

PHD21 Computational methods: Assignments

Rob Włodarski

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1 Assignment 1

Questions 1 through 4

Question:

1. Label and interpret the model ingredients properly.
2. Characterise the individual labor supply curve.
3. Characterise the aggregate labor supply curve.
4. Characterise the aggregate labor demand curve

Step 1A: Working households → intensive margin labour supply. Start with the working household's problem.

$$(1.1) \quad \begin{aligned} W(a, z) = \max_{c, n, a'} & \left\{ \log(c) - \eta \frac{1}{1 + \frac{1}{\chi}} n^{1 + \frac{1}{\chi}} + \beta v(a') \right\} \\ \text{s.t. } c + a' &= zw(1 - \tau)n + a(1 + r(1 - \tau)) + T \end{aligned}$$

Construct the Lagrangian, assuming that $v(a') = \log(a')$:

$$(1.2) \quad \mathcal{L} = \log(c) - \eta \frac{1}{1 + \frac{1}{\chi}} n^{1 + \frac{1}{\chi}} + \beta \log(a') + \lambda [zw(1 - \tau)n + a(1 + r(1 - \tau)) + T - c - a'] .$$

The first-order conditions are as follows:

$$(1.3a) \quad \mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = \frac{1}{c},$$

$$(1.3b) \quad \mathcal{L}_n = -\eta n^{\frac{1}{\chi}} + \lambda zw(1 - \tau) = 0 \implies \eta n^{\frac{1}{\chi}} = \frac{zw(1 - \tau)}{c},$$

and

$$(1.3c) \quad \mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c.$$

Combine the results of Equations (1.3b) and (1.3c) with the budget constraint to arrive at the Euler equation:

$$(1.4a) \quad \underbrace{c + a'}_{\text{Use (1.3c)}} = \underbrace{zw(1 - \tau)n + a(1 + r(1 - \tau))}_{\text{Use (1.3b)}} + T,$$

$$(WH-ILS) \quad \boxed{c(1 + \beta) = zw(1 - \tau) \left(\frac{zw(1 - \tau)}{\eta c} \right)^{\chi} + a(1 + r(1 - \tau)) + T.}$$

Equation (WH-ILS) governs the **intensive labour supply of a working household**, $c_w^*(a, z)$ is their consumption level.

Further, notice that $c_w^*(a, z)$ increases in z :

$$(1.5a) \quad \frac{\partial c_w^*(a, z)}{\partial z} (1 + \beta) = \underbrace{[w(1 - \tau)\eta^{-1}]^{1 + \chi}}_{\equiv \theta > 0} \times \frac{(1 + \chi)z^{\chi} c_w^*(a, z) - z^{\chi} \frac{\partial c_w^*(a, z)}{\partial z}}{c_w^*(a, z)^2} \implies$$

$$(1.5b) \quad \frac{\partial c_w^*(a, z)}{\partial z} \left(1 + \beta + \theta \frac{z^{\chi}}{c_w^*(a, z)^2} \right) = \frac{\theta(1 + \chi)z^{\chi}}{c_w^*(a, z)} \implies$$

$$(1.5c) \quad \frac{\partial c_w^*(a, z)}{\partial z} = \frac{\frac{\theta(1 + \chi)z^{\chi}}{c_w^*(a, z)}}{1 + \beta + \theta \frac{z^{\chi}}{c_w^*(a, z)^2}} > 0.$$

Also, differentiate Equation (1.3b):

$$(1.6a) \quad \eta\chi^{-1}n^{\frac{1}{\chi}-1}\frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} = -zw(1-\tau)c_w^*(a, z)^{-2} \implies$$

$$(1.6b) \quad \frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} = \frac{-\chi zw(1-\tau)}{\eta n^{\frac{1}{\chi}-1}c_w^*(a, z)^2}.$$

Using the combination of the Envelope Theorem and chain rule, this implies that, at the optimal consumption-hours bundle, the household sees:

$$(1.7a) \quad \frac{\partial W(a, z)}{\partial z} = \frac{1}{c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} - \eta n^{\frac{1}{\chi}} \frac{\partial n^*(a, z)}{\partial z} \implies$$

$$(1.7b) \quad \frac{\partial W(a, z)}{\partial z} = \frac{1}{c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} - \eta n^{\frac{1}{\chi}} \frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} \implies$$

$$(1.7c) \quad \boxed{\frac{\partial W(a, z)}{\partial z} > 0.}$$

Equation (1.7c) effectively means that $W(a, z)$ **monotonically increases in z** .

Step 1B: Non-working households. The problem is similar here, apart from the labour first order condition. Skipping the Lagrangian setup, I arrive at the following first-order conditions:

$$(1.8a) \quad \mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = \frac{1}{c}$$

and

$$(1.8b) \quad \mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c,$$

both of which are identical to what we see for the working household. Combining it with the budget constraint, we obtain the consumption function for the non-working household:

$$(1.9) \quad \boxed{c_{nw}^*(a) = \frac{b + a(1 + r(1 - \tau)) + T}{1 + \beta}.}$$

This effectively implies that:

$$(1.10) \quad \boxed{\frac{\partial N(a, z)}{\partial z} = 0.}$$

Step 2: Extensive margin labour supply decision. Household (a, z) enters the labour market when:

$$(1.11) \quad \mathbf{I}_n(a, z) = \begin{cases} 1 & \text{if } W(a, z) \geq N(a, z) \\ 0 & \text{if } W(a, z) < N(a, z), \end{cases}$$

where $W(\cdot, \cdot)$ and $N(\cdot, \cdot)$ are the value functions of working and not working, respectively.

Even abstracting from the Unique Point Theorem, we can see that if there exists z^* such that:

$$(1.12) \quad W(a, z^*) = N(a, z^*)$$

then for $z > z^*$, we have:

$$(1.13) \quad \mathbf{I}_n(a, z) = 1.$$

This will come handy while numerically solving the model.

Step 3: Aggregate labour supply. Assuming that $\Phi(a, z)$ is the joint distribution of ex-ante wealth and productivity, the **aggregate labour supply** is:

$$(1.14) \quad L^S = \int \mathbf{I}_n(a, z) h(a, z) d\Phi(a, z)$$

Step 4A: Aggregate labour demand. We abstract from the capital markets, which makes the representative firm's problem near-trivial:

$$(1.15a) \quad Y = \max_L \{AK^\alpha L^{1-\alpha} - wL - rK\} \implies$$

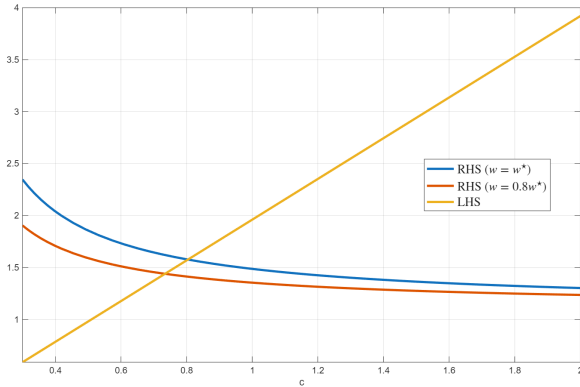
$$(1.15b) \quad w = A(1 - \alpha) \left(\frac{L^D}{K} \right)^{-\alpha},$$

or, putting L^D on the LHS:

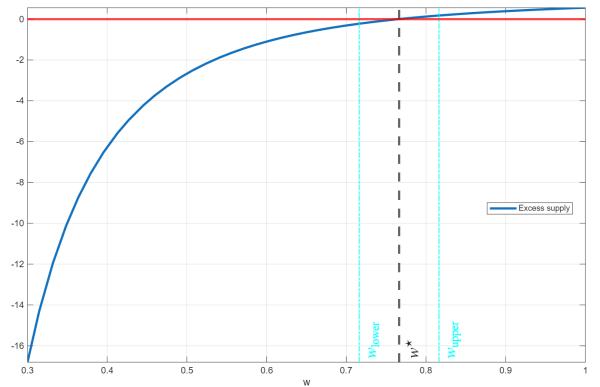
$$(1.15c) \quad L^D = \left(\frac{(1 - \alpha)A}{w} \right)^{\frac{1}{\alpha}} K.$$

Figure 1.1: Household block of the model.

(a) Euler equation & consumption decision.



(b) Excess labour supply on the aggregate level.



Question 5

Question: Suppose the following parameter levels:

$$a = 1, \quad \alpha = 0.3, \quad \tau = 0.15, \quad \bar{z} = 1, \quad A = 1, \quad r = 0.04, \quad \beta = 0.96$$

Define and characterize the stationary recursive competitive equilibrium.

Step 1: Parameters left. Ξ represents the original vector of parameters:

$$(1.16) \quad \Xi = (a, \eta, \xi, \tau, b, \beta, \sigma_z, A, \alpha, r)^T.$$

Given the pre-specified parameters, the unknown ones are:

$$(1.17) \quad \hat{\Xi} = (\eta, b, \sigma_z, \chi)^T.$$

In my further work, I already use the calibrated parameters (see question 8).

Step 2: Equilibrium. Given the distribution of labour productivity, Φ , a set of functions $\{n, c, a', z^*, L, w, T\}$ is a **stationary competitive equilibrium** if

1. (n, c, a', z^*) solves the household's problem.
2. (K, L) solves the production sector's problem.
3. The labour and capital markets clear.

Step 3: Equilibrium algorithm. I set out the **algorithm used to compute the equilibrium** given a set of parameters.

1. Guess (w_0, T_0) .
2. Compute individual decisions.
 - **fnIntensiveLabourSupply** computes the **intensive labour supply for a working household**. The following equations flesh out the approach leveraging concavity of Equation (WH-ILS). Start at the initial consumption guess, c_0 . The code follows the logic fleshed out by the equations below:

$$(1.18a) \quad RHS(c) \equiv zw(1 - \tau) \left(\frac{zw(1 - \tau)}{\eta c} \right)^x + a(1 + r(1 - \tau)) + T$$

$$(1.18b) \quad LHS(c) \equiv (1 + \beta)c \implies$$

$$(1.18c) \quad c_1 = \frac{RHS(c_0)}{1 + \beta}$$

and

$$(1.18d) \quad \epsilon_n \equiv c_n - c_{n-1} \implies$$

$$(1.18e) \quad \epsilon_1 = c_1 - c_0.$$

If it's above the tolerance level, then repeat until it works:

$$(1.18f) \quad c_n = \frac{RHS(c_{n-1})}{1 + \beta} \implies$$

$$(1.18g) \quad \epsilon_n = c_n - c_{n-1}.$$

This approach computes the individually optimal values of consumption and labour market participation, (c_w, n_w) , provided the household chooses to work.

- **fnExtensiveLabourSupply** determines if household (a, z) chooses to work based on Equation (1.11).
3. Aggregate all labour supply decisions (**fnAggregateLabourSupply**) and compare them with the aggregate labour demand. **fnSolvePrices** iterates w and T until both clear the labour market.
 - One way of doing that is following the same method as for **fnIntensiveLabourSupply**, with Equations (1.14) and (1.15c) used to compute labour supply and demand, respectively.
 - Another method is to use bisection, in **fnSolvePricesBisection**. As illustrated in Figure 1.1b, the method is based on creating a grid for different wage values, finding the negative value closest to 0, w_{lower} , and taking the weighted average of w_{lower} and w_{upper} (the next value in the grid). **Note:** I use the “naive” approach in my code, as bisection seems to produce a
 4. If the error is too large, update (w_n, T_n) and iterate until convergence.

Question 6

Question: Visualize the aggregate supply and demand curves in the labor market.

Question 7

Question: Visualize the comparative statics of the wage with respect to the change in A .

Question 8

Question: Estimate parameters $(\eta, b, \chi, \sigma_z)$ to match the following hypothetical moments in general equilibrium:

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2 Assignment 2

Questions 1 through 3

Question:

1. Label and interpret the model ingredients properly.
2. Discretise the idiosyncratic productivity process by the Tauchen method using 2 grid points on one standard deviation range.
3. Characterise the individual (probabilistic) labour supply decision analytically.

The value functions are:

$$(2.1a) \quad V_t(a, h, z) = \int \max \{W_t(a, h, z) + \xi_{Wt}, N_t(a, h, z) + \xi_{Nt}\} dG(\xi_{st}; \xi)$$

and

$$(2.1b) \quad S_t(a, h, z) = \int \max \{\varphi W_t(a, h, z) + (1 - \varphi)N_t(a, h, z) - \phi + \xi_{Wt}, N_t(a, h, z) + \xi_{Nt}\} dG(\xi_{st}; \xi)$$

Step 1A Working household problem:

$$(2.2) \quad \begin{aligned} W_t(a, h, z) &= \max_{c, a'} \{ \log(c) - \eta + \beta \mathbb{E} V_{t+1}(a', h', z') \} \\ \text{s.t. } c + a' &= w(h, z) + (1 + r)a, \quad a' \geq 0 \\ h' &= \mathbb{I}\{h < \bar{h}\} (h + 1) + \mathbb{I}\{h \geq \bar{h}\} h \end{aligned}$$

The Lagrangian:

$$(2.3) \quad \mathcal{L} = \log(c) - \eta + \beta \mathbb{E} [V_{t+1}(a', h', z')] + \lambda [w(h, z) + (1 + r)a - a' - c]$$

The FOCs:

$$(2.4a) \quad \mathcal{L}_c = \frac{1}{c} - \lambda = 0 \implies \lambda = \frac{1}{c}$$

$$(2.4b) \quad \mathcal{L}_{a'} = \beta \frac{\partial \mathbb{E} [V_{t+1}(a', h', z')]}{\partial a'} - \lambda = 0 \implies \lambda = \beta \frac{\partial \mathbb{E} [V_{t+1}(a', h', z')]}{\partial a'}.$$

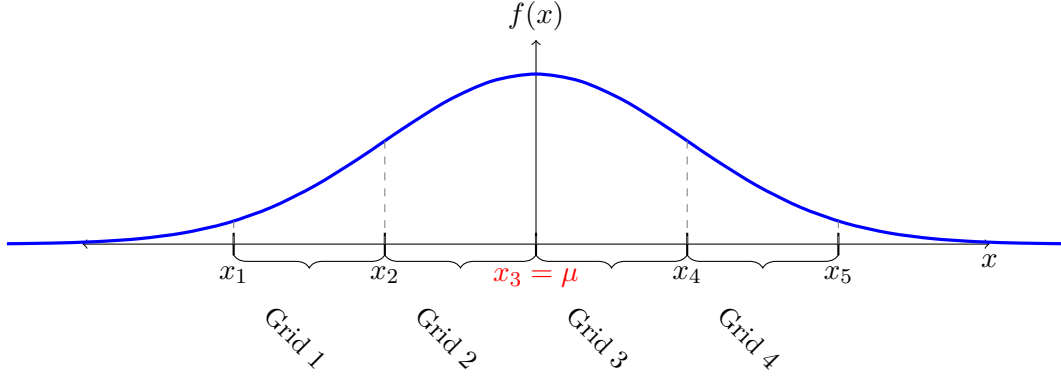


Figure 2.1: Tauchen discretisation.

Then, the optimality is given by:

$$(2.5) \quad \boxed{\frac{1}{c} = \beta \frac{\partial \mathbb{E}[V_{t+1}(a', h', z')]}{\partial a'}}.$$

Further, assuming that $V_{T+1} = \log(a')$, the terminal asset choice is:

$$(2.6a) \quad \frac{1}{c} = \frac{\beta}{a'} \implies$$

$$(2.6b) \quad a' = \beta c \implies$$

$$(2.6c) \quad \left(1 + \frac{1}{\beta}\right) a' = w(h, z) + (1 + r)a \implies$$

$$(2.6d) \quad \boxed{a' = \frac{1 + \beta}{\beta} [w(h, z) + (1 + r)a]}.$$

This can be used to recover the solution and final V_t , which can later be used to solve the problem.

Step 1B Working household problem: The non-working household faces the following problem:

$$(2.7) \quad \begin{aligned} N_t(a, h, z) &= \max_{c, a'} \log(c) + \beta \mathbb{E} S_{t+1}(a', h', z') \\ \text{s.t. } c + a' &= b + (1 + r)a, \quad a' \geq 0 \\ h' &= h \end{aligned}$$

Following the same steps, we arrive at the optimality condition:

$$(2.8) \quad \boxed{\frac{1}{c} = \beta \frac{\partial \mathbb{E}[S_{t+1}(a', h', z')]}{\partial a'}}.$$

The terminal asset choice becomes:

$$(2.9) \quad \boxed{a' = \frac{1 + \beta}{\beta} [b + (1 + r)a]}.$$

Step 2 Tauchen discretisation: `fnTauchenLogNormal` allows for conducting a quick Tauchen discretisation under the assumption that z follows $\log \mathcal{N}(0, \sigma_z)$, as visualised on Figure 2.1.

Step 3 Probabilistic labour supply decision: Given the presence of the Gumbel “shock” in the participation decision, the conditional decisions to participate, d_v and d_s , are governed by the following binary probabilities:

$$(2.10a) \quad \mathbb{P}(d_v = 1) = \frac{\exp\left(\frac{W}{\zeta}\right)}{\exp\left(\frac{W}{\zeta}\right) + \exp\left(\frac{N}{\zeta}\right)}$$

and

$$(2.10b) \quad \mathbb{P}(d_s = 1) = \frac{\exp\left(\frac{\varphi W + (1-\varphi)N - \phi}{\zeta}\right)}{\exp\left(\frac{\varphi W + (1-\varphi)N - \phi}{\zeta}\right) + \exp\left(\frac{N}{\zeta}\right)}.$$

Gumbel probabilities trick: Estimating Equations (2.10a) and (2.10b) in MATLAB is tricky. In each case, both numerators and denominators get very large (small), with the programme doing a bad job at computing $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The method I leverage in `fnGumbelTrickProbabilities` is based on the properties of the logistic sigmoid function. For notational parsimony, set, $A \equiv \frac{W}{\zeta}$ and $B \equiv \frac{N}{\zeta}$, $X \equiv \max(A, B)$, and focus on Equation (2.10a):

$$(2.11a) \quad \mathbb{P}(d_v = 1) = \frac{\exp A}{\exp A + \exp B} \implies$$

$$(2.11b) \quad \log \mathbb{P}(d_v = 1) = A - \log(\exp A + \exp B) \underbrace{- X + \log(\exp X)}_{=0} \implies$$

$$(2.11c) \quad \log \mathbb{P}(d_v = 1) = A - X - \log[\exp(A - X) + \exp(B - X)] \implies$$

$$(2.11d) \quad \mathbb{P}(d_v = 1) = \exp \left\{ \underbrace{A - X}_{\leq 0} - \log \left[\underbrace{\exp(\overbrace{A - X}^{\leq 0}) + \exp(\overbrace{B - X}^{\leq 0})}_{\leq 1} \right] \right\}.$$

By restating the problem as in Equation (2.11d) I ensure that MATLAB doesn't have to deal with excessively large numbers.

Questions 4 through 6

Question:

- Suppose the following parameter levels:

$$\zeta = 0.01, \varphi = 0.80, \phi = 0.5, \eta = 2, b = 0.2, \bar{h} = 10, r = 0.04, \beta = 0.96$$

$$\sigma_z = 0.05, \rho = 0.90, \gamma_h = 0.1, \gamma_z = 0.1, \gamma_0 = 0.2$$

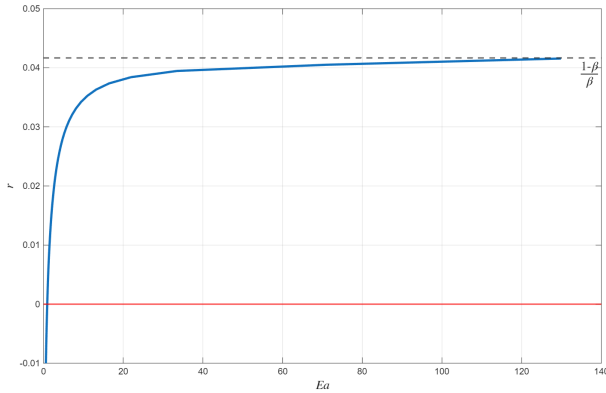
- Simulate 20,000 households (same cohort) and simulate them for the life time.
- Visualize the life-time wealth and consumption patterns.

Step 1 The algorithm: The model's solution is structured according to the following algorithm.

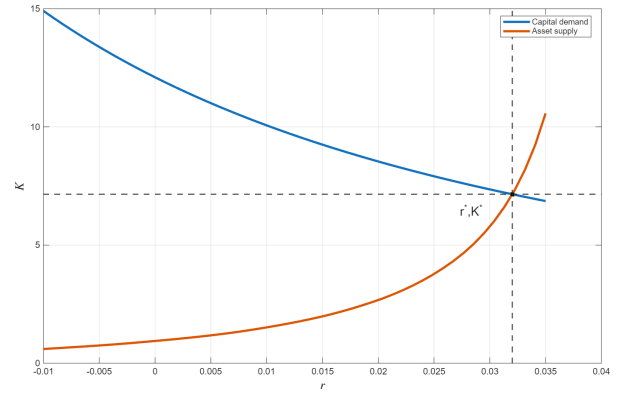
1. `LoadParameters.m` loads parameters and sets the wealth, skill, productivity, and age grids. For the skill grid, I use a standard Tauchen method from `fnTauchenLogNormal.m`.
2. In `fnValueFunctionMatrices.m`, the algorithm computes the value function for worker (a, h, z, t, a') .
 - (a) For $t = 50$, I compute $\mathbb{E}V(a', h', z') = \mathbb{E}S(a', h', z') = \log a'$.

Figure 3.1: Reproduced figures from Aiyagari (1994).

(a) Mean assets vs. interest rate.



(b) Asset supply and demand.



(b) For $t < 50$, I compute:

$$(2.12a) \quad W(a, h, z, t, a') = \log c - \eta + \beta \mathbb{E}_t V_{\max}(a', z, t, h)$$

$$(2.12b) \quad N(a, h, z, t, a') = \log c + \beta \mathbb{E}_t S_{\max}(a', z, t, h)$$

$$(2.12c) \quad V(a, h, z, t, a') = \zeta \left\{ \delta_E + \log \left[\exp \left(\frac{W}{\zeta} \right) + \exp \left(\frac{N}{\zeta} \right) \right] \right\}$$

$$(2.12d) \quad S(a, h, z, t, a') = \zeta \left\{ \delta_E + \log \left[\exp \left(\frac{\varphi W + (1 - \varphi)N - \phi}{\zeta} \right) + \exp \left(\frac{N}{\zeta} \right) \right] \right\}$$

3 Assignment 3

Questions (a) through (d)

Question:

- Why does a household need to understand the aggregate state Φ
- In the long run, will the economy reach a stationary state or not? Why?
- Given the problem, define the stationary recursive competitive equilibrium (SRCE).
- How large is the labor supply in each period?

Questions (e) and (f)

Question: Solve the general equilibrium model using the interpolated search and the histogram methods. Replicate Figures IIa and IIb from Aiyagari (1994).

Part 1 The algorithm: To be added later.

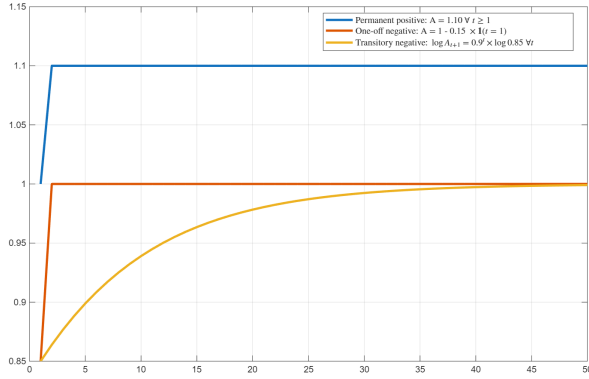
Part 2 Plots: Figures 3.1a and 3.1b reproduce Figures IIa and IIb from Aiyagari (1994).

Questions (g) through (i)

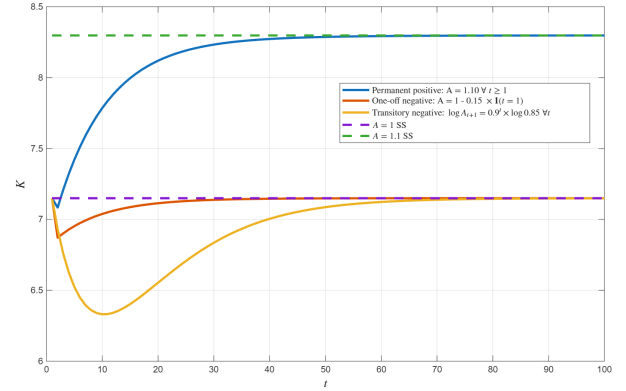
Question: Solve the general equilibrium model using the interpolated search and the histogram methods. Replicate Figures IIa and IIb from Aiyagari (1994).

Figure 4.1: Transition dynamics.

(a) Shock types.



(b) Capital dynamics.



4 Assignment 4

Questions (a) through (e)

Question:

- Solve the stationary competitive equilibrium for $A = 1$ and $A = 1.1$.
- Define the transitional competitive equilibrium for the jump in A .
- Compute the transitional competitive equilibrium.
- How long does it take for an economy to converge to the new stationary competitive equilibrium?

Part 1 Equilibrium definition: Let Θ_0 be the original SRCE distribution, for $A = 1$, and v_{T+1} be the new SRCE value function, for $A = 1.1$.

Definition 4.1 (Transitional competitive equilibrium) Given Θ_0 and v_{T+1} , $(g_{a,t}, g_{c,t}, v_t, G_t, r_t, w_t, \Theta_t)_{t=1}^T$ and $(g_K, g_{a,L})$ are **transitional competitive equilibrium** if:

1. $(g_{a,t}, g_{c,t}, v_t)_{t=1}^T$ solves the household's problem given $(\Theta_t)_{t=1}^T$.
2. $(g_K, g_{a,L})$ solves a representative firm's problem given $(\Theta_t)_{t=1}^T$.
3. $(r_t, w_t)_{t=1}^T$ clears the capital and labour markets $\forall t$:

$$(4.1a) \quad [K] : \quad g_K(\Theta_t) = \int a \, d\Theta_t$$

$$(4.1b) \quad [L] : \quad g_{a,L}(\Theta_t) = \iint \Theta_t(a, z) z \, da \, dz.$$

4. The aggregate resource constraint holds:

$$(4.2) \quad \iint g_{c,t}(a, z) + g_{a,t}(a, z) \, d\Theta_t = F(g_K, g_{a,L}) + (1 - \delta)g_K.$$

Questions (g)

Question: Calculate the Gini coefficient for each period on the transition path.

To compute the Gini coefficient, I adapt the following formula:

$$(4.3) \quad G = \frac{1}{2\mu} \iint p(x)p(y) |x - y| \, dx dy,$$

where μ is the average asset position of the households. In my Matlab code, $p(\cdot)$ comes from `iMarginalDist` in each iteration:

$$(4.4a) \quad p(x) = \text{iMarginalDist} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \dots \\ \varphi_N \end{pmatrix} \Rightarrow$$

$$(4.4b) \quad \iint p(x)p(y) \, dx dy = \iint \boldsymbol{\iota}^T \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \dots \\ \varphi_N \end{pmatrix} \times \begin{pmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_N \end{pmatrix} \boldsymbol{\iota} \, dx dy \Rightarrow$$

$$(4.4c) \quad \iint p(x)p(y) \, dx dy = \text{sum}(\text{sum}(\text{iMDist} * \text{iMDist}'))$$

which can be combined with the fact that:

$$(4.4d) \quad \iint |x - y| \, dx dy = \iint \left| \boldsymbol{\iota}^T \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_N \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & \dots & a_N \end{pmatrix} \boldsymbol{\iota} \right| \, dx dy \Rightarrow$$

$$(4.4e) \quad \iint |x - y| \, dx dy = \text{abs}(\text{repmat}(\text{vGridA2}', \text{pNA2}, 1) - \text{repmat}(\text{vGridA2}, 1, \text{pNA2})).$$

This gives:

$$(4.4f) \quad G = \text{sum}((\text{iMDist} * \text{iMDist}') .* \text{abs}(\text{repmat}(\text{vGridA2}', \text{pNA2}, 1) - \text{repmat}(\text{vGridA2}, 1, \text{pNA2})), 'all').$$

5 Assignment 5

Part 1: Re-writing the problem. Consider the problem without plugging all the functions. To solve the policy function iteration, we effectively need to “get rid off” the value function.

$$(5.1) \quad \begin{aligned} V(k) &= \max_{n, k', c} \{u(c) + u(n) + \beta V(k')\} \\ \text{s.t. } c + k' + \Psi(k, k') &= (1 + r)k + wn \end{aligned}$$

The problem's Lagrangian is:

$$(5.2) \quad \mathcal{L} = u(c) + u(n) + \beta V(k') + \lambda [(1 + r)k + wn - c - k' - \Psi(k, k')],$$

with the associated optimality conditions:

$$(5.3a) \quad \mathcal{L}_c = u_c(c) - \lambda = 0 \Rightarrow \lambda = u_c(c)$$

$$(5.3b) \quad \mathcal{L}_n = u_n(n) + \lambda n = 0 \Rightarrow u_n(n) = -u_c(c)w$$

$$(5.3c) \quad \mathcal{L}_{k'} = \beta V_{k'}(k') - \lambda [1 + \Psi_{k'}(k, k')] = 0 \Rightarrow \beta V_{k'}(k') = u_c(c) [1 + \Psi_{k'}(k, k')].$$

We want to find an expression for the first derivative of the value function, which can be simplified using the first-order conditions:

$$(5.4a) \quad V_k(k) = u_c(c) \frac{\partial c}{\partial k} + u_n(n) \frac{\partial n}{\partial k} + \beta V_{k'}(k') \frac{\partial k'}{\partial k} \implies$$

$$(5.4b) \quad V_k(k) = u_c(c) \frac{\partial c}{\partial k} - u_c(c) w \frac{\partial n}{\partial k} + u_c(c) [1 + \Psi_{k'}(k, k')] \frac{\partial k'}{\partial k} \implies$$

$$(5.4c) \quad V_k(k) = u_c(c) \left\{ \frac{\partial c}{\partial k} - w \frac{\partial n}{\partial k} + [1 + \Psi_{k'}(k, k')] \frac{\partial k'}{\partial k} \right\}.$$

The unpleasant terms from the “curly” bracket can be obtained from the budget constraint’s derivative w.r.t. k :

$$(5.5a) \quad \frac{\partial c}{\partial k} + \frac{\partial k'}{\partial k} + \Psi_k(k, k') + \Psi_{k'}(k, k') \frac{\partial k'}{\partial k} = (1 + r) + w \frac{\partial n}{\partial k} \implies$$

$$(5.5b) \quad \frac{\partial c}{\partial k} - w \frac{\partial n}{\partial k} + [1 + \Psi_{k'}(k, k')] \frac{\partial k'}{\partial k} = 1 + r - \Psi_k(k, k').$$

This implies that:

$$(5.6) \quad \boxed{V_k(k) = u_c(c) [1 + r - \Psi_k(k, k')],}$$

which is the key element of the policy function iteration.

Part 2: Global nonlinear solution method in the sequence space (Lee, 2025). I adapt the lecture algorithm to the problem.

1. I simulate $\mathbf{A} \equiv (A_t)_{t=1}^T$ and guess $\Theta^0 \equiv (K_t^0, N_t^0, C_t^0, W_t^0)_{t=1}^T$, with $T = 10,000$.
2. (**Backward solution**) For each $K_t^{(n)}$ pick up all the potential future TFPs, \tilde{A}_{t+1} (apart from the realised A_{t+1}). For each of them, find the “neighbour” candidates, called **vKlow** and **vKHigh** in the code. For simplicity, denote the higher neighbour candidate as $\bar{K}_t^{(n)}$ and the lower neighbour candidate as $\underline{K}_t^{(n)}$.
 - (a) Calculate standard weights for the candidates.
 - (b) Compute the associated wage, labour participation, interest rate, and first derivative of the adjustment cost. For the “**lower**” neighbour, they are:

$$(5.7a) \quad \underline{w}_t^{(n)} = \tilde{A}_t (1 - \alpha) \left(\underline{K}_t^{(n)} \right)^\alpha \left(N_t^{(n)} \right)^{-\alpha}$$

$$(5.7b) \quad \underline{N}_t^{(n)} = \left[\frac{\left(C_t^{(n)} \right)^{-\sigma} \underline{w}_t^{(n)}}{\eta} \right]^\chi$$

$$(5.7c) \quad \underline{r}_t^{(n)} = \tilde{A}_t \alpha \left(\underline{K}_t^{(n)} \right)^{\alpha-1} \left(\underline{N}_t^{(n)} \right)^{1-\alpha} - \delta$$

$$(5.7d) \quad \underline{\Psi}_{k,t}^{(n)} = -\mu \left(\frac{\underline{K}_{t+1}^{(n)} - \underline{K}_t^{(n)}}{\underline{K}_t^{(n)}} \right) \frac{\underline{K}_{t+1}^{(n)}}{\underline{K}_t^{(n)}} + \frac{\mu}{2} \left(\frac{\underline{K}_{t+1}^{(n)} - \underline{K}_t^{(n)}}{\underline{K}_t^{(n)}} \right)^2$$

Then, they are plugged to Equation (5.6) to find:

$$(5.7e) \quad \underline{V}_{k,t}^{(n)} = \left(C_t^{(n)} \right)^{-\sigma} \left[1 + \underline{r}_t^{(n)} - \underline{\Psi}_{k,t}^{(n)} \right]$$

- (c) Compute the interpolated derivative:

$$(5.8) \quad \mathcal{V}_{k,t}^{(n)} = \omega \underline{V}_{k,t}^{(n)} + (1 - \omega) \bar{V}_{k,t}^{(n)}.$$

- (d) Re-do the same computations, without the interpolation element, for the “true” realised shock, A_{t+1} . Combined together, I have:

$$(5.9) \quad \begin{bmatrix} \mathcal{V}_{k,t}^n(A_1) & \text{Counterfactual} \\ \mathcal{V}_{k,t}^n(A_2) & \text{Counterfactual} \\ \dots & \dots \\ V_{k,t}^n(A_m) & \text{Realised} \\ \dots & \dots \\ \mathcal{V}_{k,t}^n(A_{N_a}) & \text{Counterfactual} \end{bmatrix}.$$

- (e) These allow me to compute the expected value of the first derivative of the value function:

$$(5.10) \quad \mathbb{E}_t \left[V_{k,t+1}^n \left(K_{t+1}^{(n)} \right) \right] = \pi_{\text{Realised}} V_{k,t}^n \left(K_{t+1}^{(n)} \right) + (1 - \pi_{\text{Realised}}) \mathbb{E}_t \left[\mathcal{V}_{k,t+1}^n \left(K_{t+1}^{(n)} \right) \mid \text{Counterfactual} \right].$$

- (f) This, in turn, allows me to use the envelope condition to recover the optimal consumption level:

$$(5.11) \quad C_t^{\star(n)} = \left(\frac{1 + \mu \frac{K_{t+1}^{(n)} - K_t^{(n)}}{K_t^{(n)}}}{\beta \mathbb{E}_t \left[V_{k,t+1}^n \left(K_{t+1}^{(n)} \right) \right]} \right)^{-\frac{1}{\sigma}}$$

- (g) Update the labour allocation:

$$(5.12) \quad N_t^{\star(n)} = \left[\frac{\left(C_t^{\star(n)} \right)^{-\sigma} w_t^{(n)}}{\eta} \right]^\chi$$

3. (**Iterate forward**) Next, I allow the new decisions to “flow” to set the new capital level.

- (a) Set the investment rate:

$$(5.13) \quad I_t^{(n)} \equiv A_t \left(K_t^{(n)} \right)^\alpha \left(N_t^{\star(n)} \right)^{1-\alpha} - \Psi \left(K_{t+1}^{(n)}, K_t^{(n)} \right) - C_t^{\star(n)}.$$

- (b) Use it to update the capital level, **vKnew** in my code:

$$(5.14) \quad K_t^{\star(n)} = K_{t-1}^{(n)}(1 - \delta) + I_{t-1}^{(n)}.$$

- (c) Recover the consumption level from the budget constraint:

$$(5.15) \quad C_t^{\star\star(n)} = A_t \left(K_t^{(n)} \right)^\alpha \left(N_t^{\star(n)} \right)^{1-\alpha} - \Psi \left(K_{t+1}^{(n)}, K_t^{(n)} \right) - I_t^{(n)}.$$

4. Compute the error and set the new guesses for consumption and capital levels.

$$(5.16a) \quad C_t^{(n+1)} = \lambda C_t^{(n)} + (1 - \lambda) C_t^{\star\star(n)}$$

$$(5.16b) \quad K_t^{(n+1)} = \lambda K_t^{(n)} + (1 - \lambda) K_t^{\star(n)}$$

5. Compute the new wage and labour supply guesses based on the updated guesses:

$$(5.17a) \quad W_t^{(n+1)} = \left\{ (1 - \alpha) A_t \left(K_t^{(n+1)} \right)^\alpha \left[\frac{\left(C_t^{(n+1)} \right)^{-\sigma}}{\eta} \right]^{-\alpha\chi} \right\}^{\frac{1}{1-\alpha}}$$

$$(5.17b) \quad N_t^{(n+1)} = \left[\frac{W_t^{(n+1)} \left(C_t^{(n+1)} \right)^{-\sigma}}{\eta} \right]^\chi.$$

6. Repeat steps 2-5 until the error term is small enough.

References

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