RESTRICTED NUMERICAL RANGES OF DIGRAPH LAPLACIANS

by

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ABSTRACT

We present the restricted numerical range as a tool to characterize directed graphs by means of the graph Laplacian. We show that empty graphs, complete graphs, cycles, and regular tournaments are characterized by the restricted numerical range of their Laplacian, in addition to k-imploding stars and 3-balanced graphs, which are notably not characterized by their Laplacian spectra. We conclude by demonstrating a partial characterization of graphs with polygonal restricted numerical ranges and presenting open problems for future research.

ACKNOWLEDGMENT

Among the many men and women who have encouraged my mathematical exploration and nurtured my maturity, I offer the greatest thanks to Doug Eickman and Thomas Cameron, my Virgilian guides through the "secret world of mathematics, a hidden parallel universe of beauty and elegance, intricately intertwined with ours, invisible to most of us [5]."

Heather Smith and Yan Zhuang of the Davidson Math department joined Doug and Thomas in offering both mathematical and personal support, and I will miss discussing both mathematics and the less precise aspects of life with them dearly.

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"Now my vectors are all o'erthrown,
And what strength I have's mine own,
Which is most faint: now, 'tis true,
I must be here confined by you,
Or sent to Kyrgyzstan. Let me not,
Since I have my high honors got
And finish'd the defense, dwell
In this fare campus by your spell;
But release me from my bands
With the help of your good hands:
Gentle breath of yours my wings
Must float, or else my project fails,
Which was to please. Now I want
Theorems to enforce, figures to enchant,
And my ending is despair,
Unless I be relieved by prayer,
Which pierces so that it assaults
Gauss himself and frees all faults.
As you from errors would pardon'd be,
Let your indulgence set me free." [12]

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CHAPTER 1

INTRODUCTION

1.1 Overview

This thesis examines the ability of the numerical range to classify directed graphs by way of the graph Laplacian. We will begin by defining the numerical range from a linear algebra perspective and the graph Laplacian from a graph theory perspective. As this paper joins tools of analysis from linear algebra and graph theory, we will alternate between these mathematical perspectives and show how using these tools in tandem can provide a new and expanded perspective on graph characterization which is, in some cases, superior to the standard methods of spectral graph theory.

We first show that the empty and complete graphs can be characterized by their restricted numerical range, and then we will continue on to the characterize the directed cycle. We then characterize the classes of graphs whose restricted numerical range is a singleton, a vertical line, and entirely real. We conclude with a partial characterization of the graphs which produce polygonal numerical ranges, and present open questions for future research.

1.2 Terminology and Notation

We begin by presenting germane content from graph theory and linear algebra, and defining the terminology we will employ throughout the thesis.

1.2.1 Graph Laplacian

We follow the notation in [15] and [14] but we will reproduce common definitions below. We define a *directed graph* G to be the pair (V, E) where V is a non-empty set of vertices and E is a set of ordered pairs of distinct elements in V. The *edge set* of G is the set E in the pair (V, E), which we will refer to as $E(G)$. Similarly we refer to the *vertex set* V in (V, E) as $V(G)$. Two graphs are *isomorphic* if there exists a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$. For the sake of brevity, we will refer to the edge (v_i, v_j) as v_iv_j henceforth, and since we will only discuss simple directed graphs, without multiedges or loops, we will use the term "graph" to refer to these simple directed graphs.

The *order* of G, which is $|V|$, will be denoted by $n(G)$, and abbreviated to n when the graph is understood. For counting purposes, we assume all vertices are labeled $\{v_1, v_2, \ldots, v_n\}$ and thus distinct. A vertex v_j is a *neighbor* of another vertex v_i if $(v_i, v_j) \in E$. We note that this neighborly relation is not generally symmetric. We denote the out-degree of the vertex $v_i \in V$ by $d^+(i) = |\{v_i v_j \in E : 1 \leq j \leq n\}|$, and similarly the in-degree by $d^-(i) = |\{v_j v_i \in E : 1 \leq j \leq n\}|$.

A graph G is *complete* if for all distinct pairs of vertices $v_i, v_j \in V$, we have $v_i v_j, v_j v_i \in E$; we denote the complete graph on n vertices as K_n. We define the directed *cycle* as $C_n = (V_c, E_c)$ with $V_c = \{v_1, v_2, \ldots, v_n\}$ and $E_c = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. Since we will only work with the directed cycle, we will use "cycle" throughout to refer to the directed cycle. The *graph complement* $\overline{G}$ of a graph $G = (V, E)$ has vertex set V and edge set $E(\overline{G})$ such that $v_i v_j \in E(\overline{G})$ if and only if $v_i v_j \notin E$. The *empty graph* on n vertices, which we denote E_n, has the edge set $\emptyset$.

We say a *directed path* exists from a vertex u to another vertex v if there is a sequence of $(u_1, u_2, \ldots, u_k)$ where $u = u_1, v = u_k$, and for each $i < k$ we have $u_i u_{i+1} \in E(G)$. A *subgraph* $S = (V_s, E_s)$ of $G = (V, E)$ is a graph whose set of vertices V_s is a subset of V, and whose edge set E_s is a subset of $E \cap (V_s \times V_s)$. A graph is *strongly connected* if for all $u, v \in V$, there exists a directed path from u to v and a directed path from v to u. We say that a subgraph S of a graph G is strongly connected if S is strongly connected. Further, a *maximal* strongly connected subgraph is one such that no vertices $v \in V$ or edges $uv \in E$ could be added to V_s or E_s such that S would still be strongly connected.

To define the *graph Laplacian*, we first must define the adjacency and diagonal matrices. Let G be a graph on n vertices. The adjacency matrix $A(G)$ is the $n \times n$ matrix with entry $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$ otherwise. The degree matrix

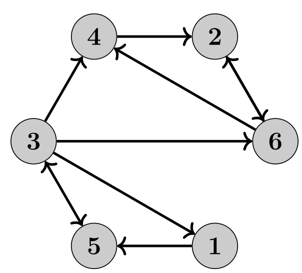
$D(G)$ is the $n \times n$ diagonal matrix with entry $a_{ii} = d^+(v_i)$ and 0 elsewhere. The *Laplacian matrix* $L(G)$ is then the $n \times n$ matrix given by

$$L(G) = D(G) - A(G)$$

or, shortening our notation,

$$L = D - A.$$

A *strongly connected component* of a graph G is a maximal strongly connected subgraph of G. A graph can be uniquely decomposed into strongly connected components [13, 1.4.12]. If its vertices are relabeled so as to group vertices belonging to the same strongly connected component, its associated Laplacian will be in *Frobenius normal form*. For example, consider the graph

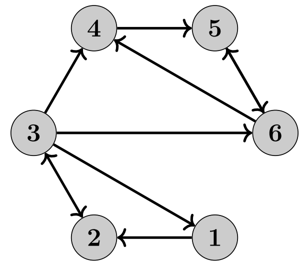


which has Laplacian

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 4 & -1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \end{bmatrix}.$$

To relabel the vertices in order to group by the two strongly connected components, we switch the labels of vertex 2 and 5, and obtain the graph which has Laplacian (in Frobenius normal form)

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$



Graph Laplacians in Frobenius normal form will have blocks along the diagonal which correspond to each strongly connected component. For the two strongly connected components in our example, we have

$$L_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

corresponding to the strongly connected component with vertices 1, 2 and 3 and

$$L_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

which corresponds to the strongly connected component with vertices 4, 5, and 6.

By grouping by strongly connected components, we can write any Laplacian L in the form

$$L = \begin{bmatrix} L_1 & L_{12} & \dots & L_{1r} \\ & L_2 & \dots & L_{2r} \\ & & \ddots & \vdots \\ & & & L_r \end{bmatrix}$$

which we call *Frobenius normal form* [3, Section 8.1] [6].

A square matrix P is a *permutation matrix* if it has exactly one 1 in every row and column and zeros elsewhere. A square matrix is *reducible* if it has dimension $n \geq 2$ and there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where B is an $(r \times r)$ matrix, D is an $(n-r) \times (n-r)$ matrix, C is a $(r \times n - r)$ and 0 is the $(n-r) \times r$ zero matrix. A matrix is irreducible if it is not reducible [8, Definition 6.2.21]. By [8, Theorem 6.2.24(d)] a matrix A with entries in $\{0, 1\}$ is irreducible if and only if the graph whose adjacency matrix is A is strongly connected.

Since we know each L_k block is the Laplacian of a strongly connected subgraph of G, we see that when a Laplacian L is written in Frobenius normal form, the L_k blocks are irreducible.

Example 1. Consider the L_1 and L_2 from our previous example:

$$L_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 4 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

By inspection, it seems clear that L_1 and L_2 cannot be reduced, and [8, Theorem 6.2.23] states that a $n \times n$ matrix A is irreducible if and only if $(I + |A|^{n-1})$, where the absolute value of A is taken entry-wise, has all positive entries.

Testing our two cases, we see

$$I + |L_1|^2 = I + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 4 \end{bmatrix}^2 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 5 \\ 5 & 6 & 18 \end{bmatrix}$$

and

$$I + |L_2|^2 = I + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 3 \\ 3 & 4 & 6 \end{bmatrix}.$$

Both matrices only have positive entries, as desired. $\Diamond$

A *Z-matrix* is a matrix that has nonpositive off diagonal entries. By construction, the Laplacian only has -1 entries off of its diagonal, so all Laplacians, and the L_k blocks of the Frobenius normal form, are Z-matrices. It follows that a Z-matrix A can be written in the form $sI - B$ for a positive scalar s and entry-wise nonnegative matrix B.

We will work with *M-matrices*, a special subclass of Z-matrices for which $s \geq \rho(B)$, where $\rho(B)$ is the spectral radius, the largest complex modulus (or absolute value, in the real case) of the eigenvalues of B. By [1, Lemma 6.4.1], we know that L is a general M-matrix if $L + \epsilon I$ is a nonsingular M-matrix for any $\epsilon > 0$. We know that for any row of a graph Laplacian, the sum of the off diagonal elements equals the negative value of the diagonal element. That is,

$$l_{ii} = \sum_{i \neq j} |l_{ij}|.$$

We can therefore say that

$$l_{ii} + \epsilon > \sum_{i \neq j} |l_{ij}|,$$

so $L + \epsilon I$ is diagonally dominant for any $\epsilon > 0$, as defined in [1, Theorem 6.2.3 M_{35}], and therefore is an invertible M-matrix. Hence, we can conclude that L is a general M matrix. We will discuss specific cases when the L_k block matrices are also nonsingular M-matrices in Section 2.2.3.

1.2.2 Numerical Range

In this section we will work with square $n \times n$ matrices, and describe the numerical range from a linear algebra perspective. Let A be an $n \times n$ matrix acting on vectors $x \in \mathbb{C}^n$. The *numerical range* of A is a set of scalar values in $\mathbb{C}$ defined as in [15]

$$W(A) = \{x^* A x : x \in \mathbb{C}^n, ||x|| = 1\},$$

where x^* is the conjugate transpose of x, so $x^* A x$ is the standard inner product of x and Ax, calculated by the rules of matrix multiplication.[1] The numerical range of a graph's Laplacian provides a representation of the graph as a set in $\mathbb{C}$.

Intuitively, $x^* A x$ can be viewed as $\langle x, Ax \rangle$, the inner product of a vector with its image under the linear transformation A. The elementary result that $W(I) = 1$ follows briefly: $Ix = x$, so $\langle x, Ix \rangle = x^* x = ||x||^2 = 1$. For a square matrix $A, \sigma(A)$ is the multiset of the eigenvalues of A.

In what follows we reproduce some short proofs of the fundamental properties of the numerical range proved in [15] for the sake of clarity for the reader.

Proposition 1.1. *For any matrix $A, \sigma(A) \subseteq W(A)$.*

Proof. Let $\lambda \in \sigma(A)$, with normalized eigenvector v. Then $v^* A v = v^* \lambda v = \lambda v^* v = \lambda ||v|| = \lambda$. Hence, $\lambda \in W(A)$, so $\sigma(A) \subseteq W(A)$. $\square$

[1] The W in the notation comes from the earlier name for the numerical range, *Wertovorrat*.

Example 2. Consider an example 3×3 matrix A and its numerical range, shown in Figure 1.1, with

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \end{bmatrix}.$$

In Figure 1.1, we can clearly see that the eigenvalues, shown in dark blue, lie within

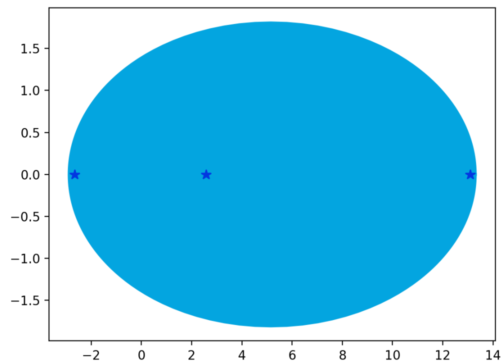


Figure 1.1: The numerical range of A from Example 2 in blue and its eigenvalues as dark blue stars.

the numerical range, shown in a lighter blue. $\diamondsuit$

In Figure 1.1, we show the numerical range by coloring it blue, and show the eigenvalues as stars of a darker blue. We will maintain this convention in all future figures unless otherwise noted, with the occasional omission of the eigenvalues.

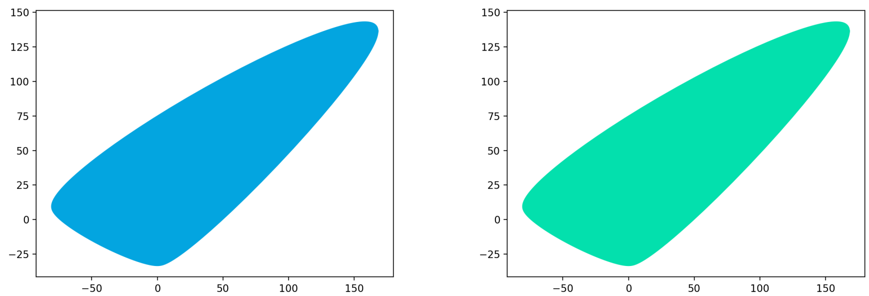
Proposition 1.2. *For any $n \times n$ matrix A, $W(A) = W(A^T)$.*

Proof. Consider the element $a \in W(A^T)$. By definition $a = x^* A^T x$, for some x such that $||x|| = 1$. If we take the transpose of both sides of the equation for a, we know $a^T = a$ because a is a scalar, but $(x^* A^T x)^T = x^T A \overline{x}$. If we let $y = \overline{x}$, we see $x^T A \overline{x} = \overline{y}^T A y = y^* A y$, and since $||y|| = ||x|| = 1$, it follows that $a = y^* A y \in W(A)$. This shows that $W(A^T) \subseteq W(A)$. Knowing that $W(A^T) \subseteq W(A)$ implies that $W((A^T)^T) \subseteq W(A^T)$, which is equivalent to saying that $W(A) \subseteq W(A^T)$ since $(A^T)^T = A$. Hence, $W(A) = W(A^T)$, as desired. $\square$

Example 3. Consider another example matrix $A \in \mathbb{C}^{3\times 3}$, this time with entries chosen randomly from real and complex parts ranging from 0 to 100. Let

$$A = \begin{bmatrix} 5+10i & 72+58i & 37+52i \\ 84+3i & 19+61i & 98+77i \\ 55+53i & 52+40i & 65+46i \end{bmatrix}.$$

Figure 1.2 shows us that $W(A) = W(A^T)$, as desired.



(a) $W(A)$ in light blue

(b) $W(A^T)$ in light green

Figure 1.2: Comparison of A and A^T in Example 3 to show that $W(A) = W(A^T)$

$\Diamond$

In the following proof and throughout, for a set of complex numbers S, the conjugate $\overline{S}$ denotes the element-wise conjugate.

Proposition 1.3. *For a matrix A, $W(A^*) = \overline{W(A)}$.*

Proof. In a similar fashion to the proof of Proposition 1.2, consider $a \in \overline{W(A^*)}$, where $a = x^*A^*x$ for some x such that $||x|| = 1$. Then $a^T = a = (x^*A^*x)^T = x^T\overline{A}\overline{x}$. If we let $y = \overline{x}$, then $x^T\overline{A}\overline{x} = \overline{y}^T\overline{A}y = y^*\overline{A}y$. Since $||y|| = ||x|| = 1$, we see that $a = y^*\overline{A}y \in W(\overline{A})$, so $W(A) \subseteq W(\overline{A})$. Knowing that $W(A) \subseteq W(\overline{A})$ implies that $W(\overline{\overline{A}}) \subseteq W(\overline{A})$, which is equivalent to saying that $W(A) \subseteq W(\overline{A})$ since $\overline{\overline{A}} = A$. Hence, $W(A^*) = W(\overline{A})$, as desired.

$\square$

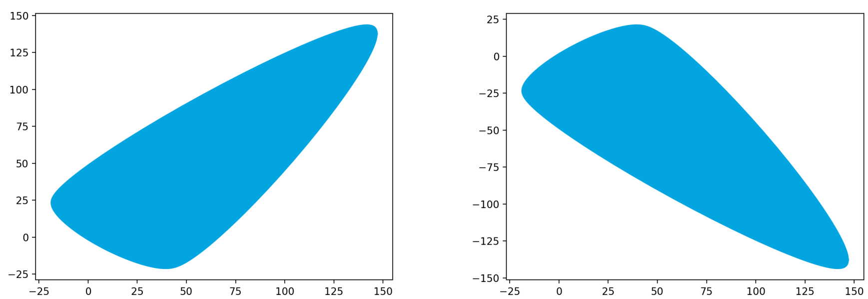
Example 4. We will randomly generate the entries of $A \in \mathbb{C}^{3\times 3}$ as above, producing

$$A = \begin{bmatrix} 64+42i & 10+62i & 40+25i \\ 51+33i & 77+57i & 53+64i \\ 78+10i & 29+56i & 30+49i \end{bmatrix}$$

and

$$A^* = \begin{bmatrix} 64-42i & 51-33i & 78-10i \\ 10-62i & 77-57i & 29-56i \\ 40-25i & 53-64i & 30-49i \end{bmatrix}.$$

We see in Figure 1.3 that $W(A^*)$ is simply a reflection of $W(A)$ over the real line, which shows that $W(A^*) = \overline{W(A)}$.



(a) $W(A)$ (b) $W(A^*)$

Figure 1.3: Lovely ponds in $\mathbb{C}$, demonstrating that using A from Example 4, $\overline{W(A)} = W(A^*)$

$\lozenge$

In the following proofs, we will make use of *unitary matrices*, defined in Definition 1.4

Definition 1.4. *A unitary matrix U is a square $n\times n$ matrix that satisfies the equation $U^*U = UU^* = I$.*

Proposition 1.5. *For any matrix A, and any unitary matrix U, the numerical range is such that $W(U^TAU) = W(A)$.*

Proof. The result follows from the well known fact that any unitary matrix U is an automorphism on $\mathbb{C}^n$. Take an element $y \in W(U^TAU)$, which implies there exists an $x \in \mathbb{C}^n$ such that $y = x^*U^TAUx = (Ux)^*A(Ux)$. Since $U : \mathbb{C}^n \to \mathbb{C}^n$, we know $Ux \in \mathbb{C}^n$, so $y \in W(A)$. Now take an element $z \in W(A)$, and $w \in \mathbb{C}^n$ such that $z = w^*Aw$. Since U is a surjection, we know that there exists an $a \in \mathbb{C}^n$ such that $w = Ua$, so we see $z = w^*Aw = (Ua)^*A(Ua) = a^*U^TAUa \in W(U^TAU)$, so $w \in W(U^TAU)$. We have shown that $W(U^TAU) \subseteq W(A)$ and $W(A) \subseteq W(U^TAU)$, so $W(U^TAU) = W(A)$. □

We will use this result to prove a similar invariance of the numerical range under permutation transformations in Proposition 1.6.

Proposition 1.6. *For any matrix A and any permutation matrix P, the numerical range satisfies $W(P^TAP) = W(P)$.*

Proof. Let P be a $n \times n$ permutation matrix. Recall from our definition of permutation matrices that they have a single 1 in each row and column, and zero elsewhere. Hence, P is an orthogonal matrix, so $P^{-1} = P^T$, so it is clear that $P^TP = PP^T = I$. This shows that P is a unitary matrix, so Proposition 1.5 tells us that $W(P^TAP) = W(A)$ for any $n \times n$ matrix A. □

Definition 1.7. *A Hermitian matrix H is one such that $H^* = H$, where A^* again denotes the conjugate transpose of A.*

Proposition 1.8. *For Hermitian matrices $A, W(A) \subseteq \mathbb{R}$.*

Proof. Let A be a Hermitian $n \times n$ matrix and let $y \in W(A)$, so $y = x^*Ax$ for some $x \in \mathbb{C}^n$. See that $y^* = (x^*Ax)^* = x^*A^*x = x^*Ax = y$. Since $y^* = y$, we know $y \in \mathbb{R}^n$, and y was arbitrarily chosen from $W(A)$, so $W(A) \subseteq \mathbb{R}$. □

Example 5. Consider the Hermitian matrix

$$A = \begin{bmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{bmatrix}$$

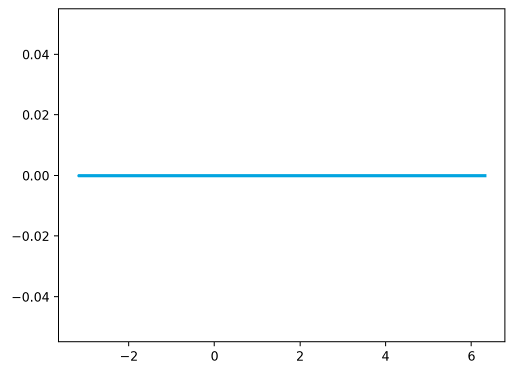


Figure 1.4: The numerical range of A in Example 5 is a subset of $\mathbb{R}$ when A is Hermitian.

In Figure 1.4, we see that the each value in the numerical range is real, as expected.

$\Diamond$

Proposition 1.9. *For real matrices A, $W(A)$ is symmetric with respect to the real axis.*

Proof. If A is a real matrix, then $A^* = A^T$. Then $\overline{W(A)} = W(A^*) = W(A^T) = W(A)$ by Propositions 1.2 and 1.3. $\square$

Example 6. Let

$$J = \begin{bmatrix} 8 & 6 & 7 \\ 5 & 3 & 0 \\ 9 & 0 & 1 \end{bmatrix}$$

and see the numerical range of J in Figure 1.5 does indeed have the desired symmetry.

$\Diamond$

Definition 1.10. *A normal matrix A is defined to commute with its conjugate transpose. That is, $A^*A = AA^*$.*

To describe the numerical ranges of normal matrices, we will first define the *convex hull* of a set of points.

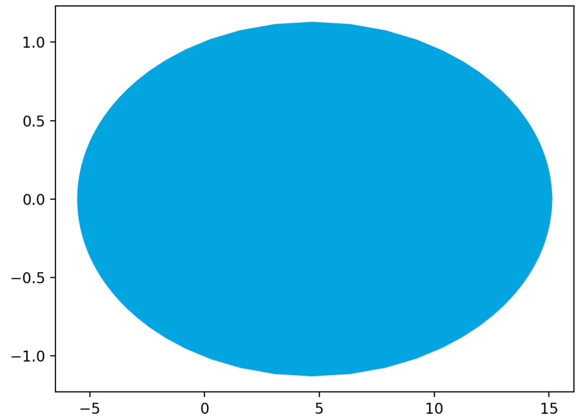


Figure 1.5: Numerical range of J in Example 6, which is symmetric across the real axis.

Definition 1.11. *The convex hull of a set $S = \{x_1, x_2, \dots, x_n\}$, with $x_i \in \mathbb{C}$ is the set of all points of the form*

$$\sum_{i=1}^{n} c_i x_i,$$

where every $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$.

We note that the convex hull of a set of complex points is a polygon in $\mathbb{C}$ which contains its interior. Throughout, we will use "polygon" to refer to polygons that contain their interior.

Proposition 1.12. *If a matrix A is normal, then $W(A)$ is the convex hull of $\sigma(L)$.*

Proof. Let A be a normal matrix. Another defining property of normal matrices is that they are similar to a diagonal matrix under a unitary similarity transformation, and we know that $W(A)$ is invariant under unitary transformations by Proposition 1.5. Assuming we have performed this unitary transformation, we can then take A in its diagonal form, so

$$A = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_n \end{bmatrix},$$

and $\{a_1, a_2, \dots a_n\} = \sigma(A)$. We now see that any element $y \in W(A)$ can be written in the form $y = x^*Ax = \sum_{i=1}^{n} a_i x_i \overline{x_i} = \sum_{i=1}^{n} a_i |x_i|^2$, where we know $\sum_{i=1}^{n} |x_i|^2 = 1$ because $\sum_{i=1}^{n} |x_i|^2 = x^*x = \langle x, x \rangle = ||x||^2 = ||x|| = 1$.

Hence the sum $\sum_{i=1}^{n} a_i x_i^2$ is a point in the convex hull of $\{a_1, a_2, \dots, a_n\}$. This shows that all points in $W(A)$ lie within the convex hull of $\sigma(A)$. Since x can vary over all the real vectors that satisfy $||x|| = 1$, we further see that every point in the convex hull of $\{a_1, a_2, \dots, a_n\}$ is in $W(A)$, so we conclude that $W(A)$ is the convex hull of $\{a_1, a_2, \dots, a_n\} = \sigma(A)$. $\square$

Example 7. Take a normal matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To verify that A is normal, we calculate

$$A^* = A^T = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

to see that

$$A^TA = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \end{bmatrix} = AA^T.$$

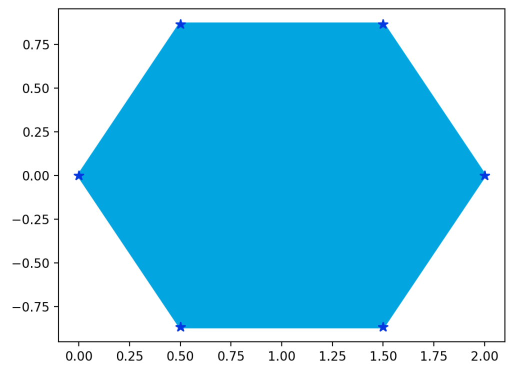


Figure 1.6: Using A from Example 7, we see $W(A)$ is the convex hull of the dark blue eigenvalues.

Since $A^*A = A^TA = AA^T = AA^*$, we see that A is indeed normal. Figure 1.6 illustrates that the numerical range is the convex hull of its eigenvalues, and therefore a polygon. $\Diamond$

CHAPTER 2

BACKGROUND

2.1 Algebraic Connectivity

In the graph Laplacian, the i^{th} row of L has a -1 in it for each edge leaving vertex i as well as a $d^+(i)$ term from the diagonal. Thus, the row sum of the i^{th} row of L is

$$\left(\sum_{ij\in E(G)} -1\right) + d^+(G) = -d^+(G) + d^+(G) = 0.$$

Since every row sum of L is zero, we see that if we take the all ones vector $e = [1, \ldots, 1]^T$, then Le will have the row sum in each of its n entries, so Le is the zero vector. This establishes that zero is an eigenvalue of L.

In his seminal 1973 paper [4], Fiedler defined the *algebraic connectivity* α for undirected graphs as the second smallest eigenvalue of the graph Laplacian.[1][2] In 2005, Wu generalized the algebraic connectivity to apply to directed graphs in [14] by defining it as

$$\alpha(G) = \min_{x\in S} x^T Lx,$$

where

$$S = \{x \in \mathbb{R}^n : x \perp e, ||x|| = 1\}.$$

For undirected graphs, this definition agrees with Fiedler's.

As its name suggests, the algebraic connectivity provides a way to measure the connectivity of the graph. A related quantity is the maximum value of $x^T Ax$, which we define as

$$\beta(G) = \max_{x\in S} x^T Lx.$$

[1]The eigenvector associated with the algebraic connectivity is known as the Fielder vector, and it provides a partition of the graph that corresponds to the *sparsest cut* of the graph. For adaptations of the sparest cut specific to the directed case, see [11].

[2]The use of "second" implies an ordering that only applies to the real spectrum of symmetric undirected graphs. The asymmetry of Laplacians for directed graphs produces complex eigenvalues, and motivates Wu's generalization.

In order to incorporate the requirement that x must be perpendicular to e, we modify the numerical range, and define the *restricted numerical range* as

$$W_r(L) = \{x^*Lx : x \perp e, ||x|| = 1, x \in \mathbb{C}^n\}.$$

Equivalently, using a notation similar to the algebraic connectivity,

$$W_r(L) = \{x^*Lx : x \in S^*\}$$

where $S^* = \{x \in \mathbb{C}^n : x \perp e, ||x|| = 1\}$.

The restriction that we are only considering vectors in the set S^* becomes difficult when we wish to directly analyze the matrix, so we construct matrices which ease computation, as described in 2.1.

Definition 2.1. *We define Q matrices as the class of real $n \times (n-1)$ orthonormal matrices whose columns are perpendicular to e. We will reserve the letter Q in the remainder of this thesis to refer to matrices in this class.*

Example 8 demonstrates what such a matrix might look like.

Example 8. Consider the matrix Q_0 below whose columns are orthogonal to each other and e, but Q_0 is not (yet) normal.

$$Q_0 = \begin{bmatrix} 1 & 1 & 1 & & \dots & 1 \\ -1 & 1 & 1 & & \dots & 1 \\ 0 & -2 & 1 & & \dots & 1 \\ 0 & 0 & -3 & & \dots & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & & 0 & -n \end{bmatrix}$$

To show that Q_0 is orthogonal, take any two distinct columns in Q_0, say q_i and q_j, where $i < j$. Then q_i has its first negative entry in the $(i+1)^{\text{th}}$ row and q_j has its first negative entry in the $(j+1)^{\text{th}}$ row. Then, in the inner product of q_i and q_j, the first i terms are all $(1)(1) = 1$, and the $(i+1)^{\text{th}}$ term is $-i(1) = -i$. Since all further entries of q_i are zero, the remaining terms of the inner product are zero. Hence, we see $\langle q_i, q_j \rangle = \left(\sum_{k=1}^{i} 1\right) - i = i - i = 0$.

To show that any of the columns of Q_0 are perpendicular to e, take q_i as above, and see that $\langle q_i, e\rangle = \left(\sum_{k=1}^{i} 1\right) + (i)(-1) = i - i = 0$.

So since $\langle q_i, q_j\rangle = \langle q_i, e\rangle = 0$ for arbitrary i and j, we see that Q_0 is an orthogonal matrix whose columns are perpendicular to e. Though the columns of Q_0 do not have unit length, we can easily make them so by replacing each column q_i with $\frac{q_i}{||q_i||}$. This scalar division does not affect their perpendicularity to each other or e, so in this way we can make the columns of Q_0 have unit length, so it is suitable to our preferences. $\diamondsuit$

Using such Q matrices as defined in 2.1 and explored in Example 8, we enumerate the useful properties of the restricted numerical range below.

Proposition 2.2. *Let L be the Laplacian matrix of a graph G on n vertices. Then:*

(a) For a Q matrix, $W_r(L) = W(Q^T LQ)$.

(b) Relabeling the vertices in G does not affect $W_r(L)$.

(c) The eigenvalues of L are contained in $W_r(L)$, except for the zero eigenvalue corresponding to the eigenvector e. Further, $\sigma(L) = \{0\} \cup \sigma(Q^T LQ)$ for any Q matrix.

(d) Connection to Algebraic Connectivity: The minimum real part of the restricted numerical range is $\alpha(G)$, and the maximum real part of the restricted numerical range is $\beta(G)$. That is

$$\alpha = \min \mathrm{Re}(W_r(L)) = \min_{x\in S} x^T Lx$$

and

$$\beta = \max \mathrm{Re}(W_r(L)) = \max_{x\in S} x^T Lx$$

where $S = \{x \in \mathbb{R}^n : x \perp e, ||x|| = 1\}$.

(e) If L is normal, then $Q^T LQ$ is normal for any Q matrix.

Proof. (a) By its definition, Q is a bijection from the $n-1$ dimensional space $\mathbb{C}^{n-1}$ to the $n-1$ dimensional subspace perpendicular to e in $\mathbb{C}^n$. Therefore for any element $w \in W_r(L)$, where $w = y^*Ly$ for some $y \in S^*$, we know that $y = Qx$ for some $x \in \mathbb{C}^{n-1}$, and $||x|| = 1$ since Q is orthonormal and preserves lengths. This implies that $y^* = (Qx)^* = x^*Q^* = x^*Q^T$, so $y^*Ly = x^*Q^TLQx$, and $x^*Q^TLQx \in W(Q^TLQ)$ by definition. Hence, $x^*Q^TLQx = w \in W(Q^TLQ)$, so $W_r(L) \subseteq W(Q^TLQ)$. A similar argument shows that any element $w \in W(Q^TLQ)$ is also in $W_r(L)$, so $W(Q^TLQ) \subseteq W_r(L)$, and we can conclude that $W(Q^TLQ) = W_r(L)$, and the proof is complete.

Note that while Q is a bijection between two $n-1$ dimensional spaces, it is not square. Instead, Q is a $n \times (n-1)$ matrix. Although its column space is perpendicular to e, it is written in the standard basis, not a basis containing e, so vectors in its image have dimension n, even though the column space has dimension $n-1$. Note further that while L is a $n \times n$ matrix, Q^TLQ is a $(n-1) \times (n-1)$ matrix.

(b) The Laplacian of G after a relabeling of vertices is equal P^TLP where L is the original Laplacian and P is a permutation matrix.

Since PQ is just the permutation of Q, PQ retains the definitional properties of Q, namely that it is a real $n \times (n-1)$ matrix with normal columns orthogonal to e, so we can use PQ instead of Q as our restricting matrix. Further note that $(PQ)^T = Q^TP^T$. Using part(a), we can say that

$$W_r(L) = W((PQ)^TLPQ) = W(Q^T(P^TLP)Q) = W_r(P^TLP).$$

Hence $W_r(L) = W_r(P^TLP)$, as desired.

(c) First, we append the column vector $\hat{e} = \frac{e}{\sqrt{n}}$ to Q, which has unit length since $||e|| = \sqrt{n}$. Since the columns of Q were defined to be orthogonal to e, the columns of our new matrix $[Q\ \ \hat{e}]$ are orthogonal, as well as normal, so $[Q\ \ \hat{e}]$ is also an orthonormal matrix. Hence, we can conclude that $[Q\ \ \hat{e}]^{-1} = [Q\ \ \hat{e}]^T$. Since $[Q\ \ \hat{e}]$ is a orthonormal matrix, it is certainly nonsingular, so we can write a similarity transformation of L to L' as $L' = [Q\ \ \hat{e}]^T L[Q\ \ \hat{e}]$. By simply expanding this matrix multiplication, we see that

$$L' = [Q\ \ \hat{e}]^T L[Q\ \ \hat{e}] = \begin{bmatrix} Q^T LQ & Q^T L\hat{e} \\ \hat{e}^T LQ & \hat{e}^T L\hat{e} \end{bmatrix} = \begin{bmatrix} Q^T LQ & 0 \\ \hat{e}^T LQ & 0 \end{bmatrix}$$

Where $Q^T L\hat{e} = \hat{e}^T L\hat{e} = 0$ because $\hat{e}$ is an eigenvector of L with eigenvalue 0. Since the right most matrix is lower block triangular, we see its eigenvalues are the eigenvalues of the blocks. Of these two blocks, one is simply 0, so 0 is an eigenvalue, and the other is $Q^T LQ$. Hence, we see that the eigenvalues of L' are $\{0\} \cup \sigma(Q^T LQ)$. Since L' and L are similar, we see that $\sigma(L) = \{0\} \cup \sigma(Q^T LQ)$ as well. To see that $\sigma(L) \setminus \{0\} \subseteq W_r(L)$, note that since $\sigma(L) = \{0\} \cup \sigma(Q^T LQ)$, it follows that $\sigma(L) \setminus \{0\} = \sigma(Q^T LQ)$. Further, since $\sigma(Q^T LQ) \subseteq W(Q^T LQ) = W_r(L)$ by Proposition 1.1, we conclude that $\sigma(L) \setminus \{0\} \subseteq W_r(L)$.

(d) We will prove that $\alpha = \min \mathrm{Re}(W_r(L)) = \min_{x \in S} x^T Lx$, but simply replacing each use of minimum below with maximum proves that $\beta = \max \mathrm{Re}(W_r(L))$, so no generality is lost between α and β in the proof below.

Recall Wu's definition in [14] of α as

$$\alpha = \min_{x \in S} x^T Lx$$

where $S = \{x \in \mathbb{R}^n : x \perp e, ||x|| = 1\}$. We see that S only contains real vectors, so $S \subset S^*$, where $S^* = \{x \in \mathbb{C}^n : x \perp e, ||x|| = 1\}$ is the set of complex vectors used to generate $W(L)$. Let x_α be the vector in S that attains the minimum in the above expression. Since Q is a bijection from $\mathbb{C}^{n-1}$ to S^*, it is surjective from $\mathbb{C}^{n-1}$ onto S^* and therefore surjective onto S as well, since $S \subset S^*$. Hence, we can take $y \in \mathbb{C}^{n-1}$ such that $x_\alpha = Qy$, by which it follows that $x_\alpha^* = y^* Q^T$. By definition, $y^* Q^T LQy \in W_r(L)$, so $y^* Q^T LQy = x^* Lx = \alpha \in W_r(L)$ as well.

Since it is possible that a complex vector x produces the minimum real part of $W_r(L)$, we must show that a real vector in S can produce the minimum real part of $W_r(L)$.

We begin with [15, Theorem 9], which states that if we take the Hermitian part of our restricted Laplacian matrix Q^TLQ,

$$Q^TLQ = \frac{Q^TLQ + Q^TL^TQ}{2} + i\frac{Q^TLQ - Q^TL^TQ}{2i} = H_1 + iH_2$$

where H_1 and H_2 are Hermitian, and iH_2 is anti-Hermitian (so $\sigma(H_1), W(H_1)$, $\sigma(H_2), W(H_2) \in \mathbb{R}$ by Propositions 1.1 and 1.8), then $\min(\sigma(H_1)) =$ $\min \mathrm{Re}(W(A))$. Let $a = \min(\sigma(H_1))$ and let v be a normalized eigenvector of H_1 associated with a. Since H_1 and a are real, we can assume that v is real as well. See that for a restricted Laplacian matrix Q^TLQ,

$$\begin{aligned} v^*Q^TLQv &= v^*(H_1 + iH_2)v \\ &= v^*H_1v + iv^*H_2v \\ &= v^*va + i(v^*H_2v) \\ &= a + i(v^*H_2v). \end{aligned}$$

Since v and Q^TLQ are real, it follows that v^*Q^TLQv is real as well. By [15], we know that v^*H_2v is real, which implies that $i(v^*H_2v)$ is entirely imaginary. Hence $i(v^*H_2v)$ must be zero, since the left hand side is entirely real and has no imaginary part. Thus, we see that $v^*Q^TLQv = a$, where v is real.

So in our Laplacian case, the vector that generates the minimum real part in $W_r(L)$ has only real entries. Thus we have found a real vector $v \in S$ such that v produces $a = \min \mathrm{Re}(W_r(L))$.

Let y be a vector that minimizes x^TLx over $x \in S$. Since Q is surjective from $\mathbb{C}^{n-1}$ onto S as mentioned above, it follows that there exists some w such that $Qw = y$ and $w^*Q^T = y^*$. Thus, $w^*Q^TLQw = y^*Ly$ minimizes x^TLx over $x \in S$.

Since we know that the real v we constructed minimizes x^*Q^TLQx over $x \in S$, we know that $v^TQ^TLQv \leq w^*Q^TLQw = y^*Ly$. But y was chosen to minimize x^TLx over $x \in S$, and it is certain that $Qv \in S$ since both Q and v are real, we know that the inequality is in fact an equality, so $v^TQ^TLQv = w^*Q^TLQw = y^*Ly$. We now see that for real $Qv \in S$, we know $v^TQ^TLQv = a$ is the minimum of x^*Q^TLQx over $x \in S$ as well, so $a = \min_{x \in S} x^TLx$. Hence, we have shown that $\min \mathrm{Re}(W_r(L)) = a = \min_{x \in S} x^TLx = \alpha$, as desired.

(e) Assume L is normal, so $L^*L = LL^*$. Then

$$\begin{aligned}(Q^TLQ)^*Q^TLQ &= Q^*L^*(Q^T)^*Q^TLQ \\ &= Q^TL^TQQ^TLQ \quad \text{(we know } L \text{ and } Q \text{ are real).}\end{aligned}$$

Since each entry in QQ^T is the product of the orthogonal columns, they will equal 1 when they are the same column and zero otherwise. Since Q is an $n \times (n-1)$ matrix and Q^T is an $(n-1) \times n$ matrix, QQ^T will be an $n \times n$ matrix with ones along the diagonal and zeros elsewhere. That is, $QQ^I = I_n$. Using this, we can continue the calculation to see that

$$\begin{aligned}Q^TL^*QQ^TLQ &= Q^TL^*LQ \\ &= Q^TLL^*Q \quad \text{because } L \text{ is normal} \\ &= Q^TLIL^*Q \\ &= Q^TLQQ^TL^*Q \\ &= Q^TLQ(Q^TL^*Q) \\ &= Q^TLQ(Q^TLQ)^*.\end{aligned}$$

Hence, $(Q^TLQ)^*Q^TLQ = Q^TLQ(Q^TLQ)^*$, so Q^TLQ is normal, as desired.

□

Example 9. As an example of Proposition 2.2(d), we will reproduce a graph from Table 1 of [14], which provides the algebraic connectivity of directed cycles. In Figure

2.1, we see the restricted numerical range of the C_6. By Wu's calculation for a cycle on n vertices, $\alpha = 2\sin^2(\frac{\pi}{n}) = 2\sin^2(\frac{\pi}{6}) = \frac{2}{4} = \frac{1}{2}$. We see in Figure 2.1 that the minimum real part of the numerical range is 0.5, as desired.

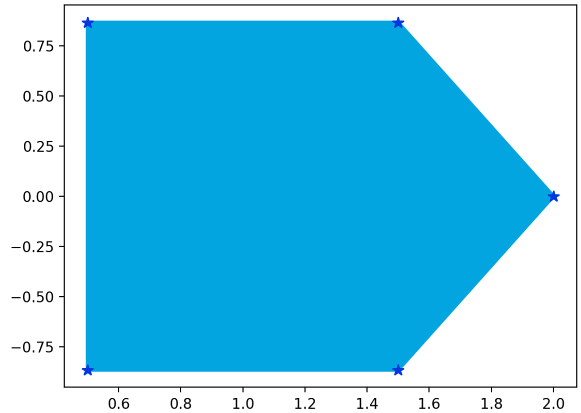


Figure 2.1: Restricted numerical range of the 6-cycle, eigenvalues in blue.

$\Diamond$

2.2 Basic Characterizations

In the spirit of Wu's generalization of Fiedler's algebraic connectivity from undirected to both directed and undirected graphs, we further generalize the ability to characterize graphs by their Laplacian matrix through the numerical range. We begin by presenting three full (if and only if) characterizations of empty, complete, and cycle graphs by the restricted numerical range.

2.2.1 Empty Graph

For the empty graph, we have the following complete characterization.

Theorem 2.3. *A graph G is the empty graph E_n if and only if $W_r(L(G)) = \{0\}$.*

Proof. For the empty graph E_n, we see that L is the zero matrix. Thus, the restricted numerical range of E_n is $W_r(L(E_n)) = W_r(0) = \{x^*0x\} = \{0\}$.

Now, given that $W_r(L) = \{0\}$, we wish to show that L is the zero matrix, which implies that it is the Laplacian of E_n.

We can use Proposition 1.1 to conclude that all eigenvalues of $Q^T LQ$ are zero since $\sigma(Q^T LQ) \subseteq W(Q^T LQ)$. By Proposition 2.2(b), we know $\sigma(L) = \{0\} \cup \sigma(Q^T LQ)$, so $\sigma(L) = \{0\}$ as well. If we consider the trace of L, we see that it is zero, since a matrix's trace is equal to the sum of its eigenvalues. However, we also know that the trace is the sum of the diagonal entries, and so it is the sum of the out-degrees. Since the out-degrees are all greater than zero and sum to zero, we can conclude they are all zero. Since the out-degree of every vertex is zero, G has no edges, so it is the empty graph.

Hence, we have a complete characterization for the empty graph: G is the empty graph if and only if $W_r(L(G)) = \{0\}$. $\square$

Example 10. We will use the empty graph on four vertices as our example. The Laplacian of the empty graph is the zero matrix. We can see the restricted numerical range of the zero matrix is simply the singleton zero in Figure 2.2. Conversely, if we are given a numerical range of a singleton zero that corresponds to a graph on four vertices, we know all three eigenvalues in the restricted numerical range must be zero, so we know that the eigenvalues of L must be zero with multiplicity four by Proposition 2.2(c). Once we see all the eigenvalues are zero, we know that the Laplacian is the 4×4 zero matrix, which corresponds to the empty graph. $\diamondsuit$

2.2.2 Complete Graph

We will prove the following analogous theorem for the complete graph.

Theorem 2.4. *A graph $G = K_n$ if and only if $W_r(L(G)) = \{n\}$.*

Proof. For K_n, we have a symmetric graph Laplacian L, so $(Q^T LQ)^T = Q^T L^T Q = Q^T LQ$, which shows that $Q^T LQ$ is also symmetric. Since $Q^T LQ$ is symmetric, it is clearly Hermitian and normal, so Proposition 1.8 and Proposition 2.2(a) tells us that $W(Q^T LQ) = W_r(L) \subseteq \mathbb{R}$ and Proposition 1.12 tells us that $W(Q^T LQ)$ is the convex hull of its spectrum.

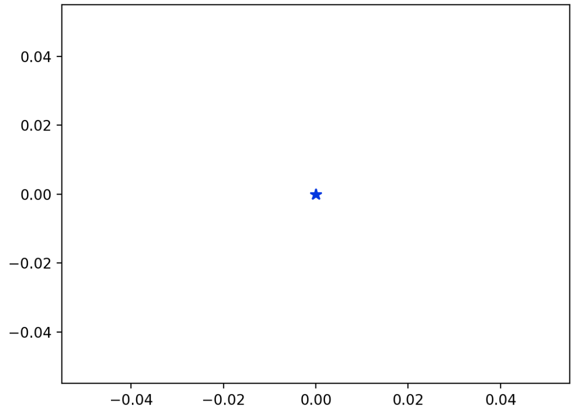


Figure 2.2: The restricted numerical range of the empty graph on four vertices consists of the value zero.

We can directly analyze the Laplacian to see that

$$L(K_n) = \begin{bmatrix} n-1 & -1 & -1\dots & -1 \\ -1 & n-1 & -1\dots & -1 \\ \vdots & \dots & \dots & \vdots \\ -1 & \dots & \dots & n-1 \end{bmatrix} = nI - J,$$

where J is the all ones matrix. Recall that the characteristic polynomial of a matrix A is the determinant of $A - \lambda I$. The determinant of $-J - \lambda I$ is $-(\lambda + n)\lambda^{n-1}$. Since the roots of a matrix's characteristic polynomial are its eigenvalues, we see the eigenvalues of $-J$ are 0 with multiplicity $n-1$ and $-n$ with multiplicity 1. To find the eigenvalues of $L = nI - J$, we simply add n to the eigenvalues of $-J$, so we see that the eigenvalues of $L(K_n)$ are $n = 0 + n$ with (algebraic) multiplicity $n-1$ and $0 = n - n$ with (algebraic) multiplicity 1. By Proposition 2.2(c), we know that $\sigma(Q^T LQ) = \sigma(L) \setminus \{0\}$, so $\sigma(Q^T LQ)$ contains n with multiplicity $n-1$. Since we already established that $W_r(L)$ is the convex hull of $\sigma(Q^T LQ)$, we can conclude that $W_r(L(K_n)) = \{n\}$.

Going in the other direction, we assume that the restricted numerical range of a graph on n vertices is the single point n. We know that $\sigma(Q^T LQ) \subseteq W(Q^T LQ)$ by Proposition 1.1, so $\sigma(Q^T LQ) = \{n\}$.

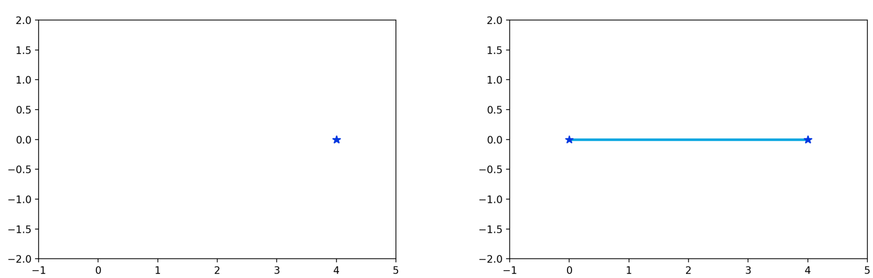
We can use Proposition 2.2(c) again to see that $\sigma(L) = \{0\} \cup \sigma(Q^T LQ)$, so $\sigma(L) =$

$\{0, n\}$, where 0 has algebraic multiplicity 1 and n has algebraic multiplicity $n-1$. This is illustrated in our example in Figure 2.3(b) by the fact that the unrestricted numerical range is a real segment between 0 and $n = 4$.

Using the fact that the trace of a matrix must equal the sum of its eigenvalues, we can see that $\text{Tr}(L(K_n)) = \sum_{i=1}^{n} \lambda_i = \left(\sum_{i=1}^{n-1} n\right) + 0 = n(n-1)$.

Considering that the maximum value of each diagonal element of L is $n-1$ (since L is a simple graph), we see that the trace of L reaches a maximum of $n(n-1)$ only when all n of the diagonal entries are $n-1$. Since we have shown that the trace is in fact equal to that maximum, we know L has $n-1$ in every diagonal entry, so each vertex in G has out-degree $n-1$. Since we have no multi-edges, this implies that each vertex is neighbors with every other vertex, so $G = K_n$. □

Example 11. We show the restricted numerical range of the complete graph on four vertices in Figure 2.3, which is a single point, 4, as expected.



(a) $W_r(L(K_4)) = \{4\}$

(b) $W(L(K_4)) = [0, 4]$

Figure 2.3: Restricted and unrestricted numerical range of the complete graph on four vertices

◊

2.2.3 Cycle

We will also prove a complete characterization for the cycle. Recall that we first defined the *convex hull* of a set of complex numbers in Definition 1.11.

Theorem 2.5. *A graph G is a cycle if and only if its restricted numerical range is a complex polygon with vertices*

$$\{1 - e^{i2\pi j/n} : j = 1, \ldots, n-1\}$$

Proof. We consider the cycle graph C_n, which has a circulant Laplacian matrix which we will call L_c. A *circulant* matrix is of the form

$$\begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix},$$

and is therefore wholly determined by any one of its columns. The eigenvalues of a circulant matrix are $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$, where

$$\lambda_j = \sum_{k=0}^{n-1} c_k p_j^{n-k},$$

$p_j = e^{i\frac{2\pi j}{n}}$ is one of the n^{th} roots of unity, and c_k is the entry in the first column and $(k-1)^{\text{th}}$ row of the circulant matrix [7, Section 3.1]. After a possible relabeling of the vertices so that $v_n v_{n+1} \in E(G)$, our circulant matrix L_c will have first column vector $c = [c_0, c_2, c_3, \ldots, c_{n-1}] = [1, 0, 0 \ldots, 0, -1]$ because every vertex in a cycle has out-degree 1. Using this, we see that

$$L_c = \begin{bmatrix} 1 & -1 & 0 & & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 \\ \vdots & & & \ddots & \ddots & -1 \\ -1 & 0 & & \dots & 0 & 1 \end{bmatrix}$$

so we can be specific about our eigenvalues, using the large number of zeros to simplify our sum:

$$\lambda_j = \sum_{k=0}^{n-1} c_k p_j^{n-k} = c_0 p_j^n + c_{n-1} p_j^{n-(n-1)} = 1 - p_j \tag{2.1}$$

Since $p_j \neq p_k$ for distinct $j, k \in \{0, 1, \ldots, n-1\}$, we have n distinct eigenvalues, which implies that there is a basis of n linearly independent eigenvectors. By [8, Section 2.2], these eigenvectors are orthogonal, so L is a normal matrix. This implies that $Q^T LQ$ is normal for any Q matrix by Proposition 2.2(e), so $W_r(L_c)$ equals the convex hull of the eigenvalues of $Q^T L_c Q$ by Proposition 1.12. Lastly, Proposition 2.2(c) implies that $\sigma(L) \setminus \{0\} = \sigma(Q^T LQ)$, so we see that $\sigma(Q^T LQ) = \{1 - p_j : j \in \{1, \ldots, n-1\}\}$.

This proves one half of our characterization: a cycle C_n has a restricted numerical range that is the (polygonal) convex hull of the set $\{1 - e^{i2\pi j/n} : j \in \{1, \ldots, n-1\}\}$. Now, we will show that if the restricted numerical range of $L(G)$ has such a form, then G is a cycle.

To begin, we assume that $W_r(L)$ is a complex polygon with vertices

$$\{1 - e^{i\frac{2\pi j}{n}} : j \in \{1, \ldots, n-1\}\}.$$

By [15, Theorem 13] and Proposition 2.2(a), every corner on the boundary of $W_r(L)$ is an eigenvalue of $Q^T LQ$, so the values in the above expression are all eigenvalues of L. This shows that the spectrum of L satisfies (2.1).

We claim and will prove that while generally $L = D - A$, under our assumption $D = I$, so $L = I - A$. To this end, assume toward a contradiction that $d^+(v_i) = 0$ for some $v_i \in V$. Without loss of generality, assume $d^+(v_n) = 0$, which implies v_n is itself a strongly connected component, so L has Frobenius normal form

$$L = \begin{bmatrix} L_1 & L_{12} & \ldots & L_{1r} \\ & L_2 & \ldots & L_{2r} \\ & & \ddots & \vdots \\ & & & L_r \end{bmatrix}$$

in which L_r, the block corresponding to v_n, is 0.

Since the determinant of a block triangular matrix is the product of the determinants of the diagonal block matrices, the characteristic polynomial of L is equal to the product of the characteristic polynomials of the block matrices, so the eigenvalues of L are precisely the union of the eigenvalues of all the L_k blocks [8, Section 0.9.4]. We would like to show that each of the L_k matrices, except $L_r = 0$, are M-matrices, as introduced at the end of Section 1.2.1, and that they are also nonsingular. For the remainder of this proof, let L_k be one of the diagonal block matrices that is not L_r.

First, from the definition of the Laplacian, we know that all the nonzero off-diagonal elements are negative, so L_k is clearly a Z-matrix. We know that in Equation 2.1, λ_j is complex except possibly for two values of p_j. When $p_j = 1$, then $\lambda_j = 0$ and when $p_j = -1$, then $\lambda_j = 2$. We know that the 0 eigenvalue is in the bottom right hand corner of the matrix, so if an eigenvalue of one of the L_k blocks is not complex, then it is 2. Since this amounts to saying that every real eigenvalue of every L_k is positive, we see that L_k is an nonsingular M-matrix [1, Theorem 6.2.3 D_{16}]. Because we showed that the diagonal blocks are irreducible in Section 1.2.1, we can now say L_k is an irreducible nonsingular M-matrix.

Since L_k is an M-matrix, L_k^{-1} has all positive entries, and since L_k is irreducible, L_k^{-1} is irreducible as well [1, Theorem 6.2.3 N_{38}]. Since L_k^{-1} has all positive entries and is irreducible, we can apply the Perron–Frobenius theorem [1, Theorem 2.1.4(b)] to say that the spectral radius $\rho(L_k^{-1})$, which is defined to be the maximum complex modulus among the eigenvalues of a matrix, is an eigenvalue. We know that every eigenvalue of L is an eigenvalue of L_k for some $1 \leq k \leq n-1$, so take L_m to be the block that has eigenvalue

$$p_{\text{min}} = 1 - e^{i\frac{2\pi}{n}},$$

which is smallest (in terms of complex modulus) eigenvalue of L other than zero. In our case, p_{min} can be taken to be either λ_1 or λ_{n-1}. Recall that λ is an eigenvalue of A if and only if, λ^{-1} is an eigenvalue of A^{-1}. By this property, since p_{min} is the smallest eigenvalue of L_k, we know p_{min}^{-1} will be the largest eigenvalue of L_k^{-1} in terms of complex modulus, so we can conclude that $\rho(L_k^{-1}) = |p_{\text{min}}^{-1}|$. Using this relationship between inverses and eigenvalues again, we see that if $|p_{\text{min}}^{-1}|$ is an eigenvalue of L_k^{-1}, then $|p_{\text{min}}|$ must be an eigenvalue of L_k, and hence an eigenvalue of L. However, this purely real eigenvalue p_{min} is not equal to zero, and if it were 2, then $p_{\text{min}} = \frac{1}{2}$, which would require n in $1-e^{i\frac{2\pi}{n}}$ to be complex, which is a contradiction. Hence, we have a real eigenvalue in $\sigma(L)$ which is not 0 or 2, which is a contradiction to our original assumption that $\sigma(L)$ satisfied Equation (2.1). We can then say that our assumption that there exists a vertex with out-degree equal to 0 is false.

To use this fact to show that $D = I$, note that if we take the sum of the eigenvalues described in Equation (2.1) we see that for even n, $p_j + p_{n-j} = 0$ when j is not $\frac{n}{2}$ or 0. However, $p_{\frac{n}{2}} + p_0 = 0$, so we see the roots of unity sum to zero, so

$$\begin{aligned}\sum_{j=0}^{n-1} \lambda_j &= \sum_{j=0}^{n-1}(1-p_j)\\ &= \sum_{j=0}^{n-1} 1 - \sum_{j=0}^{n-1} p_j\\ &= n - 0 = n.\end{aligned}$$

So we have shown that when n is even, $\sum\limits_{\lambda\in\sigma(L)} \lambda = n$. If n is odd, then we can write the sum of the eigenvalues as

$$\begin{aligned}\sum_{j=0}^{n-1}(1-p_j) &= \sum_{j=0}^{n-1} 1 - \sum_{j=0}^{n-1} p_j\\ &= n - \left(e^{i\frac{2\pi(0)}{n-1}} + \sum_{j=1}^{n-1} e^{i\frac{2\pi j}{n-1}}\right)\\ &= n - 1 - e^{\left(\sum\limits_{j=1}^{n-1} j\right)\frac{2\pi}{n-1}i}\\ &= n - 1 - e^{\frac{(n-1)(n)}{2}\frac{2\pi}{n-1}i} = n - 1 - e^{n\pi i}\\ &= n - 1 - (-1) \quad \text{(since n is odd)}\\ &= n,\end{aligned}$$

so in either case, we see that the sum of the eigenvalues of L is n, so $\mathrm{Tr}(L) = n$ as well.

By the structure of the Laplacian, we know that all of its diagonal entries are integers, and we have laboriously shown that no vertices have out-degree 0, so none of its diagonal elements are 0. Hence, if the n diagonal entries are to sum to n, and all of them are integers and greater than zero, it is clear that each $d^+(v_i) = 1 = L_{ii}$ for all $1 \le i \le n$. This shows that $D = I$.

We can now say that $L = I - A$, and we recall that

$$\sigma(L) = \{1 - p_j : j \in \{0, 1, \dots, n-1\}\}.$$

It clearly follows that $A = I - L$, so the eigenvalues of A must be of the form $1 - (1 - p_j) = p_j$, so $\sigma(A) = \{p_j : j \in \{0, 1 \dots, n-1\}\}$. When $j = 0$ we know A has eigenvalue 1 corresponding to the all ones eigenvector e, since $(I - L)e = Ie - Le = e$. The only other possible real eigenvalue for A is -1 when $j = \frac{n}{2}$, which causes $p_j = -1$. However, since A has all nonnegative entries, a negative eigenvalue must correspond to a eigenvector with at least one negative entry. Thus, A has one and only one real nonnegative eigenvector e, every entry of which is positive, so A is irreducible by [1, Theorem 2.1.3].

Since A is irreducible, G must be a strongly connected graph [8, Theorem 6.2.24(d)]. Thus each vertex in G has out-degree 1 and G is strongly connected.

Let $P = (v_1, v_2, \dots, v_k)$ be a maximal directed path in G, that is, a path such that no additional vertices in $V(G)$ can be added to the P to increase its length. Now consider which vertices are neighbors to the last vertex in the path, v_k. Since each vertex in G has out-degree 1, we know v_k must have one neighbor, and this neighbor must be in P, otherwise P would not be maximal. Since G is strongly connected, there must be a path from v_k to v_1, and we call this path $P' = (v_k, \dots, v_0, v_1)$. All the vertices in P (and therefore in G) other than v_k have a neighbor already that is not v_1, so v_1 is not a neighbor to any of them. Hence, v_0 cannot be any vertex in P other than v_k, so $P' = (v_k, v_1)$, and $v_k v_1 \in E(G)$.

Now, suppose toward a contradiction there exists a vertex $u \in G$ such that u is not on P. Since G is connected, we know there exists a path $P' = (u_1, u_2, \dots, u_t)$ with $u_1 = v_1$ and $u_t = u$. Since $u \notin P$, there must be a $u_i \in P \cap P'$ such that its neighbor, u_{i+1}, is in P' but not in P. However, we have already shown that every vertex on P has an edge to another vertex on P, and we know the out-degree of each vertex is 1. Hence, $u_{i+1} \in P$, which is a contradiction, so we see that every vertex in G is on P.

By labeling the vertices in the order the appear in P, we see $E(G) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$, so G is a cycle on n vertices. $\square$

Example 12.

We will continue in our series of examples on four vertices with a cycle on four vertices, as shown below in Figure 2.4. If we examine the numerical range and re-

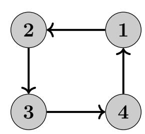


Figure 2.4: Cycle on four vertices

stricted numerical range in Figure 2.5, we see that the vertices of the numerical range on the right are $0, 1-i, 2$ and $1+i$. Recall that $1, i, -1$ and $-i$ are the fourth roots of unity p_0, p_1, p_2, p_3, and see that we can rewrite the vertices as

$$\begin{aligned} 0 &= 1 - 1 = 1 - p_0 \\ 1 - i &= 1 - i = 1 - p_1 \\ 2 &= 1 - (-1) = 1 - p_2 \\ 1 + i &= 1 - (-i) = 1 - p_3, \end{aligned}$$

as we described them in Theorem 2.5.

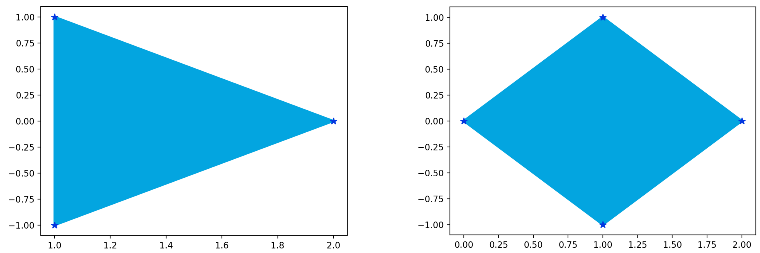


Figure 2.5: Restricted numerical range of C_4 on left, unrestricted numerical range of C_4 on right

Observe also that when we remove the vertex $1 - p_1 = 0$ of the polygon in the unrestricted numerical range from our set of eigenvalues, the convex hull of the remaining eigenvalues forms the restricted numerical range, shown in the left of Figure 2.5, which has vertices $1 - i = 1 - p_1, 2 = 1 - p_2$ and lastly $1 + i = 1 - p_3$.

$\Diamond$

CHAPTER 3

REAL RESTRICTED NUMERICAL RANGE

3.1 Singleton

We define the (noncommutative) *directed join* operator on two graphs G and H as $G\vec{\nabla}H = (V, E)$ where $V = V(G) \cup V(H)$, provided that the vertex labels on $V(G)$ and $V(H)$ have been labeled to have no overlap, so $|V(G) \cup V(H)| = |V(G)| + |V(H)|$. The edge set E of $G\vec{\nabla}H$ is then

$$E = E(G) \cup E(H) \cup \{vu : v \in V(G), u \in V(H)\}.$$

We call $G\vec{\nabla}H = (V, E)$ the directed join of G onto H. We define a *k-imploding star* on n vertices for $1 \leq k < n$ as $S_{n,k} = E_{n-k}\vec{\nabla}K_k$. That is, the directed join of the $n-k$ vertex empty graph onto the k vertex complete graph. See below for Example 13 of 1-imploding and 2-imploding stars on 6 vertices.

Example 13. Examine $S_{6,1} = E_5\vec{\nabla}K_1$, the 1-imploding star on six vertices, and $S_{6,2} = E_4\vec{\nabla}K_2$, the 2-imploding star on six vertices.

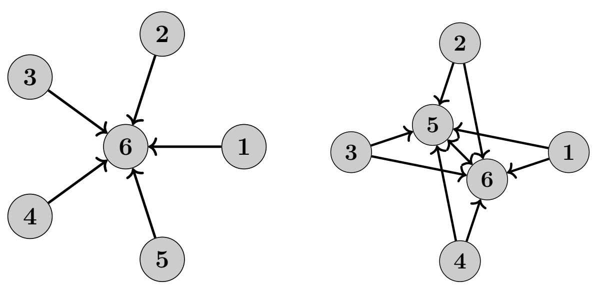


Figure 3.1: For six vertices, we have the the 1-imploding star on the left, and the 2-imploding star on the right.

We can also begin to investigate the structure of the Laplacian for imploding stars by looking at

$$L(S_{6,1}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad L(S_{6,2}) = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

$\Diamond$

Recalling the definitions of $\alpha(G)$ and $\beta(G)$, we will use Lemma 1 in [14], which states that for two graphs G and H where $V(G) = V(H)$,

$$\alpha(G) + \alpha(H) \leq \alpha(G \cup H) \leq \beta(G \cup H) \leq \beta(G) + \beta(H) \tag{3.1}$$

where $\cup$ is the edge union, defined as

$$V(G \cup H) = V(G) = V(H) \text{ and } E(G \cup H) = E(G) \cup E(H).$$

We will use Equation (3.1) to prove Theorem 3.1 by induction.

Theorem 3.1. *For an imploding star $S_{n,k}, W_r(S_{n,k}) = \{k\}$.*

Proof. We take

$$P(k) := \text{The } k\text{-imploding star has a singleton numerical range of } \{k\}$$

as our inductive statement, and see that for a base case of $k = 0$, we have an empty graph, whose restricted numerical range we have already shown to be zero in Subsection 2.2.1. We will also show directly that $P(1)$ is true.

For a 1-imploding star on n vertices, we have $Q^T LQ = [q_i^T Lq_j]_{i,j=1}^n$ by matrix multiplication, where each q_i and q_j is a column vector of Q. We know the Laplacian form of a 1-imploding star is

$$L = \begin{bmatrix} 1 & 0 & \dots & & -1 \\ & 1 & \dots & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \\ & & & & 0 \end{bmatrix}.$$

Recalling that we can multiply an $n \times 1$ row vector by a $1 \times n$ column vector to produce an $n \times n$ matrix using standard matrix multiplication, we can write $L = I - ee_n^T$, where e is again the all ones vector and e_n is the n^{th} standard basis vector, $e_n = (0, 0, \dots, 0, 1)$.

Using this formulation, we see $q_i^T L = q_i^T(I - ee_n^T) = q_i^T - q_i^T ee_n^T = q_i^T - 0 = q_i^T$, since we know that the columns of Q are defined to be perpendicular to e. Hence it follows that $q_i^T L q_j = q_i^T q_j$. Because Q is an orthonormal matrix, we know $q_i^T q_j$ is 0 when $i \neq j$ and 1 when $i = j$, so $Q^T Q_j = I$. Having shown this, we can say that $W_r(L) = x^* Q^T L Q x = x^* I x = x^* x = 1$ for any unit $x \in \mathbb{C}^{n-1}$, so we conclude that the restricted numerical range of the 1-imploding star is $\{1\}$.

To show the inductive step, we assume $P(k)$, so we know that a k-imploding star on n vertices $S_{n,k}$ has a restricted numerical range $\{k\}$.

For a given star $S_{n,k}$, which has vertex set $V(K_k) \cup V(E_{n-k})$, we will call the K_k subgraph (the vertices being imploded upon) the center, and E_{n-k} (the imploding vertices) the outside.

We will first label $V(S_{n,1}) = V(K_1) \cup V(E_{n-1})$. Let the center of $S_{n,1}$, the vertex in K_1 be labeled 1, and let $n-1$ vertices the outside of $S_{n,1}$ in $V(E_{n-1})$ be labeled $\{2, \ldots, n\}$. Second, we will label $S_{n,k} = V(K_k) \cup V(E_{n-k})$. Let the vertices in $V(E_{e-k})$ on the outside of $S_{n,k}$ be labeled $\{1, \ldots, n-k\}$, and let the vertices in the center of $S_{n,k}$ in $V(K_k)$ be labeled $\{n-k+1, \ldots, n\}$.

We consider the graph G which is the edge union of the k-imploding star $S_{n,k}$ and the 1-imploding star $S_{n,1}$,

$$G = S_{n,k} \cup S_{n,1}$$

and we claim that with our vertex labelings G is isomorphic to $S_{n,k+1}$.

Figure 3.2 gives an example of the general principle of our argument that $S_{n,k+1}$ is isomorphic to $S_{n,k} \cup S_{n,1}$.

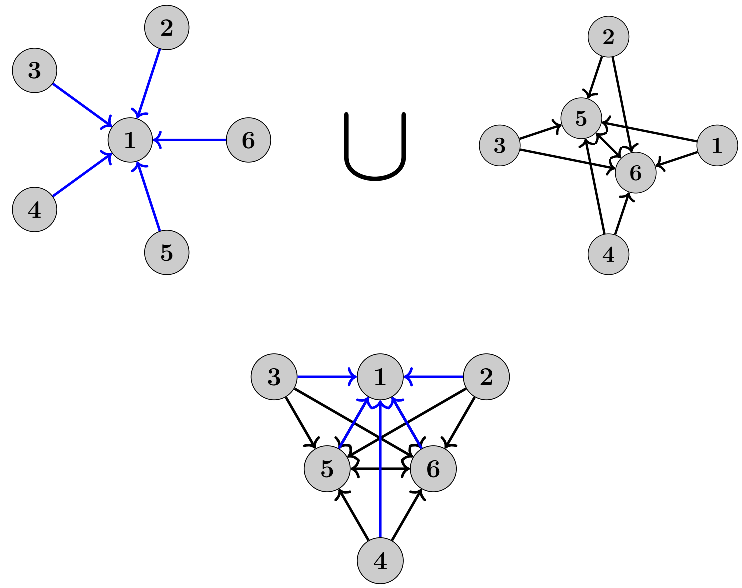


Figure 3.2: Edge union of 1-imploding star and 2-imploding star produces the graph below them, which is the 3-imploding star.

First, we will show that the vertices of G with labels $\{1, n-k+1, n-k+2, \ldots, n\}$ form K_{k+1}. For distinct $i, j \in \{1, n-k+1, n-k+2\}$, if (without loss of generality) $j > i \geq n-k+1$, then $ij, ji \in E(G)$ because ij and ji were in the center $V(K_k)$ of $S_{n,k}$. The only other possible case is when, without loss of generality, $i = 1, j \geq n-k+1$. In this case j is in $V(E_{n-1})$, the outside of $S_{n,1}$, so $ji \in E(G)$. In addition, i is in $V(E_{n-k})$, the outside of $S_{n,k}$, and j is in the center of $S_{n,k}$, so $ij \in E(G)$ as well. Hence every vertex in $\{1, n-k+1, n-k+2, \ldots, n\}$ has an edge to every other vertex, so it is the complete graph K_{k+1}.

Now consider the vertices with labels $\{2, 3, \ldots, n-k\}$, which we will show is isomorphic to the empty graph E_{n-k-1}. Let $i, j \in \{2, 3, \ldots, n-k\}$. Then in $S_{n,1}$, we see $i, j \in V(E_{n-1})$, and in $S_{n,k}$, we similarly see $i, j \in V(E_{n-k})$. Since there are no edges in E_{n-1} or E_{n-k}, we can conclude that $ij, ji \notin E(G)$, so $\{2, 3, \ldots, n-k\}$ forms the empty graph E_{n-k-1} in G.

Lastly, let $i \in \{2, 3, \ldots, n-k\}$ and $j \in \{1, n-k+1, n-k+2, \ldots, n\}$. If $j \neq 1$, then $ij \in E(G)$ because i is in the outside of $S_{n,k}$ and j is in the center of $S_{n,k}$. If $j = 1$, then i is in the outside of $S_{n,1}$, and $j = 1$ is the center. In either case $ij \in E(G)$, so we see that G is the directed join of $\{2, 3, \ldots, n-k\}$ onto $\{1, n-k+1, n-k+2, \ldots, n\}$, so $G = E_{n-k-1} \vec{\nabla} K_{k+1} = S_{n,k+1}$. To complete our inductive step, we will show that the restricted numerical range of $L(G)$ is in fact $\{k+1\}$.

Note that by our inductive hypothesis, the restricted numerical range of $S_{n,k}$ is equal to $\{k\}$, and we have directly shown as a base case that the restricted numerical range of $S_{n,1}$ is equal to $\{1\}$. Since by Proposition 2.2(d) α and β are the minimum and maximum real parts of the restricted numerical range, it is clear that $\alpha = \beta = k$ when $W_r(L) = \{k\}$. Using Equation 3.1, we know that

$$\begin{aligned} \alpha(S_{n,k}) + \alpha(S_{n,1}) \leq \alpha(S_{n,k} \cup S_{n,1}) \leq \beta(S_{n,k} \cup S_{n,1}) \leq \beta(S_{n,k}) + \beta(S_{n,1}) \\ k+1 \leq \alpha(S_{n,k+1}) \leq \beta(S_{n,k+1}) \leq k+1 \\ \alpha(S_{n,k+1}) = k+1 = \beta(S_{n,k+1}). \end{aligned}$$

Again using the fact that α and β are lower and upper bounds on the real part of the restricted numerical range, we see that the real part of every element of $W_r(S_{n,k+1})$ is $k+1$. Could some of these elements also have a nonzero complex part?

We can examine the structure of L directly to determine its eigenvalues. We see that each of the $n-(k+1)$ vertices on the outside of $S_{n,k+1}$ has 0 in-degree, so they each compose their own strongly connected component. The center is the complete graph K_{k+1}, so it is its own strongly connected component as well. Thus we have $n-(k+1)+1 = n-k$ strongly connected components, so if we place the matrix in Frobenius normal form, we will have $n-k$ blocks. We have previously shown that the eigenvalues of L are the union of the eigenvalues of its Frobenius normal form block matrices. The $n-(k+1)$ singleton blocks of L are each the single value $k+1$, so the eigenvalue of each of these singletons is clearly $k+1$. From the proof of Theorem 2.4, recall that $\sigma(L(K_n)) = \{n, 0\}$, so the block corresponding to the center K_{k+1} has spectrum $\{k+1, 0\}$. Hence, L has the eigenvalue $k+1$ with multiplicity k and eigenvalue 0 with multiplicity 1, so by Proposition 2.2(c), we know $\sigma(Q^T LQ) = \{k+1\}$.

Using [10, Theorem 2.1], we know that if a value c is a corner of $W(A)$ for some matrix A, then c is an eigenvalue of A. Now suppose toward a contradiction that $W_r(L)$ is not a single real point, in which case it must be a complex line. The largest and smallest complex value in this line is clearly a corner of $W_r(L)$, so this complex number must be an eigenvalue of $Q^T LQ$. However, we have shown that $\sigma(Q^T LQ) = \{k+1\} \subset \mathbb{R}$, so we have reached a contradiction. Hence, $W_r(L) = \{k+1\}$, so $P(k+1)$ is true and the proof is complete by induction. $\square$

We have shown that a k-imploding star has restricted numerical range $\{k\}$. We will now prove the converse in Theorem 3.2 — a singleton restricted numerical range implies an imploding star graph.

Theorem 3.2. *For $k \leq n$ If $W_r(L(G)) = \{k\}$, where G is a graph on n vertices, then $G = S_{n,k}$.*

Proof. Suppose that G is a graph on n vertices whose Laplacian satisfies $W_r(L) = \{k\}$. We choose a useful $x \perp e, x = \frac{1}{\sqrt{2}}(e_i - e_j)$, where e_i and e_j are standard basis vectors for $\mathbb{R}^n$. We see that

$$\begin{aligned} ||x||^2 = \langle x, x \rangle &= \left\langle \frac{1}{\sqrt{2}}(e_i - e_j), \frac{1}{\sqrt{2}}(e_i - e_j) \right\rangle \\ &= \frac{1}{2}(e_i^T e_i - 2e_i^T e_j + e_j^T e_j) \\ &= \frac{1}{2}(1 - 2(0) + 1) = 1, \end{aligned}$$

so x is of unit length. This fact, combined with the knowledge that $x \perp e$, tells us that $x^* Lx \in W_r(L)$, and since we know every element in $W_r(L)$ is equal to k, we know $x^* Lx = x^T Lx = k$. Now, denoting the i^{th} column of L as l_i and the ij^{th} entry of L as l_{ij}, we can then say that

$$\begin{aligned} k &= x^T L x \\ &= \left(\frac{1}{\sqrt{2}}(e_i - e_j)\right)^T L \left(\frac{1}{\sqrt{2}}(e_i - e_j)\right) \\ &= \frac{1}{2}(e_i - e_j)^T (Le_i - Le_j) \\ &= \frac{1}{2}(e_i - e_j)^T (l_i - l_j). \end{aligned}$$

By the distributive property of the inner product above, we may rewrite it as the four terms:

$$\begin{aligned} e_i^T l_i &= l_{ii} \\ e_i^T(-l_j) &= -l_{ij} \\ -e_j^T(l_i) &= -l_{ji} \\ -e_j^T(-l_j) &= l_{jj}. \end{aligned}$$

Hence, we see

$$k = \frac{1}{2}\left(l_{ii} + l_{jj} - l_{ij} - l_{ji}\right),$$

so we can write a useful relation between k and the entries of L,

$$2k = l_{ii} + l_{jj} - l_{ij} - l_{ji} \tag{3.2}$$

for $1 \leq i, j \leq n$ and $i \neq j$. We now state one of the equations in [14, Lemma 8]:

$$\alpha \leq \min_{v \in V}\left(d^+(v) + \frac{d^-(v)}{n-1}\right) \leq \max_{v \in V}\left(d^+(v) + \frac{d^-(v)}{n-1}\right) \leq \beta$$

Since the restricted numerical range is a singleton, we know that $\alpha = \beta = k$, so

$$k = \min_{v \in V}\left(d^+(v) + \frac{d^-(v)}{n-1}\right) = \max_{v \in V}\left(d^+(v) + \frac{d^-(v)}{n-1}\right).$$

Since the the minimum and maximum of the argument above are equal over $v \in V$, we can conclude that that they are identical for all v, so we know $d^+(v) + \frac{d^-(v)}{n-1} = k$ for all $v \in V$, which can be rewritten as

$$d^+(v) = k - \frac{d^-(v)}{n-1}$$

Since $d^+(v)$ and k are integers, we know that $\frac{d^-(v)}{n-1}$ must be an integer, so $(n-1)|d^-(v)$. Since G is a graph on n vertices, we know that $d^-(v) \leq n-1$, so since $(n-1)|d^-(v)$, we see that either $d^-(v) = n-1$ or $d^-(v) = 0$. Since $d^+(v)$ and $d^-(v)$ are directly related by the above equation, this implies that $d^+(v)$ must also be one of two values. When $d^-(v) = 0, d^+(v) = k$, and when $d^-(v) = n-1, d^+(v) = k-1$. Since each diagonal element of L is equal to the out-degree of a vertex, this implies that the diagonal entries l_{ii} of L must be either k or $k-1$. But for how many vertices is the out-degree k, and for how many is it $k-1$? Let s be the number of times it is k, which implies that the out-degree is $k-1$ exactly $n-s$ times.

By Proposition 1.1 we know that $\sigma(Q^T LQ) \subseteq W_r(L)$, so we see that $\sigma(Q^T LQ) = \{k\}$ since $W_r(L) = \{k\}$. By Proposition 2.2(c), we have $\sigma(L) = \{0\} \cup \sigma(Q^T LQ)$, so $\sigma(L) = \{k, 0\}$, where k has multiplicity $n-1$ and 0 has multiplicity 1. Recalling that the trace of a matrix is equal to the sum of its eigenvalues, we see that $\mathrm{Tr}(L) = k(n-1) + 0 = kn - k$. However, we assumed that there are s diagonal elements equal to k and $n-s$ diagonal elements equal to $k-1$. Since we can also calculate the trace by taking the sum of the diagonal elements, $\mathrm{Tr}(L) = sk + (n-s)(k-1) = sk + nk - sk - n + s = nk - n + s$. Equating these two expressions for the trace, we see that $nk - n + s = kn - k$, which implies that $s = n - k$, so $n - k$ vertices have out-degree k. If we relabel the vertices so that we number these $n-k$ vertices first, we can say that $l_{ii} = k$ for $1 \leq i \leq n-k$ and $l_{ii} = k-1$ for $n-k < i \leq n$. After this relabeling, we can write L in blocks as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

where L_{11} will be a $(n-k) \times (n-k)$ block corresponding to the vertices with out-degree k, and L_{22} will be a $k \times k$ block corresponding to the vertices with out-degree $k-1$. By definition of the Laplacian, the diagonal elements in L_{11} will be k, and the diagonal elements in L_{22} will be $k-1$. For $1 \leq i, j \leq n-k$ then, we see that since

$l_{ii} = k = l_{jj}$ so applying Equation (3.2) for such i and j tells us

$$2k = k + k - l_{ij} - l_{ji},$$

which implies that $l_{ij} = l_{ji} = 0$, since each must be either 0 or -1. Hence, we see that $L_{11} = kI_{n-k}$. Applying Equation (3.2) to the vertices in L_{22} tells us that

$$2k = 2(k-1) - l_{ij} - l_{ji} = 2k - 2 - l_{ij} - l_{ji},$$

so $l_{ij} + l_{ji} = -2$, which implies $l_{ij} = l_{ji} = -1$. This tells us that L_{22} has -1 on all the off-diagonal entries, so $L_{22} = L(K_k)$. By definition of the Laplacian, there are exactly as many -1 off diagonal entries as the value of the diagonal entry, and $L(K_k)$ has the entry -1 appear $k-1$ times in each row, so we know all other entries in those rows must be zero, which implies that L_{21} is the zero matrix. By a similar argument, we know that -1 must appear k times in the rows containing L_{11} and L_{22}, and since $L_{11} = kI$, we know that each row has all k of its -1 entries in L_{12}. But since L_{12} is a $(n-k) \times k$ matrix, it only has k columns, so every entry in L_{12} must be -1. To summarize, we have shown that $L_{11} = L(kI), L_{22} = L(K_k), L_{21} = 0_{k,n-k}$, and $L_{12} = -J_{n-k,k}$, so

$$L = \begin{bmatrix} kI_{n-k} & -J_{n-k,k} \\ 0_{k,n-k} & L(K_k) \end{bmatrix}$$

where we use the notation that $J_{n,m}$ is the all ones $n \times m$ matrix.

Thus, we have directly constructed $L = L(G)$ from our assumption that $W_r(L) = \{k\}$, and using $L(G)$ to construct G, we see that G is the k-imploding star. □

We can combine the results of Theorems 3.1 and 3.2 to state Theorem 3.3.

Theorem 3.3. *A graph G on n vertices is a k-imploding star if and only if $W_r(L(G)) = \{k\}$, where k is an integer between 0 and n.*

Note that this characterization generalizes Theorems 2.3 and 2.4, if we view the empty graph as the 0-imploding star and the complete graph as the n-imploding star.

Example 14. In Figures 3.3 and 3.4, we will examine the restricted and unrestricted numerical ranges of $S_{6,1}$ and $S_{6,2}$, first introduced in Example 13.

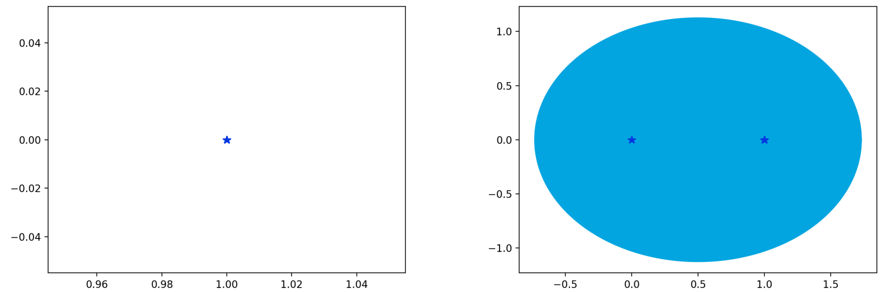


Figure 3.3: The restricted numerical range of $S_{6,1}$ is $\{1\}$, while the unrestricted numerical range is an ellipse, with eigenvalues at 0 and 1.

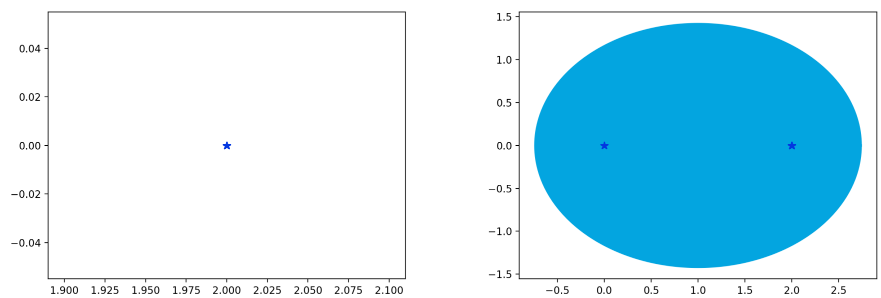


Figure 3.4: The restricted numerical range of $S_{6,2}$ is $\{2\}$, while the unrestricted numerical range is an ellipse, with eigenvalues at 0 and 2.

We see that the restricted numerical range of $S_{6,1}$ on the left of Figure 3.3 is precisely $\{1\}$, and the restricted numerical range of $S_{6,2}$ on the right of Figure 3.4 is $\{2\}$, as desired. $\diamondsuit$

Note that in Example 14, the unrestricted numerical ranges are ellipses. This relatively uninformative geometry reflects the inability of both the Laplacian spectrum and the unrestricted numerical range to characterize the k-imploding star, as Example 15 demonstrates.

Example 15. Consider $S_{6,3}$ and a graph we obtain by moving one edge in $S_{6,3}$, shown in Figure 3.5, which we will call $S^*_{6,3}$.

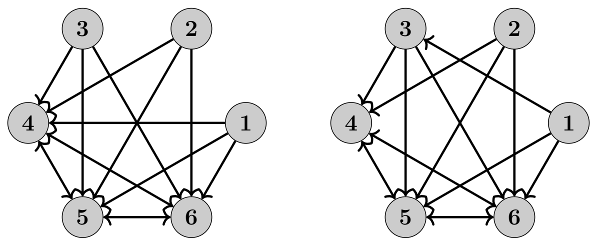
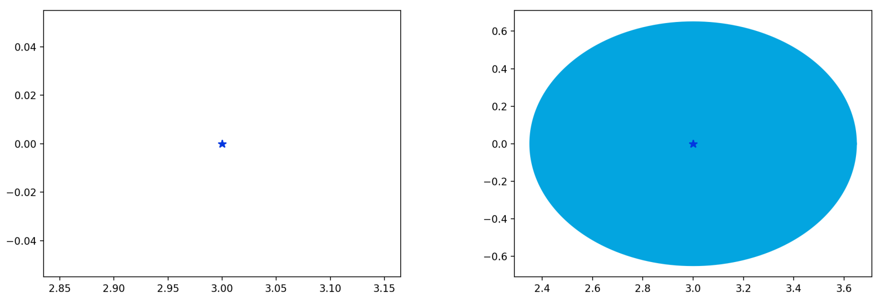


Figure 3.5: We obtain $S^*_{6,3}$ (right) from $S_{6,3}$ (left) by replacing $(1, 4)$ with $(1, 3)$.

We see that $S^*_{6,3}$ is not imploding star (and does not have singleton restricted numerical range), but is nevertheless cospectral with $S_{6,3}$, as we show in Figure 3.6.



(a) $W_r(L(S_{6,3}))$ (b) $W_r(L(S^*_{6,3}))$

Figure 3.6: A counterexample to $\sigma(L)$ characterizing k-imploding stars (eigenvalues in dark blue).

Thus, we see that though the spectrum fails to characterize the k-imploding star, its restricted numerical range does characterize it. $\Diamond$

In our characterization above, we assume that k is an integer. Theorem 3.5 strengthens Theorem 3.3. Theorem 3.5 also deals with *regular tournament graphs*, which we define in Definition 3.4.

Definition 3.4. *A graph G is a tournament if for each $i, j \in V(G)$, either $ij \in E(G)$ or $ji \in E(G)$ (but not both). Further, a regular tournament is a tournament in which for each $v \in V(G), d^-(v) = d^+(v)$.*

Theorem 3.5. *If the restricted numerical range of a graph G with n vertices contains only the singleton k, then k is an integer in $\{0, 1, \ldots, n\}$.*

Proof. Assume for a graph G and Laplacian $L = L(G)$, that $W_r(L) = \{k\}$. Since L and Q are real matrices, we know that $W(Q^T LQ) = W_r(L)$ must be symmetric with respect to the real axis by Proposition 1.9, so we can conclude that k is real. Since L is a general M-matrix (as we showed in Section 1.2.1), we know from [1, Theorem 6.4.6(E_{11})] that the real part of every eigenvalue of L must be non-negative. By Proposition 1.1, we know that k is an eigenvalue of L, so we know that $k \geq 0$.

We recall that the graph complement $\overline{G}$ of a graph G has the same vertex set as G and edge set $E(\overline{G})$ such that $v_i v_j \in E(\overline{G})$ if and only if $v_i v_j \notin E(G)$. Since $L(\overline{G})$ is also an M-matrix, we can use [1, Theorem 6.4.6(E_{11})] again to say that the real part of every eigenvalue of $L(\overline{G})$ must be positive. This implies that $\beta(\overline{G})$, which is the maximum real part of the $W_r(L(\overline{G}))$ by Proposition 2.2(d), must be positive as well, since $\sigma(L(\overline{G})) \subseteq W_r(L(\overline{G}))$. Since $\beta(\overline{G}) \geq 0$, we can use Lemma 4 of [14], which states that $\alpha(G) + \beta(\overline{G}) = n$, to conclude that $\alpha(G) \leq n$, which implies that $k \leq n$. Hence, we know that $0 \leq k \leq n$.

Assume toward a contradiction that k is not an integer. Since $\alpha = \beta = k$ as in the proof of Theorem 3.2, we can use the same reasoning as that proof to say that Lemma 8 of [14] again implies

$$d^+(v) = k - \frac{d^-(v)}{n-1}.$$

for every $v \in V$. Since $d^+(v)$ must be an integer, we know that $k - \frac{d^-(v)}{n-1}$ must be an integer. Further, since $0 \leq d^-(v) \leq n-1$, we know that $0 \leq \frac{d^-(v)}{n-1} \leq 1$, so we can conclude that $d^+(v) = \lfloor k \rfloor$ for all vertices.

Recall Equation 3.2 from the proof of Theorem 3.2, which is true because we have also assumed in this proof that $W_r(L) = \{k\}$. Reproducing Equation 3.2 below,

$$2k = l_{ii} + l_{jj} - l_{ij} - l_{ji}$$

we can use the fact that $l_{ii} = l_{jj} = \lfloor k \rfloor$ to see that $2k = 2\lfloor k \rfloor - l_{ij} - l_{ji}$, so $k = \lfloor k \rfloor - l_{ij} - l_{ji}$. Since k is not an integer, $\lfloor k \rfloor \neq k$, so we see that it is not possible for l_{ij} and l_{ji} to be zero. Further, if both l_{ij} and l_{ji} were -1, then $k - \lfloor k \rfloor = 2$, which is impossible by the definition of the floor function. Hence, we see that exactly one of l_{ij}, l_{ji} is -1. Further, since each row of L must sum to zero, the sum of all the entries in L must also be zero. Since exactly half of the off-diagonal entries of L are -1, the diagonal entries are equal to $\lfloor k \rfloor$. Thus, the diagonal entries contribute $n\lfloor k \rfloor$ to the overall sum, and the off diagonal entries, of which there are $n^2 - n = n(n-1)$, contribute $-\frac{n(n-1)}{2}$, so $n\lfloor k \rfloor - \frac{n(n-1)}{2} = 0$, which implies that $\lfloor k \rfloor = \frac{n-1}{2}$. Since we now know the value of $\lfloor k \rfloor$, we know the value of $d^+(v_i) = l_{ii} = \frac{(n-1)}{2}$ for all $v_i \in V(G)$, and we also see that n must be odd because $2|(n-1)$.

To recap, we have shown that every vertex in G has out-degree and in-degree equal to $\frac{(n-1)}{2} = \frac{1}{n}\binom{n}{2}$, and that for every pair of vertices $u, v \in V(G), uv \in E(G)$ if and only if $vu \notin E(G)$. Hence, since there is exactly one edge for every pair of two vertices, we see G is a tournament graph. Further, since the in and out-degrees of the vertices are uniform, G is a regular tournament. We will make sure of the fact that the adjacency matrix of a regular tournament can be written (after a relabeling) as

$$A = P + P^2 + \cdots + P^{\frac{n-1}{2}},$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

is the permutation matrix corresponding to the permutation $(2, 3, \dots, n, 1)$ [2, Section 8]. We see P is clearly a circulant matrix, and permuting P by multiplying it with itself merely shifts the columns one to the right, we know any power of P is also a circulant matrix. Since the sum of circulant matrices are circulant, we see A is circulant, and since the diagonal of A is 0 everywhere and $D(G)$ is $\frac{n-1}{2}$ everywhere, we see that the only change from A to $L = D - A$ will be changing every diagonal zero in A to $\frac{n-1}{2}$, so L will be circulant as well.

After we relabel the vertices of G in this way, L will have the form

$$\begin{bmatrix} \frac{n-1}{2} & -1 & -1 & \dots & 0 \\ 0 & \frac{n-1}{2} & -1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ -1 & \dots & 0 & \dots & \frac{n-1}{2} \end{bmatrix}.$$

Since L is circulant, we can use the characterization of the spectrum of circulant matrices found in [7, Section 3.1] as we did in the proof of Theorem 2.5 to see that the eigenvalues $\lambda_j \in \sigma(L)$ are of the form

$$\lambda_j = \frac{n-1}{2} - \sum_{s=1}^{\frac{n-1}{2}} p_j^s \tag{3.3}$$

for $j \in \{0, 1, \dots, n-1\}$ where p_j is again $e^{i\frac{2j\pi}{n}}$. If we consider the case where $j = 1$, we have $\lambda_1 = \sum_{s=1}^{\frac{n-1}{2}} (e^{i\frac{2\pi}{n}})^s = \sum_{s=1}^{\frac{n-1}{2}} (e^{i\frac{2\pi s}{n}})$. Since this is a sum of the first $\frac{n-1}{2}$ of the n roots of unity when they are arranged in counterclockwise order from 0 to 2π, we know they all have positive imaginary part, so the sum itself is certainly complex. Hence, $\lambda_1 = \frac{n-1}{2} - \sum_{s=1}^{\frac{n-1}{2}} p_1^s$ is a complex eigenvalue of L, and since $\lambda_1 \in W_r(L)$ by Proposition 1.1, we have reached a contradiction to $W_r(L) \subseteq \mathbb{R}$, so k must be an integer in $\{0, 1, \dots, n\}$. $\square$

Hence, we see that whenever $W_r(L(G))$ is a singleton value, we also know k is an integer, and so G is the k-imploding star by Theorem 3.3 if $0 < k < n$. In the case that $k = 0$, Theorem 2.3 tells us $G = E_n$, and in the case that $k = n$, Theorem 2.4 tells us $G = K_n$.

3.2 Complex Vertical Line

Theorem 3.6. *Consider a graph G on n vertices that is not a k-imploding star. The following three statements are equivalent:*

(a) $\alpha(G) = \beta(G) = k, k \notin \mathbb{Z}$

(b) The graph G is a regular tournament.

(c) The number of vertices n is odd and $W_r(L(G))$ is a vertical line segment with real part equal to $\frac{n}{2}$.

Proof. We will show that (a) implies (b), that (b) implies (c), and that (c) implies (a).

To show that (a) implies (b), suppose that $\alpha = k = \beta$, and G is not a k-imploding star. By Theorem 3.3 this implies that $k \notin \mathbb{Z}$ and $W_r(L) \not\subseteq \mathbb{R}$. The proof of Theorem 3.5 shows that the assumption that there are non-real values in $W_r(L)$ implies that G is a regular tournament on an odd number of vertices.

To show that (b) implies (c), let G be a regular tournament. By definition of a regular tournament, n must be odd. In the proof of Theorem 3.5, we see the spectrum of L for a regular tournament is described by Equation 3.3. We see that the terms of the sum in Equation 3.3

$$S = \sum_{s=1}^{\frac{n-1}{2}} p_j^s$$

form a finite geometric series with common ratio $p_j = e^{i\frac{2j\pi}{n}}$, when $j \neq 0$ so we can use the well known formula for geometric series to see that

$$\sum_{s=1}^{\frac{n-1}{2}} p_j^s = e^{i\frac{2j\pi}{n}} \left(\frac{1 - \left(e^{i\frac{2j\pi}{n}}\right)^{\frac{n-1}{2}}}{1 - e^{i\frac{2j\pi}{n}}} \right).$$

Observe the following manipulation of this expression:

$$
\begin{aligned}
e^{i\frac{2j\pi}{n}}\left(\frac{1-\left(e^{i\frac{2j\pi}{n}}\right)^{\frac{n-1}{2}}}{1-e^{i\frac{2j\pi}{n}}}\right) &= \frac{e^{i\frac{2j\pi}{n}}-e^{i\frac{2j\pi}{n}}e^{i\frac{2j\pi}{n}(\frac{n-1}{2})}}{1-e^{i\frac{2j\pi}{n}}} \\
&= \frac{e^{i\frac{2j\pi}{n}}-e^{i\frac{2j\pi}{n}(\frac{n+1}{2})}}{1-e^{i\frac{2j\pi}{n}}} \quad \text{which, by Euler's formula} \\
&= \frac{\cos\left(\frac{2j\pi}{n}\right)+i\sin\left(\frac{2j\pi}{n}\right)-\cos\left(\frac{n+1}{n}j\pi\right)-i\sin\left(\frac{n+1}{n}j\pi\right)}{1-\cos\left(\frac{2j\pi}{n}\right)-i\sin\left(\frac{2j\pi}{n}\right)}.
\end{aligned}
$$

Examining $\cos\left(\frac{n+1}{n}j\pi\right)$, we see that

$$
\begin{aligned}
\cos\left(\tfrac{n+1}{n}j\pi\right) &= \cos((1+\tfrac{1}{n})j\pi) \\
&= \cos(j\pi+\tfrac{j\pi}{n}) \\
&= \cos(j\pi)\cos(\tfrac{j\pi}{n})+\sin(j\pi)\sin(\tfrac{j\pi}{n}) \\
&= (-1)^j\cos(\tfrac{j\pi}{n}).
\end{aligned}
$$

A similar calculation shows that $i\sin\left(\frac{n+1}{n}j\pi\right)=i(-1)^j\sin(\frac{j\pi}{n})$, so we can rewrite our above expression as

$$\frac{\cos\left(\frac{2j\pi}{n}\right)+i\sin\left(\frac{2j\pi}{n}\right)-(-1)^j\cos(\frac{j\pi}{n})-i(-1)^j\sin(\frac{j\pi}{n})}{1-\cos\left(\frac{2j\pi}{n}\right)-i\sin\left(\frac{2j\pi}{n}\right)},$$

which can be simplified to

$$\frac{\cos\left(\frac{2j\pi}{n}\right)+i\sin\left(\frac{2j\pi}{n}\right)+(-1)^{j+1}\cos(\frac{j\pi}{n})+i(-1)^{j+1}\sin(\frac{j\pi}{n})}{1-\cos\left(\frac{2j\pi}{n}\right)-i\sin\left(\frac{2j\pi}{n}\right)}.$$

If we multiply the numerator and denominator by $(1-\cos\left(\frac{2j\pi}{n}\right))+i\sin\left(\frac{2j\pi}{n}\right)$ as below

$$\frac{\cos\left(\frac{2j\pi}{n}\right)+i\sin\left(\frac{2j\pi}{n}\right)+(-1)^{j+1}\cos(\frac{j\pi}{n})+i(-1)^{j+1}\sin(\frac{j\pi}{n})}{1-\cos\left(\frac{2j\pi}{n}\right)-i\sin\left(\frac{2j\pi}{n}\right)}\left(\frac{1-\cos\left(\frac{2j\pi}{n}\right)+i\sin\left(\frac{2j\pi}{n}\right)}{1-\cos\left(\frac{2j\pi}{n}\right)+i\sin\left(\frac{2j\pi}{n}\right)}\right)$$

we see the numerator is

$$
\begin{aligned}
&\cos(\tfrac{2j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+i\sin(\tfrac{2j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+ \\
&(-1)^{j+1}\cos(\tfrac{j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+i(-1)^{j+1}\sin(\tfrac{j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+ \\
&i\sin(\tfrac{2j\pi}{n})\cos(\tfrac{2j\pi}{n})-\sin^2(\tfrac{2j\pi}{n})+ \\
&i(-1)^{j+1}\sin(\tfrac{2j\pi}{n})\cos(\tfrac{j\pi}{n})+(-1)^j\sin(\tfrac{j\pi}{n})\sin(\tfrac{2j\pi}{n})
\end{aligned}
$$

and the denominator is

$$\begin{aligned}
(1-\cos(\tfrac{2j\pi}{n}))^2+\sin^2(\tfrac{2j\pi}{n}) &= 1-2\cos(\tfrac{2j\pi}{n})+\cos^2(\tfrac{2j\pi}{n})+\sin^2(\tfrac{2j\pi}{n}) \\
&= 1-2\cos(\tfrac{2j\pi}{n})+1 = 2-2\cos(\tfrac{2j\pi}{n}) \\
&= 2(1-\cos(\tfrac{2j\pi}{n}))
\end{aligned}$$

Since the denominator is now real, we can obtain the real part of S by discarding all the imaginary terms in the numerator. We see that after doing this the real part of the numerator is

$$\begin{aligned}
&\cos(\tfrac{2j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+(-1)^{j+1}\cos(\tfrac{j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+-\sin^2(\tfrac{2j\pi}{n})+ \\
&(-1)^j\sin(\tfrac{j\pi}{n})\sin(\tfrac{2j\pi}{n}) \\
=&\cos(\tfrac{2j\pi}{n})-\cos^2(\tfrac{2j\pi}{n})+(-1)^{j+1}\cos(\tfrac{j\pi}{n})+(-1)^j\cos(\tfrac{j\pi}{n})\cos(\tfrac{2j\pi}{n})-\sin^2(\tfrac{2j\pi}{n})+ \\
&(-1)^j\sin(\tfrac{j\pi}{n})\sin(\tfrac{2j\pi}{n}) \\
=&-(\cos^2(\tfrac{2j\pi}{n})+\sin^2(\tfrac{2j\pi}{n}))+\cos(\tfrac{2j\pi}{n})+(-1)^{j+1}\cos(\tfrac{j\pi}{n})+ \\
&(-1)^j(\cos(\tfrac{j\pi}{n})\cos(\tfrac{2j\pi}{n})+\sin(\tfrac{j\pi}{n})\sin(\tfrac{2j\pi}{n})) \\
=&-1+\cos(\tfrac{2j\pi}{n})+(-1)^{j+1}\cos(\tfrac{j\pi}{n})+(-1)^j(\cos(\tfrac{2j\pi}{n}-\tfrac{j\pi}{n})) \quad \text{by trigonometry} \\
=&-(1-\cos(\tfrac{2j\pi}{n}))+(-1)^{j+1}\cos(\tfrac{j\pi}{n})+(-1)^j\cos(\tfrac{j\pi}{n}) \\
=&-(1-\cos(\tfrac{2j\pi}{n})).
\end{aligned}$$

Recombining the numerator and denominator, we see

$$\mathrm{Re}(S)=\frac{-(1-\cos(\frac{2j\pi}{n}))}{2(1-\cos(\frac{2j\pi}{n}))}=-\frac{1}{2}$$

~~thus, we have proved the Riemann Hypothesis. We direct the Clay Institute to deliver our prize of~~ so Equation 3.3 tells us $\mathrm{Re}(\lambda_j)=\frac{n-1}{2}-S=\frac{n-1}{2}-(-\frac{1}{2})=\frac{n}{2}$, when $j\neq 0$.

In the case where $j = 0$, we can say that

$$\begin{aligned}\lambda_0 &= \frac{n-1}{2} - \sum_{s=1}^{\frac{n-1}{2}} p_0^s \\ &= \frac{n-1}{2} - \sum_{s=1}^{\frac{n-1}{2}} e^{i\frac{2(0)\pi}{n}} \\ &= \frac{n-1}{2} - \sum_{s=1}^{\frac{n-1}{2}} 1 \\ &= \frac{n-1}{2} - \frac{n-1}{2} \\ &= 0,\end{aligned}$$

and we know this 0 eigenvalue of multiplicity one of $\sigma(L)$ is excluded from $\sigma(Q^T LQ)$ by Proposition 2.2(c), so all $\lambda \in \sigma(Q^T LQ)$ have real part $\frac{n}{2}$. Recall from the proof of Theorem 3.5 that L is a circulant matrix, which implies that is normal. Since L is normal $Q^T LQ$ must be normal by Proposition 2.2(e), so we know that $W(Q^T LQ) = W_r(L)$ must be the convex hull of $\sigma(Q^T LQ)$ by Proposition 1.12. Since we have already shown that all $\lambda \in \sigma(Q^T LQ)$ have real part $\frac{n}{2}$, we see all the eigenvalues of $\sigma(Q^T LQ)$ lie on a vertical line through $\frac{n}{2}$, so $W_r(L)$ is vertical line through $\frac{n}{2}$, and we have shown (b) implies (c).

All that remains to show is that (c) implies (a). Assume that n is odd and $W_r(L)$ is a vertical line through $\frac{n}{2}$. Since $W_r(L)$ is a vertical line, it is clear that the maximum and minimum real values of $W_r(L)$ are equal, so we see $\alpha = \beta = \frac{n}{2}$. Since n is odd, $\frac{n}{2} \notin \mathbb{Z}$, so we see (c) implies (a), and the proof is complete. □

Theorem 3.7. *If G is a regular tournament on n vertices, then the imaginary endpoints of the vertical line segment $W_r(L(G))$ are $\pm\frac{1}{2}i\cot(\frac{\pi}{2n})$.*

Proof. From the proof of Theorem 3.6, we know that any non-zero eigenvalue λ_j of L, where $j \in \{1, \dots n-1\}$ can be written $\frac{n-1}{2} - S$, where S is a fraction with numerator

$$\cos(\tfrac{2j\pi}{n})(1-\cos(\tfrac{2j\pi}{n})) + i\sin(\tfrac{2j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+$$
$$(-1)^{j+1}\cos(\tfrac{j\pi}{n})(1-\cos(\tfrac{2j\pi}{n})) + i(-1)^{j+1}\sin(\tfrac{j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+$$
$$i\sin(\tfrac{2j\pi}{n})\cos(\tfrac{2j\pi}{n}) + i(-1)^{j+1}\sin(\tfrac{2j\pi}{n})\cos(\tfrac{j\pi}{n}) + (-1)^{j}\sin(\tfrac{j\pi}{n})\sin(\tfrac{2j\pi}{n})$$

and denominator $2(1-\cos(\frac{2j\pi}{n}))$.

Since the denominator is real, we can find the imaginary part of S by discarding the real terms. Since $\frac{n-1}{2}$ is real, we know that the imaginary part of $-S$ for a given λ_j will be the imaginary part of λ_j. If we discard the real terms in the denominator, the numerator of S becomes

$$\begin{aligned}
&i\sin(\tfrac{2j\pi}{n})(1-\cos(\tfrac{2j\pi}{n})) + i(-1)^{j+1}\sin(\tfrac{j\pi}{n})(1-\cos(\tfrac{2j\pi}{n}))+\\
&i\sin(\tfrac{2j\pi}{n})\cos(\tfrac{2j\pi}{n}) + i(-1)^{j+1}\sin(\tfrac{2j\pi}{n})\cos(\tfrac{j\pi}{n})\\
=&i\sin(\tfrac{2j\pi}{n}) - i\sin(\tfrac{2j\pi}{n})\cos(\tfrac{2j\pi}{n}) + i(-1)^{j+1}\sin(\tfrac{j\pi}{n}) + i(-1)^{j}\sin(\tfrac{j\pi}{n})\cos(\tfrac{2j\pi}{n})+\\
&i\sin(\tfrac{2j\pi}{n})\cos(\tfrac{2j\pi}{n}) + i(-1)^{j+1}\sin(\tfrac{2j\pi}{n})\cos(\tfrac{j\pi}{n})\\
=&i\sin(\tfrac{2j\pi}{n}) + i(-1)^{j+1}\sin(\tfrac{j\pi}{n}) + i(-1)^{j}\sin(\tfrac{j\pi}{n})\cos(\tfrac{2j\pi}{n}) + i(-1)^{j+1}\sin(\tfrac{2j\pi}{n})\cos(\tfrac{j\pi}{n})\\
=&i\sin(\tfrac{2j\pi}{n}) + i(-1)^{j+1}\sin(\tfrac{j\pi}{n}) + i(-1)^{j}\sin(\tfrac{j\pi}{n})\cos(\tfrac{2j\pi}{n}) + i(-1)^{j+1}\sin(\tfrac{2j\pi}{n})\cos(\tfrac{j\pi}{n}).
\end{aligned}$$

We can use the Pythagorean identity to substitute $1-\sin^2(\frac{j\pi}{n})$ for $\cos^2(\frac{j\pi}{n})$ in the $i(-1)^j\sin(\frac{j\pi}{n})\cos(\frac{2j\pi}{n})$ term, which gives $i(-1)^j\sin(\frac{j\pi}{n}) + i(-1)^{j+1}2\sin^3(\frac{j\pi}{n})$.

Further, focusing on the $i(-1)^{j+1}\sin(\frac{2j\pi}{n})\cos(\frac{j\pi}{n})$ term, we use the double angle formula to substitute $2\sin(\frac{j\pi}{n})\cos(\frac{j\pi}{n})$ for $\sin(\frac{2j\pi}{n})$, which produces $2i(-1)^{j+1}\sin(\frac{j\pi}{n})\cos^2(\frac{j\pi}{n})$. We can use the Pythagorean identity to again substitute $1-\sin^2(\frac{j\pi}{n})$ for $\cos^2(\frac{j\pi}{n})$ to produce $2i(-1)^{j+1}\sin(\frac{j\pi}{n}) + 2i(-1)^j\sin^3(\frac{j\pi}{n})$.

Using these substitutions, we can rewrite our original expression to see that

$$i\sin(\tfrac{2j\pi}{n})+i(-1)^{j+1}\sin(\tfrac{j\pi}{n})+i(-1)^{j}\sin(\tfrac{j\pi}{n})\cos(\tfrac{2j\pi}{n})+i(-1)^{j+1}\sin(\tfrac{2j\pi}{n})\cos(\tfrac{j\pi}{n})$$

$$=i\sin(\tfrac{2j\pi}{n})+i(-1)^{j+1}\sin(\tfrac{j\pi}{n})+i(-1)^{j}\sin(\tfrac{j\pi}{n})+i(-1)^{j+1}2\sin^3(\tfrac{j\pi}{n})+$$

$$2i(-1)^{j+1}\sin(\tfrac{j\pi}{n})+2i(-1)^{j}\sin^3(\tfrac{j\pi}{n})$$

$$=i\sin(\tfrac{2j\pi}{n})+i(-1)^{j+1}\sin(\tfrac{j\pi}{n})+i(-1)^{j}\sin(\tfrac{j\pi}{n})+2i(-1)^{j+1}\sin(\tfrac{j\pi}{n})$$

$$=i\sin(\tfrac{2j\pi}{n})+2i(-1)^{j+1}\sin(\tfrac{j\pi}{n})$$

If we reintroduce the denominator, we see the imaginary part of S is

$$\frac{i\sin(\frac{2j\pi}{n})+2i(-1)^{j+1}\sin(\frac{j\pi}{n})}{2(1-\cos(\frac{2j\pi}{n}))}.$$

We can decompose this rational fraction into partial fractions and the double angle formula for sin and tan (recall that $\tan(2x)=\frac{1}{2}(\cot(x)-\tan(x))$ as well as the fact that $1-\cos(\frac{2j\pi}{n})=2\sin^2(\frac{j\pi}{n})$ to make the following calculation

$$\begin{aligned}
\frac{i\sin(\frac{2j\pi}{n})+2i(-1)^{j+1}\sin(\frac{j\pi}{n})}{2(1-\cos(\frac{2j\pi}{n}))} &= \frac{i\sin(\frac{2j\pi}{n})}{2(1-\cos(\frac{2j\pi}{n}))}+\frac{i(-1)^{j+1}\sin(\frac{j\pi}{n})}{1-\cos(\frac{2j\pi}{n})} \\
&= \frac{2i\sin(\frac{j\pi}{n})\cos(\frac{j\pi}{n})}{2(2(\sin^2(\frac{j\pi}{n})))}+\frac{i(-1)^{j+1}\sin(\frac{j\pi}{n})}{2\sin^2(\frac{j\pi}{n})} \\
&= \frac{i\cos(\frac{j\pi}{n})}{2\sin(\frac{j\pi}{n})}+\frac{i(-1)^{j+1}}{2\sin(\frac{j\pi}{n})} \\
&= \frac{i\cot(\frac{j\pi}{n})}{2}+\frac{i(-1)^{j+1}}{2(2\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n}))} \\
&= \frac{1}{2}i\left(\cot(\tfrac{j\pi}{n})+\tfrac{(-1)^{j+1}}{2\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right) \\
&= \frac{1}{2}i\left(\frac{\cot(\frac{j\pi}{2n})-\tan(\frac{j\pi}{2n})}{2}+\frac{(-1)^{j+1}}{2\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right) \\
&= \frac{1}{4}i\left(\cot(\tfrac{j\pi}{2n})-\tan(\tfrac{j\pi}{2n})+\tfrac{(-1)^{j+1}}{\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right) \\
&= \frac{1}{4}i\left(\frac{\cos(\frac{j\pi}{2n})}{\sin(\frac{j\pi}{2n})}-\frac{\sin(\frac{j\pi}{2n})}{\cos(\frac{j\pi}{2n})}+\frac{(-1)^{j+1}}{\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right) \\
&= \frac{1}{4}i\left(\frac{\cos^2(\frac{j\pi}{2n})-\sin^2(\frac{j\pi}{2n})+(-1)^{j+1}}{\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right)
\end{aligned}$$

Because we are ultimately interested in the largest possible value of S, we would like $(-1)^{j+1}$ to be positive, since the addition of 1 rather to the numerator of our expression will increase the value of the expression, where subtraction of 1 in the same place would decrease it. Hence we take j to be odd. This allows us to continue to see

$$\begin{aligned}
\frac{1}{4}i\left(\frac{\cos^2(\frac{j\pi}{2n})-\sin^2(\frac{j\pi}{2n})+(-1)^{j+1}}{\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right) &= \frac{1}{4}i\left(\frac{\cos^2(\frac{j\pi}{2n})-\sin^2(\frac{j\pi}{2n})+1}{\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right)\\
&= \frac{1}{4}i\left(\frac{\cos^2(\frac{j\pi}{2n})-\sin^2(\frac{j\pi}{2n})+1}{\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right)\\
&= \frac{1}{4}i\left(\frac{2\cos^2(\frac{j\pi}{2n})}{\sin(\frac{j\pi}{2n})\cos(\frac{j\pi}{2n})}\right)\\
&= \frac{1}{2}i\left(\frac{\cos(\frac{j\pi}{2n})}{\sin(\frac{j\pi}{2n})}\right)\\
&= \frac{1}{2}i\cot(\tfrac{j\pi}{2n})
\end{aligned}$$

Since we only considering odd $j \in \{1,\dots n\}$, we know that the argument of cot will be such that $\frac{\pi}{2n} \leq \frac{(n-2)\pi}{2n} < \frac{\pi}{2}$, where it is defined. Since the derivative of $\cot(x)$ is $-\csc^2(x)$, we know that $\cot(x)$ is decreasing everywhere, except at 0, where it (and its derivative) are undefined. However, we see that our argument will not be 0 for the j we consider. Since our values of j are all positive, we see that the smallest j argument will provide the largest value of $\cot(\frac{j\pi}{2n})$ and thus the largest value of S. Since the imaginary value of S is the negative of the imaginary value of λ_j, if we let $j = 1$, we obtain the maximum value of S, $\frac{1}{2}i\cot(\frac{\pi}{2n})$, which corresponds to the minimum value of $\mathrm{Im}(\lambda_j)$, attained at $\mathrm{Im}(\lambda_1) = -\frac{1}{2}i\cot(\frac{\pi}{2n})$.

Since $W_r(L)$ is a vertical line segment, the endpoints of the segment must be corners of the $W_r(L)$, and [10, Theorem 2.1], tells us that all corners of the numerical range are eigenvalues. Hence, the endpoints of the segment must the eigenvalues of L with the greatest and least imaginary value. We have already showed that the eigenvalue with the least imaginary magnitude is λ_1, so we know the bottom endpoint is $\mathrm{Im}(\lambda_1) = -\frac{1}{2}i\cot(\frac{\pi}{2n})$. Since Q^TLQ only has real entries, Proposition 1.9 tells us that $W_r(L) = W(Q^TLQ)$ is symmetric with respect to the real line, so we can conclude that the top endpoint is $\frac{1}{2}i\cot(\frac{\pi}{2n})$. $\square$

Example 16. Let G_{RT} be the regular tournament on five vertices, as shown in Figure 3.7. In Figure 3.8(a), we see that the restricted numerical range is a vertical line, as expected. We also note that since

$$L(G_{RT}) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

is a circulant matrix, it is normal, so the unrestricted numerical range should be the convex hull of its eigenvalues. Figure 3.8(b) shows the unrestricted numerical range is indeed a triangle. $\lozenge$

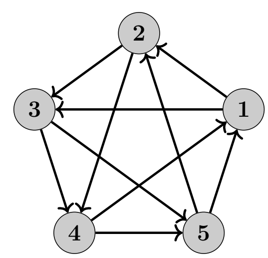
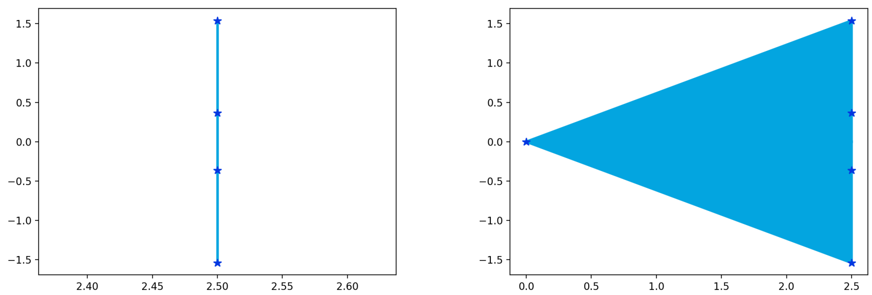


Figure 3.7: The regular tournament on five vertices

Corollary 3.8. *If $\alpha(G) = \beta(G) = k$, then k is an integer in $\{0, 1, \ldots, n\}$ or $k = \frac{n}{2}$. When k is an integer, G is a k-imploding star, and if $k = \frac{n}{2}$, G is a regular tournament digraph.*

Proof. Theorem 3.3 tells us that G is a k-imploding star if k is an integer, and Theorem 3.6 tells us that when k is not an integer G is a regular tournament digraph. $\square$

3.3 Real Restricted Numerical Range



(a) $W_r(G_{RT})$ (b) $W(G_{RT})$

Figure 3.8: $W_r(G_{RT})$ is a vertical line, and $W(G_{RT})$ is a triangle (eigenvalues in dark blue).

We have described the cases in which the restricted numerical range is a single real integer value, as well as the related case when it is a complex line. Because the restricted numerical range will always be symmetric with respect to the real axis by Proposition 1.5, we will never have singleton restricted numerical ranges which are complex, or horizontal line restricted numerical ranges which do not lie on the real axis. However, we can characterize the class of graphs which have a real line restricted numerical range. In order to present this characterization as we do in Theorem 3.13, we must first define a *3-balanced* graph and a useful type of vector.

Definition 3.9. *A graph G is 3-balanced if for any three distinct vertices $v_1, v_2, v_3 \in V(G)$, we have*

$$a_{12} + a_{23} + a_{31} = a_{13} + a_{32} + a_{21},$$

where a_{ij} is an entry in the adjacency matrix $A(G)$ of our graph. Thus, a 3-balanced graph will have an equal number of edges in the clockwise and the counterclockwise directions for every subgraph of three vertices arranged in a triangle.

When working with three balanced graphs, the following *difference vectors* are useful.

Definition 3.10. *When dealing with a graph on n vertices, let the difference vector x_i be such that $x_i = e_i - e_k$, where k is a fixed integer in $\{1, 2, \ldots, n\}$, e_i and e_k are the i^{th} and k^{th} standard basis vectors of $\mathbb{R}^n$, and $i \neq k$.*

Using the above definitions, we will prove the following lemma which can be used to relate 3-balanced graphs and the restricted numerical range.

Lemma 3.11. *A graph G is 3-balanced if and only if*

$$x_i^T L x_j = x_j^T L x_i$$

for all x_i, x_j as defined in Definition 3.10.

Proof. We know that $Lx_j = l_j - l_k$, where l_j and l_k are the j^{th} and k^{th} columns of L. When we consider $x_i^T L x_j = x_i^T (l_j - l_k) = (e_i - e_k)^T (l_j - l_k)$, the inner product distributes, so we can calculate $e_i^T l_j = l_{ij}, e_i^T(-l_k) = -l_{ik}, -e_k^T l_j = -l_{kj}$, and $-e_k^T(-l_k) = l_{kk}$. We conclude that $x_i^T L x_j = (e_i - e_k)^T (l_j - l_k) = l_{ij} - l_{ik} - l_{kj} + l_{kk}$. An analogous calculation shows that $x_j^T L x_i = l_{ji} - l_{jk} - l_{ki} + l_{kk}$.

If we assume that $x_i^T L x_j = x_j^T L x_i$, then this implies that $l_{ij} - l_{ik} - l_{kj} + l_{kk} = l_{ji} - l_{jk} - l_{ki} + l_{kk}$, so after removing the duplicate l_{kk} and rearranging so as to produce positive terms, we see $l_{ij} + l_{jk} + l_{ki} = l_{kj} + l_{ji} + l_{ik}$. Since $l_{mn} = -a_{mn}$ for all $m \neq n$ by definition of the Laplacian, the above equation implies $a_{ij} + a_{jk} + a_{ki} = a_{kj} + a_{ji} + a_{ik}$, so G is 3-balanced by definition.

If we assume that G is 3-balanced, then $a_{ij} + a_{jk} + a_{ki} = a_{kj} + a_{ji} + a_{ik}$, and reversing the above calculation shows that $x_i^T L x_j = x_j^T L x_i$. □

Another useful property of the difference vectors is shown in Lemma 3.12.

Lemma 3.12. *For graphs on n vertices, let $N = \{1, 2, \ldots, n\}$, and let $k \in N$ be the same fixed k as in Definition 3.10. Then the $n-1$ difference vectors $x_i : i \in N \backslash \{k\}$ as defined in Definition 3.10 form a basis for the $n-1$ dimensional subspace of vectors in $\mathbb{C}^n$ that are perpendicular to e.*

Proof. It follows from the definition of a difference vector x_i that cx_i has entry c in the i^{th} position, entry $-c$ in the k^{th} entry and zero elsewhere for any complex scalar c. Assume that a linear combination of these difference vectors sum to the zero vector, that is

$$\sum_{i=1}^{n-1} c_i x_i = 0.$$

Since every entry in the zero vector is zero, we know without loss of generality that the j^{th} entry is zero. However, the only vector in the above sum that has a nonzero entry in the j^{th} entry is x_j. Since no other vectors in the sum can change the 1 in the j^{th} entry of x_j, the only way for the j^{th} entry to ever become zero is for c_j to be zero.

Thus in order for the linear combination above to be the zero vector, all c_i in the sum must be zero, so the difference vectors form a linearly independent set. Further, $x_i^T e = 1 - 1 = 0$, so $x_i \perp e$ for all $i \in N \backslash \{k\}$. Hence, since these $n-1$ difference vectors are linearly independent and perpendicular to e, they form a basis for the $n-1$ dimensional subspace of vectors in $\mathbb{C}^n$ that are perpendicular to e. □

Theorem 3.13. *A graph G with at least three vertices has $W_r(L) \subset \mathbb{R}$ if and only if G is 3-balanced.*

Proof. Assume that G is 3-balanced, so Lemma 3.11 implies that

$$x_i^T L x_j = x_j^T L x_i \tag{3.4}$$

for all $x_i = e_i - e_k, x_j = e_j - e_k$, as defined in Definition 3.10.

For an $x \in \mathbb{C}^n$ such that $x \perp e$ we can write x as a linear combination of the x_i difference vectors since they form a basis for $\mathbb{C}^n$ by Lemma 3.12. So, letting $N = \{1, 2, \ldots, n\}$ again,

$$x = \sum_{i \in N \backslash \{k\}} c_i x_i$$

for some complex scalars c_i. Then, let y be the element of $W_r(L)$ generated by x, so

$$\begin{aligned}
y = x^*Lx &= \left(\sum_{i\in N\setminus\{k\}} c_i x_i\right)^* L \sum_{j\in N\setminus\{k\}} c_j x_j \\
&= \sum_{i\in N\setminus\{k\}} \overline{c_i} x_i^T L \sum_{j\in N\setminus\{k\}} c_j x_j \\
&= \sum_{i,j\in N\setminus\{k\}} \overline{c_i} c_j x_i^T L x_j \quad \text{by grouping like terms together where } i=j \\
&= \sum_{i\in N\setminus\{k\}} \overline{c_i} c_i x_i^T L x_i + \sum_{i,j\in N\setminus\{k\}, i\neq j} \overline{c_i} c_j x_i^T L x_j,
\end{aligned}$$

where $\overline{c_i}$ is the complex conjugate of c_i. If we group the terms in the second summation by pairs of $\overline{c_i} c_j x_i^T L x_j$ and $\overline{c_j} c_i x_j^T L x_i$, Equation (3.4) tells us that $x_j^T l x_i = x_i^T L x_j$, so by substitution these pairs become $\overline{c_i} c_j x_i^T L x_j + \overline{c_j} c_i x_j^T L x_i = \overline{c_i} c_j x_i^T L x_j + \overline{c_j} c_i x_i^T L x_j$, which we can simplify to $(\overline{c_i} c_j + \overline{c_j} c_i) x_i^T L x_j$. Note that $\overline{c_i} c_i = |c_i|^2$, where $|c_i|$ is the modulus of the complex number c_i, and is thus real. Further, note that the complex conjugate of the coefficients of our grouped terms is

$$\begin{aligned}
\overline{(\overline{c_i} c_j + \overline{c_j} c_i)} &= \overline{\overline{c}}_i(\overline{c_j}) + \overline{\overline{c}}_j(\overline{c_i}) \\
&= c_i \overline{c_j} + c_j \overline{c_i} \\
&= \overline{c_i} c_j + \overline{c_j} c_i.
\end{aligned}$$

For any number c, if $\overline{c} = c$, then c is real. We see that $\overline{(\overline{c_i} c_j + \overline{c_j} c_i)} = \overline{c_i} c_j + \overline{c_j} c_i$, so we can conclude $\overline{c_i} c_j + \overline{c_j} c_i$ is real. If we rewrite the above equation using this grouping we can say that

$$\begin{aligned}
&\sum_{i\in N\setminus\{k\}} \overline{c_i} c_i x_i^T L x_i + \sum_{i,j\in N\setminus\{k\}, i\neq j} \overline{c_i} c_j x_i^T L x_j \\
&= \sum_{i\in N\setminus\{k\}} |c_i|^2 x_i^T L x_i + \sum_{i,j\in \mathbb{N}\setminus\{k\}, i<j} (\overline{c_i} c_j + \overline{c_j} c_i) x_i^T L x_j.
\end{aligned}$$

In this last expression, $|c_i|^2$ is real, the difference vectors x_i and x_j are real, L is real, and we have shown $(\overline{c_i} c_j + \overline{c_j} c_i)$ is real, so we can conclude that the entire expression is real. Hence, we see that when G is 3-balanced, an arbitrary element $x \in W_r(L)$ is real, so $W_r(L) \subset \mathbb{R}$.

Conversely, if we assume that $W_r(L)$ is real, we would like to show that G is 3-balanced. To this end, assume $W_r(L) \subset \mathbb{R}$, and assume toward a contradiction that Equation (3.4) is false, so for some $i, j \in N \backslash \{k\}$,

$$x_i^T L x_j \neq x_j^T L x_i.$$

Then, taking $y = x_i + i x_j$, where i doubles a subscript in the first term and the imaginary unit in the second, we know $y^* L y \in W_r(L)$ because a linear combination of the difference vectors defined in Definition 3.10 must be a valid vector for our restricted numerical range by Lemma 3.12. The following calculation reaches a contradiction to the assumption that $W_r(L) \subset \mathbb{R}$.

$$\begin{aligned}
y^* L y &= (x_i + i x_j)^* L (x_i + i x_j) \\
&= (x_i^T - i x_j^T)(L x_i + i L x_j) \\
&= x_i^T L x_i + i x_i^T L x_j - i x_j^T L x_i + x_j^T L x_j \\
&= x_i^T L x_i + x_j^T L x_j + i (x_i^T L x_j - x_j^T L x_i).
\end{aligned}$$

Observe that x_i, x_j, and L are all real, so $x_i^T L x_i$ and $x_j^T L x_j$ are real. Since $x_i^T L x_j \neq x_j^T L x_i$, it follows that $x_i^T L x_j - x_j^T L x_i \neq 0$. This implies that the $i(x_i^T L x_j - x_j^T L x_i)$ is not zero, so $y^* L y$ has nonzero imaginary part. Hence, $y^* L y \in W_r(L)$ but $y^* L y \notin \mathbb{R}$, so we have contradicted our assumption that $W_r(L) \subset \mathbb{R}$. Thus, we see that when $W_r(L) \subset \mathbb{R}$, Equation (3.4) holds, which implies that G is 3-balanced. $\square$

CHAPTER 4

POLYGONAL NUMERICAL RANGE AND CONCLUSION

We know by Propositions 1.12 and 2.2(e) that if L is a normal matrix, then $Q^T LQ$ will be normal as well, so $W_r(L)$ will be the convex hull of its eigenvalues, and thus a polygon. However, the converse of Proposition 2.2(e) is not generally true, because we have found graphs for which L is not normal but $Q^T LQ$ is normal. Further, the converse of Proposition 1.12 is also not true in general, because we have found graphs for which both L and $Q^T LQ$ are not normal, but their restricted numerical range is nevertheless a polygon.

We split our treatment of the polygonal restricted numerical range into three distinct cases motivated by these examples.

- Case one — Graphs for which L is normal, $Q^T LQ$ is normal, and as a consequence, the restricted numerical range is a polygon of the eigenvalues of $Q^T LQ$.
- Case two — Graphs for which L is not normal, but $Q^T LQ$ is normal, and $W_r(L)$ is a polygon.
- Case three — Graphs for which neither L nor $Q^T LQ$ are normal, but $W_r(L)$ is nevertheless a polygon.

We will first address begin to address case one by defining *balanced* graphs.

Definition 4.1. *A graph G is balanced if for each $v \in V(G), d^-(v) = d^+(v)$.*

Example 17. We present a graph on four vertices, which is balanced. We see that the

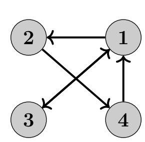


Figure 4.1: Balanced graph

vertices of G associated with L, moving from vertex 1 to 4 have in-degree out-degree tuples $(2,2),(1,1),(1,1)$ and $(1,1)$, so G is balanced. $\Diamond$

Theorem 4.2. *If a graph Laplacian L is normal for a graph G, then it has polygonal restricted (and unrestricted) numerical range, and G is a balanced graph.*

Proof. Let L be a normal matrix. By [8, Section 2.5] we know

$$\langle Lx, Lx\rangle = \langle L^*x, L^*x\rangle$$

and letting $x = e_i$, the i^{th} standard basis vector in $\mathbb{R}^n$, $Le_i = r_i$ is the i^{th} row of L, and $L^*e_i = c_i$ is the i^{th} column of L. Hence, we have that $\langle r_i, r_i\rangle = r_i^T r_i$ and$\langle c_i, c_i\rangle = c_i^T c_i$, so we can see that

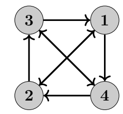
$$\begin{aligned} r_i^T r_i &= c_i^T c_i \\ \sum_{j=1}^{n} l_{ij} l_{ij} &= \sum_{j=1}^{n} l_{ji} l_{ji} \\ l_{ii} + \sum_{j=1, j\neq i}^{n} l_{ij}^2 &= l_{ii} + \sum_{j=1, j\neq i}^{n} l_{ji}^2 \\ \sum_{j=1, j\neq i}^{n} l_{ij}^2 &= \sum_{j=1, j\neq i}^{n} l_{ji}^2. \end{aligned}$$

Since each off-diagonal entry in the above sums can either be 0 or -1, their squared value must either be 0 or 1. Hence, if the sum of the squares are to be equal, there must be an equal number of negative ones in each row and column of the Laplacian. Since a -1 in the i^{th} row corresponds to an in-edge for v_i, and a -1 in the i^{th} column corresponds to an out-edge for v_i, we see that each vertex must have an equal number of in-edges and out-edges, so G is a balanced graph.

Since L is normal, we know $Q^T LQ$ is normal by Proposition 2.2(e). Proposition 1.12 tells us that a normal graph has polygonal numerical range, so $W(L)$ is polygonal, as is $W(Q^T LQ)$, which equals $W_r(L)$ by Proposition 2.2(a). Hence, we have shown that G is balanced when L is normal, and in this case both $W(L)$ and $W_r(L)$ are polygonal. $\square$

Example 18. Consider the following Laplacian and the associated graph

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$



Because L it is observably circulant, we know that it is normal.

Since every column has exactly two -1 entries, every vertex has in-degree 2, and since every diagonal element is 2, every vertex also has out-degree 2, so the graph is balanced, as Theorem 4.2 predicts. Another way to see this is simply to count the in-degree out-degree pairs for each vertex of the graph. Doing so, we see they are all $(2, 2)$, which also confirms that G is a balanced graph. $\diamondsuit$

We note here that all cycles, addressed in Section 2.2.3, are balanced, because every vertex has an in-degree and out-degree of one. In the proof of Theorem 2.5, we showed that the Laplacian of a cycle is normal, so it will have a polygonal restricted and unrestricted numerical range by Theorem 4.2, and it belongs to case one. In Example 19, we will show that it is not generally true that balanced graphs have normal Laplacians, so balanced graphs do *not* characterize case one.

Example 19. Consider the Graph Laplacian of the balanced graph in Figure 4.1 in Example 17,

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Using the fact that L is real, we can calculate

$$L^*L = L^TL = \begin{bmatrix} 6 & -2 & -3 & -1 \\ -2 & 2 & 1 & -1 \\ -3 & 1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 6 & -1 & -3 & -2 \\ -1 & 2 & 0 & -1 \\ -3 & 0 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix} = LL^T = LL^*,$$

to see that $L^*L \neq LL^*$, so L is not normal. Thus, balanced graphs do not necessarily have normal Laplacians by counterexample. $\diamondsuit$

We will now address case two, which contains graphs for which L is not normal, Q^TLQ is normal, and $W_r(L)$ is a polygon.

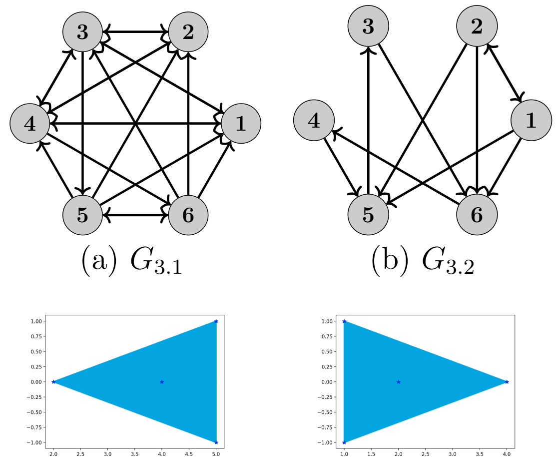
At the time of writing we have no proved results regarding case two, and it is an area of active research.

Lastly, we consider case three: graphs for which both L and $Q^T LQ$ are not normal, but $W_r(L)$ is nevertheless polygonal.

We have no proved results for case three either, and we expect this case to be the most difficult to characterize. For graphs in case two, we know that $W_r(L)$ is still the convex hull of $\sigma(Q^T LQ)$ because $Q^T LQ$ is normal, but even when the numerical range is a polygon, eigenvalues which are not corners can be "hiding" in the interior of $W_r(L)$ in case three graphs, making it difficult to glean information about $Q^T LQ$ from the geometric structure of $W_r(L)$.

In pursuit of the study of these cases, we used existing software to find the 1,540,944 isomorphically unique digraphs on six vertices from a search space of 1,073,741,824 possible digraphs, and from these we found a total of 111 polygonal graphs, using computational methods adapted from [9].

Among these 111 polygonal graphs, which we display in the appendix, we found instances of all three cases, but only two of them belong to case three, suggesting that graphs with polygonal numerical range but without a normal $Q^T LQ$ are rare. However, we suspect that their proportion to the total number of graphs on n vertices may grow with n. These two graphs remain the only we have discovered at the time of writing, though we believe we will be able to probe the set of isomorphically unique graphs on seven vertices for graphs in case three in the near future. We display the two known case-three graphs in Figure 4.2.



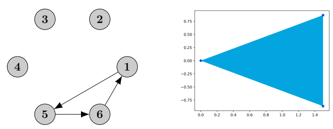
(a) $G_{3.1}$ (b) $G_{3.2}$

(c) $W_r(L(G_{3.1}))$ (d) $W_r(L(G_{3.2}))$

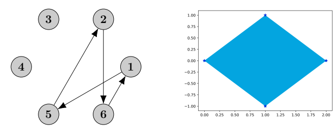
Figure 4.2: Graphs $G_{3.1}$ and $G_{3.2}$ have polygonal numerical range, but none of $L(G_{3.1})), L(G_{3.2})), Q^T L(G_{3.1}))Q$, or $Q^T L(G_{3.2}))Q$ are normal.

In future research we hope to discover more about the relationship between balanced graphs and normal Laplacians, develop a characterization of a class of graphs based on polygonal numerical ranges that incorporates cases one and two, and more ambitiously, case three, and continue to refine our computational methods in order to expand our search space to graphs on seven vertices and greater.

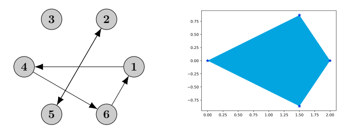
APPENDIX: Graphs on Six Vertices with Polygonal Restricted Numerical Range



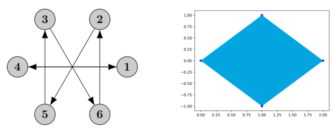
G_1 $W_r(L(G_1))$



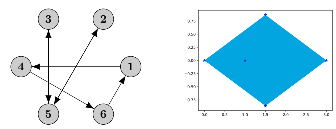
G_2 $W_r(L(G_2))$



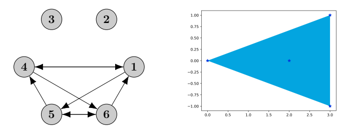
G_3 $W_r(L(G_3))$



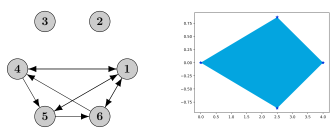
G_4 $W_r(L(G_4))$



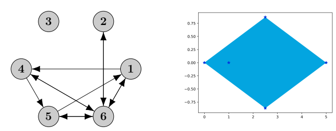
G_5 $W_r(L(G_5))$



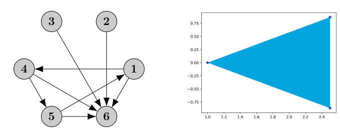
G_6 $W_r(L(G_6))$



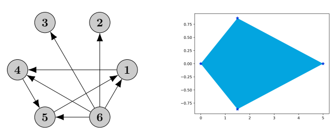
G_7 $W_r(L(G_7))$



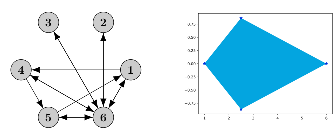
G_8 $W_r(L(G_8))$



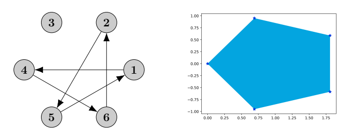
G_9 $W_r(L(G_9))$



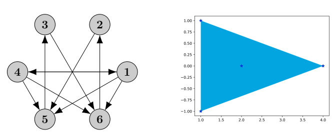
G_{10} $W_r(L(G_{10}))$



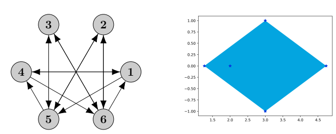
G_{11} $W_r(L(G_{11}))$



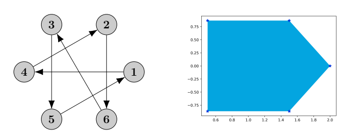
G_{12} $W_r(L(G_{12}))$



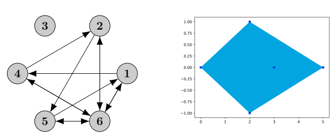
G_{13} $W_r(L(G_{13}))$



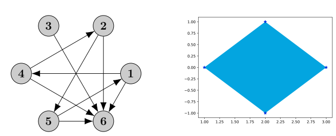
G_{14} $W_r(L(G_{14}))$



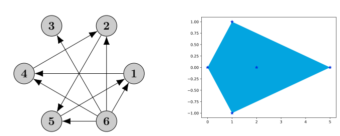
G_{15} $W_r(L(G_{15}))$



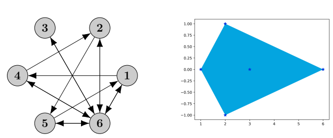
G_{16} $W_r(L(G_{16}))$



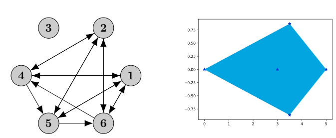
G_{17} $W_r(L(G_{17}))$



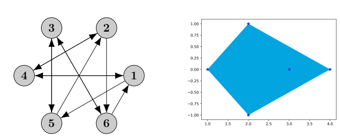
G_{18} $W_r(L(G_{18}))$



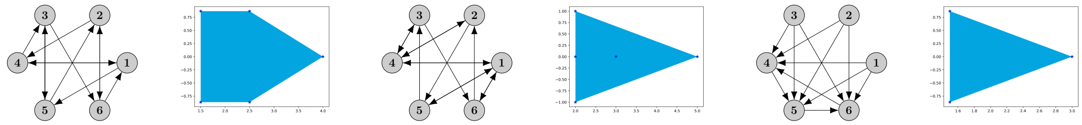
G_{19} $W_r(L(G_{19}))$



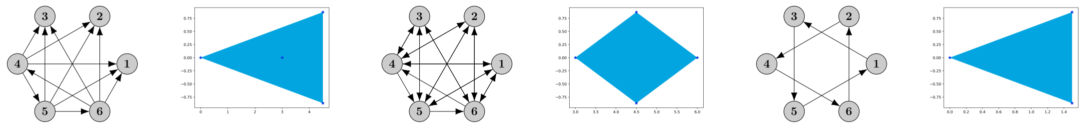
G_{20} $W_r(L(G_{20}))$



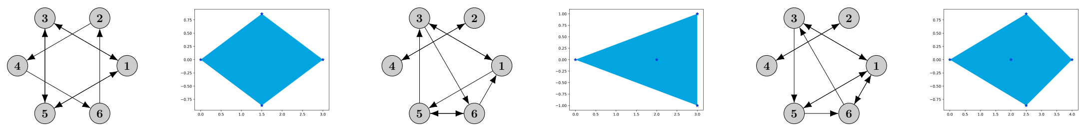
G_{21} $W_r(L(G_{21}))$



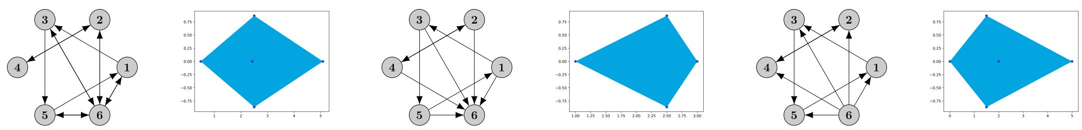
G_{22} $W_r(L(G_{22}))$ G_{23} $W_r(L(G_{23}))$ G_{24} $W_r(L(G_{24}))$



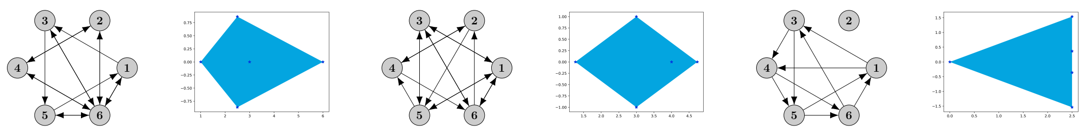
G_{25} $W_r(L(G_{25}))$ G_{26} $W_r(L(G_{26}))$ G_{27} $W_r(L(G_{27}))$



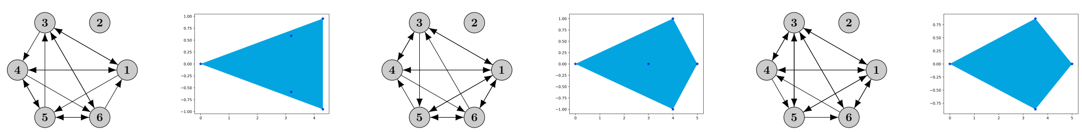
G_{28} $W_r(L(G_{28}))$ G_{29} $W_r(L(G_{29}))$ G_{30} $W_r(L(G_{30}))$



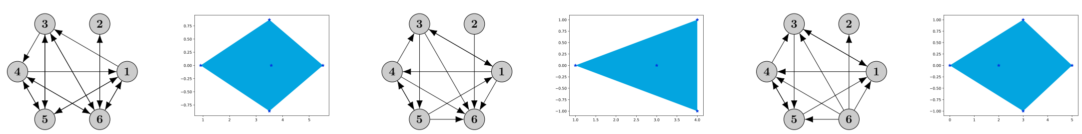
G_{31} $W_r(L(G_{31}))$ G_{32} $W_r(L(G_{32}))$ G_{33} $W_r(L(G_{33}))$



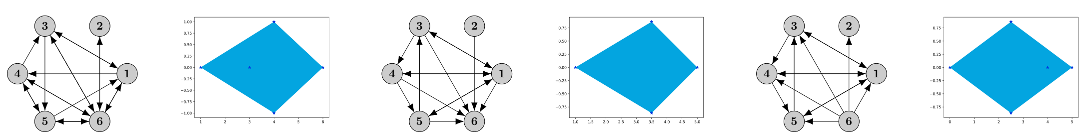
G_{34} $W_r(L(G_{34}))$ G_{35} $W_r(L(G_{35}))$ G_{36} $W_r(L(G_{36}))$



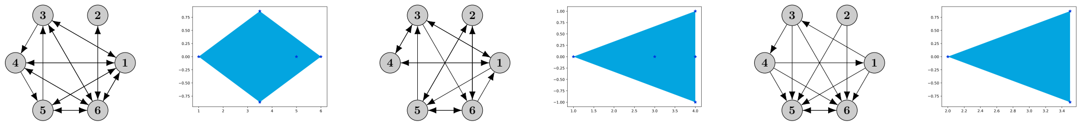
G_{37} $W_r(L(G_{37}))$ G_{38} $W_r(L(G_{38}))$ G_{39} $W_r(L(G_{39}))$



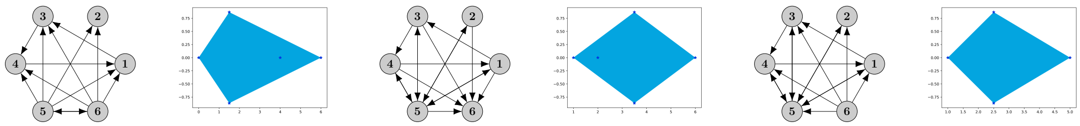
G_{40} $W_r(L(G_{40}))$ G_{41} $W_r(L(G_{41}))$ G_{42} $W_r(L(G_{42}))$



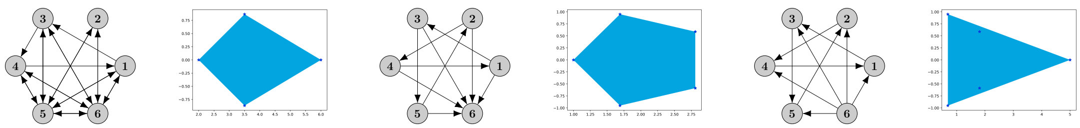
G_{43} $W_r(L(G_{43}))$ G_{44} $W_r(L(G_{44}))$ G_{45} $W_r(L(G_{45}))$



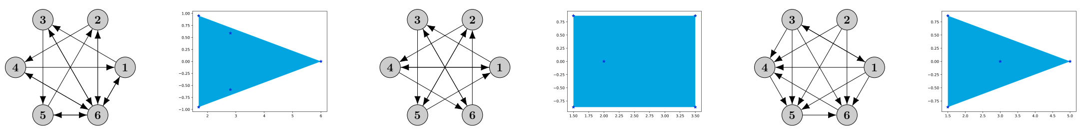
G_{46} $W_r(L(G_{46}))$ G_{47} $W_r(L(G_{47}))$ G_{48} $W_r(L(G_{48}))$



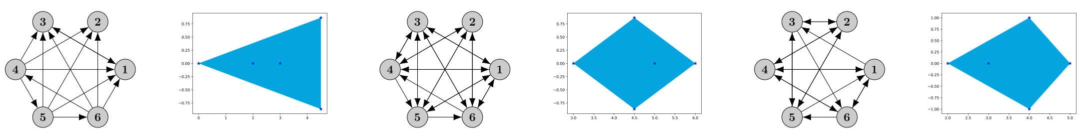
G_{49} $W_r(L(G_{49}))$ G_{50} $W_r(L(G_{50}))$ G_{51} $W_r(L(G_{51}))$



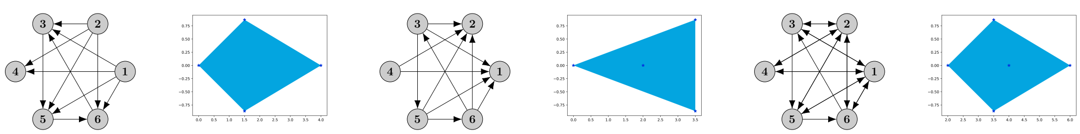
G_{52} $W_r(L(G_{52}))$ G_{53} $W_r(L(G_{53}))$ G_{54} $W_r(L(G_{54}))$



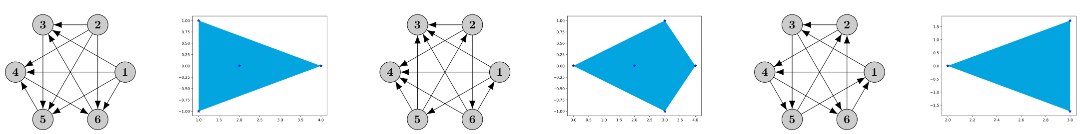
G_{55} $W_r(L(G_{55}))$ G_{56} $W_r(L(G_{56}))$ G_{57} $W_r(L(G_{57}))$



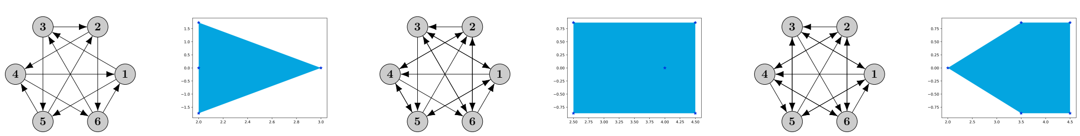
G_{58} $W_r(L(G_{58}))$ G_{59} $W_r(L(G_{59}))$ G_{60} $W_r(L(G_{60}))$



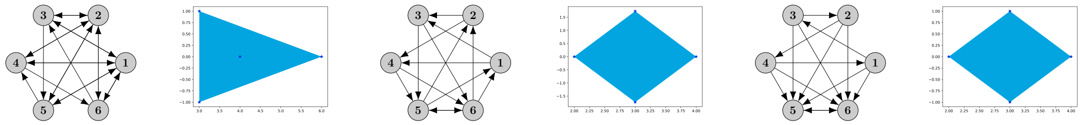
G_{61} $W_r(L(G_{61}))$ G_{62} $W_r(L(G_{62}))$ G_{63} $W_r(L(G_{63}))$



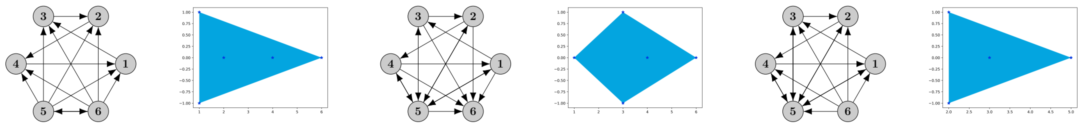
G_{64} $W_r(L(G_{64}))$ G_{65} $W_r(L(G_{65}))$ G_{66} $W_r(L(G_{66}))$



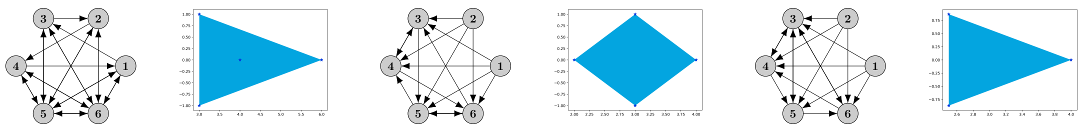
G_{67} $W_r(L(G_{67}))$ G_{68} $W_r(L(G_{68}))$ G_{69} $W_r(L(G_{69}))$



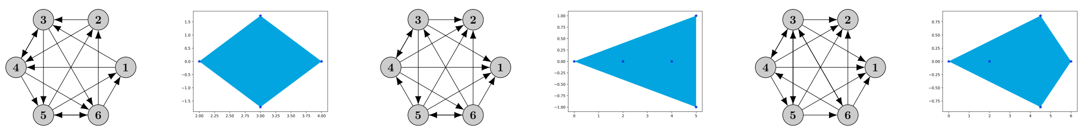
G_{70} $W_r(L(G_{70}))$ G_{71} $W_r(L(G_{71}))$ G_{72} $W_r(L(G_{72}))$



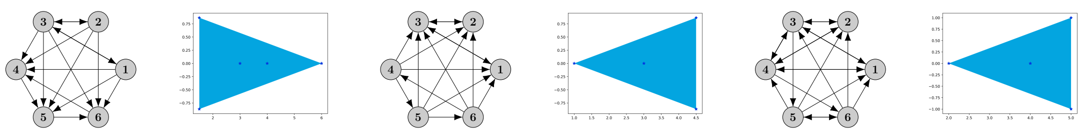
G_{73} $W_r(L(G_{73}))$ G_{74} $W_r(L(G_{74}))$ G_{75} $W_r(L(G_{75}))$



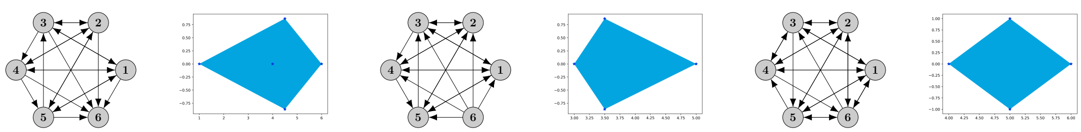
G_{76} $W_r(L(G_{76}))$ G_{77} $W_r(L(G_{77}))$ G_{78} $W_r(L(G_{78}))$



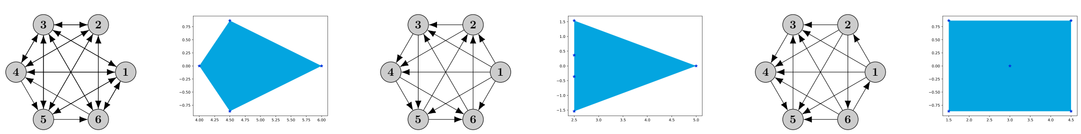
G_{79} $W_r(L(G_{79}))$ G_{80} $W_r(L(G_{80}))$ G_{81} $W_r(L(G_{81}))$



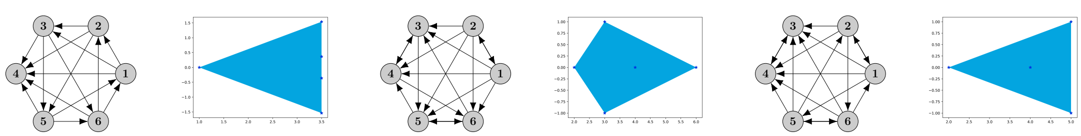
G_{82} $W_r(L(G_{82}))$ G_{83} $W_r(L(G_{83}))$ G_{84} $W_r(L(G_{84}))$



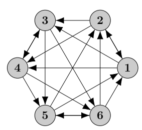
G_{85} $W_r(L(G_{85}))$ G_{86} $W_r(L(G_{86}))$ G_{87} $W_r(L(G_{87}))$



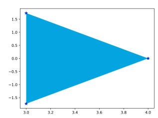
G_{88} $W_r(L(G_{88}))$ G_{89} $W_r(L(G_{89}))$ G_{90} $W_r(L(G_{90}))$



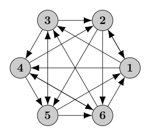
G_{91} $W_r(L(G_{91}))$ G_{92} $W_r(L(G_{92}))$ G_{93} $W_r(L(G_{93}))$



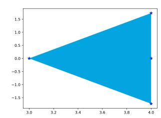
G_{94}



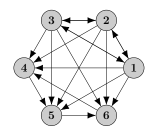
$W_r(L(G_{94}))$



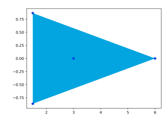
G_{95}



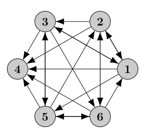
$W_r(L(G_{95}))$



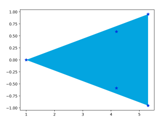
G_{96}



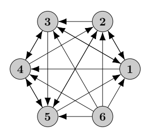
$W_r(L(G_{96}))$



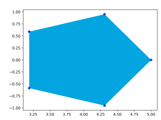
G_{97}



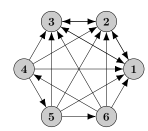
$W_r(L(G_{97}))$



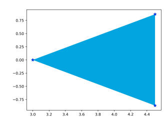
G_{98}



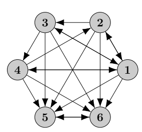
$W_r(L(G_{98}))$



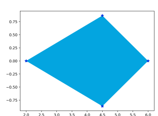
G_{99}



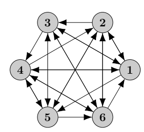
$W_r(L(G_{99}))$



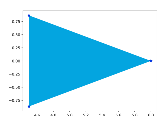
G_{100}



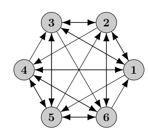
$W_r(L(G_{100}))$



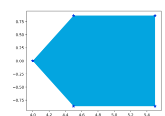
G_{101}



$W_r(L(G_{101}))$



G_{102}



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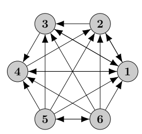
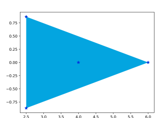
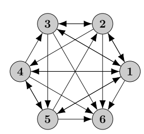


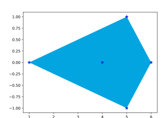
G_{103}



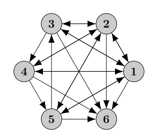
$W_r(L(G_{103}))$



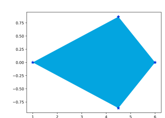
G_{104}



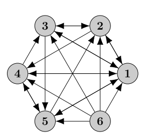
$W_r(L(G_{104}))$



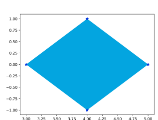
G_{105}



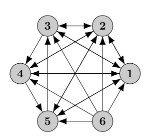
$W_r(L(G_{105}))$



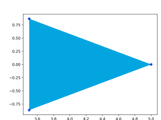
G_{106}



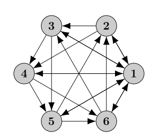
$W_r(L(G_{106}))$



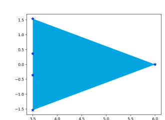
G_{107}



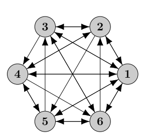
$W_r(L(G_{107}))$



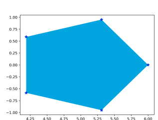
G_{108}



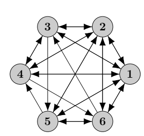
$W_r(L(G_{108}))$



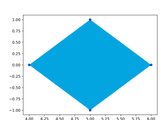
G_{109}



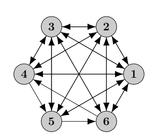
$W_r(L(G_{109}))$



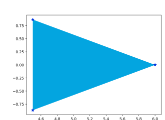
G_{110}



$W_r(L(G_{110}))$



G_{111}



$W_r(L(G_{111}))$

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