PHD21 Computational methods: Assignments

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Contents

1 Assignment 1

Questions 1 through 4

Question:

1. *Label and interpret the model ingredients properly.*
2. *Characterise the individual labor supply curve.*
3. *Characterise the aggregate labor supply curve.*
4. *Characterise the aggregate labor demand curve*

Step 1A: Working households → intensive margin labour supply. Start with the working household's problem.

$$\begin{aligned} W(a,z) &= \max_{c,n,a'} \left\{ \log(c) - \eta \frac{1}{1+\frac{1}{\chi}} n^{1+\frac{1}{\chi}} + \beta v\left(a'\right) \right\} \\ &\text{s.t. } c + a' = zw(1-\tau)n + a(1+r(1-\tau)) + T \end{aligned} \tag{1.1}$$

Construct the Lagrangian, assuming that $v\left(a'\right) = \log\left(a'\right)$:

$$\mathcal{L} = \log(c) - \eta \frac{1}{1+\frac{1}{\chi}} n^{1+\frac{1}{\chi}} + \beta \log\left(a'\right) + \lambda\left[zw(1-\tau)n + a(1+r(1-\tau)) + T - c - a'\right]. \tag{1.2}$$

The first-order conditions are as follows:

$$\mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = \frac{1}{c}, \tag{1.3a}$$

$$\mathcal{L}_n = -\eta n^{\frac{1}{\chi}} + \lambda zw(1-\tau) = 0 \implies \eta n^{\frac{1}{\chi}} = \frac{zw(1-\tau)}{c}, \tag{1.3b}$$

and

$$\mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c. \tag{1.3c}$$

Combine the results of Equations (1.3b) and (1.3c) with the budget contraint to arrive at the Euler equation:

$$\underbrace{c + a'}_{\text{Use (1.3c)}} = \underbrace{zw(1-\tau)n}_{\text{Use (1.3b)}} + a(1+r(1-\tau)) + T, \tag{1.4a}$$

$$\boxed{c(1+\beta) = zw(1-\tau)\left(\frac{zw(1-\tau)}{\eta c}\right)^{\chi} + a(1+r(1-\tau)) + T.} \tag{WH-ILS}$$

Equation (WH-ILS) governs the **intensive labour supply of a working household**, $c_w^\star(a,z)$ is their consumption level.

Further, notice that $c_w^\star(a,z)$ increases in z:

$$\frac{\partial c_w^\star(a,z)}{\partial z}(1+\beta) = \underbrace{\left[w(1-\tau)\eta^{-1}\right]^{1+\chi}}_{\equiv \theta > 0} \times \frac{(1+\chi)z^{\chi} c_w^\star(a,z) - z^{\chi}\frac{\partial c_w^\star(a,z)}{\partial z}}{c_w^\star(a,z)^2} \implies \tag{1.5a}$$

$$\frac{\partial c_w^\star(a,z)}{\partial z}\left(1+\beta+\theta\frac{z^{\chi}}{c_w^\star(a,z)^2}\right) = \frac{\theta(1+\chi)z^{\chi}}{c_w^\star(a,z)} \implies \tag{1.5b}$$

$$\frac{\partial c_w^\star(a,z)}{\partial z} = \frac{\frac{\theta(1+\chi)z^{\chi}}{c_w^\star(a,z)}}{1+\beta+\theta\frac{z^{\chi}}{c_w^\star(a,z)^2}} > 0. \tag{1.5c}$$

Also, differentiate Equation (1.3b):

$$\eta\chi^{-1}n^{\frac{1}{\chi}-1}\frac{\partial n^\star(a,z)}{\partial c_w^\star(a,z)} = -zw(1-\tau)c_w^\star(a,z)^{-2} \implies \tag{1.6a}$$

$$\frac{\partial n^\star(a,z)}{\partial c_w^\star(a,z)} = \frac{-\chi zw(1-\tau)}{\eta n^{\frac{1}{\chi}-1}c_w^\star(a,z)^2}. \tag{1.6b}$$

Using the combination of the Envelope Theorem and chain rule, this implies that, at the optimal consumption-hours bundle, the household sees:

$$\frac{\partial W(a,z)}{\partial z} = \frac{1}{c_w^\star(a,z)}\frac{\partial c_w^\star(a,z)}{\partial z} - \eta n^{\frac{1}{\chi}}\frac{\partial n^\star(a,z)}{\partial z} \implies \tag{1.7a}$$

$$\frac{\partial W(a,z)}{\partial z} = \frac{1}{c_w^\star(a,z)}\frac{\partial c_w^\star(a,z)}{\partial z} - \eta n^{\frac{1}{\chi}}\frac{\partial n^\star(a,z)}{\partial c_w^\star(a,z)}\frac{\partial c_w^\star(a,z)}{\partial z} \implies \tag{1.7b}$$

$$\boxed{\frac{\partial W(a,z)}{\partial z} > 0.} \tag{1.7c}$$

Equation (1.7c) effectvely means that **$W(a,z)$ monotonically increases in z**.

Step 1B: Non-working households. The problem is similar here, apart from the labour first order condition. Skipping the Lagrangian setup, I arrive at the following first-order conditions:

$$\mathcal{L}_c = \frac{1}{c} + \lambda = 0 \implies \lambda = \frac{1}{c} \tag{1.8a}$$

and

$$\mathcal{L}_{a'} = \frac{\beta}{a'} - \lambda = 0 \implies a' = \beta c, \tag{1.8b}$$

both of which are identical to what we see for the working household. Combining it with the budget constraint, we obtain the consumption function for the non-working household:

$$\boxed{c_{nw}^\star(a) = \frac{b + a(1 + r(1-\tau)) + T}{1+\beta}.} \tag{1.9}$$

This effectively implies that:

$$\boxed{\frac{\partial N(a,z)}{\partial z} = 0.} \tag{1.10}$$

Step 2: Extensive margin labour supply decision. Household (a,z) enters the labour market when:

$$\mathbf{I}_n(a,z) = \begin{cases} 1 & \text{if} \quad W(a,z) \geq N(a,z) \\ 0 & \text{if} \quad W(a,z) < N(a,z), \end{cases} \tag{1.11}$$

where $W(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are the value functions of working and not working, respectively.

Even abstracting from the Unique Point Theorem, we can see that if there exists $z^\star$ such that:

$$W(a,z^\star) = N(a,z^\star) \tag{1.12}$$

then for $z > z^\star$, we have:

$$\mathbf{I}_n(a,z) = 1. \tag{1.13}$$

This will come handy while numerically solving the model.

Step 3: Aggregate labour supply. Assuming that $\Phi(a, z)$ is the joint distribution of ex-ante wealth and productivity, the **aggregate labour supply** is:

$$L^S = \int \mathbf{I}_n(a, z) h(a, z) \, \mathrm{d}\Phi(a, z) \tag{1.14}$$

Step 4A: Aggregate labour demand. We abstract from the capital markets, which makes the representative firm's problem near-trivial:

$$Y = \max_L \left\{ AK^\alpha L^{1-\alpha} - wL - rK \right\} \implies \tag{1.15a}$$

$$w = A(1-\alpha)\left(\frac{L^D}{K}\right)^{-\alpha}, \tag{1.15b}$$

or, putting L^D on the LHS:

$$\boxed{L^D = \left(\frac{(1-\alpha)A}{w}\right)^{\frac{1}{\alpha}} K.} \tag{1.15c}$$

Question 5

Question: *Suppose the following parameter levels:*

$$a = 1, \quad \alpha = 0.3, \quad \tau = 0.15, \quad \bar{z} = 1, \quad A = 1, \quad r = 0.04, \quad \beta = 0.96$$

Define and characterize the stationary recursive competitive equilibrium.

Step 1: Parameters left. $\boldsymbol{\Xi}$ represents the original vector of parameters:

$$\boldsymbol{\Xi} = (a, \eta, \xi, \tau, b, \beta, \sigma_z, A, \alpha, r)^T. \tag{1.16}$$

Given the pre-specified parameters, the unknown ones are:

$$\hat{\boldsymbol{\Xi}} = (\eta, b, \sigma_z,)^T. \tag{1.17}$$

Step 2: Equilibrium. Given the distribution of labour productivity, Φ, a set of functions $\{n, c, a', z^\star, L, w, T\}$ is a **stationary competitive equilibrium** if

1. $(n, c, a', z^\star)$ solves the household's problem.
2. (K, L) solves the production sector's problem.
3. The labour and capital markets clear.

Step 3: Simplification. Given that we have only one wealth level in the economy, $a = 1$, the aggregate labour supply in Equation (1.14) becomes:

$$L^s = \int_{z^\star} n(z) \, \mathrm{d}\Phi(z), \tag{1.18}$$

where I set $\Phi(z) \equiv \Phi(z, 1)$ for expositional clarity.

Step 4: Equilibrium algorithm. I set out the **algorithm used to compute the equilibrium** given a set of parameters.

1. Guess (w_0, T_0).
2. Compute individual decisions.
   - `fnIntensiveLabourSupply` computes the **intensive labour supply for a working household**. The following equations flesh out the approach leveraging concavity of Equation (WH-ILS). Start at the initial consumption guess, c_0. The code follows the logic fleshed out by the equations below:

$$RHS\,(c) \equiv zw(1-\tau)\left(\frac{zw(1-\tau)}{\eta c}\right)^{\chi} + a(1+r(1-\tau)) + T \tag{1.19a}$$

$$LHS\,(c) \equiv (1+\beta)c \implies \tag{1.19b}$$

$$c_1 = \frac{RHS\,(c_0)}{1+\beta} \tag{1.19c}$$

     and

$$\epsilon_n \equiv c_n - c_{n-1} \implies \tag{1.19d}$$

$$\epsilon_1 = c_1 - c_0. \tag{1.19e}$$

     If it's above the tolerance level, then repeat until it works:

$$c_n = \frac{RHS\,(c_{n-1})}{1+\beta} \implies \tag{1.19f}$$

$$\epsilon_n = c_n - c_{n-1}. \tag{1.19g}$$

     This approach computes the indiviually optimal values of consumption and labour market participation, (c_w, n_w), provided the household chooses to work.
   - `fnExtensiveLabourSupply` determines if household (a, z) chooses to work based on Equation (1.11).
3. Aggregate all labour supply decisions (`fnAggregateLabourSupply`) and compare them with the aggregate labour demand.