Welcome!

'One of the marvelous things about community is that it enables us to welcome and help people in a way we couldn't as individuals. When we pool our strength and share the work and responsibility, we can welcome many people, even those in deep distress.' Jean Vanier

Restricted Numerical Ranges of Digraph Laplacians: Defense

Michael Robertson

Davidson College

April 30, 2020

Mathematical Background

Definitions

- For a graph G on n vertices, the adjacency matrix $A(G)$ is the $n \times n$ matrix with entry $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise. The degree matrix $D(G)$ is the $n \times n$ diagonal matrix with entry $a_{ii} = d^+(v_i)$ and 0 elsewhere. The *Laplacian Matrix* $L(G)$ is then the $n \times n$ matrix given by

$$L(G) = D(G) - A(G) \tag{1}$$

or, shortening our notation, L = D - A.

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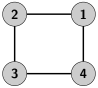
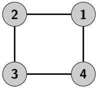
- Let A be an $n \times n$ matrix acting on vectors $x \in \mathbb{C}^n$. The *numerical range* of A is a set of scalar values in $\mathbb{C}$ defined as in

$$W(A) = \{x^* Ax : x \in \mathbb{C}^n, ||x|| = 1\}, \tag{2}$$

Example



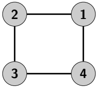
Example



Adjacency

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Example



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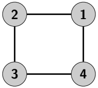

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Historical Background

Algebraic Connectivity α for undirected graphs [1]

Miroslav Fiedler (1926-2015)

[1] M. Fiedler. Algebraic connectivity of graphs. Czechoslovak Mathematical Journal, 23(2):298–305, 1973.

Algebraic Connectivity

Fiedler defined α as the second smallest eigenvalue of the Laplacian matrix for a graph.

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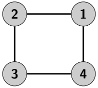
"Smallest" well defined

Second smallest

$$e = (1, 1, \ldots, 1), e \in \mathrm{Null}(L)$$

Example



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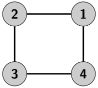

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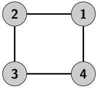

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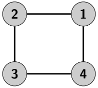
$$\sigma(L) = \{0, 2, 2, 4\}$$

Example



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$$\alpha(L) = 2$$

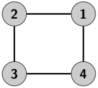
Undirected Graphs vs Directed Graphs

Undirected Graph

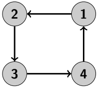
Directed Graph (Digraph)

Undirected Graphs vs Directed Graphs

Undirected Graph

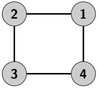


Directed Graph (Digraph)



Undirected Graphs vs Directed Graphs

Undirected Graph



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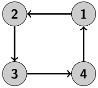

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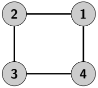
Undirected Graphs vs Directed Graphs

Undirected Graph



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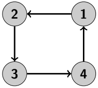


$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma(L) = \{0, 1+i, 1-i, 2\}$$

*A*lgebraic connectivity α for digraphs [2004].[2]

Chai Wah Wu (1968-)

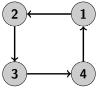
[1]C. W. Wu. Algebraic connectivity of directed graphs. Linear and Multilinear Algebra, 53:3:203–223, 2005

Definition:

For directed graphs on n vertices, let $S = \{x \in \mathbb{R}^n : x \perp e, ||x|| = 1\}$. Then

$$\alpha = \min_{x \in S} x^T L x$$

Example



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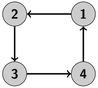

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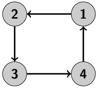

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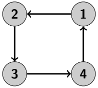

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$$\alpha(G) = \min_{x \in S} x^T L x = 1$$

Our work

Fielders's Definition:

The algebraic connectivity α is the second smallest eigenvalue of L.

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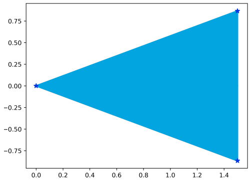
0.75
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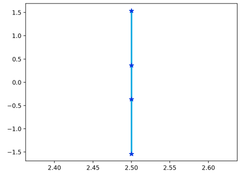
−0.75
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4


1.5
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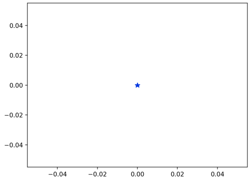
2.40 2.45 2.50 2.55 2.60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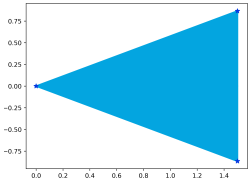
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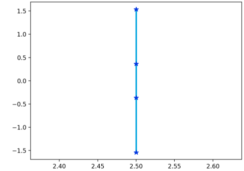
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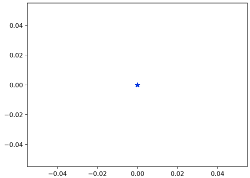
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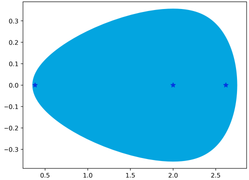
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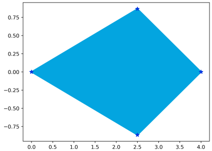
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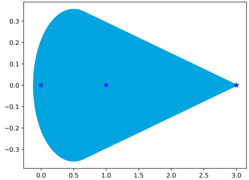
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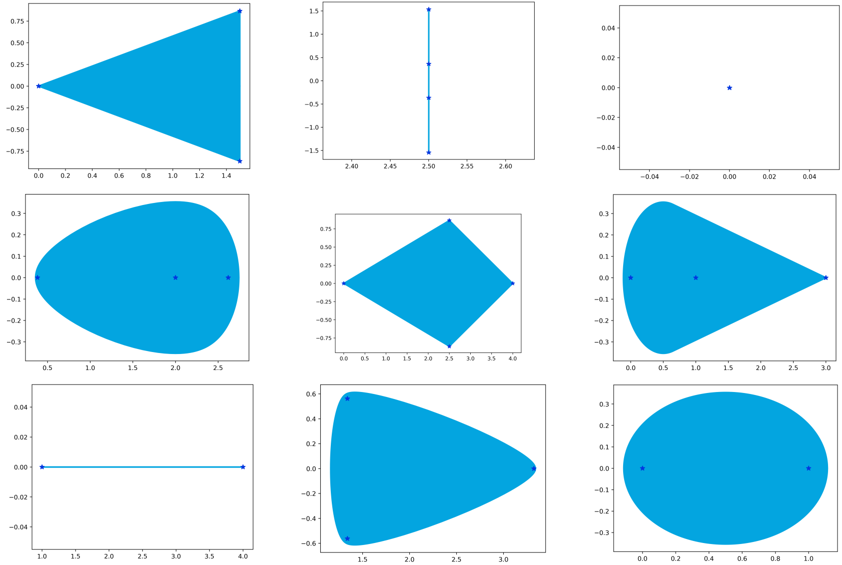
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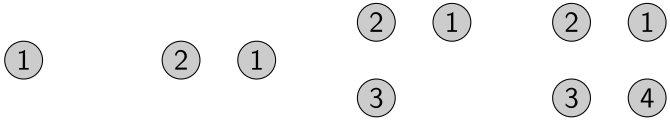
Characterizations

Graph Characterizations — Empty Graph

One vertex | Two vertices | Three | Four



Graph Characterizations — Empty Graph

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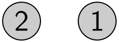
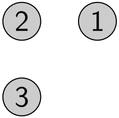
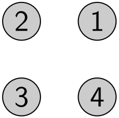
Two vertices

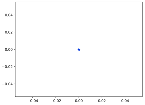


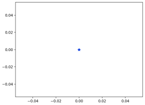
Three

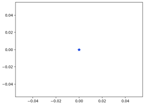


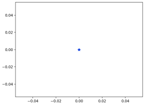
Four











Theorems

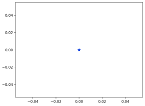
In the following theorems let G be a graph on n vertices.

Theorem — Empty Graph

The graph G is the empty graph if and only if the restricted numerical range of $L(G)$ is $\{0\}$. That is, $G = E_n$ if and only if $W_r(L(G)) = \{0\}$.

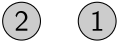
Graph Characterizations — Empty Graph

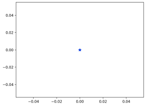
One vertex



$\{0\}$

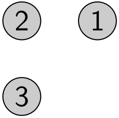
Two vertices

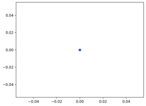




$\{0\}$

Three





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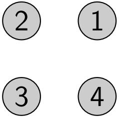
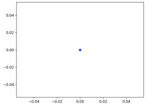
Four

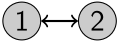




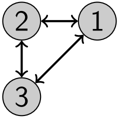
$\{0\}$

Graph Characterizations — Complete Graph

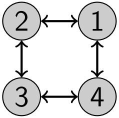
Two vertices



Three

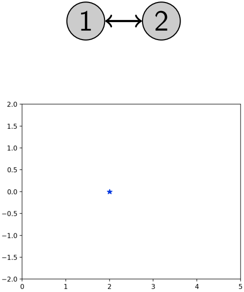


Four

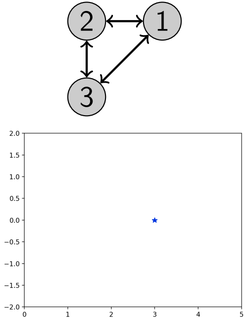


Graph Characterizations — Complete Graph

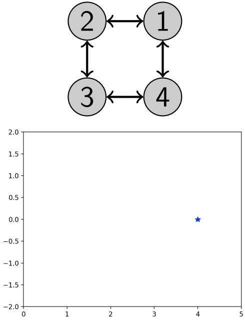
Two vertices



Three



Four



Theorems

In the following theorems let G be a graph on n vertices.

Theorem — Empty Graph

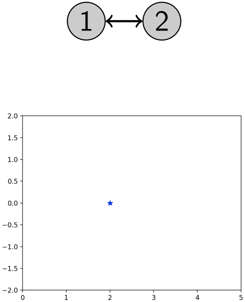
... That is, $G = E_n$ if and only if $W_r(L(G)) = \{0\}$.

Theorem — Complete Graph

For $n > 1$, G is the complete graph if and only if the restricted numerical range of $L(G)$ is $\{n\}$. That is, $G = K_n$ if and only if $W_r(L(G)) = \{n\}$.

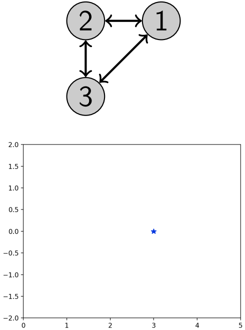
Graph Characterizations — Complete Graph

Two vertices



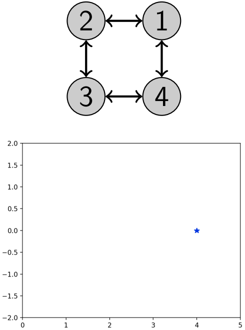
$\{2\}$

Three



$\{3\}$

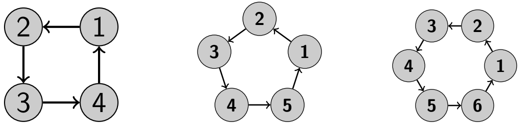
Four



$\{4\}$

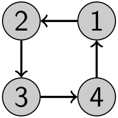
Graph Characterizations — Cycle

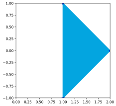
Four vertices Five Six



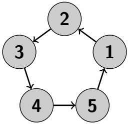
Graph Characterizations — Cycle

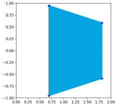
Four vertices



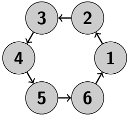


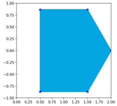
Five





Six





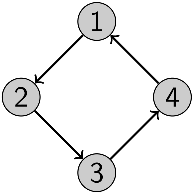
Woah...

The cycle...

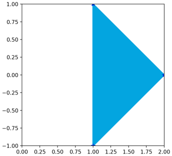
Woah...

The cycle...

C_4



$W_r(L(C_4))$



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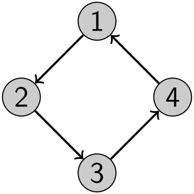

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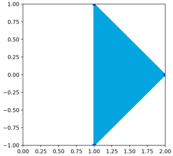

is a *Cycle*!

Woah...

The cycle...

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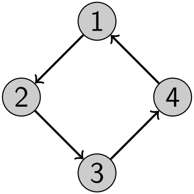
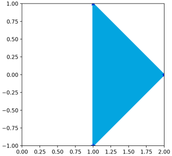


$W_r(L(C_4))$



is a *Cycle*!

Convex hull of $\{1 - e^{(2\pi j/n)i} : j = 1, \ldots, n-1\}$

Theorems

In the following theorems let G be a graph on n vertices.

Theorem — Empty Graph

... That is, $G = E_n$ if and only if $W_r(L(G)) = \{0\}$.

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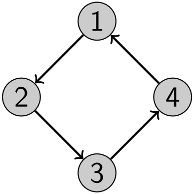
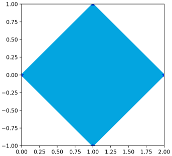


$W(L(C_4))$

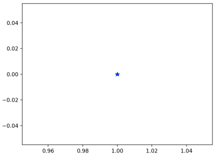


is a *Cycle*!

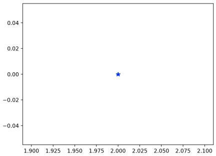
Wonder

'The union of the mathematician with the poet, fervor with measure, passion with correctness, this surely is the ideal.' William James

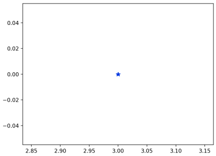
Graph Characterizations — Singleton Restricted Numerical Range



$\{1\}$



$\{2\}$



$\{3\}$

Definition — Directed Join

We define the directed join of G onto H, $G\vec{\nabla}H = (V, E)$ where $V = V(G) \cup V(H)$ and

$$E = E(G) \cup E(H) \cup \{vu : v \in V(G), u \in V(H)\}.$$

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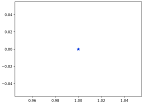
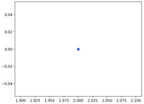
Definition — k-Imploding Star vertices

The k-imploding star on n vertices is the directed join of the empty graph on $n-k$ onto the complete graph on k vertices. We write $S_{n,k} = E_{n-k} \vec{\nabla} K_k$.

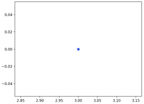
Graph Characterizations — Singleton Restricted Numerical Range



$\{1\}$

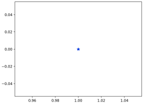


$\{2\}$



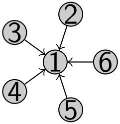
$\{3\}$

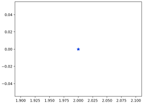
Graph Characterizations — Singleton Restricted Numerical Range



$\{1\}$

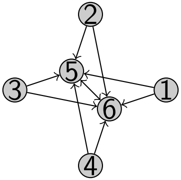
$S_{5,1}$

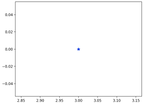




$\{2\}$

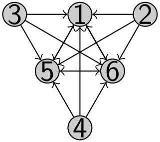
$S_{5,2}$





$\{3\}$

$S_{5,3}$



Theorems

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Theorem — Imploding Stars

For $k \geq 0$, G is a k-imploding star on $n \geq k$ vertices if and only if the restricted numerical range of $L(G)$ is the singleton $\{k\}$. That is, $G = S_{n,k}$ if and only if $W_r(L(G)) = \{k\}$.

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All singleton numerical ranges are integers

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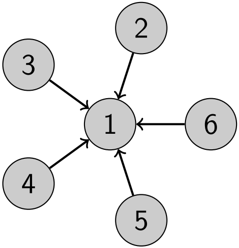
Theorem — Singleton Numerical Ranges

All singleton numerical ranges are integers (and so they correspond to k-imploding stars).

Graph Characterizations — Singleton Restricted Numerical Range

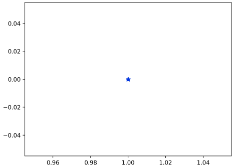
1-Imploding Star: $S_{6,1}$



Spectrum: $\sigma(L(S_{6,1}))$

$$\sigma = \{0, 1, 1, 1, 1, 1\}$$

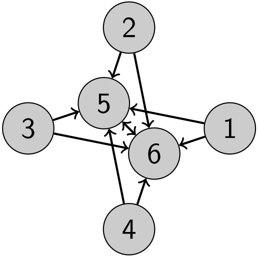
$W_r(L(S_{6,1}))$



$$\{1\}$$

Graph Characterizations — Singleton Restricted Numerical Range

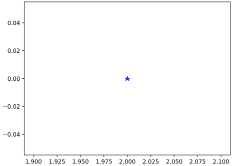
$S_{6,2}$



$\sigma(L(S_{6,2}))$

$$\sigma = \{0, 2, 2, 2, 2, 2\}$$

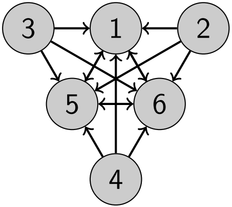
$W_r(L(S_{6,2}))$



$$\{2\}$$

Graph Characterizations — Singleton Restricted Numerical Range

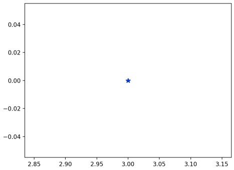
$S_{6,3}$



$\sigma(L(S_{6,3}))$

$$\sigma = \{0, 3, 3, 3, 3, 3\}$$

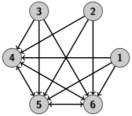
$W_r(L(S_{6,3}))$



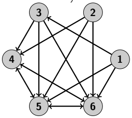
$$\{3\}$$

Great Success!

$S_{6,3}$



$S^*_{6,3}$



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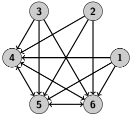
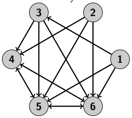


$\sigma(L(S_{6,3})) = \{0, 3, 3, 3, 3, 3\}$

$S_{6,3}^*$



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Great Success!

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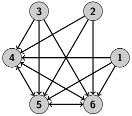

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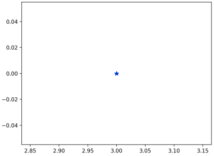
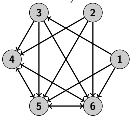
$W_r(S_{6,3})$



$S^*_{6,3}$



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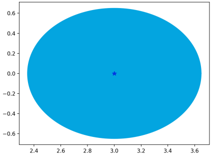
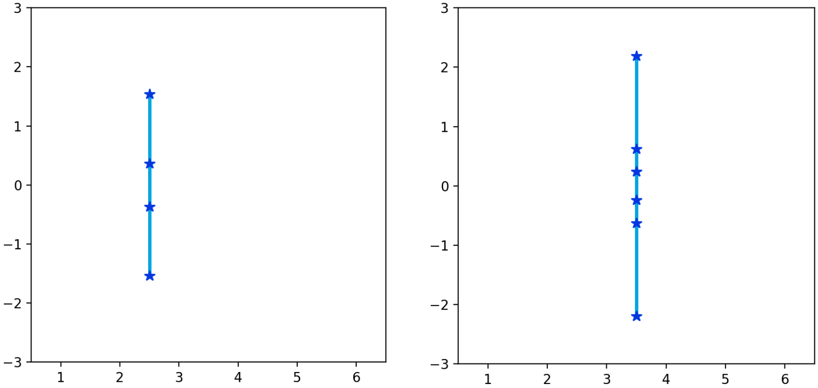


Graph Characterizations — Vertical Line Restricted Numerical Range



Definition — Tournament Graph

A graph G is a tournament graph if for any two vertices $v, u \in G$, exactly one of (u, v) and (v, u) are in E.

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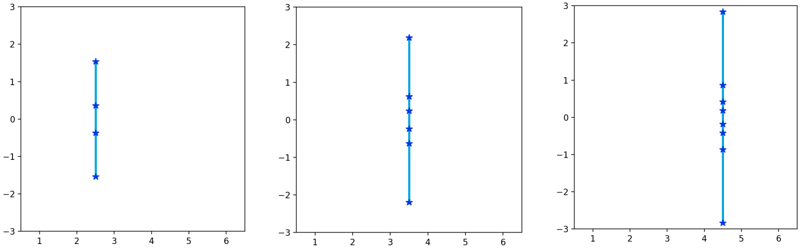
Definition — Regular Tournament Graph

A tournament graph G on an odd number of vertices is regular if every vertex has equal in-degree and out-degree.

Graph Characterizations — Vertical Line Restricted Numerical Range



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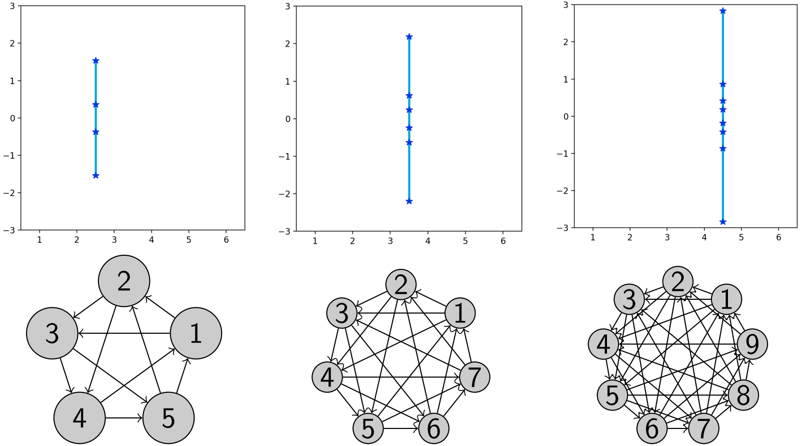

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Theorem — Regular Tournaments

A graph G on n vertices is a regular tournament if and only if $W_r(G)$ is a vertical line with real part $\frac{n}{2}$.

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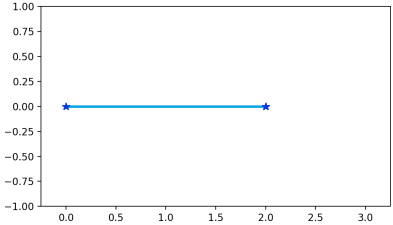
Theorem — Imploding Stars

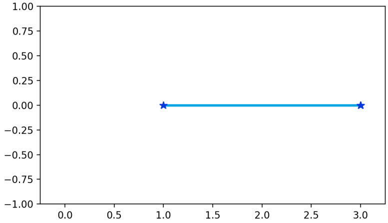
... That is, $G = S_{n,k}$ if and only if $W_r(L(G)) = \{k\}$.

Theorem — Regular Tournaments

A graph G on n vertices is a regular tournament if and only if $W_r(G)$ is a vertical line with real part $\frac{n}{2}$. Further, all vertical lines are of this form (and thus correspond to regular tournament graphs). So all vertical lines correspond to regular tournament graphs, and vice-versa.

Graph Characterizations — Real Numerical Range



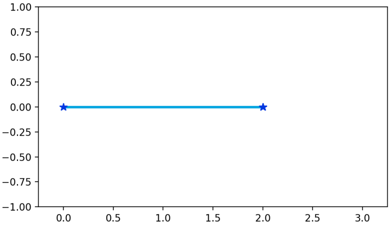


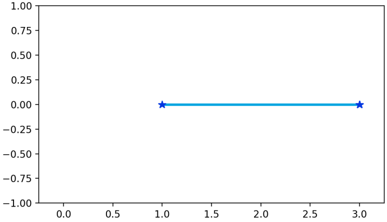
Definition — 3-Balanced

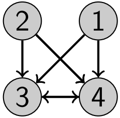
A graph G is 3-balanced if for any three distinct vertices $i, j, k \in V(G)$, we have

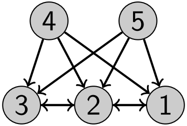
$$a_{ij} + a_{jk} + a_{ki} = a_{ji} + a_{ik} + a_{kj}, \tag{4}$$

Graph Characterizations — Real Numerical Range









Theorems

In the following theorems let G be a graph on n vertices.

Theorem — Imploding Stars

. . . That is, $G = S_{n,k}$ if and only if $W_r(L(G)) = \{k\}$.

Theorem — Regular Tournaments

. . . So all vertical lines correspond to regular tournament graphs, and vice-versa.

Theorems

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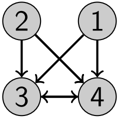
Theorem — Regular Tournaments

...So all vertical lines correspond to regular tournament graphs, and vice-versa.

Theorem — 3-Balanced Graphs

A graph G is 3-balanced if and only if its restricted numerical range lies entirely on the real line. That is, G is 3-balanced if and only if $W_r(L(G)) \subset \mathbb{R}$

Three Balanced example



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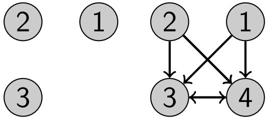

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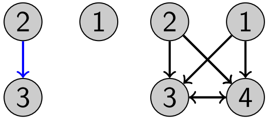

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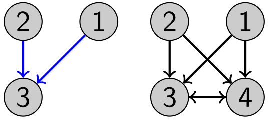

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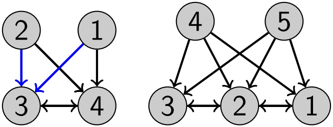

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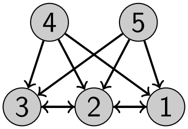

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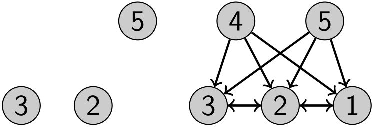

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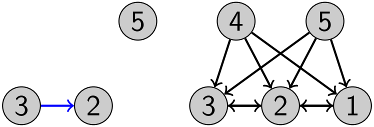

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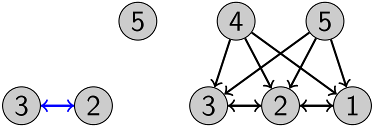

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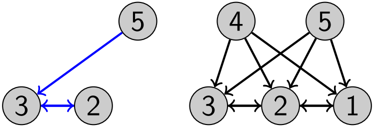

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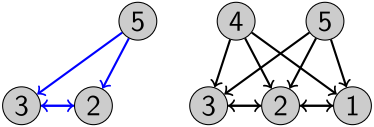

Current Work

Graph Characterizations — Polygonal Numerical Range

- Case 1: Normal Laplacian L

Graph Characterizations — Polygonal Numerical Range

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Graph Characterizations — Polygonal Numerical Range

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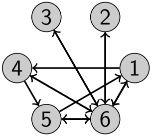
1: Normal L



2: Normal $Q^T LQ$

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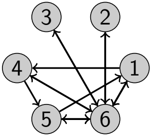

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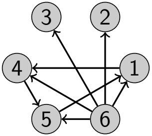

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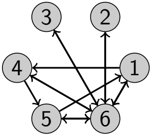

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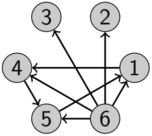

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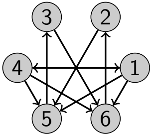

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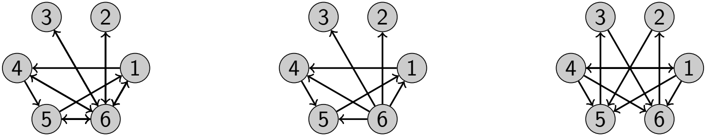

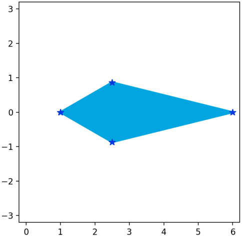

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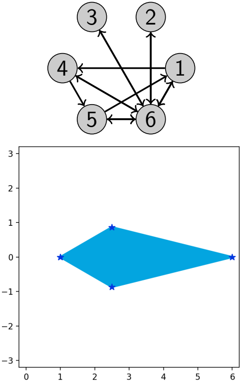

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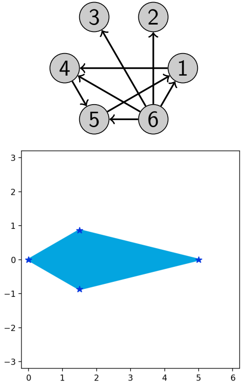

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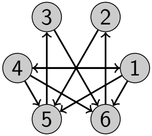

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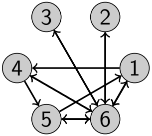

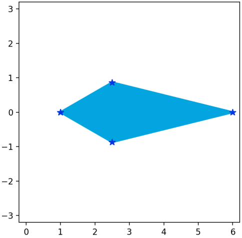

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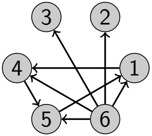

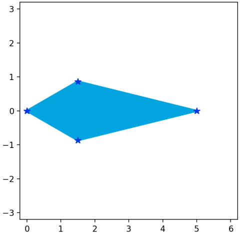

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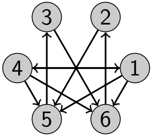

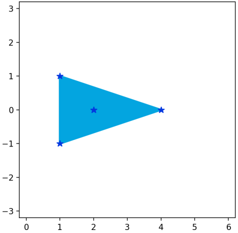

Thanks

- Thomas Cameron, Advisor, Co-author

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- Carl Yerger, Chair

Thanks for coming! 😄