Handout: Restricted Numerical Ranges of Digraph Laplacians

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- For a graph G on n vertices, the adjacency matrix $A(G)$ is the $n \times n$ matrix with entry $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise. The degree matrix $D(G)$ is the $n \times n$ diagonal matrix with entry $a_{ii} = d^+(v_i)$ and 0 elsewhere. The *Laplacian Matrix* $L(G)$ is then the $n \times n$ matrix given by

  $$L(G) = D(G) - A(G)$$

  or, shortening our notation,

  $$L = D - A. \tag{1}$$

- Let A be an $n \times n$ matrix acting on vectors $x \in \mathbb{C}^n$. The *numerical range* of A is a set of scalar values in $\mathbb{C}$ defined as in

  $$W(A) = \{x^*Ax : x \in \mathbb{C}^n, ||x|| = 1\}, \tag{2}$$

- We define Q *matrices* as the class of real $n \times (n-1)$ orthonormal matrices whose columns are perpendicular to $e = (1, 1 \ldots 1)$.

- Using the above definition of Q matrices, the *restricted numerical range* (of a graph Lapacian L), defined as

  $$W_r(L) = \{x^*Lx : x \perp e, ||x|| = 1, x \in \mathbb{C}^n\},$$

  may be written as

  $$W_r(L) = W(Q^T LQ). \tag{3}$$

- A graph G is 3-balanced if for any three distinct vertices $i, j, k \in V(G)$, we have

  $$a_{ij} + a_{jk} + a_{ki} = a_{ji} + a_{ik} + a_{kj}, \tag{4}$$