Estimation of causal effects in macroeconometrics

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Introduction

• VARs approximate the data linearly by finding, given a lag length, the best 1 -step ahead prediction error (global approximation).

• IRF are nonlinear functions of estimated VAR coefficients.

• Local projections (LP) directly approximate IRF at each horizon (local approximation). Could be useful if true IRF are not smooth.

• Features of LP:
- single equation OLS/IV is enough (with qualifications).
- robust to parametric misspecification of the estimated model.
- potential to make IRFs non-linear.
- lag length of the projection (and instruments) may depend on horizon.

Average causal effects in microeconometrics

- Computation of IRFs in macro similar to computation of average causal effects (ACE) in microeconometrics.

- In micro, ACE are $E(Y|X=1) - E(Y|X=0)$, where X is a binary control (e.g. medicine intake), **randomly assigned across individuals**, Y an outcome variable (e.g. disease), and the average is across individuals in each group (those who take the medicine and those who do not).

- ACE are assessed by the magnitude and significance of β in:

$$Y = \beta X + u \quad E(u|X) = 0 \tag{1}$$

- If there are covariates, condition on $W = w$ (group the data so that is the case) and the ACE is β in the regression

$$Y = \beta X + \gamma W + u \quad E(u|X,W) = 0, \tag{2}$$

• If X is endogenous (likely case), define $X^* = X - E(X|W)$, $u^* = u - E(u|W)$ and consider instruments $Z^* = Z - E(Z|W)$. We need
1) Instrument Relevance $E(X^{*'} Z^*) = \alpha' \neq 0$
2) Instrument Exogeneity $E(u^{*'} Z^*) = 0$,

• If the instrument are "strong" (i.e. α is large), X, Z are scalars, $\beta_{IV} = (Z^* X^*)^{-1} (Z^* Y)$, is consistent in (2).

• If instruments are "weak", need different asymptotics (see Kleiberger, 2005)

• Problems: what is a good Z? How do we make sure they are strong?

• Good general reference: Athey and Imbens (2017).

Dynamic average causal effects in macroeconometrics

• In macro, we care about **dynamic** average causal effects (DACE), which are measured through interventions that propagates over time (see Frisch, 1933, Slutzky, 1937). Consider the MA model:

$$y_t = \theta(L)\epsilon_t \tag{3}$$

ϵ_t is a $N_e \times 1$ vector of structural shocks, iid over $i = 1, \ldots N_e$ and time, with zero mean, constant variance, and $\theta(L) = \theta_0 + \theta_1 L + \theta_2 L^2 + \ldots$.

• DACE at horizon $h = 0, 1, \ldots$ is estimated via the dynamic counterfactual

$$E(Y_{i,t+h}|\epsilon_{jt} = 1) - E(Y_{i,t+h}|\epsilon_{j,t} = 0) = \theta_{ij}^h = \frac{\partial Y_{i,t+h}}{\partial \epsilon_{j,t}} \tag{4}$$

$i = 1, 2 \ldots, N_y, h = 1, \ldots, H,\ j = 1, \ldots, N_\epsilon$. As $T \to \infty$ and if $\epsilon_{t,j}$ is observable, (4) could be estimated with

$$Y_{i,t+h} = \theta_{ij}^h \epsilon_{j,t} + u_{i,t+h} \tag{5}$$

• **In macro there are no randomized controlled experiment.** We can think of doing a randomized experiment by drawing a shock $\epsilon_{j,t}$ that perturbs the economy from the steady state at some t.

• **The average is constructed across time episodes where $\epsilon_{jt} = 1$ and** $\epsilon_{jt} = 0$ (rather than across individuals).

• $\epsilon_{j,t}$ **must be independent of** $u_{i,t+h}$ (this is the definition of a structural shock).

• $\epsilon_{j,t}$**, in general, is non-observable.**

Two ways of computing DACE

- VAR approach (Sims, 1980).

- Setup and estimate a VAR(q): $Y_t = A(L)Y_{t-1} + e_t$.

- Invert the system and linearly relate VAR innovations e_t to structural shocks ϵ_t, i.e. $Y_t = \theta(L)e_t, e_t = R\epsilon_t, \theta(L) = (1 - A(L))^{-1}$.

- Impose identification restrictions (typically on R or $\theta(1)R$, via zeros, signs, etc.) to identify ϵ_t. Estimate structural responses using $\theta(L)R$.

- Two step method. Consistent (but inefficient) estimation of VAR parameters and IRFs.

• Local Projections (Jorda, 2005)

- Assume a linear system linking Y_t and ϵ_t (equation (5)).

- Project Y_{t+h} on ϵ_{jt}, for each $h = 1, 2, \ldots$, some j.

- Correct standard errors of the estimates of θ^h for the presence of MA components in u_{t+h} (i.e. use a HAC stardard errors).

• Trade-off 1: Local projections non-parametric. VAR is parametric and misspecification could be present.

• Trade-off 2: If the DGP is a VAR, local projections inefficient (there is a relationship between θ^h and θ^{h-1}).

General idea of LP

- The idea comes from **direct forecasting** methodology.

- Suppose $Y_{t+h} = AY_{t+h-1} + e_{t+h}$; e_{t+h} iid $(0, \Sigma)$, Σ general matrix; A could be the companion form of a VAR(q).

- Repeatedly substituting backward

$$
\begin{aligned}
Y_{t+h} &= A^{h+1}Y_{t-1} + A^h e_t + A^{h-1}e_{t+1} + \ldots Ae_{t+h-1} + e_{t+h} \\
&= A^{h+1}Y_{t-1} + (A_1^h e_{1t} + A_-^h e_{-1,t}) + A^{h-1}e_{t+1} + \ldots Ae_{t+h-1} + e_{t+h}
\end{aligned}
\tag{6}
$$

where e_{1t} is the shock of interest and $e_{-1,t}$ is the vector of all other shocks.

- If $\Sigma = \text{diag}\{\sigma_i\}$, $e_{1t} = \epsilon_{1t}$, the response of Y_{t+h} to a ϵ_{1t} impulse is

$$E(Y_{t+h}|\epsilon_{1t} = 1) - E(Y_{t+h}|\epsilon_{1t} = 0) = A_1^h \quad (7)$$

Thus, A^h in (7) can be estimated via the regression

$$Y_{t+h} = A^{h+1}Y_{t-1} + A_1^h\epsilon_{1t} + v_{t+h} \quad (8)$$

where $v_{t+h} = f(\epsilon_{-1,t}, e_{t+1}, \ldots, e_{t+h-1}, e_{t+h})$.

- If $\Sigma \neq \text{diag}\{\sigma_i\}$, the response of Y_{t+h} to an impulse in ϵ_{1t} is

$$E(Y_{t+h}|\epsilon_{1t} = 1) - E(y_{t+h}|\epsilon_{1t} = 0) = B_1^h \quad (9)$$

where $B^h = A^hR$, and $e_t = R'\epsilon_t$, for some square matrix R. Thus, B^h in (9) can be estimated via the regression

$$Y_{t+h} = A^{h+1}Y_{t-1} + B_1^h\epsilon_{1t} + u_{t+h} \quad (10)$$

where $u_{t+h} = f(\epsilon_{-1,t}, e_{t+1}, \ldots, e_{t+h-1}, e_{t+h})$ and given R, find A^h.

• **Problem:** since ϵ_{1t} is unobservable, there is an identification problem: for any invertible matrix T such that $B^h\epsilon_t = (B^hT)(T^{-1}\epsilon_t) = D^hv_t$. How do remedy to the non-observability of ϵ_{1t}?

• Traditional approaches (Kuttner, 2001, Faust, et al., 2003, Gurkaynak et al., 2005, Gertler and Karadi, 2015). Substitute ϵ_{1t} with a **proxy** m_t.

$$Y_{t+h} = \beta^h Y_{t-1} + \gamma^h m_t + u_{t+h} \tag{11}$$

where m_t comes from external information (e.g. high frequency data, announcement dates etc.) and "directly" captures ϵ_{1t}.

• If m_t is exogenous (can we make such an assumption?), can omit Y_{t-1} from (11).

• Narrative approaches (e.g. Romer and Romer, 2004, Ramey, 2016). Similar setup: plug m_t from narrative introspection for ϵ_{1t}.

- Extended local projections:

$$Y_{t+h} = \delta^h X_t + \gamma^h m_t + u_{t+h} \tag{12}$$

where $X_t = (Y_{t-1}, W_t, W_{t-1})$, W_t are additional variables not in Y_t: factors, expectations, news, VIX, etc.

- **Adding X_t useful to make m_t and u_{t+h} independent.**

- Can use a lot more variables in X_t than those appearing in a VAR.

• Alternative: (normalization) $B^0_{11} = 1$. This means, e.g. that a 100 basis points monetary policy shock is such that the nominal interest rate increases by 100 basis points; or that a government spending shock increases expenditure by 1 percent of GDP.

• If $B^0_{11} = 1$, one can run local projections, $h = 1, 2, \ldots$:

$$Y_{t+h} = \beta^h Y_{t-1} + \gamma^h Y_{1t} + u_{t+h} \tag{13}$$

i.e. rather than using a (non-observable) shock can use an (observable) variable in the regression. Still Y_{1t} is a proxy for ϵ_{1t}.

• Can estimate γ^h by OLS as long as (Y_{t-1}, Y_{1t}) are uncorrelated with $e_{t+p},\ p = 1, 2, \ldots, h$ and $\epsilon_{-1,t}$.

• In general, the second restriction does not hold (nominal interest rate may be responding contemporaneously to shocks other than monetary policy).

Advantages of LP over VARs

• No need to have the correct A (typical problem with a small scale VAR).

• No need for the linear model to span the DGP information set (typical problem if states are omitted from the VAR or the VAR is of smaller dimension as the DGP).

• Can be obtained variable by variable and horizon by horizon (single equation regressions).

• Conditioning variables X_t may be different for different h.

• If ϵ_{1t} is observable, LP has good properties (Kilian and Kim, 2011).

• Variance decomposition estimates are also OK. (Gorodnichenko and Lee, 2020; Plagborg-Moller and Wolf, 2022). Could be improved by bootstrap corrections, if sample is short.

Problems of LP

• m_t is a proxy for ϵ_{1t} (e.g. changes in federal fund futures around policy announcement dates fail to account that monetary policy is made public also via speeches at different dates). OLS may not have the right asymptotic properties in (11)-(12).

• OLS in (13) is **inconsistent** if Y_{1t} is endogenous (e.g. Y_{1t} depends on $\epsilon_{-1,t}$).

• How do we deal with these problems?

LP-IV

- Suppose we have variables Z_t satisfying

- $E(\epsilon^*_{1t} Z^*_t) \neq 0 = \alpha$ (relevance)
- $E(\epsilon^*_{-1,t} Z^*_t) = 0$ (contemporaneous uncorrelation)
- $E(\epsilon^*_{t+h} Z^*_t) = 0, \; h > 0$ (lead uncorrelation)

where $\epsilon^*_{1t} = \epsilon_{1t} - P(\epsilon_{1t}|X_t); Z^*_t = Z_t - P(Z_t|X_t)$.

Then γ^h in (13) can be estimated as

$$\hat{\gamma}^h = \frac{E(Y_{t+h} Z_t)\Omega E(Z_t Y_{1t})}{E(Y_{1t} Z_t)\Omega E(Z_t Y_{1t})} \tag{14}$$

for any positive definite matrix Ω. If $\Omega = \sigma_u^2 E(Z_t' Z_t)^{-1}$, $\hat{\gamma}^h$ is the same as the one obtained running the two stage regression

$$\begin{aligned} Y_{1t} &= \alpha_0 + \alpha_1 Z_t + \zeta_{1t} \\ Y_{t+h} &= \delta^h X_t + \gamma^h \hat{Y}_{1t} + \beta(h) Y_{t-1} + u_{t+h} \end{aligned} \tag{15}$$

A few details

• If X_t are excluded from (15), $u_{t+h} = f(e_{t-q}, \ldots, e_{t-1}, \epsilon_{-1,t}, e_{t+1}, \ldots, e_{t+h-1}, e_{t+h})$. Thus, we also need the condition $E(e^*_{t-k} Z^*_t) = 0,\ k > 0$. **Without** X_t, Z_t **have to be exogenous**. Much stronger requirement.

• Efficiency of the IV estimator can be improved by taking into account that u_{t+h} has a MA structure. Use HAC/HAR standard errors.

• If Z_t are weak (i.e. α is small), use weak instrument IV regression methods (see Kleiberger, 2005).

• Impulse responses to news shock: if e_{1t} affects Y_{1t+k}, the normalization is $B^k_{11} = 1$ and the instruments should be relevant for Y_{1t+k} (not for Y_t).

• Impulse responses to an anticipated shock: if e_{1t+k} affects Y_{1t}, the normalization is $B^1_{11} = 1$ and future instruments should be relevant for Y_{1t}.

Improvements

• Since LP are estimated separately for each h, $\hat{\gamma}^h$ is typically jagged. One possibility is to estimate (15) jointly for all h, since u_{t+h} is correlated across horizons.

• Penalized (ridge) estimation (set $Y_{t-1} = 0$):

$$\min(Y_{t+h} - \delta^h X_t - \gamma^h Y_{1t})(Y_{t+h} - \delta^h X_t - \gamma^h Y_{1t})' \tag{16}$$

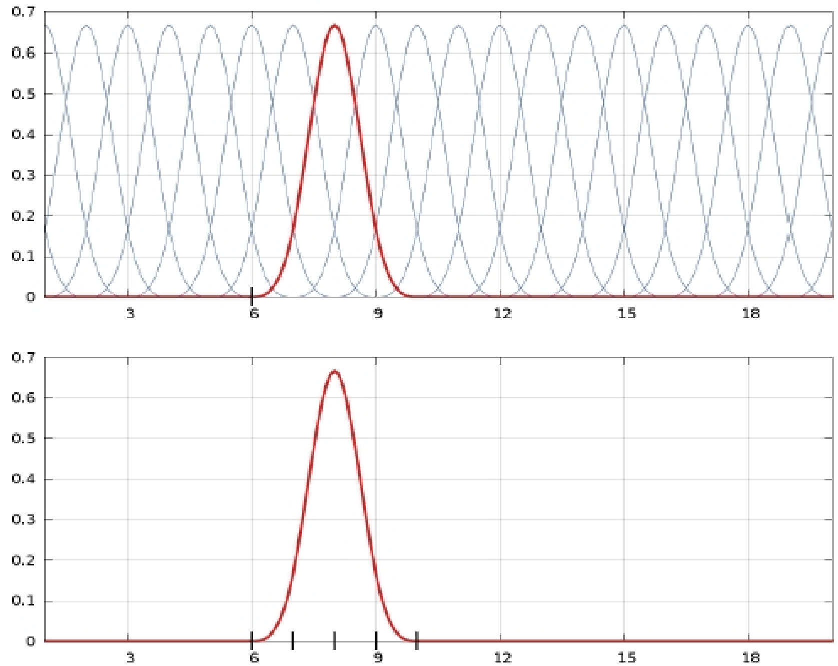
subject to $(\delta^h - \delta^{h-1})^2 \leq \zeta$. Use ζ as the Lagrange multiplier on the constraints; if large, it forces smoothness on δ^h.

• Spline restrictions (Barnichon and Bronwlees, 2019): Let $\chi^h = (\delta^h, \gamma^h)$. Assume $\chi^h = \sum_k \beta_k B(k)^h$, where $B(k)^h$ are B-splines (orthgonal basis).

- A B-spline is made up of $q + 1$ polynomial pieces of order q. The polynomial pieces join on a set of q $+$ 2 inner knots and are calibrated so that derivatives up to the order q -1 are continuous at the inner knots.

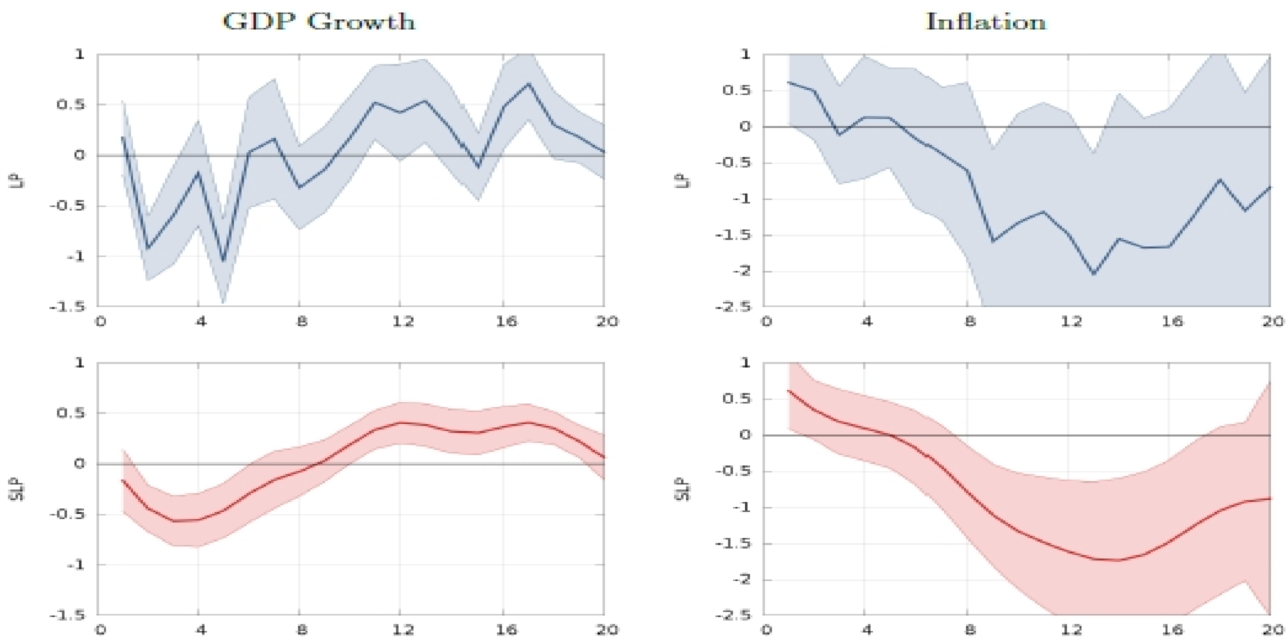
- A B-splines basis function is nonzero over the domain spanned by the q $+$2 inner knots and zero elsewhere

Figure 1: B-Spline Basis



The figure shows the graph of the B-splines basis functions. The top panel displays the set of B-splines basis used in this work. The bottom panel shows in detail the B-splines basis of knot 6.

Figure 9: IR to a monetary shock using instrumental variables



The figure displays the IR of GDP growth (left panels) and inflation (right panels) to a monetary shock identified using instrumental variables and estimated using LP (top panels) and SLP (bottom panels). The shaded area denotes the 90% confidence interval.

IV-SVARs

• Standard setup: $Y_t = A(L)Y_{t-1} + e_t$.

• Estimate A(L) by OLS assuming some lag length and deciding the variables to be included in Y_t.

• Assume $A(L)$ is invertible (i.e. compute $B(L) = (1 - A(L))^{-1}$).

• Impose restrictions, identify shocks, i.e. estimate R in $e_t = R\epsilon_t$. Compute B^h using estimates of A(L) and R

• Parametric assumptions imply that B^h are restricted across h.

SVAR-IV

• Standard Cholesky identification approach given an "internal" IV approach: the VAR innovations in equation i are instrumented by the VAR innovations in equation $i-1, i-2, ...$

• Earlier external SVAR-IV setup (see Ramey, 2016). Add the (external) instrument to the VAR and order it first in a Cholesky decomposition. Here instruments are predetermined not exogenous.

• In analogy with LP-IV, one can also use external instruments to estimate R. Let Z_t satisfy $E(\epsilon_{1t}Z_t') = \alpha' \neq 0$; $E(\epsilon_{-1,t}Z_t') = 0$.

• In terms of VAR residuals e_t these conditions imply

$$E(e_t Z_t') = R \begin{pmatrix} \alpha' \\ 0 \end{pmatrix} = \begin{pmatrix} R_{11}\alpha' \\ R_{.1}\alpha' \end{pmatrix} \tag{17}$$

If $R_{11} = 1$ (normalization), then $R_{i1}, i = 2, \ldots, N$ can be obtained by IV

$$\hat{R}_{i1} = \frac{E(e_{it}Z_t)\Omega E(Z_t e_{1t})}{E(e_{1t}Z_t)\Omega E(Z_t e_{1t})} \tag{18}$$

where again Ω is any positive definite matrix.

In general, one can think of running the regression:

$$e_{it} = R_{i1}\epsilon_{1t} + u_{it} \tag{19}$$

• Since ϵ_t are not observables two approaches are possible:
1) (internal instruments) Use $\hat{e}_{1t}$ for ϵ_{1t}, where $\hat{e}_{1t}$ is the first element of a Cholesky decomposition. Consistent estimates of R_{i1} are obtained (since $\hat{e}_t$ is consistent) but standard errors need to be adjusted for generated regressors ($Z_t \equiv \hat{e}_{1t}$ may be correlated with $Y_{t-k},\ k > 0$)

2) (external instruments) Using the definition of innovations write (19) as

$$Y_{it} = R_{i1}Y_{1t} + \gamma_i(L)Y_{t-1} + u_{it} \tag{20}$$

where $\gamma_i(L)$ are the coefficients of $P(Y_{it} - R_{i1}Y_{1t}|Y_{t-1}, Y_{t-2}, \ldots)$. Use 2SLS with Z_t as instruments for Y_{1t} in (20) to estimate R_{i1}. Standard errors are correct here.

• Split estimation problem in two steps: use VAR to estimate A(L); use (20) to estimate R_{i1} each i.

• Advantage: $A(L), R_{i1}$ can be estimated over different samples (impossible with LP-IV).

- For h=0 LP-IV and SVAR-IV are the same if X$_t$ are the same.

- **For $h > 0$ they may differ. LP-IV is non parametric and estimates each h separately. SVAR-IV makes parametric assumptions and has restrictions across h.**

- To identify news shocks (with k period ahead lead) (20) becomes:

$$Y_{it} = R_{i1}\hat{W}_{t+h} + \gamma_i(L)Y_{t-1} + u_{it} \tag{21}$$

where $\hat{W}_{t+k} = \hat{B}_k Y_t$ and $\hat{B}(L) = (1 - \hat{A}(L))^{-1}$.

• Stock and Watson (2018): If the DGP is a VAR(q) and invertibily holds, SVAR-IV and LP-IV are consistent and SVAR-IV more efficient. If DGP is not a VAR(q) or invertibility does not hold, SVAR-IV inconsistent, local projections consistent.

• Can use Hausmann test to check misspecification (if we assume a VAR(q) DGP, this becomes a test of invertibility): under the null both SVAR and LP are consistent but SVAR more efficient; under the alternative only LP is consistent.

• Stock and Watson (2018): Assuming $E(e^*_{t+h} Z^*_t) = 0,\ h > 0$ in LP-IV is equivalent to assuming that the VAR is invertible.

• Plagborg-Muller and Wolf (2022): under standard assumptions SVAR-LP they are equivalent asymptotically (they estimate the same IRF).

Relationship IV-narrative sign restrictions

• Narrative sign restrictions (Antolin and Rubio, 2019): can be though as IV approach in the sign restriction space.

• Sign restrictions identify sets which can be very large and may include responses which are quantitatively unreasonable.

• Narrative sign restrictions pick responses which are consistent (correlated with) restrictions on the sign or the relative importance of shocks in terms of variance decomposition in particular historical episodes.

• Sign of shocks are particular instruments!

• Same idea with non-normality restrictions applied to sign identified responses.

SVAR one-step likelihood estimation

- SVAR-IV standard errors computed with bootstrap method too small (see Jentsch and Lundsford, 2019). Overstating significance of IRFs.

- Typically use one instrument per shock. What if there are more than one instrument?

- Angelini and Fanelli (2019): SVAR-AC. Want to identify k shocks using $r \geq k$ instruments. Assume $E(v_t \epsilon_{1t}) = F, rank(F) = k$ and $E(v_t \epsilon_{2t}) = 0_{r \times (n-k)}$.

$$\begin{pmatrix} Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} A(L) & 0_{n\times r} \\ \Gamma(L) & \Theta(L) \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ Z_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} B_1 & B_2 & 0_{n\times r} \\ \Phi & 0_{r\times(n-k)} & \Sigma_\omega^{0.5} \end{pmatrix} \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \omega_t \end{pmatrix} \tag{23}$$

- Instruments Z_t purged i.e. $v_t = z_t - E_t(z_t|F_{t-1})$

- Instrument orthogonality imposed (see zeros in (23)); instrument relevance can be tested: null is $\Phi = 0$.

- Can rewrite the (22)-(23) as

$$
\begin{aligned}
W_t &= D(L)W_{t-1} + \eta_t && (24) \\
\eta_t &= G\zeta_t && (25)
\end{aligned}
$$

- Restriction: $GG' = \mathbf{\Sigma}_\eta$ because $E(\zeta_t\zeta_t' = I)$.

- G can be estimated by ML

- Orthogonality of v_t can be tested via a LR test.

Bayesian LP

- LP defines a regression model so can use Bayesian methods to obtain posterior distributions of IRFs, given a prior for regression parameters.

- No new issue relative to VARs. Posterior intervals easily computed by simulations.

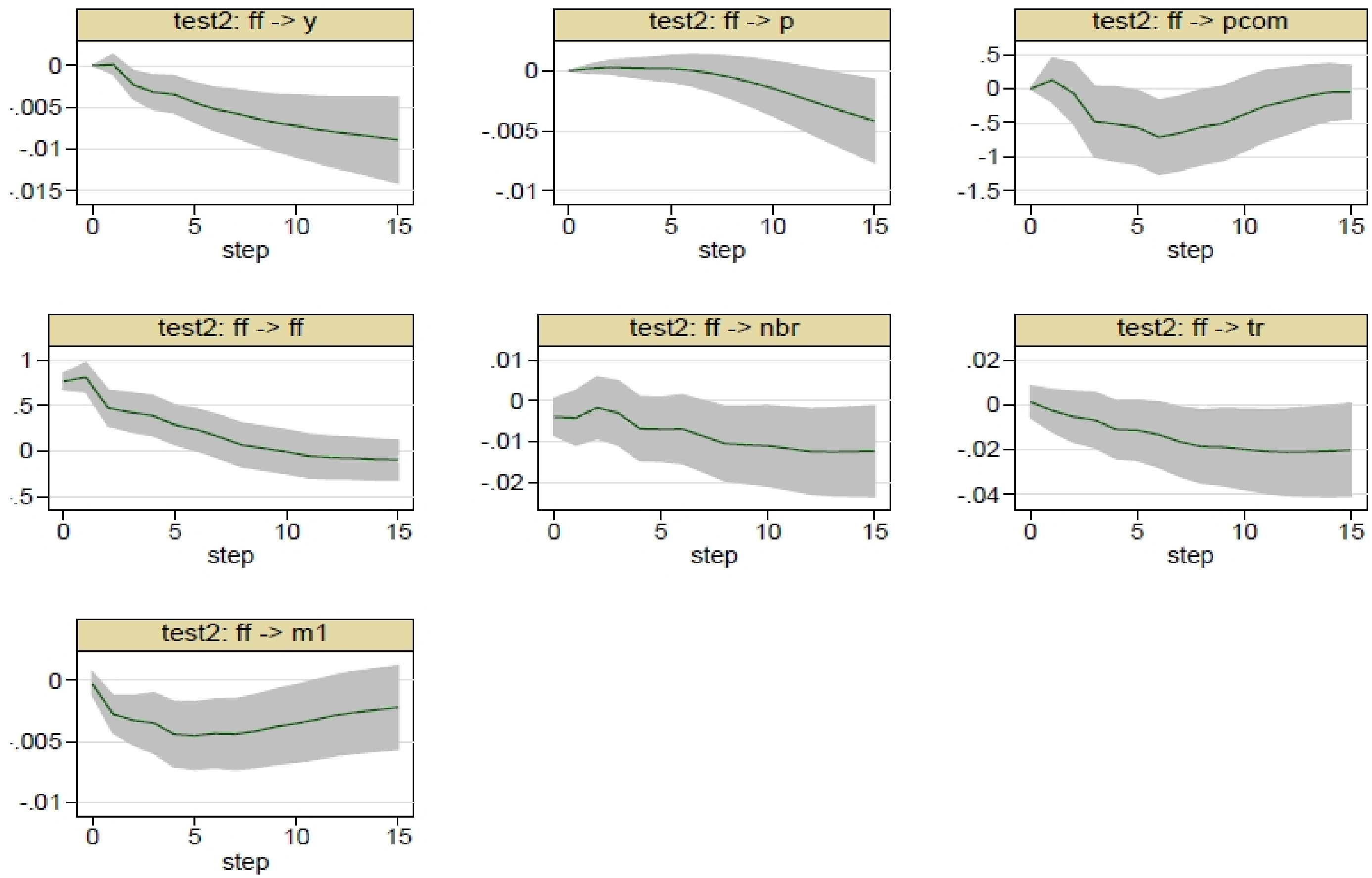
- Miranda Agrippino-Ricco (2021) Use the VAR-LP relationship to setup a prior for LP parameters.

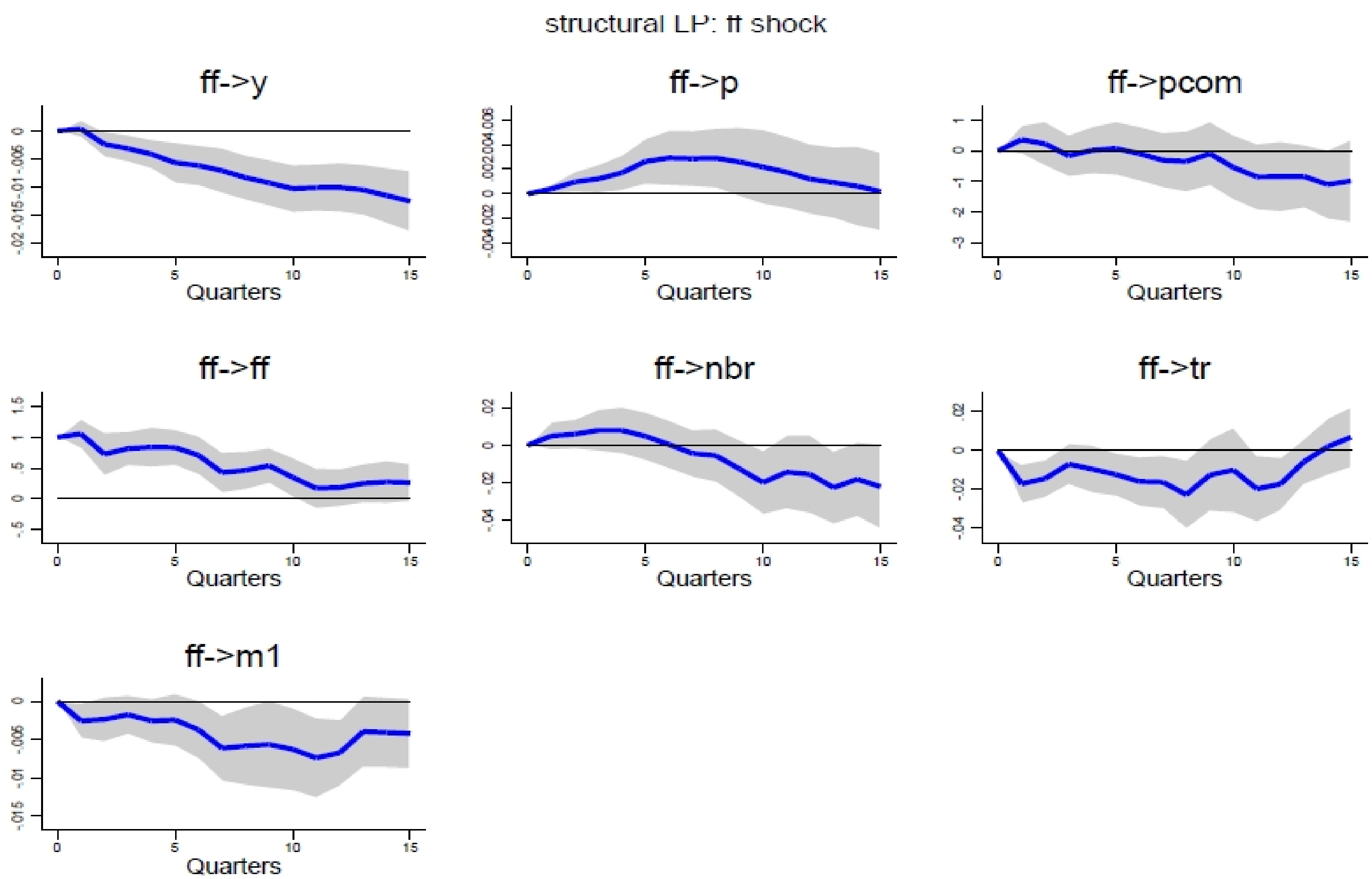
- Empirical macro toolkit uses this approach to setup priors for LP coefficients.

Example 1 *(Jorda, 2005) Consider estimation of the impact of a 100 basis points increase in the FFR on a number of macrovariables. Use SVAR and local projections (without instruments).*

- *Monetary policy shock* ϵ_{1t}. *Identified with Cholesky.*

- *Dynamic ACE:* $E(Y_{i,t+h}|\epsilon_{1t} = 1) - E(Y_{i,t+h}|\epsilon_{1,t} = 0) \equiv \theta_{i1}^{h}$.


test2: ff -> y
test2: ff -> p
test2: ff -> pcom
test2: ff -> ff
test2: ff -> nbr
test2: ff -> tr
test2: ff -> m1
step


structural LP: ff shock
ff->y
ff->p
ff->pcom
ff->ff
ff->nbr
ff->tr
ff->m1
Quarters

Example 2 *(Stock-Watson, 2017) Gertler and Karadi (2015) VAR with* $(R_t, 100 * \Delta \ln(IP), 100 * \Delta \ln(CPI)$, *External bond premium (EBP)).*

- Z_t: Δ *FFR futures in a short window around FOMC announcements.*

- $E(Z_t \epsilon_{1t}) \neq 0$;

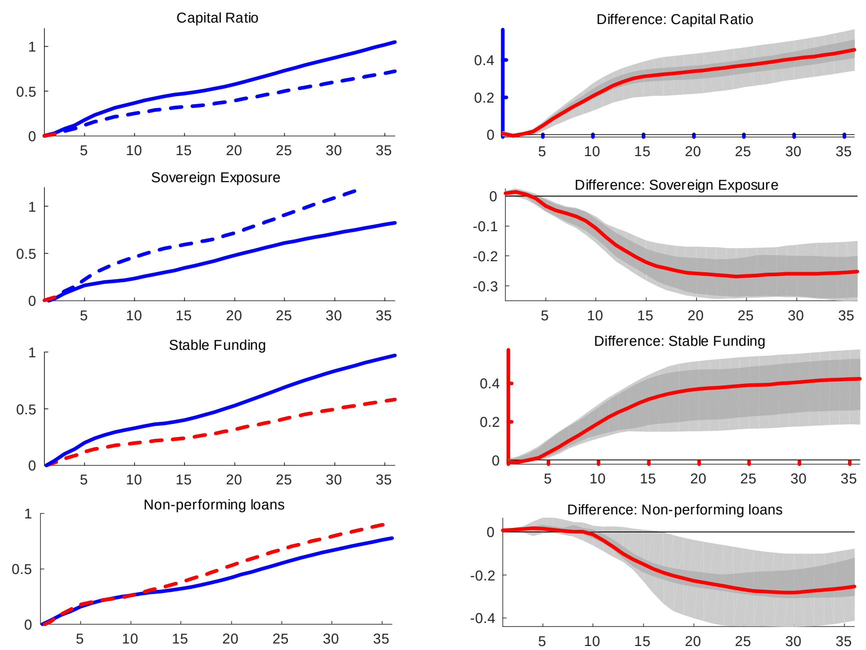
- $E(Z_t \epsilon_{-1,t} = 0)$ *(hopefully).*

Table 1: Estimated causal effect of monetary policy shocks on selected economic variables: Gertler-Karadi (2015) variables, instrument and sample period

		LP-IV			SVAR		SVAR – LP
	lag (*h*)	(a)	(b)	(c)	(d)		(d)-(b)
R	0	1.00 (0.00)	1.00 (0.00)	1.00 (0.00)	1.00 (0.00)		0.00 (0.00)
	6	-0.07 (1.34)	1.12 (0.52)	0.67 (0.57)	0.89 (0.31)		-0.23 (1.19)
	12	-1.05 (2.51)	0.78 (1.02)	-0.12 (1.07)	0.78 (0.46)		0.00 (1.79)
	24	-2.09 (5.66)	-0.80 (1.53)	-1.57 (1.48)	0.40 (0.49)		1.19 (2.57)
IP	0	-0.59 (0.71)	0.21 (0.40)	0.03 (0.55)	0.16 (0.59)		-0.06 (0.35)
	6	-2.15 (3.42)	-3.80 (3.14)	-4.05 (3.65)	-0.81 (1.19)		3.00 (2.32)
	12	-3.60 (6.23)	-6.70 (4.70)	-6.86 (5.49)	-1.87 (1.54)		4.83 (4.00)
	24	-2.99 (10.21)	-9.51 (7.70)	-8.13 (7.62)	-2.16 (1.65)		7.35 (6.40)
P	0	0.02 (0.07)	-0.08 (0.25)	-0.04 (0.25)	0.02 (0.23)		0.10 (0.13)
	6	0.16 (0.42)	-0.39 (0.52)	-0.79 (0.83)	0.31 (0.41)		0.71 (0.98)
	12	-0.26 (0.88)	-1.35 (1.03)	-1.37 (1.23)	0.45 (0.54)		1.80 (1.53)
	24	-0.88 (3.08)	-2.26 (1.31)	-2.58 (1.69)	0.50 (0.65)		2.76 (2.60)
EBP	0	0.51 (0.61)	0.67 (0.40)	0.82 (0.49)	0.77 (0.29)		0.09 (0.24)
	6	0.22 (0.30)	1.33 (0.81)	1.66 (1.04)	0.48 (0.20)		-0.85 (0.51)
	12	0.56 (0.91)	0.84 (0.65)	0.91 (0.80)	0.18 (0.13)		-0.66 (0.55)
	24	-0.44 (1.29)	0.94 (0.66)	0.85 (0.76)	0.06 (0.07)		-0.88 (0.62)
Controls		none	4 lags of (*z*,*y*)	4 lags of (*z*,*y*,*f*)	12 lags of *y* 4 lags of *z*		na
First-stage F^{Hom}		1.7	23.7	18.6	20.5		na
First-stage F^{HAC}		1.1	15.5	12.7	19.2		na

Notes: The instrument, Z_t, is available from 1990m1-2012m6; the other variables are available from 1979m1-2012m6. The LP-IV estimates in (a)-(c) use data from 1990m1-2012m6. The VAR for (d) is computed over 1980m7-2012m6; and the IV-regression computed over 1990m5-2012m6. The numbers in parentheses are standard errors computed by Newey-West HAC with h+1 lags for the local projections, and using a parametric Gaussian bootstrap for the SVAR and the SVAR – LP differences shown in (e). In the final two rows F^{Hom} is the standard (conditional homoscedasticity, no serial correlation) first-stage F-statistic, while F^{HAC} is the Newey-West version using 12 lags in (a) and heteroskedasticity-robust (no lags) in (b), (c), and (d).

Example 3 *Altavilla et al. (2020) SVAR-IV. Identify conventional monetary policy shocks (2007-2014) using announcement dates as instruments.*



Upper and lower quantiles of the distribution of lending rate responses to MP shocks.

Using lag augmentation

• Common to run LP using:

$$y_{t+h} = \beta_h y_t + \text{controls} + u_{t+h}, \quad h = 1, 2, \ldots \tag{26}$$

• Interest is typically in large horizon h.

• Typically y_t is persistent (close to a unit root).

• Combination of persistent y_t and large h create problems to classical inference.

• Montiel Olea and Plagborg Muller (2021). Use lag augmentation

$$y_{t+h} = \beta_h y_t + \sum_{\ell=1}^{p} \gamma_{\ell h} y_{t-\ell} + \text{controls} + e_{t+h}, \quad h = 1, 2, ... \qquad (27)$$

• Lag-augmented LP inference is uniformly valid when the DGP features a unit root and h is large.

• Valid also when $h = h_T \propto T^{\eta}$ for $\eta \in [0, 1)$ (and $\propto T$ if no unit root).

• Lag augmentation obviates need for HAC/HAR standard error (Heteroskedasticity-robust standard errors suffice).

• Simple. No need to choose tuning parameters.

AR(1) example

$$y_{t+h} = \rho^h y_t + \sum_{\ell=1}^{h} \rho^{h-\ell} u_{t+\ell} \equiv \beta(\rho, h) y_t + \epsilon_t(\rho, h) \tag{28}$$

- OLS $\hat{\beta}$ consistent and asymptotically normal for $|\rho| < 1$.

- Non-normal if $\rho \approx 1$.

- HAR estimates needed even when $|\rho| < 1$ since ϵ_t is serially correlated. Inference difficult in small samples. Need tuning parameters.

- With lag augmentation inference is robust.

- Let $x_t = (y_t, y_{t-1})'$. Let $\alpha(h) = (\beta(\rho, h), \gamma(\rho, h))$. Then

$$\hat{\alpha}(h) = (\sum_t x_t x_t')^{-1} (\sum x_t y_{t+h}) \tag{29}$$

- $\hat{\alpha}(h)_{LA}$ is the same as the one obtained using $u_t = y_t - \rho y_{t-1}$ and y_{t-1}.

- $\hat{\beta}(h)_{LA}$ has uniform normal limit, since

$$y_{t+1} = \beta(h) u_t + \beta(\rho, h+1) y_{t-1} + \epsilon_t \tag{30}$$

- First term stationary. Inference is OK. y_{t-1} is non-stationary if $\rho \approx 1 \rightarrow \gamma(h) = \beta(1, h+1)$ non-normal, but we do not care.

• Lag augmentation simplies computation of standard errors. Leading term in asymptotic expansion

$$\hat{\beta}(h) = \beta(\rho, h) + \frac{\sum_{t=1}^{T-1} \epsilon_t(\rho, h) u_t}{u_t^2} \tag{31}$$

• ϵ_t is serially correlated but the score of $\epsilon_t(\rho, h)u_t$ are not as long as $E(u_t|u_s) = 0, s > t$ since for $s < t$

$$E[\epsilon_t(\rho, h)u_t\epsilon_s(\rho, h)u_s] = E[\epsilon_t(\rho, h)u_t\epsilon_s(\rho, h)E(u_s|u_{s+1}, u_{s+2}, \ldots)] \tag{32}$$

and the last expectation is zero.

• Sufficient to use (Eicker-Huber-White) standard errors

$$\hat{s}(h) = \frac{(\sum_t \hat{\epsilon}(h)^2 \hat{u}(h)^2)^{0.5}}{\sum_t \hat{u}(h)^2} \tag{33}$$

where $\hat{\epsilon}_t = y_{t+h} - \hat{\beta}(h)y_t - \hat{\gamma}(h)y_{t-1}$; $\hat{u}_t = y_t - \hat{\rho}(h)y_{t-1}$;$\hat{\rho}(h) = \frac{\sum_t y_t y_{t-1}}{\sum_t y_{t-1}^2}$.

- No need of tuning parameters.

- Confidence interval:

$$\hat{C}(h, \alpha) = [\hat{\beta}(h) \pm z_{(1-0.5\alpha)}\hat{s}(h)] \tag{34}$$

is uniformely valid, i.e. as $T \longrightarrow \infty$:

$$\inf_{\rho \in [-1,1]} \inf_{1 \leq h \leq h_T} P_\rho(\beta(\rho, h) \in \hat{C}(h, \alpha) \longrightarrow 1 - \alpha$$

for any h_T sequence such that $\frac{h_T}{T} \longrightarrow 0$.

- If $\rho < 1 - a$ one could even choose $h_T \propto T$.

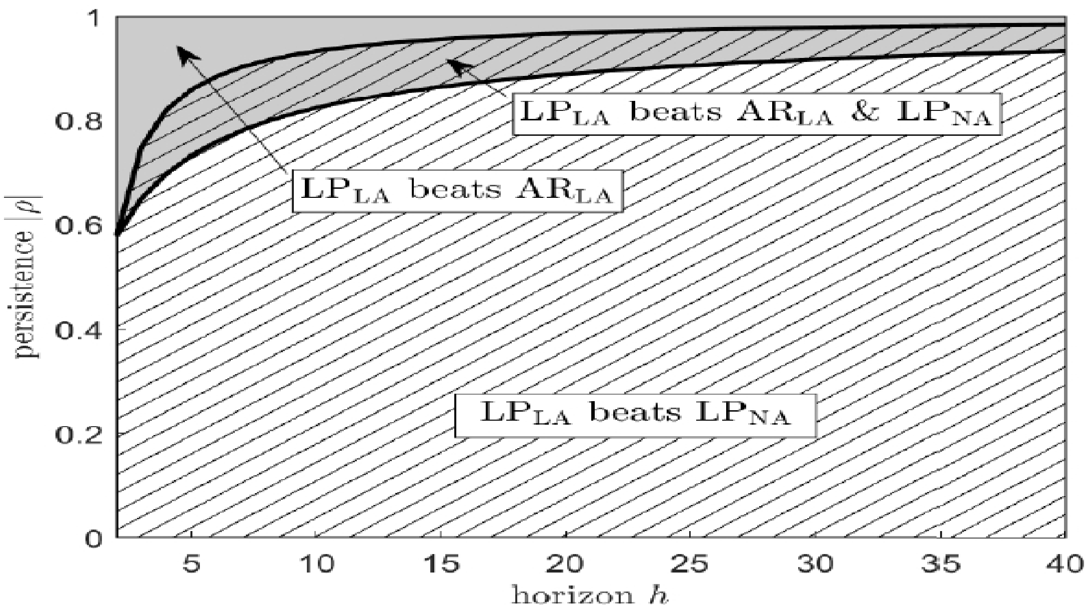


Figure 1: Efficiency ranking of three different estimators of the fixed impulse response $\beta(\rho, h) = \rho^h$ in the homoskedastic AR(1) model: lag-augmented LP ($\mathrm{LP_{LA}}$), non-augmented LP ($\mathrm{LP_{NA}}$), and lag-augmented AR ($\mathrm{AR_{LA}}$). Gray area: combinations of $(|\rho|, h)$ for which $\mathrm{LP_{LA}}$ is more efficient than $\mathrm{AR_{LA}}$. Thatched area: $\mathrm{LP_{LA}}$ is more efficient than $\mathrm{LP_{NA}}$. See Appendix B.2.1 for analytical derivations of the indifference curves (thick lines).

Small sample biases

• Herbst et al (2021): LP is ok in large samples but very biased in small samples.

• Biases present even when relevant regressor is iid.

• LP bias at horizon h is a weighted sum of the (population) impulse response function at other h's. Thus, if LP estimators across horizons have the same sign, the least-squares estimators are biased toward zero at every horizon.

• Small sample LP estimates are not ''local'' because the small-sample biases depend on the true impulse responses at other horizons.

• Standard error also biased. HAR correction still biased in standard macro samples (mostly downward).

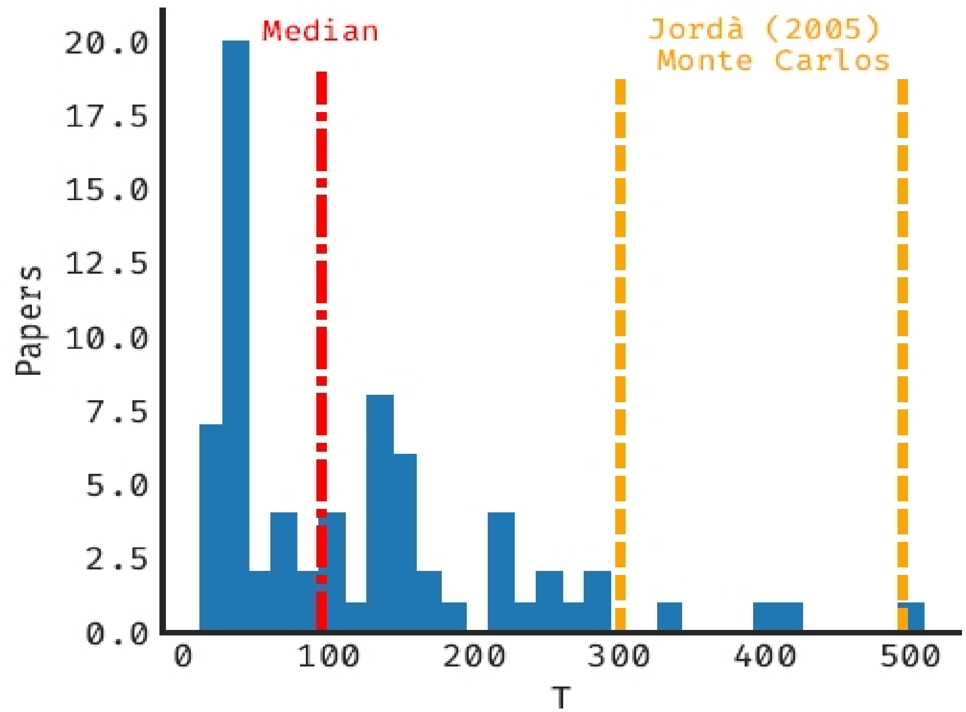


Figure 1: T is small in the literature using LPs.

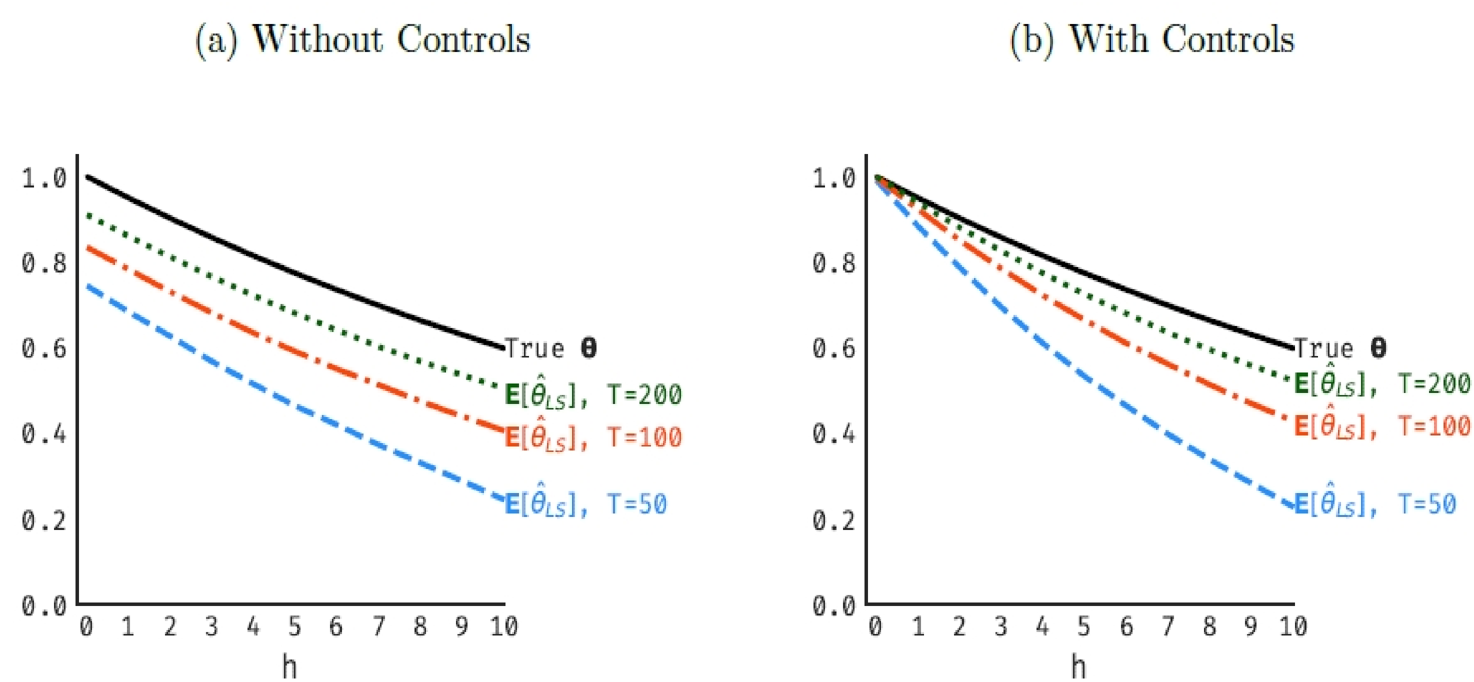


Figure 2: LP estimators are biased in empirically-relevant samples when y_t is an AR(1) with $\rho = 0.95$.

• Expression for the asymptotic bias.

$$\mathbb{E}\left[\widehat{\theta}_{h,LS}\right] - \theta_h = -\frac{1}{T-h}\sum_{j=1}^{T-h-1}\left(1-\frac{j}{T-h}\right)(\theta_{h+j}+\theta_{h-j}) + O\left(T^{-3/2}\right).$$

• Given estimates of $\theta_{h\pm j}$ the responses at horizons $h \pm j$, one can correct estimates by adding back (an estimate of) the bias.

(a) $T = 50$ (b) $T = 100$

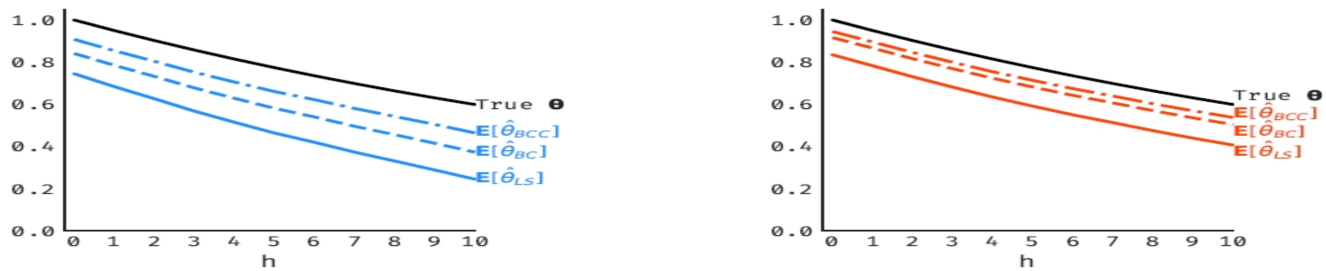


Figure 4: $\widehat{\theta}_{BC}$ and $\widehat{\theta}_{BCC}$ are closer than $\widehat{\theta}_{LS}$ to θ, on average, in our LPs without controls when y_t is an AR(1) with $\rho = 0.95$.

(a) $T = 50$ (b) $T = 100$

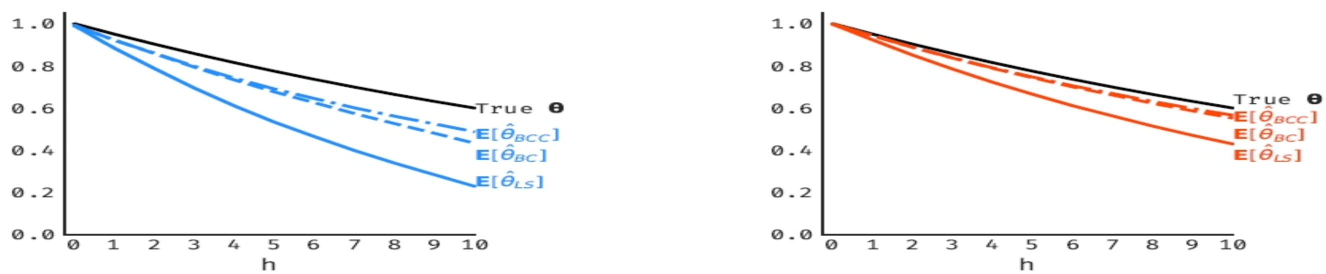


Figure 5: $\widehat{\theta}_{BC}$ and $\widehat{\theta}_{BCC}$ are closer than $\widehat{\theta}_{LS}$ to θ, on average, in our LPs with controls when y_t is an AR(1) with $\rho = 0.95$.

LP or VAR in small samples

• Li et al. (2021): use a general DFM for DGP. Compute variance-bias tradeoff for LP and VAR dynamic responses.

• Least-squares LP and VAR estimators lie on opposite ends of the bias-variance spectrum: small bias and large variance for LPs, and large bias and small variance for VARs.

• Given h, there exists a weight ω on squared bias relative to variance in the loss function that would make a researcher indifferent between the two estimators (should care around four times more about squared bias than about variance).

• Shrinkage methods dramatically lower the variance of LP and VAR methods, at a moderate cost in bias. Unless researchers care almost exclusively about bias, penalized LP and Bayesian VAR preferable.

• Given a loss function, no single method dominates at all horizons. Penalized LP best at short horizons, while Bayesian VAR at long horizons.

• In the case of IV identification, the SVAR-IV estimator is heavily (median) biased, but provides substantial reduction in dispersion. Depending on the weight attached to bias, justifiable to use external IV methods despite their lack of robustness to non-invertibility (unlike internal IV methods).

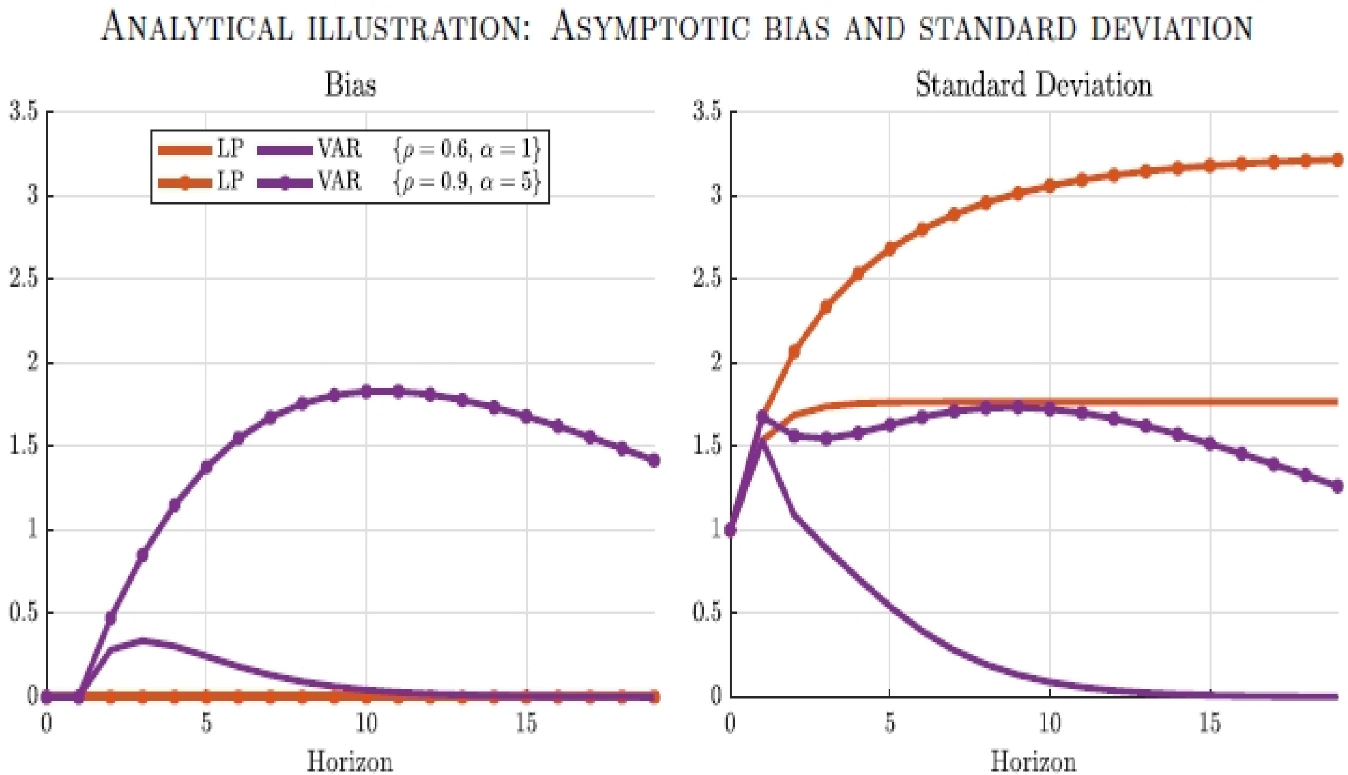


Figure 1: Asymptotic bias and standard deviation for LP (red) and VAR (purple) in the DGP (1) with $\sigma_1 = 1$ and $\{\rho = 0.6, \alpha = 1\}$ (no markers) or $\{\rho = 0.9, \alpha = 5\}$ (markers).

ANALYTICAL ILLUSTRATION: INDIFFERENCE WEIGHT ω_h^*

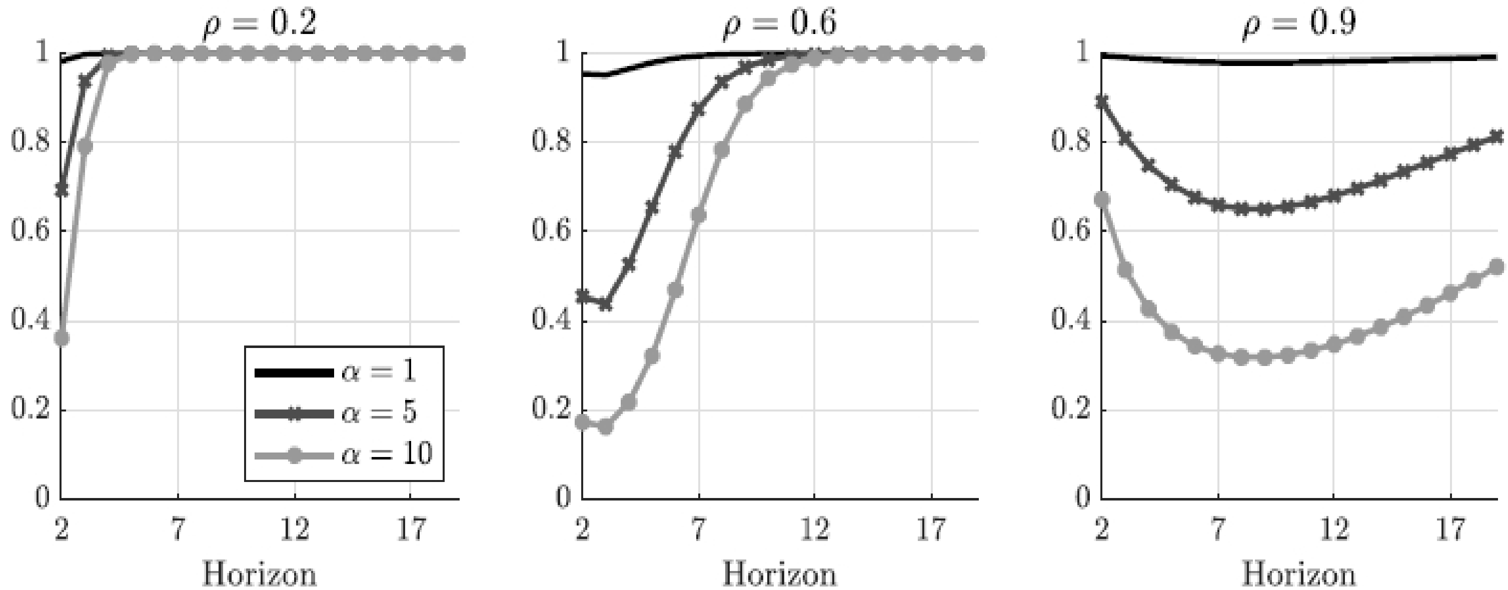


Figure 2: Weight $\omega = \omega_h^*$ in the asymptotic loss function $\omega \times \text{aBias}^2_{\hat{\theta}_h} + (1-\omega) \times \text{aVar}_{\hat{\theta}_h}$ that yields indifference between the LP and VAR estimator. LP is preferred whenever $\omega \geq \omega_h^*$. The three panels correspond to different values of $\rho \in \{0.2, 0.6, 0.9\}$; the three curves in each panel correspond to different values of $\alpha \in \{1, 5, 10\}$. All results are computed with $\sigma_2 = 1$. The figure omits the horizons $h \in \{0, 1\}$, at which the two estimation methods are (asymptotically) equivalent.

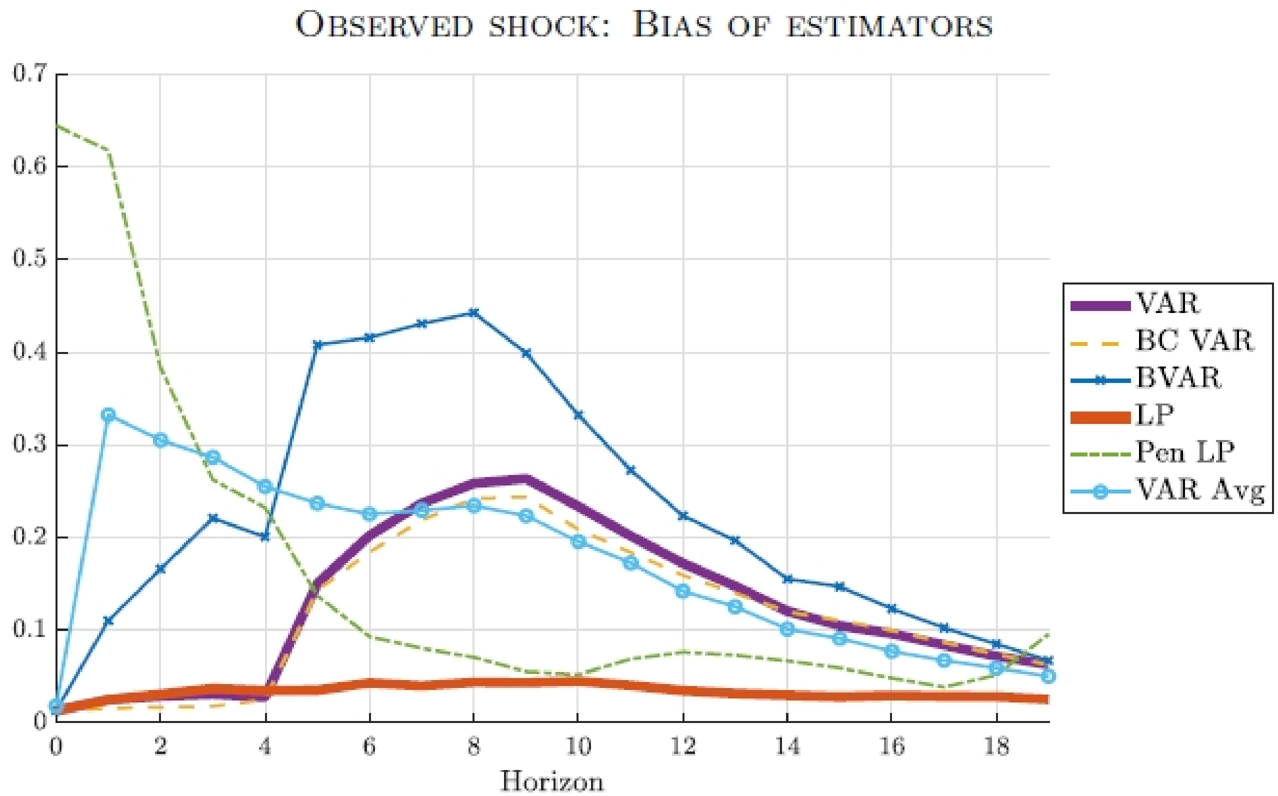


Figure 4: Median (across DGPs) of absolute bias of the different estimation procedures, relative to $\sqrt{\frac{1}{20}\sum_{h=0}^{19}\theta_h^2}$.

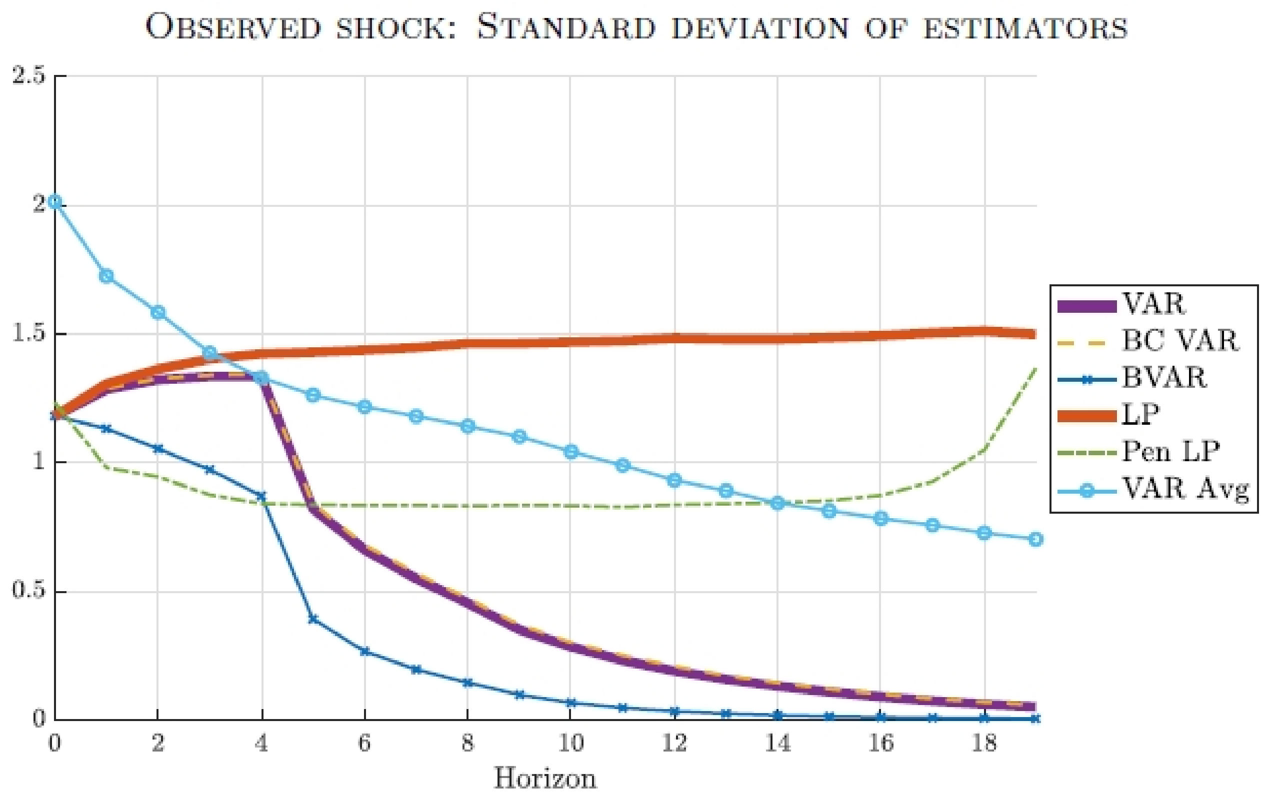


Figure 5: Median (across DGPs) of standard deviation of the different estimation procedures, relative to $\sqrt{\frac{1}{20}\sum_{h=0}^{19}\theta_h^2}$.

Measuring non-linear effects

$$Y_{t+h} = A^{h+1}Y_{t-1} + B_1^h \epsilon_{1t} + u_{t+h} \tag{35}$$

Want the effects of ϵ_t^+ and ϵ_t^- where ϵ_t^+ could be a positive or a large shock or $\epsilon_t^+ = \epsilon(s_t^+)$ a shock in s_t^+ and ϵ_t^- its complement.

- Normalization:

$$\begin{aligned} Y_{t+h} &= A^{h+1}Y_{t-1} + \gamma^{h+}Y_{1t}^+ + u_{t+h} && (36) \\ Y_{t+h} &= A^{h+1}Y_{t-1} + \gamma^{h-}Y_{1t}^- + u_{t+h} && (37) \end{aligned}$$

If instruments $Z_t^+(Z_t^-)$ for $Y_{1t}^+(Y_{1t}^-)$ are available, same LP-IV technology can be used. Are there reliable $Z_t^+(Z_t^-)$?

- Can also consider non-linear effects in a SVAR-IV using the same idea.

- Different from allowing dynamics to be non-linear, i.e. $A^{h+\mathbf{1},+} \neq A^{h+\mathbf{1},-}$.

- Can test $\gamma^{h+} = \gamma^{h-}$ with a t-test (F-test).

- State dependent Lical projections: Cloyne et al. (2023).

General nonlinear setup

- $y_t = \phi(\epsilon_t, \epsilon_{t-1}, \ldots)$. Non-linear MA.

- Volterra (Wold) expansion:

$$y_t = \sum_i \phi_i \epsilon_{t-i} + \sum_i \sum_j \phi_{ij} \epsilon_{t-i} \epsilon_{t-j} + \sum_i \sum_j \sum_k \phi_{ijk} \epsilon_{t-i} \epsilon_{t-j} \epsilon_{t-k} + \ldots \tag{38}$$

- VAR: stop at first order. $y_t = \sum_i \phi_i \epsilon_{t-i}$

- Responses symmetric.
- Responses state, history independent.

• General non-linear LP:

$$\begin{aligned} y_{t+h} &= a_h + b_1^h y_t + \ldots + b_p^h y_{t-p} + u_{t+h} \\ &+ q_1^h y_t^2 + r_1^h y_t^3 + \ldots \end{aligned} \tag{39}$$

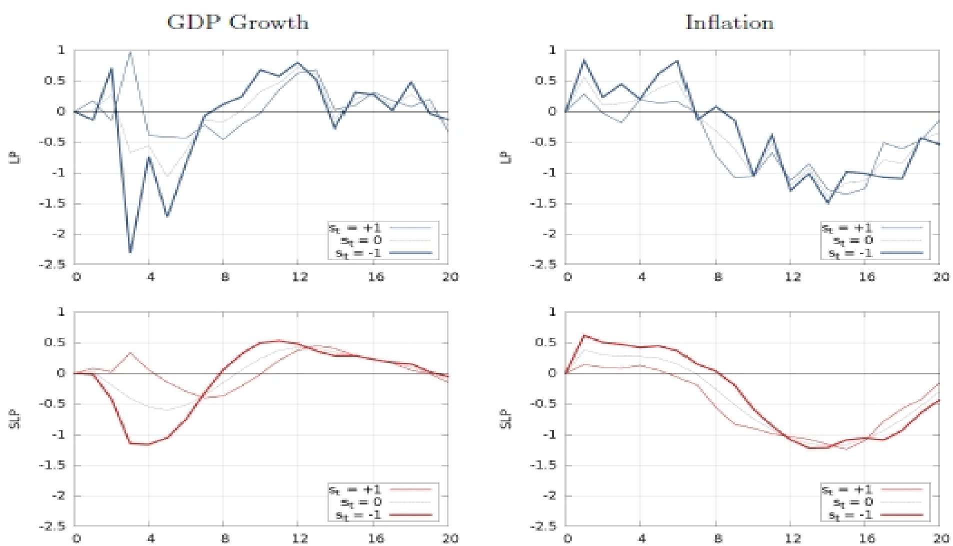
(Note: only y_t matter for IRFs)

• $IRF(t, h, d) = b_1^h d + q_1^h(2y_t d + d^2) + r_1^h(3y_t^2 d + 3y_t d^2 + d^3) + \ldots$, t=time, h=horizon, d=size of the shock.

• Non-linear LP could be asymmetric, state and history dependent, since y_t enters $IRF(t, h, d)$.

Example 4 *Barnichon and Brownlees (2019)*

Figure 10: State-dependent IR to a monetary shock



The figure displays the state-dependent IR of GDP growth (left panels) and inflation (right panels) to a monetary shock estimated using LP (top panels) and SLP (bottom panels). The state variable s_t respectively takes the values -1 (a recession), 0, and +1 (an expansion) units of $\sigma(s_t)$.

LP and model misspecification

- Use one sector growth model. How does LP works with this DGP?

$$\max_{C_t} \sum_{t=1}^{\infty} \beta^t U(C_t)$$

$$\begin{aligned} C_t/B_t + I_t &= O_t = A_t K_{t-1}^{\alpha} \\ K_t &= (1-\delta)K_{t-1} + V_t I_t \end{aligned}$$

where O_t is output, C_t is consumption, I_t investment and K_t is the capital stock, $0 < \alpha, \beta, \delta < 1$.

- Assume that (A_t, V_t, B_t) are iid with unitary means and standard deviation $\sigma_i, i = A, V, B$, uncorrelated with each other at all leads and lags.

• When $U(C_t) = \log C_t$ and $\delta = 1$, the solution is:

$$\log O_t = \alpha \log K_{t-1} + \log A_t \tag{40}$$

$$\log C_t = \log(1 - \alpha\beta) + \alpha \log K_{t-1} + \log B_t + \log A_t \tag{41}$$

$$\log K_t = \log(\alpha\beta) + \alpha \log K_{t-1} + \log V_t + \log A_t \tag{42}$$

Three endogenous variables and three disturbances (two supply (A_t, V_t) and one demand B_t).

• In a VAR with $(\log O_t, \log C_t, \log K_t)$, all structural disturbances are identifiable from the innovations using theory-based recursive restriction ($\log A_t$ can be obtained from the innovations in $\log O_t$; given $\log A_t$, the other two innovations determine $\log V_t$ and $\log B_t$).

- The steady states are:

$$\begin{aligned}\log K &= \frac{\log(\alpha\beta)}{1-\alpha} \\ \log C &= \log(1-\alpha\beta) + \frac{\alpha}{1-\alpha}\log(\alpha\beta) \\ \log O &= \frac{\alpha}{1-\alpha}\log(\alpha\beta) \qquad (43)\end{aligned}$$

- Use lower case letters to indicate log deviation from steady states. Then:

$$k_t = \alpha k_{t-1} + v_t + a_t \qquad (44)$$

$$c_t = \alpha k_{t-1} + b_t + a_t \qquad (45)$$

$$o_t = \alpha k_{t-1} + a_t \qquad (46)$$

or $w_t = Aw_{t-1} + Be_t$, where $w_t = [k_t, c_t, o_t]'$ and $e_t = [a_t, v_t, b_t]'$.

- The MA representation of (44)-(46) is $w_t = B(L)e_t$ where:

$$k_t = \sum_{i=0}^{\infty} \alpha^i a_{t-i} + \sum_{i=0}^{\infty} \alpha^i v_{t-i} \tag{47}$$

$$c_t = \sum_{i=0}^{\infty} \alpha^i a_{t-i} + \sum_{i=0}^{\infty} \alpha^i b_{t-i} + \alpha \sum_{i=0}^{\infty} \alpha^i v_{t-i-1} \tag{48}$$

$$o_t = \sum_{i=0}^{\infty} \alpha^i a_{t-i} + \alpha \sum_{i=0}^{\infty} \alpha^i v_{t-i-1} \tag{49}$$

Writing out the output equation explicitly:

$$\begin{aligned} o_{t+h} &= a_{t+h} + \alpha a_{t+h-1} + \ldots + \alpha^{h-1} a_{t+1} + \alpha^h a_t + \alpha^{h+1} a_{t-1} + \ldots \\ &+ \alpha v_{t+h-1} + \ldots + \alpha^{h-1} v_{t+1} + \alpha^h v_t + \alpha^{h+1} v_{t-1} + \ldots \end{aligned} \tag{50}$$

- Theory projection of output at horizon h on a technology shock a_t.

$$o_{t+h} = b_h a_t + u_{t+h} \tag{51}$$

- Matching (50) and (51):

$$\begin{aligned} b_h &= \alpha^h && (52) \\ u_{t+h} &= (a_{t+h} + \alpha a_{t+h-1} + \ldots + \alpha^{h-1} a_{t+1} + \alpha^{h+1} a_{t-1} + \alpha^{h+2} a_{t-2} + \ldots \\ &+ \alpha \sum_{i=0}^{\infty} \alpha^i v_{t+h-1-i} && (53) \end{aligned}$$

- Error term u_{t+h} in the projection is a function of all values of v_t and past and future values of a_t.

- a_t is not observable; (51) can not be used. Suppose one employs the residuals of the VAR in place of a_t.

OLS estimation

• (No misspecification) If the VAR uses w_t and, at least, one lag, then $\hat{e}_t = w_t - \hat{A} w_{t-1}$ and $b_{h,OLS}$ is **consistent** for a^h, because $\hat{e}_{3,t} = a_t$.

• (Deformation) If the VAR is **deformed**, for example , only c_t, y_t are used to compute $\hat{e}_t$, $b_{h,OLS}$ is still **consistent**. The system is

$$\begin{aligned} c_t &= \alpha c_{t-1} + a_t + \alpha v_{t-1} + b_t - \alpha b_{t-1} \\ o_t &= \alpha o_{t-1} + a_t + \alpha v_{t-1} \end{aligned} \tag{54}$$

(54) allows identification of $\tilde{e}_{2t} = a_t + \alpha v_{t-1}$ from o_t, and given $\tilde{e}_{2t}$, $\tilde{e}_{1t} = b_t - \alpha b_{t-1}$ from c_t, given $\tilde{e}_{2t}$. The MA representation is

$$c_t = \sum_{i=0}^{\infty} \alpha^i a_{t-i} + \sum_{i=0}^{\infty} \alpha^i (b_{t-i} - \alpha b_{t-i-1}) + \alpha \sum_{i=0}^{\infty} \alpha^i v_{t-i-1} \tag{55}$$

$$o_t = \sum_{i=0}^{\infty} \alpha^i a_{t-i} + \alpha \sum_{i=0}^{\infty} \alpha^i v_{t-i-1} \tag{56}$$

• Compare (49) and (56): the MA for o_t is unchanged. Then

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}(\tilde{e}_{2t}\tilde{e}_{2t}) &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}(a_t+\alpha v_{t-1})(a_t+\alpha v_{t-1}) \\
&= \sigma_a^2+\alpha^2\sigma_v^2 \qquad (57)\\
\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}(\tilde{e}_{2t}o_{t+h}) &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}(a_t+\alpha v_{t-1})(\ldots+\alpha^h a_t+\ldots+\alpha^{h+1}v_{t-1}+\ldots) \\
&= \alpha^h(\sigma_a^2+\alpha^2\sigma_v^2) \qquad (58)
\end{aligned}$$

Thus $b_{h,OLS}=\alpha^h$.

• Numerator and denominator are biased, but the bias is in the same direction and cancels out. LP solves some of the IRF problems existing with deformed VARs, see Canova and Ferroni (2022).

• (Measurement error) If the VAR includes w_t but o_t is measured with error, say $y_t = o_t + \nu_t$, ν_t iid uncorrelated with the structural shocks, $b_{h,OLS}$ is **inconsistent**. This is because $\tilde{e}_{3,t} = a_t + \nu_t$ and

$$\begin{aligned}
b_{h,OLS} &= \left(\sum_{t=1}^{T} \tilde{e}_{3,t} y_{t+h}\right) \left(\sum_{t=1}^{T} (\tilde{e}_{3,t})^2\right)^{-1} \\
&= \left(\sum_{t=1}^{T} (a_t + \nu_t)(\sum_i \alpha^i z_{t+h-i} + \alpha \sum_i \alpha^i v_{t+h-i-1} + \nu_{t+h})\right) \left(\sum_{t=1}^{T} (a_t + \nu_t)^2\right)^{-1} \\
&= \left(\sum_{t=1}^{T} (a_t + \nu_t)(... + \alpha^h a_t + ...)\right) \left(\sum_{t=1}^{T} (a_t + \nu_t)^2\right)^{-1} \\
&\xrightarrow{T\to\infty} \alpha^h \frac{\sigma_a^2}{\sigma_a^2 + \sigma_\nu^2} < \alpha^h
\end{aligned} \tag{59}$$

• Classical contamination bias because of a proxy regressor. Asymptotic downward bias. The bias is larger, the larger is the standard deviation of the measurement error σ_ν^2.

• (Normalization) Using y_t in place of $\tilde{e}_{3,t}$, as in Stock and Watson (2018), change nothing:

$$
\begin{aligned}
b_{h,OLS} &= \left(\sum_{t=1}^{T} y_t y_{t+h}\right)\left(\sum_{t=1}^{T} y_t^2\right)^{-1} \\
&= \left(\sum_{t=1}^{T}(\sum_i \alpha^i a_{t-i} + \alpha \sum_i \alpha^i v_{t-i-1} + \nu_t)(\sum_i \alpha^i a_{t+h-i} + \alpha \sum_i \alpha^i v_{t+h-i-1} + \nu_{t+h})\right) \times \\
&\quad \left(\sum_{t=1}^{T}(\sum_i \alpha^i a_{t-i} + \alpha \sum_i \alpha^i v_{t-i-1} + \nu_t)^2\right)^{-1} \qquad (60)
\end{aligned}
$$

The denominator limit is

$$
\lim_{T\to\infty} \frac{1}{T}\sum_t (o_t + \nu_t)^2 = \sum_{i=0}^{\infty} \alpha^{2i}\sigma_a^2 + \alpha^2 \sum_{i=0}^{\infty} \alpha^{2i}\sigma_v^2 + \sigma_\nu^2
$$

The numerator limit is

$$\lim_{T\to\infty}\frac{1}{T}\sum_t((o_t+\nu_t)(o_{t+h}+\nu_{t+h}) = \alpha^h(\sum_{t=0}^{\infty}\alpha^{2i}\sigma_a^2+\alpha^2\sum_{i=0}^{\infty}\alpha^{2i}\sigma_v^2)$$

Hence

$$b_{h,OLS}\xrightarrow{T\to\infty}\alpha^h\frac{\frac{\sigma_a^2}{1-\alpha^2}+\frac{\alpha^2\sigma_v^2}{1-\alpha^2}}{\frac{\sigma_a^2}{1-\alpha^2}+\frac{\alpha^2\sigma_v^2}{1-\alpha^2}+\sigma_\nu^2} = \alpha^h\frac{\sigma_a^2+\alpha^2\sigma_v^2}{\sigma_a^2+\alpha^2\sigma_v^2+(1-\alpha^2)\sigma_\nu^2}\neq\alpha^h \tag{61}$$

• Bias remains. The size of the bias depends also on the magnitude of α and σ_v^2 (Note: before it did not).

• (Incorrect timing) What if $\tilde{e}_{3t} = a_{t-1} + \nu_t$. This could be produced, for example, if $y_t = o_{t-1} + \nu_t$. Here OLS is **biased**.

$$b_{h,OLS} = \left(\sum_{t=1}^{T} \tilde{e}_{3,t} y_{t+h}\right) \left(\sum_{t=1}^{T} \tilde{e}_{3,t}^2\right)^{-1} \tag{62}$$

$$= \left(\sum_{t=1}^{T} (a_{t-1} + \nu_t)(\sum_i \alpha^i a_{t+h-i} + \alpha \sum_i \alpha^i v_{t+h-i-1} + \nu_{t+h})\right) \left(\sum_{t=1}^{T} (a_{t-1} + \nu_t)^2\right)^{-1}$$

$$= \left(\sum_{t=1}^{T} (a_{t-1} + \nu_t)(... + \alpha^{h+1} a_{t-1} + ...)\right) \left(\sum_{t=1}^{T} (a_{t-1} + \nu_t)^2\right)^{-1} \tag{63}$$

$$\xrightarrow{T \to \infty} \alpha^{h+1} \frac{\sigma_a^2}{\sigma_a^2 + \sigma_\nu^2} \neq \alpha^h \tag{64}$$

• Downward bias. If $\alpha < 1$ bias is larger than with correct timing.

• (Measurement error and incorrect timing). If $\tilde{e}_{3,t} = \gamma a_t + (1-\gamma) a_{t-1} + \nu_t$ then

$$\begin{aligned} b_{h,OLS} & \xrightarrow{T\to\infty} (\gamma\alpha^h + (1-\gamma)\alpha^{h+1})\frac{\sigma_a^2}{\sigma_a^2+\sigma_\nu^2} \\ & = \alpha^h(\gamma + (1-\gamma)\alpha)\frac{\sigma_a^2}{\sigma_a^2+\sigma_\nu^2} \end{aligned} \tag{65}$$

General expression (here $\gamma(L)$ could be two-sided):

$$\begin{aligned} \tilde{e}_{3,t} & = \gamma(L) a_t + \nu_t \\ b_{h,OLS} & \xrightarrow{T\to\infty} \alpha^h \left(\sum_{i=0}^{\infty} \gamma_i \alpha^i\right) \frac{\sigma_a^2}{\sigma_a^2+\sigma_\nu^2} \end{aligned} \tag{66}$$

IV estimation: does it help?

- Suppose we have an instrument z_t, a contaminated measure of a_t, e.g.

$$z_t = \gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 v_t + \nu_t \tag{67}$$

Let $\gamma_1 >> 0$ (relevance condition), ν_{1t} be iid and suppose we run the regression with o_t as dependent variable.

- z_t is not necessarily orthogonal to e_{t+h}.

$$\begin{aligned} z_t o_{t+h} &= (\gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 v_t + \nu_t) \\ &\times (\ldots + \alpha^{h-1} a_{t+1} + \alpha^h a_t + \alpha^{h+1} a_{t-1} + \ldots + \alpha^{h-1} v_{t+1} + \alpha^h v_t + \alpha^{h+1} v_{t-1} + \ldots) \\ z_t o_t &= (\gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 v_t + \nu_t) \times (a_t + \alpha a_{t-1} + \ldots + \alpha v_{t-1} + \ldots) \end{aligned} \tag{68}$$

For $h > 0$ and $h = 0$, we have respectively

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} z_t o_{t+h} = (\gamma_3 \alpha^{h-1} + \gamma_1 \alpha^h + \gamma_2 \alpha^{h+1}) \sigma_a^2 + \alpha^h \gamma_4 \sigma_v^2 \tag{69}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} z_t o_t = (\gamma_1 + \alpha\gamma_2)\ \sigma_a^2 \tag{70}$$

Hence

$$\begin{aligned} b_{h,IV} &= \left(\sum_{t=1}^{T} z_t o_t\right)^{-1} \left(\sum_{t=1}^{T} z_t o_{t+h}\right) \\ &\xrightarrow{T\to\infty} \frac{(\gamma_3 \alpha^{h-1} + \gamma_1 \alpha^h + \gamma_2 \alpha^{h+1}) \sigma_a^2 + \alpha^h \gamma_4 \sigma_v^2}{(\gamma_1 + \alpha\gamma_2)\sigma_a^2} \\ &= \alpha^h \left[\frac{(\gamma_3 \alpha^{-1} + \gamma_1 + \gamma_2 \alpha)}{\gamma_1 + \alpha\gamma_2} + \frac{\gamma_4 \sigma_v^2}{(\gamma_1 + \alpha\gamma_2)\sigma_a^2}\right] \neq \alpha^h \end{aligned} \tag{71}$$

A few special cases

- No news (i.e. $\gamma_3 = 0$)

$$b_{h,IV} = \frac{(\gamma_1 + \gamma_2 \alpha)\alpha^h \sigma_a^2 + \ \alpha^h \gamma_4 \sigma_v^2}{(\gamma_1 + \alpha\gamma_2)\sigma_a^2} = \alpha^h \left(1 + \frac{\gamma_4}{(\gamma_1 + \alpha\gamma_2)} \frac{\sigma_v^2}{\sigma_a^2} \right) \quad (72)$$

Asymptotic bias present because of invalid instrument.

- No lags (i.e. $\gamma_2 = 0$)

$$b_{h,IV} = \frac{(\gamma_1 + \gamma_3 \alpha^{-1})\alpha^h \sigma_a^2 + \alpha^h \gamma_4 \sigma_v^2}{\gamma_1 \sigma_a^2} = \alpha^h + \frac{\gamma_3}{\gamma_1}\alpha^{h-1} + \alpha^{h+1}\frac{\gamma_4 \sigma_v^2}{\gamma_1 \sigma_a^2} \quad (73)$$

Presence or absence of lags does not affect the consistency of $b_{h,IV}$.

- No news and no contamination (i.e. $\gamma_4 = \gamma_3 = 0$)

$$b_{h,IV} = \alpha^h \quad (74)$$

• What if the projection is run with $\tilde{e}_{3,t}$ and y_{t+h} (in place of o_t and o_{t+h}?

$$
\begin{aligned}
z_t \tilde{e}_{3t} &= (\gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 v_t + \nu_{1t}) \times (a_t + \nu_{2t}) && (75)\\
z_t y_{t+h} &= (\gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 v_t + \nu_{1t}) \\
&\times (\ldots + \alpha^{h-1} a_{t+1} + \alpha^h a_t + \alpha^{h+1} a_{t-1} + \ldots + \alpha^h v_t + \alpha^{h+1} v_{t-1} + \ldots + \nu_{2t+h}) && (76)
\end{aligned}
$$

Sample moments:

$$
\lim_{T\to\infty} \quad \frac{1}{T}\sum_{t=1}^{T} z_t \tilde{e}_{3t} = \gamma_1 \sigma_a^2 \tag{77}
$$

$$
\lim_{T\to\infty} \quad \frac{1}{T}\sum_{t=1}^{T} z_t y_{t+h} = (\gamma_3 \alpha^{h-1} + \gamma_1 \alpha^h + \gamma_2 \alpha^{h+1})\sigma_a^2 + \alpha^h \gamma_4 \sigma_v^2 \tag{78}
$$

Hence

$$
b_{h,IV} = \frac{\frac{1}{T}\sum_t z_t y_{t+h}}{\frac{1}{T}\sum_t z_t \tilde{e}_{3t}} \xrightarrow{T\to\infty} \alpha^h \left[\frac{(\gamma_3 \alpha^{-1} + \gamma_1 + \gamma_2 \alpha)}{\gamma_1} + \frac{\gamma_4 \sigma_v^2}{\gamma_1 \sigma_a^2}\right] \neq \alpha^h \tag{79}
$$

• Assume that the first stage regression uses k_t instead of o_t; that is, we first regress k_t on z_t and then we regress o_{t+h} on the fitted values. Since

$$z_t k_t = (\gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 v_t + \nu_t)(a_t + \alpha a_{t-1} + \ldots + v_t + \alpha v_{t-1} + \ldots) \tag{80}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} z_t k_t = (\gamma_1 + \alpha\gamma_2)\ \sigma_a^2 + \gamma_4 \sigma_v^2 \tag{81}$$

Hence

$$\begin{aligned} b_{h,IV} &= \left(\sum_{t=1}^{T} z_t k_t\right)^{-1} \left(\sum_{t=1}^{T} z_t o_{t+h}\right) \\ &\xrightarrow{T\to\infty} \frac{(\gamma_3\alpha^{h-1} + \gamma_1\alpha^h + \gamma_2\alpha^{h+1})\sigma_a^2 + \alpha^h\gamma_4\sigma_v^2}{(\gamma_1 + \alpha\gamma_2)\sigma_a^2 + \gamma_4\sigma_v^2} \end{aligned} \tag{82}$$

• If there are no news ($\gamma_3 = 0$), $b_{h,IV} \to \alpha^h$.

• Suppose the instrument z_t is related to a_t and the demand shock b_t:

$$z_t = \gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 b_t + \nu_{1t} \tag{83}$$

• Keep $\gamma_1 >> 0$ (relevance condition), ν_{1t} is iid and assume we run the regression with c_t as dependent variable. Then

$$\begin{aligned} z_t c_{t+h} &= (\gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 b_t + \nu_t) \\ &\times \ (a_{t+h} + \ldots + \alpha^{h-1} a_{t+1} + \alpha^h a_t + \alpha^{h+1} a_{t-1} + \ldots + \alpha^h v_t + \ldots + \alpha^h b_t + \ldots) \\ z_t c_t &= (\gamma_1 a_t + \gamma_2 a_{t-1} + \gamma_3 a_{t+1} + \gamma_4 b_t + \nu_t)(a_t + \alpha a_{t-1} + \ldots + \alpha v_{t-1} + \ldots + b_t + \alpha b_{t-1} + \ldots \end{aligned}$$

For $h > 0$ and $h = 0$, we have respectively

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} z_t c_{t+h} = (\gamma_3 \alpha^{h-1} + \gamma_1 \alpha^h + \gamma_2 \alpha^{h+1}) \sigma_a^2 + \alpha^h \gamma_4 \sigma_b^2 \tag{84}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} z_t c_t = (\gamma_1 + \alpha\gamma_2)\, \sigma_a^2 + \gamma_4 \sigma_b^2 \tag{85}$$

Hence

$$\begin{aligned} b_{h,IV} &= \left(\sum_{t=1}^{T} z_t c_t\right)^{-1} \left(\sum_{t=1}^{T} z_t c_{t+h}\right) \\ &\xrightarrow{T\to\infty} \alpha^h \frac{(\gamma_3\alpha^{-1} + \gamma_1 + \gamma_2\alpha)\sigma_a^2 + \gamma_4\sigma_b^2}{(\gamma_1 + \alpha\gamma_2)\sigma_a^2 + \gamma_4\sigma_b^2} \neq \alpha^h \qquad (86) \end{aligned}$$

If no news ($\gamma_3 = 0$), $b_{h,IV} \to \alpha^h$. Similar conclusions can be derived using o_t instead of c_t.

- **The presence of news makes IV inconsistent!**