STA414 assignment 3

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1.

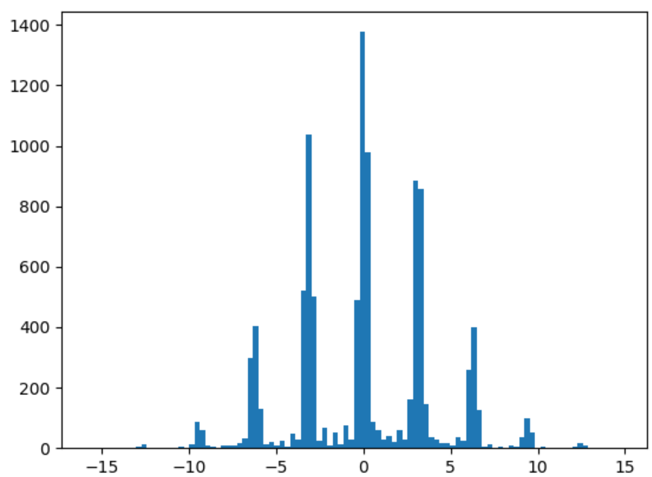
Code:

```
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt
from numpy import sin, sqrt, exp, pi

def d0(x):
    return (1 - (sin(5 * x)) ** 2 / (25 * (sin(x) ** 2))) / sqrt(32 * pi) * exp(-x ** 2 / 32)

def d1(x):
    return (sin(5 * x)) ** 2 / (25 * (sin(x) ** 2)) / sqrt(32 * pi) * exp(-x ** 2 / 32)

def d2(x):
    return (sin(5 * x))** 2 / (25 * (sin(x) ** 2))


def q1a():
    x = np.linspace(-20, 20, 10000)
    y = (1 - sin(5 * x) ** 2 / (25 * sin(x) ** 2 )) / sqrt(32 * pi) * exp(-x**2 / 32)
    fig = plt.figure()
    plt.scatter(x, y, s=1)
    plt.show()
    estimation = np.sum(y)
    return estimation * 40 / 10000


def q1b():
    accept = []
    count = 0
    total = 0
    while count <= 10000:
        s = np.random.uniform(0, 1)
        t = np.random.uniform(-20, 20)
        y = (sin(5*t)**2) / (25 * sin(t) ** 2) / sqrt(32 * pi) * exp(-t**2/32)
        if s <= y:
            accept.append(t)
            count += 1
        total += 1
    plt.hist(accept, 100)
    plt.show()
    return count / total


def q1c():
    w = np.random.normal(0, scale=4, size=1000)
    denom = 0
    for i in range(1000):
        x = w[i]
        y1 = d0(x)
        denom += y1 / norm.pdf(x, 0, scale=4)
    result = 0
    for i in range(1000):
        x = w[i]
        result += (1-d2(x))*d0(x) / norm.pdf(x, 0, scale=4) / denom
    return result


def density(theta):
    px = norm.pdf(1.7, loc=theta, scale=4)
    pg = (np.sin(5 * (1.7 - theta)) ** 2) / (25 * sin(1.7 - theta) ** 2)
    return px*pg/((10*pi)*(1+(theta/10)**2))


def q1d():
    theta = np.linspace(-20, 20, 10000)
    d = density(theta)
    plt.plot(theta, d)
    plt.show()
```

```python
def metropolis(x, iter):
    lst = []
    for i in range(iter):
        u = np.random.normal(x, scale=4)
        accept = min(1, density(u)/density(x))
        r = np.random.uniform(0, 1)
        if r <= accept:
            x = u
        lst.append(x)
    return lst

def q1e():
    lst = metropolis(0, 10000)
    plt.hist(lst, 100)
    plt.show()

def q1f():
    lst = metropolis(0, 10000)
    count = 0
    for i in lst:
        if -3 < i < 3:
            count += 1
    print(count / 10000)

if __name__ == '__main__':
    print(q1a())
    print(q1b())
    print(q1c())
    q1d()
    q1e()
    q1f()
```

a. 0.799919471407323

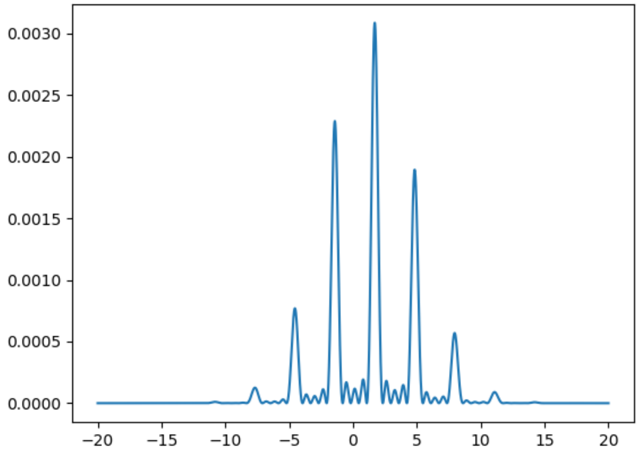
b.



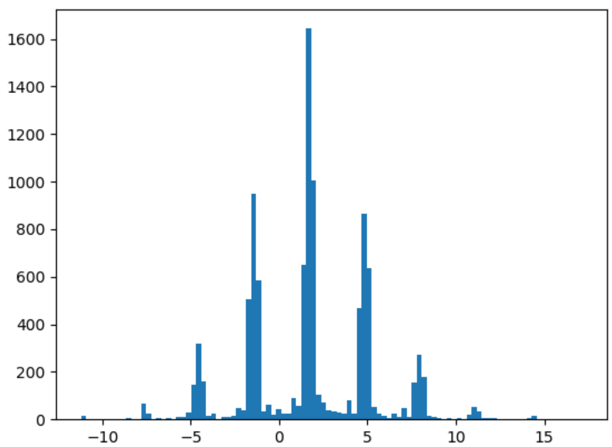
Accepted rate: 0.004998263277718346

c. 0.9253319948922147

d.


0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
−20
−15
−10
−5
0
5
10
15
20

e.


1600
1400
1200
1000
800
600
400
200
0
−10
−5
0
5
10
15

f.

0.6212

2.

a.

First apply the log derivation, then exchange the order of differentiation and integration.

$E_{p(x;\theta)}[\nabla_\theta \log p(x;\theta)] = E_{p(x;\theta)}\left[\frac{\nabla_\theta p(x;\theta)}{p(x;\theta)}\right]$

$= \int p(x;\theta)\frac{\nabla_\theta p(x;\theta)}{p(x;\theta)}dx$

$= \nabla_\theta \int p(x;\theta)dx = \nabla_\theta 1 = 0$

b.

$E_{p(b|\theta)}\left[f(b)\frac{\partial}{\partial\theta}\log p(b|\theta)\right] = \int p(b|\theta)\frac{\partial}{\partial\theta}\log p(b|\theta)\,f(b)db$

$= \int \frac{p(b|\theta)}{p(b|\theta)}\frac{\partial}{\partial\theta}p(b|\theta)f(b)db$

$= \frac{\partial}{\partial\theta}p(b|\theta)f(b)db$

$= \frac{\partial}{\partial\theta}E_{p(b|\theta)}[f(b)]$

Therefore, the reinforce is unbiased.

c.

$E_{p(b|\theta)}\left[[f(b)-c]\frac{\partial}{\partial\theta}\log p(b|\theta)\right] = E_{p(b|\theta)}\left[f(b)\frac{\partial}{\partial\theta}\log p(b|\theta) - c\frac{\partial}{\partial\theta}\log p(b|\theta)\right]$

$= E_{p(b|\theta)}\left[f(b)\frac{\partial}{\partial\theta}\log p(b|\theta)\right] - cE_{p(b|\theta)}\left[\frac{\partial}{\partial\theta}\log p(b|\theta)\right]$

$= \frac{\partial}{\partial\theta}E_{p(b|\theta)}[f(b)] - 0$ from part a and part b

$= \frac{\partial}{\partial\theta}E_{p(b|\theta)}[f(b)]$

Therefore, the reinforce with a fixed baseline is unbiased.

d.

Let $c(b) = f(b)$

$E_{p(b|\theta)}\left[[f(b)-c(b)]\frac{\partial}{\partial\theta}\log p(b|\theta)\right] = E_{p(b|\theta)}\left[0 * \frac{\partial}{\partial\theta}\log p(b|\theta)\right]$

$= 0 \neq \frac{\partial}{\partial\theta}E_{p(b|\theta)}[f(b)]$

Therefore, the reinforce is biased.

3.

a.

$Var(\hat{L}_{MC}) = Var(\sum_{d=1}^{D} x_d)$

$since\ x_i \sim iid\ N(\theta_i, 1)$

$Var(\sum_{d=1}^{D} x_d) = \sum_{d=1}^{D} Var(x_d) = \sum_{d=1}^{D} 1 = D$

b.

$x_d \sim p(x_d|\theta_d) = \frac{1}{2\pi}\exp\left(-\frac{(x_d-\theta_d)^2}{2}\right)$

$\log p(x_d|\theta_d) = -\frac{(x_d-\theta_d)^2}{2} - \log(2\pi)$

$\frac{\partial}{\partial\pi}\log p(x_d|\theta_d) = (x_d - \theta_d)$

$f(x) - c(\theta) = \sum_{d=1}^{D}(x_d - \theta_d)$

$\widehat{g_\iota}^{SF}(f) = \sum_{d=1}^{D}(x_d - \theta_d)(x_i - \theta_i)$ where $i \in \{1, \ldots, D\}$

$x_d - \theta_d = \varepsilon_d$ since $x_d \sim N(\theta d, 1)$, $\varepsilon_d \sim N(0,1)$

$\widehat{g_\iota}^{SF}(f) = \varepsilon_d \varepsilon_i$

$\hat{g}^{SF} = (\widehat{g_1}^{SF}(f), \widehat{g_2}^{SF}(f), \ldots, \widehat{g_D}^{SF}(f))$

c.

$Var\left(\widehat{g_1}^{SF}(f)\right) = Var(\varepsilon_1 \sum_{d=1}^{D} \varepsilon_d) = Var(\varepsilon_1 * \varepsilon_1 + \varepsilon_1 \sum_{d=2}^{D} \varepsilon_d) = Var(\varepsilon_1^2 + \varepsilon_1 \sum_{d=2}^{D} \varepsilon_d)$

$= E\left(\left(\varepsilon_1^2 + \varepsilon_1 \sum_{d=2}^{D} \varepsilon_d\right)^2\right) - E\left(\varepsilon_1^2 + \varepsilon_1 \sum_{d=2}^{D} \varepsilon_d\right)^2$

$E\left(\left(\varepsilon_1^2 + \varepsilon_1 \sum_{d=2}^{D} \varepsilon_d\right)^2\right) = E\left(\varepsilon_1^4 + 2\varepsilon_1^3 \sum_{d=2}^{D} \varepsilon_d + \varepsilon_1^2\left(\sum_{d=2}^{D} \varepsilon_d\right)^2\right)$

$= E(\varepsilon_1^4) + 2E\left(\varepsilon_1^3 \sum_{d=2}^{D} \varepsilon_d\right) + E\left(\varepsilon_1^2\left(\sum_{d=2}^{D} \varepsilon_d\right)^2\right)$

since $\varepsilon_d \sim iid\ N(0,1), E(\varepsilon_i \varepsilon_j) = E(\varepsilon_i)E(\varepsilon_j)\ for\ all\ i \neq j,$

$= E(\varepsilon_1^4) + 2E(\varepsilon_1^3)E\left(\sum_{d=2}^{D} \varepsilon_d\right) + E(\varepsilon_1^2) + E\left(\left(\sum_{d=2}^{D} \varepsilon_d\right)^2\right)$

based on hint, $E[\varepsilon_i^4] = 3, E[\varepsilon_i^3] = 0, E[\varepsilon_i^2] = Var(\varepsilon_i) + E(\varepsilon_i)^2 = 1, E(\varepsilon_i) = 0$ for $i \in \{1, \ldots, D\}$

$= 3 + 2 * 0 * 0 + 1 * \sum_{d=2}^{D} E(\varepsilon_d^2)$

$= 2 + D$

$E\left(\varepsilon_1^2 + \varepsilon_1 \sum_{d=2}^{D} \varepsilon_d\right)^2 = \left(E(\varepsilon_1^2) + E(\varepsilon_1)E\left(\sum_{d=2}^{D} \varepsilon_d\right)\right)^2 = (1+0)^2 = 1$

$Var\left(\widehat{g_1}^{SF}(f)\right) = 2 + D - 1 = 1 + D$

d.

$\hat{g}^{REPARAM}(f) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}$

$f = \sum_{d=1}^{D} x_d$

$\frac{\partial f}{\partial x_i} = 1$

$x_i = \theta_i + \varepsilon_i$

$\frac{\partial x_i}{\partial \theta_i} = 1$

$\widehat{g_\iota}^{REPARAM}(f) = 1 * 1 = 1$

$\hat{g}^{REPARAM} = (1, \ldots, 1)$, size (1xD)

$Var\left(\widehat{g_\iota}^{REPARAM}\right) = Var(1) = 0$