Homework Set 2

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1.Prove that $\overline{A\cap B}=\overline{A}\cup\overline{B}$ by giving a proof using logical equivalence.
Solution: We can prove this identity with the following steps.

$$\begin{aligned}
\overline{A\cap B} &= \{x|x\notin A\cap B\} \\
&= \{x|\neg(x\in(A\cap B))\} \\
&= \{x|\neg(x\in A\wedge x\in B)\} \\
&= \{x|x\notin A\vee x\notin B\} \\
&= \{x|x\in\overline{A}\vee x\in\overline{B}\} \\
&= \{x|x\notin\overline{A}\cup\overline{B}\} \\
&= \overline{A}\cup\overline{B}
\end{aligned}$$

In Problems 2-4 mark each statement TRUE or FALSE. Assume that the statement applies to all sets.

2. *A-(B-C)=(A-B)-C.*
Solution: FALSE

Prove:

$$\begin{aligned}
\textit{A-(B-C)} &= A\cap\overline{(B-C)} \\
&= A\cap\overline{B\cap\overline{C}} \\
&= A\cap(\overline{B}\cup C)
\end{aligned}$$

while

$$\begin{aligned}
\textit{(A-B)-C} &= (A\cap\overline{B})\cap\overline{C} \\
&= A\cap(\overline{B}\cap\overline{C})
\end{aligned}$$

As $\overline{B}\cup C\neq\overline{B}\cap\overline{C}$
So $A-(B-C)\neq(A-B)-C$

3.*(A-C)-(B-C)=A-B.*
Solution: FALSE

Prove:

$$\begin{aligned}
\textit{(A-C)-(B-C)} &= (A\cap\overline{C})\cap\overline{(B\cap\overline{C})} \\
&= (A\cap\overline{C})\cap(\overline{B}\cup C) \\
&= (A\cap\overline{C}\cap\overline{B})\cup(A\cap\overline{C}\cap C) \\
&= (A\cap\overline{C}\cap\overline{B})\cup\emptyset \\
&\neq A\cap\overline{B} \\
&= A-B
\end{aligned}$$

4. $\overline{A\cup\overline{B}}\cup\overline{A}=\overline{A}$
Solution: TRUE

Prove:

$$\overline{A\cup\overline{B}}\cup\overline{A} = (\overline{A}\cap B)\cup\overline{A}$$

As $(\overline{A}\cap B)\subseteq\overline{A}$
So $(\overline{A}\cap B)\cup\overline{A}=\overline{A}$
That is $\overline{A\cup\overline{B}}\cup\overline{A}=\overline{A}$

5. Find $\bigcap\limits_{i=1}^{+\infty}[1-\frac{1}{i},1]$.

Solution: $\forall i,j \in [1,+\infty)$, i and j are integers, and $i<j$

so that $1-\frac{1}{i}<1-\frac{1}{j}$

we can prove that $[1-\frac{1}{i},1]\bigcap[1-\frac{1}{j},1]=[1-\frac{1}{j},1]$

So $\bigcap\limits_{i=1}^{+\infty}[1-\frac{1}{i},1]=\lim\limits_{i\to+\infty}[1-\frac{1}{i},1]=[1-\frac{1}{\lim\limits_{i\to+\infty}i},1]=1$

6. Suppose $A=\{x,y\}$ and $B=\{x,\{x\}\}$. Mark the statement TRUE or FALSE: $\{x\}\subseteq A-B$.

Solution: It is FALSE.

$$\begin{aligned} A-B &= \{x|x\in A\cap\overline{B}\} \\ &= \{x|x\in A\wedge x\notin B\} \\ &= \{y\} \end{aligned}$$

So $\{x\}\nsubseteq A-B$

In the meanwhile:

$B-A=\{\{x\}\}$

So that $\{x\}\nsubseteq B-A$ either

7. Suppose $A=\{1,2,3,4,5\}$. Mark the statement TRUE or FALSE:

$\{\{3\}\}\subseteq P(A)$.

Solution: It is TRUE.

Because $P(A)=\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,3\},...,\{1,2,3,4,5\}\}$

So $\{\{3\}\}\subseteq P(A)$ is TRUE.

8. Determine whether the set is finite or infinite. If the set is finite, find its size:

$\{x|x\in N \ and \ 4x^2-8=0\}$.

Solution: As there are only two solutions to the equation $4x^2-8=0$, that is {2,-2}.

while it should also satisfy $\{x|x\in N\}$ which means x is an integer and $x\geq 0$.

So the set $\{x|x\in N \ and \ 4x^2-8=0\}$ is finite and its size $|A|=1$.

9. Determine whether the following set is countable or uncountable. If it is countably infinite, exhibit a one-to-one correspondence between the set of positive integers and the set:
The set of irrational numbers between $\sqrt{2}$ and $\frac{\pi}{2}$.

Solution: This set is uncountable.

Let :

Set $A=\{rational\ numbers\ between\ \sqrt{2}\ and\ \frac{\pi}{2}\}$

Set $B=\{irrational\ numbers\ between\ \sqrt{2}\ and\ \frac{\pi}{2}\}$

Set $C=\{real\ numbers\ between\ \sqrt{2}\ and\ \frac{\pi}{2}\}$

We know that all real numbers are uncountable, and real numbers between $\sqrt{2}$ *and* $\frac{\pi}{2}$ is uncountable either, which means set C is uncountable.

And we also know rational numbers is countable and rational numbers between $\sqrt{2}$ *and* $\frac{\pi}{2}$ is countable too, which means set A is countable.

We suppose that irrational numbers between $\sqrt{2}$ *and* $\frac{\pi}{2}$ is countable, that is set B is countable.

According to the Theorem "If A and B are countable sets, then $A\cup B$ is also countable".

$C=A\cup B$, so C is countable. It is contradict with the known knowledge. So the hypothesis is false, which means irrational numbers between $\sqrt{2}$ *and* $\frac{\pi}{2}$ is uncountable.

10. Show that (0,1) has the same cardinality as (0,2).

Solution:

According to SCHRDER-BERNSTEIN THEOREM: If A and B are sets with $|A|\leq|B|$ and $|B|\leq|A|$, then $|A|=|B|$.

So we only have to find a one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

(i)Let f be the function $f(x)=2x$, where the domain is $x\in(0,1)$

So $\forall x \in (0,1), f(x) \in (0,2)$
And $\forall x_1, x_2 \in (0,1),\ x_1 \neq x_2$,
$f(x_1) - f(x_2) = 2x_1 - 2x_2 = 2(x_1 - x_2) \neq 0$
which means function f is a one-to-one function,
so that $|(0,1)| \leq |(0,2)|$
(ii)In the same way, let g be the function $g(x) = \frac{x}{2}$, where the domain is $x \in (0,2)$
we can draw a conclusion that $|(0,2)| \leq |(0,1)|$
From above, we know $|(0,2)| = |(0,1)|$, that's "(0,1) has the same cardinality as (0,2)".
Done

11. Suppose $f : N \to N$ has the rule $f(n) = 4n + 1$. Determine whether f is 1-1.
Solution: As f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.
So $\forall a, b \in N$, while $a \neq b$, it is obviously that $f(a), f(b) \in N$
$f(b) - f(a) = (4b + 1) - (4a + 1) = 4(b - a) \neq 0$
that is $f(b) \neq f(a)$
We can determine that f is 1-1.

12. Suppose $f : N \to N$ has the rule $f(n) = 4n + 1$. Determine whether f is onto N.
Solution: According to the definition:
a function f is onto if $\forall y \exists x (f(x) = y)$
So $\forall b \in N$,we suppose there exists an $a \in N$ so that $f(a) = b$
namely, $f(a) = 4a + 1 = b$
$\therefore\ a = \frac{b-1}{4}$
However, a will not always be natural numbers, for example, while $b = 2$, then $a = \frac{1}{4}$
So, this function f is not onto N.

13. Suppose $f : Z \to Z$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is 1-1.
Solution: As f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.
$\forall a, b \in Z$ and $a \neq b$

$$\begin{aligned} f(b) - f(a) &= (3a^2 - 1) - (3b^2 - 1) \\ &= 3a^2 - 3b^2 \\ &= 3(a^2 - b^2) \\ &= 3(a + b)(a - b) \end{aligned}$$

as $a, b \in Z$,when $a = (-b) \neq 0, \quad f(b) - f(a) = 0$
So $f(n) = 3n^2 - 1$ is not 1-1.

14. Suppose $f : Z \to Z$ has the rule $f(n) = 3n - 1$. Determine whether f is onto Z.
Solution: According to the definition:
a function f is onto if $\forall y \exists x (f(x) = y)$
$\forall b \in Z$, we suppose there exists an $a \in Z$ so that $f(a) = b$
namely, $f(a) = 3a - 1 = b$
$\therefore\ a = \frac{b+1}{3}$
However, a is not always integer. For example, when $b = 1$, $a = \frac{2}{3}$
So $f(n) = 3n - 1$ is not onto Z.

15. Suppose $f : N \to N$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is 1-1.
Solution: As f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.
$\forall a, b \in N$ and $a \neq b$

$$\begin{aligned} f(b) - f(a) &= (3a^2 - 1) - (3b^2 - 1) \\ &= 3a^2 - 3b^2 \\ &= 3(a^2 - b^2) \\ &= 3(a + b)(a - b) \end{aligned}$$

Because $a, b \in N$, and $a \neq b$, so $(a + b) > 0,\ (a - b) \neq 0$
So that $f(b) - f(a)$= $3(a + b)(a - b) \neq 0$

So $f(n) = 3n^2 - 1$ is 1-1.

16. Suppose $f : N \to N$ has the rule $f(n) = 4n^2 - 1$. Determine whether f is onto N.
Solution: According to the definition:
a function f is onto if $\forall y \exists x (f(x) = y)$
$\forall b \in N$, we suppose there exists an natural number a that $b = 4a^2 - 1$
$a = \sqrt{\frac{b+1}{4}}$
However a is not always natural number.
For example, $b = 1$, then $a = \sqrt{\frac{1}{2}}$
So, we can say that $f(n) = 4n^2 - 1$ is not onto N.

17. Suppose $g : R \to R$ where $g(x) = \lfloor \frac{x-1}{2} \rfloor$.
(i)Is g 1-1?
(ii)Is g onto R
Solution:
(i)g is not 1-1.
For example, $a = 1.1,\ b = 1.2$, obviously, $a \neq b$
then $g(a) = \lfloor \frac{1.1-1}{2} \rfloor = 0 = \lfloor \frac{1.2-1}{2} \rfloor = g(b)$
It is contradict with the definition of one-to-one function.So g is not 1-1.
(ii)g is not onto R
Because the range of function g is Z, so we can choose a fraction from R.
For example, $g(a) = \lfloor \frac{a-1}{2} \rfloor = \frac{3}{2}$
when $a < 3$, $g(a) < \lfloor \frac{3-1}{2} \rfloor = 1$
when $3 \leq a < 5$, $g(a) = 1$
when $a > 5$, $g(a) > \lfloor \frac{5-1}{2} \rfloor = 2$
so there is no a that can let function g be $\frac{3}{2}$, which $\frac{3}{2} \in R$.

In questions 18-19 suppose $g : A \to B$ and $f : B \to C$ where $A = \{a, b, c, d\}, B = \{1, 2, 3\}, C = \{2, 3, 6, 8\}$, and g and f are defined by $g = \{(a, 2), (b, 1), (c, 3), (d, 2)\}$ and $f = \{(1, 8), (2, 3), (3, 2)\}$.

18. Find $f \circ g$.
Solution: For all $x \in A$
$f \circ g(x) = f(g(x))$
$\therefore \quad f \circ g(a) = f(g(a)) = f2 = 3$
$f \circ g(b) = f(g(b)) = f(1) = 8$
$f \circ g(c) = f(g(c)) = f(3) = 2$
$f \circ g(d) = f(g(d)) = f(2) = 3$
So $f \circ g$=$\{(a, 3), (b, 8), (c, 2), (d, 3)\}$

19. Find f^{-1}.
Solution: We can see that f is one-to-one, so there exists an f^{-1} that $f^{-1} \circ f(x) = x$ when $x \in B$
$\therefore \quad f^{-1}(f(1)) = f^{-1}(8) = 1$
$f^{-1}(f(2)) = f^{-1}(3) = 2$
$f^{-1}(f(3)) = f^{-1}(2) = 3$
So $f^{-1} = \{(8, 1), (3, 2), (2, 3)\}$

20. Suppose $g : A \to B$ and $f : B \to C$, where $f \circ g$ is 1-1 and g is 1-1. Must f be 1-1?
Solution: The statement that "f be 1-1 " is FALSE.
We can create a counterexample:
$A = \{1\},\ B = \{2, 3\},\ C = \{4\}$ and $g : A \to B$ and $f : B \to C$ are defined by $g = \{(1, 2)\}$ and $f = \{(2, 4),\ (3, 4)\}$
So $\quad f \circ g = \{(1, 4)\}$

Obviously, g and $f \circ g$ are 1-1, but f is not 1-1.

21. Suppose $g : A \to B$ and $f : B \to C$, where $f \circ g$ is 1-1 and f is 1-1. Must g be 1-1?
Solution: We suppose that g is not 1-1.
Which means there exists $a, b \in A$ and $a \neq b$, but $g(a) = g(b) = c$
So that $\quad f \circ g(a) = f(g(a)) = f(c) = f(g(b)) = f \circ g(b)$
It is contradict with the premise that $f \circ g$ is 1-1.
So our hypothesis is false, which means g is 1-1.

22. Verify that $a_n = 3^n + 1$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
Solution: when $n > 2$

$$\begin{aligned} 4a_{n-1} - 3a_{n-2} &= 4 \times (3^{n-1} + 1) - 3 \times (3^{n-2} + 1) \\ &= 4 \times 3^{n-1} + 4 - 3 \times 3^{n-2} - 3 \\ &= (4 - 1) \times 3^{n-1} + 1 \\ &= 3^n + 1 \\ &= a_n \end{aligned}$$

23. You take a job that pays \$25,000 annually.
(a) How much do you earn n years from now if you receive a three percent raise each year?
(b) How much do you earn n years from now if you receive a five percent raise each year?
(c) How much do you earn n years from now if each year you receive a raise of \$1000 plus two percent of your previous year's salary.
Solution:

(a).Consider the salary sequence $\{a_n\}$:
$a_1 = 25000$
$a_2 = 25000 \times (1 + 3\%)$
...
$a_n = 25000 \times (1 + 3\%)^{n-1}$
...
So it is a geometric progression.
$S_n = 25000 \times \frac{1-(1+3\%)^n}{1-(1+3\%)} = 25000 \times \frac{100}{3} \times ((1 + 3\%)^n - 1)$

(b).It is the same condition with problem(a), so the answer is:
$S_n = 25000 \times \frac{1-(1+5\%)^n}{1-(1+5\%)} = 25000 \times \frac{100}{5} \times ((1 + 5\%)^n - 1)$

(c).Consider the salary sequence $\{a_n\}$:
$a_1 = 25000$
when $n > 1$, $a_n = a_{n-1} + 1000 + 2\% \times a_{n-1}$=$1.02a_{n-1} + 1000$
So we suppose $a_n + t = 1.02(a_{n-1} + t)$
$\therefore\ a_n = 1.02a_{n-1} + 0.02t$
$\therefore\ t = 50000$
$\therefore\ a_n + 50000 = 1.02(a_{n-1} + 50000) = 1.02(1.02(a_{n-2} + 50000))$
=...=$1.02^{n-1}(a_1 + 50000)$=$1.02^{n-1} \times 75000$
So $\quad a_n = 1.02^{n-1} \times 75000 - 50000$
So $\quad S_n = 75000 \times \frac{1-1.02^n}{1-1.02} - 50000n$

24. Let $A = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$.Find A^n where n is a positive integer.
Solution 1:

$$\because\ A = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix}$$

$$\therefore\ A^n = \left(\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix}\right)^n$$

$$= C_n^0 \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}^n + C_n^1 \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}^{n-1} \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix} + \cdots + C_n^{n-1} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix}^{n-1} + C_n^n \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix}^n$$

Solution 2:

$\because$ *let* $|\lambda E - A| = \begin{vmatrix} \lambda - 3 & -5 \\ -2 & \lambda - 4 \end{vmatrix} = 0$

we can get that $\lambda_1 = \frac{7+\sqrt{41}}{2}$ and $\lambda_2 = \frac{7-\sqrt{41}}{2}$ is eigenvalues of A

we let $(\lambda_1 E - A)v = 0$ and $v = (x_1, x_2)^T$

we can get one solution that $v_1 = (10,\ 1+\sqrt{41})^T$ is an eigenvectors of A

In the same way, we let $(\lambda_2 E - A)v = 0$ and $v = (x_1, x_2)^T$

we can get one solution that $v_2 = (10,\ 1-\sqrt{41})^T$ is another eigenvectors of A

So, we let $P = (v_1^T,\ v_2^T) = \begin{pmatrix} 10 & 10 \\ 1+\sqrt{41} & 1-\sqrt{41} \end{pmatrix}$

We can get $adj(P) = \begin{pmatrix} 1-\sqrt{41} & -10 \\ -1-\sqrt{41} & 10 \end{pmatrix}$

So $P^{-1} = \frac{1}{p} adj(P) = \begin{pmatrix} \frac{1-\sqrt{41}}{-20\sqrt{41}} & \frac{-10}{-20\sqrt{41}} \\ \frac{-1-\sqrt{41}}{-20\sqrt{41}} & \frac{10}{-20\sqrt{41}} \end{pmatrix}$

$\therefore\ A = P^{-1} \begin{pmatrix} \frac{7+\sqrt{41}}{2} & 0 \\ 0 & \frac{7-\sqrt{41}}{2} \end{pmatrix} P$

$\therefore A^n = \left[P^{-1} \begin{pmatrix} \frac{7+\sqrt{41}}{2} & 0 \\ 0 & \frac{7-\sqrt{41}}{2} \end{pmatrix} P \right]^n = \underbrace{P^{-1} Diag(A) P \cdot P^{-1} Diag(A) P \cdot \cdots \cdot P^{-1} Diag(A) P}_{n}$

$= P^{-1} \begin{pmatrix} \frac{7+\sqrt{41}}{2} & 0 \\ 0 & \frac{7-\sqrt{41}}{2} \end{pmatrix}^n P$

$= \begin{pmatrix} \frac{1-\sqrt{41}}{-20\sqrt{41}} & \frac{-10}{-20\sqrt{41}} \\ \frac{-1-\sqrt{41}}{-20\sqrt{41}} & \frac{10}{-20\sqrt{41}} \end{pmatrix} \begin{pmatrix} (\frac{7+\sqrt{41}}{2})^n & 0 \\ 0 & (\frac{7-\sqrt{41}}{2})^n \end{pmatrix} \begin{pmatrix} 10 & 10 \\ 1+\sqrt{41} & 1-\sqrt{41} \end{pmatrix}$

$= \begin{pmatrix} \frac{10(1-\sqrt{41})}{-20\sqrt{41}} \times (\frac{7+\sqrt{41}}{2})^n - \frac{10(1+\sqrt{41})}{-20\sqrt{41}} \times (\frac{7-\sqrt{41}}{2})^n & \frac{10(1-\sqrt{41})}{-20\sqrt{41}} \times (\frac{7+\sqrt{41}}{2})^n - \frac{10(1-\sqrt{41})}{-20\sqrt{41}} \times (\frac{7-\sqrt{41}}{2})^n \\ \frac{10(-1-\sqrt{41})}{-20\sqrt{41}} \times (\frac{7+\sqrt{41}}{2})^n + \frac{10(1+\sqrt{41})}{-20\sqrt{41}} \times (\frac{7-\sqrt{41}}{2})^n & \frac{10(-1-\sqrt{41})}{-20\sqrt{41}} \times (\frac{7+\sqrt{41}}{2})^n + \frac{10(1-\sqrt{41})}{-20\sqrt{41}} \times (\frac{7-\sqrt{41}}{2})^n \end{pmatrix}$

$= \begin{pmatrix} -\frac{1-\sqrt{41}}{2\sqrt{41}} \times (\frac{7+\sqrt{41}}{2})^n + \frac{1+\sqrt{41}}{2\sqrt{41}} \times (\frac{7-\sqrt{41}}{2})^n & -\frac{1-\sqrt{41}}{2\sqrt{41}} \times (\frac{7+\sqrt{41}}{2})^n + \frac{1-\sqrt{41}}{2\sqrt{41}} \times (\frac{7-\sqrt{41}}{2})^n \\ \frac{1+\sqrt{41}}{2\sqrt{41}} \times (\frac{7+\sqrt{41}}{2})^n - \frac{1+\sqrt{41}}{2\sqrt{41}} \times (\frac{7-\sqrt{41}}{2})^n & \frac{1+\sqrt{41}}{2\sqrt{41}} \times (\frac{7+\sqrt{41}}{2})^n - \frac{1-\sqrt{41}}{2\sqrt{41}} \times (\frac{7-\sqrt{41}}{2})^n \end{pmatrix}$