PHD21 Computational methods: Assignments

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1 Assignment 1

Questions 1 through 4

Question:

1. *Label and interpret the model ingredients properly.*
2. *Characterise the individual labor supply curve.*
3. *Characterise the aggregate labor supply curve.*
4. *Characterise the aggregate labor demand curve*

Step 1A: Working households → intensive margin labour supply. Start with the working household's problem.

$$W(a,z)=\max_{c,n,a'}\left\{\log(c)-\eta\frac{1}{1+\frac{1}{\chi}}n^{1+\frac{1}{\chi}}+\beta v\left(a'\right)\right\} \tag{1.1}$$
$$\text{s.t. } c+a'=zw(1-\tau)n+a(1+r(1-\tau))+T$$

Construct the Lagrangian, assuming that $v\left(a'\right)=\log\left(a'\right)$:

$$\mathcal{L}=\log(c)-\eta\frac{1}{1+\frac{1}{\chi}}n^{1+\frac{1}{\chi}}+\beta\log\left(a'\right)+\lambda\left[zw(1-\tau)n+a(1+r(1-\tau))+T-c-a'\right]. \tag{1.2}$$

The first-order conditions are as follows:

$$\mathcal{L}_c=\frac{1}{c}+\lambda=0 \implies \lambda=\frac{1}{c}, \tag{1.3a}$$

$$\mathcal{L}_n=-\eta n^{\frac{1}{\chi}}+\lambda zw(1-\tau)=0 \implies \eta n^{\frac{1}{\chi}}=\frac{zw(1-\tau)}{c}, \tag{1.3b}$$

and

$$\mathcal{L}_{a'}=\frac{\beta}{a'}-\lambda=0 \implies a'=\beta c. \tag{1.3c}$$

Combine the results of Equations (1.3b) and (1.3c) with the budget contraint to arrive at the Euler equation:

$$\underbrace{c+a'}_{\text{Use (1.3c)}}=\underbrace{zw(1-\tau)n}_{\text{Use (1.3b)}}+a(1+r(1-\tau))+T, \tag{1.4a}$$

$$\boxed{c(1+\beta)=zw(1-\tau)\left(\frac{zw(1-\tau)}{\eta c}\right)^{\chi}+a(1+r(1-\tau))+T.} \tag{WH-ILS}$$

Equation (WH-ILS) governs the **intensive labour supply of a working household**, $c_w^\star(a,z)$ is their consumption level.

Further, notice that $c_w^\star(a,z)$ increases in z:

$$\frac{\partial c_w^\star(a,z)}{\partial z}(1+\beta)=\underbrace{\left[w(1-\tau)\eta^{-1}\right]^{1+\chi}}_{\equiv\theta>0}\times\frac{(1+\chi)z^\chi c_w^\star(a,z)-z^\chi\frac{\partial c_w^\star(a,z)}{\partial z}}{c_w^\star(a,z)^2} \implies \tag{1.5a}$$

$$\frac{\partial c_w^\star(a,z)}{\partial z}\left(1+\beta+\theta\frac{z^\chi}{c_w^\star(a,z)^2}\right)=\frac{\theta(1+\chi)z^\chi}{c_w^\star(a,z)} \implies \tag{1.5b}$$

$$\frac{\partial c_w^\star(a,z)}{\partial z}=\frac{\frac{\theta(1+\chi)z^\chi}{c_w^\star(a,z)}}{1+\beta+\theta\frac{z^\chi}{c_w^\star(a,z)^2}}>0. \tag{1.5c}$$

Also, differentiate Equation (1.3b):

$$\eta\chi^{-1}n^{\frac{1}{\chi}-1}\frac{\partial n^{\star}(a,z)}{\partial c_w^{\star}(a,z)} = -zw(1-\tau)c_w^{\star}(a,z)^{-2} \implies \tag{1.6a}$$

$$\frac{\partial n^{\star}(a,z)}{\partial c_w^{\star}(a,z)} = \frac{-\chi zw(1-\tau)}{\eta n^{\frac{1}{\chi}-1}c_w^{\star}(a,z)^2}. \tag{1.6b}$$

Using the combination of the Envelope Theorem and chain rule, this implies that, at the optimal consumption-hours bundle, the household sees:

$$\frac{\partial W(a,z)}{\partial z} = \frac{1}{c_w^{\star}(a,z)}\frac{\partial c_w^{\star}(a,z)}{\partial z} - \eta n^{\frac{1}{\chi}}\frac{\partial n^{\star}(a,z)}{\partial z} \implies \tag{1.7a}$$

$$\frac{\partial W(a,z)}{\partial z} = \frac{1}{c_w^{\star}(a,z)}\frac{\partial c_w^{\star}(a,z)}{\partial z} - \eta n^{\frac{1}{\chi}}\frac{\partial n^{\star}(a,z)}{\partial c_w^{\star}(a,z)}\frac{\partial c_w^{\star}(a,z)}{\partial z} \implies \tag{1.7b}$$

$$\boxed{\frac{\partial W(a,z)}{\partial z} > 0.} \tag{1.7c}$$

Equation (1.7c) effectvely means that **$W(a,z)$ monotonically increases in z**.

Step 1B: Non-working households. The problem is similar here, apart from the labour first order condition. Skipping the Lagrangian setup, I arrive at the following first-order conditions:

$$\mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = \frac{1}{c} \tag{1.8a}$$

and

$$\mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c, \tag{1.8b}$$

both of which are identical to what we see for the working household. Combining it with the budget constraint, we obtain the consumption function for the non-working household:

$$\boxed{c_{nw}^{\star}(a) = \frac{b + a(1+r(1-\tau)) + T}{1+\beta}.} \tag{1.9}$$

This effectively implies that:

$$\boxed{\frac{\partial N(a,z)}{\partial z} = 0.} \tag{1.10}$$

Step 2: Extensive margin labour supply decision. Household (a,z) enters the labour market when:

$$\mathbf{I}_n(a,z) = \begin{cases} 1 & \text{if} \quad W(a,z) \geq N(a,z) \\ 0 & \text{if} \quad W(a,z) < N(a,z), \end{cases} \tag{1.11}$$

where $W(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are the value functions of working and not working, respectively.

Even abstracting from the Unique Point Theorem, we can see that if there exists $z^{\star}$ such that:

$$W(a,z^{\star}) = N(a,z^{\star}) \tag{1.12}$$

then for $z > z^\star$, we have:

$$\mathbf{I}_n(a, z) = 1. \tag{1.13}$$

This will come handy while numerically solving the model.

Step 3: Aggregate labour supply. Assuming that $\Phi(a, z)$ is the joint distribution of ex-ante wealth and productivity, the **aggregate labour supply** is:

$$L^S = \int \mathbf{I}_n(a, z) h(a, z) \, \mathrm{d}\Phi(a, z) \tag{1.14}$$

Step 4A: Aggregate labour demand. We abstract from the capital markets, which makes the representative firm's problem near-trivial:

$$Y = \max_L \left\{ AK^\alpha L^{1-\alpha} - wL - rK \right\} \implies \tag{1.15a}$$

$$w = A(1-\alpha) \left(\frac{L^D}{K} \right)^{-\alpha}, \tag{1.15b}$$

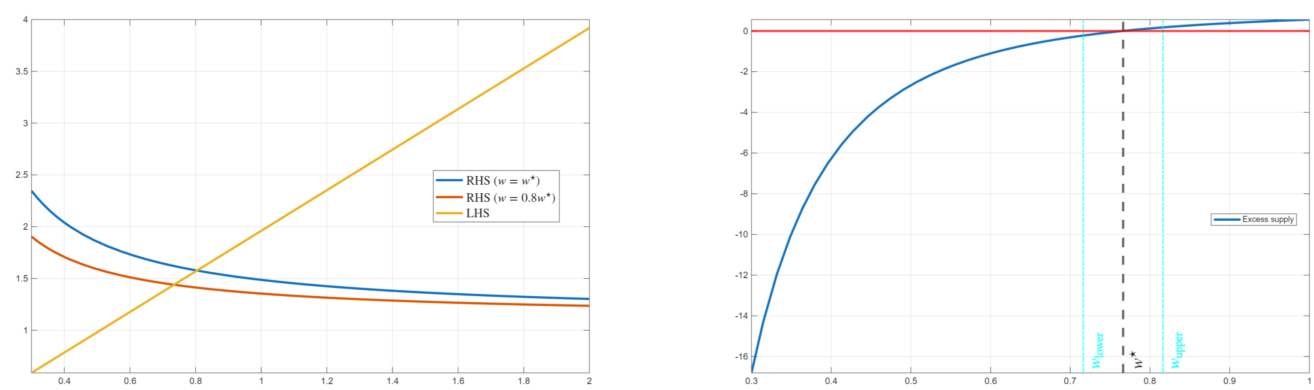
or, putting L^D on the LHS:

$$\boxed{L^D = \left(\frac{(1-\alpha)A}{w} \right)^{\frac{1}{\alpha}} K.} \tag{1.15c}$$

Figure 1: Household block of the model.

(a) Euler equation & consumption decision.

(b) Excess labour supply on the aggregate level.



Question 5

Question: *Suppose the following parameter levels:*

$$a = 1, \quad \alpha = 0.3, \quad \tau = 0.15, \quad \bar{z} = 1, \quad A = 1, \quad r = 0.04, \quad \beta = 0.96$$

Define and characterize the stationary recursive competitive equilibrium.

Step 1: Parameters left. $\mathbf{\Xi}$ represents the original vector of parameters:

$$\mathbf{\Xi} = (a, \eta, \xi, \tau, b, \beta, \sigma_z, A, \alpha, r)^T. \tag{1.16}$$

Given the pre-specified parameters, the unknown ones are:

$$\hat{\boldsymbol{\Xi}} = (\eta, b, \sigma_z, \chi)^T . \tag{1.17}$$

In my further work, I already use the calibrated parameters (see question 8).

Step 2: Equilibrium. Given the distribution of labour productivity, Φ, a set of functions $\{n, c, a', z^\star, L, w, T\}$ is a **stationary competitive equilibrium** if

1. $(n, c, a', z^\star)$ solves the household's problem.
2. (K, L) solves the production sector's problem.
3. The labour and capital markets clear.

Step 3: Equilibrium algorithm. I set out the **algorithm used to compute the equilibrium** given a set of parameters.

1. Guess (w_0, T_0).
2. Compute individual decisions.
   - `fnIntensiveLabourSupply` computes the **intensive labour supply for a working household**. The following equations flesh out the approach leveraging concavity of Equation (WH-ILS). Start at the initial consumption guess, c_0. The code follows the logic fleshed out by the equations below:

$$RHS\,(c) \equiv zw(1-\tau)\left(\frac{zw(1-\tau)}{\eta c}\right)^{\chi} + a(1 + r(1-\tau)) + T \tag{1.18a}$$

$$LHS\,(c) \equiv (1+\beta)c \implies \tag{1.18b}$$

$$c_1 = \frac{RHS\,(c_0)}{1+\beta} \tag{1.18c}$$

   and

$$\epsilon_n \equiv c_n - c_{n-1} \implies \tag{1.18d}$$

$$\epsilon_1 = c_1 - c_0. \tag{1.18e}$$

   If it's above the tolerance level, then repeat until it works:

$$c_n = \frac{RHS\,(c_{n-1})}{1+\beta} \implies \tag{1.18f}$$

$$\epsilon_n = c_n - c_{n-1}. \tag{1.18g}$$

   This approach computes the indiviually optimal values of consumption and labour market participation, (c_w, n_w), provided the household chooses to work.
   - `fnExtensiveLabourSupply` determines if household (a, z) chooses to work based on Equation (1.11).
3. Aggregate all labour supply decisions (`fnAggregateLabourSupply`) and compare them with the aggregate labour demand. `fnSolvePrices` iterates w and T until both clear the labour market.
   - One way of doing that is following the same method as for `fnIntensiveLabourSupply`, with Equations (1.14) and (1.15c) used to compute labour supply and demand, respectively.
   - Another method is to use bisection, in `fnSolvePricesBisection`. As illustrated in Figure 1b, the method is based on creating a grid for different wage values, finding the negative value closest to 0, w_{lower}, and taking the weighted average of w_{lower} and w_{upper} (the next value in the grid). **Note**: I use the "naive" approach in my code, as bisection seems to produce a
4. If the error is too large, update (w_n, T_n) and iterate until convergence.

Question 6

Question: *Visualize the aggregate supply and demand curves in the labor market.*

Question 7

Question: *Visualize the comparative statics of the wage with respect to the change in A.*

Question 8

Question: *Estimate parameters $(\eta, b, \chi, \sigma_z)$ to match the following hypothetical moments in general equilibrium:*

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2 Assignment 2

Questions 1 through 3

Question:

1. *Label and interpret the model ingredients properly.*
2. *Discretise the idiosyncratic productivity process by the Tauchen method using 2 grid points on one standard deviation range.*
3. *Characterise the individual (probabilistic) labour supply decision analytically.*

The value functions are:

$$V_t(a,h,z) = \int \max\left\{W_t(a,h,z) + \xi_{Wt}, N_t(a,h,z) + \xi_{Nt}\right\} dG\left(\xi_{st};\xi\right) \tag{2.1a}$$

and

$$S_t(a,h,z) = \int \max\left\{\varphi W_t(a,h,z) + (1-\varphi)N_t(a,h,z) - \phi + \xi_{Wt}, N_t(a,h,z) + \xi_{Nt}\right\} dG\left(\xi_{st};\xi\right) \tag{2.1b}$$

Step 1A Working household problem:

$$\begin{aligned} W_t(a,h,z) &= \max_{c,a'}\left\{\log(c) - \eta + \beta\mathbb{E}V_{t+1}\left(a',h',z'\right)\right\} \\ \text{s.t. } & c + a' = w(h,z) + (1+r)a, \quad a' \geq 0 \\ & h' = \mathbb{I}\left\{h < \bar{h}\right\}(h+1) + \mathbb{I}\left\{h \geq \bar{h}\right\} h \end{aligned} \tag{2.2}$$

The Lagrangian:

$$\mathcal{L} = \log(c) - \eta + \beta\mathbb{E}\left[V_{t+1}\left(a',h',z'\right)\right] + \lambda\left[w(h,z) + (1+r)a - a' - c\right] \tag{2.3}$$

The FOCs:

$$\mathcal{L}_c = \frac{1}{c} - \lambda = 0 \implies \lambda = \frac{1}{c} \tag{2.4a}$$

$$\mathcal{L}_{a'} = \beta\frac{\partial\mathbb{E}\left[V_{t+1}\left(a',h',z'\right)\right]}{\partial a'} - \lambda = 0 \implies \lambda = \beta\frac{\partial\mathbb{E}\left[V_{t+1}\left(a',h',z'\right)\right]}{\partial a'}. \tag{2.4b}$$

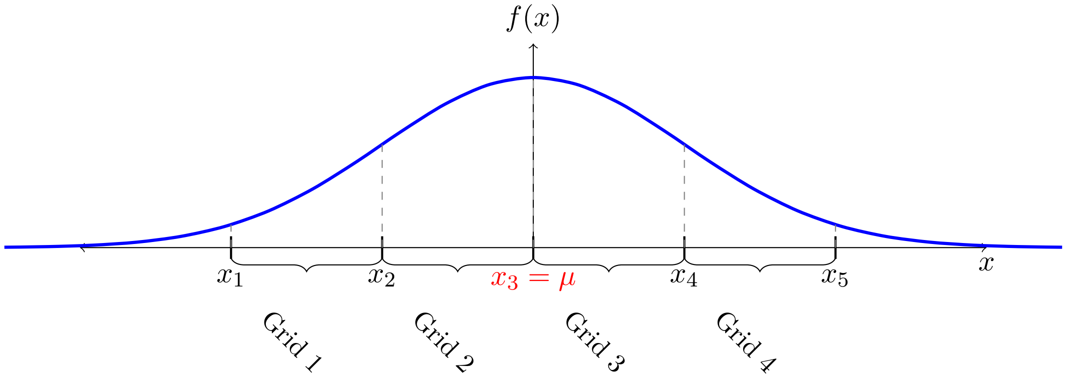

Figure 2: Tauchen discretisation.

Then, the optimality is given by:

$$\boxed{\frac{1}{c} = \beta \frac{\partial \mathbb{E}\left[V_{t+1}\left(a', h', z'\right)\right]}{\partial a'}.} \tag{2.5}$$

Further, assuming that $V_{T+1} = \log\left(a'\right)$, the terminal asset choice is:

$$\frac{1}{c} = \frac{\beta}{a'} \implies \tag{2.6a}$$

$$a' = \beta c \implies \tag{2.6b}$$

$$\left(1 + \frac{1}{\beta}\right) a' = w(h, z) + (1 + r)a \implies \tag{2.6c}$$

$$\boxed{a' = \frac{1 + \beta}{\beta}\left[w(h, z) + (1 + r)a\right].} \tag{2.6d}$$

This can be used to recover the solution and final V_t, which can later be used to solve the problem.

Step 1B Working household problem: The non-working household faces the following problem:

$$\begin{aligned} N_t(a, h, z) = \max_{c, a'} & \log(c) + \beta \mathbb{E} S_{t+1}\left(a', h', z'\right) \\ \text{s.t. } & c + a' = b + (1 + r)a, \quad a' \geq 0 \\ & h' = h \end{aligned} \tag{2.7}$$

Following the same steps, we arrive at the optimality condition:

$$\boxed{\frac{1}{c} = \beta \frac{\partial \mathbb{E}\left[S_{t+1}\left(a', h', z'\right)\right]}{\partial a'}.} \tag{2.8}$$

The terminal asset choice becomes:

$$\boxed{a' = \frac{1 + \beta}{\beta}\left[b + (1 + r)a\right].} \tag{2.9}$$

Step 2 Tauchen discretisation: `fnTauchenLogNormal` allows for conducting a quick Tauchen discretisation under the assumption that z follows $\log \mathcal{N}\left(0, \sigma_z\right)$, as visualised on Figure 2.

Step 3 Probabilistic labour supply decision: Given the presence of the Gumbel "shock" in the participation decision, the conditional decisions to participate, d_v and d_s, are governed by the following binary probabilities:

$$\mathbb{P}\left(d_v=1\right)=\frac{\exp\left(\frac{W}{\zeta}\right)}{\exp\left(\frac{W}{\zeta}\right)+\exp\left(\frac{N}{\zeta}\right)} \tag{2.10a}$$

and

$$\mathbb{P}\left(d_s=1\right)=\frac{\exp\left(\frac{\varphi W+(1-\varphi)N-\phi}{\zeta}\right)}{\exp\left(\frac{\varphi W+(1-\varphi)N-\phi}{\zeta}\right)+\exp\left(\frac{N}{\zeta}\right)}. \tag{2.10b}$$

Gumbel probabilities trick: Estimating Equations (2.10a) and (2.10b) in MATLAB is tricky. In each case, both numerators and denominators get very large (small), with the programme doing a bad job at computing $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The method I leverage in `fnGumbelTrickProbabilities` is based on the proprtis of the logistic sigmoid function. For notational parsimony, set, $A\equiv\frac{W}{\zeta}$ and $B\equiv\frac{N}{\zeta}$, $X\equiv\max(A,B)$, and focus on Equation (2.10a):

$$\mathbb{P}\left(d_v=1\right)=\frac{\exp A}{\exp A+\exp B}\implies \tag{2.11a}$$

$$\log\mathbb{P}\left(d_v=1\right)=A-\log\left(\exp A+\exp B\right)\underbrace{-X+\log(\exp X)}_{=0}\implies \tag{2.11b}$$

$$\log\mathbb{P}\left(d_v=1\right)=A-X-\log\left[\exp(A-X)+\exp(B-X)\right]\implies \tag{2.11c}$$

$$\boxed{\mathbb{P}\left(d_v=1\right)=\exp\left\{\underbrace{A-X}_{\leq 0}-\log\left[\underbrace{\exp(\overbrace{A-X}^{\leq 0})+\exp\,(\overbrace{B-X}^{\leq 0})}_{\leq 1}\right]\right\}}. \tag{2.11d}$$

By restating the problem as in Equation (2.11d) I ensure that MATLAB doesn't have to deal with excessively large numbers.

Questions 4 through 6

Question:

- *Suppose the following parameter levels:*

$$\zeta=0.01,\varphi=0.80,\phi=0.5,\eta=2,b=0.2,\bar{h}=10,r=0.04,\beta=0.96$$
$$\sigma_z=0.05,\rho=0.90,\gamma_h=0.1,\gamma_z=0.1,\gamma_0=0.2$$

- *Simulate 20,000 households (same cohort) and simulate them for the life time.*
- *Visualize the life-time wealth and consumption patterns.*

Step 1 The algorithm: The model's solution is structured according to the following algorithm.

1. `LoadParameters.m` loads parameters and sets the wealth, skill, productivity, and age grids. For the skill grid, I use a standard Tauchen method from `fnTauchenLogNormal.m`.

2. In `fnValueFunctionMatrices.m`, the algorithm computes the value function for worker (a,h,z,t,a').

   (a) For $t=50$, I compute $\mathbb{E}V\left(a',h',z'\right)=\mathbb{E}S\left(a',h',z'\right)=\log a'$.

(b) For $t < 50$, I compute:

$$W\left(a,h,z,t,a'\right) = \log c - \eta + \beta \mathbb{E}_t V_{\max}\left(a',z,t,h\right) \tag{2.12a}$$

$$N\left(a,h,z,t,a'\right) = \log c + \beta \mathbb{E}_t S_{\max}\left(a',z,t,h\right) \tag{2.12b}$$

$$V\left(a,h,z,t,a'\right) = \zeta\left\{\delta_E + \log\left[\exp\left(\frac{W}{\zeta}\right) + \exp\left(\frac{N}{\zeta}\right)\right]\right\} \tag{2.12c}$$

$$S\left(a,h,z,t,a'\right) = \zeta\left\{\delta_E + \log\left[\exp\left(\frac{\varphi W + (1-\varphi)N - \phi}{\zeta}\right) + \exp\left(\frac{N}{\zeta}\right)\right]\right\} \tag{2.12d}$$