

PHD21 Computational methods: Assignments

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Questions 1 through 4

Question:

1. Label and interpret the model ingredients properly.
2. Characterise the individual labor supply curve.
3. Characterise the aggregate labor supply curve.
4. Characterise the aggregate labor demand curve

Step 1A: Working households \rightarrow intensive margin labour supply. Start with the working household's problem.

$$(1.1) \quad \begin{aligned} W(a, z) = \max_{c, n, a'} & \left\{ \log(c) - \eta \frac{1}{1 + \frac{1}{\chi}} n^{1 + \frac{1}{\chi}} + \beta v(a') \right\} \\ \text{s.t. } c + a' &= zw(1 - \tau)n + a(1 + r(1 - \tau)) + T \end{aligned}$$

Construct the Lagrangian, assuming that $v(a') = \log(a')$:

$$(1.2) \quad \mathcal{L} = \log(c) - \eta \frac{1}{1 + \frac{1}{\chi}} n^{1 + \frac{1}{\chi}} + \beta \log(a') + \lambda [zw(1 - \tau)n + a(1 + r(1 - \tau)) + T - c - a'] .$$

The first-order conditions are as follows:

$$(1.3a) \quad \mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = -\frac{1}{c},$$

$$(1.3b) \quad \mathcal{L}_n = -\eta n^{\frac{1}{\chi}} + \lambda zw(1 - \tau) = 0 \implies \eta n^{\frac{1}{\chi}} = \frac{zw(1 - \tau)}{c},$$

and

$$(1.3c) \quad \mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c.$$

Combine the results of Equations (1.3b) and (1.3c) with the budget constraint to arrive at the Euler equation:

$$(1.4a) \quad \underbrace{c + a'}_{\text{Use (1.3c)}} = \underbrace{zw(1 - \tau)n + a(1 + r(1 - \tau))}_{\text{Use (1.3b)}} + T,$$

$$(WH-ILS) \quad \boxed{c(1 + \beta) = zw(1 - \tau) \left(\frac{zw(1 - \tau)}{\eta c} \right)^{\chi} + a(1 + r(1 - \tau)) + T.}$$

Equation (WH-ILS) governs the **intensive labour supply of a working household**, $c_w^*(a, z)$ is their consumption level.

Further, notice that $c_w^*(a, z)$ increases in z :

$$(1.5a) \quad \frac{\partial c_w^*(a, z)}{\partial z} (1 + \beta) = \underbrace{[w(1 - \tau)\eta^{-1}]^{1 + \chi}}_{\equiv \theta > 0} \times \frac{(1 + \chi)z^{\chi} c_w^*(a, z) - z^{\chi} \frac{\partial c_w^*(a, z)}{\partial z}}{c_w^*(a, z)^2} \implies$$

$$(1.5b) \quad \frac{\partial c_w^*(a, z)}{\partial z} \left(1 + \beta + \theta \frac{z^{\chi}}{c_w^*(a, z)^2} \right) = \frac{\theta(1 + \chi)z^{\chi}}{c_w^*(a, z)} \implies$$

$$(1.5c) \quad \frac{\partial c_w^*(a, z)}{\partial z} = \frac{\frac{\theta(1 + \chi)z^{\chi}}{c_w^*(a, z)}}{1 + \beta + \theta \frac{z^{\chi}}{c_w^*(a, z)^2}} > 0.$$

Also, differentiate Equation (1.3b):

$$(1.6a) \quad \eta\chi^{-1}n^{\frac{1}{\chi}-1}\frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} = -zw(1-\tau)c_w^*(a, z)^{-2} \implies$$

$$(1.6b) \quad \frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} = \frac{-\chi zw(1-\tau)}{\eta n^{\frac{1}{\chi}-1}c_w^*(a, z)^2}.$$

Using the combination of the Envelope Theorem and chain rule, this implies that, at the optimal consumption-hours bundle, the household sees:

$$(1.7a) \quad \frac{\partial W(a, z)}{\partial z} = \frac{1}{c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} - \eta n^{\frac{1}{\chi}} \frac{\partial n^*(a, z)}{\partial z} \implies$$

$$(1.7b) \quad \frac{\partial W(a, z)}{\partial z} = \frac{1}{c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} - \eta n^{\frac{1}{\chi}} \frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} \implies$$

$$(1.7c) \quad \boxed{\frac{\partial W(a, z)}{\partial z} > 0.}$$

Equation (1.7c) effectively means that $W(a, z)$ **monotonically increases in z** .

Step 1B: Non-working households. The problem is similar here, apart from the labour first order condition. Skipping the Lagrangian setup, I arrive at the following first-order conditions:

$$(1.8a) \quad \mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = -\frac{1}{c}$$

and

$$(1.8b) \quad \mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c,$$

both of which are identical to what we see for the working household. Combining it with the budget constraint, we obtain the consumption function for the non-working household:

$$(1.9) \quad \boxed{c_{nw}^*(a) = \frac{b + a(1 + r(1 - \tau)) + T}{1 + \beta}}.$$

This effectively implies that:

$$(1.10) \quad \boxed{\frac{\partial N(a, z)}{\partial z} = 0.}$$

Step 2: Extensive margin labour supply decision. Household (a, z) enters the labour market when:

$$(1.11) \quad \mathbf{I}_n(a, z) = \begin{cases} 1 & \text{if } W(a, z) \geq N(a, z) \\ 0 & \text{if } W(a, z) < N(a, z), \end{cases}$$

where $W(\cdot, \cdot)$ and $N(\cdot, \cdot)$ are the value functions of working and not working, respectively.

Even abstracting from the Unique Point Theorem, we can see that if there exists z^* such that:

$$(1.12) \quad W(a, z^*) = N(a, z^*)$$

then for $z > z^*$, we have:

$$(1.13) \quad \mathbf{I}_n(a, z) = 1.$$

This will come handy while numerically solving the model.

Step 3: Aggregate labour supply. Assuming that $\Phi(a, z)$ is the joint distribution of ex-ante wealth and productivity, the **aggregate labour supply** is:

$$(1.14) \quad L^S = \int \mathbf{I}_n(a, z) h(a, z) d\Phi(a, z)$$

Step 4A: Aggregate labour demand. We abstract from the capital markets, which makes the representative firm's problem near-trivial:

$$(1.15a) \quad Y = \max_L \{AK^\alpha L^{1-\alpha} - wL - rK\} \implies$$

$$(1.15b) \quad w = A(1 - \alpha) \left(\frac{L^D}{K} \right)^{-\alpha},$$

or, putting L^D on the LHS:

$$(1.15c) \quad L^D = \left(\frac{(1 - \alpha)A}{w} \right)^{\frac{1}{\alpha}} K.$$

Question 5

Question: Suppose the following parameter levels:

$$a = 1, \quad \alpha = 0.3, \quad \tau = 0.15, \quad \bar{z} = 1, \quad A = 1, \quad r = 0.04, \quad \beta = 0.96$$

Define and characterize the stationary recursive competitive equilibrium.

Step 1: Parameters left. Ξ represents the original vector of parameters:

$$(1.16) \quad \Xi = (a, \eta, \xi, \tau, b, \beta, \sigma_z, A, \alpha, r)^T.$$

Given the pre-specified parameters, the unknown ones are:

$$(1.17) \quad \hat{\Xi} = (\eta, b, \sigma_z,)^T.$$

Step 2: Equilibrium. Given the distribution of labour productivity, Φ , a set of functions $\{n, c, a', z^*, L, w, T\}$ is a **stationary competitive equilibrium** if

1. (n, c, a', z^*) solves the household's problem.
2. (K, L) solves the production sector's problem.
3. The labour and capital markets clear.

Step 3: Equilibrium algorithm. I set out the **algorithm used to compute the equilibrium** given a set of parameters.

1. Guess (w_0, T_0) .

2. Compute individual decisions.

- `fnIntensiveLabourSupply` computes the **intensive labour supply for a working household**. The following equations flesh out the approach leveraging concavity of Equation (WH-ILS). Start at the initial consumption guess, c_0 . The code follows the logic fleshed out by the equations below:

$$(1.18a) \quad RHS(c) \equiv zw(1-\tau) \left(\frac{zw(1-\tau)}{\eta c} \right)^x + a(1+r(1-\tau)) + T$$

$$(1.18b) \quad LHS(c) \equiv (1+\beta)c \implies$$

$$(1.18c) \quad c_1 = \frac{RHS(c_0)}{1+\beta}$$

and

$$(1.18d) \quad \epsilon_n \equiv c_n - c_{n-1} \implies$$

$$(1.18e) \quad \epsilon_1 = c_1 - c_0.$$

If it's above the tolerance level, then repeat until it works:

$$(1.18f) \quad c_n = \frac{RHS(c_{n-1})}{1+\beta} \implies$$

$$(1.18g) \quad \epsilon_n = c_n - c_{n-1}.$$

This approach computes the individually optimal values of consumption and labour market participation, (c_w, n_w) , provided the household chooses to work.

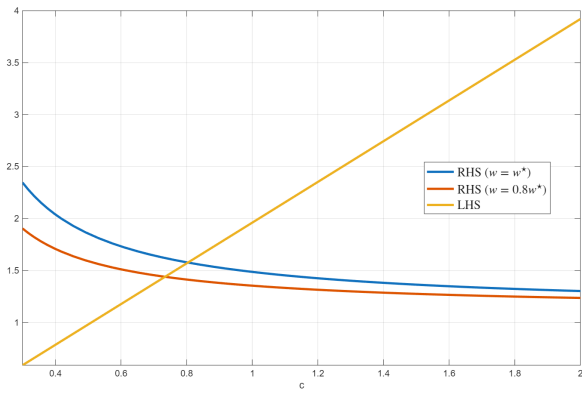
- `fnExtensiveLabourSupply` determines if household (a, z) chooses to work based on Equation (1.11).

3. Aggregate all labour supply decisions (`fnAggregateLabourSupply`) and compare them with the aggregate labour demand. `fnSolvePrices` iterates w and T until both clear the labour market.

- One way of doing that is following the same method as for `fnIntensiveLabourSupply`, with Equations (1.14) and (1.15c) used to compute labour supply and demand, respectively.
- Another method is to use bisection, in `fnSolvePricesBisection`. As illustrated in Figure 1b, the method is based on creating a grid for different wage values, finding the negative value closest to 0, w_{lower} , and taking the weighted average of w_{lower} and w_{upper} (the next value in the grid). **Note:** I use the “naive” approach in my code, as bisection seems to produce a

Figure 1: Household block of the model.

(a) Euler equation & consumption decision.



(b) Excess labour supply on the aggregate level.

