

# PHD21 Computational methods: Assignments

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# 1 Assignment 1

## Questions 1-4

### Question:

1. Label and interpret the model ingredients properly.
2. Characterise the individual labor supply curve.
3. Characterise the aggregate labor supply curve.
4. Characterise the aggregate labor demand curve

**Step 1A:** Working households → intensive margin labour supply. Start with the working household's problem.

$$W(a, z) = \max_{c, n, a'} \left\{ \log(c) - \eta \frac{1}{1 + \frac{1}{\chi}} n^{1 + \frac{1}{\chi}} + \beta v(a') \right\} \quad (1.1)$$

s.t.  $c + a' = zw(1 - \tau)n + a(1 + r(1 - \tau)) + T$

Construct the Lagrangian, assuming that  $v(a') = \log(a')$ :

$$\mathcal{L} = \log(c) - \eta \frac{1}{1 + \frac{1}{\chi}} n^{1 + \frac{1}{\chi}} + \beta \log(a') + \lambda [zw(1 - \tau)n + a(1 + r(1 - \tau)) + T - c - a'] \quad (1.2)$$

The first-order conditions are as follows:

$$\mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = -\frac{1}{c}, \quad (1.3a)$$

$$\mathcal{L}_n = -\eta n^{\frac{1}{\chi}} + \lambda zw(1 - \tau) = 0 \implies \eta n^{\frac{1}{\chi}} = \frac{zw(1 - \tau)}{c}, \quad (1.3b)$$

and

$$\mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c. \quad (1.3c)$$

Combine the results of Equations (1.3b) and (1.3c) with the budget constraint to arrive at the Euler equation:

$$\underbrace{c + a'}_{\text{Use (1.3c)}} = \underbrace{zw(1 - \tau)n + a(1 + r(1 - \tau)) + T}_{\text{Use (1.3b)}}, \quad (1.4a)$$

$$c(1 + \beta) = zw(1 - \tau) \left( \frac{zw(1 - \tau)}{\eta c} \right)^{\chi} + a(1 + r(1 - \tau)) + T. \quad (\text{WH-ILS})$$

Equation (WH-ILS) governs the **intensive labour supply of a working household**,  $c_w^*(a, z)$  is their consumption level.

Further, notice that  $c_w^*(a, z)$  increases in  $z$ :

$$\frac{\partial c_w^*(a, z)}{\partial z} (1 + \beta) = \underbrace{[w(1 - \tau)\eta^{-1}]^{1 + \chi}}_{\equiv \theta > 0} \times \frac{(1 + \chi)z^{\chi} c_w^*(a, z) - z^{\chi} \frac{\partial c_w^*(a, z)}{\partial z}}{c_w^*(a, z)^2} \implies \quad (1.5a)$$

$$\frac{\partial c_w^*(a, z)}{\partial z} \left( 1 + \beta + \theta \frac{z^{\chi}}{c_w^*(a, z)^2} \right) = \frac{\theta(1 + \chi)z^{\chi}}{c_w^*(a, z)} \implies \quad (1.5b)$$

$$\frac{\partial c_w^*(a, z)}{\partial z} = \frac{\frac{\theta(1 + \chi)z^{\chi}}{c_w^*(a, z)}}{1 + \beta + \theta \frac{z^{\chi}}{c_w^*(a, z)^2}} > 0. \quad (1.5c)$$

Also, differentiate Equation (1.3b):

$$\eta\chi^{-1}n^{\frac{1}{\chi}-1}\frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} = -zw(1-\tau)c_w^*(a, z)^{-2} \implies \quad (1.6a)$$

$$\frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} = \frac{-\chi zw(1-\tau)}{\eta n^{\frac{1}{\chi}-1}c_w^*(a, z)^2}. \quad (1.6b)$$

Using the combination of the Envelope Theorem and chain rule, this implies that, at the optimal consumption-hours bundle, the household sees:

$$\frac{\partial W(a, z)}{\partial z} = \frac{1}{c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} - \eta n^{\frac{1}{\chi}} \frac{\partial n^*(a, z)}{\partial z} \implies \quad (1.7a)$$

$$\frac{\partial W(a, z)}{\partial z} = \frac{1}{c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} - \eta n^{\frac{1}{\chi}} \frac{\partial n^*(a, z)}{\partial c_w^*(a, z)} \frac{\partial c_w^*(a, z)}{\partial z} \implies \quad (1.7b)$$

$$\boxed{\frac{\partial W(a, z)}{\partial z} > 0.} \quad (1.7c)$$

Equation (1.7c) effectively means that  $W(a, z)$  **monotonically increases in  $z$** .

**Step 1B:** Non-working households. The problem is similar here, apart from the labour first order condition. Skipping the Lagrangian setup, I arrive at the following first-order conditions:

$$\mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = -\frac{1}{c} \quad (1.8a)$$

and

$$\mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c, \quad (1.8b)$$

both of which are identical to what we see for the working household. Combining it with the budget constraint, we obtain the consumption function for the non-working household:

$$\boxed{c_{nw}^*(a) = \frac{b + a(1 + r(1 - \tau)) + T}{1 + \beta}.} \quad (1.9)$$

This effectively implies that:

$$\boxed{\frac{\partial N(a, z)}{\partial z} = 0.} \quad (1.10)$$

**Step 2:** Extensive margin labour supply decision. Household  $(a, z)$  enters the labour market when:

$$\mathbf{I}_n(a, z) = \begin{cases} 1 & \text{if } W(a, z) \geq N(a, z) \\ 0 & \text{if } W(a, z) < N(a, z), \end{cases} \quad (1.11)$$

where  $W(\cdot, \cdot)$  and  $N(\cdot, \cdot)$  are the value functions of working and not working, respectively.

Even abstracting from the Unique Point Theorem, we can see that if there exists  $z^*$  such that:

$$W(a, z^*) = N(a, z^*) \quad (1.12)$$

then for  $z > z^*$ , we have:

$$\mathbf{I}_n(a, z) = 1. \quad (1.13)$$

This will come handy while numerically solving the model.

**Step 3:** Aggregate labour supply. Assuming that  $\Phi(a, z)$  is the joint distribution of ex-ante wealth and productivity, the **aggregate labour supply** is:

$$L^S = \int \mathbf{I}_n(a, z) h(a, z) d\Phi(a, z) \quad (1.14)$$

**Step 4A:** Aggregate labour demand. We abstract from the capital markets, which makes the representative firm's problem near-trivial:

$$Y = \max_L \{AK^\alpha L^{1-\alpha} - wL - rK\} \implies \quad (1.15a)$$

$$w = A(1 - \alpha) \left( \frac{L^D}{K} \right)^{-\alpha}, \quad (1.15b)$$

or, putting  $L^D$  on the LHS:

$$\boxed{L^D = \left( \frac{(1 - \alpha)A}{w} \right)^{\frac{1}{\alpha}} K.} \quad (1.15c)$$

## Question 5

**Question:** Suppose the following parameter levels:

$$a = 1, \quad \alpha = 0.3, \quad \tau = 0.15, \quad \bar{z} = 1, \quad A = 1, \quad r = 0.04, \quad \beta = 0.96$$

Define and characterize the stationary recursive competitive equilibrium.

**Step 1:** Parameters left.  $\Xi$  represents the original vector of parameters:

$$\Xi = (a, \eta, \xi, \tau, b, \beta, \sigma_z, A, \alpha, r)^T. \quad (1.16)$$

Given the pre-specified parameters, the unknown ones are:

$$\hat{\Xi} = (\eta, b, \sigma_z,)^T. \quad (1.17)$$

**Step 2:** Equilibrium. Given the distribution of labour productivity,  $\Phi$ , a set of functions  $\{n, c, a', z^*, L, w, T\}$  is a **stationary competitive equilibrium** if

1.  $(n, c, a', z^*)$  solves the household's problem.
2.  $(K, L)$  solves the production sector's problem.
3. The labour and capital markets clear.

**Step 3:** Simplification. Given that we have only one wealth level in the economy,  $a = 1$ , the aggregate labour supply in Equation (1.14) becomes:

$$L^S = \int_{z^*} n(z) d\Phi(z), \quad (1.18)$$

where I set  $\Phi(z) \equiv \Phi(z, 1)$  for expositional clarity.