

# **PHD21 Computational methods: Assignments**

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## **Contents**

# 1 Assignment 1

## Questions 1 through 4

### Question:

1. Label and interpret the model ingredients properly.
2. Characterise the individual labor supply curve.
3. Characterise the aggregate labor supply curve.
4. Characterise the aggregate labor demand curve

**Step 1A:** Working households  $\rightarrow$  intensive margin labour supply. Start with the working household's problem.

$$(1.1) \quad W(a, z) = \max_{c, n, a'} \left\{ \log(c) - \eta \frac{1}{1 + \frac{1}{\chi}} n^{1 + \frac{1}{\chi}} + \beta v(a') \right\}$$

s.t.  $c + a' = zw(1 - \tau)n + a(1 + r(1 - \tau)) + T$

Construct the Lagrangian, assuming that  $v(a') = \log(a')$ :

$$(1.2) \quad \mathcal{L} = \log(c) - \eta \frac{1}{1 + \frac{1}{\chi}} n^{1 + \frac{1}{\chi}} + \beta \log(a') + \lambda [zw(1 - \tau)n + a(1 + r(1 - \tau)) + T - c - a'].$$

The first-order conditions are as follows:

$$(1.3a) \quad \mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = \frac{1}{c},$$

$$(1.3b) \quad \mathcal{L}_n = -\eta n^{\frac{1}{\chi}} + \lambda zw(1 - \tau) = 0 \implies \eta n^{\frac{1}{\chi}} = \frac{zw(1 - \tau)}{c},$$

and

$$(1.3c) \quad \mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c.$$

Combine the results of Equations (??) and (??) with the budget constraint to arrive at the Euler equation:

$$(1.4a) \quad \underbrace{c + a'}_{\text{Use (??)}} = \underbrace{zw(1 - \tau)}_{\text{Use (??)}} + a(1 + r(1 - \tau)) + T,$$

$$\boxed{(WH-ILS) \quad c(1 + \beta) = zw(1 - \tau) \left( \frac{zw(1 - \tau)}{\eta c} \right)^{\chi} + a(1 + r(1 - \tau)) + T.}$$

Equation (??) governs the **intensive labour supply of a working household**,  $c_w^*(a, z)$  is their consumption level.

Further, notice that  $c_w^*(a, z)$  increases in  $z$ :

$$(1.5a) \quad \frac{\partial c_w^*(a, z)}{\partial z}(1 + \beta) = \underbrace{[w(1 - \tau)\eta^{-1}]^{1+\chi}}_{\equiv \theta > 0} \times \frac{(1 + \chi)z^\chi c_w^*(a, z) - z^\chi \frac{\partial c_w^*(a, z)}{\partial z}}{c_w^*(a, z)^2} \implies$$

$$(1.5b) \quad \frac{\partial c_w^*(a, z)}{\partial z} \left( 1 + \beta + \theta \frac{z^\chi}{c_w^*(a, z)^2} \right) = \frac{\theta(1 + \chi)z^\chi}{c_w^*(a, z)} \implies$$

$$(1.5c) \quad \frac{\partial c_w^*(a, z)}{\partial z} = \frac{\frac{\theta(1 + \chi)z^\chi}{c_w^*(a, z)}}{1 + \beta + \theta \frac{z^\chi}{c_w^*(a, z)^2}} > 0.$$

Also, differentiate Equation (??):

$$(1.6a) \quad \eta\chi^{-1}n^{\frac{1}{\chi}-1}\frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} = -zw(1-\tau)c_w^*(a, z)^{-2} \implies$$

$$(1.6b) \quad \frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} = \frac{-\chi zw(1-\tau)}{\eta n^{\frac{1}{\chi}-1}c_w^*(a, z)^2}.$$

Using the combination of the Envelope Theorem and chain rule, this implies that, at the optimal consumption-hours bundle, the household sees:

$$(1.7a) \quad \frac{\partial W(a, z)}{\partial z} = \frac{1}{c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} - \eta n^{\frac{1}{\chi}} \frac{\partial n^*(a, z)}{\partial z} \implies$$

$$(1.7b) \quad \frac{\partial W(a, z)}{\partial z} = \frac{1}{c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} - \eta n^{\frac{1}{\chi}} \frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} \implies$$

$$(1.7c) \quad \boxed{\frac{\partial W(a, z)}{\partial z} > 0.}$$

Equation (??) effectively means that  $W(a, z)$  monotonically increases in  $z$ .

**Step 1B:** Non-working households. The problem is similar here, apart from the labour first order condition. Skipping the Lagrangian setup, I arrive at the following first-order conditions:

$$(1.8a) \quad \mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = \frac{1}{c}$$

and

$$(1.8b) \quad \mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c,$$

both of which are identical to what we see for the working household. Combining it with the budget constraint, we obtain the consumption function for the non-working household:

$$(1.9) \quad \boxed{c_{nw}^*(a) = \frac{b + a(1 + r(1 - \tau)) + T}{1 + \beta}.}$$

This effectively implies that:

$$(1.10) \quad \boxed{\frac{\partial N(a, z)}{\partial z} = 0.}$$

**Step 2:** Extensive margin labour supply decision. Household  $(a, z)$  enters the labour market when:

$$(1.11) \quad \mathbf{I}_n(a, z) = \begin{cases} 1 & \text{if } W(a, z) \geq N(a, z) \\ 0 & \text{if } W(a, z) < N(a, z), \end{cases}$$

where  $W(\cdot, \cdot)$  and  $N(\cdot, \cdot)$  are the value functions of working and not working, respectively.

Even abstracting from the Unique Point Theorem, we can see that if there exists  $z^*$  such that:

$$(1.12) \quad W(a, z^*) = N(a, z^*)$$

then for  $z > z^*$ , we have:

$$(1.13) \quad \mathbf{I}_n(a, z) = 1.$$

This will come handy while numerically solving the model.

**Step 3:** Aggregate labour supply. Assuming that  $\Phi(a, z)$  is the joint distribution of ex-ante wealth and productivity, the **aggregate labour supply** is:

$$(1.14) \quad L^S = \int \mathbf{I}_n(a, z) h(a, z) d\Phi(a, z)$$

**Step 4A:** Aggregate labour demand. We abstract from the capital markets, which makes the representative firm's problem near-trivial:

$$(1.15a) \quad Y = \max_L \{ AK^\alpha L^{1-\alpha} - wL - rK \} \implies$$

$$(1.15b) \quad w = A(1-\alpha) \left( \frac{L^D}{K} \right)^{-\alpha},$$

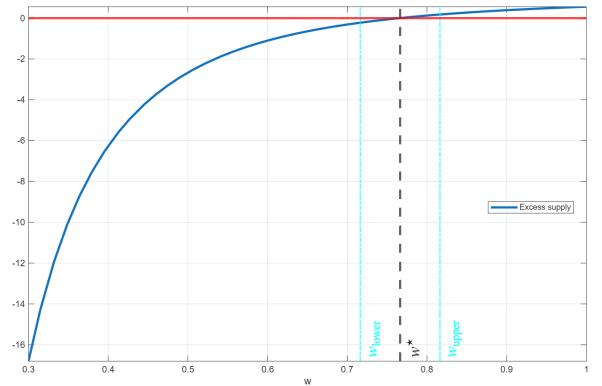
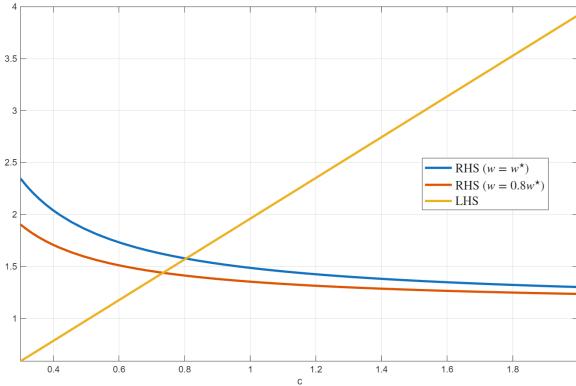
or, putting  $L^D$  on the LHS:

$$(1.15c) \quad L^D = \left( \frac{(1-\alpha)A}{w} \right)^{\frac{1}{\alpha}} K.$$

Figure 1.1: Household block of the model.

(a) Euler equation & consumption decision.

(b) Excess labour supply on the aggregate level.



## Question 5

**Question:** Suppose the following parameter levels:

$$a = 1, \quad \alpha = 0.3, \quad \tau = 0.15, \quad \bar{z} = 1, \quad A = 1, \quad r = 0.04, \quad \beta = 0.96$$

Define and characterize the stationary recursive competitive equilibrium.

**Step 1:** Parameters left.  $\Xi$  represents the original vector of parameters:

$$(1.16) \quad \Xi = (a, \eta, \xi, \tau, b, \beta, \sigma_z, A, \alpha, r)^T.$$

Given the pre-specified parameters, the unknown ones are:

$$(1.17) \quad \hat{\Xi} = (\eta, b, \sigma_z, \chi)^T.$$

In my further work, I already use the calibrated parameters (see question 8).

**Step 2:** Equilibrium. Given the distribution of labour productivity,  $\Phi$ , a set of functions  $\{n, c, a', z^*, L, w, T\}$  is a **stationary competitive equilibrium** if

1.  $(n, c, a', z^*)$  solves the household's problem.
2.  $(K, L)$  solves the production sector's problem.
3. The labour and capital markets clear.

**Step 3:** Equilibrium algorithm. I set out the **algorithm used to compute the equilibrium** given a set of parameters.

1. Guess  $(w_0, T_0)$ .
2. Compute individual decisions.

- `fnIntensiveLabourSupply` computes the **intensive labour supply for a working household**. The following equations flesh out the approach leveraging concavity of Equation (??). Start at the initial consumption guess,  $c_0$ . The code follows the logic fleshed out by the equations below:

$$(1.18a) \quad RHS(c) \equiv zw(1 - \tau) \left( \frac{zw(1 - \tau)}{\eta c} \right)^\chi + a(1 + r(1 - \tau)) + T$$

$$(1.18b) \quad LHS(c) \equiv (1 + \beta)c \implies$$

$$(1.18c) \quad c_1 = \frac{RHS(c_0)}{1 + \beta}$$

and

$$(1.18d) \quad \epsilon_n \equiv c_n - c_{n-1} \implies$$

$$(1.18e) \quad \epsilon_1 = c_1 - c_0.$$

If it's above the tolerance level, then repeat until it works:

$$(1.18f) \quad c_n = \frac{RHS(c_{n-1})}{1 + \beta} \implies$$

$$(1.18g) \quad \epsilon_n = c_n - c_{n-1}.$$

This approach computes the individually optimal values of consumption and labour market participation,  $(c_w, n_w)$ , provided the household chooses to work.

- `fnExtensiveLabourSupply` determines if household  $(a, z)$  chooses to work based on Equation (??).
3. Aggregate all labour supply decisions (`fnAggregateLabourSupply`) and compare them with the aggregate labour demand. `fnSolvePrices` iterates  $w$  and  $T$  until both clear the labour market.
    - One way of doing that is following the same method as for `fnIntensiveLabourSupply`, with Equations (??) and (??) used to compute labour supply and demand, respectively.
    - Another method is to use bisection, in `fnSolvePricesBisection`. As illustrated in Figure ??, the method is based on creating a grid for different wage values, finding the negative value closest to 0,  $w_{lower}$ , and taking the weighted average of  $w_{lower}$  and  $w_{upper}$  (the next value in the grid). **Note:** I use the “naive” approach in my code, as bisection seems to produce a
  4. If the error is too large, update  $(w_n, T_n)$  and iterate until convergence.

## Question 6

**Question:** Visualize the aggregate supply and demand curves in the labor market.

## Question 7

**Question:** Visualize the comparative statics of the wage with respect to the change in  $A$ .

## Question 8

**Question:** Estimate parameters  $(\eta, b, \chi, \sigma_z)$  to match the following hypothetical moments in general equilibrium:

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## 2 Assignment 2

### Questions 1 through 3

**Question:**

1. Label and interpret the model ingredients properly.
2. Discretise the idiosyncratic productivity process by the Tauchen method using 2 grid points on one standard deviation range.
3. Characterise the individual (probabilistic) labour supply decision analytically.

The value functions are:

$$(2.1a) \quad V_t(a, h, z) = \int \max \{W_t(a, h, z) + \xi_{Wt}, N_t(a, h, z) + \xi_{Nt}\} dG(\xi_{st}; \xi)$$

and

$$(2.1b) \quad S_t(a, h, z) = \int \max \{\varphi W_t(a, h, z) + (1 - \varphi) N_t(a, h, z) - \phi + \xi_{Wt}, N_t(a, h, z) + \xi_{Nt}\} dG(\xi_{st}; \xi)$$

**Step 1A** Working household problem:

$$(2.2) \quad \begin{aligned} W_t(a, h, z) &= \max_{c, a'} \{ \log(c) - \eta + \beta \mathbb{E} V_{t+1}(a', h', z') \} \\ \text{s.t. } c + a' &= w(h, z) + (1 + r)a, \quad a' \geq 0 \\ h' &= \mathbb{I}\{h < \bar{h}\}(h + 1) + \mathbb{I}\{h \geq \bar{h}\}h \end{aligned}$$

The Lagrangian:

$$(2.3) \quad \mathcal{L} = \log(c) - \eta + \beta \mathbb{E} [V_{t+1}(a', h', z')] + \lambda [w(h, z) + (1 + r)a - a' - c]$$

The FOCs:

$$(2.4a) \quad \mathcal{L}_c = \frac{1}{c} - \lambda = 0 \implies \lambda = \frac{1}{c}$$

$$(2.4b) \quad \mathcal{L}_{a'} = \beta \frac{\partial \mathbb{E} [V_{t+1}(a', h', z')]}{\partial a'} - \lambda = 0 \implies \lambda = \beta \frac{\partial \mathbb{E} [V_{t+1}(a', h', z')]}{\partial a'}.$$

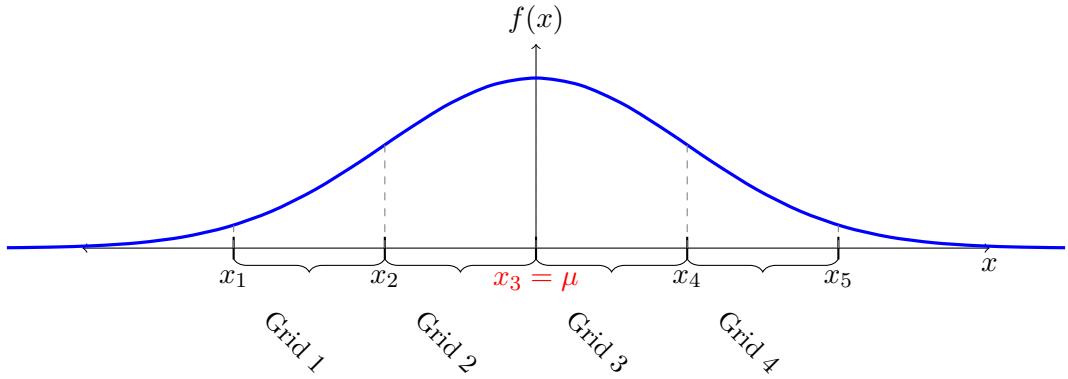


Figure 2.1: Tauchen discretisation.

Then, the optimality is given by:

$$(2.5) \quad \boxed{\frac{1}{c} = \beta \frac{\partial \mathbb{E}[V_{t+1}(a', h', z')]}{\partial a'}}.$$

Further, assuming that  $V_{T+1} = \log(a')$ , the terminal asset choice is:

$$(2.6a) \quad \frac{1}{c} = \frac{\beta}{a'} \implies$$

$$(2.6b) \quad a' = \beta c \implies$$

$$(2.6c) \quad \left(1 + \frac{1}{\beta}\right) a' = w(h, z) + (1+r)a \implies$$

$$(2.6d) \quad \boxed{a' = \frac{1+\beta}{\beta} [w(h, z) + (1+r)a].}$$

This can be used to recover the solution and final  $V_t$ , which can later be used to solve the problem.

**Step 1B** Working household problem: The non-working household faces the following problem:

$$(2.7) \quad \begin{aligned} N_t(a, h, z) &= \max_{c, a'} \log(c) + \beta \mathbb{E} S_{t+1}(a', h', z') \\ \text{s.t. } c + a' &= b + (1+r)a, \quad a' \geq 0 \\ h' &= h \end{aligned}$$

Following the same steps, we arrive at the optimality condition:

$$(2.8) \quad \boxed{\frac{1}{c} = \beta \frac{\partial \mathbb{E}[S_{t+1}(a', h', z')]}{\partial a'}}.$$

The terminal asset choice becomes:

$$(2.9) \quad \boxed{a' = \frac{1+\beta}{\beta} [b + (1+r)a].}$$

**Step 2** Tauchen discretisation: `fnTauchenLogNormal` allows for conducting a quick Tauchen discretisation under the assumption that  $z$  follows  $\log \mathcal{N}(0, \sigma_z)$ , as visualised on Figure ??.

**Step 3** Probabilistic labour supply decision: Given the presence of the Gumbel “shock” in the participation decision, the conditional decisions to participate,  $d_v$  and  $d_s$ , are governed by the following binary probabilities:

$$(2.10a) \quad \mathbb{P}(d_v = 1) = \frac{\exp\left(\frac{W}{\zeta}\right)}{\exp\left(\frac{W}{\zeta}\right) + \exp\left(\frac{N}{\zeta}\right)}$$

and

$$(2.10b) \quad \mathbb{P}(d_s = 1) = \frac{\exp\left(\frac{\varphi W + (1-\varphi)N - \phi}{\zeta}\right)}{\exp\left(\frac{\varphi W + (1-\varphi)N - \phi}{\zeta}\right) + \exp\left(\frac{N}{\zeta}\right)}.$$

**Gumbel probabilities trick:** Estimating Equations (??) and (??) in MATLAB is tricky. In each case, both numerators and denominators get very large (small), with the programme doing a bad job at computing  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ . The method I leverage in `fnGumbelTrickProbabilities` is based on the properties of the logistic sigmoid function. For notational parsimony, set,  $A \equiv \frac{W}{\zeta}$  and  $B \equiv \frac{N}{\zeta}$ ,  $X \equiv \max(A, B)$ , and focus on Equation (??):

$$(2.11a) \quad \mathbb{P}(d_v = 1) = \frac{\exp A}{\exp A + \exp B} \implies$$

$$(2.11b) \quad \log \mathbb{P}(d_v = 1) = A - \log(\exp A + \exp B) \underbrace{- X + \log(\exp X)}_{=0} \implies$$

$$(2.11c) \quad \log \mathbb{P}(d_v = 1) = A - X - \log[\exp(A - X) + \exp(B - X)] \implies$$

$$(2.11d) \quad \boxed{\mathbb{P}(d_v = 1) = \exp \left\{ \underbrace{A - X}_{\leq 0} - \log \left[ \underbrace{\exp(A - X) + \exp(B - X)}_{\leq 1} \right] \right\}}.$$

By restating the problem as in Equation (??) I ensure that MATLAB doesn't have to deal with excessively large numbers.

## Questions 4 through 6

### Question:

- Suppose the following parameter levels:

$$\begin{aligned} \zeta &= 0.01, \varphi = 0.80, \phi = 0.5, \eta = 2, b = 0.2, \bar{h} = 10, r = 0.04, \beta = 0.96 \\ \sigma_z &= 0.05, \rho = 0.90, \gamma_h = 0.1, \gamma_z = 0.1, \gamma_0 = 0.2 \end{aligned}$$

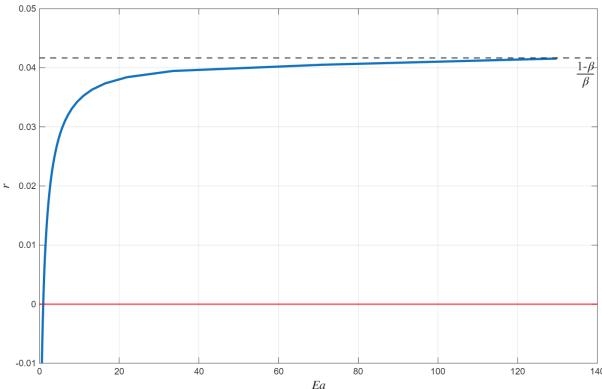
- Simulate 20,000 households (same cohort) and simulate them for the life time.
- Visualize the life-time wealth and consumption patterns.

**Step 1** The algorithm: The model's solution is structured according to the following algorithm.

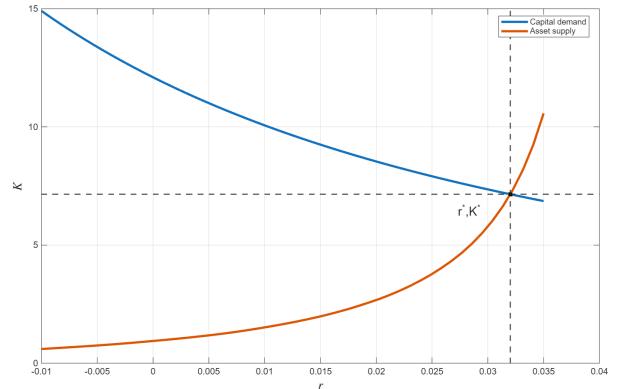
1. `LoadParameters.m` loads parameters and sets the wealth, skill, productivity, and age grids. For the skill grid, I use a standard Tauchen method from `fnTauchenLogNormal.m`.
2. In `fnValueFunctionMatrices.m`, the algorithm computes the value function for worker  $(a, h, z, t, a')$ .
  - (a) For  $t = 50$ , I compute  $\mathbb{E}V(a', h', z') = \mathbb{E}S(a', h', z') = \log a'$ .

Figure 3.1: Reproduced figures from **aiyagari'uninsured'1994**.

(a) Mean assets vs. interest rate.



(b) Asset supply and demand.



(b) For  $t < 50$ , I compute:

$$(2.12a) \quad W(a, h, z, t, a') = \log c - \eta + \beta \mathbb{E}_t V_{\max}(a', z, t, h)$$

$$(2.12b) \quad N(a, h, z, t, a') = \log c + \beta \mathbb{E}_t S_{\max}(a', z, t, h)$$

$$(2.12c) \quad V(a, h, z, t, a') = \zeta \left\{ \delta_E + \log \left[ \exp \left( \frac{W}{\zeta} \right) + \exp \left( \frac{N}{\zeta} \right) \right] \right\}$$

$$(2.12d) \quad S(a, h, z, t, a') = \zeta \left\{ \delta_E + \log \left[ \exp \left( \frac{\varphi W + (1 - \varphi)N - \phi}{\zeta} \right) + \exp \left( \frac{N}{\zeta} \right) \right] \right\}$$

### 3 Assignment 3

Questions (a) through (d)

**Question:**

- Why does a household need to understand the aggregate state  $\Phi$
- In the long run, will the economy reach a stationary state or not? Why?
- Given the problem, define the stationary recursive competitive equilibrium (SRCE).
- How large is the labor supply in each period?

Questions (e) and (f)

**Question:** Solve the general equilibrium model using the interpolated search and the histogram methods. Replicate Figures IIa and IIb from **aiyagari'uninsured'1994**.

**Part 1** The algorithm: To be added later.

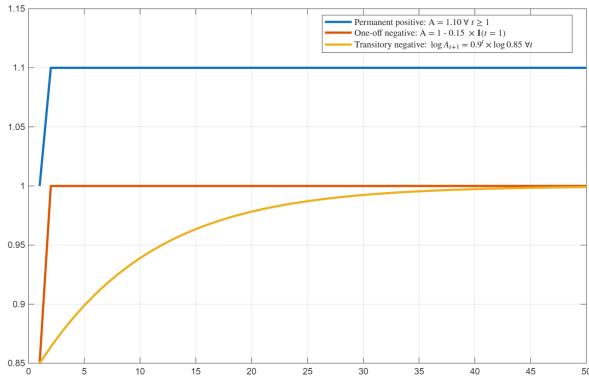
**Part 2** Plots: Figures ?? and ?? reproduce Figures IIa and IIb from **aiyagari'uninsured'1994**.

Questions (g) through (i)

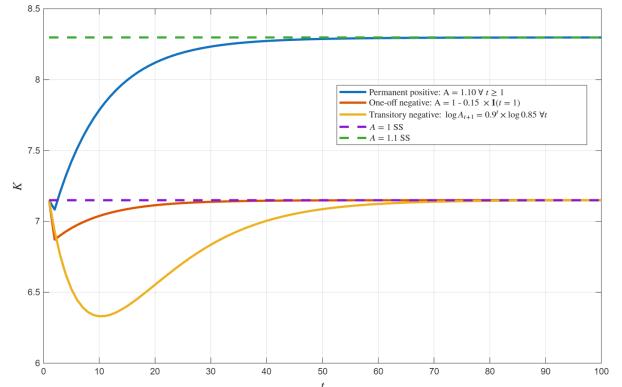
**Question:** Solve the general equilibrium model using the interpolated search and the histogram methods. Replicate Figures IIa and IIb from **aiyagari'uninsured'1994**.

Figure 4.1: Transition dynamics.

(a) Shock types.



(b) Capital dynamics.



## 4 Assignment 4

Questions (a) through (e)

**Question:**

- Solve the stationary competitive equilibrium for  $A = 1$  and  $A = 1.1$ .
- Define the transitional competitive equilibrium for the jump in  $A$ .
- Compute the transitional competitive equilibrium.
- How long does it take for an economy to converge to the new stationary competitive equilibrium?

**Part 1** Equilibrium definition: Let  $\Theta_0$  be the original SRCE distribution, for  $A = 1$ , and  $v_{T+1}$  be the new SRCE value function, for  $A = 1.1$ .

**Definition 4.1** (Transitional competitive equilibrium) Given  $\Theta_0$  and  $v_{T+1}$ ,  $(g_{a,t}, g_{c,t}, v_t, G_t, r_t, w_t, \Theta_t)_{t=1}^T$  and  $(g_K, g_{a,L})$  are **transitional competitive equilibrium** if:

1.  $(g_{a,t}, g_{c,t}, v_t)_{t=1}^T$  solves the household's problem given  $(\Theta_t)_{t=1}^T$ .
2.  $(g_K, g_{a,L})$  solves a representative firm's problem given  $(\Theta_t)_{t=1}^T$ .
3.  $(r_t, w_t)_{t=1}^T$  clears the capital and labour markets  $\forall t$ :

$$(4.1a) \quad [K] : \quad g_K(\Theta_t) = \int a \, d\Theta_t$$

$$(4.1b) \quad [L] : \quad g_{a,L}(\Theta_t) = \iint \Theta_t(a, z) z \, da \, dz.$$

4. The aggregate resource constraint holds:

$$(4.2) \quad \iint g_{c,t}(a, z) + g_{a,t}(a, z) \, d\Theta_t = F(g_K, g_{a,L}) + (1 - \delta)g_K.$$

Questions (g)

**Question:** Calculate the Gini coefficient for each period on the transition path.

To compute the Gini coefficient, I adapt the following formula:

$$(4.3) \quad G = \frac{1}{2\mu} \iint p(x)p(y) |x - y| \, dx dy,$$

where  $\mu$  is the average asset position of the households. In my Matlab code,  $p(\cdot)$  comes from `iMarginalDist` in each iteration:

$$(4.4a) \quad p(x) = \text{iMarginalDist} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \dots \\ \varphi_N \end{pmatrix} \Rightarrow$$

$$(4.4b) \quad \iint p(x)p(y) \, dx dy = \iint \boldsymbol{\iota}^T \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \dots \\ \varphi_N \end{pmatrix} \times (\varphi_1 \ \varphi_2 \ \dots \ \varphi_N) \boldsymbol{\iota} \, dx dy \Rightarrow$$

$$(4.4c) \quad \iint p(x)p(y) \, dx dy = \text{sum}(\text{sum}(\text{iMDist} * \text{iMDist}'))$$

which can be combined with the fact that:

$$(4.4d) \quad \iint |x - y| \, dx dy = \iint \left| \boldsymbol{\iota}^T \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_N \end{pmatrix} - (a_1 \ a_2 \ \dots \ a_N) \boldsymbol{\iota} \right| \, dx dy \Rightarrow$$

$$(4.4e) \quad \iint |x - y| \, dx dy = \text{abs}(\text{repmat}(\text{vGridA2}', \text{pNA2}, 1) - \text{repmat}(\text{vGridA2}, 1, \text{pNA2})).$$

This gives:

$$(4.4f) \quad G = \text{sum}((\text{iMDist} * \text{iMDist}') * \text{abs}(\text{repmat}(\text{vGridA2}', \text{pNA2}, 1) - \text{repmat}(\text{vGridA2}, 1, \text{pNA2}))), 'all').$$