Time series and structural trend cycle decompositions. Turning point dating

Fabio Canova
Norwegian Business School and CEPR
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Outline

- Intro and generics of the decomposition.
- Burns and Mitchell: turning point analysis.
- Lucas 1: LT, SEGM, FOD, Hamilton, UC, BN.
- Lucas 2: HP, BP, Wavelets, Butterworth.
- Economic model-based decomposition: BQ, KPSW.
- Collecting cyclical information: does it matter?
- Business and financial cycles.
- Measuring trends in real rates and in gender gap
- Fitting DSGE models to cyclical data.

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1 Introduction

• Why we care about business cycles and not about seasonal cycles or longer (Kitchin, Juglar, Kutnets, Kondriatev) cycles?

• How do we measure business cycles?

• What are the main features of business cycles? Are they different from, say, financial cycles? Or service cycles? Why?

- Burns and Mitchell (BM) (1943):

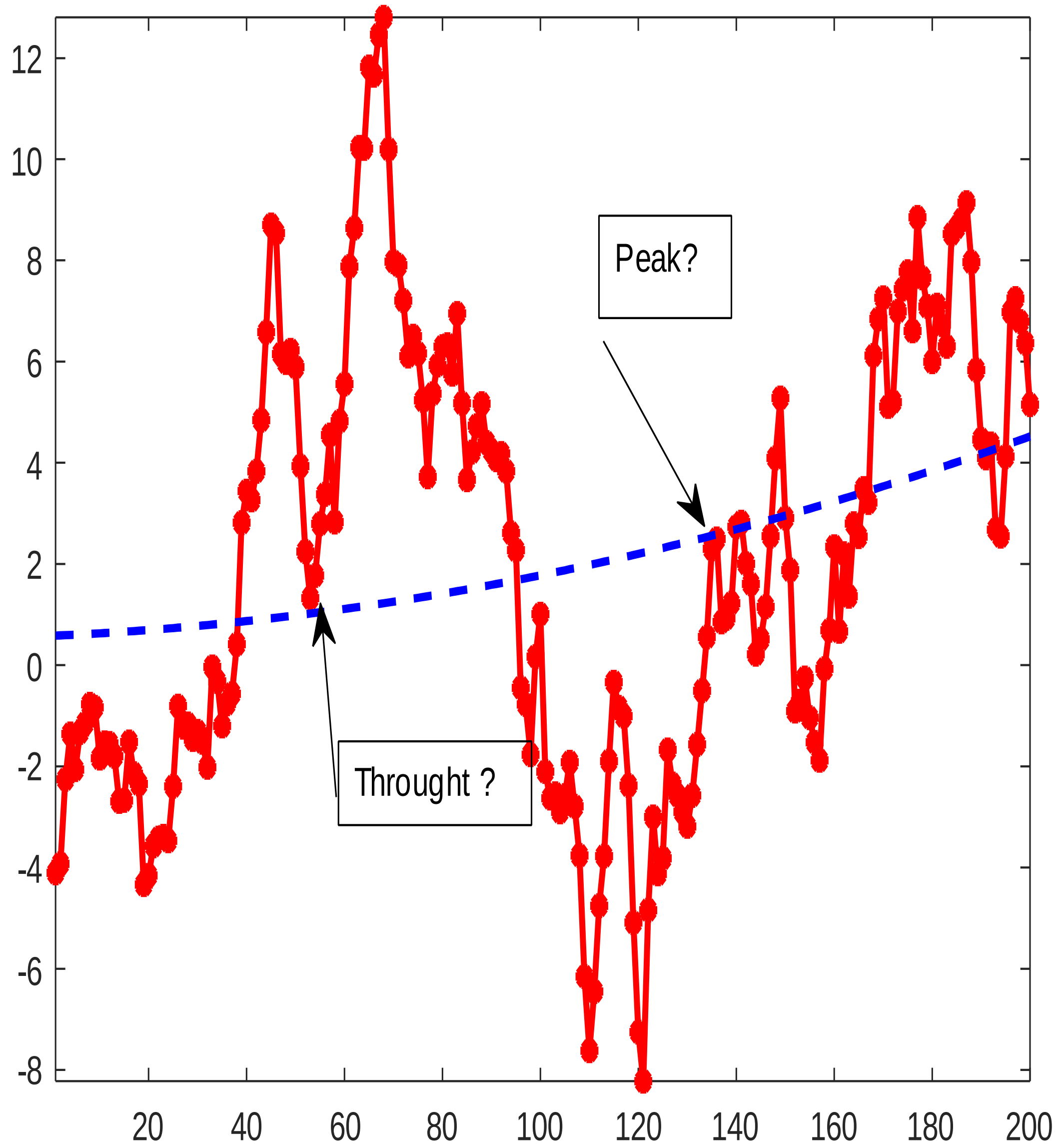
"Business cycles are a type of fluctuations found in the aggregate economic activity of nations that organize their work mainly in business enterprises: a cycle consists of expansions **occurring at about the same time in many economic activities**, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle; **this sequence of changes is recurrent but not periodic**; in duration business cycles vary from **more than one year to ten or twelve years**; they are not divisible into shorter cycles of similar characters with amplitudes approximating their own."

- Lucas (1977):

" **Movements about trend in gross national product** in any country can be well described by a stochastically disturbed difference equation of very low order. These movements do not exhibit uniformity of either period or amplitude, which is to say, **they do not resemble the deterministic wave motions which sometimes arise in the natural sciences**. Those regularities which are observed are **in the co-movements among different aggregative time series** (....). There is, as far as I know, **no need to qualify these observations by restricting them to particular countries or time periods**: they appear to be regularities common to all decentralized market economies. Though there is absolutely no theoretical reason to anticipate it, one is led by the facts to conclude that, with respect to the qualitative behavior of co-movements among series, **business cycles are all alike**.

• Burns and Mitchell consider level data. Characterize the business cycle of a nation by specifying interesting durations and looking at comovements across series within a country.

• Lucas looks at detrended data (growth cycles). Seeks commonality across series, time periods and, potentially, countries. Does not specify interesting periodicities.


Peak?
Throught ?
12
10
8
6
4
2
0
-2
-4
-6
-8
20
40
60
80
100
120
140
160
180
200

NBER (methodology):

“The NBER’s Business Cycle Dating Committee maintains a chronology of the U.S. business cycle. The chronology comprises alternating dates of peaks and troughs in economic activity. **A recession is a period between a peak and a trough, and an expansion is a period between a trough and a peak. During a recession, a significant decline in economic activity spreads across the economy and can last from a few months to more than a year.** Similarly, during an expansion, economic activity rises substantially, spreads across the economy, and usually lasts for several years. ... The Committee applies its judgment based on the above definitions of recessions and expansions and **has no fixed rule** to determine whether a contraction is only a short interruption of an expansion, or an expansion is only a short interruption of a contraction. .... The Committee **does not have a fixed definition of economic activity**. It examines and compares the behavior of various measures of broad activity: real GDP measured on the product and income sides, economy-wide employment, and real income. The Committee also may consider indicators that do not cover the entire economy, such as real sales and the Federal Reserve's index of industrial production (IP) ... a well-defined peak or trough in real sales or IP might help to determine the overall peak or trough dates, particularly if the economy-wide indicators are in conflict or do not have well-defined peaks or troughs".

CEPR (recessions):

"a significant decline in the level of economic activity, spread across the economy of the euro area, usually visible in **two or more consecutive quarters of negative growth in GDP, employment and other measures of aggregate economic activity** for the euro area as a whole"

- https://cepr.org/about/news/latest-findings-cepr-eabcn-euro-area-business-cycle-dating-committee-eabcdc and https://eabcn.org/dc/news.

- http://asesec.org/en/committees/spanish-business-cycle-dating-committee/

Why does one take deviations from trend?

• Data has more than business cycle fluctuations; it displays growth, seasonal, and irregular (high frequency) variations.

• Economic models are typically stationary and built to explain only business cycle fluctuations.

• To collect cyclical facts (Lucas definition) need to detrend the data (since second moments may not exist).

• Note: detrending and filtering are different operations!

Basic Questions I

• How does one **define** the business cycle?

i) Burns-Mitchell/Harding-Pagan: the sequence of alternating, irregularly spaced turning points and repetition of expansion/recession phases of 2 quarters minimum duration.

ii) Majority of macroeconomists: the presence variability, serial and cross correlation in a vector of detrended aggregate macroeconomic variables.

iii) Time series econometricians: spectral peak at cyclical frequencies in one or more time series.

iv) Policymakers: fluctuations in "gaps".

Basic questions II

• How does one **measure** the business cycle?

i) Use a statistical or an economic model?

ii) If a statistical model: use a univariate or a multivariate approach?

iii) If an economic model: what frictions should be included?

• If detrend: use a deterministic or a stochastic trend? With or without breaks?

• If filtering: which filter? Which frequencies (cycles) to keep?

Basic questions III

• How does one **interpret** the facts in light of economic theories?

• Theories talk about permanent and transitory shocks. How do they relate to statistical "trends" and "cycles"?

• How do " neutrality" propositions (e.g. long run money neutrality) are linked to "trend and cycle" decompositions?

• Theories discuss "potential" ("efficient") levels of the variables and "gaps" are deviations of actual from potential (efficient) levels. How do we link this discussions to trend-cycle decompositions?

• How do we relate ''saving glut" stories or gender gap trends to statistical trends in real rates?

2 Generics

Assume (for simplicity) that the "trend" is everything that it is not the "cycle"and that the two components are additive, i.e., $y_t = y_t^x + y_t^c$.

- If additivity is suspect, take logs.

- Trend and cycle are latent.

- How we split y_t depends:

i) Assumed properties (definition) of y_t^x.

ii) Correlation trend-cycle (call it ρ).

3 Burns-Mitchell/Harding-Pagan approach

• Pattern recognition exercise: find turning points in y_{it}.

• Use mechanical rules (Bry and Boschen (BB) algorithm): find local max and min of the series. Early application of AL algorithms.

• Example of a mechanical rule: Let $S_t = 1$ if upturn occurs and zero otherwise (from some external information).Then $S_t(1 - S_{t+1}) = 1$ if there is a peak and $(1 - S_t)S_{t+1} = 1$ if there is a through.

• Use judgemental rules (NBER/CEPR committees): persistent (at least two quarters) increases/decreases (positive/negative growth). Arbitrary.

• With turning points: measure duration and amplitude of expansions and contractions; the concordance of turning points or cyclical phases.

• BB algorithm rules:

1. Peaks and throughs must alternate.

2. Each phase (peak to through or through to peak) must have a duration of at least six months (two quarters).

3. A cycle (peak to peak or through to through) must have a duration of at least 15 months (5 quarters).

4. Turning points within six months (2 quarters) of the beginning or end of the series are eliminated. Peaks or throughs within 24 months (8 quarters) of the beginning or end of the sample are eliminated if any of the points after or before are higher (or lower) than the peak (through).

• BB turning point dates may be different than NBER/CEPR turning point dates. Why?

- There may be consecutive peaks (throughs). Judgment should be used. See, e.g., Covid19 recession.

- Peaks (throughs) may occur at negative (positive) values.

- Expansions (recessions) may be uniformly small; no sharp through (peak).

Turning point dates: Euro area

Phase	CEPR	Length	BB (GDP)	Length
Peak	1974:3		1973:4	
Through	1975:1	2	1974:1	2
Peak	1980:1	20	1979:2	21
Through	1982:3	10	1979:4	2
Peak	1992:1	38	1991:2	46
Through	1993:3	6	1992:2	4
Peak	2008:1	58	2007:2	60
Through	2009:2	5	2008:3	5
Peak	2011:3	9	2010:4	9
Through	2013:1	6	2012:2	6

- High concordance of dates and duration, except for 1980 and 1990.

• BB algorithm is applied one series at a time. How do you construct a synthetic indicator of BC dates and phases? Two possibilities:

- Average-and-date (use one aggregate indicator to date turning points)

- Date-and-average (date individual series and average turning points).

• Average-and-date. Use a standard coincident indicator (e.g. Conference Board (TCB) Indicator in US); or pick one relevant series (GDP, IP). Compare turning point dates with standard classification (NBER/CEPR) to check reasonableness of dates (as we have done above). Alternatives:

• Estimate a dynamic factor model (DFM). Compute turning points in f_t.

$$\begin{aligned} y_{it} &= \lambda f_t + e_{it} \\ f_t &= a(L) f_{t-1} + u_t \quad u_t \sim iid(0, \sigma_u^2) \\ e_{it} &= b(L) e_{it-1} + v_{it} \quad v_{it} \sim iid(0, \sigma_v^2) \end{aligned} \tag{1}$$

• Construct an index standard deviation (ISD) weighting indicator:

$$I_t = \mathsf{exp}(\sum_{i=1}^{N} \omega_i y_{it}) \tag{2}$$

where $\omega_i = \frac{s_i^{-1}}{\sum_{j=1}^{N} s_j^{-1}}$ and s_i is the standard deviation of y_{it}.

• Stock and Watson (2014) for the US:

- The time series of TBC and ISD produce similar turning points.

- Coincident indicators produce smaller MSE (relative to NBER dates) than a single (GDP) series.

Table 2
Average-then-date chronologies computed using three monthly coincident indexes and four measures of monthly GDP, as a lead (positive value) or lag (negative value) of the NBER turning point.

NBER		Coincident indexes			Monthly GDP			
		CI-TCB	CI-ISD	CI-DFM	GDP(E)	GDP(I)	GDP(Avg)	GDP-MA
1960:4	P	−2	0	-	−1	−2	−1	
1961:2	T	0	0	0	−2	−2	−2	
1969:12	P	−2	−2	−4	−4	-	−4	
1970:11	T	0	0	0	−10	-	0	
1973:11	P	0	0	0	1	0	1	
1975:3	T	1	1	1	0	−1	0	
1980:1	P	0	0	−10	-	0	-	
1980:7	T	0	0	0	-	−1	-	
1981:7	P	0	1	0	2	1	2	
1982:11	T	0	0	0	−6	0	−3	
1990:7	P	−1	−1	0	0	0	0	
1991:3	T	0	0	0	0	−2	−2	
2001:3	P	−6	−6	−6	-	0	-	-
2001:11	T	4	0	0	-	−1	-	-
2007:12	P	−1	0	0	1	−12	0	1
2009:6	T	0	0	0	0	1	0	0
Mean		−0.44	−0.44	−1.27	−1.58	−1.36	−0.75	0.50
MAE		1.06	0.69	1.40	2.25	1.64	1.25	0.50

Notes: Entries are the NBER turning point minus the series-specific Bry–Boschan turning point, in months. Episodes for which the series is available but does not have a Bry–Boschan turning point are denoted by "–". The GDP(E), GDP(I), and GDP(Avg) monthly GDP series are from Stock and Watson (2010a). The GDP-MA series is the Macroeconomic Advisors Monthly GDP series, which starts in 1992:4. The mean and mean absolute error (MAE) in the final two rows summarize the discrepancies of the chronology for the column series, relative to the NBER chronology; episodes in which a series does not have a Bry–Boschan recession are excluded from the summary statistics.

• Date-and-average. Compute turning points τ_{is} for series $i = 1, \ldots, n_s$ in episode s (e.g. a recession). Compute a location measure of the turning points distribution for each episode s.

• If turning points are iid:

$$n_s^{0.5}(\hat{\tau}_s^{mean} - \tau_s^{mean}) \xrightarrow{D} N(0, var(\tau_{is})) \tag{3}$$

$$n_s^{0.5}(\hat{\tau}_s^{median} - \tau_s^{median}) \xrightarrow{D} N(0, \frac{1}{4(g_s(\tau_s))^2}) \tag{4}$$

$$(h^3 n_s)^{0.5}(\hat{\tau}_s^{mode} - \tau_s^{mode}) \xrightarrow{D} N(0, \frac{g_s(\tau_s^{mode}) \int [K'(z)]^2 dz}{g_s''(\tau_s^{mode})}) \tag{5}$$

where $K(.)$ is a kernel, h the length of the kernel, $g_s(\tau)$ is the distribution of τ in episode s (see Stock and Watson, 2014).

• Careful in blindly computing location averages! Look at the distribution!

• If certain types of series are over-represented in the sample relative to the population (e.g. there too many IP series and too few employment series) use weighted location measures; it helps also if series do not have the same lengths.

• Typical weights :

$$w_{i,s} = \frac{\pi_{m_i}}{p_{m_i,s}} \tag{6}$$

where π_m is the population probability of class m series (IPs, employments, interest rates, etc.) and $p_{m,s}$ is the sample probability of class m in business cycle episode s.

• Stock and Watson (2014): if the mean is used, NBER turning points are recognized with a lag; with the mode, they are recognized with a lead.

Table 3
Date-then-average chronologies and standard errors computed using turning points of 270 disaggregated series, as a lead (positive value) or lag (negative value) of the NBER turning point.

NBER Dates		No adjustments			Class lag-adjusted			Weighted estimation		
		Mean	Median	Mode	Mean	Median	Mode	Mean	Median	Mode
1960: 4	P	−1.8 (0.6)	−2.0 (0.7)	−1.4 (0.5)	−2.5 (0.7)	−2.3 (0.8)	−2.5 (0.3)	−2.0 (0.6)	−2.0 (0.3)	−1.4 (0.4)
1961: 2	T	−0.3 (0.4)	0.0 (0.6)	−0.5 (0.7)	−0.8 (0.3)	−1.1 (0.5)	−0.5 (0.2)	−0.3 (0.3)	0.0 (0.3)	−0.6 (0.2)
1969:12	P	−2.2 (0.7)	−2.0 (0.6)	−2.3 (0.4)	−1.7 (0.6)	−1.8 (0.7)	−1.3 (0.5)	−1.7 (0.8)	−2.0 (0.4)	−2.4 (5.9)
1970:11	T	1.2 (0.6)	0.0 (0.7)	−0.2 (0.4)	1.7 (0.6)	1.2 (0.7)	0.7 (0.3)	1.9 (0.7)	1.0 (0.6)	0.1 (2.7)
1973:11	P	1.3 (0.6)	2.0 (0.6)	1.6 (0.3)	1.9 (0.6)	3.0 (0.7)	2.2 (0.3)	2.4 (0.7)	3.0 (0.7)	1.7 (1.0)
1975: 3	T	1.0 (0.3)	0.0 (0.3)	0.4 (0.3)	1.6 (0.3)	1.2 (0.3)	1.0 (0.1)	1.3 (0.3)	1.0 (0.4)	0.8 (0.8)
1980: 1	P	−1.8 (0.7)	−1.0 (0.8)	−0.3 (0.4)	−1.3 (0.7)	−1.2 (0.9)	0.3 (0.2)	−1.8 (0.9)	−2.0 (0.8)	−0.1 (0.2)
1980: 7	T	−0.9 (0.5)	0.0 (0.4)	−0.5 (0.2)	−0.1 (0.5)	0.2 (0.3)	0.3 (0.1)	−0.5 (0.7)	0.0 (0.4)	0.0 (0.2)
1981: 7	P	−0.7 (0.5)	0.0 (0.5)	−0.1 (0.3)	−0.2 (0.5)	0.2 (0.5)	0.5 (0.1)	−0.1 (0.5)	0.0 (0.4)	0.1 (4.4)
1982:11	T	−0.6 (0.6)	0.0 (0.6)	1.1 (0.4)	−0.2 (0.6)	0.9 (0.6)	1.9 (0.2)	−0.5 (0.6)	0.0 (0.5)	0.9 (0.9)
1990: 7	P	−0.8 (0.6)	0.0 (0.7)	0.3 (0.5)	−0.3 (0.6)	−1.2 (0.8)	1.8 (0.4)	−1.1 (0.6)	−1.0 (0.5)	−0.3 (0.2)
1991: 3	T	2.1 (0.5)	1.0 (0.4)	0.4 (0.3)	2.1 (0.4)	1.1 (0.4)	0.4 (0.1)	2.0 (0.4)	1.0 (0.4)	0.2 (2.0)
2001: 3	P	−3.7 (0.5)	−3.0 (0.6)	−2.2 (0.3)	−4.1 (0.5)	−4.8 (0.6)	−3.2 (0.2)	−3.7 (0.6)	−3.0 (0.6)	−2.3 (4.4)
2001:11	T	0.2 (0.5)	1.0 (0.5)	0.6 (0.2)	0.5 (0.5)	1.2 (0.5)	1.5 (0.1)	0.6 (0.7)	1.0 (0.7)	0.6 (0.9)
2007:12	P	−1.0 (0.5)	−1.0 (0.9)	−6.1 (0.5)	−1.4 (0.5)	−1.8 (0.7)	−2.8 (0.8)	−1.4 (0.5)	−2.0 (0.9)	−6.0 (1.1)
2009:6	T	1.7 (0.3)	1.0 (0.5)	−0.1 (0.2)	1.5 (0.3)	1.7 (0.4)	1.4 (0.2)	1.6 (0.3)	1.0 (0.5)	−0.2 (0.2)
Mean		−0.39	−0.25	−0.59	−0.20	−0.20	0.10	−0.21	−0.25	−0.56
MAE		1.34	0.88	1.12	1.37	1.56	1.38	1.44	1.25	1.11

Notes: Entries are the NBER turning point minus the date-then-average chronology for that column, in months. Standard errors appear in parentheses. The mean and mean absolute error (MAE) in the final two rows summarize the discrepancies of the chronology for the column series, relative to the NBER chronology.

Alternative procedures to aggregate

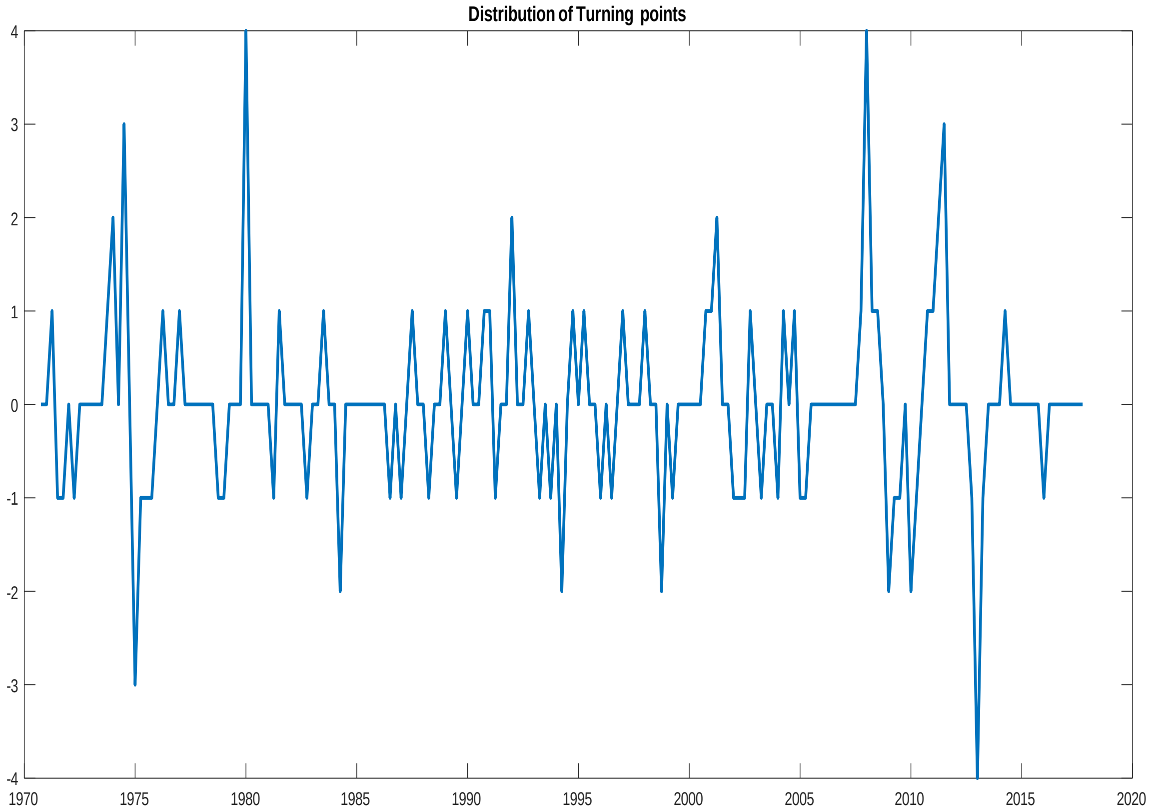
• Pagan and Harding (2016). Construct a '**reference phase**: want at least 50 per cent of the series to be in a particular BC phase.

• Pagan (2019): Construct reference turning points by minimizing the discrepancy among individual series turning points, i.e. if peaks occur at 1973:1, 1973:5, 1973:9, reference peak is 1973:5.

• Construct a weighted average of turning points; weight may depend on the (subjective) importance of individual series (GDP turning points may have more weights than, say, labor productivity turning points).

• **It is still an open question what is the best way to aggregate turning points and cyclical phases.**

- Euro data 1970:1-2017:4. Series: Y, C, Inv, Y/N, N, R, π.

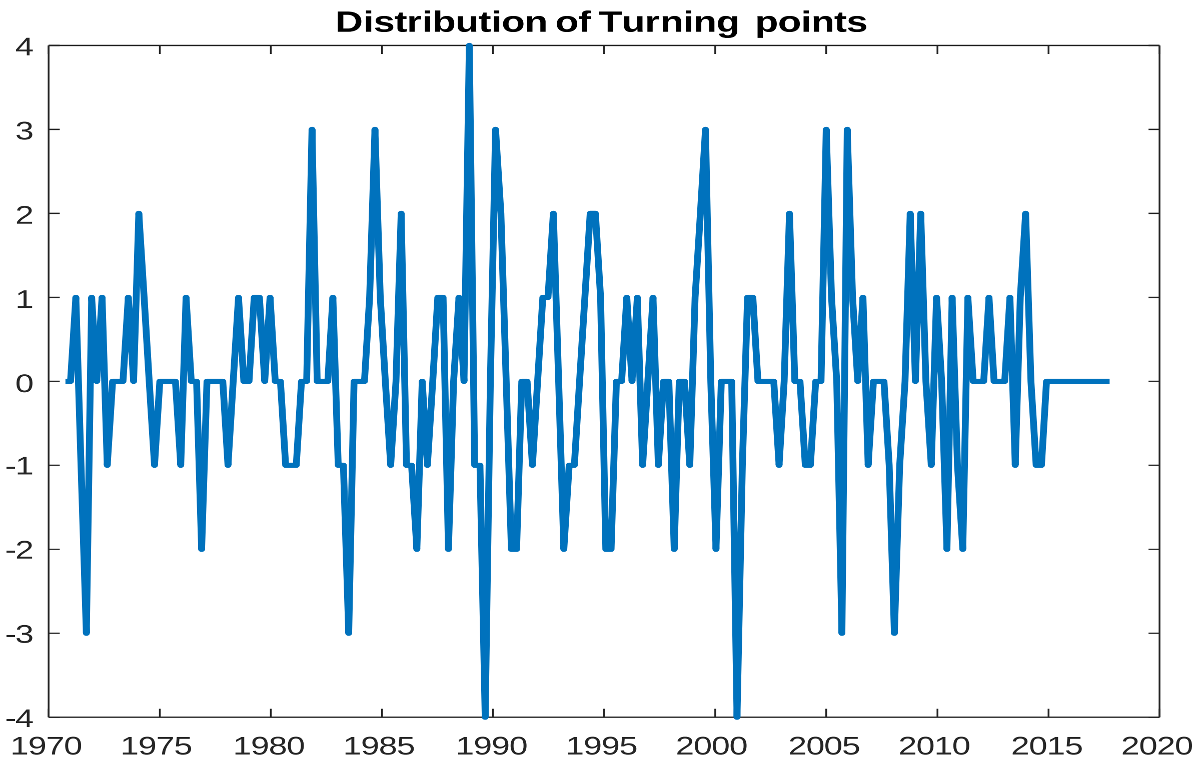


- 1975:1 is it a though? Less than 50% of the series are in a downturn.

- 2008 is it a through?. Minimal distance through is 2009:2. (two series have minimum in 2008:1 and two in 2010:3).

- **Important to have a good number of (coincident) series when performing the aggregation exercise.**

• US data 1970:1-2019:3. Series: Y, U, C, Inv, CapU, R1, R10, π, C/Y, I/Y, Term spread.



Business cycle statistics

• Average durations (AD), i.e. the average length of time spent between throughs and peaks or peaks and throughs.

• Average amplitudes (AA), i.e. the average size of the drop between peaks and troughs or of the gain between troughs and peaks.

• Concordance index $CI_{j,j'} = n^{-1}[\sum \mathcal{I}_{jt}\mathcal{I}_{it} - (1-\mathcal{I}_{jt})(1-\mathcal{I}_{it})]$. Measures comovements over business cycle phases of two variables, where n is the number of complete cycles and $\mathcal{I}_{it} = 1$ in expansions and $\mathcal{I}_{it} = 0$ in contractions. $CI = 1 (= -1)$ if the two series are perfectly positively (negatively) correlated.

• Average cumulative changes over phases ($CM = 0.5 * (AD * AA)$) and excess average cumulative changes $ECM = ((CM - CM^A + 0.5 * AA)/AD)$, where CM^A is the actual average cumulative change.

Euro area business cycle statistics

	AD (quarters)		AA (percentage)		ECM(percentage)		$CI_{j,j'}$ (phase)
	PT	TP	PT	TP	PT	TP	
GDP	3.8	33.7	-2.5	20.9	6.7	1.9	
C	5	36.6	-1.5	19.2	9.8	4.4	0.57
Inv	6.7	14.7	-7.2	14.7	14.9	1.1	0.52
Y/N	2.0	18.6	-1.2	8.9	1.7	10.15	0.61
N	9.0	22.8	-1.8	6.13	7.0	11.82	0.45
R	8.4	6.6	-3.1	2.69	10.5	7.80	0.04
π	9.0	6.9	-6.1	5.57	0.34	12.01	0.15

- Big asymmetries in durations and amplitudes.

- Output and consumption expansions longer and stronger.

- Low concordance of real and nominal series.

US business cycle statistics

	AD (quarters)		AA (percentage)		ECM(percentage)		$CI_{j,j'}$ (phase)
	PT	TP	PT	TP	PT	TP	
GDP	3.4	27.4	-0.02	0.2	3.3	13.2	
C	3.7	42.6	-0.01	0.3	-15.7	7.5	0.41
Inv	4.9	10.2	-0.1	0.2	-15.8	8.4	-0.37
U	14.0	7.8	-2.2	2.8	19.3	5.6	0.29
capU	6.2	8.9	-6.6	6.0	-13.2	2.4	-0.02
π	5.3	6.8	-2.7	2.4	8.2	2.4	0.12
R	7.2	6.5	-3.6	2.8	15.2	2.8	-0.04

- Durations in U different than durations in C, I, Y, capacity utilization.
- Asymmetries large except for nominal variables.
- Concordance low (negative for I and capacity utilization).

Features of the approach

• No need to measure y_{it}^c to compute stylized business cycle facts.

• Can collect statistics even if no econometrician cycles are present.

• Allows for asymmetries of cyclical phases.

• Results sensitive to dating rule and to minimum duration of phases (Typically: two or three quarters - so that complete cycles should be at least 5 to 7 quarters long) and to minimum amplitude restrictions (e.g. peaks to troughs drops of less than one percent should be excluded).

Questions

- How does the approach work with regional data, say, in a monetary union?

- How does one adapt the procedure for international business cycle comparisons?

- How does one compute a global cycle? Or examine business cycle convergence?

- Should one construct separate sectoral (labor market, financial markets, credit markets, banking, etc.) cycles and aggregate them? Or should one go at even more micro level?

3.1 Mixed frequency data

• BB algorithm can be applied to series of different frequencies.

• If you want to aggregate dates need to collapse high frequency dates to lower frequency dates.

• For example, if you have weekly, monthly and quarterly data, you need to transform weekly (monthly) dates into quarterly dates: e.g. 1975:11=1975:4 or 1975:47=1975:4) and then aggregate.

• To compute business cycle statistics use the frequency you have and then transform them into the lower frequency, i.e. if the duration of expansions is 36 months, that is, 9 quarters.

• Can use mixed frequency data also in aggregate-and-date approach. In a DFM y_{it} could be series at different frequencies. Need to put the model into a state space framework to reconstruct, say a weekly indicator.

$$\begin{aligned} y^o_{it} &= M y_{it} \\ y_{it} &= \lambda f_t + e_{it} \\ f_t &= a(L) f_{t-1} + u_t \quad u_t \sim iid(0, \sigma^2_u) \\ e_{it} &= b(L) e_{it-1} + v_{it} \quad v_{it} \sim iid(0, \sigma^2_v) \end{aligned} \tag{7}$$

where the first equation links the weekly series (some may be observable and same may not) to the, say, quarterly observable variables y^o_t and M is a matrix (see Dallas Fed indicator).

• Under normality of the error terms the weekly factor can be estimated with the Kalman filter (smoother).

3.2 Predicting Downturns

• Big business in financial markets is the prediction of downturns (recessions).

• Typically use probit/logit model: $\mathsf{P}(1 - S_t = 0|F_{t-1})$; F_{t-1} information available at $t-1$ to the econometrician.

• Borio et al. (2018): $F_{t-1} =$ use financial cycle information. Debt to service ratio (DSR), alone or in addition to term spread, does the job.

Financial cycle proxies help in evaluating recession risk

Regression coefficients from panel probit models — Table 1

Horizon		Financial cycle[1]	DSR	Term spread	Financial cycle and term spread	DSR and term spread
			Advanced economies			
1 year	Financial cycle	0.69***			0.62***	
	DSR		0.61***			0.57***
	Spread			−0.35***	−0.21***	−0.28***
2 year	Financial cycle	0.63***			0.60***	
	DSR		0.38***			0.35***
	Spread			−0.23***	−0.09*	−0.17***
3 year	Financial cycle	0.43***			0.44***	
	DSR		0.16***			0.15***
	Spread			−0.08	0.03	−0.06

• Berge and Jorda, 2011. To measure predictive power report area under the Receiver Operating characteristic (ROC) curve.

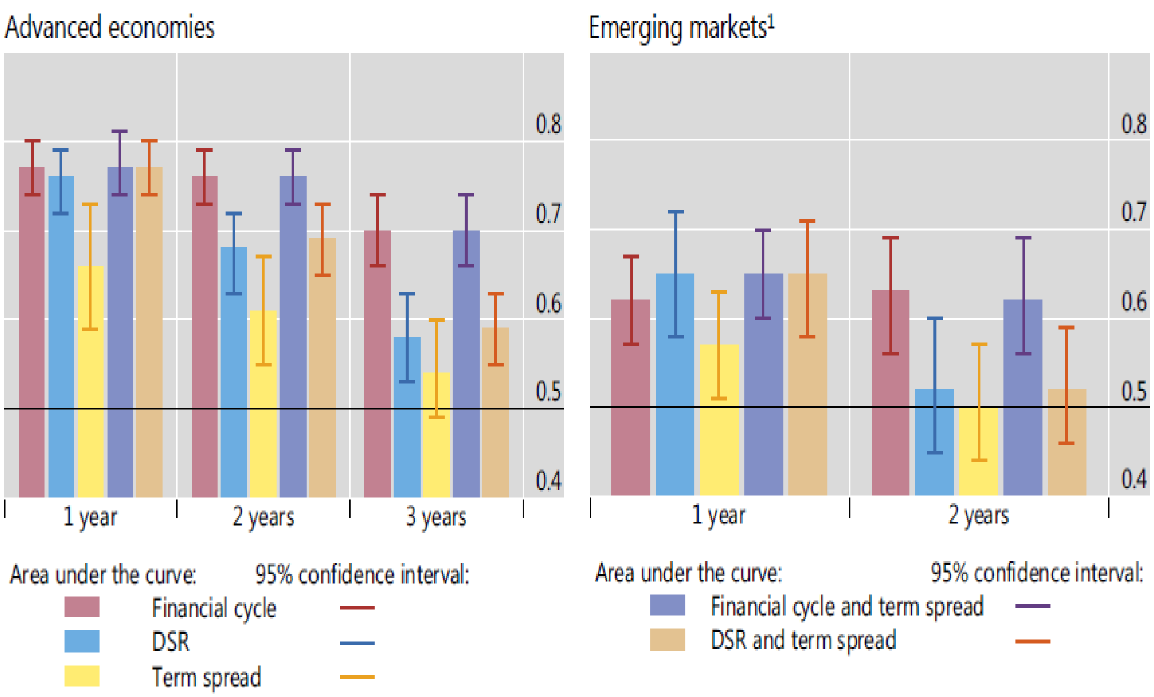
• The curve maps out combinations of type I errors (missed recessions) and type II errors (false alarms). The area under the curve (AUC) measures the indicator's signaling quality.

• If AUC=0.5: indicator uninformative; if AUC=1.0: indicator is perfect. The AUC of an "informative" indicator must be statistically different from 0.5.

Financial cycle measures are useful for assessing recession risk around the globe

AUCs for different forecast horizons

Graph 3



The horizontal lines at 0.5 indicate the area under the curve (AUC) of an uninformative, random variable.

Problems:

- Predicting $1-S_t$ (downturn) different than predicting the sign of Δy_{t+1}.

- What are we measuring? P$(1 - S_t = 0|1 - S_{t-1} = 1)$= probability of entering a recession; P$(1 - S_t = 0|1 - S_{t-1} = 0)$= probability of staying in a recession. With F_{t-1} in the regression, mixing these two probabilities.

- $1 - S_t$ is a generated variable that depends on $y_{t+k}, k = 1, 2$ because of the way BB algorithm (judgement) works. Incorrect to use it as conditioning variable in a VAR to see if,e.g., responses differ in recession and expansions - generated regression problem.

- Canova (1994): Predicting financial crisis in the preFed period.

4 How do macroeconomists think about cycles?

- Use a procedure to remove y^x_{it}.

- Compute second moments: $var(y^c_{it})$; $E(y^c_{it}, y^c_{it-k})$, i=1,2...,N; $corr(y^c_{it-k}, y^c_{1t})$, $i = 2, ..., N, k = 0, 1, \ldots$, where y^c_{1t} is cyclical output. Any pattern on average over time ?

- Alternative: fix an interval $t_0 < t < t_1$ (financial crisis, recession, etc.): compute variability, auto, and cross correlations in the episode. Is the episode different than usual? How?

- Macro research program: develop theoretical models that can produce data 'patterns' (stylized facts) (Pagan, 2013, 2019).

- **How does one remove y^x_{it}.**

Univariate (detrending) approaches

- Polynominal trend, $\rho = 0$.
- Segmented linear trend, $\rho = 0$.
- Differencing: periodic trend, $\rho = 0$.
- Hamilton local projection, $\rho = 0$ (multivariate option).
- Unobservable components, ρ may be non-zero (multivariate option).
- Beveridge Nelson: $\rho = 1$ (multivariate option).

Univariate (filtering) approaches

- Hodrick and Prescott, $\rho = 0$

- Band pass, $\rho = 0$.

- Wavelets, $\rho = 0$.

- Butterworth, $\rho = 0$.

Multivariate (economic) approches

- Blanchard and Quah; KPSW, $\rho \neq 0$ (structural shocks could be uncorrelated or correlated).

4.1 Deterministic Polynomial Trend

$$y_t^x = a + bt + ct^2 + ...$$

Estimate $a, b, c, \ldots$ in the regression

$$y_t = a + bt + ct^2 + ... + e_t$$

by OLS. Set $\hat{y}_t^c = y_t - a_{OLS} - b_{OLS}t - c_{OLS}t^2 - \ldots$.

• **Can perfectly predict trend in the future**.

• **No trend acceleration/deceleration is possible**.

• y_t^c is typically nearly non-stationary (lots of low frequency variations).

• Unless a, b, c recursively estimated, timing of information in $\hat{y}_t^c$ and y_t may differ.

4.2 Deterministically Breaking Linear Trend

$$y_t^x = a_1 + b_1 t \quad if\ t \ \le t_1 \tag{8}$$
$$y_t^x = a_2 + b_2 t \quad if\ t \ > t_1 \tag{9}$$

• Estimate a_i, b_i by OLS. Set $\hat{y}_t^c = y_t - a_{1OLS} - b_{1OLS}t,\ t \le t_1$; $\hat{y}_t^c = y_t - a_{2OLS} - b_{2OLS}t,\ t > t_1$.

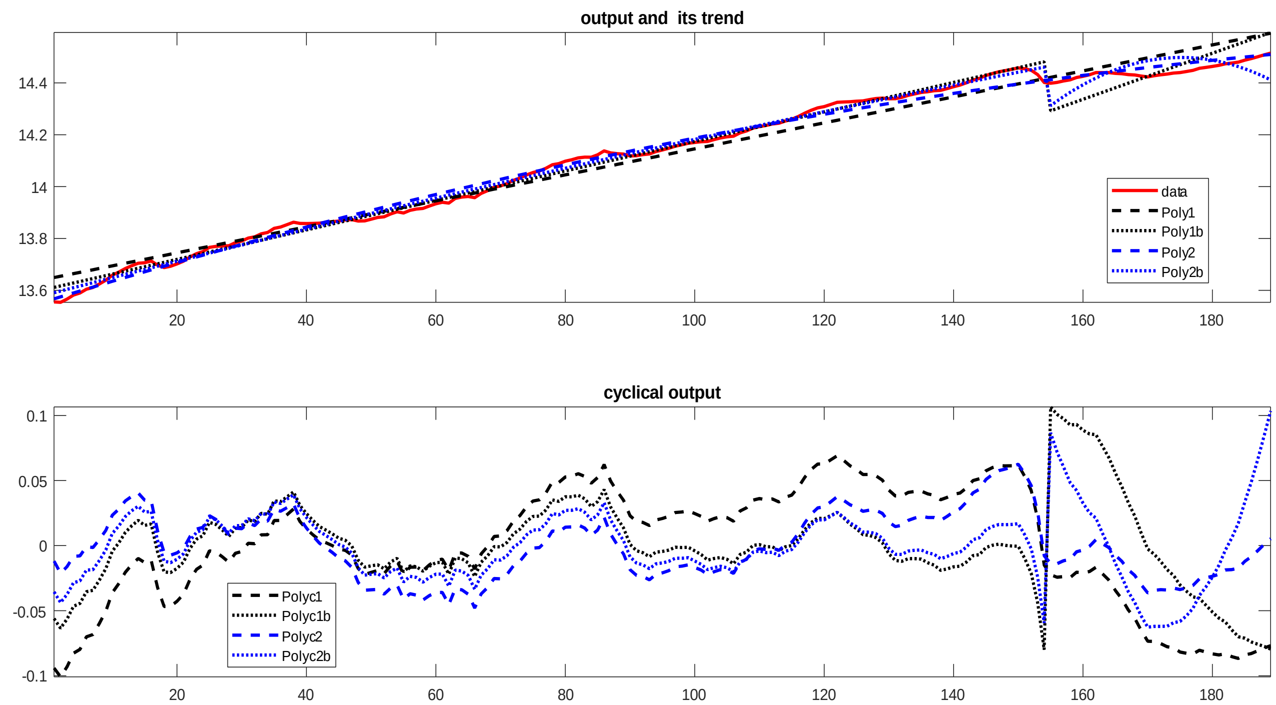
• What if t_1 unknown? Select $[t_a, t_b]$. Run OLS for every $t_1 \in [t_a, t_b]$. Use F-test to check $a_1 = a_2, b_1 = b_2$ each t_1. Break point is the t_1 producing max $F(t_1) \rightarrow$ QML statistics (see Stock and Watson, 2002).

• bf Can still perfectly predict y_{t+h} after the break.

• To avoid last problem, use Markov switching trend specification (Hamilton, 1989).

$$
\begin{aligned}
y_t^x &= a_1 + a_2 s_t + (b_1 + b_2 s_t)t && (10)\\
p &= P(s_t = 1 | s_{t-1} = 1) && (11)\\
q &= P(s_t = 0 | s_{t-1} = 0) && (12)
\end{aligned}
$$

• Estimation more complicated. It requires ML and particular filtering algorithm to compute the likelihood.



Trend and cycle: polynomial and break trends

- No slope break after 2008? Only intercept break?

4.3 Differencing

- Trend process:

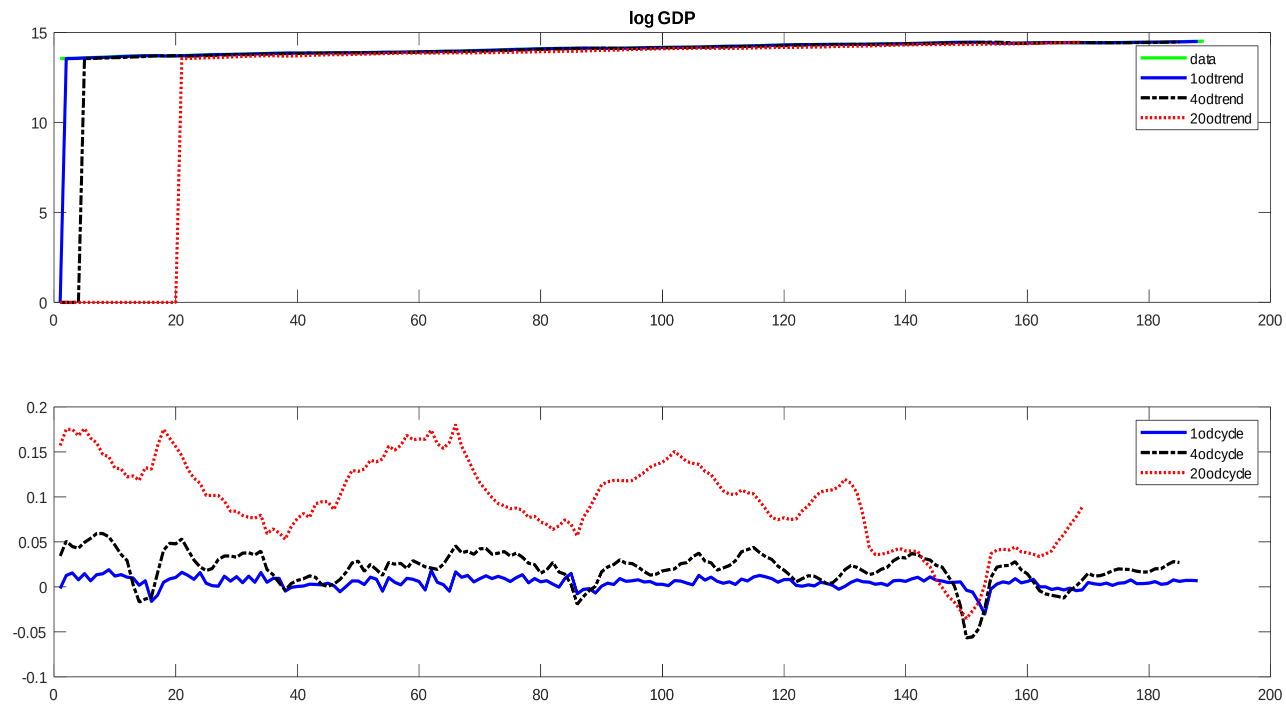
$$y_t^x = y_{t-d}^x \quad d = 1, 4, 8, 24, ... \tag{13}$$

- Cycle estimate: $y_t^c = \Delta^d y_t$,

- How do you chose d? Long or short differencing?

- For d=1 (period-on-period growth rates) cycles very volatile. Difficult to build models to explain them.

- If $d > 1$ artificial MA(d-1) components in y_t^c (artifically high serial correlation).



Trend and cycle: differencing

- Long differencing leaves a downward trend in filtered output data

4.4 Hamilton: local projection technique

• Trend defined as the medium term forecastable path of the variable.

• No parametric assumptions: trend could be deterministic, stochastic, smooth, jagged; but uncorrelated with cycle.

• Decomposition implemented via the regression:

$$y_{t+h} = \kappa_{1h}\Delta y_t + \kappa_{2h}\Delta y_{t-1} + \ldots + \kappa_{dh}\Delta^{d-1} y_t + w_{t+h} \qquad (14)$$

where, for quarterly data, $h = 8$ and $d = 4$. In practice, run OLS on:

$$y_{t+h} = \alpha_{1h} y_t + \alpha_{2h} y_{t-1} + \alpha_{3h} y_{t-2} + \ldots + \alpha_{dh} y_{t-d+1} + w_{t+h} \qquad (15)$$

• $\hat{w}_{t+h} = y_{t+t} - \sum_{j=0}^{d-1} \hat{a}_{j+1} y_{t-j}$ is an estimate of y^c_{t+h}. It is a function of (h, d).

Properties:

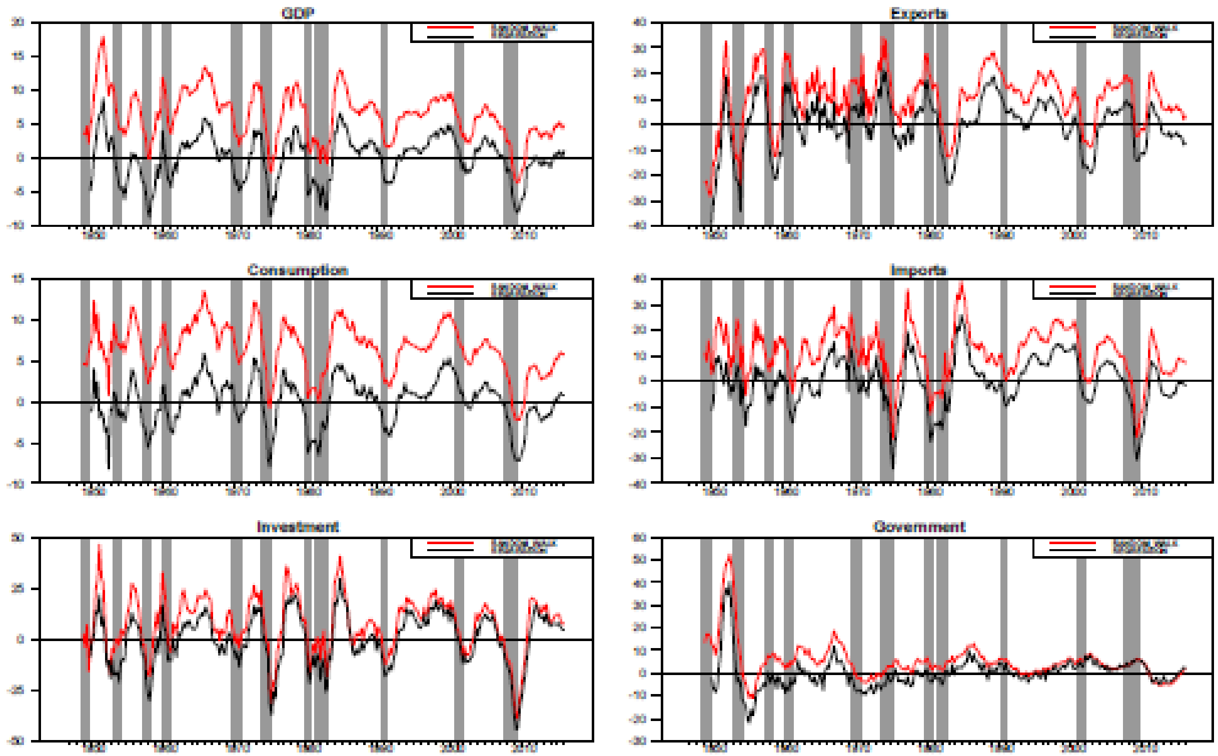
• $\hat{w}_{t+h}$ is model free (no parametric specification for trend and cycle). Robust to misspecification of the properties of the trend process.

• $\hat{w}_{t+h}$ is stationary if y_t has up to d unit roots.

• Can be used with seasonally non-adjusted data; with data of any frequency (quarter, month, week: adjust d and h).

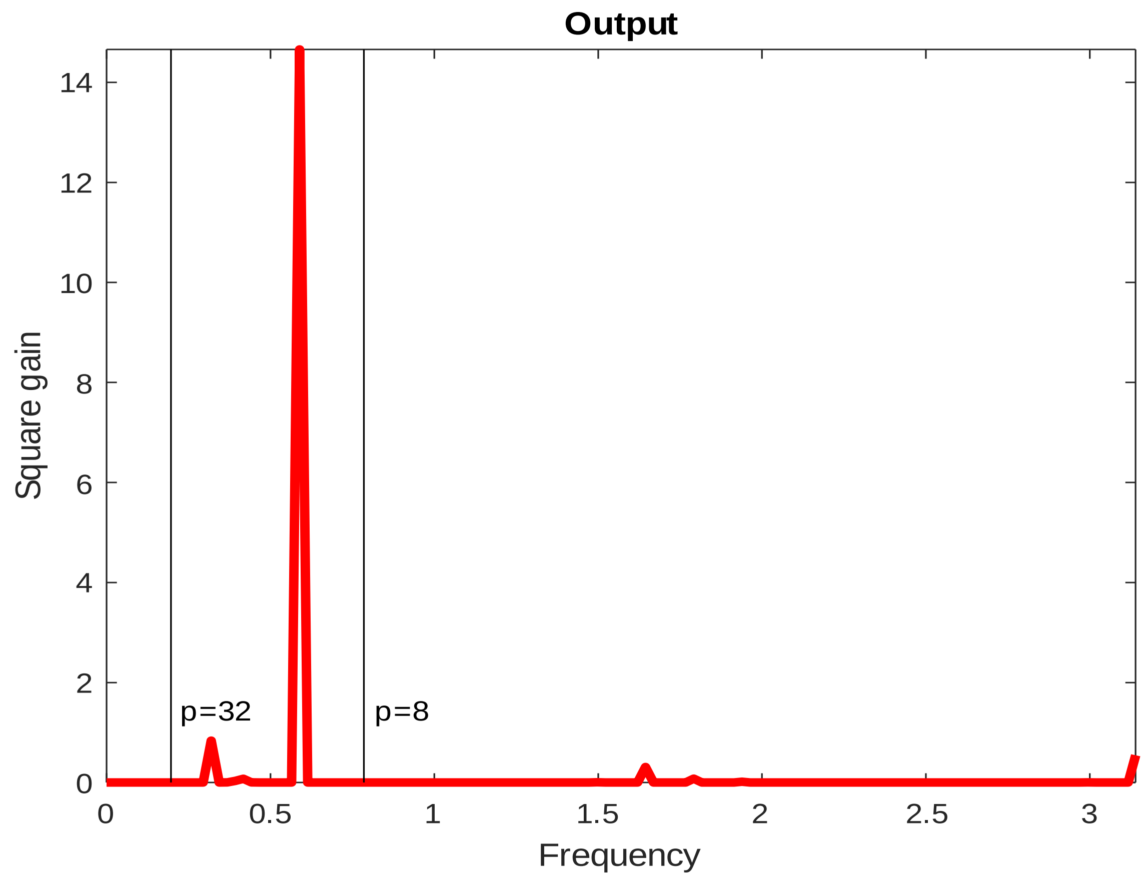
• Hamilton (2018): time series properties of $\hat{w}_{t+h}$ similar to those obtained with a h-differencing operator.

Figure 6. Results of applying regression (black) and 8-quarter-change (red) filters to 100 times the log of components of U.S. national income and product accounts.



Questions

- What are the properties of the Hamilton trend? (Schuler, 2019)
- Are any cycle features dependent on (h, d)?
- Do estimated cycles have standard durations, amplitudes, and concordances?
- Does it produce a ‘’business cycle ‘’ filter?



• Hamilton approach does not produce a standard business cycle filter. It has a squared gain function a large peak at 10.66 quarters.

4.5 Unobservable component methods

• Based on a state space model representation.

• Use a parametric representation for the trend and cycle, e.g. trend is a random walk, cycle is an AR(2).

• Can be enriched with observable regressors, additional features for the error process, see e.g. Stock and Watson, 2016.

• Can be made multivariate, see e.g. Astrudillo and Roberts, 2016; Grant and Chan, 2017a, 2017b. Can restrict latent components to be common, see Del Negro et al. 2019.

Two typical (univariate) setups:

$$
\begin{aligned}
y_t &= \tau_t + c_t + u_t \\
\tau_t &= \tau_{t-1} + \mu + \eta_t \\
c_t &= \theta_1 c_{t-1} + \theta_2 c_{t-2} + \epsilon_t
\end{aligned}
\tag{16}
$$

Estimate $(\theta_1, \theta_2, \mu, \sigma_u^2, \sigma_\eta^2, \sigma_\epsilon^2), \rho = corr(\eta_t, \epsilon_t)$ by KF-ML approach or by MCMC with flat prior.

$$
\begin{aligned}
y_t &= \tau_t + c_t + u_t \\
\tau_t &= \tau_{t-1} + \mu + \eta_t \\
c_{1t} &= \theta((\cos\omega)c_{1t-1} + (\sin\omega)c_{2t}) + \epsilon_{1t} \\
c_{2t} &= \theta(-(\sin\omega)c_{1t-1} + (\cos\omega)c_{2t}) + \epsilon_{2t}
\end{aligned}
\tag{17}
$$

$c_t = [c_{1t}, c_{2t}]$. Fix $0 < \omega < \pi$, estimate $(\theta, \mu, \sigma_u^2, \sigma_\eta^2, \sigma_{\epsilon_1}^2, \sigma_{\epsilon_1}^2)$ by ML, see Runstler and Vlekke, 2018. ω regulates the cyclical frequency of interest.

- If KF-ML estimation is used, $\rho = 0$ must be imposed.

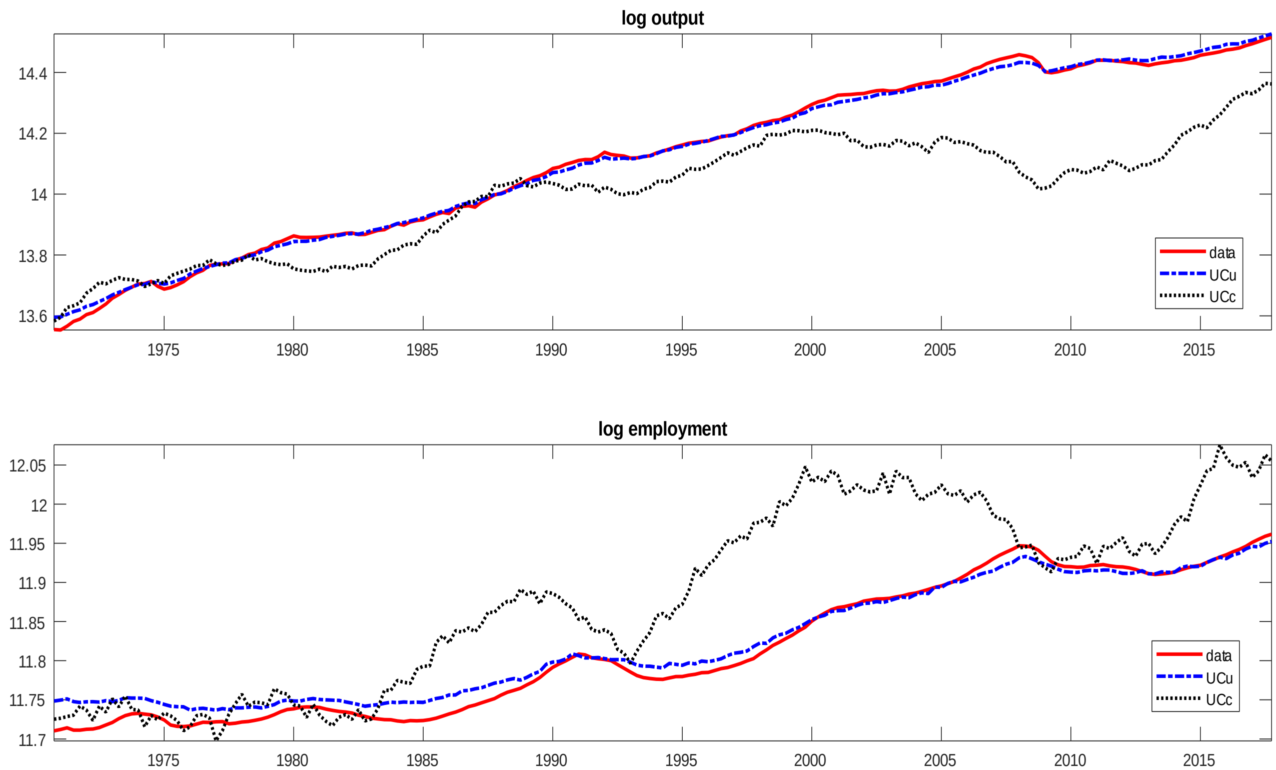
- Can also use a flexible local linear trend specification (see later)

- Can pick up more than one ω for the cycles in (17).

- Often omit u_t (measurement error).

- Can allow breaks in the trend $(\mu_1(t < t_0), \mu_2(t \geq t_0))$, Markov switching in μ, rare events (jumps in σ_η), and stochastic volatility in σ^2_ϵ.

- Flexible setup.



Data and Trends, UC $\rho = 0$, and $\rho \neq 0$

- Contemporaneous correlation for $\rho = 0$ and $\rho \neq 0$ is low (0.19579)

- AR(1) of output cycles: 0.96784 ; 0.99363

- Variabilities output cycles: 0.00015967; 0.015188.

- Quite a lot of differences! Which one to choose? Usually eyeball-econometrics!

Multivariate decompositions

$$\begin{aligned}
y_t &= \tau_t + c_t + u_t \\
\tau_t &= \tau_{t-1} + \mu_t + \eta_t \\
\mu_t &= \mu_{t-1} + \nu_t \\
c_{it} &= \sum_{j_i} \rho_j c_{it-j} + \epsilon_{it}, \quad \text{i=1,2,...N}
\end{aligned} \tag{18}$$

y_t is $N \times 1$, τ_t is a scalar. Common (stochastic) local-linear trend, series specific specification for the cycle.

$$\begin{aligned}
y_t &= \tau_t + c_t + u_t \\
\tau_{it} &= \tau_{it-1} + \mu_{it} + \eta_{it}, \quad \text{i=1,2,...N} \\
c_t &= \sum_{j} \rho_j c_{t-j} + \epsilon_t
\end{aligned} \tag{19}$$

where y_t is $N \times 1$, c_t is a scalar. Common cyclical components but there are variable-specific local linear trends.

4.6 Beveridge-Nelson decomposition

- Trend defined as the long run forecastable component of y_t.

- Compare with Hamilton's medium term definition.

- Some components of y_t must have a unit root (otherwise, long run forecastable component is the mean of y_t).

- Features of estimated y_t^c depend on **the lag length of the estimating model and sample size**.

Univariate setup

$$\Delta y_t - \bar{y} = A(\ell)\Delta(y_{t-1} - \bar{y}) + e_t$$

$e_t \sim iid(0, \Sigma_e)$, $\bar{y}$ is the mean of y_t, and all the roots of $det(A(\ell))$ are less than one in absolute value.

- MA: $(\Delta y_t - \bar{y}) \equiv \Delta y_t^* = D(\ell)e_t$, where $D(\ell) = (1 - A(\ell))^{-1}, D_0 = I$.

- If $D(1) = D_0 + D_1 + \ldots + D_m \neq 0$, can rewrite MA as:

$$\Delta y_t^* = D(1)e_t + (1-\ell)D^\dagger(\ell)e_t \tag{20}$$

where $D^\dagger(\ell) = \frac{D(\ell)-D(1)}{1-\ell}$. Cumulating:

$$y_t = (\bar{y} + D(1)\sum_{j=1}^{t} e_j) + D^\dagger(\ell)e_t \equiv y_t^x + y_t^c \tag{21}$$

Features

• Trend and cycle perfectly correlated, $\rho = 1$ (e_t drive both)

• Trend component is a random walk with drift: $(\bar{y} + D(1)\sum_{j=1}^{t} e_j) \equiv \mu + y_{t-1} + e_t$.

• Quality of the decomposition depends on the estimate of $\bar{y}$. Better to demean the data if the sample is short.

• Can be cast into a state space framework (see Morley, et al., 2003).

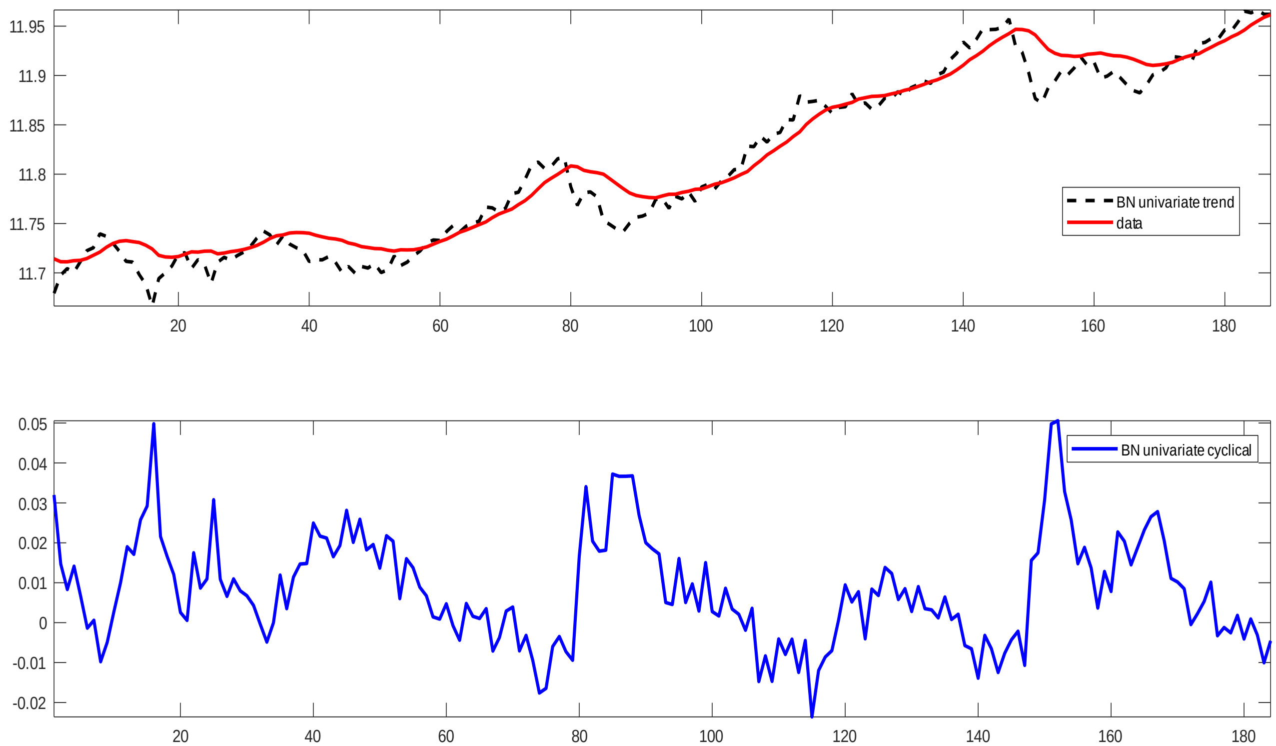
Implementation

- Choose the lag lenght of the AR model optimally (LR test, AIC, BIC, etc.).

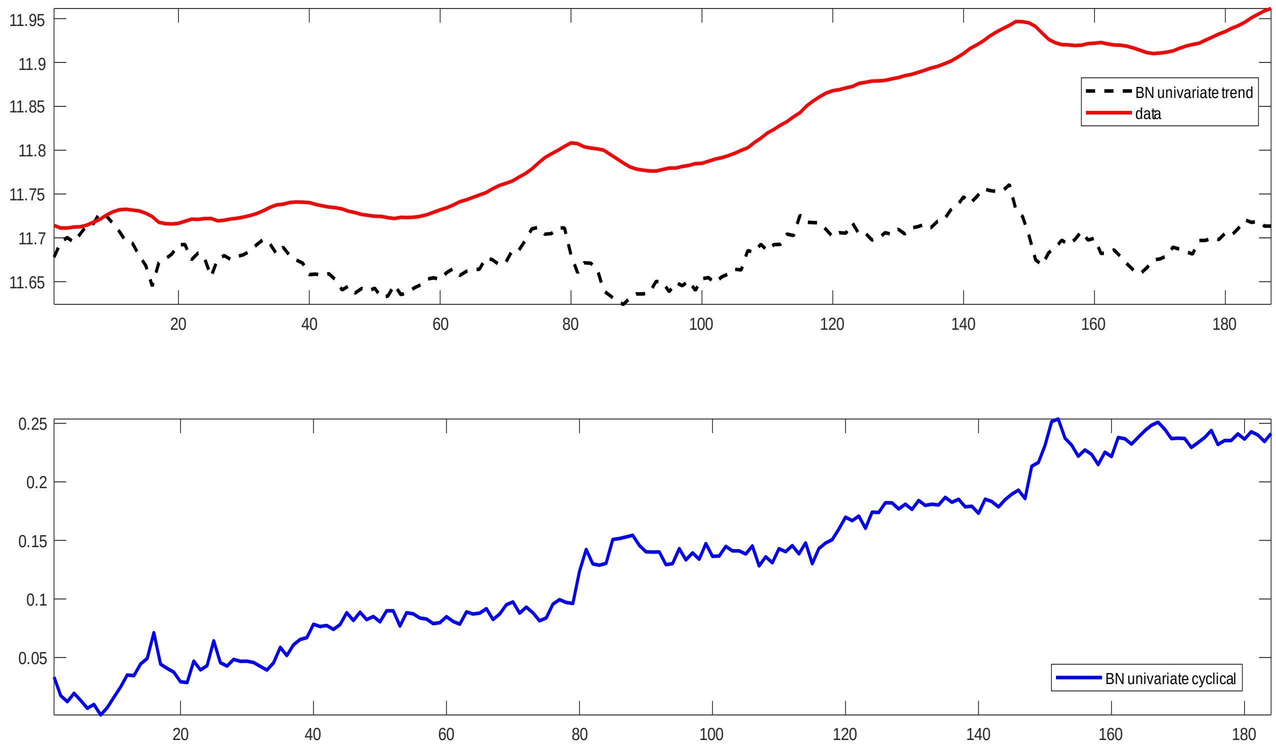
- Compute MA coefficients D_j.

- Construct estimate trend from estimates of $\bar{y}$, D_j and e_t.

- Compute cycle estimate as residual $y_t^c = y_t - y_t^x$.



Log Employment BN; demeaned Δy_t



Log Employment BN; estimated long run mean: $c/(1 - A(\ell))$

Multivariate Setup

• Let $y_t = [\Delta y_{1t}, y_{2t}]$ $(m \times 1)$; where y_{1t} are I(1); and y_{2t} are I(0);

• Suppose $y_t = \bar{y} + D(\ell)e_t$, where $e_t \sim iid(0, \Sigma_e)$ and $D_0 = I$, the roots of $det(D(\ell))$ are equal or greater than one; and that $D_1(1) \neq 0$, where $D_1(\ell)$ is $m_1 \times 1$ (first m_1 rows of $D(\ell)$). Then

$$\begin{pmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + \begin{pmatrix} D_1(1) \\ 0 \end{pmatrix} e_t + \begin{pmatrix} (1-\ell)D_1^{\dagger}(\ell) \\ (1-\ell)D_2^{\dagger}(\ell) \end{pmatrix} \Delta e_t \qquad (22)$$

$D_1^{\dagger}(\ell) = \frac{D_1(\ell)-D_1(1)}{1-\ell}$ $D_2^{\dagger}(\ell) = \frac{D_2(\ell)}{1-\ell}$, $0 <$ rank$[D_1(1)] \leq m_1$ and $y_t^x = [\bar{y}_1 + D_1(1)\sum_s e_s, \bar{y}_2]'$ is the permanent component of y_t.

• Kambler et al., 2018: impose smoothness restrictions to avoid jagged y_t^x path (add a penalty to the VAR estimation)

Implementation

- Select I(0) and I(1) variables.
- Run a VAR on Δy_t, Compute MAR coefficients D_j.
- Construct (22) separately for Δy_{1t} and Δy_{2t}; cumulate the result.
- As before, preferable to demean the data prior to running the VAR.

Summary

• With Polynomial (polynomial break): the trend is the permanent deterministic component of the data.

• With Hamilton: the trend is medium term component of the data.

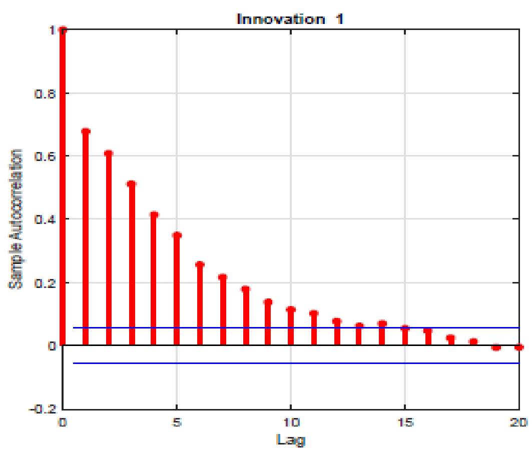
• With UC, BN: the trend is permanent stochastic component (unit root) of the data.

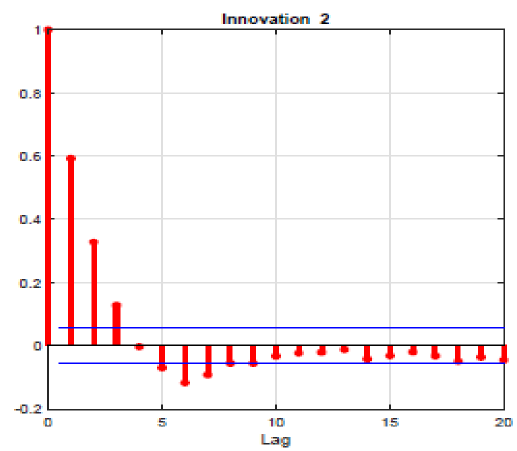
5 How do econometricians think about cycles?

• Stationary data can be summarized with autocovariance function (ACF):

$$ACF(\tau) = E_t(y_t - E_t y_t)(y_{t-\tau} - E_t y_{t-\tau}) \tag{23}$$

• ACF is symmetric but has correlated elements $(E(ACF(\tau), ACF(\tau')) \neq 0, \tau \neq \tau')$.





- Alternatively, stationary data can be summarized with spectral density:

$$\mathcal{S}(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{-\infty} ACF(\tau) e^{-i\omega\tau}, \tag{24}$$

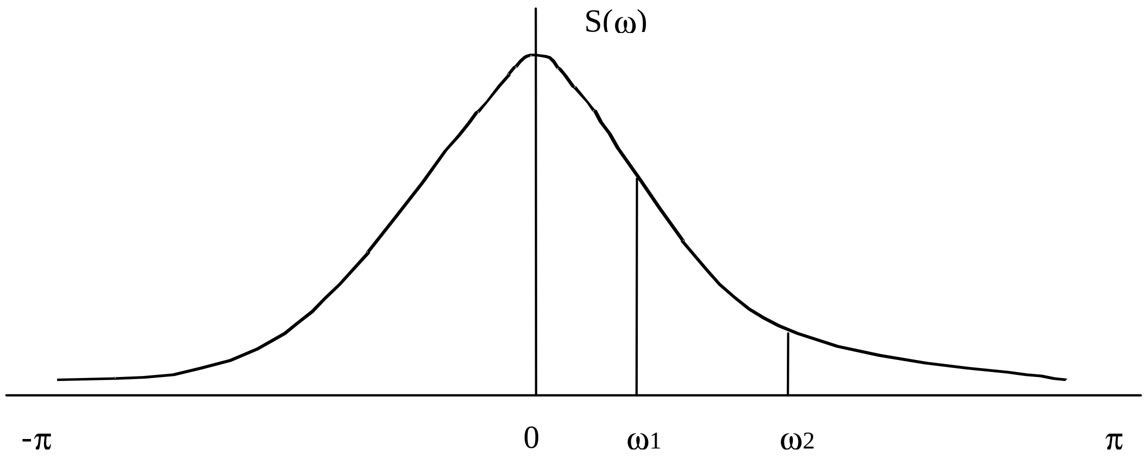
where $\omega \in [0, 2\pi]$, $i = (-1)^{0.5}$, $e^{-i\omega_\tau} = \cos(\omega_\tau) - i \sin(\omega_\tau)$.

- Spectral density changes coordinates relative to ACF.

- If $S(\omega)$ is evaluated at $\omega_\tau = \frac{2\pi\tau}{T}$, $\tau = 0, \ldots, T-1$ (Fourier frequencies):

i) $\mathcal{S}(\omega_\tau) = \mathcal{S}(\omega_{-\tau})$ (symmetry around $\omega_\tau = 0$).

ii) $E(\mathcal{S}(\omega_\tau)\mathcal{S}(\omega_{\tau'})) = 0$ (uncorrelatedness at two different ω_τ's)



• Area under the spectral density ($\sum_\omega S(\omega)$) is the **variance** of the process. Given orthogonality (by i. of previous slide), can perform variance decomposition by frequencies.

• $\mathcal{S}(\omega = 0) = \sum_{\tau=-\infty}^{-\infty} ACF(\tau)$ measures of the persistence of y_t.

• If y_t has a unit root, $S_y(\omega = 0) \uparrow \infty$ and for $x_t = \Delta y_t$, $S_x(\omega = 0) = 0$; i.e. **differencing a unit root process wipes out the variance of the series at $\omega = 0$** .

• One can associate a frequency $\omega_\mathcal{T}$ with the length of the fluctuations. The length of the fluctuations at Fourier frequency $\omega_\mathcal{T}$ is $p = \frac{2\pi}{\omega_\mathcal{T}}$.

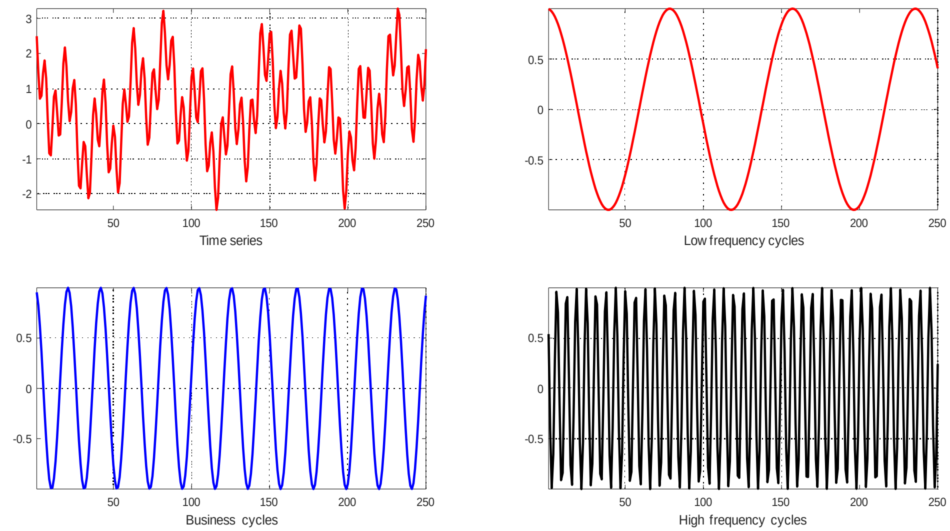
Example 5.1 $\omega_\mathcal{T} = \frac{\pi}{16} \rightarrow p = 32$; $\omega_\mathcal{T} = \frac{\pi}{2} \rightarrow p = 4$ *(quarters, years,etc.)*

• Orthogonal spectral density components:

(1) Low frequencies $\omega_\mathcal{T} \in (0, \omega_1)$: trends (Not just $S(0)$).

(2) Mid frequencies: $\omega_\mathcal{T} \in (\omega_1, \omega_2)$: cycles.

(3) High frequencies : $\omega_\mathcal{T} \in (\omega_2, \pi)$: seasonals, irregulars, noise.



• Low frequencies components associated with cycles featuring long periods of oscillations (time series moves infrequently from peaks to throughs).

• High frequencies components are associated with short cycles (time series move frequently from peaks to throughs).

Multivariate analysis

• The spectral density matrix of a stationary $N \times 1$ vector $\{y_t\}_{t=-\infty}^{\infty}$ is $\mathcal{S}(\omega) = \frac{1}{2\pi} \sum_\tau ACF(\tau) \exp(-i\omega\tau)$ where

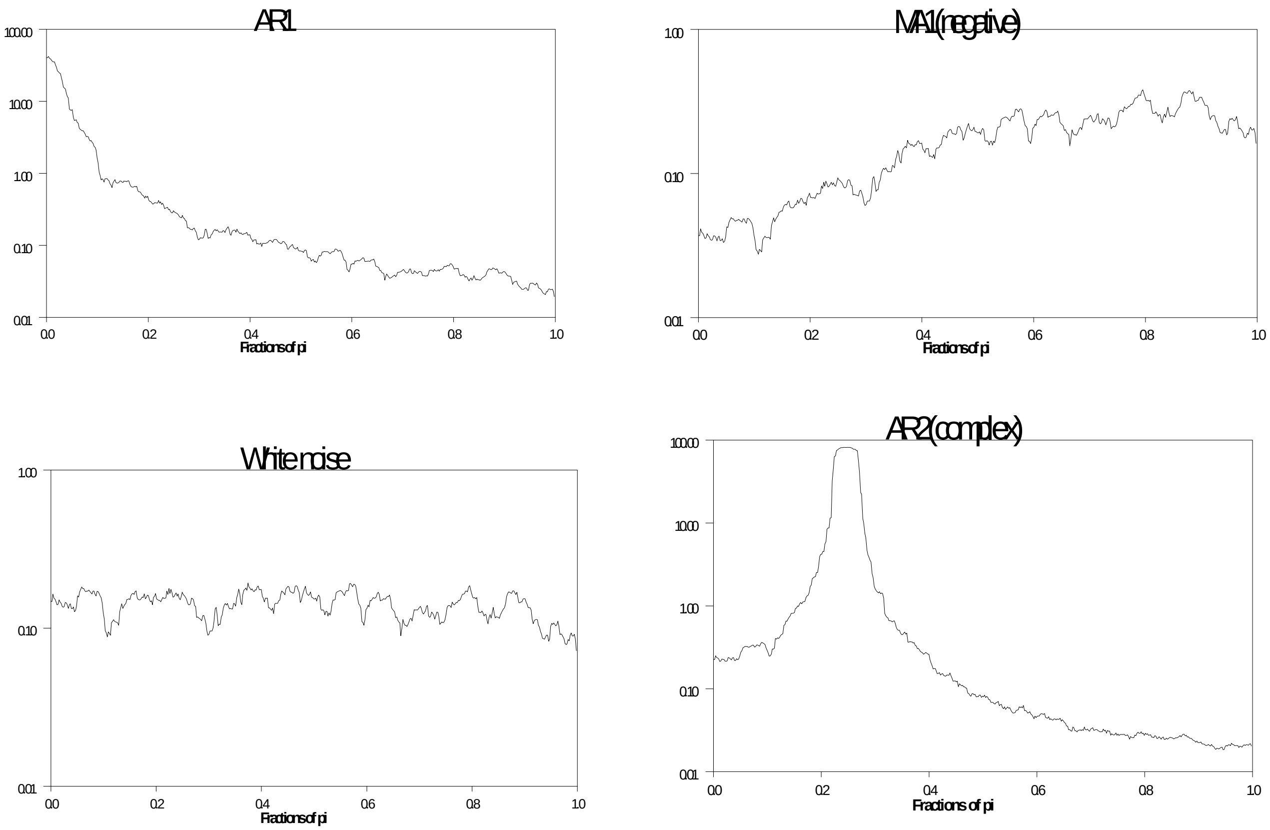
$$\mathcal{S}(\omega) = \left[\begin{array}{cccc} \mathcal{S}_{y_1y_1}(\omega) & \mathcal{S}_{y_1y_2}(\omega) & \ldots & \mathcal{S}_{y_1y_m}(\omega) \\ \mathcal{S}_{y_2y_1}(\omega) & \mathcal{S}_{y_2y_2}(\omega) & \ldots & \mathcal{S}_{y_2y_m}(\omega) \\ \ldots & \ldots & \ldots & \ldots \\ \mathcal{S}_{y_Ny_1}(\omega) & \mathcal{S}_{y_Ny_2}(\omega) & \ldots & \mathcal{S}_{y_Ny_N}(\omega) \end{array} \right]$$

• Diagonal of the spectral density matrix real; off-diagonal complex.

• The coherence between y_{it} and y_{jt} is $Co_{y_i,y_j}(\omega) = \frac{|\mathcal{S}_{y_i,y_j}(\omega)|}{(\mathcal{S}_{y_i,y_i}(\omega)\mathcal{S}_{y_j,y_j}(\omega))^{0.5}}$.

• It measures the strength of the association between y_{it}, y_{jt} at frequency ω. $\int Co(\omega) d\omega = \rho_{y_1,y_2}$: decomposition of correlation by frequency. $Co(\omega)$ is real since $|y| =$ real part of complex number y.

Examples of univariate spectral densities



Conclusions

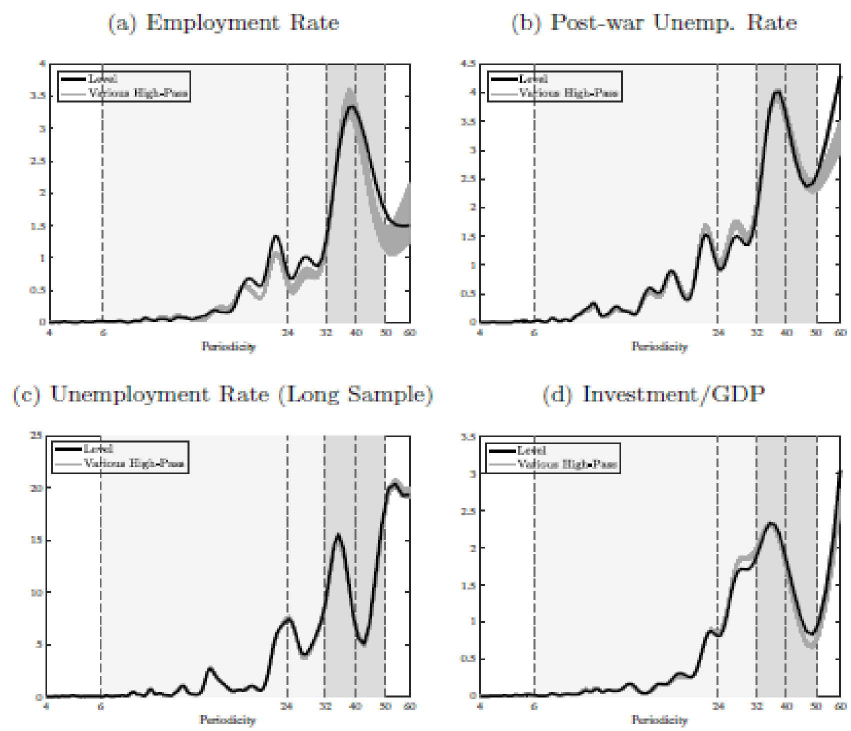
• Displaying variability and serial correlation (e.g. AR(1) or MA(1) dynamics) does not generate cycles for time series econometricians.

• Alternating sequence of irregularly sparse turning points does not necessarily imply cycles for time series econometricians

• To have econometrician cycles, there should be a large portion of the variance of the process in the business cycle frequencies (need at least a AR(2) with complex roots)

• We also need large coherence at business cycle frequencies to have y_{it} and y_{jt} moving together.

• Beaudry et al. (2019): labor market variables have econometrician cycles.



Note: horizontal axis reversed (to the left high frequencies, to the right low frequencies).

Table 1: Variance Decomposition US 1954 - 2017

	Short Term		Business Cycle		Medium Term		Long Term
	D1: 2-4Q	D2: 4-8Q	D3: 8-16Q	D4: 16-32Q	D5: 32-64Q	D6: 64-128Q	S6: >128Q
GDP Growth	32.7	22.8	19.0	12.6	6.1	2.5	4.3
Unemployment	1.3	3.8	9.6	18.5	25.4	20.3	21.1
Inflation	8.6	7.4	7.7	9.9	9.3	15.7	41.3
Federal Funds	1.5	2.8	5.6	10.2	10.5	12.8	56.6
3-Month Rate	1.3	2.4	4.8	8.9	10.1	13.1	59.3
10-Year Rate	0.6	1.3	2.5	4.0	7.5	15.5	68.5
Term Spread	5.5	9.7	17.1	26.7	19.1	5.4	16.5

- Lubik et al. (2019): Several variables have important medium term (larger than business cycle) variability.

Filters

- Spectral densities defined only for stationary series.

- Often interested in variability at certain frequencies (Why? Electrical engineers arguments?)

- Filters may render y_t stationary under certain assumptions.

- Filters eliminate variability at certain frequencies.

- Catch two birds with one stone!

• A filter is a linear transformation of a primitive stochastic process y_t.

$$y_t^f = \sum_{-J}^{J} \mathcal{B}_j y_{t-j} = \mathcal{B}(\ell) y_t \tag{25}$$

• The filter is symmetric if $\mathcal{B}_j = \mathcal{B}_{-j}$. Symmetric filters have the property that the timing of the cycles in y_t and y_t^f is the same (zero phase shift).

• If $\sum_{-J}^{J} \mathcal{B}_j = 0$ and y_t is non-stationary, y_t^f is stationary (filtering detrends/stationarize time series with unit roots).

• Two simple filters

1) $y_t^f = y_t - D_1 y_{t-1}$ (MA(1) filter).

2) $y_t^f = \sum_{j=-J}^{J} y_{t-j}$. The larger is J the smoother is y_t^f (the longer cycles you will extract)

• If S_y is the spectral density of y_t, and $y_t^f = \mathcal{B}(\ell) y_t$, then $S_{y^f} = \mathcal{B}(\omega)\mathcal{B}(-\omega) S_y$. When univariate $S_{y^f} = |\mathcal{B}(\omega)|^2 S_y$, where $|\mathcal{B}(\omega)|$ is the real part (modulus) of $\mathcal{B}(\omega)$.

Example 5.2 *Let e_t be a white noise. Its spectrum is $\mathcal{S}_e(\omega) = \frac{\sigma^2}{2\pi}$. Let $y_t = \mathcal{B}(\ell) e_t$ where $a(\ell) = \mathcal{B}_0 + \mathcal{B}_1 \ell + \mathcal{B}_2 \ell^2 + \ldots$. The spectrum of y_t is $\mathcal{S}_y(\omega) = |\mathcal{B}(e^{-i\omega})|^2 \mathcal{S}_e(\omega)$, where $|\mathcal{B}(e^{-i\omega})|^2 = \mathcal{B}(e^{-i\omega})\mathcal{B}(e^{i\omega})$.*

Terminology

• **The frequency response function** of the filter is $\mathcal{B}(\omega) = \mathcal{B}_0 + 2\sum_j \mathcal{B}_j \cos(\omega j)$ (i.e. set $\ell^j = e^{i\omega j}$); it measures the effect of a change in y_t on y_t^f at frequency ω (IRF in frequency domain).

• $|\mathcal{B}(\omega)|$ is **the gain (transfer) function**; it measures how much the **amplitude** of the fluctuations y_t^f changes relative to the amplitude of y_t at frequency ω.

• $|\mathcal{B}(\omega)|^2$ is the **squared gain function**; it measures how much the **variance** of y_t^f changes relative to the variance of y_t at frequency ω.

5.1 The Hodrick and Prescott (HP) Filter

• Trends are smooth (their variations are small; could be almost deterministic or stochastic). Constrained problem:

$$\min_{y_t^x}\{\sum_{t=1}^{T}(y_t - y_t^x)^2 + \lambda \sum_{t=2}^{T}((y_{t+1}^x - y_t^x) - (y_t^x - y_{t-1}^x))^2\} \tag{26}$$

If $\lambda = 0$, the solution is $y_t^x = y_t$. As $\lambda \uparrow, y_t^x$ becomes smoother. If $\lambda \to \infty$, y_t^x becomes deterministic (no variations).

• Typically value for quarterly data: $\lambda = 1600$. Ravn and Uhlig (2002): if $\lambda = 129000$ for monthly data and $\lambda = 6.25$ for annual data, HP filters picks cycles with similar periodicity for monthly, quarterly and annual data.

(Closed form) solution to the constrained optimization:

$$\hat{y}^x = Ay = (H'H + \lambda Q'Q)^{-1} Hy \tag{27}$$
$$\hat{y}^c = y - \hat{y}^x = (I - A)y \tag{28}$$

where $y = [y_T, \ldots, y_1]'$ is a $T \times 1$ vector, $y^x = [y^x_T, \ldots, y^x_1, y^x_0, y^x_{-1}]'$ is a $(T+2) \times 1$ vector, $H = [I,\ 0]$ where I is a $T \times T$ identity matrix and 0 a $T \times 2$ matrix of zeros and

$$Q = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & -0 & 1 & -2 & 1 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 \end{bmatrix}$$

- (27) is a "ridge" estimator.

- Bayesian interpretation: $\Delta^2 y^x_{t+1} = \epsilon_t$ is a prior with $\epsilon_t \sim N(0, \lambda * \sigma^2_c)$.

• Alternative (state space) setup:

$$
\begin{aligned}
y_t &= y_t^x + y_t^c \\
\Delta y_t^x &= \epsilon_t
\end{aligned}
\tag{29}
$$

where both ϵ_t and y_t^c are white noise, uncorrelated with y_0^x, y_{-1}^x. Two solutions (see literature on curve fitting, e.g. Wabha, 1980).

i) If $C_0^{-1} = var(y_0^x, y_{-1}^x)^{-1} \longrightarrow 0$, find a_t such that $\tilde{y}_t^x = a_t y_t$ by min $E(y_t^x - a_t y_t)^2$. Solution: $\tilde{y}^x = E(y^x y')E(yy')^{-1}y = \tilde{A}y$. If $\lambda = \frac{\sigma_c^2}{\sigma_\epsilon^2}$, then $\tilde{A} = A$.

ii) Use the Kalman smoother to solve the signal extraction problem (still assuming large C_0).

• In a state space framework $\lambda = 1600$ means that σ_C, the standard deviation of the cycle, is 40 times larger than σ_ϵ the standard deviation of the second difference of the trend.

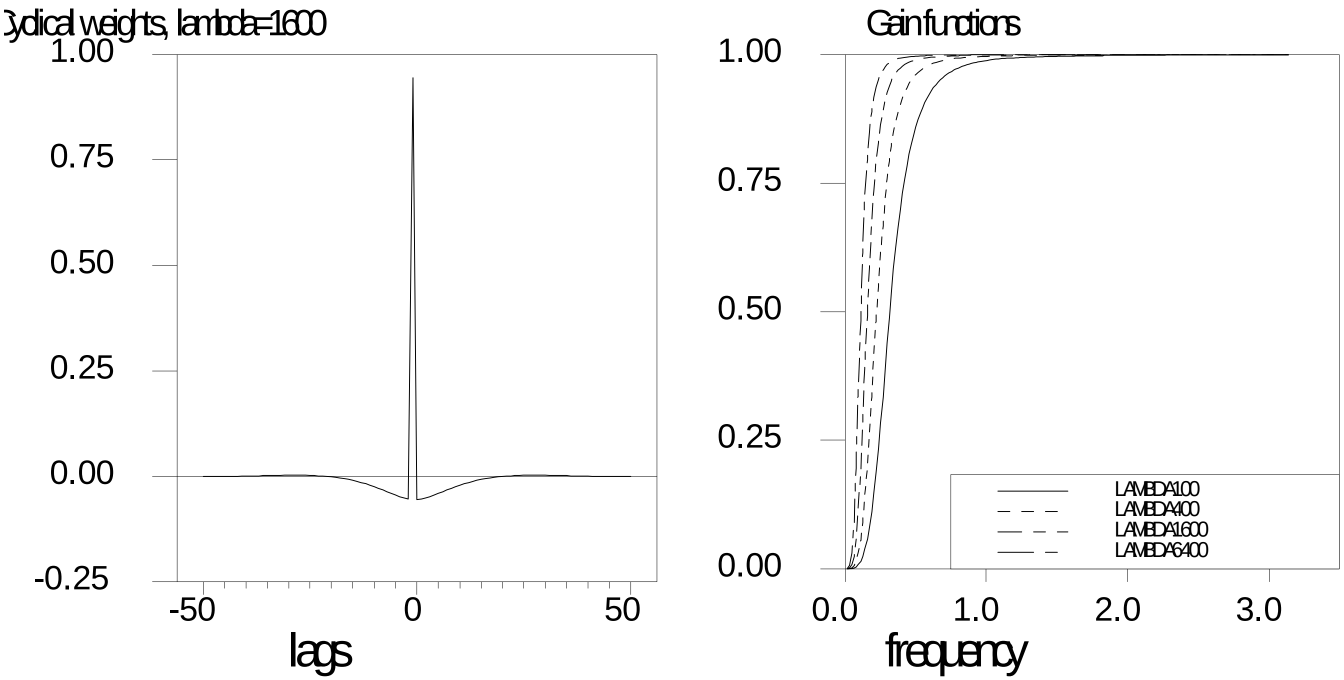
• HP solution is optimal when the cycle is a white noise.

• HP solution is time dependent (the cycle at t depends on how large is T). Beginning and end-of-sample problems.

• Premultiplying (27) by $(H'H + \lambda Q'Q)^{-1}$ and letting T grow to infinity one can show that $y_t^c = \mathcal{B}^c(\ell)y_t$, where

$$\mathcal{B}^c(\ell) \simeq \frac{(1-\ell)^2(1-\ell^{-1})^2}{\frac{1}{\lambda} + (1-\ell)^2(1-\ell^{-1})^2} \tag{30}$$

- When $\lambda = 1600$, $\mathcal{B}_j^c$ and the $|\mathcal{B}^c(\omega)|^2$ looks like in the picture below.



- Properties of HP filter:

(i) It eliminates linear and quadratic trends from y_t.

(ii) Stationarize y_t with up to 4 unit roots (King and Rebelo, 1993).

- What happens if y_t has less than 4 unit roots? Overdifferencing.

- **HP filter may create spurious autocorrelation in** y_t^f (Slutzky effect).

- Intuition: Suppose $y_t = e_t \sim iid(0, \sigma^2)$. Then

$$
\begin{aligned}
\Delta y_t &= e_t - e_{t-1} \;\; \text{correlation of order 1} \\
\Delta^2 y_t &= e_t - e_{t-1} - (e_{t-1} - e_{t-2}) \quad \text{correlation of order 2} \\
\Delta^3 y_t &= e_t - 3e_{t-1} + 3e_{t-2} - e_{t-3} \; \text{correlation of order 3} \\
\Delta^3 y_t &= e_t - 4e_{t-1} + 6e_{t-2} - 4e_{t-3} + e_{t-4}) \; \text{correlation of order 4}
\end{aligned}
$$

- Over-differencing a process y_t induces spurious serial correlation in the filtered series.

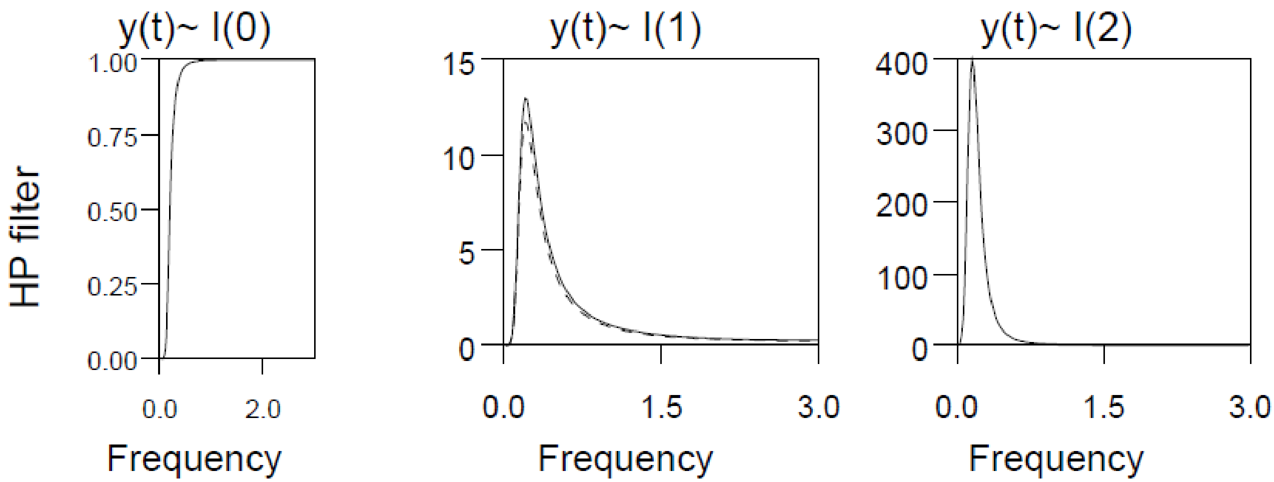
• **It can create spurious variability in the filtered data**.

• If y_t is stationary, the squared gain function is:

$$\mathcal{B}^c(\omega) \simeq \frac{16\sin^4(\frac{\omega}{2})}{\frac{1}{\lambda} + 16\sin^4(\frac{\omega}{2})} = \frac{4(1-\cos(\omega))^2}{\frac{1}{\lambda} + 4(1-\cos(\omega))^2}$$

• $\mathcal{B}^c(\omega)$ damps fluctuations with periodicity $\geq$ 24-32 quarters per cycle, it passes short cycles without changes.

• If y_t is I(1) $\mathcal{B}^c(\ell)$ is a combination of two filters: $(1-\ell)$ stationarizes y_t; $\frac{\mathcal{B}^c(\ell)}{1-\ell}$ filters Δy_t. When $\lambda = 1600$ the gain function of $\frac{\mathcal{B}^c(\ell)}{1-\ell}$ is $\simeq$ $2(1-\cos(\omega))B(\omega)$, which peaks at $\omega^* = \arccos[1-(\frac{0.75}{1600})^{0.5}] \simeq 30$ periods:



- If y_t is I(1) HP damps long and short run growth cycles and amplifies business cycle frequencies (e.g. the variance of the cycles with average duration of 7.6 years is multiplied by 13).

- Problem even larger if y_t is I(2).

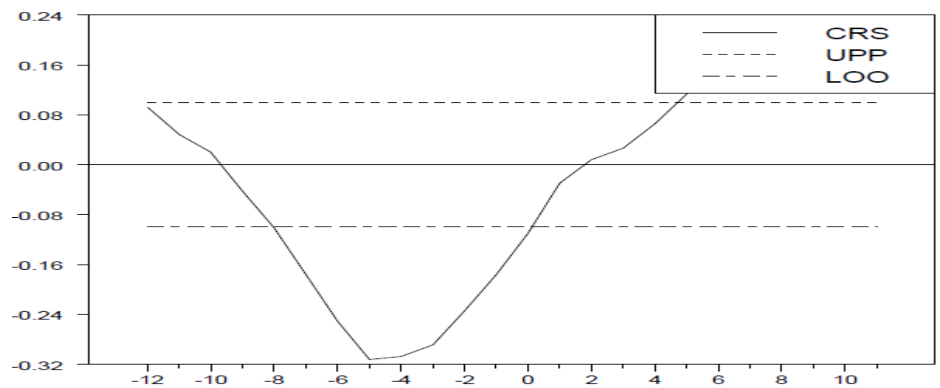
- Same problem if y_t nearly integrated ($\rho_y = 0.95$)? (see dotted line)

- What is the intuition for the increased variability?

- Suppose $\Delta y_t = e_t \sim iid(0, \sigma^2)$. Then

$$\begin{aligned} var(\Delta^2 y_t) &= var(e_t - e_{t-1}) = var(e_t) + var(e_{t-1}) = 2\sigma^2 \\ var(\Delta^3 y_t) &= var(e_t - 2e_{t-1} + e_{t-2}) = 4\sigma^2 \end{aligned}$$

etc.. So the filter $\frac{\mathcal{B}(\ell)}{1-\ell}$ can augment the variability of Δy_t.



- **It can produce spurious comovements among series.**

Example 5.3 *y_{1t} and y_{2t} are two uncorrelated random walks. Pass them through a HP filter. The figure plots the cross correlation function of y_{1t}^c, y_{2t}^c and a 95 percent asymptotic tunnel for the hypothesis of no correlation.*

- What is the intuition for this result?

The two filtered series have similar spectrum. Therefore, it is possible that they go up and down together (Note: this does not happen all the times).

- **Conclusions: The HP filter has the potential to generate spurious variability, spurious serial and cross variable correlations.**

Other properties of the HP filter:

iii) It leaves high frequency variability unchanged (high pass filter).

iv) HP cyclical component predicts the future. Alternative to (30):

$$y_t^c = \frac{\lambda(1-\ell)^4}{1+\lambda(1-\ell)^2(1-\ell^{-1})^2} y_{t+2} \tag{31}$$

v) $\lambda = 1600$ inconsistent with KF estimates of $\sigma_c^2, \sigma_\epsilon^2$ and UC setups.

Table 1. Maximum likelihood estimates of parameters of state-space formalization of the HP filter for assorted quarterly macroeconomic series.

	σ^2_c	σ^2_v	λ
GDP	0.115	0.468	0.245
Consumption	0.163	0.174	0.940
Investment	4.187	12.196	0.343
Exports	5.818	3.341	1.741
Imports	4.423	4.769	0.927
Government spending	0.221	1.160	0.191
Employment	0.006	0.250	0.023
Unemployment rate	0.014	0.092	0.152
GDP Deflator	0.018	0.081	0.216
S&P 500	21.284	15.186	1.402
10-year Treasury yield	0.135	0.054	2.486
Fed Funds Rate	0.633	0.116	5.458
Real Rate	0.875	0.091	9.596

vi) Two-sided filter (can not use y_t^c as a VAR variable!).

vii) Cross county comparisons difficult because cycles may have different length. Marcet-Ravn (2000) solve

$$\min_{y_t^x} \quad \sum_{t=1}^{T}(y_t - y_t^x)^2 \tag{32}$$

$$\mathcal{V} \geq \frac{\sum_{t=1}^{T-2}(y_{t+1}^x - 2y_t^x + y_{t-1}^x)^2}{\sum_{t=1}^{T}(y_t - y_t^x)^2} \tag{33}$$

where $\mathcal{V} \geq 0$ is a constant to be chosen by the researcher, $\mathcal{V}$ measures the relative variability of the acceleration in the trend and the cycle, and may be country specific.

Example 5.4 *200 data points from a stationary RBC model with utility* $U(c_t, c_{t-1}, N_t) = \frac{c_t^{1-\varphi}}{1-\varphi} + log(1 - N_t)$ *assuming* $\beta = 0.99, \varphi_c = 2.0, \delta = 0.025, \eta = 0.64$, *steady state hours equal to 0.3,* $\rho_\zeta = 0.9, \rho_g = 0.8, \sigma_\zeta = 0.0066, \sigma_g = 0.0146$. *Table reports average unconditional moments across 100 simulations, before and after HP filtering.*

Simulated statistics

	Raw			*HP filtered*		
	K	*W*	*LP*	*K*	*W*	*LP*
cross (GDP_t, x_t)	*0.49*	*0.65*	*0.09*	*0.84*	*0.95*	*-0.20*
cross (GDP_{t+1}, x_t)	*0.43*	*0.57*	*0.05*	*0.60*	*0.67*	*-0.38*
St. Dev	*1.00*	*1.25*	*1.12*	*1.50*	*0.87*	*0.50*

5.2 One sided HP filter

• The HP-filter is two-sided and thus not very useful for real analysis and forecasting. In addition, by construction, y_t^c artificially predicts the future.

• There is a version of the HP filter which is one-sided and does not feature future predictability.

• The trend and the cycle can be estimated with standard Kalman filter/ EM algorithm iterations, MCMC, or by serial implementation.

- The model is:

$$y_t = y_t^x + y_t^c \tag{34}$$

$$y_t^x = 2y_{t-1}^x - y_{t-2}^x + \epsilon_t \tag{35}$$

where ϵ_t, y_t^c are white noise sequences.

- State space representation (see Stock and Watson, 1999):

1. State Equation

$$\left[\begin{array}{c} y^x_{t|t} \\ y^x_{t-1|t} \end{array}\right] = \left[\begin{array}{cc} 2 & -1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} y^x_{t-1|t-1} \\ y^x_{t-2|t-1} \end{array}\right] + \left[\begin{array}{c} \epsilon_t \\ 0 \end{array}\right] \tag{36}$$

2. Observation Equation

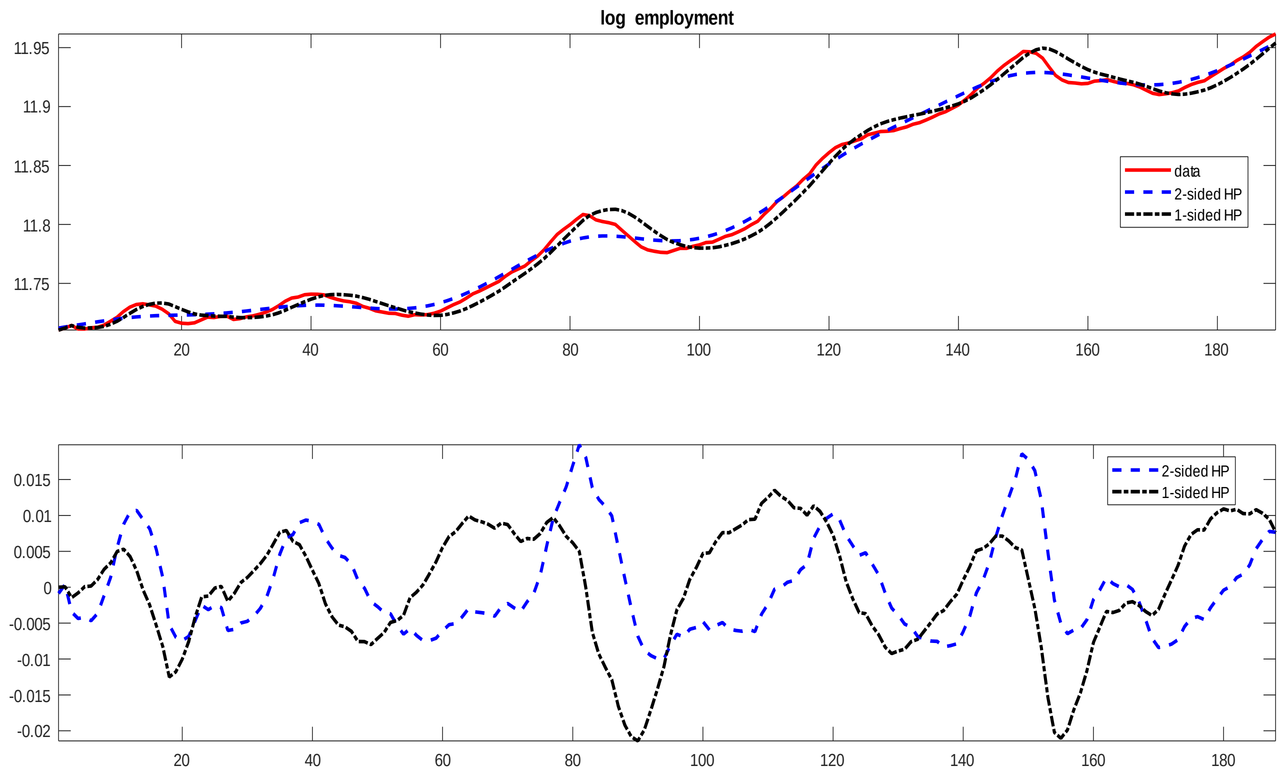
$$y_t = \left[\begin{array}{cc} 1 & 0 \end{array}\right] \left[\begin{array}{c} y^x_{t|t} \\ y^x_{t-1|t} \end{array}\right] + \left[\begin{array}{c} y^c_t \\ 0 \end{array}\right] \tag{37}$$

- Can restrict $\lambda = \frac{\sigma^2_c}{\sigma^2_\epsilon}$ with a prior, e.g. $\lambda \sim N(1600, 10)$.

- Serial implementation (Meyer-Gohde, 2010).

- Much faster than KF; gives almost identical results.

- $\{y_t^x\}_{t=1}^T$ is obtained calculating the standard HP filtered trend using data up to t and equating y_t^x with trend value for period t (i.e. compute T two-sided HP filters trends).



Log Employment: one and two sided HP

5.3 L1-HP filter

- Standard problem:

$$\min_{y_t^x}\{\sum_{t=1}^{T}(y_t - y_t^x)^2 + \lambda \sum_{t=1}^{T}((y_{t+1}^x - y_t^x) - (y_t^x - y_{t-1}^x))^2\} \tag{38}$$

- L1 problem (Kim et al.,2009):

$$\min_{y_t^x}\{\sum_{t=1}^{T}(y_t - y_t^x)^2 + \lambda \sum_{t=1}^{T}|(y_{t+1}^x - y_t^x) - (y_t^x - y_{t-1}^x)|\} \tag{39}$$

- Same features as standard HP.

- Non-linear filter.

- Gives rise to piecewise linear segments:

$$y_t^x = a_k + b_k t, \quad t_k \le t \le t_{k+1}, \quad k = 1, \ldots, p-1 \tag{40}$$

and

$$a_k + b_k t_{k+1} = a_{k+1} + b_{k+1} t_{k+1} \quad k = 1, \ldots, p-1 \tag{41}$$

- p is the number of break points where the estimated trend changes slope.

- The number of break points in y_t^x typically decreases as λ increases.

- Used in (business) finance to signal "changes in market trends".

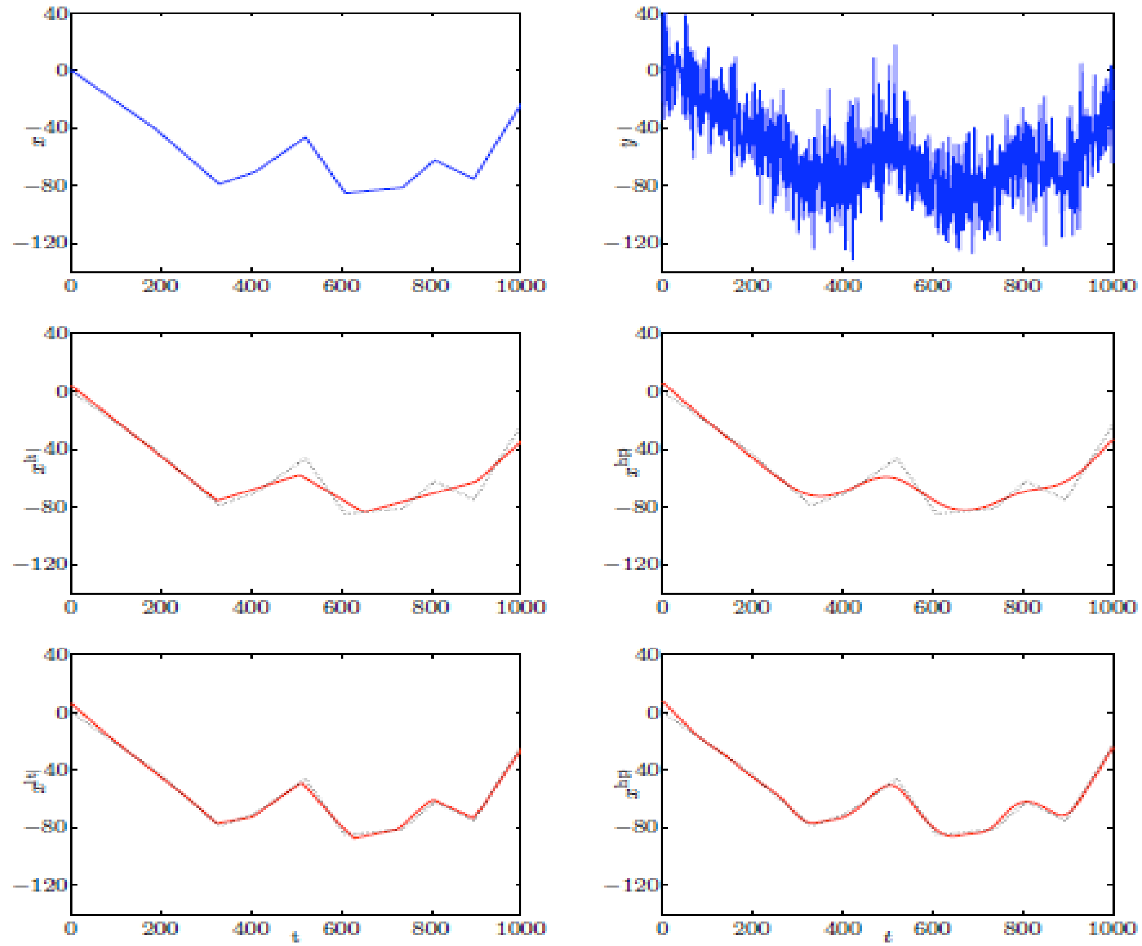


Fig. 1 *Trend estimation on synthetic data. Top left: The true trend x_t. Top right: Observed time series data y_t. Middle left: ℓ_1 trend estimate $x^{\rm lt}$ with four total kinks ($\lambda = 35000$). Middle right: H-P trend estimate $x^{\rm hp}$ with same fitting error. Bottom left: $x^{\rm lt}$ with seven total kinks ($\lambda = 5000$). Bottom right: H-P trend estimate $x^{\rm hp}$ with same fitting error.*

5.4 Exponential smoothing filter

- Similair properties and feature to HP filter.

- The ES trend solves the problem

$$\min_{y_t^x}\{\sum_{t=1}^{T}(y_t - y_t^x)^2 + \lambda \sum_{t=2}^{T}(y_t^x - y_{t-1}^x)^2\} \tag{42}$$

It penalizes the growth of the trend rather than its accelleration.

- $\mathcal{B}^c = \frac{(1-\ell)(1-\ell^{-1})}{\lambda^{-1}+(1-\ell)(1-\ell^{-1})}$

- Distortions of ES are generally larger than with HP

5.5 Other MA filters.

$$y_t^f = \sum_{-J}^{J} \mathcal{B}_j y_{t-j} = \mathcal{B}(\ell) y_t \tag{43}$$

• Symmetric MA filters ($\mathcal{B}_j = \mathcal{B}_{-j}$) with $\lim_{J \to \infty} \sum_{-J}^{J} \mathcal{B}_j = 0$ preferred because they maintain lead/lag relationships and eliminate unit roots.

• HP is a two-sided, symmetric, truncated MA filter. Other filters?

• Symmetric (truncated) MA filter: $\mathcal{B}_j = \frac{1}{2J+1}$, $0 \leq j \leq |J|$ and $\mathcal{B}_j = 0, j > |J|$.

• Tent filter: $\mathcal{B}_j = 1 - |j|$ for $0 < j < J$.

• Gaussian filter: $\mathcal{B}_j = \frac{1}{2\pi} e^{-j^2/2}$ for $-\infty < j < \infty$

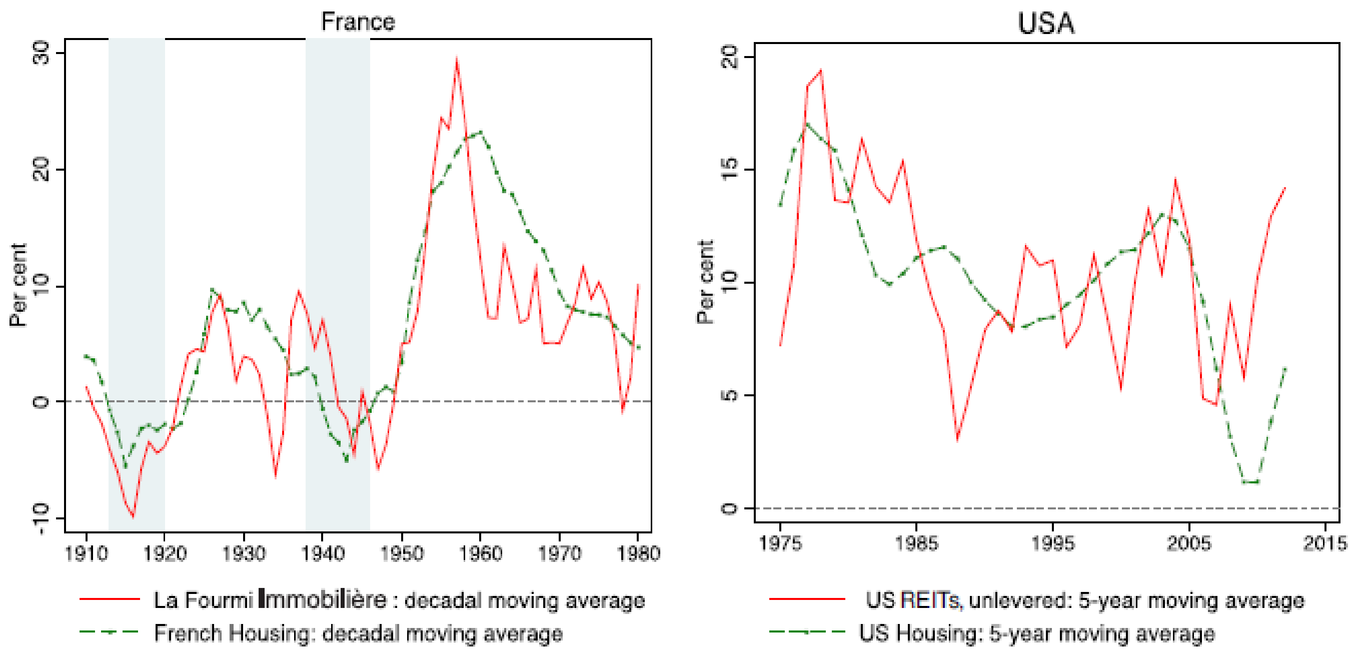
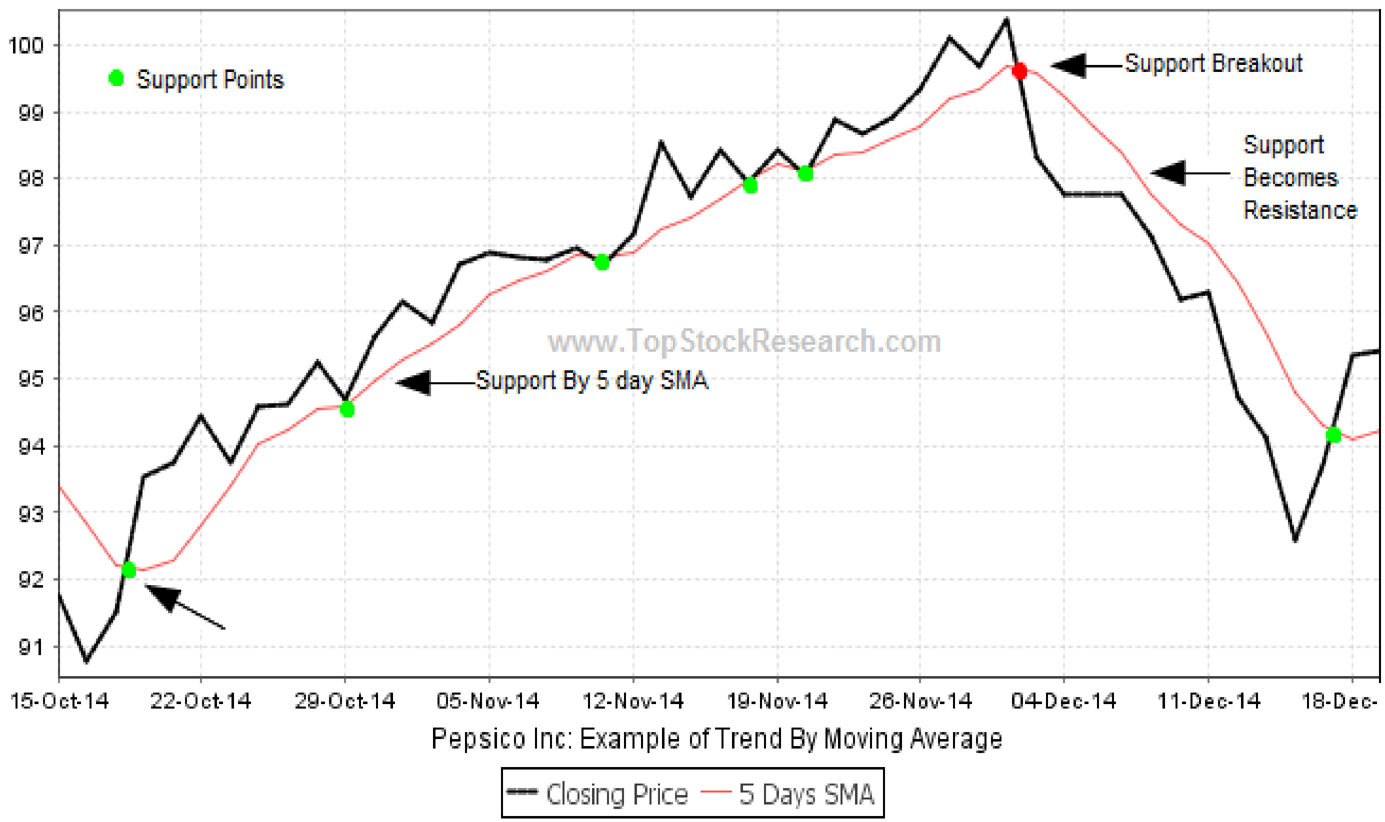


FIGURE VI

Returns on Housing Compared to Real Estate Investment Funds

Jorda et al (2019, QJE): trends in rate of returns



Support Points
Support Breakout
Support
Becomes
Resistance
Support By 5 day SMA
www.TopStockResearch.com
100
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98
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93
92
91
15-Oct-14
22-Oct-14
29-Oct-14
05-Nov-14
12-Nov-14
19-Nov-14
26-Nov-14
04-Dec-14
11-Dec-14
18-Dec-
Pepsico Inc: Example of Trend By Moving Average
Closing Price
5 Days SMA

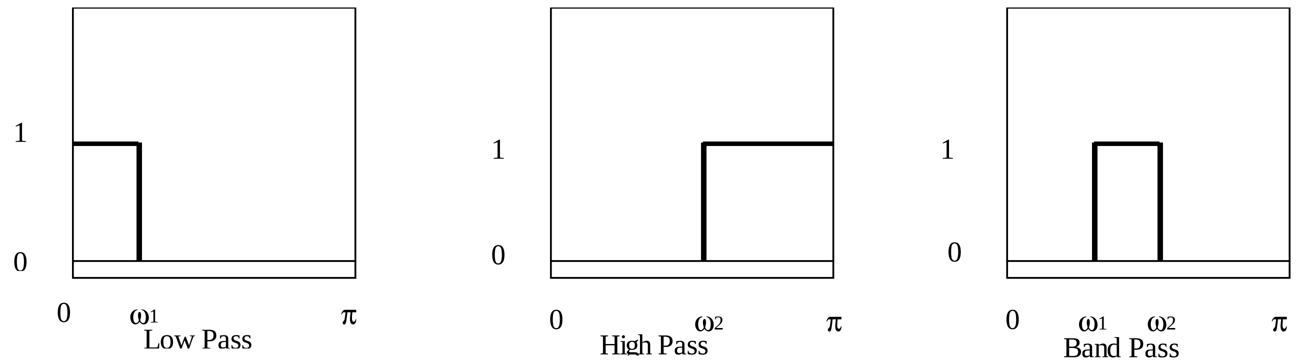
Band Pass (BP) Filters

- Combination of high pass and low pass MA filters.

- Low pass filter: $\mathcal{B}(\omega) = 1$ for $|\omega| \leq \omega_1$ and 0 otherwise.

- High pass filter: $\mathcal{B}(\omega) = 0$ for $|\omega| \leq \omega_1$ and 1 otherwise.

- Band pass filter: $\mathcal{B}(\omega) = 1$ for $\omega_1 \leq |\omega| \leq \omega_2$ and 0 otherwise.



Time series representation of the weights of the filters:

Low pass: $\mathcal{B}_0^{lp}(\omega_1) = \frac{\omega_1}{\pi}$; $\mathcal{B}_j^{lp} = \frac{sin(j\omega_1)}{j\pi}$; $\;0 < j < \infty$, some ω_1.

High pass: $\mathcal{B}_0^{hp}(\omega_1) = 1 - \mathcal{B}_0^{lp}$; $\mathcal{B}_j^{hp} = -\mathcal{B}_j^{lp}$; $\;0 < j < \infty$ (given ω_1

Band pass: $\mathcal{B}_0^{bp}([\omega_1, \omega_2]) = \mathcal{B}_j^{lp}(\omega_2) - \mathcal{B}_j^{lp}(\omega_1)$; $\;0 < j < \infty,\; \omega_2 > \omega_1$.

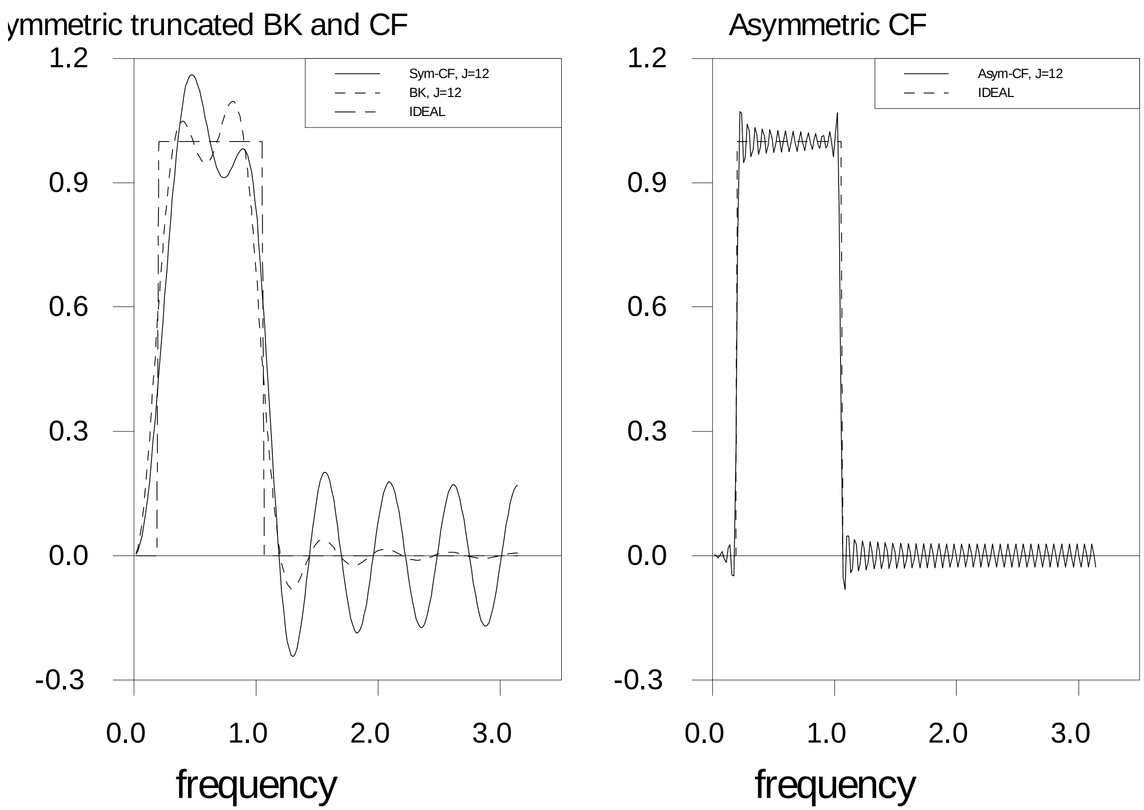
- j must go to infinity. Hence, these filters are **not realizable** for T finite.

• Baxter and King (1994): with finite T, cut at some $\bar{J} < \infty$.

• If the filter is symmetric and $\sum_{-\bar{J}}^{\bar{J}} \mathcal{B}_J = 0$ a truncated BP makes stationary series with quadratic trends and with up to two unit roots.

• BK approximation has the same problems of HP filter if y_t is (nearly) integrated.

• J needs to be large for the approximation to be good, otherwise leakage and compression.


ymmetric truncated BK and CF
Sym-CF, J=12
BK, J=12
IDEAL
1.2
0.9
0.6
0.3
0.0
-0.3
0.0
1.0
2.0
3.0
frequency
Asymmetric CF
Asym-CF, J=12
IDEAL
1.2
0.9
0.6
0.3
0.0
-0.3
0.0
1.0
2.0
3.0
frequency

• Christiano and Fitzgerald (2003): use a **non-stationary, asymmetric** approximation for finite T, which is optimal in the sense of making the approximation error as small as possible.

• Filter coefficients depend on t and change magnitude and even sign.

• Better spectral properties (see picture) but:

a) Need to know the properties of input series before taking the approximation (need to know if it is a I(0) or I(1)).

b) Because of the asymmetries, phase shifts may occur.

• Christiano and Fitzgerald approximation is the same as Baxter and King if y_t is a white noise. In general, they will differ at the beginning and end of the sample.

5.6 Wavelets filter

Similar idea as Band Pass filters but:

- Implementation is in time domain and MA is one sided.
- Size of the MA window depends on the cycles one wants to extract.
- Can be used on stationary and non-stationary input series.
- Implementation: Haar wavelet filter (see Lubik et al., 2019).

$$
\begin{aligned}
y_t &= \sum_{j=1}^{J} D_{jt} + S_{J,t} && (44) \\
D_{jt} &= 1/(2^j) * \Big(\sum_{i=0}^{2^{j-1}-1} y_{t-i} - \sum_{i=2^{j-1}}^{2^j-1} y_{t-i} \Big) && (45) \\
S_{J,t} &= 1/(2^J) * \Big(\sum_{i=0}^{2^J-1} y_{t-i} \Big) && (46)
\end{aligned}
$$

- Typically $J = 6$. Low j's capture high frequency; j=3,4 business cycles and j=5 low frequencies.

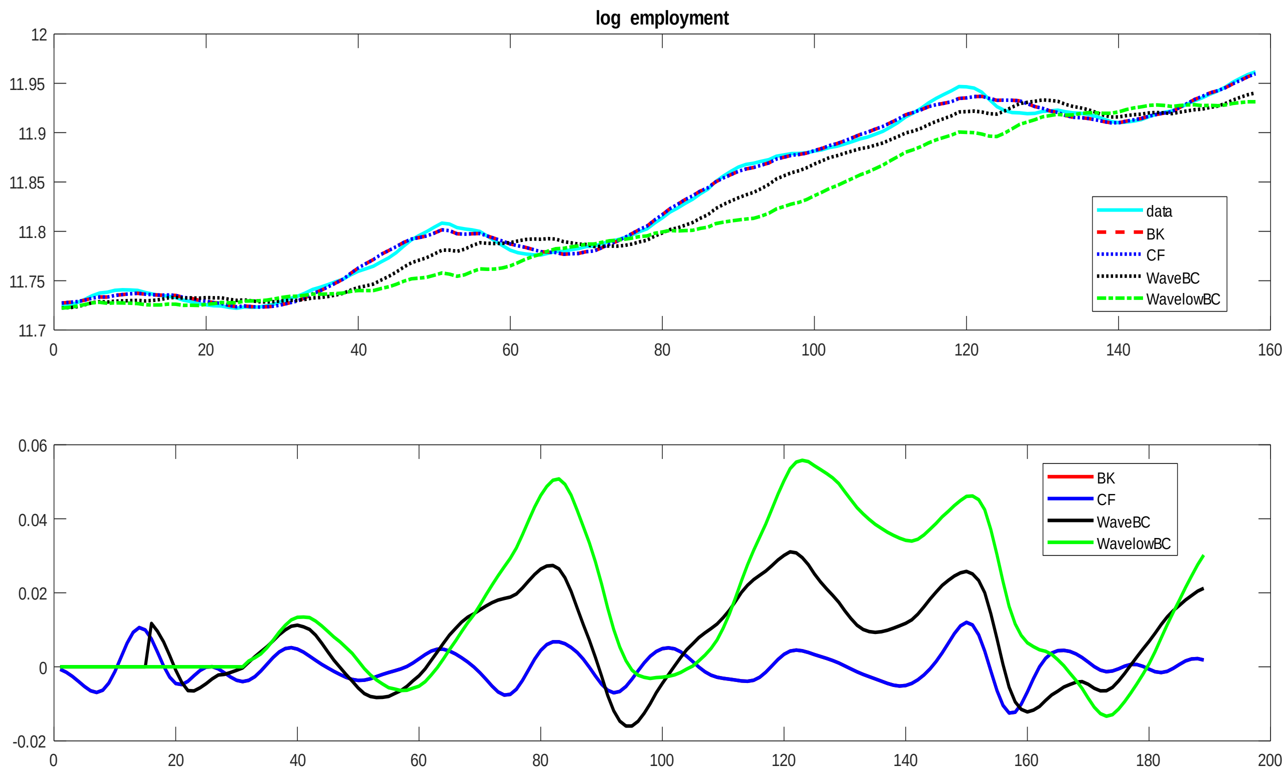
- S_{Jt} captures the long run component.

- 8-16 quarters cycles $D_{3t} = (1/8) * (y_t + y_{t-1} + y_{t-2} + y_{t-3} - y_{t-4} - y_{t-5} - y_{t-6} - y_{t-7})$.

- 16-32 quarters cycles $D_{4t} = (1/16) * (y_t + y_{t-1} + y_{t-2} + y_{t-3} + y_{t-4} + y_{t-5} + y_{t-6} + y_{t-7} - y_{t-8} - y_{t-9} - y_{t-10} - y_{t-11} - y_{t-12} - y_{t-13} - y_{t-14} - y_{t-15})$.

- 32-64 quarters cycles $D_{5t} = (1/32) * (y_t + y_{t-1} + y_{t-2} + y_{t-3} + y_{t-4} + y_{t-5} + y_{t-6} + y_{t-7} + y_{t-8} + y_{t-9} + y_{t-10} + y_{t-11} + y_{t-12} + y_{t-13} + y_{t-14} + y_{t-15} - y_{t-16} - y_{t-17} - y_{t-18} - y_{t-19} - y_{t-20} - y_{t-21} - y_{t-22} - y_{t-23} + y_{t-24} - y_{t-25} - y_{t-26} - y_{t-27} - y_{t-28} - y_{t-29} - y_{t-30} - y_{t-31})$.

- Window changes with the components.



Log Employment; Wavelet and BP filters

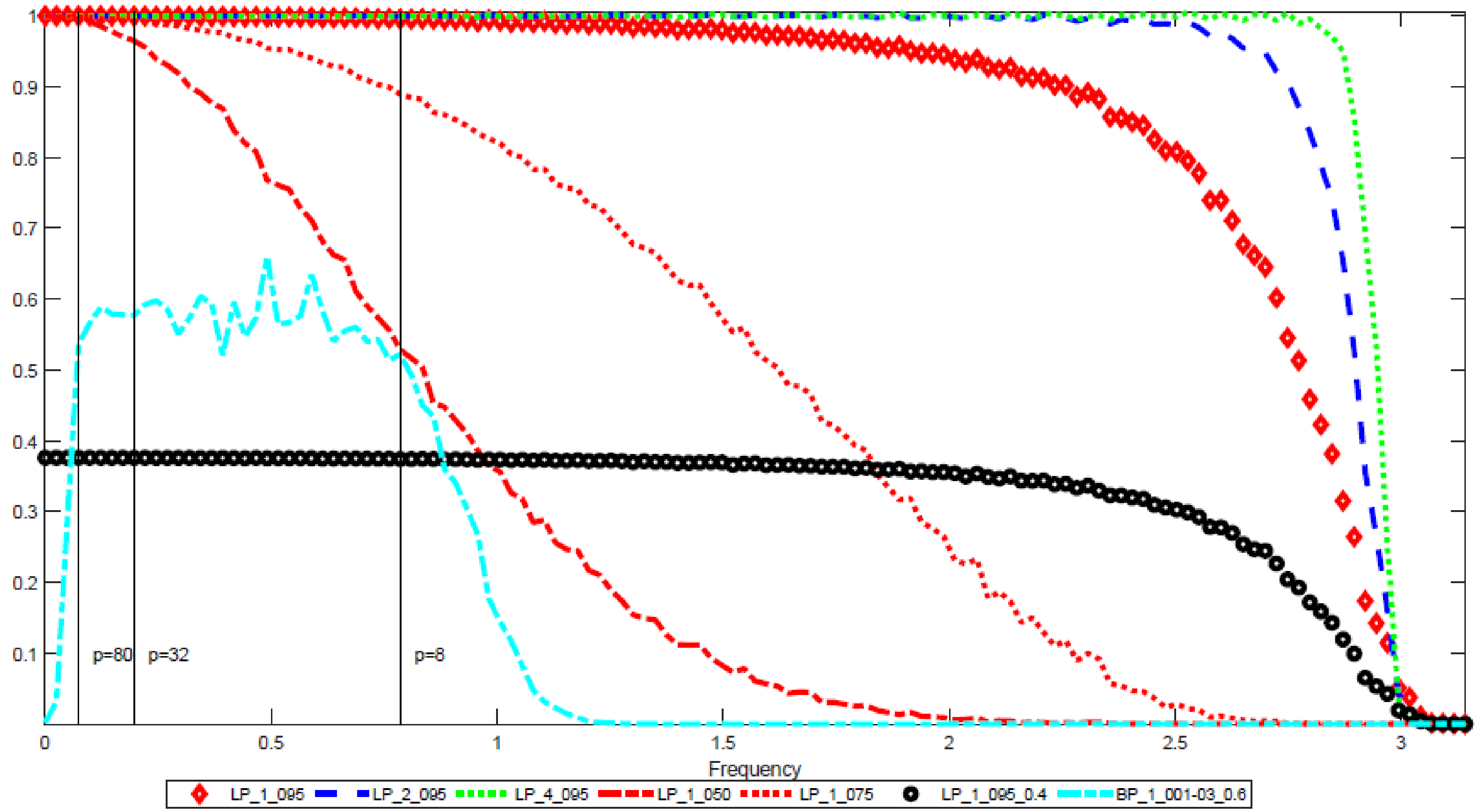
5.7 Butterworth filters

• Designed as low pass; can be adapted to high pass, band pass and stop pass. Used in electical engeneering to build radio, tv, phone antennas.

• Butterworth (1937): 'An ideal electrical filter should not only completely reject the unwanted frequencies but should also have uniform sensitivity for the wanted frequencies'

• Squared gain function: $G(\omega) = \frac{G_0}{1+(\frac{\omega}{\omega_c})^{2n}}$, where G_0 is the gain at the zero frequency, n is the (polynomial) order of the filter and ω_c a reference cutoff frequency (typically $\omega_c = 1$).

• Flexible: can be designed to capture medium and low frequency variations. Can be designed to eliminate unit roots without affecting medium frequencies.



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.5
1
1.5
2
2.5
3
p=80
p=32
p=8
Frequency
LP_1_095
LP_2_095
LP_4_095
LP_1_050
LP_1_075
LP_1_095_0.4
BP_1_001-03_0.6

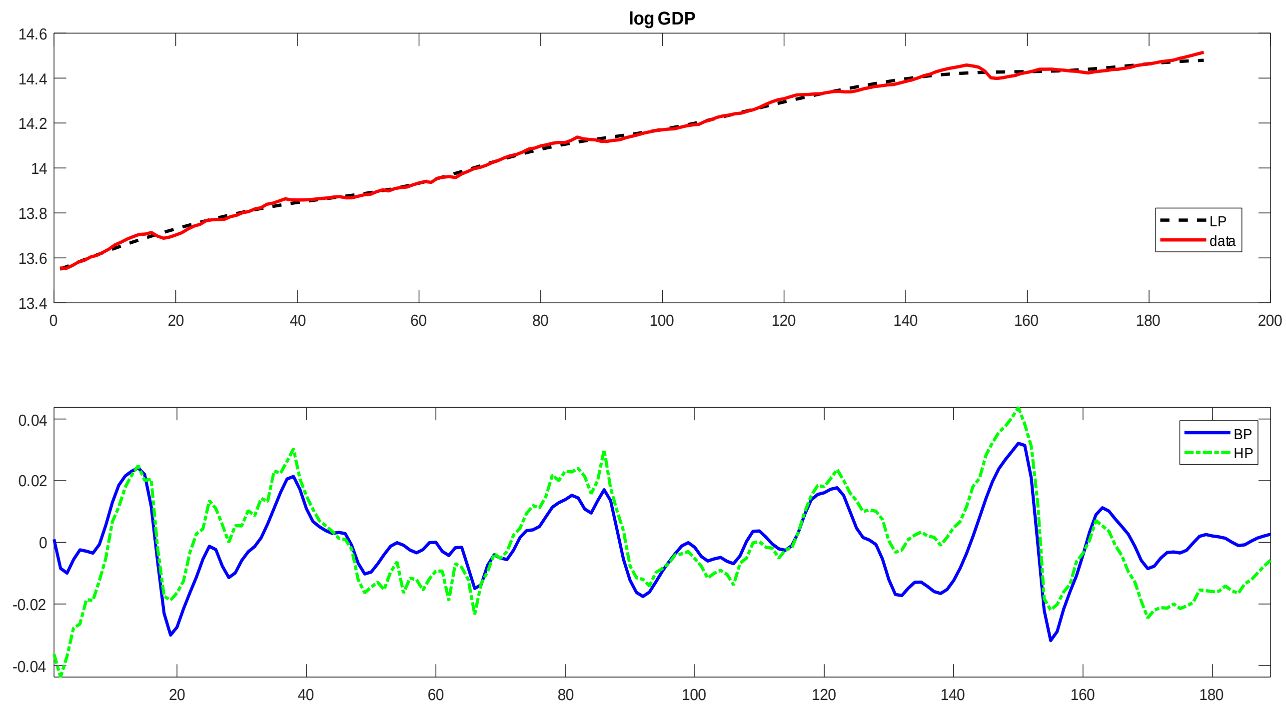
- Gain has different decays depending on n.
- Scale of the gain depends on G_0.
- Starting of decay depends on ω_c
- Useful to extract latent components with power at all frequencies.

Matlab commands to build a Butterworth filter

• [a,b]=butter(n,cutoff,type), where n is the degree of the polynomial, `cutoff` is where the squared gain falls, and `type` could be low, high, stop-pass. If cutoff is a vector with two values, the command computes band pass weights.

• y1=filtfilt(a,b,y). Creates the filtered series using an ARMA(a,b) with y as input.

• Default normalizes $G_0 = 1$. Rescale the b coefficients to change G_0 (if coefficients go up, you get lower squared gain).



Log output: Low pass, Band pass, High pass Butterworth filters

• Canova (2022) BW good to extract gaps produced by economic models. How does it perform on real data?

6 Economic Decompositions

- Idea: use economic models to split y_t into latent components.

- Leading examples: Blanchard and Quah (1989), random walk trend, $\rho \neq 0$. King, Plosser, Stock and Watson (KPSW) (1991), cointegrated trend, $\rho \neq 0$.

- Recovers permanent-transitory components (not trend/cycle because permanent may have cyclical features; not potential/gap: because gap may have permanent features).

- Results sensitive to model specification used to compute the decomposition and sample size.

- Example of a BQ decomposition: Fisher's model

$$gdp_t = gdp_{t-1} + a(\epsilon_t^s - \epsilon_{t-1}^s) + \epsilon_t^s + \epsilon_t^d - \epsilon_{t-1}^d \quad (47)$$
$$un_t = N_t - N^{fe} = -\epsilon_t^d - a\epsilon_t^s \quad (48)$$

$d =$ demand, $s =$ supply. This model implies that un_t has no trend; the trend in gdp_t is $gdp_t^x = gdp_{t-1}^x + a(\epsilon_t^s - \epsilon_{t-1}^s)$; and the cycle is $gdp_t^c = \epsilon_t^d - \epsilon_{t-1}^d + \epsilon_t^s$.

- Only supply shocks have long run effects on gdp_t.

- Both supply and demand shocks have cyclical effects on gdp_t.

- gdp_t^x and gdp_t^c correlated (ϵ^s drives both) but not perfectly.

Implementation

• Run a VAR ($y_t = [\Delta y_{1t}, y_{2t}]$, Δy_{1t} variable with unit root (output, labor productivity, etc.). Call $B(L)$ the estimated coefficients.

• Compute MA: $y_t = \bar{t} + D(L)e_t$. Proceed as in Beveridge-Nelson decomposition to compute $D(1)e_t$.

• Restrict D(1) to be lower triangular (only first shock has long run effects on y_{1t}).

• Compute D(1) from the Cholesky decomposition of $B(1)\Sigma_e B(1)'$ where Σ_e is the estimated covariance of the innovation vector.

• Implementation similar to multivariate BN but need identification at the last stage (e_t^x is a supply (technology) disturbance and not a reduced form shock).

• Example of KPSW decomposition: RBC model with contegration trend $y_t = [gdp_t, inv_i, C_t]$.

$$y_t = y_t^x + y_t^c \tag{49}$$

y_t^x a scalar, y_t^c a 3×1 vector. Δy_t has a MA representation

$$\Delta y_t \;\; = \;\; \bar{y} + D(\ell) e_t \tag{50}$$

• Trend component of y_t identified using $D(1)e_t = [1, 1, 1]' e_t^x$, where e_t^x is a permanent innovation (use Cholesky decomposition of $D(1)\Sigma_e D(1)'$).

• Cyclical component $y_t - y_t^x$.

Alternative identification assumptions

• The BQ decomposition normalizes the variance of structural shocks to one and assumes that structural shocks are uncorrelated.

• Morley et et. (2003), Grant and Chan (2017a): long run and short run disturbances may be correlated.

• Normalization chosen may matters (Waggoner and Zha, 2003).

• Cover et al. (2003): use alternative normalization plus identification assumptions that allow demand and supply shocks to be correlated.

- Structural model ($\alpha > 0$, unitary AD slope)

$$gdp_t = E_{t-1} gdp_t + \alpha(p_t - E_{t-1} p_t) + \epsilon_{1t} \tag{51}$$
$$gdp_t = p_t + E_{t-1}(gdp_t + p_t) + \epsilon_{2t} \tag{52}$$

$\epsilon_{1t}, \epsilon_{2t}$ potentially correlated; (51) is AS; (52) is AD.

- VAR : $y_t = a_0 + B(L) y_{t-1} + e_t$, $y_t = [gdp_t, p_t]'$.

- Relationship VAR-structural model

$$e_{1t} = \frac{1}{1+\alpha}\epsilon_{1t} + \frac{\alpha}{1+\alpha}\epsilon_{2t} \tag{53}$$
$$e_{2t} = -\frac{1}{1+\alpha}\epsilon_{1t} + \frac{1}{1+\alpha}\epsilon_{2t} \tag{54}$$

or $e_t = \Psi \epsilon_t$.

• Identification:

i) Normalization: $\epsilon_{it}, i = 1, 2$ has a unitary effect on y_t; ii) long run demand shock neutrality. i)-ii) together with the assumption that slope of aggregate demand is unit (demand shocks may be persistent) imply:

$$\alpha = -\frac{B_{12}(1)}{1 - B_{22}(1)} \tag{55}$$

• Given α from (55), use (53) and (54), to recover structural shocks given reduced form shocks.

• Permanent/transitory components correlated. Permanent component:

$$y_t = a_0 + B(L)y_{t-1} + \Psi_1 \epsilon_{1t} \tag{56}$$

where Ψ_1 is the first column of Ψ.

• Contrast with BQ setup:

$$e_{1t} = c_{11}\epsilon_{1t} + c_{12}\epsilon_{2t} \tag{57}$$

$$e_{2t} = c_{21}\epsilon_{1t} + c_{22}\epsilon_{2t} \tag{58}$$

Identification: $\sigma^2_{\epsilon_i} = 1; \sigma_{\epsilon_1,\epsilon_2} = 0$; long run demand shock neutrality imply:

$$c_{12}(1 - a_{22}(1)) + c_{22}a_{12}(1) = 0 \tag{59}$$

• Given (59), use (57), (58) to get the structural shocks (3 unknowns in 3 moments).

• In Cover et al. permanent and transitory components correlated because supply shocks drive both even if supply and demand shocks are uncorrelated.

6.1 Are BQ estimates robust?

• Coibion et al (2018): BQ estimates of output potentials (permanent component) only depend on supply shocks. Traditional estimates depend on both supply and demand shocks.

• Canova and Ferroni (2022): VAR estimates typically subject to *deformation*. Inference about latent variables depends on the dimensionality of the VAR model used.

• Deformation occurs if the DGP has more shocks than the number of variables in the VAR.

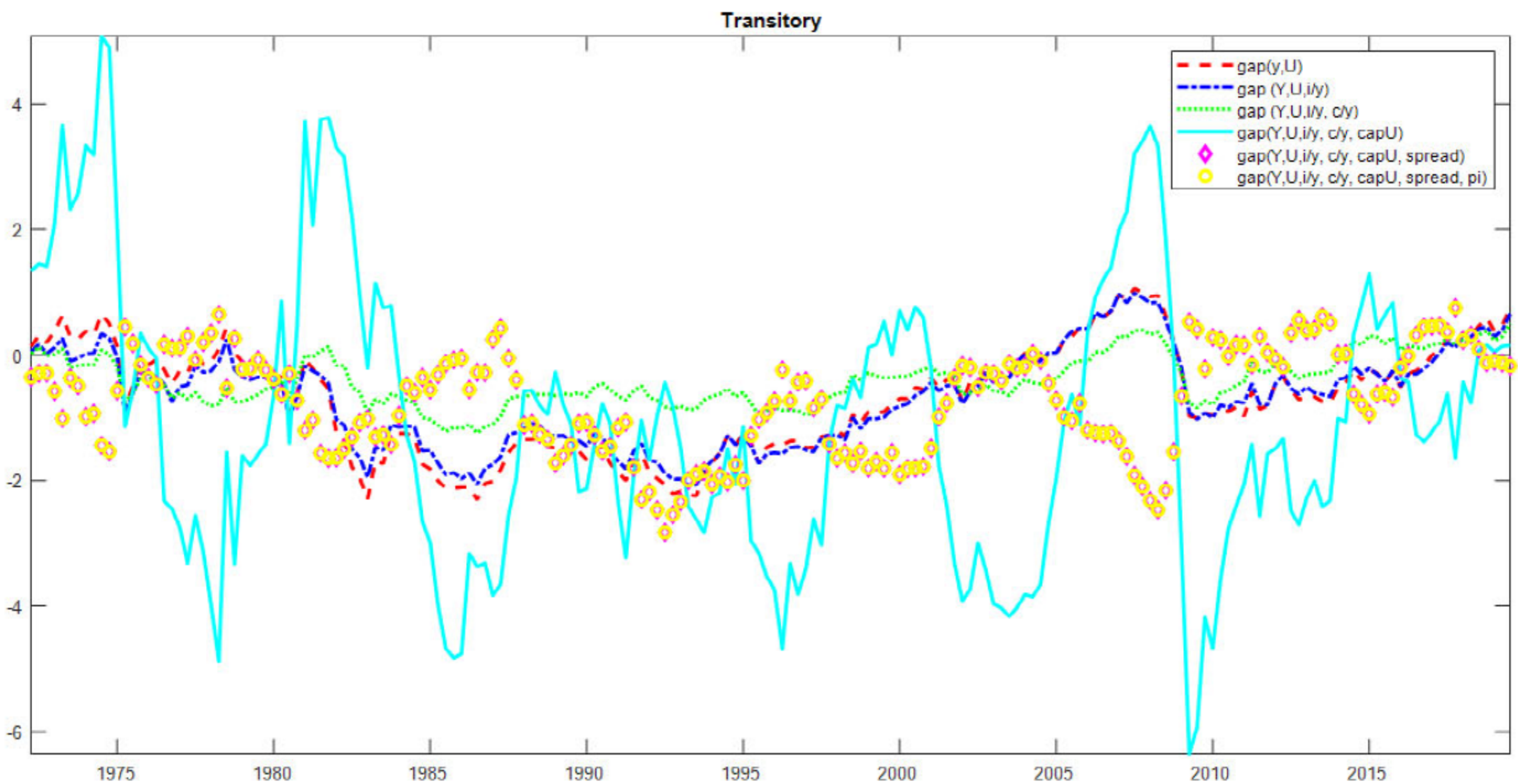
• Cross sectional and time deformation could be present.

• Illustration of deformation for permanent and transitory decompositions:

• Run a VAR with US output growth and unemployment. Compute the permanent and the transitory components of output.

• Sequentially add to the VAR:

- investment/output ratio
- consumption/output ratio
- capacity utilization
- term spread (10 years bond rate -call rate)
- inflation



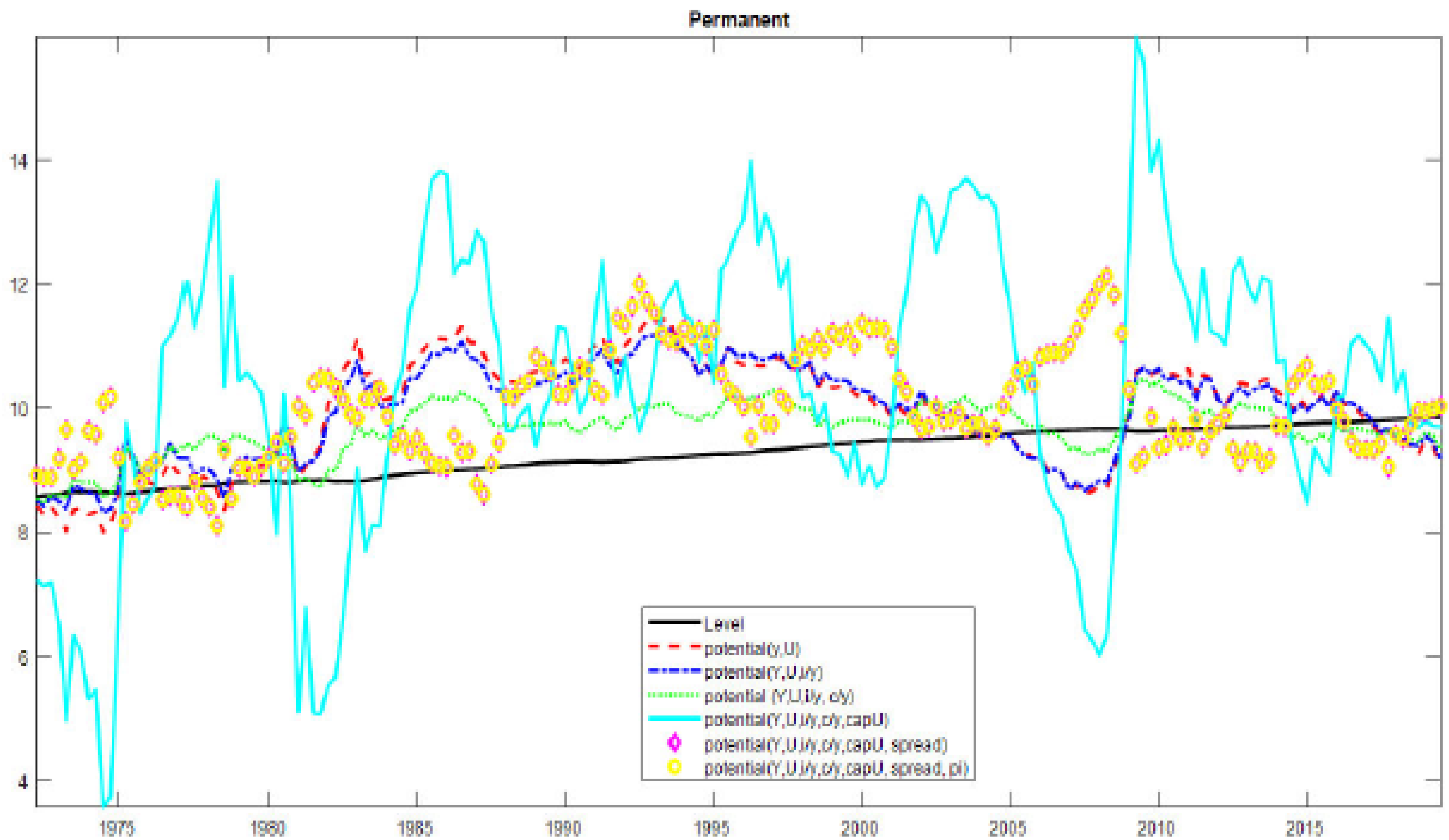
Transitory
gap(y,U)
gap (Y,U,i/y)
gap (Y,U,i/y, c/y)
gap(Y,U,i/y, c/y, capU)
gap(Y,U,i/y, c/y, capU, spread)
gap(Y,U,i/y, c/y, capU, spread, pi)
4
2
0
-2
-4
-6
1975
1980
1985
1990
1995
2000
2005
2010
2015

- Timing of peaks and throughs varies with dimension of the VAR.

- Amplitude of cycles changes.

- End of the sample: is the transitory component positive or negative?

- Which cycle estimate should we use/trust?



Permanent
14
12
10
8
6
4
1975
1980
1985
1990
1995
2000
2005
2010
2015
Level
potential(y,U)
potential(Y,U,i/y)
potential (Y,U,i/y, c/y)
potential(Y,U,i/y,c/y,capU)
potential(Y,U,i/y,c/y,capU, spread)
potential(Y,U,i/y,c/y,capU, spread, pi)

• Permanent component driven only by supply shocks but which one should we believe?

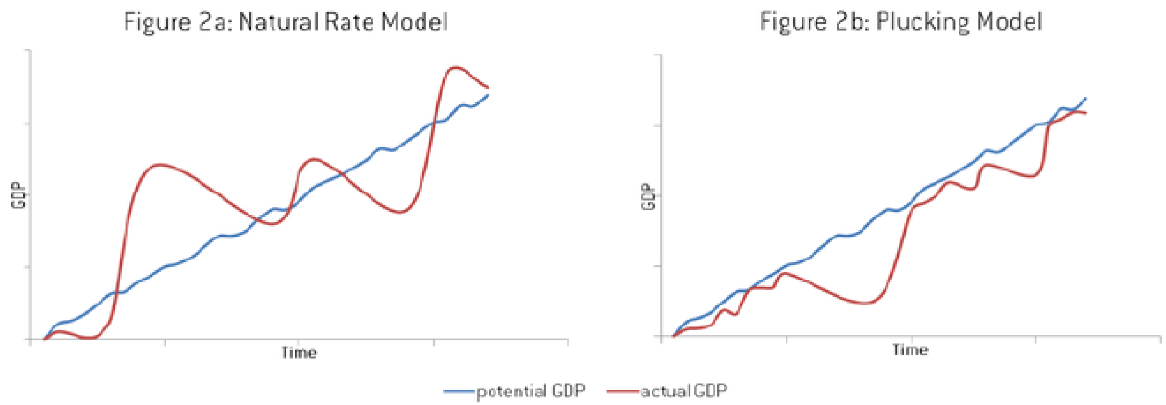
• General insight: a two variable VAR is too small to perform a robust decomposition. Shocks are contaminated. Inference whimsical.

7 How policymakers think about cycles?

• Policymakers typically interested in gaps. Care about because they want to react to gaps but not to potentials. But **gaps very loose defined**.

• Okun (1962): Potential gap is the maximum level of production with full employment that does not trigger inflationary pressures above ‘’the social desire for price stability and free markets” (point of balance between more output and greater price stability).

• Kuttner (1994): Potential is the production level generating constant inflation. Different from the maximum output level generated with any amount of aggregate demand.

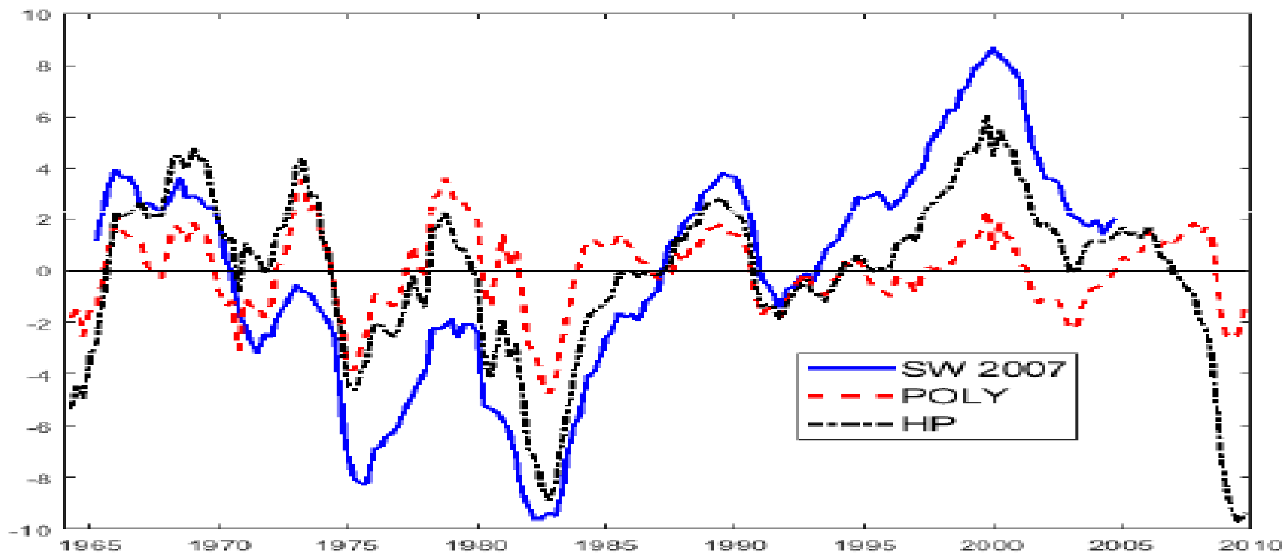


• Friedman (1964) ''plucking model'': the business cycle is a cyclical contraction (due to negative demand shocks) from the maximum feasible output (driven by supply considerations).

• DSGEs potential is the path of the variables when nominal frictions are absent.

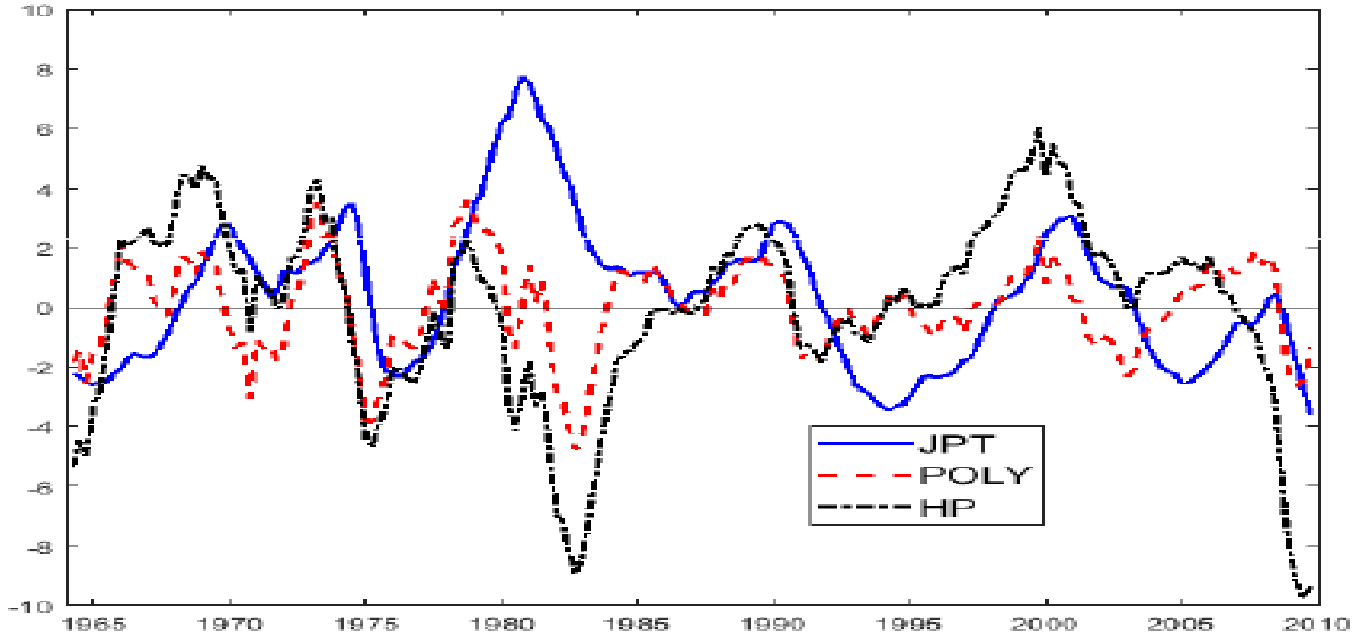
• **Gaps are meaningful only in terms of a model**.

• How do DSGE-based estimates of gaps look like relative to, say, Polynomial or HP cycle estimates?



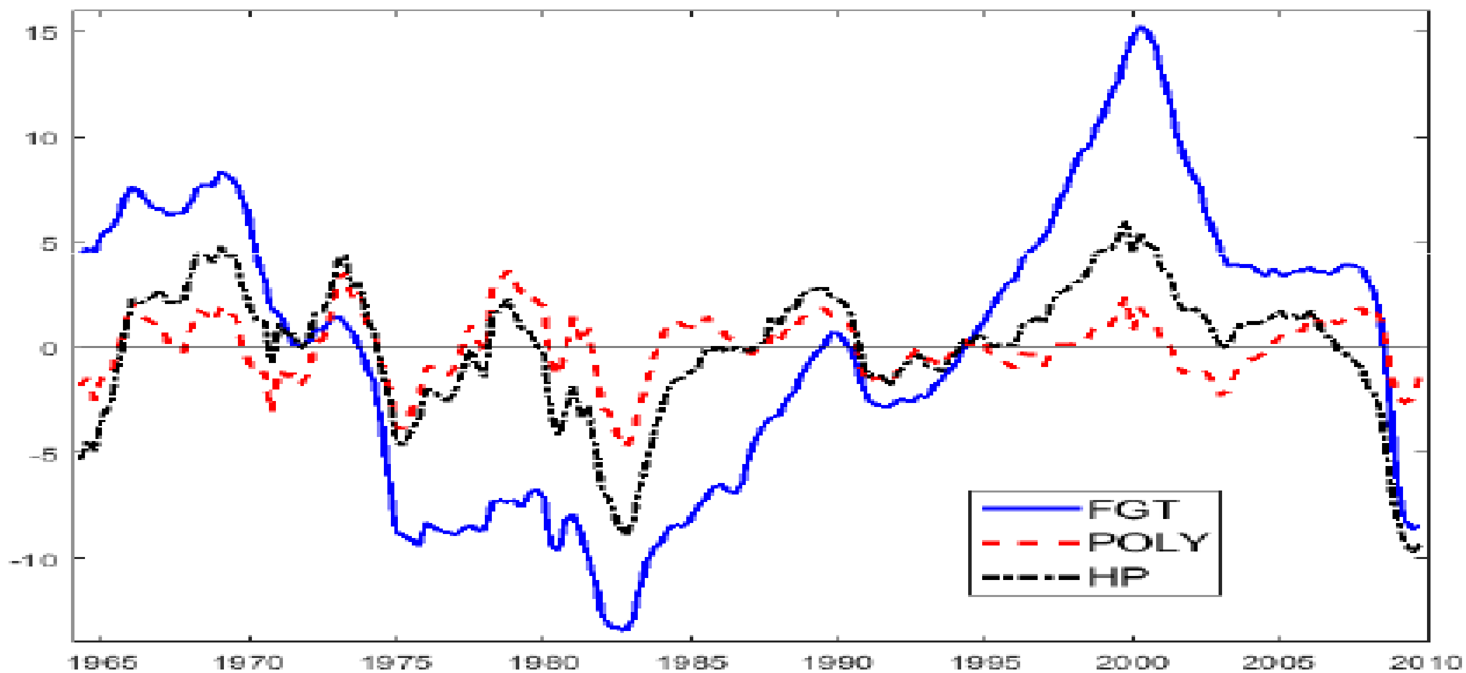
Smets and Wouter (2007) gap

• Ups and downs similar. But SW has more variability and more pessimistic about gap in 1970-s and 1980s.



JPT (2013) gap: model with two observable hours series

- Quite a big change in the 1980s. Volatility is now similar.



FGT (2020) gap: model with financial frictions

- Again, quite different properties.

- Model chosen matters for measures of output gap.

- Tend to have larger/longer swings than traditional statistical estimates: amplitude (and sometimes duration) of phases differ.

- If you do not trust a model, what do you do? Canova and Matthes (2020) use robust CL approach. Jointly estimate all the models on the table. Structural parameters will reflect the restrictions of all models. Measurement error filtered out.

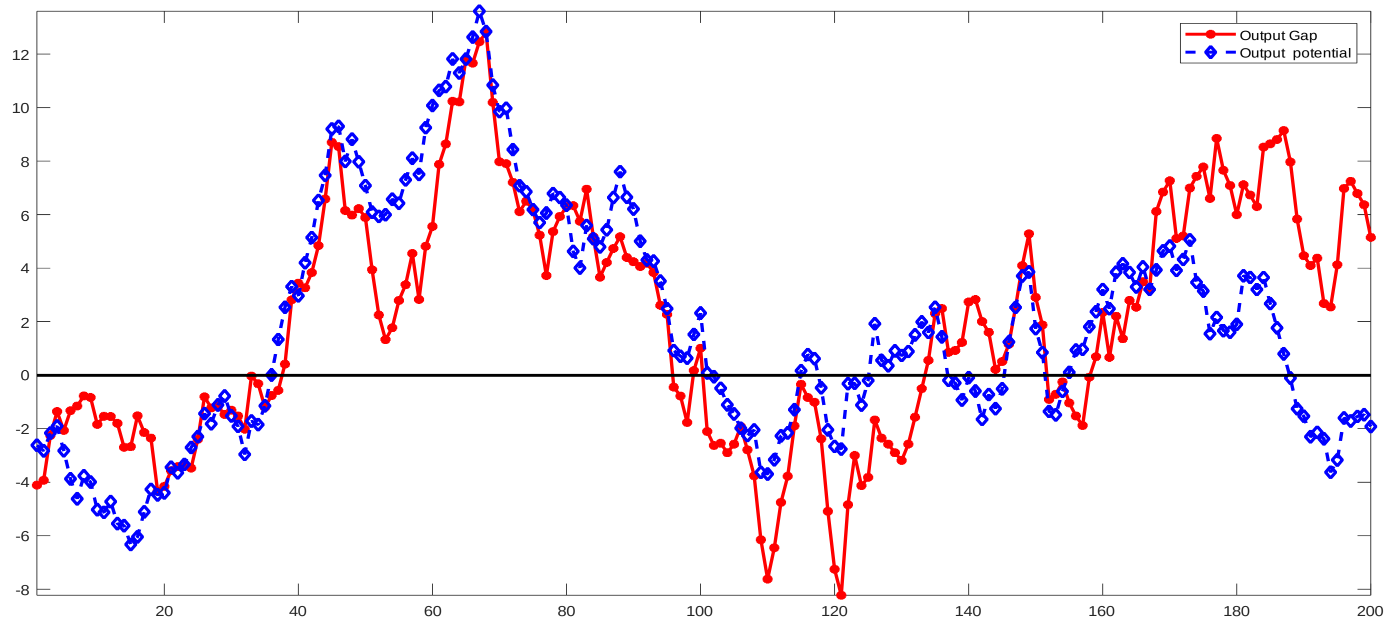
Do theory based gaps look like estimated cyclical components?

• Canova (2022): NO.

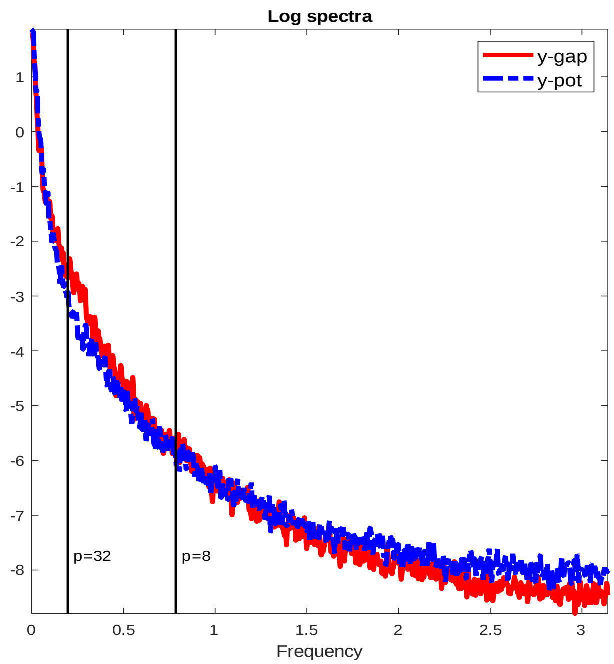
i) Theory gaps highly persistent; feature important low frequency variations.

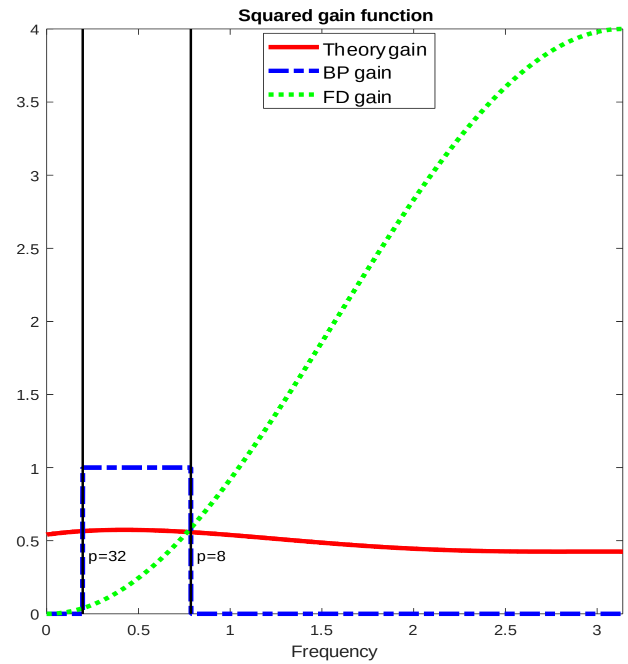
ii) Business cycle variability smaller than low frequency variability.

iii) Gaps correlated with potentials (driven by similar shocks).



Output gap and potential SW model





- No filter assumes that latent components have similar spectral properties.

Table 1: Relative variance

	All frequencies	Low frequencies	BC frequencies	Own variance low frequencies	Own variance BC frequencies
Gap (SW stationary)	**0.48**	**0.48**	**0.54**	**0.09**	**0.07**
Gap (SW 5 shocks)	0.46	0.49	0.67	0.06	0.04
Gap (SW-FF)	1.15	1.12	1.04	0.16	0.07
Gap (CMR)	< 0.01	0.21	0.16	0.09	0.01
Gap (ECB-Base)	< 0.01	0.82	0.92	0.30	0.35
Transitory (SW unitoot)	< 0.01	**0.68**	**0.79**	**0.09**	**0.08**
Transitory (SW trendcycle)	< 0.01	0.48	0.80	0.21	0.38

Horse race results 1: Gap extraction

- The *least distorting* traditional is **Polynomial filtering**. Why?

- Frequency distribution of the variance of gaps and potentials similar.

- Gaps have important low frequency components. Potentials significant business cycle frequency components.

- With Polynomial filtering the frequency distribution of the variance of the gaps not distorted. Estimated cycles display some low frequency variations.

- Conclusions independent of sample size and filters' parameters.

- Butterworth filter dominates all traditional methods.

Horse race results 2: Transitory fluctuations extraction

• The *least distorting* traditional is **differencing**, **Polynomial filtering** close second.

• Distortions larger because at business cycle and high frequencies permanent fluctuations matter a lot.

• Small samples affect the ranking; the parameters of the filters do not.

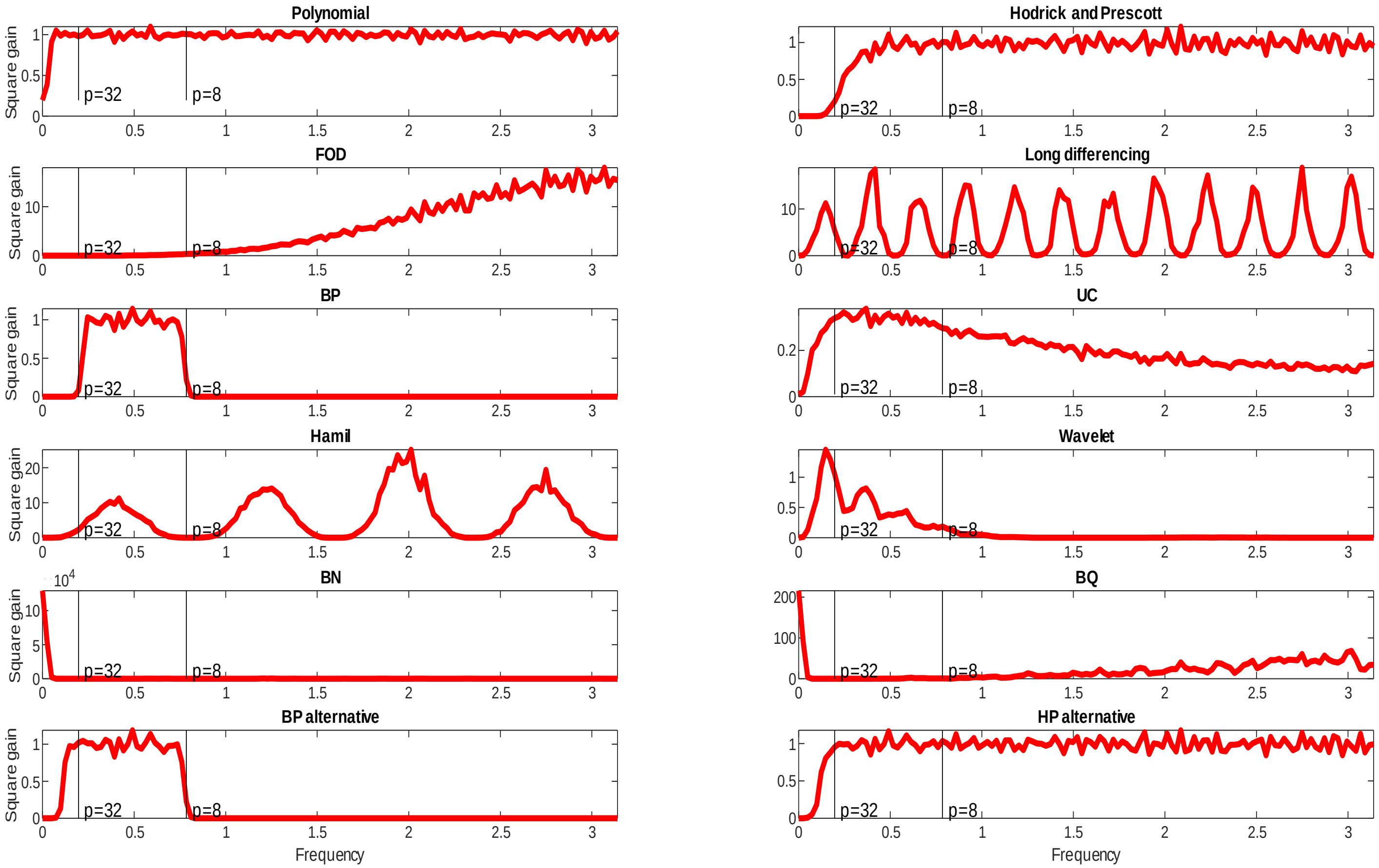
• Butterworth beats less often traditional filters.

Table 2: Summary results across variables, SW DGP, T=750

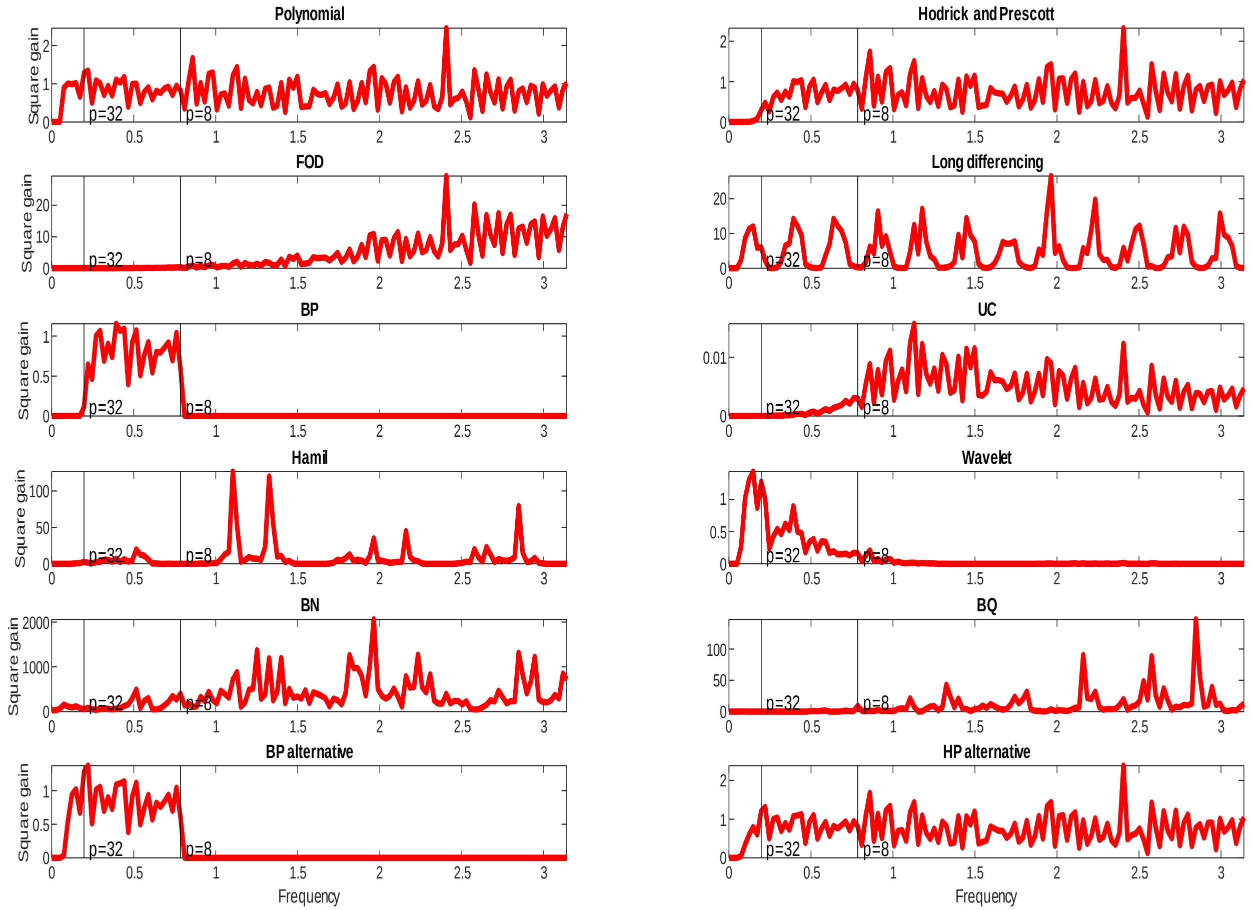
Statistic	POLY	HP	FOD	LD	BP	Wa	Ham	UC	BN	BQ	BW
						Gap					
MSE	5	3						1	0.5	0.5	8
Corr	9								0.5	0.5	8
AR1	4			3		3					6
Var	4			2		3	1				
TP	1.5	5	2	1.5							3
RT-MSE		1				3	2	3	0.5	0.5	8
PC	2										2
OL				1			1				
Total	25.5	9	2	8.5	0	9	4	4	1.5	1.5	35
						Transitory					
MSE			9					1			
Corr											
AR1	4					5.5		0.5			1
Var	3			6			1				1
TP	4	4		2							3
RT-MSE			4					6			
PC				1						1	
OL				2							
Total	11	4	13	11	0	5.5	1	7.5	0	1	5

Table 3: Summary results across statistics, different DGPs T=750

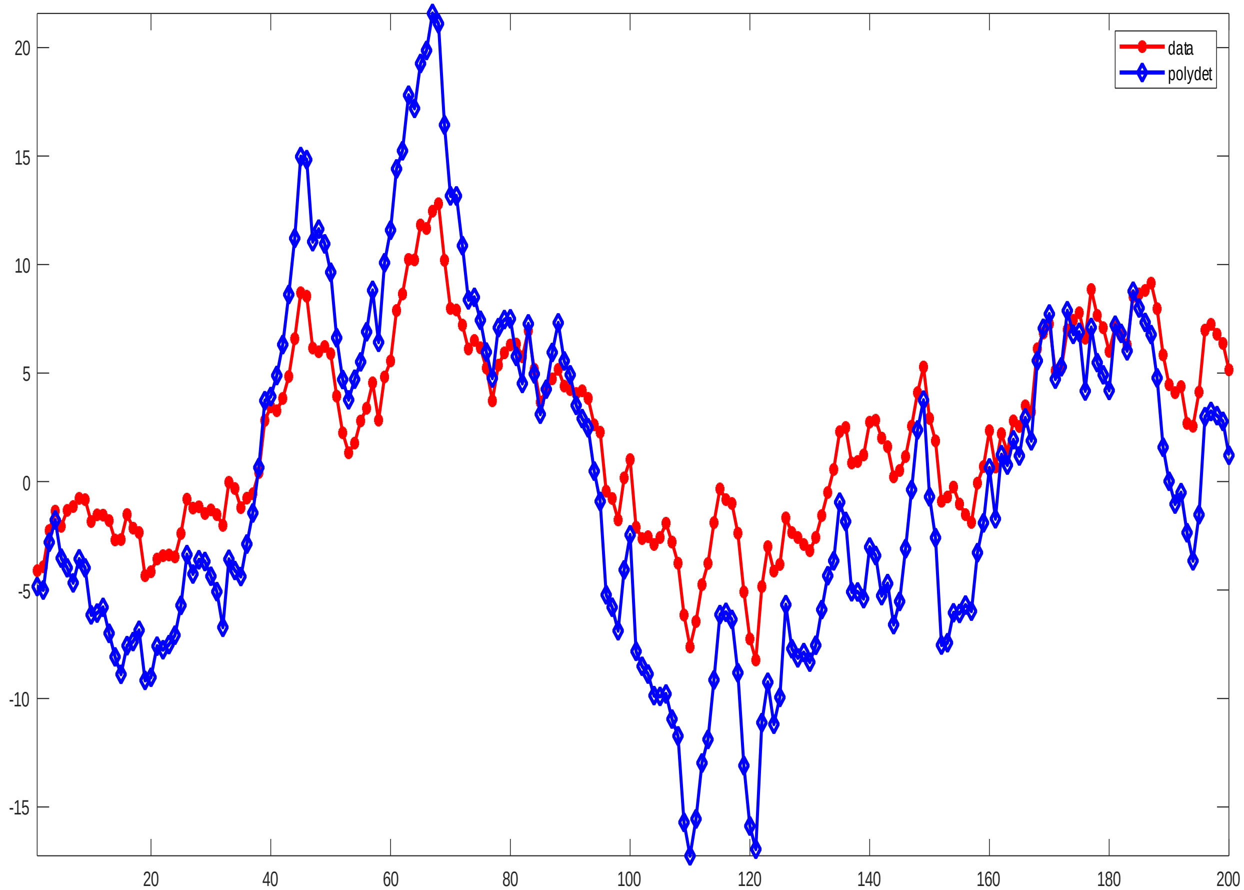
	Output Gap										
DGP	POLY	HP	FOD	LD	BP	Wa	Ham	UC	BN	BQ	BW
SW	6	3	1	1			2	1			7
SW_FF	10	1	1	2							7
CMR	2	2	1	2	0.5	2		1		2.5	
SW_5	4	6	1	1				1		1	7
Total	22	12	4	6	0.5	2	2	3		3.5	21
	Transitory output										
SW(unitroot)	2	3	2	4	1					1	3
SW(trendcycle)	1	4	2			2	1			3	7
Total	3	7	4	4	1	2	1		3	1	10



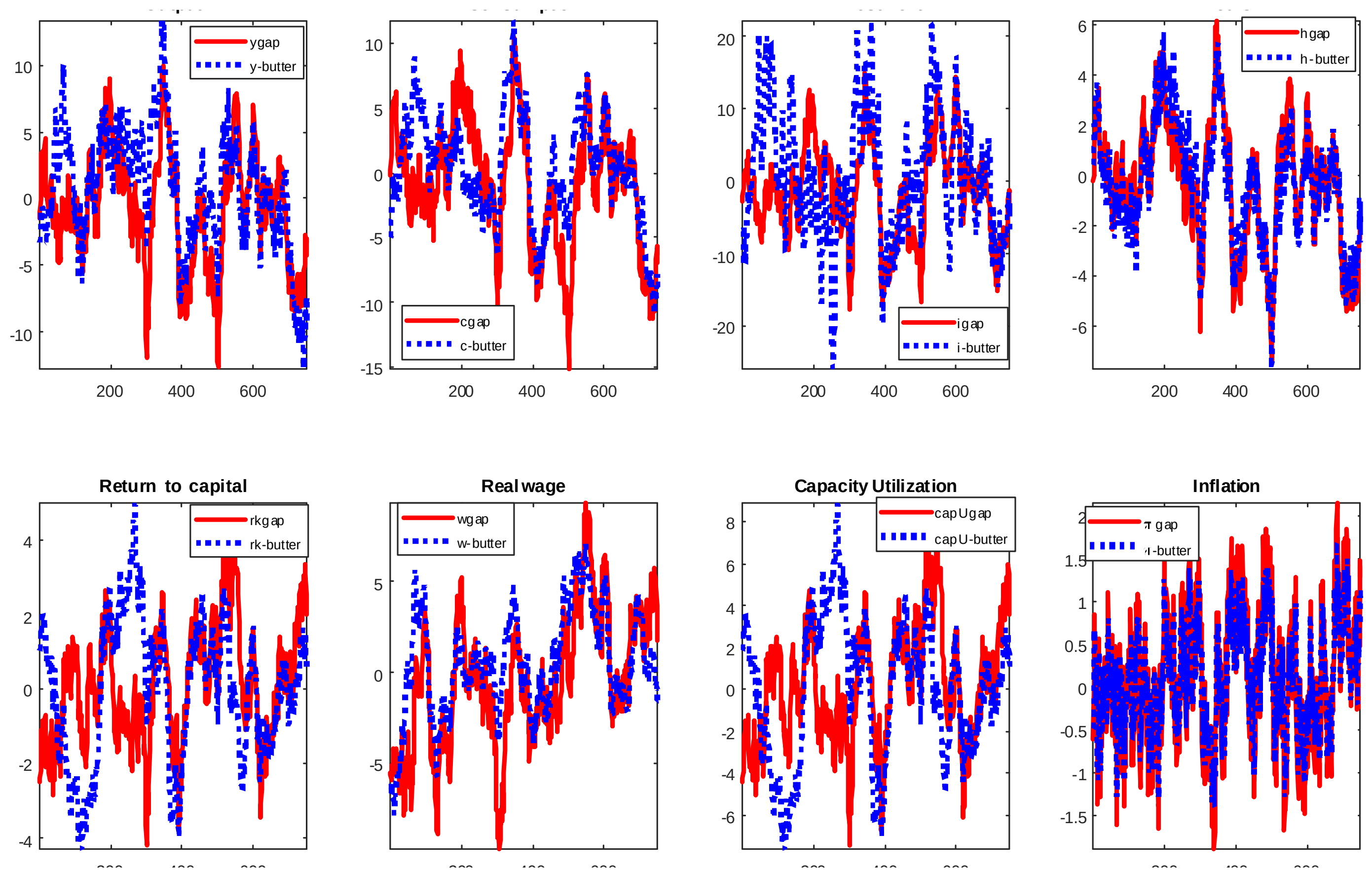
Polynomial
Hodrick and Prescott
FOD
Long differencing
BP
UC
Hamil
Wavelet
BN
BQ
BP alternative
HP alternative
Square gain
Frequency
p=32
p=8
$\times 10^4$



Polynomial
Hodrick and Prescott
FOD
Long differencing
BP
UC
Hamil
Wavelet
BN
BQ
BP alternative
HP alternative
Square gain
p=32
p=8
Frequency



Gap and polydet estimate


ygap
y-butter
cgap
c-butter
igap
i-butter
hgap
h-butter
Return to capital
rkgap
rk-butter
Real wage
wgap
w-butter
Capacity Utilization
capUgap
capU-butter
Inflation
π gap
π-butter

8 Collecting cyclical information: are there stylized facts?

• Are cyclical statistics **similar** across detrending/filtering approaches?

• Typically focus on 8-32 quarter cycles? But a unit root trend may a lot of power at these frequencies (Aguiar and Gopinath (2007)). Furthermore, variability of cycles in macro data 8-32 quarters need not be large (Beaudry, et al. 2019, Lubik et al., 2019).

• How do you compare cyclical statistics in countries (regions) where cycles have different length?

• Filtering and detrending are subject to specification errors, small sample or truncation biases.

Canova (1998)-(1999) Business cycle facts depend:

- Assumptions about the trend and procedures used to remove it.

- Whether the decomposition is univariate or multivariate.

- Whether components are assumed orthogonal or non-orthogonal.

- What portions of spectrum are emphasized.

- Sample size (in small samples cyclical coefficients poorly estimated)

Summary statistics

	Variability	Relative Variability		Contemporaneous Correlations			Periodicity
Method	GDP	Consumption	Real wage	(GDP,C)	(GDP,Inv)	(GDP, W)	(quarters)
HP1600	1.76	0.49	0.70	0.75	0.91	0.81	24
HP4	0.55	0.48	0.65	0.31	0.65	0.49	7
BN	0.43	0.75	2.18	0.42	0.45	0.52	5
BP	1.14	0.44	1.16	0.69	0.85	0.81	28
KPSW	4.15	0.71	1.68	0.83	0.30	0.89	6

• Differences present also in other statistics, e.g. dating of cyclical turning points or measuring business cycle phases.

Conclusion

• Measurement of growth cycles and calculation of robust statistics problematic.

• Empirical facts should be collected **without growth removal and should be conditional (rather than unconditional).**

• If you care about gaps, use models. If you do not trust models use composite methods. If you go for statistical approaches use Butterworth filters.

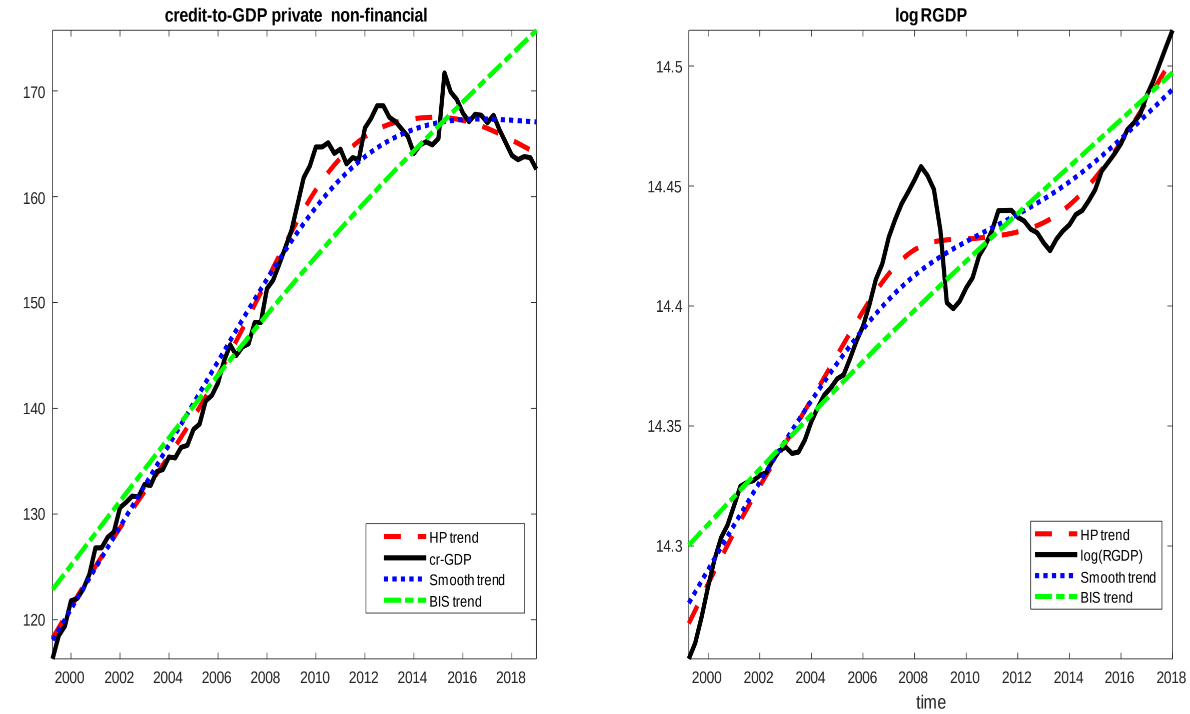
9 Business and financial cycles. Are they different?

- BIS: Financial cycles are longer than business cycles, see Borio (2012)

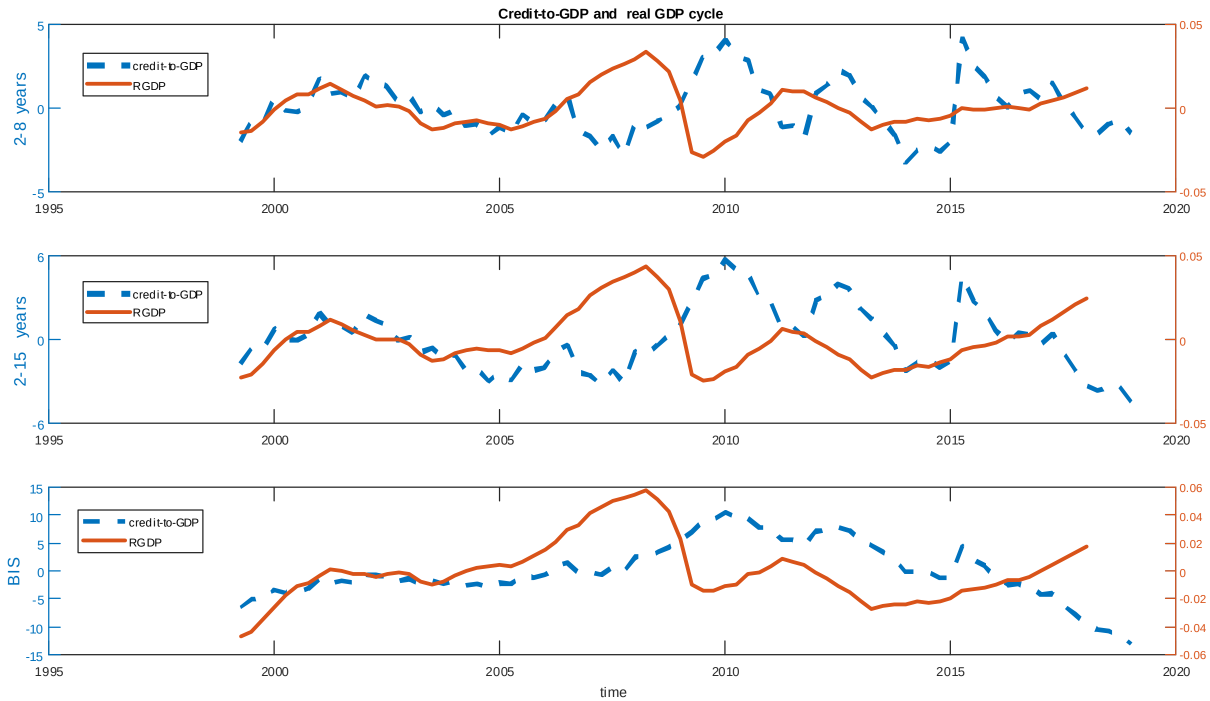
- Lots of literature on the topic, see e.g. Runstler and Vlekke (2018).

- Compare credit to nonfinancial corporations to GDP and GDP dynamics for illustration.

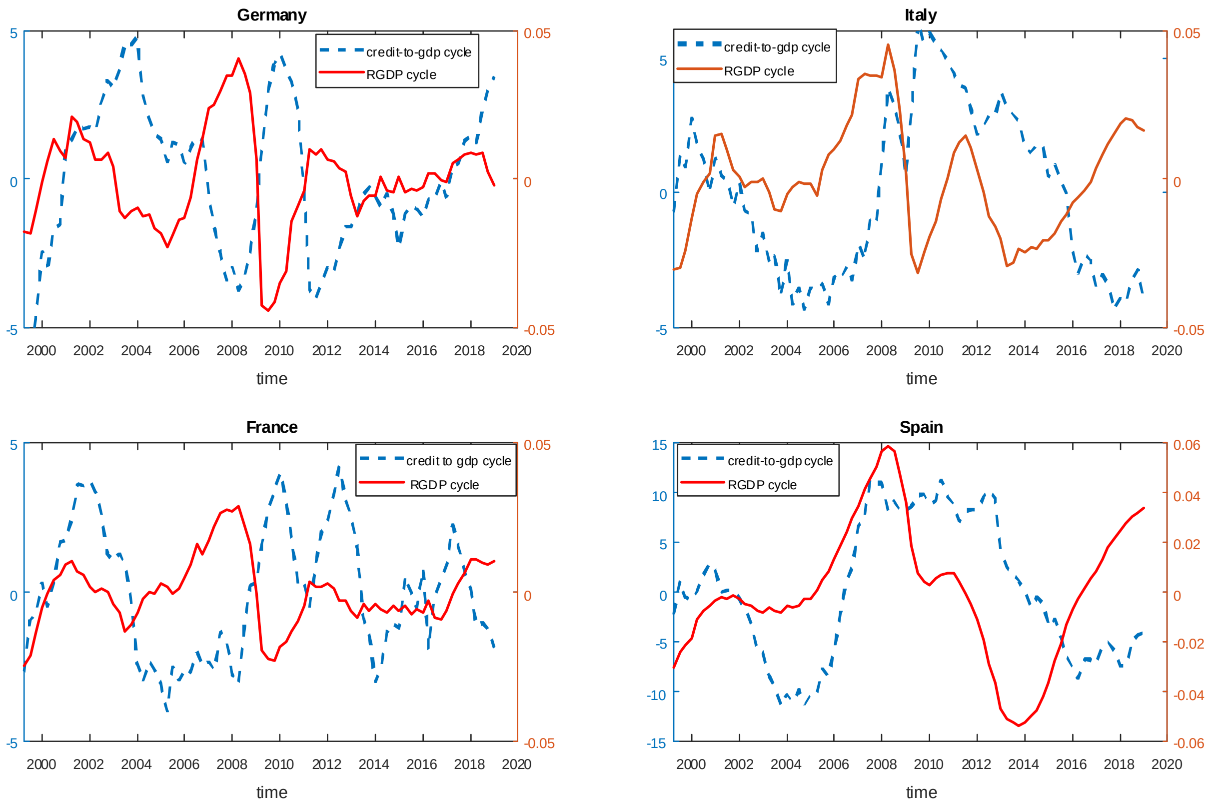
Euro area data



Variable	% of variance 2-8 years cycles	% of variance 8-15 years cycles	Persistence AR1
Credit/GDP total	1.5	18.3	0.99
Credit/GDP households	1.6	19.0	0.99
Credit/GDP private non financial	1.7	19.1	0.99
log(real GDP)	2.1	20.3	0.99
Labor Productivity	2.2	20.4	0.99
Unemployment rate	1.6	18.1	0.98

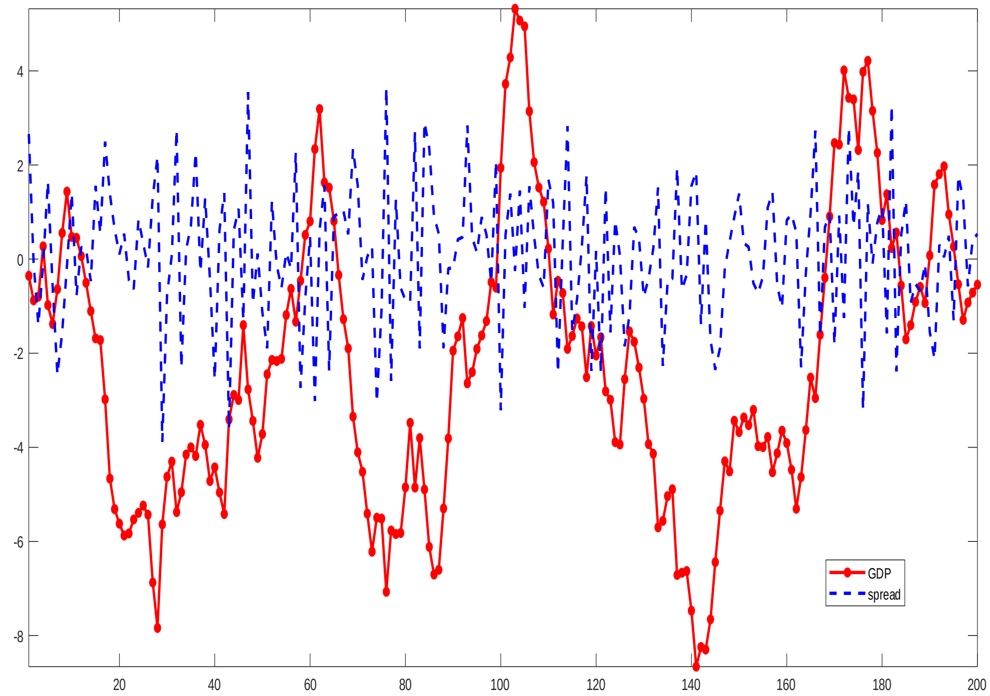


- Unclear which one is more highly correlated.

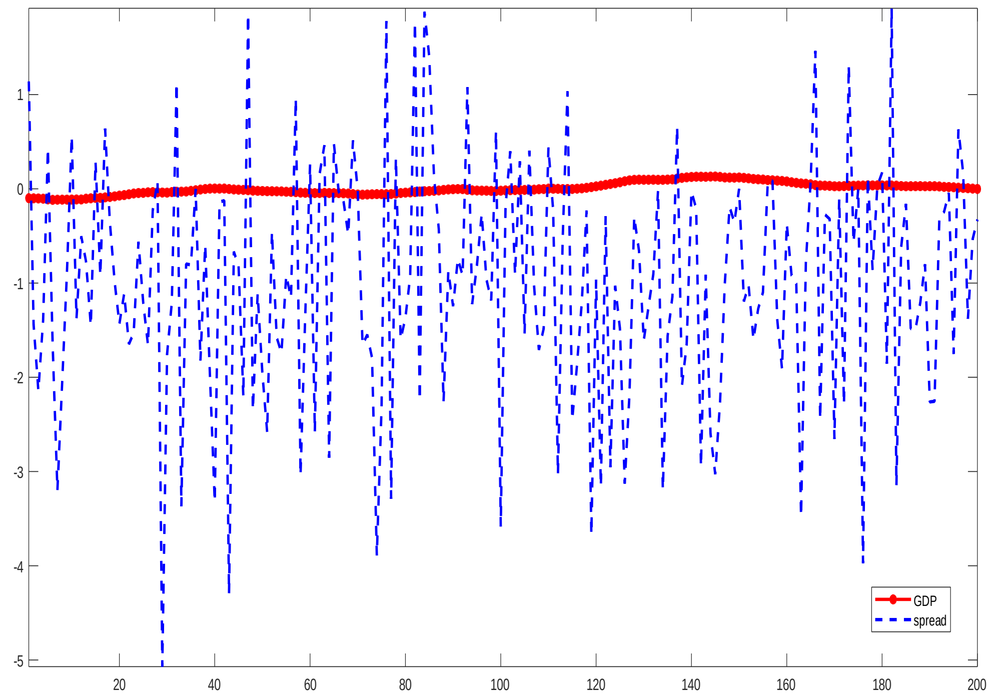

Germany
credit-to-gdp cycle
RGDP cycle
Italy
credit-to-gdp cycle
RGDP cycle
France
credit to gdp cycle
RGDP cycle
Spain
credit-to-gdp cycle
RGDP cycle
time
time
time
time

Real and Financial cycles in models

- Use SW-FF (Del Negro et al, 2015) and CMR (Cristiano, et al. 2011)
- Do model based cycles look like those in the data?

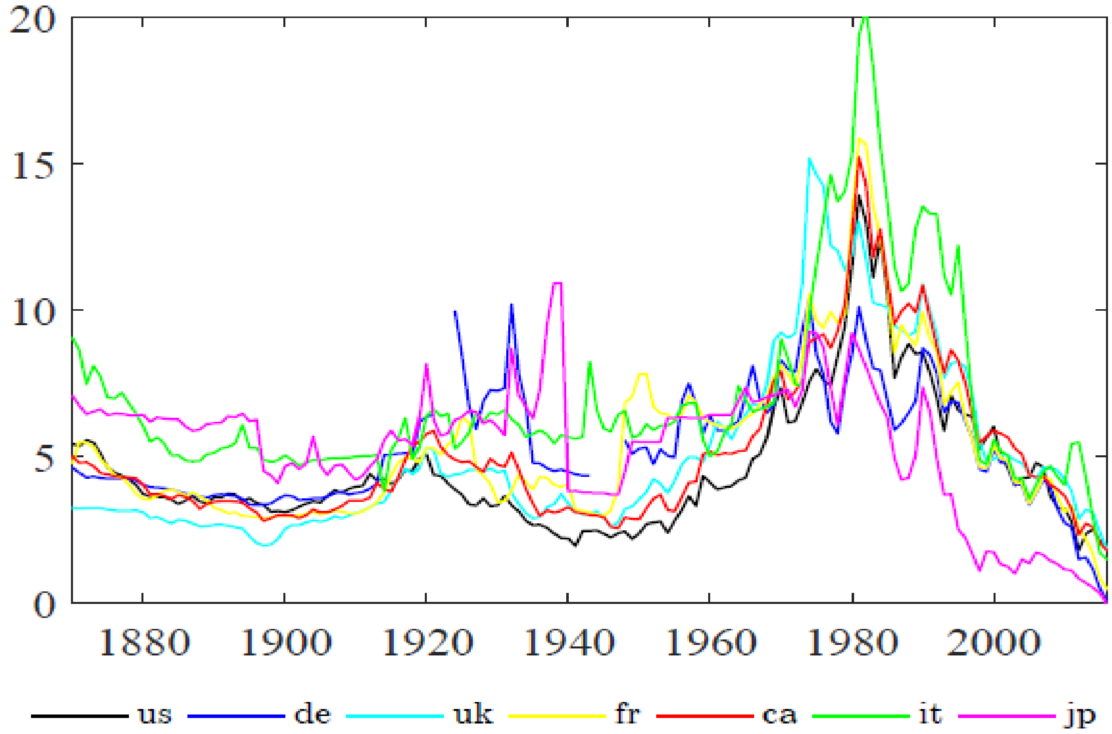


Output and spread gaps, SWFF



Output and spread gaps, CMR

10 Global trends in interest rates



- Real interest rates are at their historical low around the world.

- Secular decline? There should be a trend.

- Global phenomena? There should be a common trend across countries.

- What are the causes of the decline? Declining inflation? Low global growth? Convenience yield (Del Negro et al., 2019)?

- Del Negro et al. (2019): Use historical Jorda' et al (2017) dataset for seven countres for inflation, short and long rates.

- Estimate a trend in real rates for each country.

- Assess convergence of country specific trends.

- Study the drivers of the convergence process.

• Extent of the decline in interest rates: Holston et al. (2016); Hamilton et al., (2018); Borio et al. (2017); Fiorentini et al. (2018); Gourinchas and Rey (2016); Jorda' et al. (2017).

• Reasons:

- Convenience yield. Khrisnamurty and Vissing Jorgensen (2012), Del Negro et al. (2017)

- Saving glut: Bernanke (2005), Caballero (2010), Caballero and Farhi (2017), Caballero et al. (2016) (2017).

- Estimated model:

$$
\begin{aligned}
Y_t &= \Lambda Y_t^x + Y_t^c && (60)\\
Y_t^x &= Y_{t-1}^x + e_t && (61)\\
\phi(L) Y_t^c &= u_t && (62)
\end{aligned}
$$

Y_t is a $n \times 1$ vector. Y_t^x is a $q \times 1, q < n$, e_t, u_t iid and orthogonal.

- Format of Λ is restricted by theory

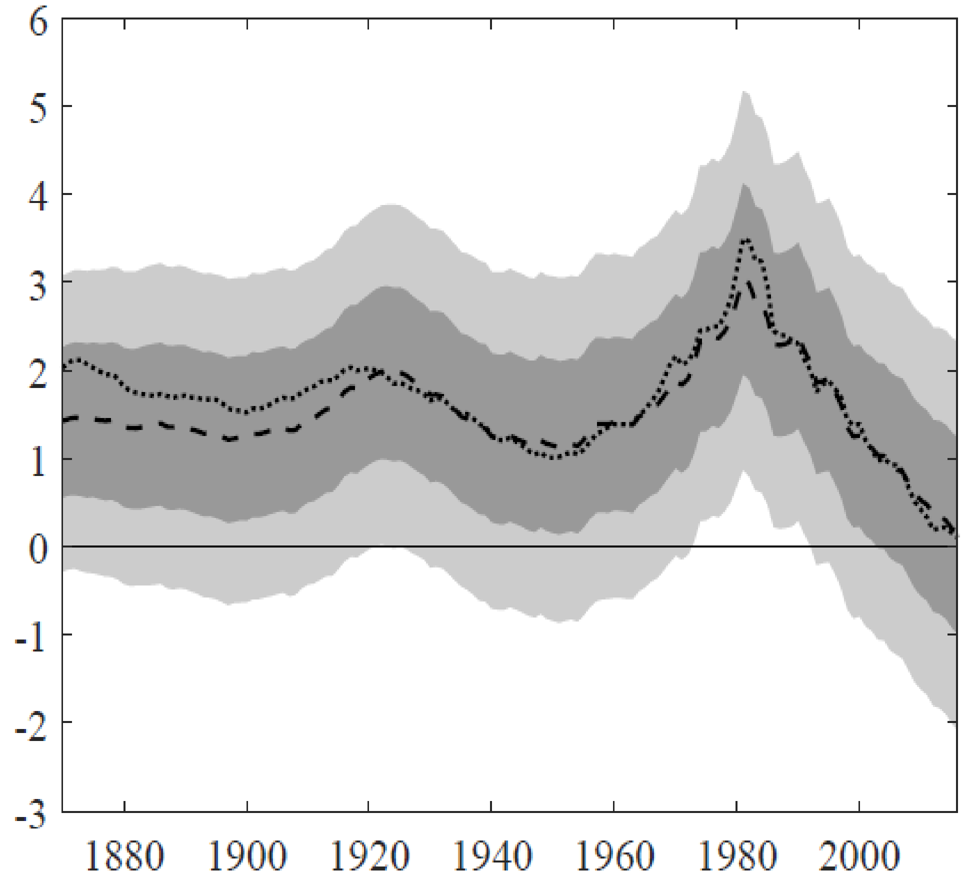
Trends

- Inflation: $\lambda_c \pi^x_t + \pi^x_{it}$. Common and idiosyncratic part.

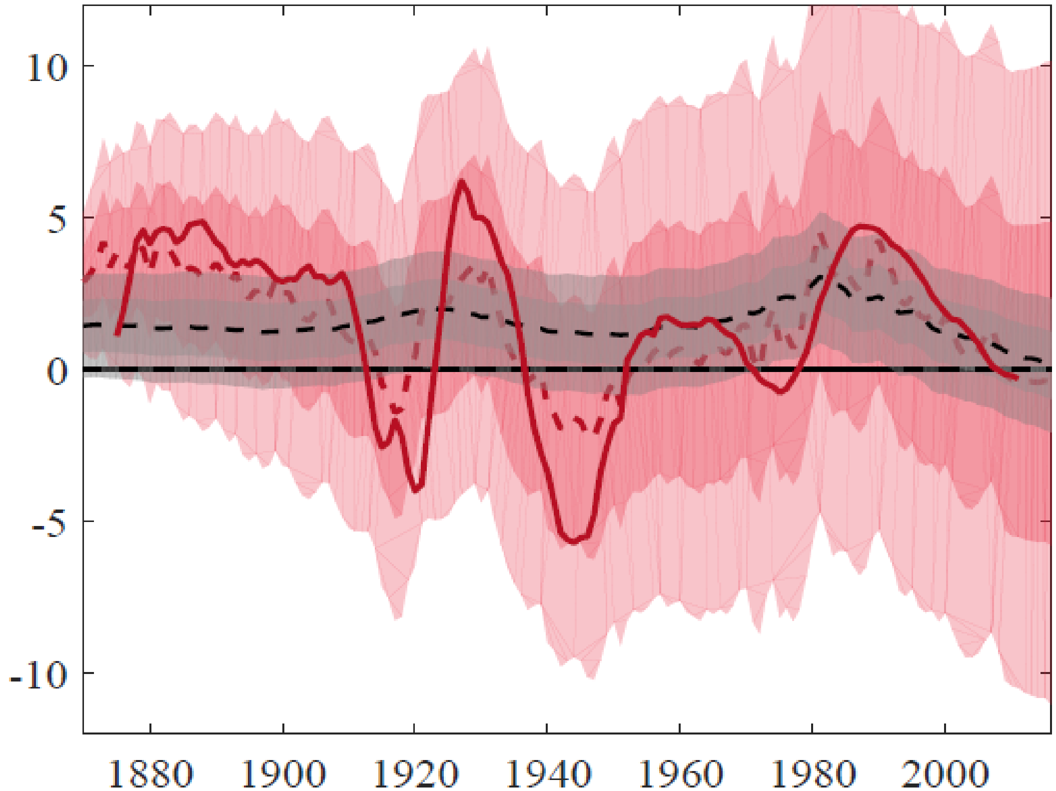
- Short rate: $\pi^x_{it} + r^x_t - cy^x_{it}$. cy^x_{it} trend in country specific convenience yield.

- World rate: $r^x_t = m^x_t + cy^x_t$. m^x_t trend in world discount factor (identified from US BAA spread), cy^x_t trend in world convenience yield.

- Long rate: $\pi^x_{it} + r^x_t - cy^x_{it} + ts^x_t + ts^x_{it}$. ts^x_t global trend in term spread; ts^x_{it} country specific trend in term spread.

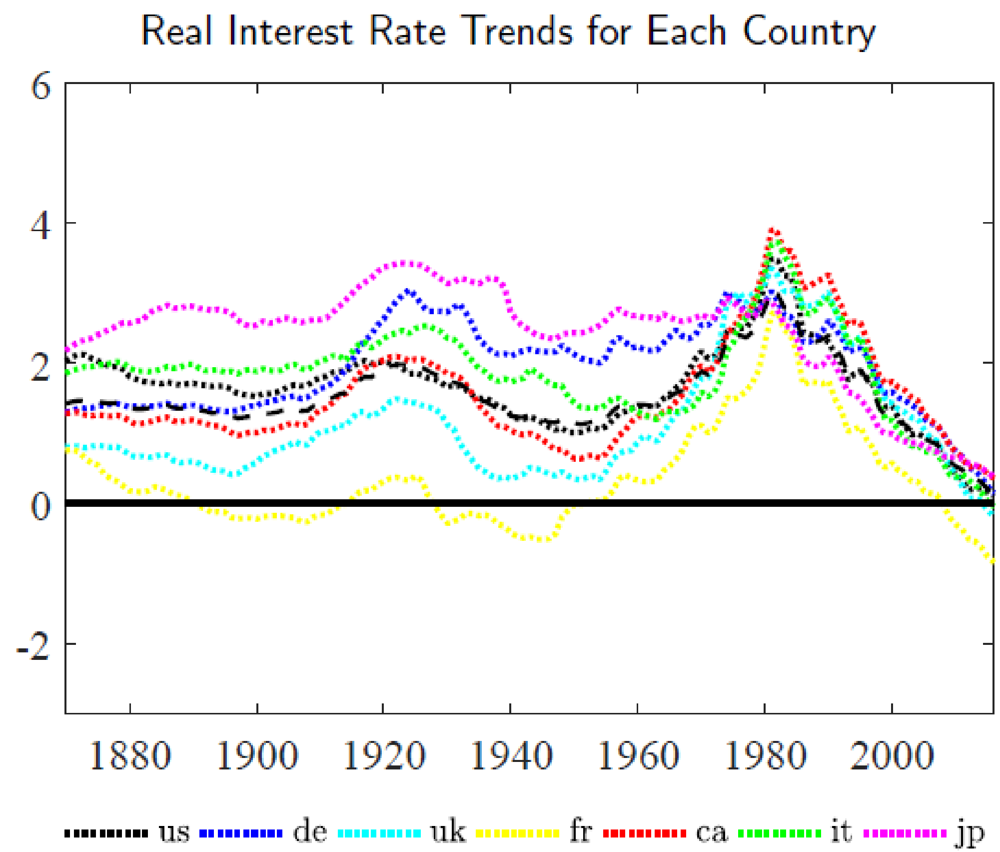


Trend in global and US real rates

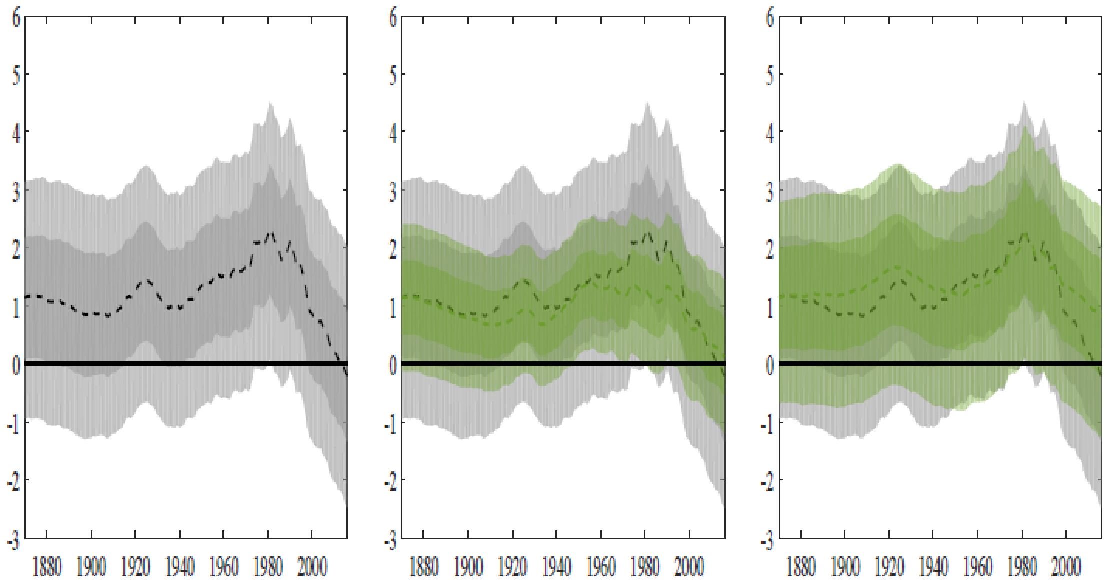


MAs of world interest rates

- Estimates are different from simple MA estimates.



- Convergence since 1980.



[spreads]

World trend, world trend and convenience yield trend, world trend and growth trend

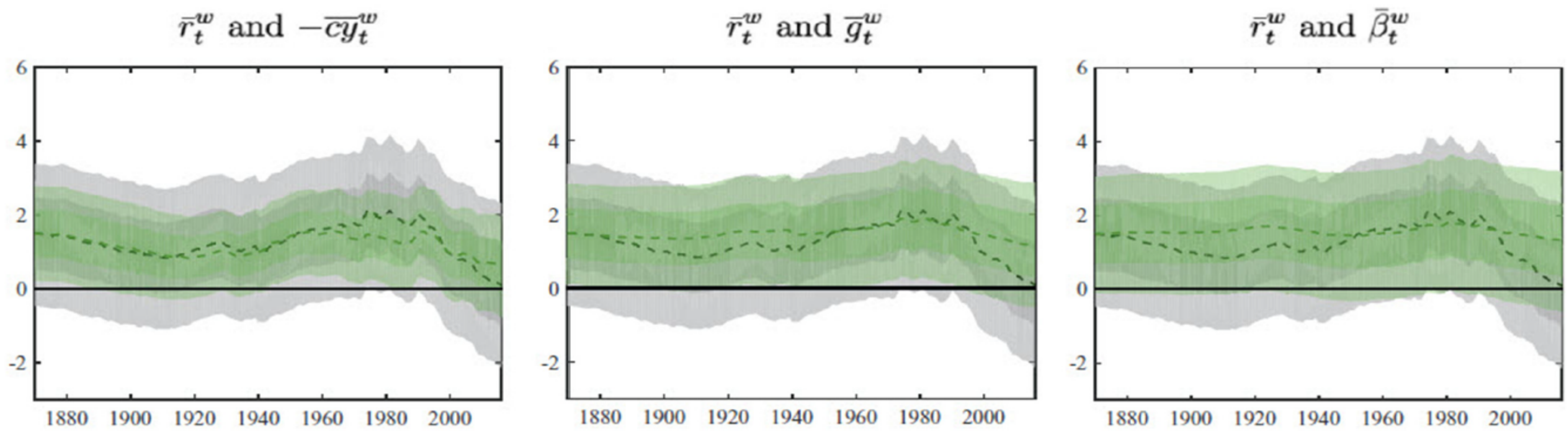


Figure 1: World rate and convenience yield trends; world rate and growth trends; world rate and beta trends

- Use consumption data to decompose $m_t^x = g_t^x + \beta_t$, growth of consumption and of the discount factor.

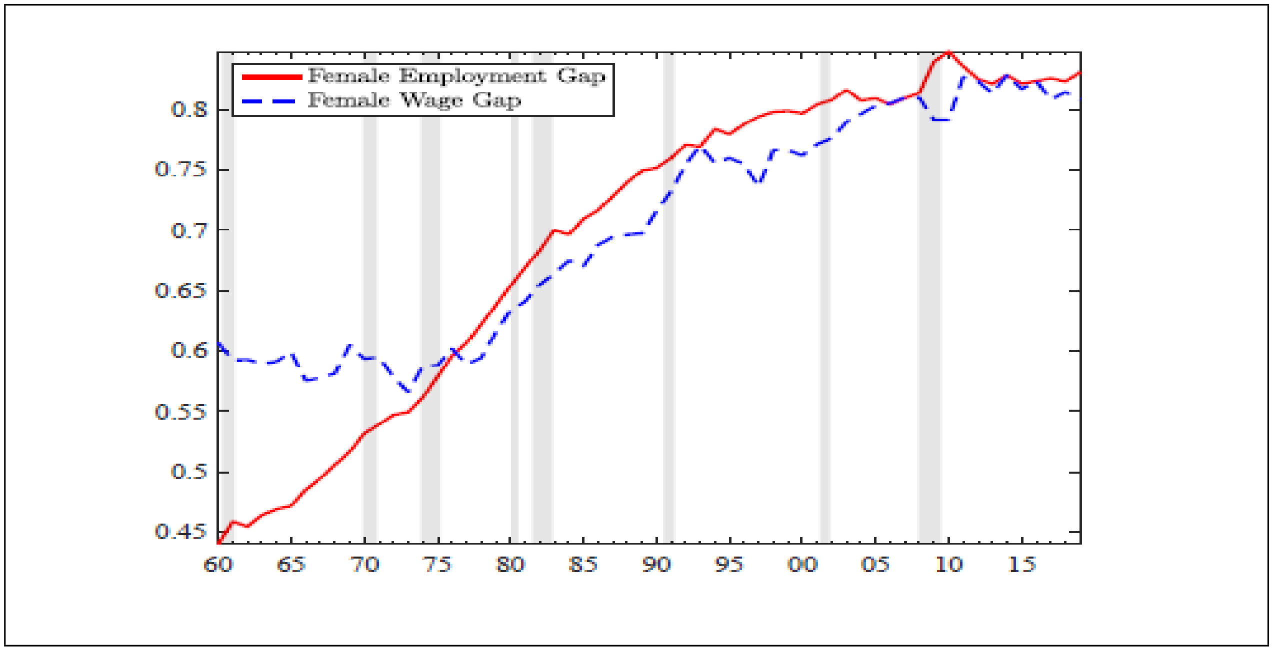
Conclusions

- Trend in world real rate declining about 200bsp in 40 years after fluctuating around 2 percent for 60 years.

- Convenience yield trend key driver in the decline, especially in the 1990.

- Lower global growth also important after 1980.

11 Gender gap trends



- What drives these gender-specific trends in the US labor market?
- How important are for the macroeconomy?

• Fukui, Nakamura and Steinsson (2020): the secular rise of female workers in US labor markets has not crowded out men.

1. Female employment trends are important for the macroeconomy;

2. Standard theoretical models, in contrast, predict substantial crowding out and consequently small macroeconomic effects;

3. This is true regardless of the underlying shocks' nature.

Bergholt, Fosso, Furlanetto (2021)

• Decompose macroeconomic and gender-specific trends into selected structural trends and quantify their role

• Disentangle two *gender-specific* trends.

Approach:

1. Retrieve theory-based identification assumptions from a standard neoclassical model with gender (version of Albanesi (2019)).

2. Decompose the data into permanent-transitory components using the model restrictions on trends (as in Del Negro et al (2019)).

Basic neoclassical model with gender

- Production structure:

$$Y_t = (A_t L_t)^{\alpha} K_{t-1}^{1-\alpha}$$

Aggregate labor L_t is a CES function of male and female labor, $L_{m,t}$ and $L_{f,t}$:

$$L_t = \left[\alpha_l \left(A_{m,t} L_{m,t}\right)^{\frac{\gamma-1}{\gamma}} + (1-\alpha_l)\left(A_{f,t} L_{f,t}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}$$

$A_{m,t}$ and $A_{f,t}$ are gender-specific productivity shocks, $\gamma > 1$ governs the degree of substitution between genders when firms produce.

- Households maximize expected lifetime welfare $\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \mathcal{U}_s$, where

$$\mathcal{U}_t = \frac{C_t^{1-\sigma}}{1-\sigma} \exp\left(-\Psi_t^{-1} \frac{(1-\sigma)\, \tilde{L}_t^{1+\varphi}}{1+\varphi}\right).$$

• Aggregate labor dis-utility $\tilde{L}_t$ is increasing in male and female labor:

$$\tilde{L}_t = \left[\left(\frac{L_{m,t}}{\Psi_{m,t}} \right)^{\frac{1+\lambda}{\lambda}} + \left(\frac{L_{f,t}}{\Psi_{f,t}} \right)^{\frac{1+\lambda}{\lambda}} \right]^{\frac{\lambda}{1+\lambda}}$$

$\Psi_{m,t}$ and $\Psi_{f,t}$ are gender-specific utility shocks, $\lambda > 0$ governs the willingness to substitute female with male labor.

• Denote the female wage gap by $w_{f,t} = \frac{W_{f,t}}{W_{m,t}}$, and the female employment gap by $l_{f,t} = \frac{L_{f,t}}{L_{m,t}}$. In the $\left(w_{f,t}, l_{f,t}\right)$-space:

- Firms' labor demand has a downward slope equal to $-\gamma$:

$$l_{f,t} = \left(\frac{1-\alpha_l}{\alpha_l}\right)^{\gamma} w_{f,t}^{-\gamma} a_{f,t}^{\gamma-1}$$

  The "ratio shock" $a_{f,t} = \frac{A_{f,t}}{A_{m,t}}$ is a demand shifter.

- Households' labor supply has an upward slope equal to λ:

$$l_{f,t} = w_{f,t}^{\lambda} \psi_{f,t}^{1+\lambda}$$

  The "ratio shock" $\psi_{f,t} = \frac{\Psi_{f,t}}{\Psi_{m,t}}$ is a supply shifter.

Analytical solutions:

$$w_{f,t} = \left(\frac{1-\alpha_l}{\alpha_l}\right)^{\frac{\gamma}{\gamma+\lambda}} a_{f,t}^{\frac{\gamma-1}{\gamma+\lambda}} \psi_{f,t}^{-\frac{1+\lambda}{\gamma+\lambda}}$$

$$l_{f,t} = \left(\frac{1-\alpha_l}{\alpha_l}\right)^{\frac{\gamma\lambda}{\gamma+\lambda}} a_{f,t}^{\frac{(\gamma-1)\lambda}{\gamma+\lambda}} \psi_{f,t}^{\frac{(1+\lambda)\gamma}{\gamma+\lambda}}$$

- "ratio" shocks' $a_{f,t}$ and $\psi_{f,t}$ affect the wage and employment gaps. "Macro" shocks A_t and Ψ_t play no role.

- Model nests Albanesi (2019) (i.e.: $\lambda = \frac{1}{\phi}$) and Fukui et al. (2020) (i.e.: $\gamma = \infty$) as special cases.

- Model-implied relationships between observables and structural trends:

$$\begin{bmatrix} \bar{W}_t \\ \bar{E}_t \\ \bar{w}_{f,t} \\ \bar{l}_{f,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \nu_{13} & \nu_{14} \\ 0 & -1 & \nu_{23} & \nu_{24} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda & -\gamma \\ & & & \end{bmatrix} \begin{bmatrix} A_t \\ \boldsymbol{\Psi}_t \\ a_{f,t} \\ \psi_{f,t} \end{bmatrix}$$

In compact form: $\bar{Y}_t = \mathcal{V} X_t$.

Let $Y_t = [W_t \;\; E_t \;\; w_{f,t} \;\; l_{f,t}]'$. Y_t can be decomposed into

$$Y_t = \hat{Y}_t + \bar{Y}_t$$

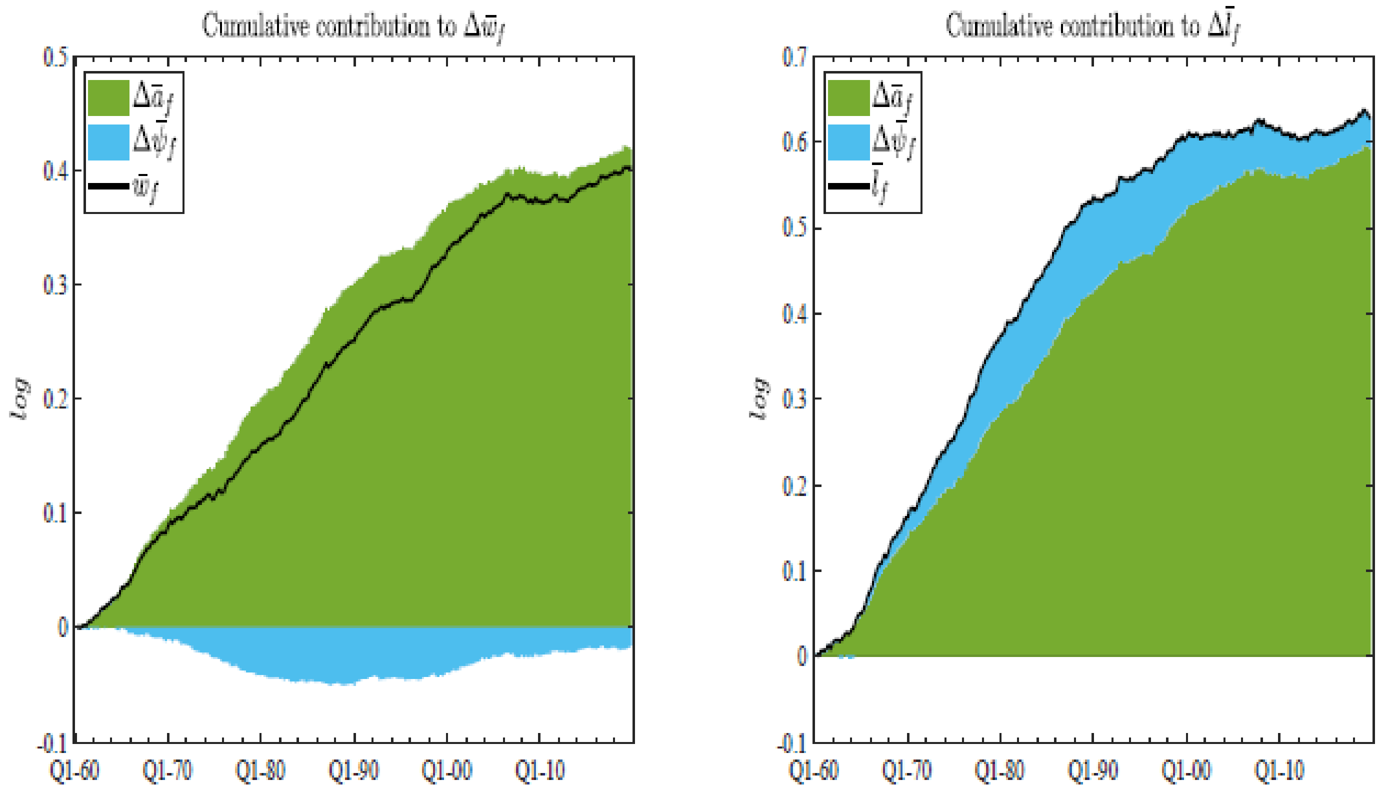
where $\bar{Y}_t = \mathcal{V} X_t$. Structural trends are independent random walks:

$$X_t = c + X_{t-1} + u_t$$

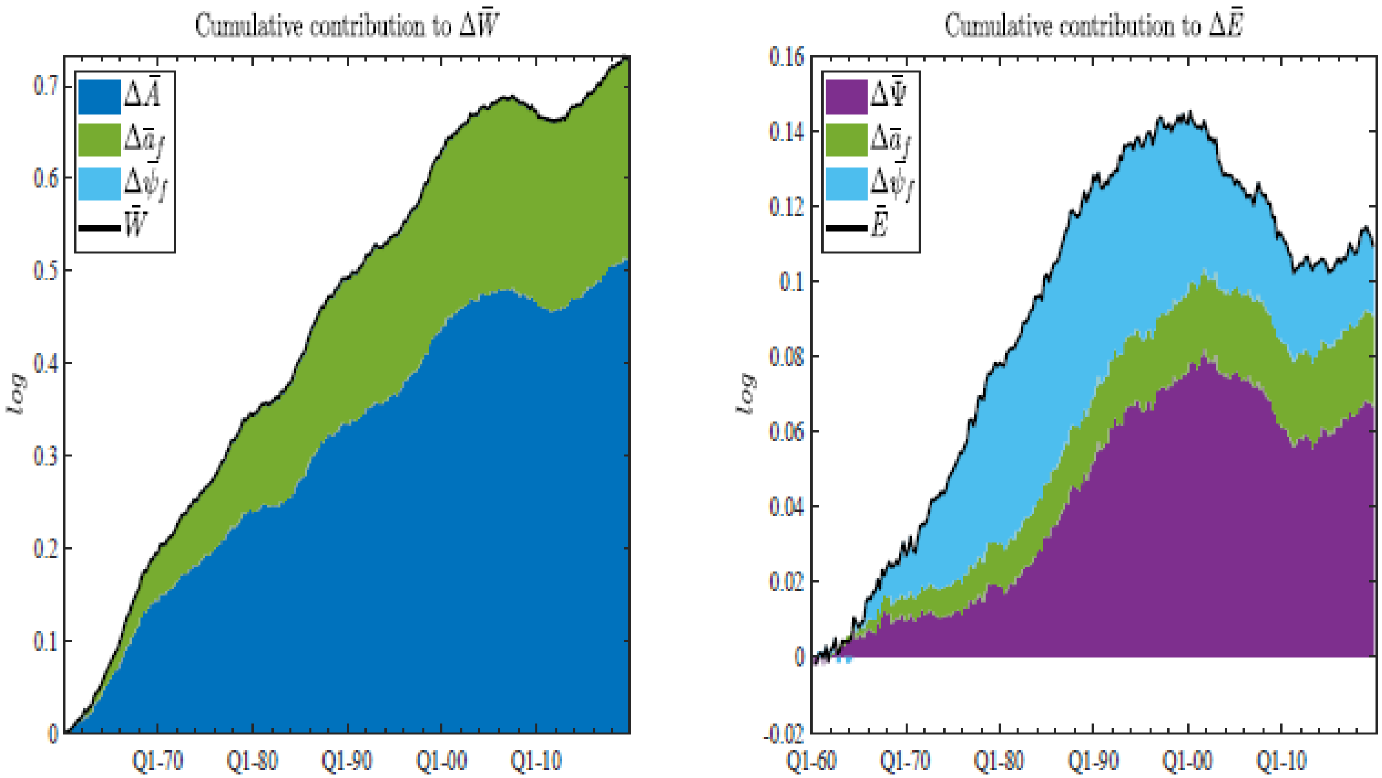
The transitory component block is a reduced-form VAR:

$$\hat{Y}_t = A\hat{Y}_{t-1} + e_t$$

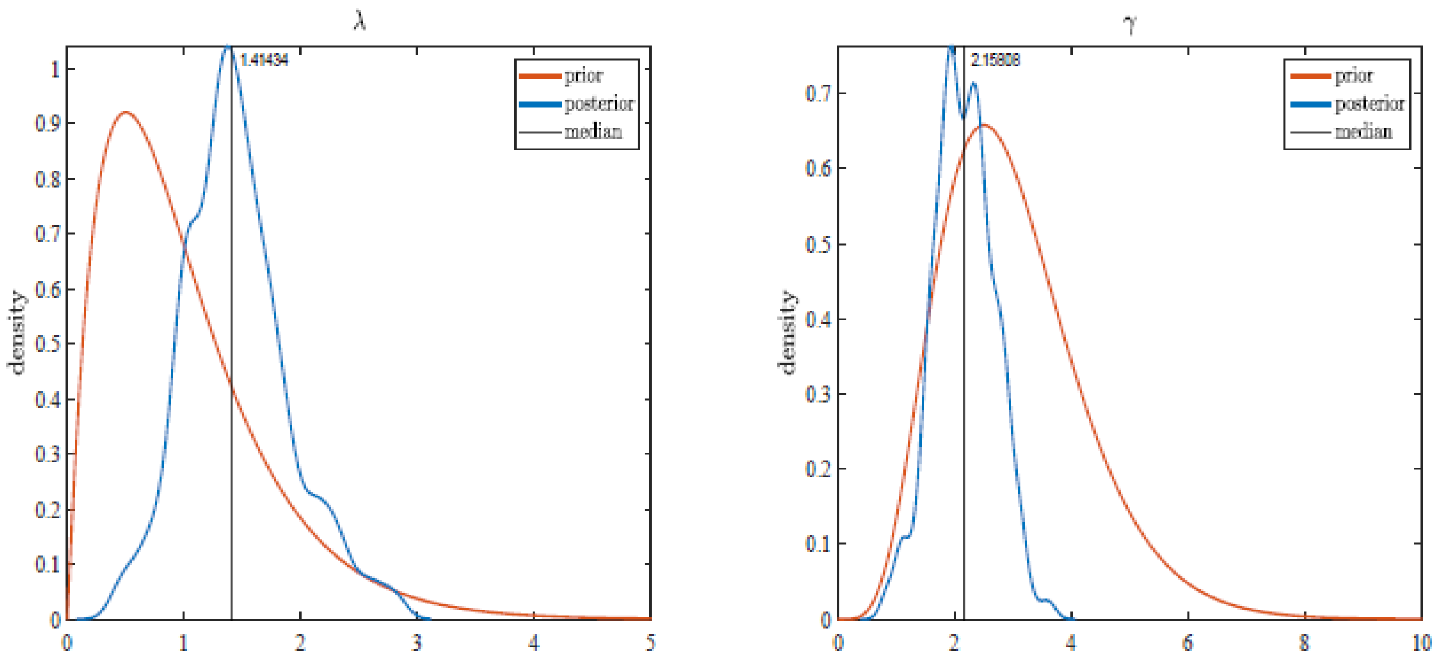
Assumptions: u_t diagonal; $cov(u_t, e_t) = 0$.



Cumulative contribution to $\Delta\bar{w}_f$
$\Delta\bar{a}_f$
$\Delta\bar{\psi}_f$
$\bar{w}_f$
log
0.5
0.4
0.3
0.2
0.1
0
-0.1
Q1-60
Q1-70
Q1-80
Q1-90
Q1-00
Q1-10
Cumulative contribution to $\Delta\bar{l}_f$
$\Delta\bar{a}_f$
$\Delta\bar{\psi}_f$
$\bar{l}_f$
log
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
Q1-60
Q1-70
Q1-80
Q1-90
Q1-00
Q1-10


Cumulative contribution to $\Delta\bar{W}$
$\Delta\bar{A}$
$\Delta\bar{a}_f$
$\Delta\bar{\psi}_f$
$\bar{W}$
log
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Q1-70
Q1-80
Q1-90
Q1-00
Q1-10
Cumulative contribution to $\Delta\bar{E}$
$\Delta\bar{\Psi}$
$\Delta\bar{a}_f$
$\Delta\bar{\psi}_f$
$\bar{E}$
log
-0.02
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
Q1-60
Q1-70
Q1-80
Q1-90
Q1-00
Q1-10

Estimates of λ, γ



Conclusions:

- Both gender-specific trends are important for the macroeconomy.

- Compared to Fukui, et el. (2020), the " type" of gender-specific trend is important in determining the degree of crowding-out:

1. $\psi_{f,t}$ generates a small crowding-out;

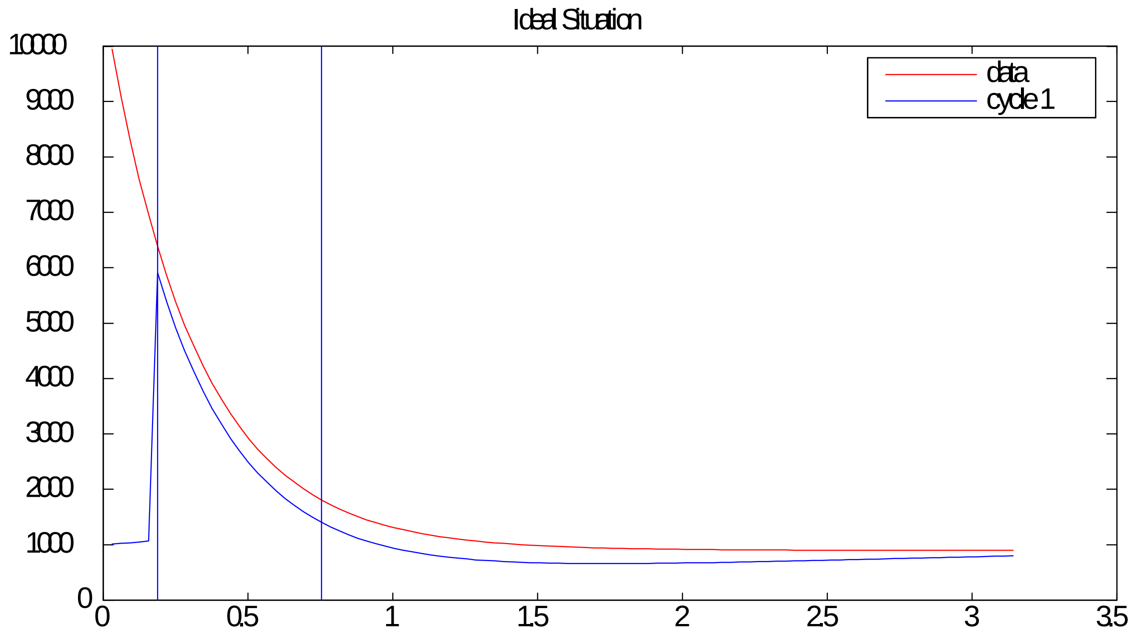
2. $a_{f,t}$ yields a large crowding-out.

12 Fitting structural models to filtered data

• Statistical filtering: Find B_j such that y_t^f has $\mathcal{S}(\omega_\tau) \neq 0$ only for certain $\omega_\tau \in (\omega_1, \omega_2)$.

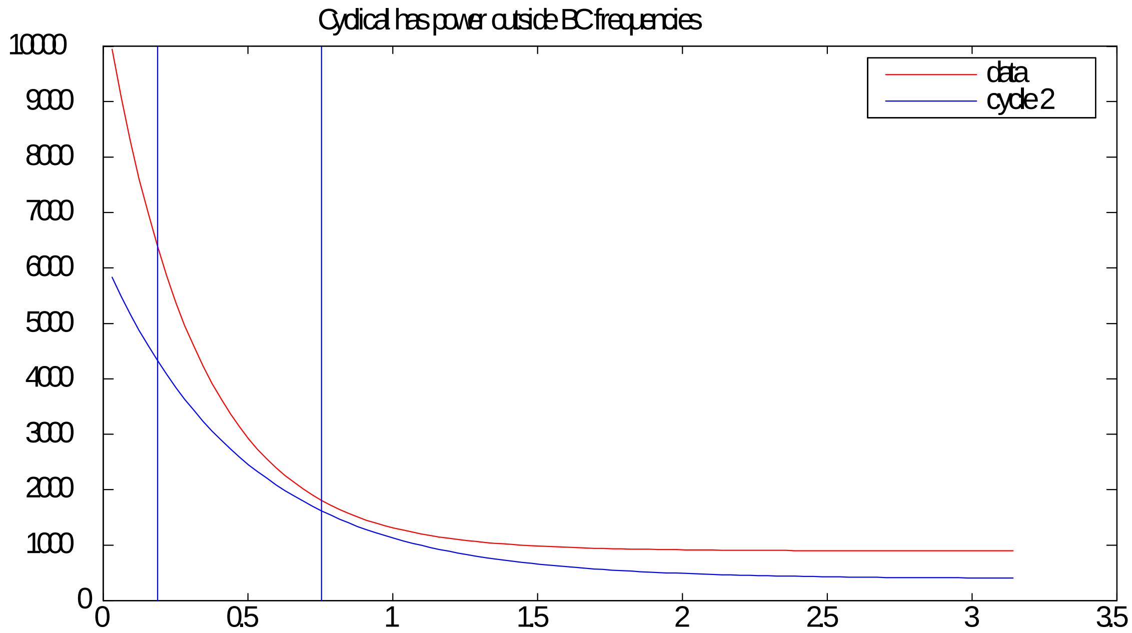
• Economic filtering: $y_t = y_{1t} + y_{2t} = A(L)e_t + B(L)u_t$, where e_t are permanent shocks, u_t are transitory shocks or e_t are disturbances entering the potential and u_t disturbances entering the gap. Note u_t and e_t may overlap.

• In general, $y_{2t} \neq y_t^f$ since y_{1t}, y_{2t} have $\mathcal{S}(\omega_\tau) \neq 0$ for all $\omega_\tau \in (0, \pi)$.



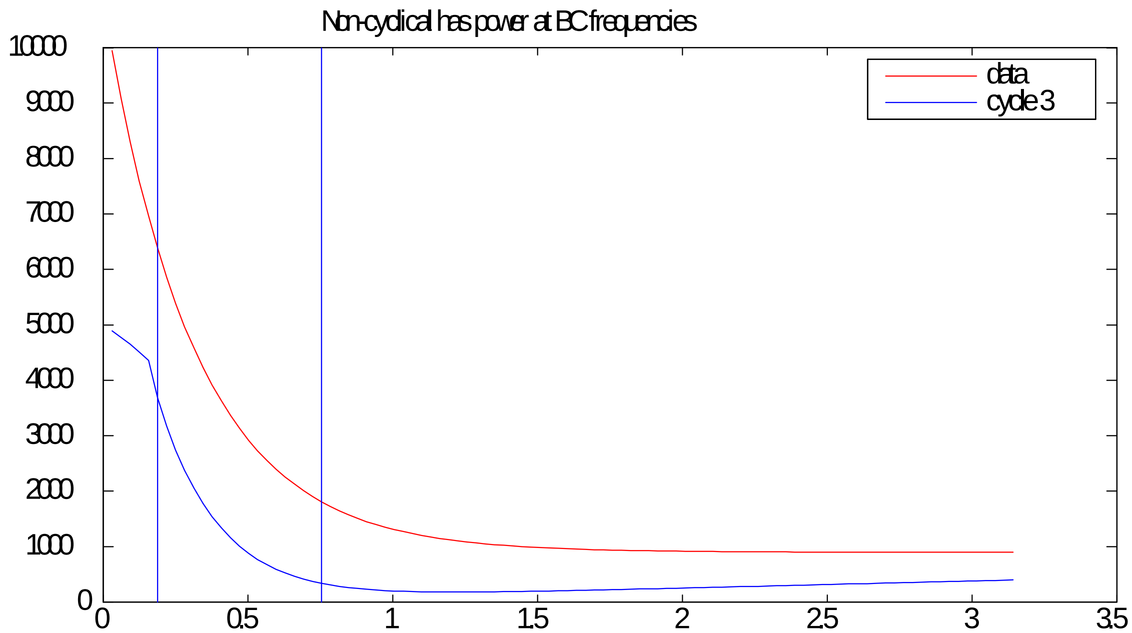
Ideal case

- (Cyclical) model has most of the variability located at business cycle frequencies. Statistical filtering would ok.



Realistic case

• If (cyclical) model is driven by persistent AR(1) shocks, lots of variability in the low frequencies. Filtering throws away information.



General case

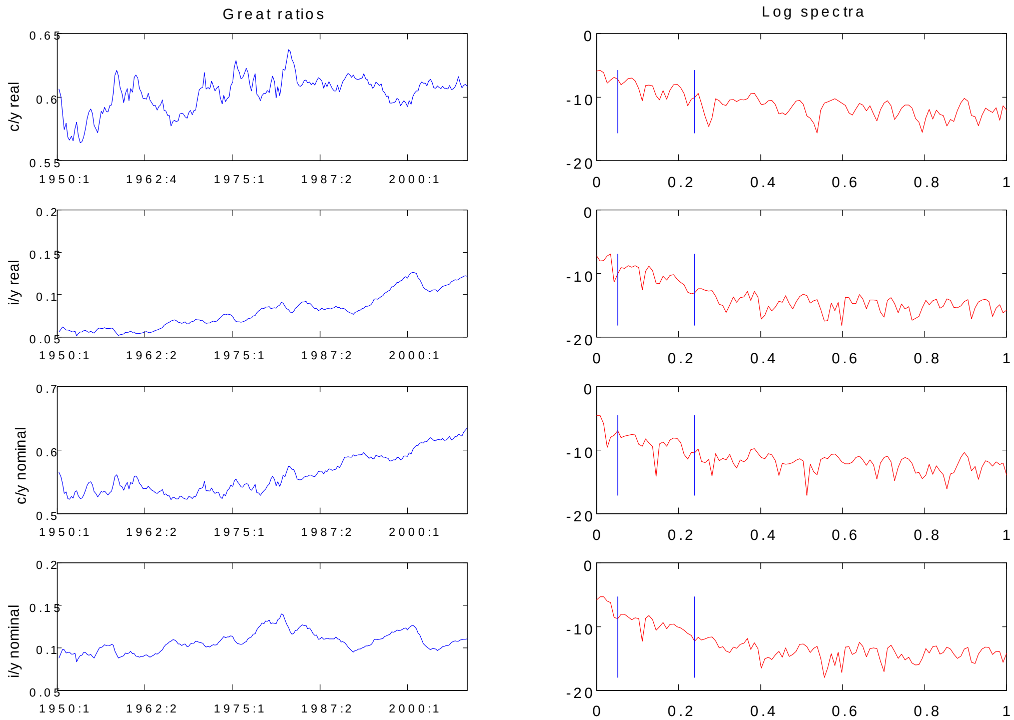
• If (cyclical) model is driven by persistent AR(1) shocks, and permanent shocks are cyclical, filtering is distortive. Different filters will give different results.

- Typical solution: Build in a trend in a (cyclical) model. Transform the data with a model consistent approach. Problems:

- Models with trends (in technology) imply balanced growth path. Typically violated in the data.

- Where do we put the trend (e.g. technology or preferences) matters for estimates of the structural parameters - nuisance parameter problem.

- Should we use a unit root or trend stationary specification?



Real and nominal Great ratios in US, 1950-2008.

Filter	LT		HP		FOD		BP		Ratio	
	Median	(s.d.)	Median	(s.d.)	Median	(s.d.)	Median	(s.d.)	Median	(s.d.)
σ_c	2.19	(0.10)	2.25	(0.12)	2.54	(0.16)	2.21	(0.10)	1.69	(0.11)
σ_n	1.79	(0.08)	1.57	(0.10)	1.90	(0.19)	1.78	(0.08)	2.16	(0.10)
h	0.67	(0.01)	0.59	(0.03)	0.44	(0.03)	0.66	(0.02)	0.64	(0.02)
α	0.17	(0.03)	0.12	(0.02)	0.12	(0.03)	0.16	(0.02)	0.13	(0.02)
ϵ	3.90	(0.12)	4.27	(0.14)	2.92	(0.11)	3.72	(0.05)	4.09	(0.12)
ρ_r	0.16	(0.04)	0.52	(0.04)	0.22	(0.06)	0.49	(0.04)	0.22	(0.04)
ρ_π	1.36	(0.08)	1.67	(0.04)	1.74	(0.05)	1.77	(0.08)	1.71	(0.05)
ρ_y	-0.15	(0.02)	0.35	(0.06)	0.13	(0.07)	0.44	(0.05)	-0.02	(0.01)
ζ_p	0.81	(0.01)	0.60	(0.03)	0.33	(0.03)	0.56	(0.03)	0.81	(0.01)
ρ_χ	0.76	(0.02)	0.59	(0.04)	0.29	(0.04)	0.82	(0.03)	0.82	(0.02)
ρ_z	0.96	(0.01)	0.54	(0.05)	0.87	(0.05)	0.46	(0.05)	0.92	(0.01)
σ_χ	0.23	(0.04)	0.37	(0.05)	0.23	(0.04)	0.20	(0.03)	0.95	(0.16)
σ_z	0.12	(0.02)	0.08	(0.01)	0.09	(0.01)	0.09	(0.01)	0.08	(0.01)
σ_{mp}	0.11	(0.01)	0.08	(0.01)	0.12	(0.02)	0.08	(0.01)	0.12	(0.01)
σ_μ	30.54	(1.17)	1.01	(0.)	0.16	(0.03)	0.63	(0.21)	34.70	(1.04)

Posterior estimates NK model. For LT, HP, FOD and BP real variables detrended, nominal demeaned. For Ratio, real variables are in terms of hours,all variables demeaned.

Which column should be trusted?

Alternatives:

- Use a data rich environment (Canova and Ferroni, 2011).

Let y_t^i be the actual data filtered with method $i = 1, 2, ..., I$ and $y_t^d = [y_t^1, y_t^2, \ldots]$. Assume:

$$y_t^d = \lambda_0 + \lambda_1 y_t(\theta) + u_t \tag{63}$$

where $\lambda_j, j = 0, 1$ are matrices of parameters, measuring bias and correlation between data and model based quantities, u_t measurement errors and θ the structural parameters.

- Factor model setup a-la Boivin and Giannoni (2005).

- Can jointly estimate θ and λ's.

- Same interpretation as GMM with many instruments.

• Bridge cyclical model and the raw data with a flexible specification (Canova, 2014).

$$y_t^d = c + y_t^T + y_t^m(\theta) + u_t \tag{64}$$

where $y_t^d \equiv \tilde{y}_t^d - E(\tilde{y}_t^d)$ the log demeaned vector of observables, $c = \bar{y} - E(\tilde{y}_t^d)$, y_t^T is the non-cyclical component, $y_t^m(\theta) \equiv S[y_t, x_t]'$, where S is a selection matrix, is the model based- cyclical component (the solution of a DSGE model), u_t is a iid $(0, \Sigma_u)$ (measurement) noise, $y_t^T, y_t^m(\theta)$ and u_t are mutually orthogonal.

• Non cyclical component

$$y_t^T = y_{t-1}^T + \bar{y}_{t-1} + e_t \qquad e_t \sim \ iid\ (0, \Sigma_e^2) \tag{65}$$

$$\bar{y}_t = \bar{y}_{t-1} + v_t \qquad v_t \sim \ iid\ (0, \Sigma_v^2) \tag{66}$$

- $\Sigma_v^2 > 0$ and $\Sigma_e^2 = 0$, y_t^T is a vector of I(2)processes.

- $\Sigma_v^2 = 0$, and $\Sigma_e^2 > 0$, y_t^T is a vector of I(1) processes.

- $\Sigma_v^2 = \Sigma_e^2 = 0$, y_t^T is deterministic.

- $\Sigma_v^2 > 0$ and $\Sigma_e^2 > 0$ and $\frac{\sigma_{v_i}^2}{\sigma_{e_i}^2}$ is large, y_{it}^T is "smooth" and nonlinear (as in HP).

- Jointly estimate structural θ and non-structural parameters (joint estimation and filtering)

- Equivalent to assume a rich measurement error structure.

How does the procedure do in a simple experimental design?

	True	Small variance Median	(s.e)	True	Large variance Median	(s.e)
σ_c	3.00	3.68	(0.40)	3.00	3.26	(0.29)
σ_n	0.70	0.54	(0.14)	0.70	0.80	(0.13)
h	0.70	0.55	(0.04)	0.70	0.77	(0.04)
α	0.60	0.19	(0.03)	0.60	0.41	(0.04)
ϵ	7.00	6.19	(0.07)	7.00	6.95	(0.09)
ρ_r	0.20	0.16	(0.04)	0.24	0.31	(0.04)
ρ_π	1.30	1.30	(0.04)	1.30	1.25	(0.03)
ρ_y	0.05	0.07	(0.03)	0.05	0.08	(0.10)
ζ_p	0.80	0.78	(0.04)	0.80	0.72	(0.02)
ρ_χ	0.50	0.53	(0.04)	0.50	0.69	(0.05)
ρ_z	0.80	0.71	(0.03)	0.80	0.90	(0.03)
σ_χ	0.011	0.012	(0.0003)	0.011	0.012	(0.0003)
σ_z	0.005	0.006	(0.0001)	0.005	0.007	(0.0001)
σ_{mp}	0.001	0.002	(0.0004)	0.001	0.002	(0.0004)
σ_μ	0.206	0.158	(0.0006)	0.206	0.1273	(0.0004)
σ_χ^{nc}	0.02			0.23		

Parameters estimates using flexible specification. σ_χ^{nc} is the standard error of the shock to the non-cyclical component.

Appendix: Other elements of Spectral Analysis

• The periodogram of y_t is $Pe(\omega) = \sum_\tau \widehat{ACF}(\tau)e^{-i\omega\tau}$ where $\widehat{ACF}(\tau) = \frac{1}{T}\sum_t(y_t - \bar{y})(y_{t-\tau} - \bar{y})$ and $\bar{y} = \frac{1}{T}\sum_t y_t$.

• Periodogram is inconsistent estimator of the spectrum. Periodogram consistently estimate only an average of the frequencies of the spectrum. For consistency need to "smooth" periodogram with a filter (kernel).

• A filter is a kernel (denoted by $\mathcal{K}_T(\omega)$) if, as $T \to \infty$, $\mathcal{K}(\omega_\tau) = 1, \quad$ for $\omega_\tau = \omega$ and $\mathcal{K}(\omega_\tau) = 0$ otherwise.

• Kernels eliminate bias in $\widehat{ACF}(\tau)$. Since as $T \to \infty$ bias disappears, wants kernels to converge to $\delta-$function as $T \to \infty$.

Two useful Kernels.

• Bartlett kernel: tent shaped, width $2J(T)$; $\mathcal{K}(\omega_j) = 1 - \frac{|\omega|}{J(T)}$. $J(T)$ chosen so that $\frac{J(T)}{T} \to 0$ as $T \to \infty$.

• Quadratic spectral kernel: wave with infinite loops;
$\mathcal{K}(\omega_j) = \frac{25}{12\pi^2 j^2} \frac{\sin(6\pi j/5)}{(6\pi j)/5} - cos(\frac{6\pi j}{5})$.