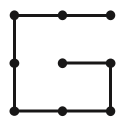
PART II

HINTS, SOME SOLUTIONS, AND FURTHER THOUGHTS

1 Distribution Dilemmas

1.1 A Shepherd and His Sheep

Many puzzles require the reader to think beyond the boundaries suggested (but not enforced!) by the problem. Connecting dots in a 3×3 array with contiguous straight line segments is a prime example of such a puzzle: *It is possible to connect nine dots in a* 3×3 *array with five contiguous line segments. It can also be done in four. Can you see how*?



These inheritance puzzles similarly require stepping beyond the boundaries implied. As a hint, what is the lowest common multiple of 2, 3, and 9? What is the sum $\frac{1}{2} + \frac{1}{3} + \frac{1}{9}$?

Comment. The three sons could, of course, convert the flock to mutton and divvy up the meat according to weight. But what should the sons do with the leftover mutton—divvy it up again according to the same proportions?

1.2 Iterated Sharing

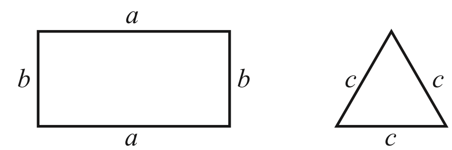
At the end of each iteration do two things. First, count the number of people with the smallest amount of candy. Second, note the largest amount of candy any one person possesses. What do you notice?

2 Weird Shapes

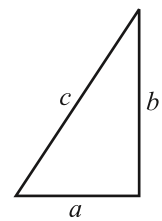
2.1 Plucky Perimeters

Ignoring units, their perimeters equal their areas!

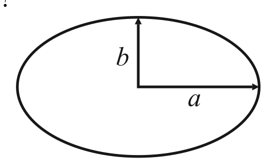
Taking it Further. Is there a rectangle with this property? An equilateral triangle?



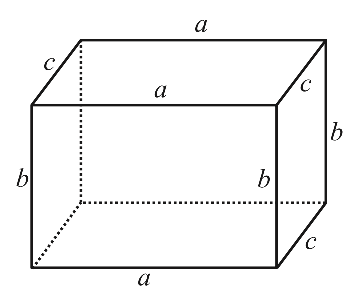
Only one other right triangle with integer side lengths has this property. What is it?



Is there a (non-circular) ellipse with this property?



Taking it Even Further. Is there a rectangular box whose volume equals both its surface area and the total sum of its edge lengths?



2.2 Weird Wheels

(*WARNING:* I used some sophisticated college mathematics!) The parametric equations of a circular wheel, unit radius, are:

$$(0 + \cos\theta, 0 + \sin\theta) \quad 0 \le \theta \le 360^\circ.$$

In these equations I have made the center of the circle, (0, 0), explicit. This certainly has the property that the distance between any two points 180º apart is always 2.

We can modify these equations by constantly changing the location of the center of the circle. As long as we ensure that the center returns to the same place after 180º, the distance between two points that are this angle apart will again always be 2. For example,

$$\left(\tfrac{1}{8}\sin(2\theta)+\cos\theta, \tfrac{1}{10}(2\theta)+\sin\theta\right)\ 0\le\theta\le 360^\circ$$

does the trick. (In fact these were the equations I used to generate the shape given.)

How could you achieve the same effect *without* the use of a computer and sophisticated mathematics?

2.3 Square Pegs and Not-so-Round Holes

Given the hint and solution to Problem 2.2, modifying the equation of a circle so that its center "wobbles" could conceivably produce new shaped holes with the desired property. We would want the wobbling center to return to the location after each 90º rotation, given the rotational symmetry of a square. Is there a simple ruler and string construction that would do this for us? Can we produce these shapes with the aid of a computer?

3 Counting the Odds … and Evens

3.1 A Coin Trick

When the coins are first tossed, Han quickly checks to see whether an even or odd number of heads is showing. He counts the number of times the word "flip" is uttered and again checks the evenness or oddness (that is, the *parity*) of the number of heads when he opens his eyes. This gives Han enough information to determine the state of the coin covered by John's hand.

3.2 Let's Shake Hands

Suppose N people are in the room. Let n_1 be the number of times the first person shakes hands, n_2 the number of times the second person shakes hands, and so on. What can you say about the sum

$$n_1 + n_2 + \cdots + n_N?$$

Can all the numbers n_i be odd?

3.3 Forty Five Cups

Initially the number of upright cups is odd. What do you notice about the number of upright cups after each move?

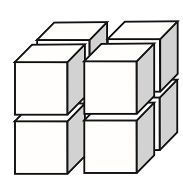
3.4 More Plastic Cups

A cup in the nth position is turned over once for every divisor d of n. Which numbers have an odd number of divisors?

4 Dicing, Slicing, and Avoiding the Bad Bits

4.1 Efficient Tofu Cutting

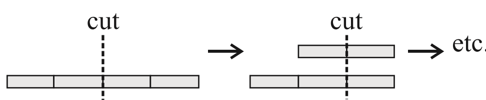
A $2 \times 2 \times 2$ cube of tofu, for example, cannot be diced into eight smaller cubes in fewer than three planar slices. Each corner cube has three



faces that must be cut, and no single planar slice will cut more than one of them.

4.2 Efficient Paper Slicing

It is helpful to take the problem down yet another dimension! With stacking allowed, what is the minimal number of cuts needed to divide a piece of string *n* units long into *n* unit segments?



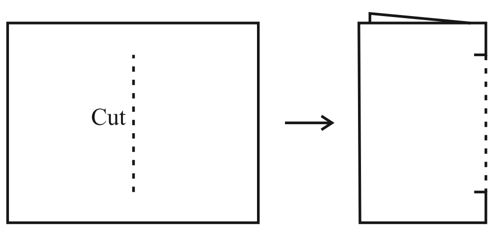
4.3 Bad Chocolate (Impossible!)

For the game with the 4 × 8 bar, I advise you to be the first player. In the 4 × 4 game, however, I strongly recommend your being the second player.

5 "Impossible Paper Tricks

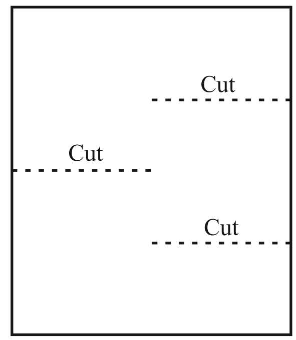
5.1 A Big Hole

Begin by cutting a slit along a center line and then folding the card in half. Make some more cuts. Think about it.

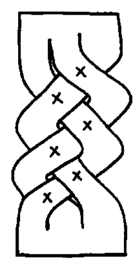


5.2 A Mysterious Flap

Begin by taking an ordinary sheet of paper and make three cuts to the center line as shown. Now what?

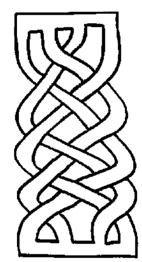


5.3 Bizarre Braids



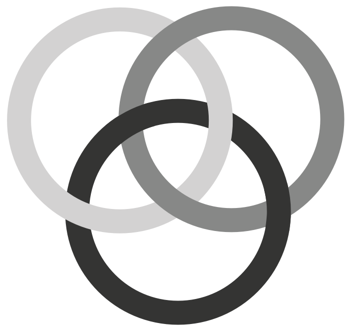
Notice that in the braid there are six places where two strands cross over. This is necessarily the case if the left strand is to return to the left position, the middle strand to the middle, and the right strand to the right. Hold the paper (or felt) in front of you with both hands and begin braiding from the top, ignoring whatever happens to the bottom half of the paper. Perform five crossovers and hold the fifth one firmly with one hand. *Do not let go!* With the other hand, do whatever it takes to untangle the bottom half. In the process a sixth crossover point will occur.

Taking it Further. Is it possible to make a four-strand braid with no free ends?



5.4 Linked Unlinked Rings

Yes it is! The diagram on the next page is flat. In three dimensions, however, you can place one ring flat in the *xy*-plane, the second flat in the *yz*-plane and the third flat in the *xz*-plane. This leads to a particularly pleasing arrangement. It is fun to construct this design with the aid of two



friends. One person forms a ring with the thumbs and index fingers from each hand. The other two people then do the same but along orthogonal planes about the first person's hands. (Try it!)

Comment. These three rings are known as the Borromean rings. They appeared on the coat of arms of the famous Italian Renaissance family of Borromeo.

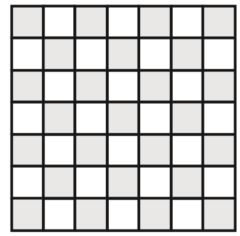
Taking it Further. Is it possible to link *four* rings of paper so that on any single cut the whole configuration will separate into four pieces?

Aside. Ring snipping reminds me of a puzzle. You have a gold chain of six links and have agreed to give to your friend one or more or perhaps even all of the links. The roll of a die will decide the number. Which one link could you cut that would ensure your ability to pay your friend no matter the outcome on the die?

6 Tiling Challenges

6.1 Checkerboard Tiling I

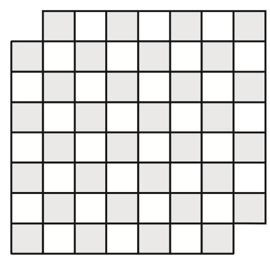
Color the 7×7 array according to a standard checkerboard scheme of 25 black cells and 24 white cells. As each domino covers one cell of each color, any tilable configuration of 48 squares must be composed of an equal number of black and white squares. Excising a white



cell thus leaves an **un**tilable diagram. What can be said about removing a black cell?

6.2 Checkerboard Tiling II

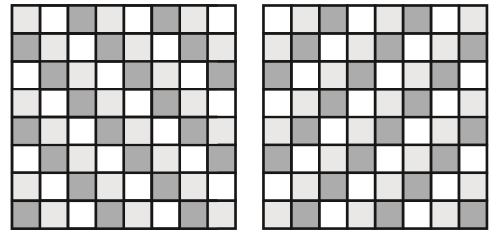
Again consider the standard checkerboard coloring scheme. Two cells of the same color have been excised leaving a configuration with an unequal number of black and white cells, rendering it untilable.



Taking it Further. Two arbitrary cells of opposite color are excised from an 8×8 checkerboard. Is the remaining configuration guaranteed to be tilable?

6.3 Checkerboard Tiling III

It is possible. The key lies in the placement of the single 1×1 tile. Consider these two coloring schemes of the 8×8 grid to help figure out just where that single tile should go.



6.4 Checkerboard Tiling IV

Having trouble? Maybe such a tiling does not exist. How many dominoes would you need to ensure that each potential separating line is crossed by a tile?

7 Things That Won't Fall Down

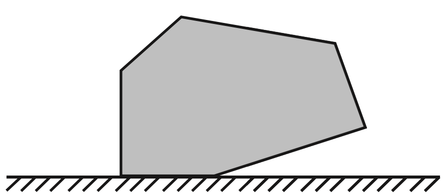
7.1 Wildly Wobbly

Of course not. Any such figure would perpetually topple from one face to the next and never come to rest. Perpetual motion devices do not exist.

Taking It Further. A cube has six stable faces; it sits at rest no matter which face it is on. The figure on page 15 has five stable faces and one unstable face (assuming of course that the figure is made from uniformly dense material with no hidden weights or hollows).

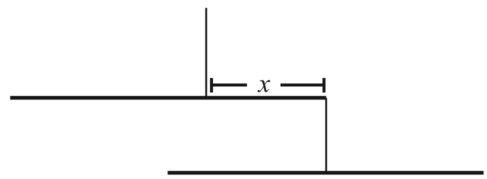
Is it possible to design a six-faced polyhedron with precisely two unstable faces? How about one with three? Four? Does there exist such a figure with five unstable faces? Try carving models of six-faced polyhedra out of florist's foam.

Taking It Even Further. Experiment with the two-dimensional analog of this problem by studying the motion of polygonal wheels. Given how wheels roll, it is appropriate to assume our wheels are convex in shape. As perpetual motion devices do not exist, every wheel must possess at least one stable edge. Is it possible to design a polygonal wheel with *precisely one* stable edge? The wheel may have as many sides as you wish.

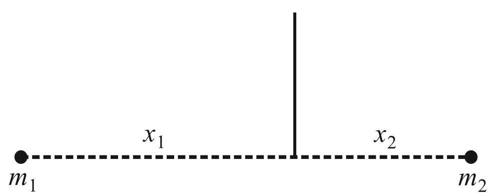


7.2 A Troubling Mobile

A single wire is balanced with a thread attached to its midpoint. The first challenge is to find where a two-wire system will balance if the other end of the thread is attached to a second wire.



Archimedes' Law of the Lever says that two weights m_1 and m_2 at distances x_1 and x_2 from the fulcrum will balance precisely when $x_1 m_1 = x_2 m_2$. If each wire is one foot long and

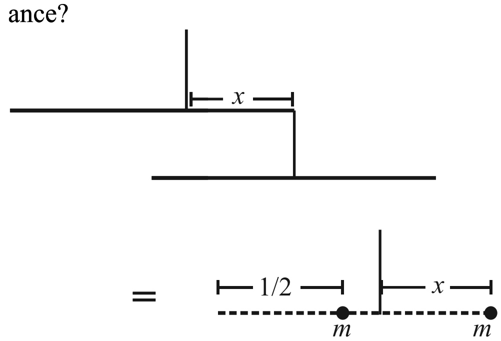


has mass m the two-wire system is equivalent to two balanced masses, one being the center of mass of the top wire at its midpoint, and the other the mass of the lower wire at the end. (See figure below.) Archimedes' Law thus dictates

$$\left(\frac{1}{2} - x\right) m = xm,$$

forcing $x = 1/4$.

At which point will a three-wire system balance?



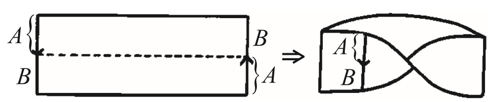
7.3 A Troubling Tower

This problem is an upside-down version of 7.2, and its solution is essentially the same.

8 Möbius Madness: Tortuous Twists on a Classic Theme

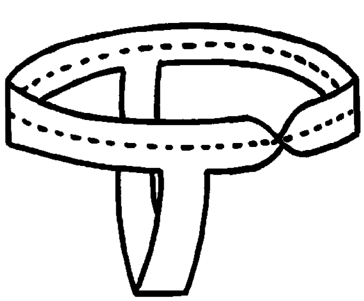
8.1 Möbius Basics

A Möbius band is formed by gluing together the ends of a long strip of paper in reverse orientation. Notice that the two segments marked A are glued together, as are the remaining two line segments labelled B. How many pieces will result, then, when this diagram is cut along the center line?



8.2 A Diabolical Möbius Construction

This construction is really a simple Möbius band with the addition of a single connecting strip. Under what conditions will this figure separate into two distinct pieces when cut?



8.3 Another Diabolical Möbius Construction

Cut around the non-twisted band first.

9 The Infamous Bicycle Problem

9.1 Which Way Did the Bicycle Go?

Think of how a bicycle is constructed. The back wheel is fixed in its frame yet the front wheel can turn and even wobble. We thus deduce that the more stable track is the back wheel track and the other the front wheel track. Moreover, the back wheel of a bicycle is fixed in its frame so as to *always point towards the front wheel.* What does this imply about the structure of the two curves?

9.2 Pedal Power

The pedal is being pushed in a direction that would normally drive the bicycle forward. So the bicycle moves forward, right? Try it on a bicycle and see what happens!

9.3 Yo-Yo Quirk

Again, try it! See what happens.

10 Making Surfaces in 3- and 4-Dimensional Space

10.1 Making a Torus

First form a cylinder by gluing the top edge A to the bottom edge A. What must you then do

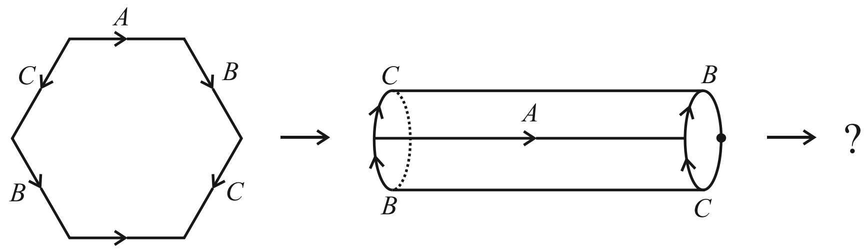
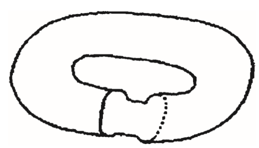


Figure 10.1

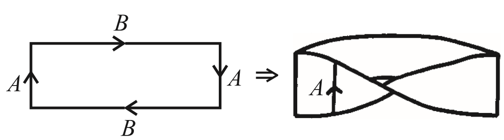
to this cylinder to complete the edge identifications? See Figure 10.1.

10.2 A Torus with a Serious Twist

This half twist changes the topology of the problem significantly. If you attempt this feat, you will notice that the two ends of the tube being created end up lying on opposite sides of the band. It is not clear what to do next to complete the edge identification.



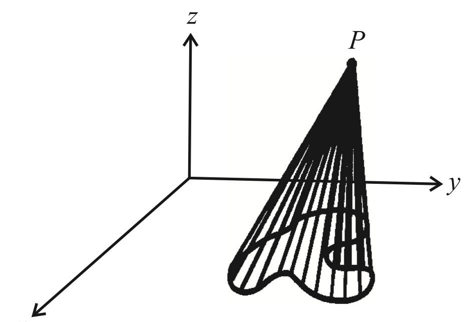
Here's another way to think about the problem. A *Möbius band* is obtained from a rectangular strip of paper by gluing the two ends together, here labelled A, with reverse orientation.



In this exercise we are asked to then glue together the edges marked B. Try instead gluing the B edges together first to form a cylinder, and then the edges marked A. Again, make sure the directions of arrows match. See Figure 10.2.

10.3 Capping Möbius

Consider the following question: Is it always possible to sew the boundary of a flexible disc onto a curve drawn in a plane? The answer is yes, but you may have to move into the third dimension to do it. Regarding a disc as com-



posed of a center point P and a dense set of spokes reaching out to the circular boundary, one can always create a circus tent construction by placing P in a third dimension to the plane. This obviates all possible self-intersections no matter how complex the shape of the curve. How then would you sew a disc to the boundary of a Möbius band?

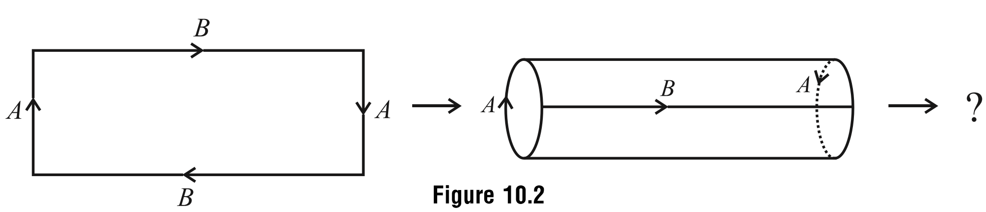


Figure 10.2

11 Paradoxes in Probability Theory

11.1 The Money or the Goat?

Try simulating this game with a friend using inverted plastic cups as doors and a piece of candy as the fabulous prize. Play the game several times, taking turns being the game show host and the contestant. What do you notice about the host's choices as to which "door" to reveal in the intermediate part of the game? How often does the contestant seem to win if a "never switch" strategy is adopted? What about an "always switch" approach?

11.2 Double or … Double!!

There is a 50% chance that the other bag contains half as many Tootsie Rolls® as you just counted and a 50% chance it is double the number. By switching you could gain far more than you could lose. It is to your advantage then to switch. But wait! Wouldn't you follow the same line of reasoning no matter which bag you chose first? You would always opt to switch. Why then didn't you just take the other bag in the first place? What's going on?

11.3 Discord among the Chords

If you perform the experiment suggested, you will find that about half of the wires cross the circle at lengths greater than the side length of the triangle. Joi's reasoning is absolutely correct.

But there are other ways to select chords at random. What if you spin a bottle in the center of the circle to select points on the perimeter to connect with a chord, or throw a dart at a circle to determine the midpoint of a chord? Both Jennifer and Bill's lines of reasoning are absolutely correct, too! What's going on?

11.4 Alternative Dice

Yes, it is possible. Label one die with 1, 3, 4, 5, 6, and 8.

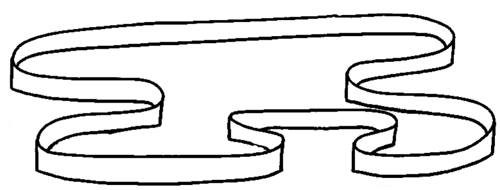
12 Don't Turn Around Just Once

12.1 Teacup Twists

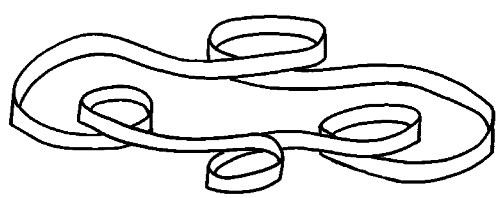
This puzzle is interesting only if you have three or more strings attached to the cup. It is easy to untangle two strings no matter how many times the cup is rotated. Given just one rotation, it is impossible to untangle three or more strings. (Try the case with just three strings. Also see section 5.2.) But with two rotations this is no longer the case! It suddenly becomes possible to untangle all strings. Try lining the strings in a row above the teacup as shown in the photo on the facing page before rotating it 720°.

12.2 Rubber Bands and Pencils

Lay a large band of paper on a table top. How many loops can you make along this band with-



out producing any twists in the paper — that is, in such a way that along every part of the band the "wall" of paper remains essentially vertical?



Diane Dixon and Lusine Ayrapetian hold up the strings before rotating the system 720°.

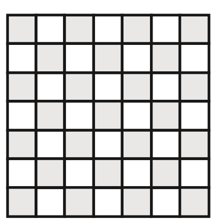
13 It's All in a Square

13.1 Square Maneuvers

When trying this out, after the fun of first letting everyone move simultaneously, try one person at a time, as follows. The first person moves to an occupied square. Its occupant then moves either to another occupied square or to an empty square — her choice. Continue in this fashion. Once someone moves to an empty square, a "circuit" of people is complete, and someone else must be nominated to take the next step. As you do this, count the number of people that constitute each circuit. What do you observe?

13.2 Path Walking

Color the grid according to the standard checkerboard coloring scheme. Any path you walk alternates between black and white cells. Is it possible then to commence a path on a white cell?



13.3 Square Folding

Does "63" sound familiar? What's going on?

14 Bagel Math

14.1 Slicing a Bagel

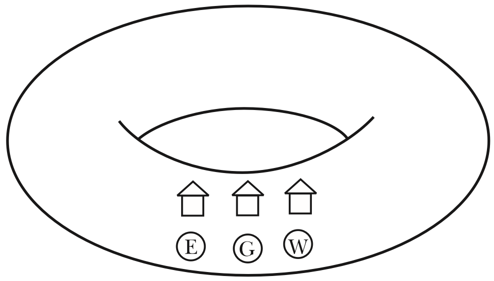
There is one other way to do this. Think diagonally!

14.2. Disproving the Obvious

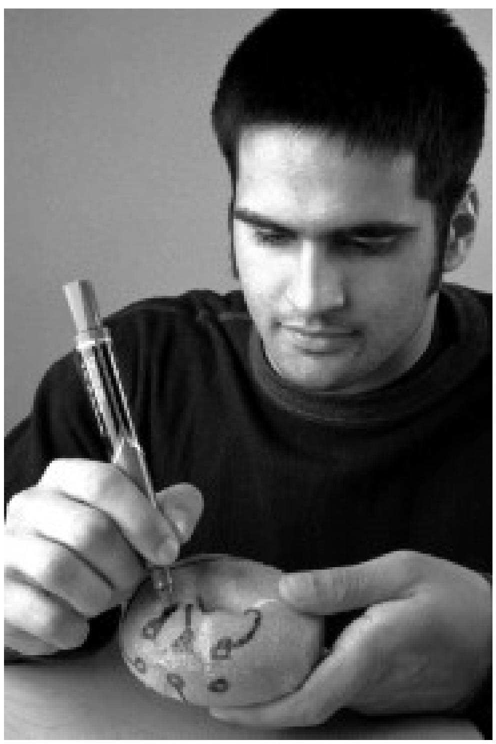
Think of Christopher Columbus!

14.3. Housing on a Bagel

Alas, no! If there were a solution on the surface of a sphere, simply puncturing the sphere and stretching the surface flat would yield a planar solution. As no such planar solution exists (see [Char], Chapter 9 and [Gard15], Chapter 11) there can be no spherical solution.



Taking It Further. What if the earth were the shape of a bagel? Could the problem be solved on a toroidal planet instead?

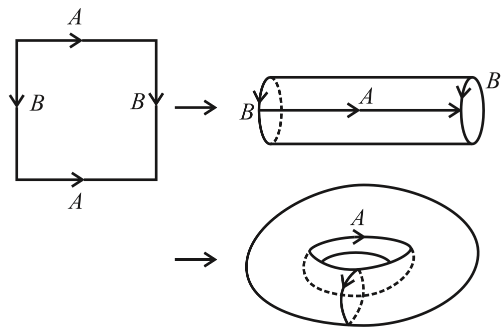


Alex Alapatt examines the housing problem on a bagel.

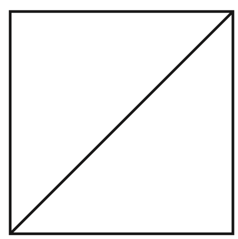
14.4. Tricky Triangulations

To answer the first question, count the number of edges in a theoretical triangulation by counting the number of triangles. (Each triangle has three edges. If you know that there are t triangles in all, do you now know the total number of edges?)

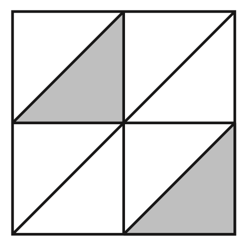
With regard to the second question, think of a torus as being formed from a square piece of paper by identifying (gluing) opposite edges. (See chapter 10.) Try experimenting with tri-



angulations on a square rather than the fully formed three-dimensional object. But keep in mind that in a valid triangulation of the torus the triangles can touch only at a single vertex or along a single edge. For instance, here are two pictures that *do not* represent valid triangulations of the torus.

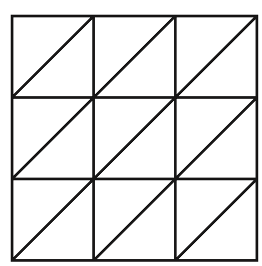


These two triangles touch along all three edges.



The two shaded triangles touch at two vertices.

The following drawing is, however, a triangulation of the torus. It uses 18 triangles — but you can do much better!



14.5. Platonic Bagels

Suppose m edges meet at each vertex. Count the number of edges in a theoretical picture in two different ways: If we are told there are a total of v vertices, do we now know the total number of edges? If we are told there are r regions, do we know the total number of edges?

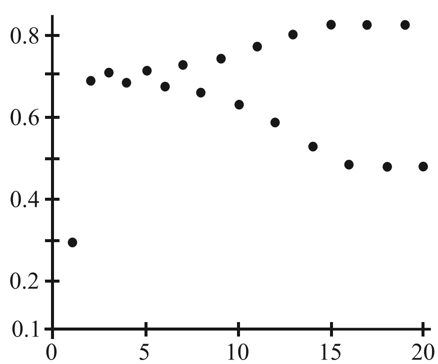
15 Capturing Chaos

15.1 Feedback Frenzy

Why do you sometimes hear those horrible screeching sounds when people are setting up microphones and amplifying systems? The situation is analogous.

15.2 Creeping up on Chaos

With $r = 3.3$ the sequence no longer converges to a single value (as for $r = 2.5$). Rather a steady oscillation between two values results.



Challenge. By experimenting with intermediate values of r, can you determine precisely where this transition, or *bifurcation,* takes place? Keep going to see what happens with higher values of r.

16 Who has the Advantage?

16.1 A Fair Game?

Can you tell from experimenting who wins most often? Try to analyze mathematically a game where one player has three coins and the other two, and then one involving four and three coins.

16.2 Voting for Pizza

It really is in Alice's best interest to vote pepperoni. She reasons as follows: "Either Brad and Cassandra will agree and submit the same vote, or they will disagree in their choices. In the first case, it does not matter what I vote, for their choice will win. In the second case, however, pepperoni will win — either by a majority vote or by my veto power within a three-way tie. It is to my advantage then to vote pepperoni."

But Brad and Cassandra are aware that Alice must reason this way and she will vote pepperoni. Knowing Alice's vote, how could Brad and Cassandra now turn the situation to their advantage?

16.3 A Three Way Duel

Notice that Alberto survives about 31% of the time, Bridget 54% of the time, and Case about 15%. Repeat the experiment again, but this time have Alberto *deliberately miss* his first shot, killing no one even if his shot would have been successful. Have Bridget and Case play as before. What do you notice this time? What's going on?

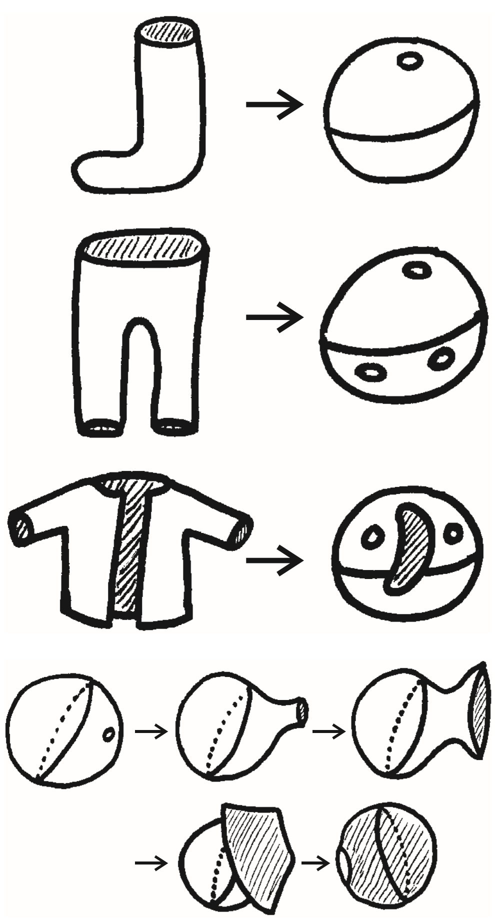
16.4 Weird Dice

Notice die A will beat B two thirds of the time. How often will B beat C? C beat D? D beat A? If you choose a die first, which die do you think I will choose in response?

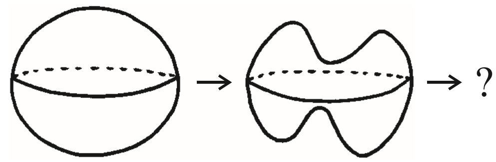
17 Laundry Math

17.1 Turning Clothes Inside Out

All clothes are basically the same shape: they are spheres; punctured spheres, to be precise. Socks are basically deformed spheres with the ankle holes providing punctures in their sides. Trousers, sweaters, and even button-down shirts are spheres with three punctures (ignoring the buttonholes). The eversion of such objects is straightforward: Everting a punctured sphere clearly yields another sphere, and the shape and structure of clothing thus does not change.

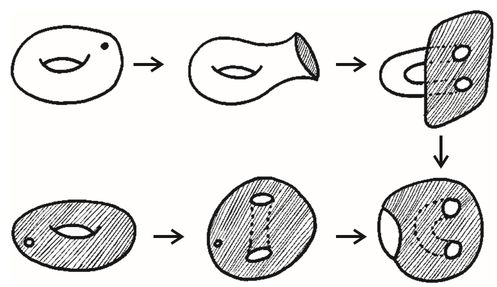


Taking it Further. Is it possible to evert a *non*-punctured sphere?

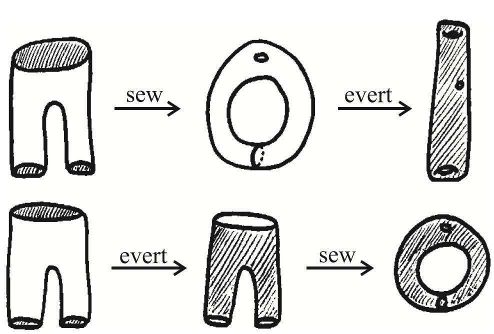


17.2 Mutilated Laundry

As we saw in 17.1, turning a sphere inside out yields another sphere. Surprisingly, turning a punctured donut inside out yields another donut! What happens for multi-donuts?



Taking it Further. Notice that the material forming the tube or *handle* of the original donut before eversion becomes the material forming the hole of the donut after eversion. For a pair of trousers this is significant. Two trouser legs sewn together form a very long and skinny donut handle. Upon eversion this must become the donut hole and the result is an elongated donut difficult to recognize as such at first. (Did you see it as another donut when you tried it? Perhaps try instead everting a punctured donut sewn together from a square piece of material.)



The surprising thing is that the outcome is *not* the same as first turning a pair of trousers inside out and then sewing the two legs together. The actual eversion produces an alternative result. We have discovered then two, possibly different, eversions of the same donut! To what extent are these two eversions distinct? Is there some way to manipulate one eversion into the other with the same pair of trousers? Try experimenting with a pair of trousers again.

17.3 Cannibalistic Clothing

Try it! See what happens.

18 Get Knotted

18.1 Party Trick I: Two Linked Rings?

I have deliberately misled you in my note! It is imperative that the strings be tied around the participants' wrists rather than held by hand if they are to escape.

18.2 Party Trick II: A T-Shirt Trick

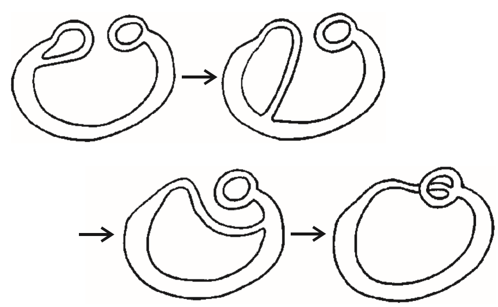
In both cases it is possible!

18.3 Party Trick III: A Waistcoat Trick

Begin by slipping both arms through the arm holes of the waistcoat.

18.4 Two More Linked Rings?

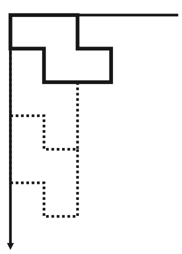
Begin by molding the stem of one loop across the body of Play-Doh® and then to the base of the other ring. (In terms of a human figure this would mean molding the base of a thumb down the length of one arm, across the chest and then up the other arm.)



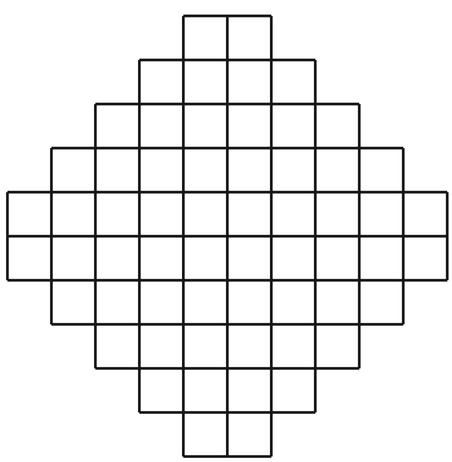
19 Tiling and Walking

19.1 Skew Tetrominoes

There is essentially only one way to place a skew tetromino in the topleftmost corner of a square or rectangular grid while staying within the boundary of the grid. The placement of this tile then forces the placement of a tile immediately below it, which in turn forces the placement of another tile below that, and so on, all the way down to the bottom leftmost corner. Assuming we want to stay within the boundary of the rectangle (which we do!), there is no room to place the final tile at the end of this chain. Thus it is impossible to tile any size rectangular grid with skew tetrominoes.



Taking it Further. Is it possible to tile this region with skew tetrominoes?

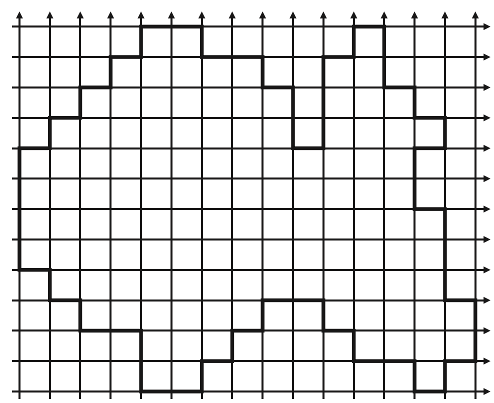


19.2 Map Walking

Such paths do exist. Given the inherent symmetry in the diagrams, none of them depend on the choice of starting intersection for either participant.

19.3 Bringing it Together

Place a copy of the boundary of the region on the map for city A. In walking the path of this boundary, will the inhabitant of city B, following a shadow journey, also walk a closed loop? What does this tell us about the tilability of the region?



20 Automata Antics

20.1 Basic Ant Walking

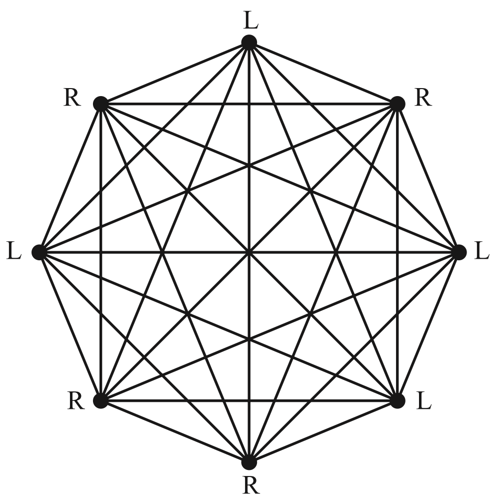
How many steps must the ant take to complete a loop?

20.2 Ant Antics

A 7×7 grid is a bit unwieldy. Try the problem on a 3×3 grid instead.

20.3 Ball Throwing

We can view this problem as one of ant motion on a *complete graph*. Such a graph consists of finitely many vertices with edges connecting all possible pairs of vertices. Each vertex is labelled "L" or "R" and the ant wanders from vertex to vertex turning either the sharpest possible left or sharpest possible right according to the label of the vertex it visits (and then changing the label of that vertex). Notice that the sharpest possible left turn could actually be a right turn if the ant is travelling along an outer edge approaching a vertex labelled L (hence our peculiar convention for the special case of ball throwing).

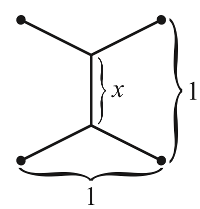


Another way to experiment with this ball throwing problem is to draw a large complete graph on the floor and have one person move along its edges like the ant. People standing at the vertices help direct the ant's motion by calling out the appropriate left or right turn. Note that the role of L's and R's is reversed in this interpretation.

21 Bubble Trouble

21.1 Road Building

Try a symmetrical design as shown. If the towns are situated on a square one (large) unit wide, what value for x is best? (*Warning:* Calculus is needed!)



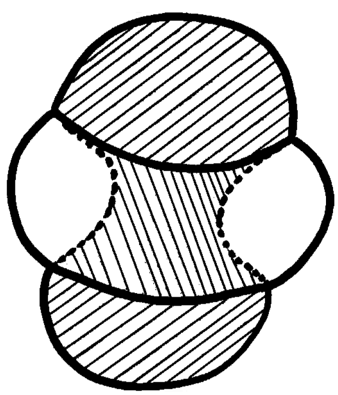
Merrimack College students perform the ant walking experiment.

21.2 Higher Dimensional "Road Building"

Don't be afraid to dip your elbows into soap! Try it!

21.3 Donut Bubbles

Actually, no! It has long been known [Hild] that the sphere is the surface of least area that traps a given amount of air. Any donut bubble you create (did you succeed, even for an instant?) will wobble, deform, and likely burst before pulling itself into a sphere. However, is it possible to create a donut bubble as part of a double bubble? Think about a donut bubble that enwraps a single dumbbell shaped bubble.



22 Halves and Doubles

22.1 Freaky Wheels I

The arrows are again aligned and pointing upwards! Can we conclude that rolling one wheel halfway along the circumference of another is the same as rolling it along the whole? What's going on?

22.2 Freaky Wheels II

Have you ever parked a car too close to the curb? The terrible screeching sound you hear is the hubcap of your back wheel scraping against the curb. The tire rolls, but the hubcap scrapes!

22.3 Breaking a Necklace

No matter how the pearls are arranged, only two cuts are ever needed, and these cuts can always be placed at opposite points on the circle! Try the experiment several times. Can you begin to explain why this is always the case?

22.4 Congruent Halves

They all can!

23 Playing with Playing Cards

23.1 A Pastiche of Card Surprises

Surprise 1. They are always equal!

Surprise 2: Six! (Is there anything special about the numbers 32 and 20?)

Surprise 3: The number of foreign cards in each pile is always the same! (Does anything special about the number ten make this work?)

Surprise 4: The card you first noted!

Surprise 5: Each pile consists of the same numbered cards!

Surprise 6: The magic card!

Surprise 7: Your friend will be handed the card he mentally selected!

Why do these tricks work?

23.2 Curious Piles

No hints, here. Instead …

Taking It Further. Suppose a shuffled deck of cards is divided into 13 piles of four cards each. Is it always possible to select an ace from one pile, a two from another, a three from a third and so on, all the way down to a king from a thirteenth pile? That is, can you select 13 distinct numbered cards from 13 distinct piles? Try it and see if you can accomplish this feat. Or, try to produce an arrangement of cards for which this cannot be done.

23.3. On Perfect Shuffling

Denote an in-shuffle by I and an out-shuffle by O and read a string of these letters from left to right as a sequence of instructions. Thus I I O, for example, means to perform two in-shuffles followed by an out-shuffle. It is convenient to regard the top position of the deck as position 0, the next card down as position 1, and so on.

Make a table of the sequence of steps required to move a card in position 0 to another position. (I have completed the first three entries for you.) Is there any connection here to binary numbers?

Destination	Moves
1	I
2	I O
3	I I
4	
5	
6	
7	

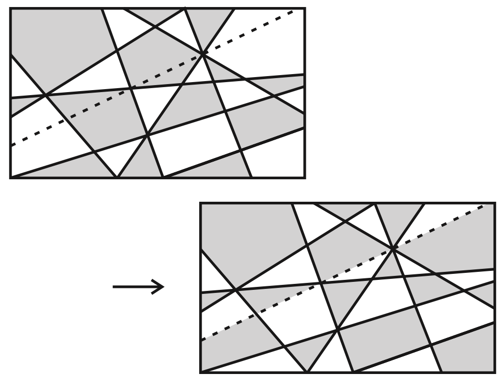
24 Map Mechanics

24.1 Cartographer's Wisdom

I forgot to mention that I wanted the outside region of the map, that is, the border between the map and the edge of the page, colored as well. Does this disrupt your coloring scheme?

24.2 Simple Map

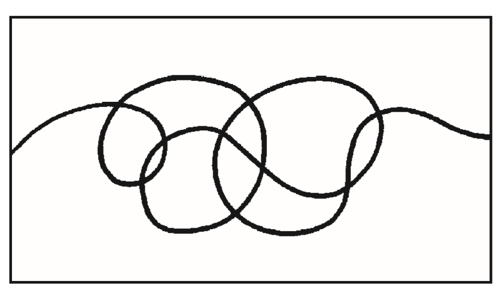
Ignoring the outside region, two colors suffice to color this map.

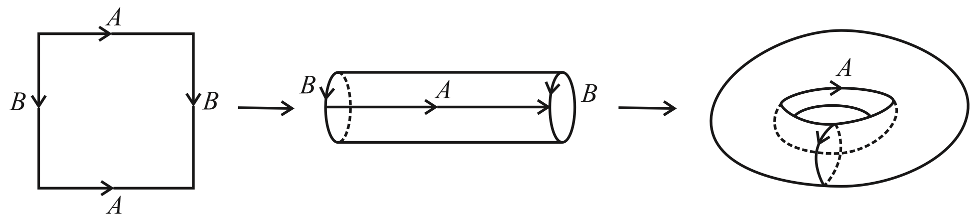


Taking it Further Answer. When adding an extra line to the diagram, just reverse the colors of the regions to one side of it. This produces a satisfactory two-coloring of the new map.

Comment. This trick works as the basis of an induction argument to prove that all maps constructed from straight lines drawn across a page are two-colorable. The same trick also works when slicing a large three-dimensional cube by planes, subdividing its interior into separate regions: it is always possible to fill the volumes of these regions with red and blue liquids so that no two regions sharing a common face are assigned the same colored liquid.

Taking it Even Further. A single curly line is drawn from one end of a page to the other. The line intersects itself at isolated points but not along entire segments of the curve. Is the resulting map necessarily two-colorable?

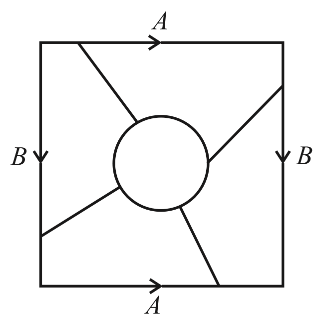




24.3 Toroidal Maps

Rather than work with a fully formed torus, it is easier to draw maps on a square piece of paper, taking note of the proper *edge identifications*. (When forming the torus, the top edge is actually glued to the bottom, and the left edge to the right.) See the figure above.

Here's a toroidal map requiring a minimum of five colors to paint.



Taking it Further. Design a map on a torus that requires a minimum of *six* colors to paint. How about one requiring a minimum of *seven*?

25 Weird Lotteries

25.1 Winning Cake

A number of possible strategies are worth considering. Would you choose a large number in order to win, or choose a small one and hope that the entries above you cancel out? Would you want to announce to the group what number you are going to write? Would you want to announce a number but secretly write a different one instead? Or is it best to keep silent? Does the size of the group affect your strategy? What if only 2, 3, or 4 people are with you?

The best thing is to try running this game and see how people operate. There is a clear trade-off, and it is interesting to see how people respond to it.

Taking it Further. Suppose you are allowed to hand in as many entries as you like. What will you do in this case?

Taking it Even Further. Suppose you are told that after the winner takes his or her share, the rest of the cake will be divided equally among the remaining participants. Will you now adopt a different strategy?

25.2 Unexpected Winner

If the professor is truthful, John has indeed won the cake. But then the announcement of the winner wouldn't be a surprise, contradicting the assumption that the professor told the truth! Has the professor lied?

25.3 Winning Tootsie Rolls®

It is hard to resist the temptation to defect. Do you think in a small group everyone would be likely to cooperate? Or would everyone defect?

Taking it Further. Try this variant scheme. If everyone cooperates, everyone receives five Tootsie Rolls® apiece. If everyone defects, no-one receives any Tootsie Rolls®. If there is a mixture of cooperators and defectors, the co-operators each receive two Tootsie Rolls® and the defectors twenty. What is your response?

Outcomes	C	D
All Cooperate	5	–
All Defect	–	0
Mixture	2	20

25.4 Buying Tootsie Rolls®

Many people feel that investing in the stock market is like entering into a lottery! This financial problem is modelled on the mathematical idea of *dollar averaging*. Is it better, in the long run, to invest a fixed amount of money every month in a stock or to buy a fixed number of shares every month? If the price variations are truly random, one method is indeed better than the other. Can you detect which is better by experimenting?

26 Flipped Out

26.1 A Real Cliff-Hanger

After simulating the experiment a few times you might start to feel that things don't look good for Dorothy.

26.2 Too Big a Difference

After a large number of trials, the average difference between the number of heads and tails is about 2.46. Of course this average difference is *not* zero since I asked you to record all differences as positive quantities. (If, however, the appearance of more heads than tails is recorded as a positive difference, and the appearance of more tails as heads as a negative difference, then the average value of these differences would be zero.)

Repeat the experiment, again performing 10 tosses many times, but this time record the square of the difference. (In an experiment yielding three heads and seven tails, for example, the square of the difference is $4^2 = 16$.) Compute the average value of the square of the difference. What do you notice?

The result is even more startling if you repeat the exercise by tossing a coin 25 times.

26.3 A Surprise

Count the number of heads and tails that appear. Repeat the experiment several times if nothing striking occurs right away.

27 Parts That Do Not Add Up to Their Whole

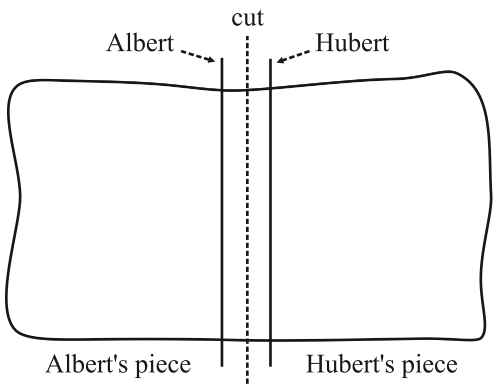
27.1 A Fibonacci Mismatch

Try doing the same trick to transform a 5 × 5 square into a 3 × 8 rectangle (that is, choose $F_5 = 5$ to begin with). Make a 3 × 3 square into a 2 × 5 rectangle. What do you observe?

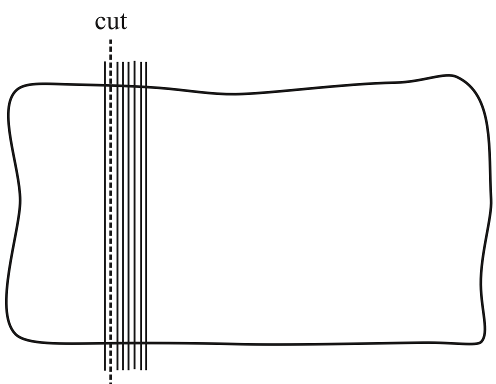
27.2 Cake Please

Have each brother place a knife, in parallel directions, across the cake, at the position each believes divides the cake in two. It is unlikely both will choose exactly the same line. (Use extraordinarily thin knives!) Cutting the cake anywhere between these two lines guarantees each brother a piece, in his estimation, greater than half.

This method generalizes to more than two participants. For instance, it is possible to divide a cake among seven people so that each person believes he is receiving more than one seventh of the cake! First have each person mark a line, parallel to one fixed direction, that he believes cuts off exactly one seventh of the cake from the left. Then make a cut between



the two leftmost lines and hand that piece to the person who marked the line closest to the end. This person is receiving more than one seventh of the cake in his estimation, and the remaining six people all believe that more than six sevenths of the cake remains. These six folk now repeat this procedure, each estimating one sixth of this portion of cake, and so on, reducing the problem to smaller numbers of people until eventually every one ends up with a piece of the cake.



In case you are wondering, it is not to anyone's advantage to be greedy in this procedure and deliberately mark a line larger than the estimated one seventh. That risks receiving too small a piece in the end!

Comment. It is fun to try this experiment with a group of friends. Draw on a piece of paper a curved figure roughly resembling a rectangle. Make photocopies of this shape. Paper "cake" has the advantage that bold lines representing knife cuts can be drawn and seen when copies are stacked on top of one another. This way people can draw knife lines free of the visual clutter of other people's score marks.

Hand copies to groups of two. See if they can devise their own cake-sharing methods. Next hand out copies to groups of three and see what schemes they devise for fair distribution. There are several techniques for sharing cake and it is interesting to see the varied methods different groups devise.

Warning. If you illustrate these sharing techniques over real cake, be careful to choose a cake that is plain and free of decoration. Battles often ensue over iced flowers, for example, and little hope remains for rational mathematical analysis!

Taking it Further. The cake-sharing method described above is said to be *proportional*: everyone receives in her estimation at least $1/n$ of the cake (if there are n people). But it is not *envy-free*: Not everyone is likely to feel that she actually received the largest piece cut! (Except in the $n = 2$ case.)

Devise a method among three players that guarantees each participant, in her estimation, the largest (or at least tied for largest) piece cut.

27.3 Sharing Indivisible Goods

Bjorn should keep the bar and give Elaina 16 Tootsie Rolls®. Both come out the equivalent of two Tootsie Rolls® ahead!

Taking It Further. Neal, Janice, and Sheryl, each possessing a large supply of TootsieRolls®, are faced with the challenge of sharing four desserts: an apricot dacquoise, a hazelnut torte, an Australian pavlova, and a single American brownie. Each person agrees the desserts should not be cut in any way, so they can take a dessert home intact to share with the family, but each is willing to trade tootsie rolls for desserts. They make the bids shown in the table:

Dessert	Neal	Janice	Sheryl
Dacquoise	105	120	132
Torte	90	80	64
Pavlova	196	75	112
Brownie	5	7	4

Who should take home which dessert? Who should trade Tootsie Rolls® and how many? Devise a scheme so that everyone, in his or her measure, comes out ahead!

28 Making the Sacrifice

28.1 The Josephus Flavius Story

Number the participants 1 through n in a clockwise direction. Assume a game begins the count with player 1 and goes in a clockwise direction. Let $W(n)$ denote the position of the winner in an n-person game (counting every third person). The following table shows the position of the winner for the games involving $n = 1$ through $n = 15$ people:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$W(n)$	1	2	2	1	4	1	4	7	1	4	7	10	13	2	5

What do you notice about these numbers?

28.2 Soldiers in the Desert

Configuration A (below) will move a penny three lines into the desert. Configuration B (below) will push a scout four lines into the desert. Is it possible to push a penny five lines in?

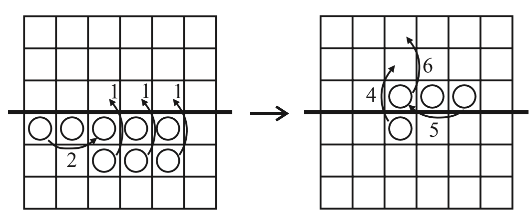
28.3 Democratic Pirates

In a game with ten pirates, the cabin boy can survive and end up with six coins! Can you see how?

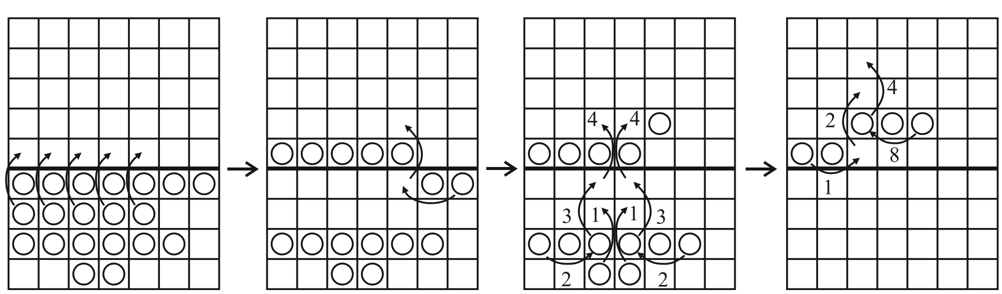
29 Problems in Parity

29.1 Magic Triangles

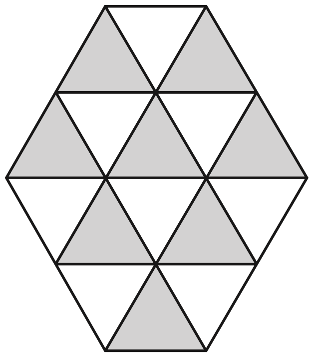
The instructions become more restrictive as you follow them, so it is hardly a surprise that I was



Configuration A. Moving a penny three lines into the desert.



Configuration B. Moving a penny four lines into the desert.



able to squeeze your available options down to the one choice of moving into cell 23. To check this, try crossing out all the disqualified cells as you proceed through the instructions. However, my certainty that you avoided all inappropriate cells (all multiples of 3 after step 1, all multiples of 13 after step 5, for example) took a little more subtlety. I relied on the above picture. Do you see how?

29.2 Walking a Loop.

Consider the journey I described. Write out the letters alternately above and below a horizontal line, ignoring the final A (it was already mentioned as the starting point of the journey):

$$\frac{\text{A B C J K D G F I L H E}}{\text{D J H G I A L K F C E B}}$$

What do you notice?

29.3 Catch Me If You Can

Joy had better head to Port Moresby right away before doing anything else. Can you see why?

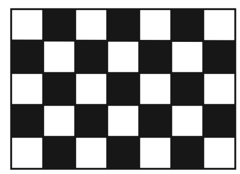
29.4 A Game of Solitaire

First try playing a game with a line of just one, two, or three pennies. Then try one with four pennies. Under what circumstances can you win the game? What do you notice about the number of heads showing initially for these winning games?

30 Chessboard Maneuvers

30.1 Grid Walking

Color the cells of the grid black and white according to a standard checkerboard scheme.



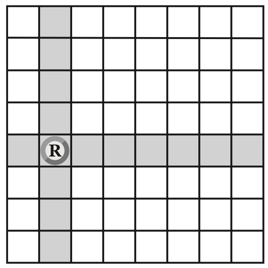
Each step Greta or Peter takes leads to a cell of opposite color. Thus an even number of color changes (that is, an even number of steps) is required to visit a cell of the initial color. In particular, an even number of steps is required to return to the initial cell. What can be said for Lashana's and Lopsided Charlie's motion?

30.2 Kingly Maneuvers

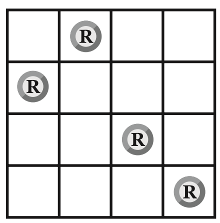
After trying a number of times (or playing the two-person game for a while), you probably suspect that it is impossible to avoid simultaneously the creation of such black and white paths. Your feeling is correct. To see why, imagine the puzzle as one of trying to dam a river. View the black cells as logs attempting to link across a whitewater river to block the flow.

30.3 Mutual Non-Attack: Rooks

The placement of a rook in a particular cell of an $n \times n$ grid "knocks out" all cells in that row and column as possible placements of other rooks. This leaves an $(n-1) \times (n-1)$ grid to analyze. Placing one more rook leaves an $(n-2) \times (n-2)$ grid to analyze, and so on. This shows that a maximum of n rooks can be placed on the board.



Note that each row and column of the grid must contain precisely one rook. When construct-



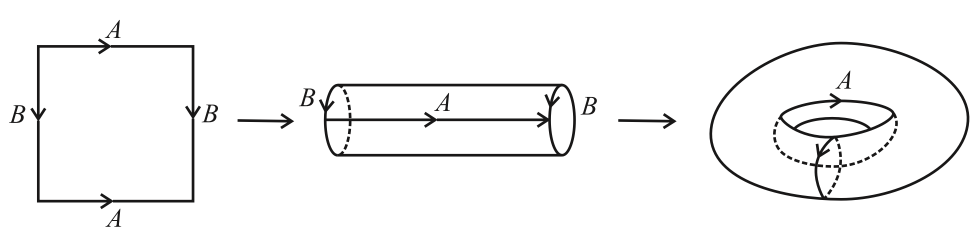
ing a configuration of rooks, there are n choices for where to place a rook in the first row. Once this first rook is in place, there are $n - 1$ choices for where to place a rook in the second row, and so on. This gives a total of

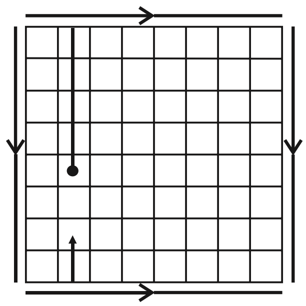
$$n \times (n-1) \times \cdots \times 2 \times 1 = n!$$

possible arrangements of n rooks on an $n \times n$ chessboard.

For a $p \times q$ rectangular array, with $p < q$, only p rooks can be placed on the board. The "one rook per row" rule dictates this. There are $q \times (q-1) \times \cdots \times (q-p+1) = q!/(q-p)!$ ways to arrange these rooks.

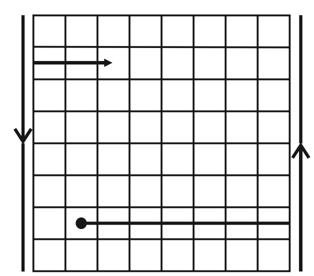
Taking It Further 1: Toroidal Chessboards. One can form a torus from a square piece of paper by gluing the top edge of the square to the bottom edge to first form a cylinder, and then the left edge to the right to form a torus (see the figure below).



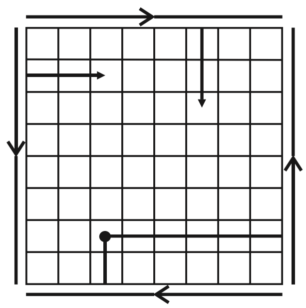


Imagine that an 8×8 chessboard has been folded to form a torus. Still regard the board as a planar square grid, but keep in mind that it is possible to move beyond the top edge to reappear at the corresponding neighboring cell on the bottom edge. The same is true for the left and right edges. What is the maximal number of rooks that can be placed on an 8×8 toroidal chessboard in mutual non-attack? Given this maximal number, what is the total number of different configurations possible?

Taking it Further 2: Möbius Chessboards. This time glue just the left edge of an 8×8 checkerboard to the right edge, but insert a half twist. This forms a Möbius chessboard. What is the maximal number of rooks that can be placed on this twisted board in mutual non-attack? Given this maximal number, what is the total number of different configurations possible?

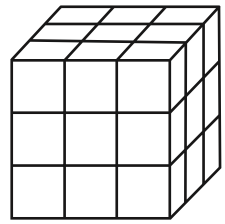


Taking it Further 3. Projective Plane Chessboards. Suppose the left and right edges, and the top and bottom edges, of an 8×8 chessboard have been glued together with a half twist. This forms a surface, difficult to draw, called a *projective plane* (see section 10.2). What is the maximal number of rooks that can be placed on this board in mutual non-attack? In how many different ways can they be placed?



Challenge. Develop solutions to the previous three problems for arbitrary $p \times q$ chessboards!

Taking it Further 4: Three-Dimensional Chessboards. Imagine a rook standing within one cell of a cubical lattice. Assume it can move



in straight lines up, down, left, right, and back or forth to attack other pieces. In a $3 \times 3 \times 3$ cubical lattice, what is the maximal number of rooks that can be placed in a configuration of mutual non-attack? In how many different ways can they be placed? Can you generalize this analysis to $n \times n \times n$ cubical lattices?

30.4 Mutual Non-Attack: Queens

Given what we know about rooks, it is not possible to place more than n queens on an $n \times n$ chessboard in mutual non-attack. It is possible to place four queens on a 4×4 board, five on a 5×5 board and eight on an 8×8 board. (Did you succeed with this?) Do you think this pattern persists?