matrix whose columns span P, then

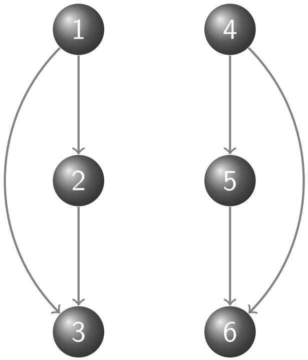
$$\alpha(G) = \lambda_{\min}\left(\frac{1}{2}Q^T\left(L + L^T\right)Q\right)$$

Properties

- If the graph G is undirected, then the definition of $\alpha(G)$ coincides with Fiedler's definition of algebraic connectivity [Fie73].
- For a graph G with Laplacian L, we have

$$\lambda_1\left(\frac{1}{2}\left(L+L^T\right)\right) \leq \alpha(G) \leq \lambda_2\left(\frac{1}{2}\left(L+L^T\right)\right).$$

Examples



Eigenvalues of the Laplacian:

$$\sigma(L) = \{2, 1, 0, 2, 1, 0\}$$

Spectral Rankability:

$$\text{rankH}(G) = 0.6$$

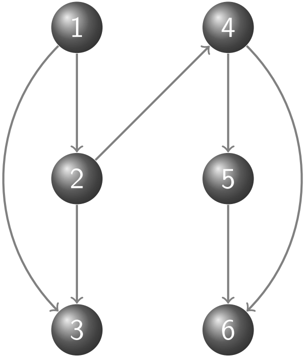
Algebraic Connectivity:

$$\alpha(G) = -0.389$$

Connectivity Rankability:

$$\text{rankA}(G) = 1.226$$
$$\sim 1.0$$

Examples



Eigenvalues of the Laplacian:

$$\sigma(L) = \{2, 2, 0, 2, 1, 0\}$$

Spectral Rankability:

$$\text{rankH}(G) = 0.6$$

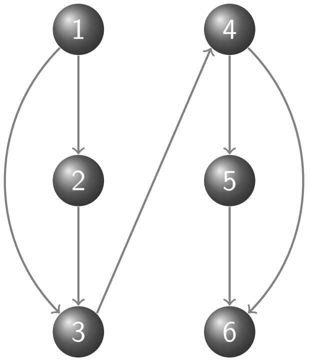
Algebraic Connectivity:

$$\alpha(G) = -0.303$$

Connectivity Rankability:

$$\text{rankA}(G) = 1.139$$
$$\sim 0.929$$

Examples



Eigenvalues of the Laplacian:

$$\sigma(L) = \{2, 1, 1, 2, 1, 0\}$$

Spectral Rankability:

$$\mathrm{rankH}(G) = 0.6$$

Algebraic Connectivity:

$$\alpha(G) = -0.068$$

Connectivity Rankability:

$$\begin{aligned} \mathrm{rankA}(G) &= 0.905 \\ &\sim 0.738 \end{aligned}$$

Bibliography I

- M. Fiedler, *Algebraic connectivity of graphs*, Czechoslovak Mathematical Journal **23** (1973), no. 98, 298–305.
- C. W. Wu, *Algebraic connectivity of directed graphs*, Linear Multilinear Algebra **53** (2005), no. 3, 203–223.
- ______, *On Rayleigh-Ritz ratios of a generalized Laplacian matrix of drected graphs*, Linear Algebra Appl. **402** (2005), 207–227.