PART I

ACTIVITIES AND PROBLEM STATEMENTS

1

Distribution Dilemmas

1.1 A Shepherd and his Sheep

Here is a classic puzzle.

An elderly shepherd died and left his entire estate to his three sons. To his first son, whom he favored the most, he bequeathed $\frac{1}{2}$ his flock of sheep, to the second son $\frac{1}{3}$, and to the third son, whom he liked the least, $\frac{1}{9}$ of his flock. (Is there a problem with these proportions?)

Not wishing to contest their father's will, the three sons went to the pasture to begin divvying up the flock. They were alarmed to count a total of 17 sheep! Is there a means for the three sons to successfully carry out their father's wishes?

Taking it Further. Meanwhile, three daughters of a recently deceased shepherdess faced a similar dilemma. Their mother, very wealthy, but also possessing a flawed understanding of fractions, had bequeathed her estate of 495 sheep according to the proportions $\frac{1}{5}$ to her first daughter, $\frac{1}{33}$ to her second, and $\frac{1}{2145}$ to her third! Can her will be successfully honored?

1.2 Iterated Sharing

A group of friends sits in a circle, each with a pile of wrapped candies. (Wrapped candy is used because each piece will be handled by many people before being eaten.) Some people have 20 or more pieces, others none, and the rest some number in between. The distribution is quite arbitrary except for the fact that everyone has been given an *even* number of pieces. A reserve supply is set aside.

The friends now follow these instructions: Give half your candy to the person on your left (and hence receive a supply of candy from the person your right). Do this simultaneously. Now recount your candy supply. If you now have an odd number of pieces, take an extra piece of candy from the reserve supply. This boosts your pile back up to an even number of pieces and everyone is ready to perform the maneuver again.

What happens to the distribution of candy among these friends if this maneuver is performed over and over again? Will people be forever taking extra pieces from the center, so everyone's amount of candy will grow without bound? Or will the distribution stabilize or equalize in some

124 wrapped candies are distributed with a reserve supply of 100 placed in the center.

sense? Will one person end up with all the candy? Might "clumps" of candy move around the circle with each iteration or some strange oscillatory pattern merge? Is it possible to predict what the result will be?

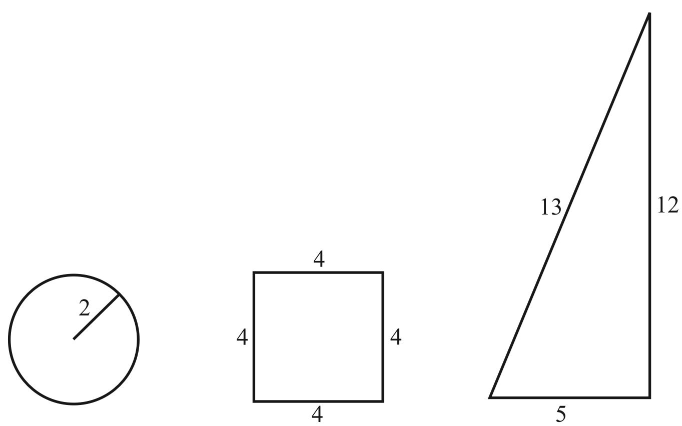
Taking it Further. What happens if instead of adding pieces, you *eat* any odd piece of candy to bring your pile back *down* to an even number? What happens if the sharing pattern is varied; say, you all give half your candy to the person on your left and the other half to the person on your right?

2

Weird Shapes

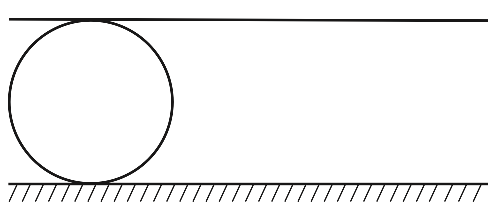
2.1 Plucky Perimeters

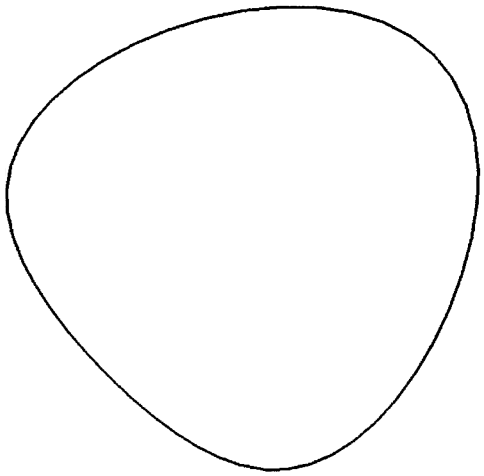
These figures sharea curious property. What is it?



2.2 Weird Wheels

A circular wheel has constant height as it rolls along the ground.





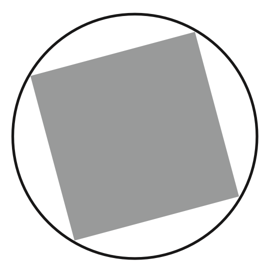
Angela Dellano examines the weird wheel.

Photocopy the irregular shape shown, enlarge the picture, and trace that shape on cardboard to make a wheel. Verify that this wheel also has constant height as it rolls along the ground. How did I make this shape?

2.3 Square Pegs and Not-so-Round Holes

You might want to read the solution to the previous problem before attempting this one.

Square pegs can fit in round holes! For a snug-fitting square peg, all four corners just touch the walls of the hole. A circular hole has the property that this is the case no matter how the square peg is oriented: all four corners always just touch. Is a circle the only shape of a hole that accommodates square pegs in this way?



3

Counting the Odds … and Evens

3.1 A Coin Trick

Han tosses 12 coins onto a table top. He closes his eyes and instructs John to turn over as many coins as he likes. John can, if he wishes, turn over the same coin every time or any number of times, but there is one proviso: Every time a coin is turned John must say out loud the word “flip.”

When finished, John covers one coin with his hand and tells Han it is okay to open his eyes. Han then swiftly, and correctly, announces the state of the coin under John’s hand, whether it is heads up or tails up. Han is able to do this correctly every time the game is played, even if a different number of coins is used. What is Han’s trick?

Comment. When performing this trick in front of a large group, consider using chips colored black on one side and white on the other rather than coins for better visibility.

3.2 Let’s Shake Hands

With an odd number of people in the room, ask everyone to shake hands an odd number of times. No person need shake everyone’s hand. In fact, each person could just shake hands with the same small selection of people multiple times. All that is required is that every person be involved in an odd number of handshakes. Noting that it is impolite to refuse a handshake when offered (and that shaking hands with yourself is considered invalid), what curious predicament do folks find themselves in when they attempt this experiment?

Comment. To ensure an odd number of people are involved in this exercise the leader can participate, or not. But keep the motive for your involvment (or lack of involvement) secret!

3.3 Forty-five Cups

Forty-five plastic cups are placed upright on a table top. Turning over six at a time (no more, no less) you can flip all the cups upside down. Try it! (You may have to invert the same cups multiple times in order to accomplish this feat.)

3.4 More Plastic Cups

Twenty-six plastic cups are placed in a row upside down on a table top. Angela turns over every cup. Barry then comes along and turns over every second cup, followed by Cane who turns over every third cup, and so on, all the way down to Zachary who turns over every 26th cup — that is, just the last one!. At the end of this process, which cups are left upright?

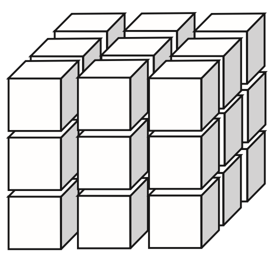
John Dockstader attempts to invert 45 cups.

4

Dicing, Slicing, and Avoiding the Bad Bits

4.1 Efficient Tofu Cutting

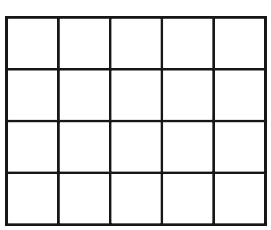
We can subdivide a cube of tofu into 27 smaller cubes with six planar cuts. Is it possible to complete the same task with fewer than six cuts if we allow stacking the pieces of tofu and slicing through entire stacks with planar cuts?



Taking it Further. What is the minimal number of planar cuts needed to dice a $4 \times 4 \times 4$ cube of tofu into 64 smaller cubes? What about a $5 \times 5 \times 5$ cube?

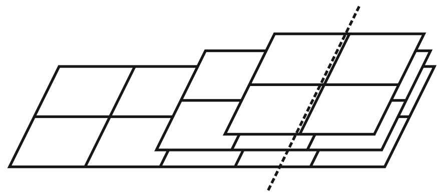
James Taylor attempts to dice a cube of tofu with fewer than six cuts.

4.2 Efficient Paper Slicing

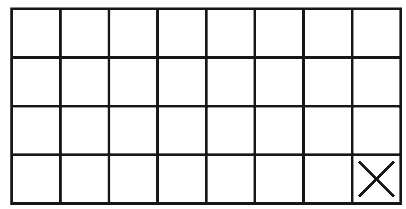


Let's take the tofu cube problem down a dimension: It is possible to slice a $4'' \times 5''$ piece of paper into 20 unit squares with just seven straight line cuts. The same feat can be accomplished with fewer straight line slices if we stack cut pieces of paper during the slicing process (see diagram on p. 10). What is the minimal number of slices required to completely slice a $4'' \times 5''$ piece of paper? What about an arbitrary $n'' \times m''$ piece of paper?

Stacking cut pieces of paper during the slicing process allows you to slice a 4″ × 5″ piece of paper into 20 uhnit squares with fewer than seven straight line cuts.

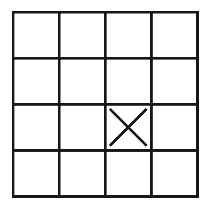


4.3 Bad Chocolate (Impossible!)



Dan and James are presented with a rectangular 4 × 8 chocolate bar with score marks for breaking it into 32 individual square pieces. They note that the bottom right square of chocolate is spoiled and cannot be eaten.

These gentlemen decide to play the following game: Dan will break the bar along one entire score line, hand the piece containing the bad square to James, and place the remaining piece aside. James will then break the piece handed to him into two sections, again along an entire score line, and hand the portion containing the bad square to Dan, placing the other piece aside; and so on. They will do this until someone is handed a lone square of bad chocolate. That person will then be declared the loser and will keep only the single rotten piece of chocolate, while the other person gets all the rest to enjoy. If you were to play this game, what strategy would you employ?



Taking it Further. A second chocolate bar of different dimensions has a bad square located as shown. Would you want to play the game with this bar?

Kelly Ogden and Angela Dellano of Merrimack College play the bad chocolate game.

5

"Impossible" Paper Tricks

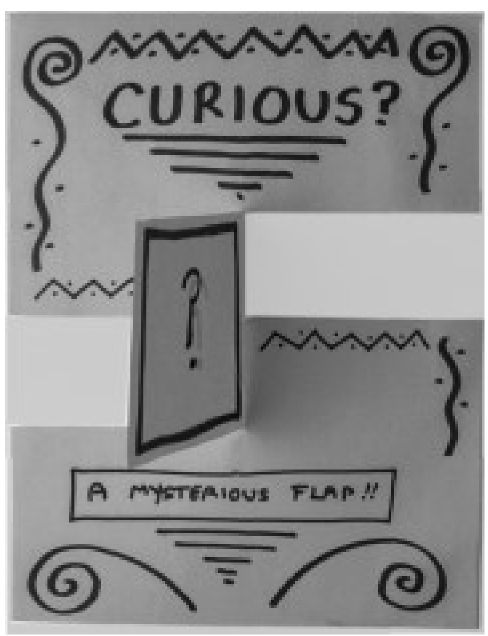
5.1 A Big Hole

Can you cut a hole in an index card big enough to walk through? I can!

5.2 A Mysterious Flap

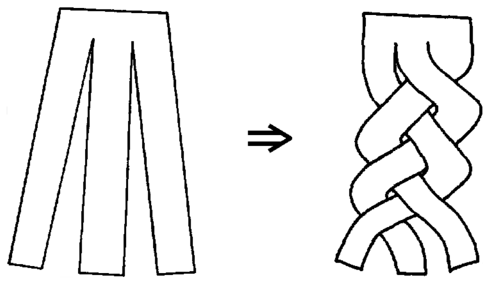
Paper is manufactured flat and two-dimensional. How then is it possible to construct a piece of paper with a flap extending into the third dimension?

The flap is contiguous with the planar base of paper. No adhesive was used to make this configuration.

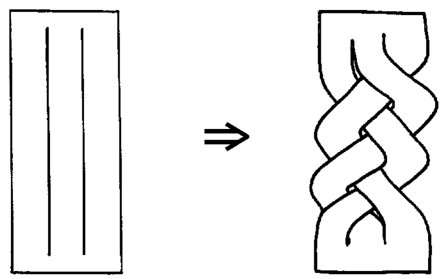


5.3 Bizarre Braids

One usually makes a braid with three strands that are joined at one end but free at the other.



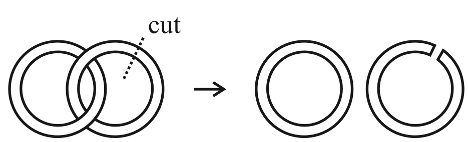
Is it possible to make a braid with no free ends?



Comment. You can use a slit rectangle of paper, but felt is more flexible and easier to manipulate.

5.4 Linked Unlinked Rings

Two linked rings have the property that if you cut either of them, the configuration falls into two separate pieces.



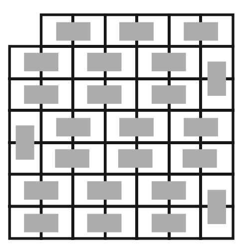
Is it possible to interlink three rings of paper in such a way that if you cut any one of them (once) the configuration is guaranteed to fall into three separate pieces?

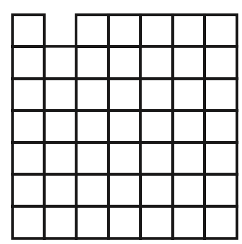
6

Tiling Challenges

6.1 Checkerboard Tiling I

It is impossible to tile a 7 × 7 grid of squares with 2 × 1 dominoes in such a way that each square of the grid is covered by one domino and no domino hangs over the edge of the diagram. (Why?) However, if we excise one corner of the grid the surviving configuration of 48 squares is tilable.





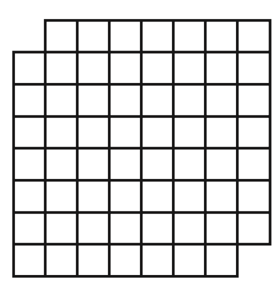
Suppose instead we excise the cell next to the corner. Does this also leave a tilable configuration? (Try tiling it! Draw a grid of squares on paper and use paper clips as dominoes.)

Precisely which cells can be excised from the 7 × 7 square grid to leave a tilable arrangement of 48 squares?

6.2 Checkerboard Tiling II

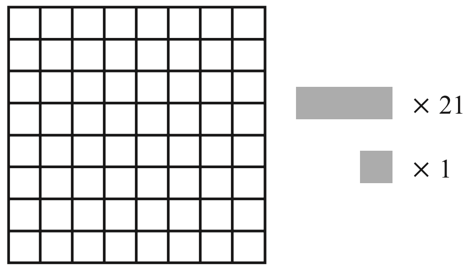
This is a classic puzzle from domino tiling theory. If you have worked through section 6.1, its solution should be straightforward.

Two diagonally opposite corners of an 8 × 8 checkerboard have been excised. Is it possible to tile the remaining configuration of 62 squares with 31 dominoes?



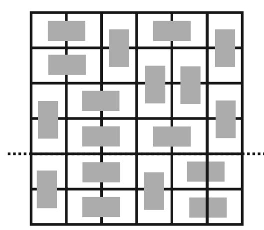
6.3 Checkerboard Tiling III

Is it possible to tile an 8×8 grid of squares with 21 3×1 tiles and one 1×1 tile? If so, how? If not, why not?



6.4 Checkerboard Tiling IV

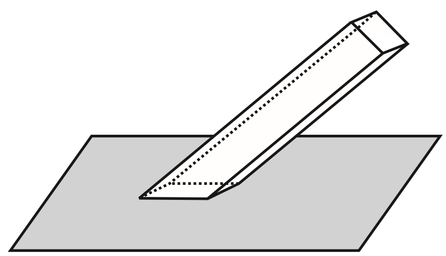
Here is a tiling of a 6×6 square array with 18 2×1 dominoes. Notice that this diagram contains a horizontal line that separates the tiles into two disjoint groups (it also contains a vertical line with this property.) Present a tiling of the 6×6 array that avoids such separating lines.



7

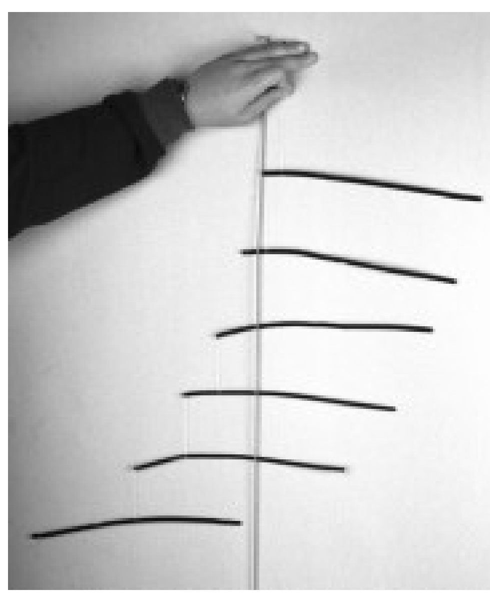
Things That Won't Fall Down

7.1 Wildly Wobbly



This six-faced polyhedral figure will surely fall over when placed on a table top as shown. Is it possible to design a polyhedral figure that will always topple over, no matter on which face it is placed?

7.2 A Troubling Mobile



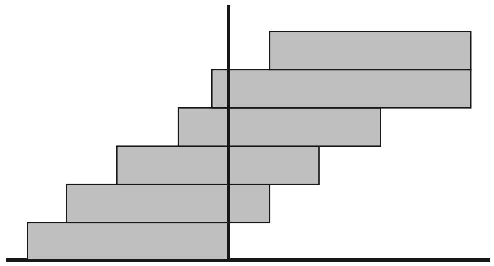
How do you make a perfectly balanced mobile with the property that the lowest wire completely extends beyond the length of the top wire? The mobile shown was made with pipe cleaners and thread (florist wire also works well). Notice that no part of the bottom wire is beneath the top wire.

Notice that the end points of the wires in the photograph form a beautiful curve. Approximately what curve is it?

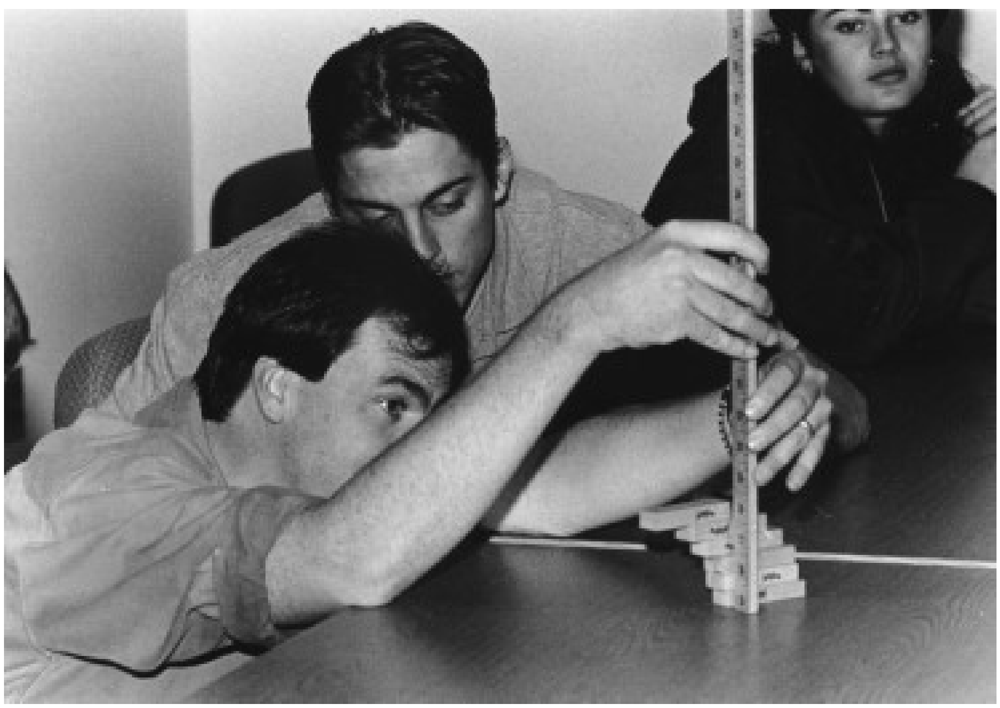
Taking it Further. Is it possible to make a perfectly balanced mobile with the property that the lowest wire extends outwards by more than *two* lengths?

7.3 A Troubling Tower

It is possible to stack wooden building blocks in a staircase fashion so that the top block completely extends beyond the end point of the bottom block. How?



Comment. Yard sticks, cassette tapes and even playing cards also work well for this demonstration.



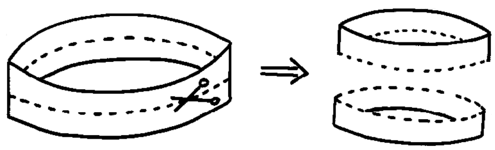
I check Josh Davis' impossible tower.

8

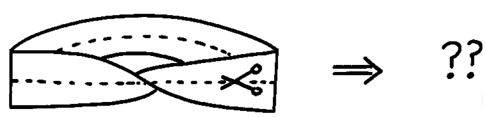
Möbius Madness: Tortuous Twists on a Classic Theme

8.1 Möbius Basics

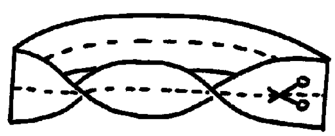
If a band of paper is cut along its center line it will separate into two pieces. No surprises here!



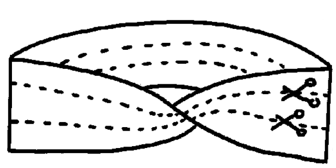
Now take a strip of paper, draw the center line on both sides and form a *Möbius band* by taping the ends together with a half twist. What happens if you cut this figure in half?



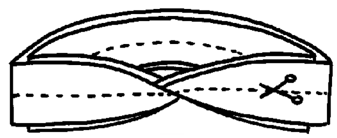
Taking It Further 1. Suppose you cut in half a band with two, three, or more half twists? Can you predict what will result?



Taking It Further 2. How many pieces result if a Möbius band is cut into thirds? How many pieces result if a band with five half twists is cut into fifths?

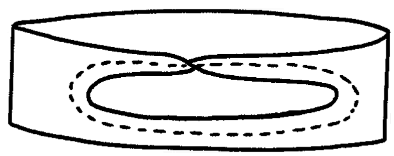


Taking It Further 3. Take a long strip of paper and bring the ends together, with a half twist, to form a Möbius band. But before taping the ends together slide one end of the paper along the strip all the way round back to the other end to form a "double layered" Möbius band. (Equivalently, lay two strips of paper on top of one another, simultaneously give them a half twist, and tape their respective ends together.) What happens if this object is cut along its center line?

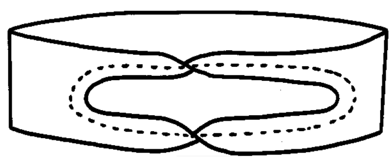


8.2 A Diabolical Möbius Construction

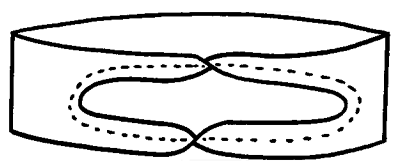
Each of these figures has a center hole. What happens when you cut around the center hole of this figure?



What about in this predicament, with the two half twists in the same direction?

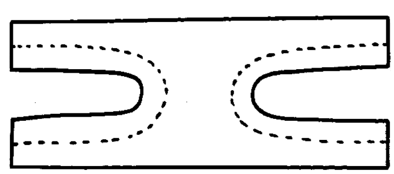


Or this one, with the two half twists in oppositie directions?



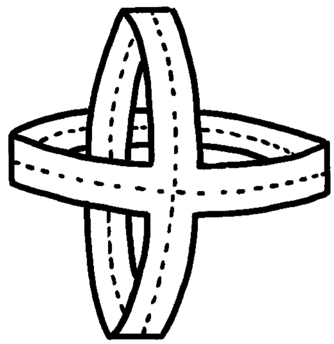
Taking It Further. Experiment with different configurations of twists and multi-twists in the same and opposite directions. Explore what happens and see if you can explain any patterns that occur.

Comment. To make these curious bands, begin with a long wide strip of paper, and cut out half ovals from both ends. Draw guide lines for cutting around each half oval, on both sides of the paper. Then roll up the paper and tape the appropriate ends together with the desired number of half twists.

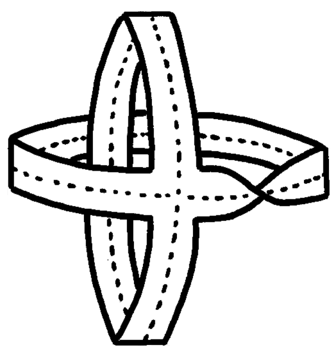


8.3 Another Diabolical Möbius Construction

Cut a piece of paper into an **X** shape to construct two bands of equal length and width attached perpendicularly to one another. What happens when each is cut along its center line?



How does the result change when one band is given a half twist? Two half twists? Seven half twists?



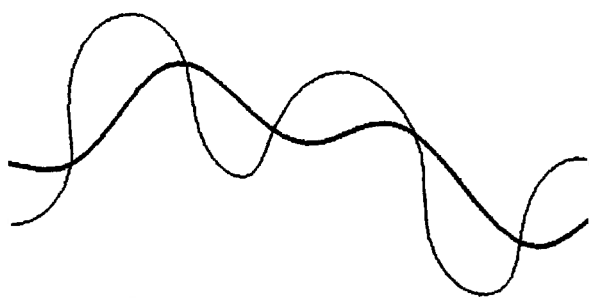
9

The Infamous Bicycle Problem

9.1 Which Way Did the Bicycle Go?

Here's a problem gaining some notoriety.

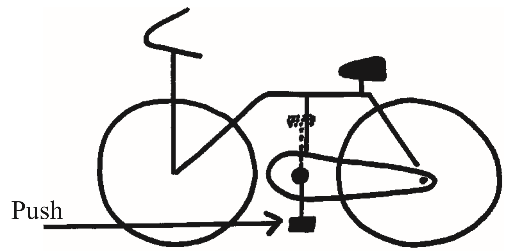
These tracks were left by a bicycle whose wheels were colored with sidewalk chalk. Which way did the bicycle go, and what was the length of the bicycle?



Dortheanne Roberts rides along a 15-foot length of paper. The wheels were colored with sidewalk chalk and the tracks left on the paper were later traced over with permanent markers to make them bolder.

9.2 Pedal Power

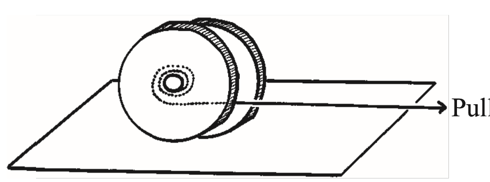
A bicycle is held gently by the seat to keep it balanced while the pedal situated at its lowest position is pushed backwards. Which way does the bicycle move? Forward or backward?



Josh Dixon holds the bicycle while Diane Dixon pushes on the pedal.

9.3 Yo-Yo Quirk

A yo-yo, with its string wound around the spool as usual, is placed on its edge on a table top. If the string is gently pulled, which way will the yo-yo move: forward with the pull or backward with the unwinding of the string?

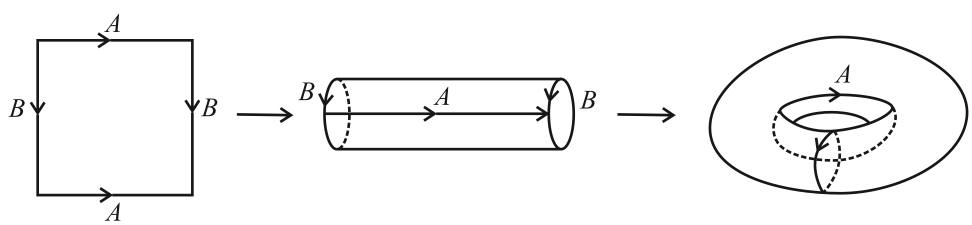


10

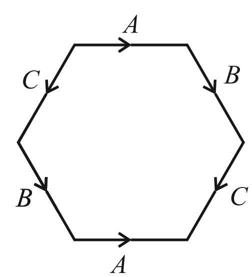
Making Surfaces in 3- and 4-Dimensional Space

10.1 Making a Torus

To form a *torus* (donut shape) from a square piece of paper, simply glue the top edge to the bottom edge to form a cylinder and then bend this cylinder to glue the left edge to the right edge. (Topologists say that the opposite edges have thus been *identified*.)

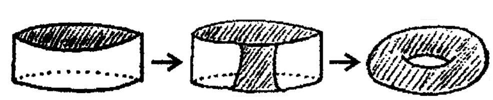


What surface results if you identify the opposite edges of a regular hexagon? Make sure the directions of the arrows match when you glue the edges together.



10.2 A Torus with a Serious Twist

Another way to form a torus from a band of paper is to glue the top edge to the bottom edge all the way around. (In practice, however, this is tricky to do. Paper is very unforgiving and won't stretch. The donut you obtain, after much perseverance, will be quite crumpled!)

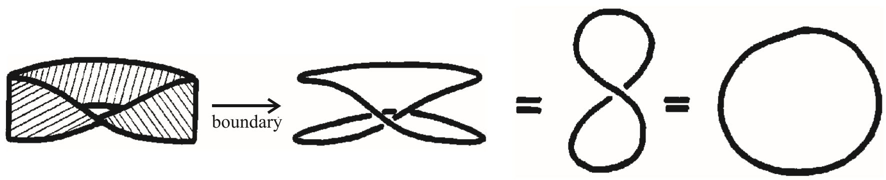


Suppose this band is given a half twist in its construction. Can we still obtain a torus by gluing the top edge to the bottom edge?

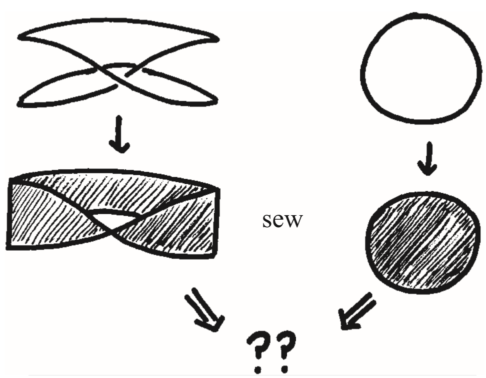
Comment. Try this exercise with paper and tape, fabric, needle and thread, or better yet, Pla-Doh.

10.3 Capping Möbius

The boundary edge of a Möbius band is simply a circle: one that has been bent into a squashed figure eight.



The boundary of a circular disc is also a circle. What happens if we sew these two figures together along their common boundary circles? What surface results?



11

Paradoxes in Probability Theory

11.1 The Money or the Goat?

This classic puzzle from probability theory is known as the Monty Hall problem after the host of "Let's Make a Deal!" It was played as part of that TV game show.

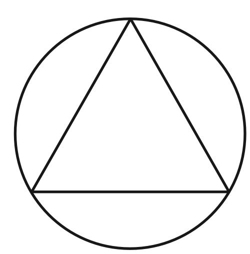
Imagine you are a game show contestant with three closed doors before you. You are told that behind one of these doors is a prize of one million dollars in cash. Behind each of the other two are goats. You select a door, but before you open it, the host opens one of the two remaining doors to reveal—a goat! He now gives you the opportunity to stay with your original choice or switch to the remaining unopened door. What should you do? Switch doors or stay with your initial choice? Does it make any difference?

11.2 Double or ... Double!!

A friend places before you two paper bags, both containing Tootsie Rolls®. He tells you that one contains twice as many as the other and that you may keep the contents of one of the bags. You select a bag, open it up, and count the number of Tootsie Rolls® it contains. Your friend then gives you the option to change your mind and take instead (without peeking inside!) the contents of the other bag. Assuming you would like as many Tootsie Rolls® as possible (and you don't feel it is an insult to his generosity to switch), is it to your advantage to switch bags? Or is it better to stay with your first choice?

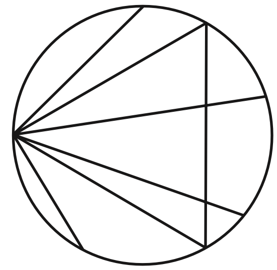
11.3 Discord Among the Chords

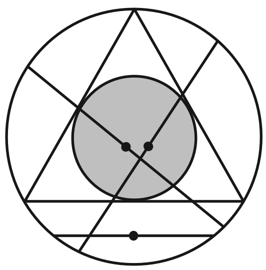
This puzzler is known as Bertrand's Paradox.



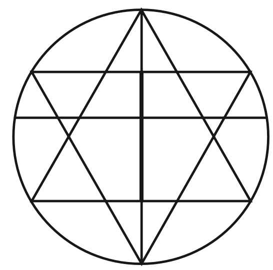
Consider a circle of radius R. Inside this circle inscribe an equilateral triangle. This triangle has side length $\sqrt{3}R$. Suppose a chord of the circle is selected at random. What is the probability P that the length of this chord is greater than $\sqrt{3}R$, the side length of the triangle?

Jennifer answers this question in the following way: Once a chord is drawn, we can always rotate the picture of the circle so that one end of the selected chord is placed at the leftmost position of the circle. We may as well assume then that all chords considered in this problem have one end point at this left-most position. Now draw the equilateral triangle as shown. It is clear that the length of the chord will be greater than the side length of the triangle if the other end point lies in the middle third of the perimeter of the circle. Thus $P = 1/3$.





Bill reasons: If the midpoint of the chord lies anywhere in the shaded region shown below, its length will be greater than the side length of the triangle. Thus P equals the probability that the midpoint lies in this region. A quick calculation shows that the area of this region is one quarter of the area of the entire circle, thus $P = 1/4$.



Joi, on the other hand, argues this way: Rotating the picture of the circle and the selected chord, we may assume that the chord chosen is horizontal. If the midpoint of this chord lies on the solid segment of the vertical line shown, its length will be greater than the side length of the triangle. Thus $P = 1/2$.

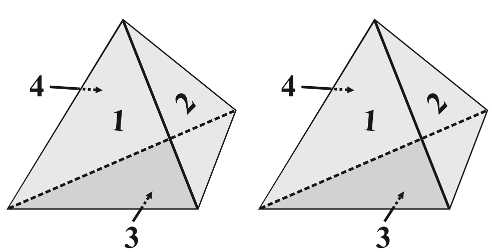
Whose reasoning is correct?

Taking It Further. What would happen if you threw a handful of wires onto a circle drawn on the ground and measured the lengths of the chords crossing that circle? Approximately how many would be of length greater than $\sqrt{3}R$?

11.4 Alternative Dice

An ordinary six-sided die has faces labelled 1 through 6. Thus, given a pair of dice, the probability of rolling a sum of 12 is $1/36$; a sum of 8 is $5/36$; a sum of 4 is $3/36$; and so on. Is it possible to relabel the faces of two six-sided dice with alternative positive integers so as to produce two dice with the same sum probabilities as ordinary dice?

Taking It Further. Two "ordinary" tetrahedral dice have sides labelled 1, 2, 3, and 4. Is there a clever relabelling that yields the same sum probabilities as the ordinary dice?



12

Don't Turn Around Just Once

12.1 Teacup Twists

Andy holds up a teacup in the center of a room while his friends tape several strings from the teacup to various points about the room, leaving plenty of slack for later maneuverability. Next, Andy carefully rotates the cup 360°, tangling the strings in the process, and then holds the cup

Andy Furey holds a teacup while club members attach strings to various points around the room.

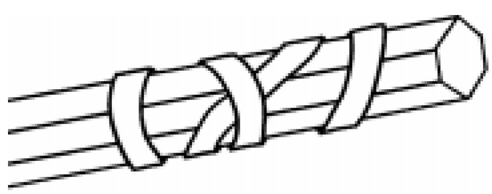
Andy and the tangled cup.

firmly at that point in space never to be moved again! Is it possible for his friends to maneuver the strings around the teacup and untangle them?

Later they decide to try the experiment again. This time Andy gives the cup two full turns, 720°, tangling the strings even more than before! In this situation, is it possible for his friends to maneuver the strings around the teacup, again held in place, and untangle them?

12.2 Rubber Bands and Pencils

Wrap a rubber band around the end of a pencil so that the band always lies flat against the wood. How many times *must* the band wrap around the pencil to achieve this?



13

It's All in a Square

13.1 Square Maneuvers

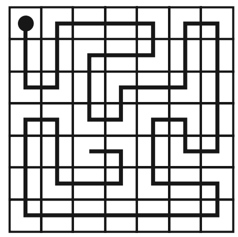
Twenty-five people stand in a large 5×5 square grid, one person per cell. Each person is asked to take one step (vertical or horizontal, but not diagonal) into a neighboring cell, that is, a cell sharing one entire edge with their current cell. People are allowed to exchange squares, but no one may share a square. Is it possible to end up with a new arrangement of all 25 in the square grid?

Taking It Further. Suppose everyone is energetic and decides instead to all leap to a square *two* places away, either in a vertical or horizontal direction. Is the puzzle solvable?

Elbows fly as 25 St. Mary's College students seek new squares.

13.2 Path Walking

Starting at the top left corner of a 7×7 square grid, it is possible to walk a path, using vertical and horizontal motions only, that visits each and every cell of the grid precisely once. Is such a path possible starting one square over from the top left one? From which cells is it possible to commence such a path?



Taking It Further. Is it ever possible, when walking a path, to return to your initial cell and form a loop of steps that visits each and every cell precisely once?

Comment. To get a feel for this problem try walking in the large 5×5 grid you drew in section 13.1.

13.3 Square Folding

Construct a 4×4 square grid and label the cells as shown. Repeatedly fold the grid along its straight line markings to reduce it to a 1×1 wad of paper 16 layers thick. Be as devilish with your folding as you like, sneakily tucking folds within each other, turning the square over multiple times in the process, and the like. Once done, trim away the edges of this unit square, and without disturbing the orientation of the 16 square layers, spread them out over the table. Sum all the numbers you see. What do you get?

4	12	10	4
5	2	9	13
4	11	11	8
7	16	7	3

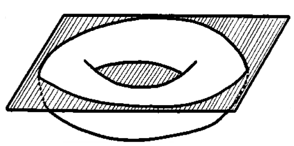
Repeat this experiment several times or have several people do it at the same time. What do you notice? (You are in for a surprise!)

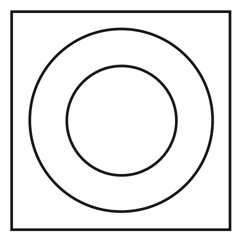
14

Bagel Math

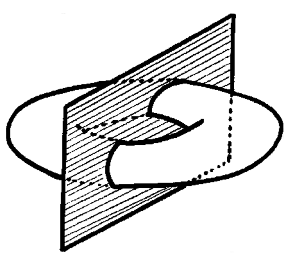
14.1 Slicing a Bagel

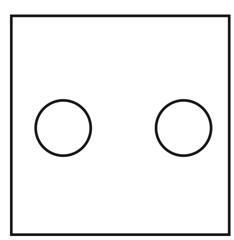
One normally cuts a bagel with a horizontal planar slice. The image of this cut on the slicing plane consists of two perfect circles bounding a region of dough.





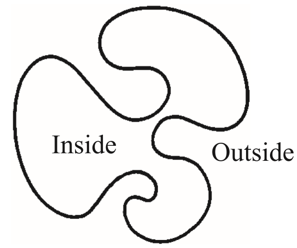
When sharing a bagel with a friend you might cut the bagel in half through a vertical plane. This also produces an image of two perfect circles on the slicing plane.





How else could one slice a bagel so as to produce the image of two perfect circles on the slicing plane? Assume we are working with perfectly toroidal bagels!

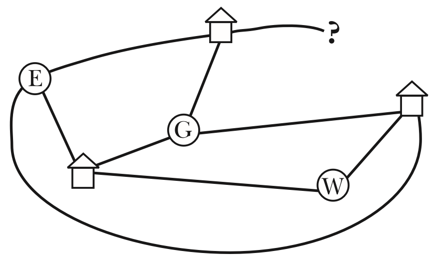
14.2 Disproving the Obvious



It seems completely and utterly obvious that a closed loop divides whatever it is drawn on into two distinct pieces: an inside and an outside. Show that this "utterly obvious" theorem is in fact false for some curves drawn on the surface of a bagel!

14.3 Housing on a Bagel

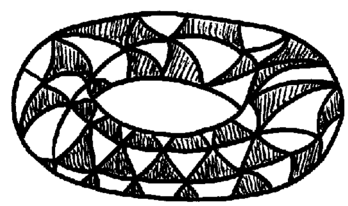
Here's a classic problem from graph theory.



Three houses must be connected to three utility companies (for electricity, water and gas) in such a way that no lines or mains cross. Is this possible?

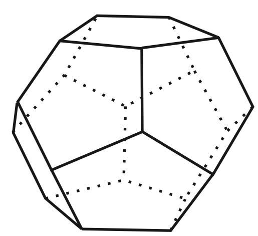
Surprisingly, for buildings situated on a plane this problem is never solvable. (Try it!) However, on our spherical earth we have the (theoretical) option of allowing pipes to circumnavigate the entire globe. Is this problem solvable on a sphere?

14.4 Tricky Triangulations



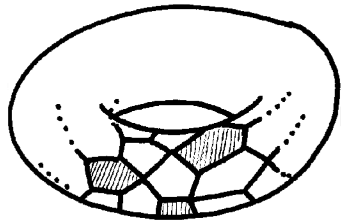
Here I have covered a bagel with 64 warped triangular regions in such a way that any two neighboring triangles touch either in a *single* vertex or along just *one entire* edge. Could I have accomplished this feat with an *odd* number of triangles? I could have done this with less than 64 triangles. Show that it is possible to cover a bagel with just 14 triangles but no fewer!

14.5 Platonic Bagels



A dodecahedron, the fourth platonic solid, can be formed by sewing together 12 pentagons. Notice that the same number of edges meet at each vertex.

Is it possible to cover a bagel with five-sided regions so that the same number of edges meet at each vertex?



15

Capturing Chaos

15.1 Feedback Frenzy

Most video cameras can be hooked up to a TV to display what the video camera sees in real time. Thus if you point the video camera at the TV screen (being careful to exclude the sides of the TV) the screen displays an image of the screen itself. The image is blank. This remains so even if you focus on only a small portion of the screen or if you hold the camera at a tilt.

Now hold the camera at an angle to horizontal. Dim the lights in the room and place a lit candle between the camera and the TV. The camera will see the candle and display its image on the screen, which it then sees and displays again, which again it sees and again displays, and so on. What appears is a beautiful swirling image of a flame dancing around the screen. (Try it!)

If you experiment carefully with the camera and TV alignment, you can obtain a situation where the image of the candle does not disappear even if you blow out the flame. At this point you have captured the image on screen. Since the flame and its slight motion are no longer present, you

Lane Anderson of St. Mary's College points a video camera at a screen showing its own output.

would expect a stable image, frozen in time, to be left on the screen, but this is not the case! Instead the spectacular swirling dance continues, forever captured on the screen! What is going on?

Comment. This is quite finicky to set up and much perseverance is required. Place the camera on a tripod and turn off all automatic features. Adjust the zoom on the camera until the image of the screen is almost the size of the screen itself. Set the TV brightness on low and then light the candle. Adjust the color, focus, zoom, and brightness until interesting effects occur. You may even want to place tape with a small pinpoint hole over the lens of the camera.

15.2 Creeping up on Chaos

Setting $a_0 = 0.1$ and $r = 2$, consider the *recursive relation*

$$a_{n+1} = ra_n(1 - a_n).$$

This defines a sequence of values

$$\begin{aligned} a_1 &= ra_0\left(1-a_0\right) = 0.180 \\ a_2 &= ra_1\left(1-a_1\right) = 0.295 \\ &\vdots \end{aligned}$$

Using a calculator or computer we can easily determine the first ten terms of this sequence:

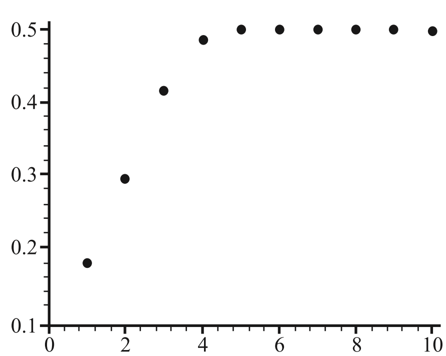
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
0.180	0.295	0.416	0.486	0.500	0.500	0.500	0.500	0.500	0.500

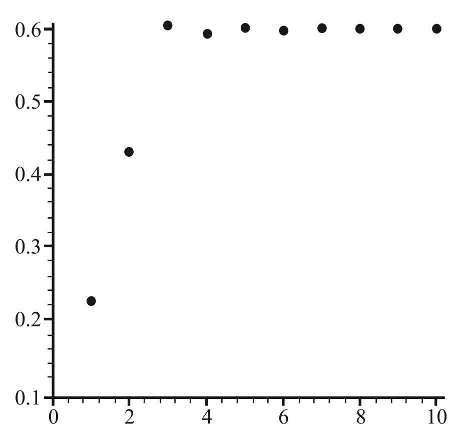
The sequence appears to converge to the value 0.500.

The same type of behavior occurs if we repeat the exercise with the value $r = 2.5$, though the limit of the sequence appears to be different.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
0.225	0.436	0.615	0.592	0.604	0.598	0.601	0.600	0.600	0.600

However, curious things occur if you repeat this exercise with higher values of r. For instance, set $r = 3.3$. If you have the patience, calculate the first 20 terms of the sequence. What do you notice? Now set $r = 3.4$ and then 3.5 and calculate the first 20 terms of each sequence. What do you observe? Check out what's happening for $r = 3.54$ and $r = 3.55$. (You may need to calculate at least 30 terms of the sequence here.) Finally, given the first 20 terms of the sequence for $r =$ 3.70, can you predict what the 21st term will be?





16

Who has the Advantage?

16.1 A Fair Game?

Peter has ten coins, Pennelope has nine. Peter and Pennelope agree to toss all their coins simultaneously. Whoever receives the largest number of heads will win. In case of a tie Pennelope will be declared the winner, so as to offset the advantage Peter has to begin with. Given this agreement, who is most likely to win?

Find a partner and experiment with this game a few times. Which player appears to be favored?

16.2 Voting for Pizza

Alice, Brad, and Cassandra decide to order a pizza to share. Alice will pay for the pizza but she only has enough money to order one topping: pepperoni, anchovies, or olives. Alice prefers pepperoni, is indifferent about anchovies, but simply detests olives. Brad and Cassandra have equally strong opinions, Brad preferring anchovies to olives, and olives to pepperoni, and Cassandra olives to pepperoni, and pepperoni to anchovies.

	Alice	Brad	Cassandra
1	Pepperoni	Anchovies	Olives
2	Anchovies	Olives	Pepperoni
3	Olives	Pepperoni	Anchovies

Seeing no clear group preference, they decide to vote. They will write their choice of topping on a slip of paper, and the topping listed the greatest number of times among the ballots will be the one chosen. In case of a three-way tie, pepperoni will prevail. This voting advantage is given to Alice, who is paying for the pizza, after all. If each person is a savvy player, what topping will the group end up ordering? Play this game with a group of three and see what happens.

John Dockstader, Angela Dellano, and Kelly Ogden of Merrimack College engage in a three-way duel with dice.

16.3 A Three Way Duel

Here's a classic puzzle from probability theory with a counterintuitive conclusion.

Three people, armed with pistols but unequal in marksmanship, enter into a three-way "duel." Alberto can hit his target on average one third of the time. Bridget, on average, hits her target two thirds of the time, but Case is a perfect shooter — he always hits his target.

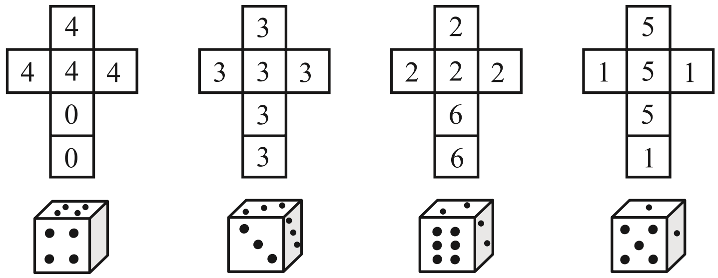
Being gentle folk, these participants agree on a shooting order that reflects their shooting abilities. Alberto will shoot first, aiming in any direction he desires, then Bridget will shoot (if she is still alive), to be followed by Case (if alive), then back to Alberto, and so on in cyclic order until just one person is left standing. What is the optimal strategy for each player?

For Case, clearly his best strategy is always to shoot the more competent opponent remaining — namely Bridget, if she is still alive. Thus Bridget should always aim for Case, knowing she is his prime target. But what should Alberto do? Should Alberto follow the same strategy and aim for the most competent player alive? What are his chances of survival?

Comment. Use dice for guns, where rolling a 1 or a 2 is a successful shot for Alberto, a 1, 2, 3, or 4 is a successful shot for Bridget. Play out this game a large number of times and compute the average survival rate for each player. Have everyone always aim for the more competent opponent alive.

16.4 Weird Dice

Consider the unusual numbering schemes illustrated on these dice. You pick one die, and then I will pick another. We each will roll our chosen die and the larger number wins. Do you want to play with me?



A B C D

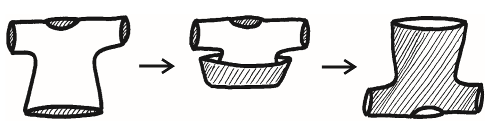
Comment. Try experimenting with dice made of colored paper.

17

Laundry Math

17.1 Turning Clothes Inside Out

A T-shirt inside out looks the same in shape and structure as a T-shirt right-side out; only the seamwork and the patterns on the material tell you something is amiss. The same thing is true for socks, trousers, skirts, and jackets. Is this always the case? Does the process of *eversion*, turning things inside out, preserve the general structure and shape of all clothes? If so, why?

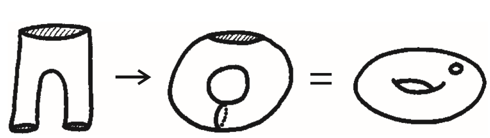


Aside. For awhile it was fashionable to deliberately wear sweatshirts inside out. In taking off a sweatshirt do you grab the neck opening and pull it over your head, preserving the orientation of the shirt throughout the process, or do you grab from the waist and turn the shirt inside out as you take it off? Do you take note of the effect when you next put it on? What does this reveal about you?

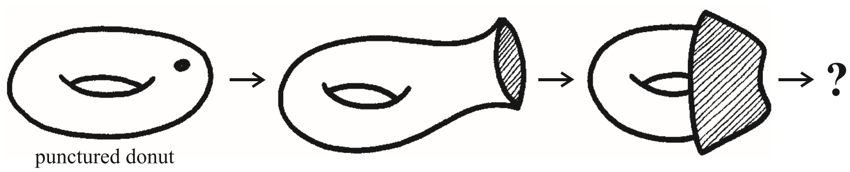
17.2 Mutilated Laundry

Let's attack our laundry with needle and thread.

Sewing together the two leg openings of a pair of trousers yields a punctured donut.

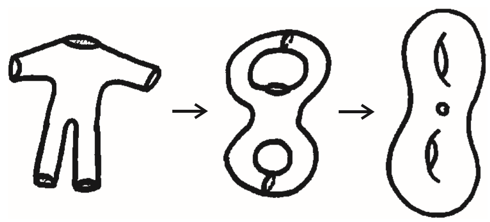


Dennis Horton attempts to turn a pair of mutilated trousers inside out.



What happens if you turn a punctured donut inside out? Is this possible? If so, what shape do you obtain?

A jumpsuit (that is, a one-piece shirt and trouser outfit) provides a means for creating a *double-donut*. Using sufficiently bizarre items of clothing (or sewing multiple pieces of clothing together) we can also create *triple-donuts*, *quadruple-donuts* and other *multi-donuts*.

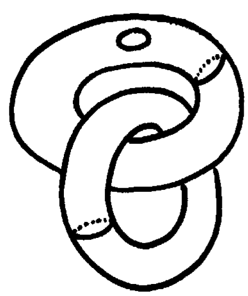


What happens if you turn a punctured double-donut inside out? An arbitrary multi-donut?

Comment. Stapling is quicker and easier than sewing.

17.3 Cannabalistic Clothing

Take two pairs of trousers and connect them to form two linked donuts. Open up the puncture of one donut (as in eversion) and "swallow" the second. What happens? Is the result complicated or elegantly simple? How exactly does the second donut sit inside the stomach of the first?



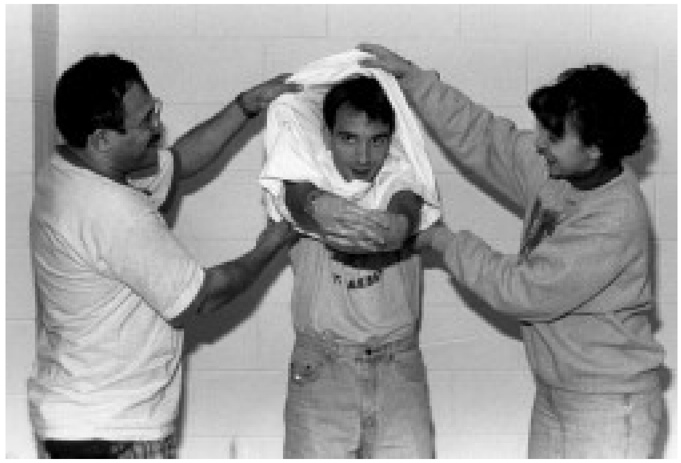
18

Get Knotted

18.1 Party Trick I: Two Linked Rings?

One end of a long piece of string is tied around Jason's left wrist, and the other end is tied around his right wrist. Paul's wrists are also tied, but his string passes through the loop created by Jason's arms and his string. Can these two gentlemen separate themselves from their linked dilemma? They may move the string any way they like, step through any loops, or wrap around themselves in any clever way.

Jason Summers and Paul Ogle are trapped as two linked rings.



Moncef Boufaida and Dortheanne Roberts try to turn George Hinkel's shirt inside out.

Comment. Tying the strings to their wrists is easier than simply holding the strings in their hands. This leaves their hands free for complex maneuvering.

18.2 Party Trick II: A T-Shirt Trick

George, wearing a baggy T-shirt over his clothes, holds his clasped hands out in front of him. Is it possible for his friends to remove his T-shirt, turn it inside out, and put it back on him while George *keeps his hands firmly clasped together*?

Taking It Further. George is again wearing the baggy T-shirt over his clothes and Aliza is firmly holding each of his wrists. Is it possible to remove the T-shirt from George and place it on Aliza all the while Aliza keeps her grip?

Aliza Steurer holds George Hinkel's wrists.

Sten-Ove Uva wears a waistcoat and jacket. Josh Davis tries to turn the waistcoat inside out.

18.3 Party Trick III: A Waistcoat Trick

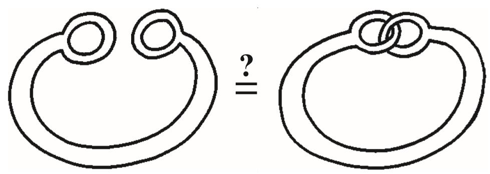
Sten-Ove is wearing a baggy waistcoat, unbuttoned, underneath his jacket, also unbuttoned. Is it possible for Josh to take off Sten-Ove's waistcoat, turn it inside out, and place it back on underneath the jacket *without* slipping any material down inside the sleeves of the jacket?

18.4 Two More Linked Rings?

John can touch the tip of each thumb with his index fingers to form two unlinked rings or two linked rings. If John were composed entirely of soft clay and if we only stretch and mold the clay (no puncturing or tearing of material allowed: keep neighboring molecules of clay neighbors)

John Dockstader forms rings with his thumbs and index fingers.

can we smoothly transform the picture on the left to that on the right? Are these two linked rings, in some sense, no different from two unlinked rings?

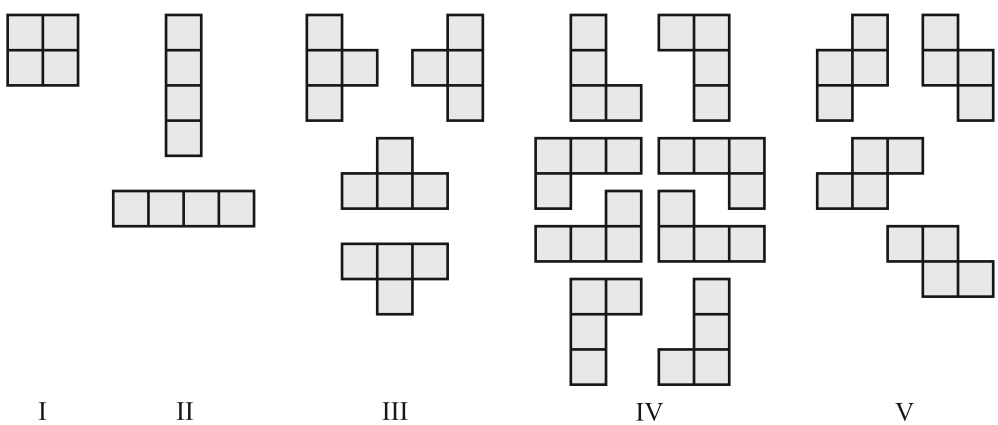


19

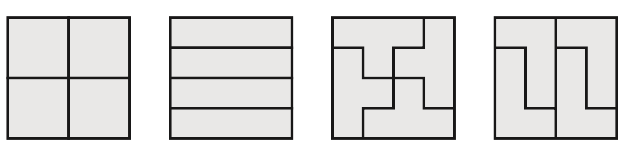
Tiling and Walking

19.1 Skew Tetrominoes

A *tetromino* is any tile composed of four connected cells from a square lattice (a *domino*, on the other hand, is a tile composed of just two). There are 19 tetrominoes, each being a reflection or rotation of one of five basic configurations, called types I, II, III, IV, and V. Those of type V are called the *skew tetrominoes.*

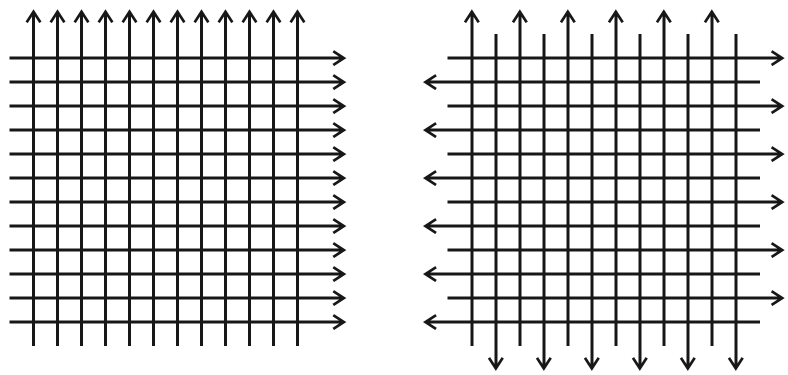


It is possible to tile a 4 × 4 square with four tiles of type I, four of type II, four of type III, or four of type IV. No four skew tetrominoes, however, will tile this square. Is there a square or rectangular grid of any size that can be completely tiled with non-overlapping skew tetrominoes?



19.2 Map Walking

The diagrams represent the location of streets and avenues in Adelaide (city A) and Brisbane (city B). The thoroughfares divide each downtown area into square city blocks, with the streets running east to west and the avenues north to south. Traffic is allowed to move only in the directions indicated. (Do the inhabitants of city A have a problem?)

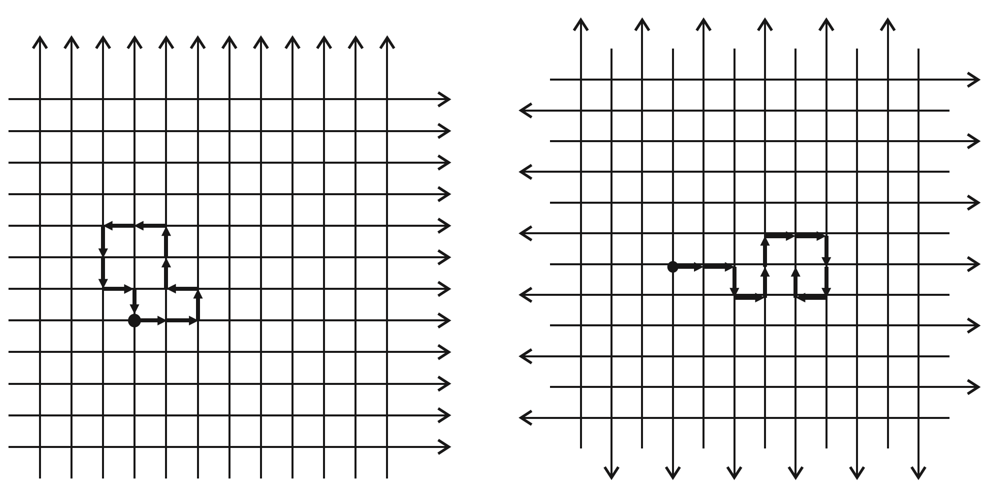


City A City B

Point to an intersection in city A and have a friend point to one in city B. Imagine, as a pedestrian, you are walking through city A, following the streets and avenues from intersection to intersection, going either with or against the traffic along the roads. Describe your journey to your friend simply by stating whether you are moving along a street or an avenue, with or against the traffic, as you move along the city blocks. For example, the path illustrated below takes you back to your starting point and would be described as

$$SSAS^{-1}AAS^{-1}S^{-1}A^{-1}A^{-1}SA^{-1}$$

where S means "move one block along a street with the traffic," A^{-1} means "move one block along an avenue against the traffic," and so on.



City A Journey City B Shadow Journey

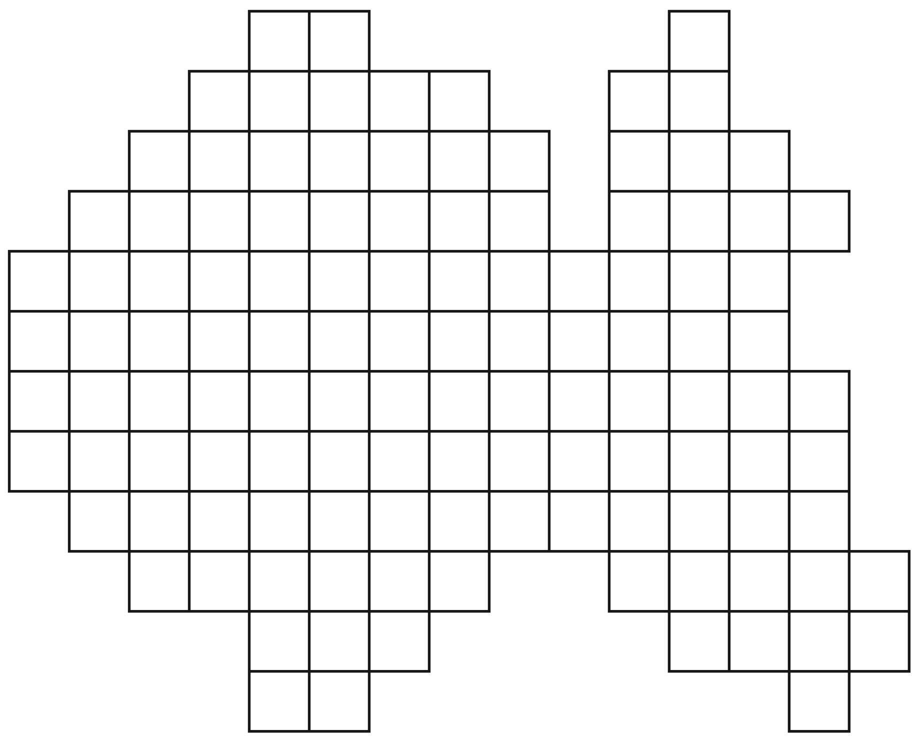
Your friend, hearing the description of your journey, will follow suit on her City B map and walk a “shadow journey,” moving along streets and avenues, with or against the traffic, as per your instructions. In the example presented here, your friend’s journey is very different from your own: it does not even form a closed loop, for example.

Does there exist a closed loop journey in city A that results in a closed loop shadow journey in city B? If you find one, does it depend on the particular intersection at which your friend starts her journey?

19.3 Bringing It Together

Can this region be tiled with skew tetrominoes?

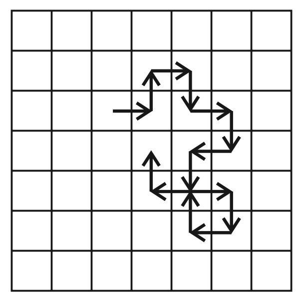
Hint. Consider the title of this section!



20

Automata Antics

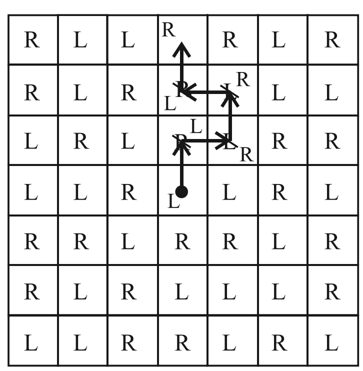
20.1 Basic Ant Walking



An ant moves about a 7 × 7 grid of squares, taking single steps in alternating vertical and horizontal directions. If the ant enters a cell from a horizontal direction, can it ever visit that cell again from a vertical direction?

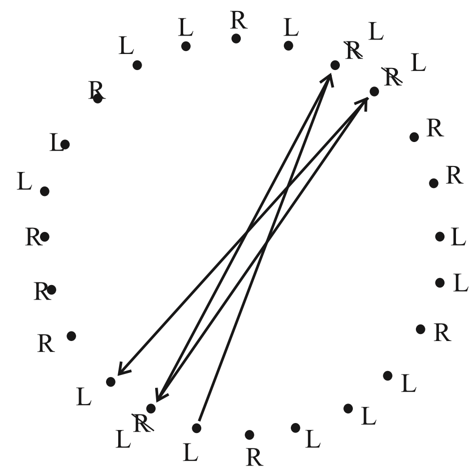
20.2 Ant Antics

An ant moves about a 7 × 7 square grid beginning in the central square, facing north. Each cell is labelled L for left or R for right. The ant moves according to a set procedure: It takes a step forward and looks at the label of its new cell. If the cell is labelled L the ant turns left 90°, right 90° if it is labelled R. The ant then changes the label of the cell (from R to L, or L to R), takes its next step forward, and repeats this procedure over and over again. Given these rules of motion, is it possible to devise an initial labelling scheme of the cells so that the ant is *not* forced to leave the 7 × 7 grid?

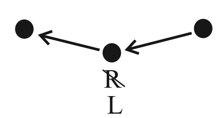


20.3 Ballthrowing

A number of students stand in a circle, each with the word left or right in mind. A student begins a ball game by tossing a ball across the circle. Whoever catches the ball throws it back across the circle one place to the *left* (in their perspective) of the first tosser if the second tosser is thinking left, or one place to the *right* if thinking right. Then the second tosser switches words ("right" becomes "left" and vice versa) and waits for another turn. Each person receiving the ball operates in this way. Here is the question: In this game is everyone guaranteed a turn? Will the ball eventually reach everyone, no matter the choice of word held initially in everyone's mind?



There is one complication: If a student is thinking "right" and receives the ball from the person directly to her right, the rules do not make sense. We make the convention that a student in this predicament tosses the ball to the person directly on her left. The student still changes the word she holds in her mind. Similarly, if a student is thinking "left" when he receives the ball from the person to his left, he throws the ball to the person on his direct right. This is a little confusing at first, but it doesn't take long to get the hang of it.



Comment. I recommend a lightweight ball if playing this game indoors!

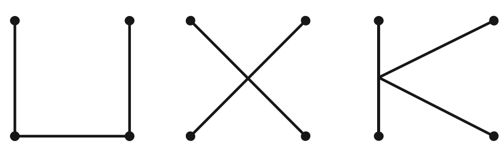
21

Bubble Trouble

21.1 Road Building

Here's a classic puzzle.

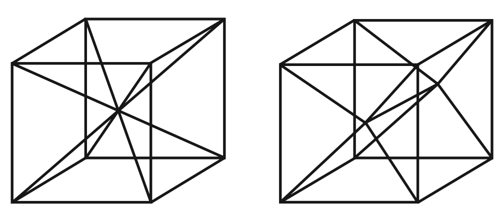
Four towns, situated on a plane at the vertices of a square, are to be connected by a road system using the minimum total length of road. Costs of construction are of importance here, not the convenience of the towns' inhabitants. Should the local government settle on one of the designs below? If so, which one? Is there an even better solution?



21.2 Higher Dimensional "Road Building"

Let's take the problem up a dimension.

What design of surfaces, meeting somewhere in the center, connects the skeleton of a cube (namely its 12 edges and 8 vertices)with minimal total surface area?

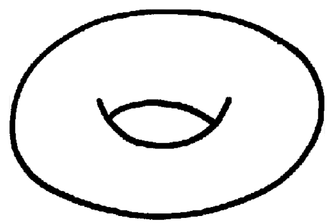


This problem is very difficult to analyze mathematically, but with the aid of soap solution the answer can be determined experimentally. Using pliable wire make a frame of a cube and dip it into soap solution. The surface tension of the liquid film acts to minimize surface area, so careful dipping (making sure a film is attached to every edge of the cube) will result in the desired solution to the problem.

What happens if you gently shake the structure of film you obtain? How does the solution to the problem change?

21.3 Donut Bubbles

Is it possible to make a stable donut-shaped bubble? (Try it!)



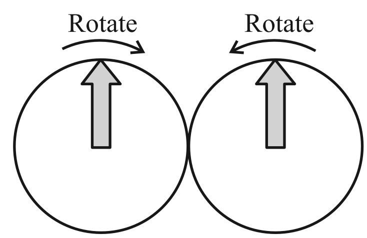
Frank Francisco dips a wire frame into soap solution.

Making wire frames and soap solution. Florist's wire, or 18 gauge aluminum wire works well. Toy stores sell bubble solution in large containers. Or you can make your own: Mix together one gallon of hot water, one cup of Dawn liquid detergent, and one tablespoon of glycerin; let the mixture sit overnight.

22

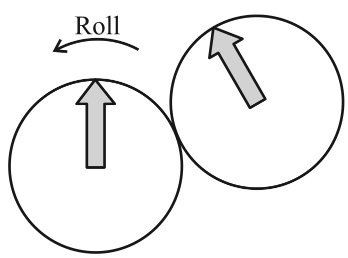
Halves and Doubles

22.1 Freaky Wheels I



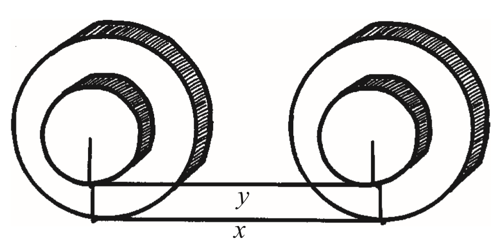
Cut out two large circles of equal size from thick cardboard and mark on each an arrow emanating from the center. Place the circles side by side with arrows pointing up and rotate each circle in the directions indicated. It takes a full rotation from each wheel before the two arrows are again parallel and pointing upwards. Notice each wheel "rolls" along the entire circumference of the other in this process.

Now hold one wheel fixed and roll the other wheel half way along its circumference. What happens?

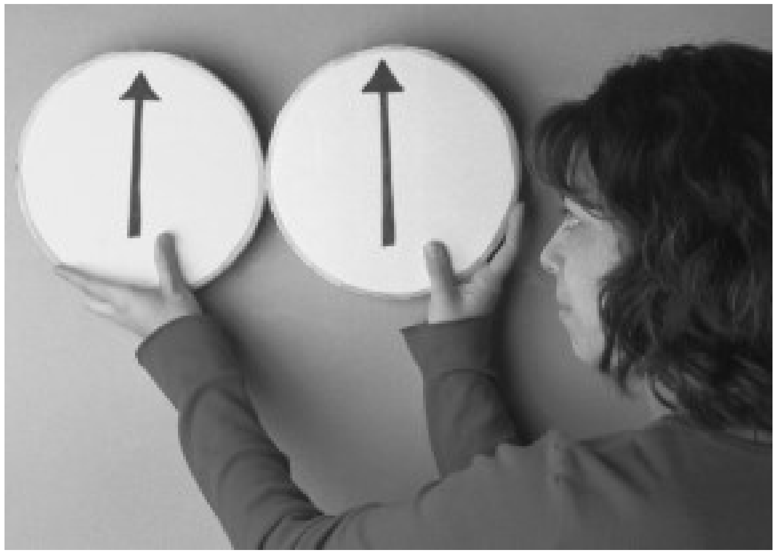


22.2 Freaky Wheels II

This puzzler was known to Aristotle more than 2000 years ago.



Cut two circles of different sizes from thick cardboard and glue the smaller wheel onto the larger so that their centers align. Mark a common radius on both wheels. Imagine these two wheels rolling along a double track as shown.

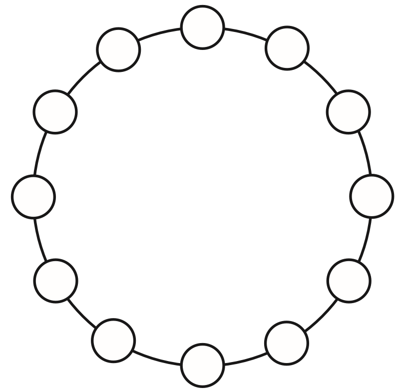


Kelly Ogden holds up a pair of freaky wheels.

Now consider one revolution of the entire system. Both wheels move along the same distance of track, so, in the diagram, $x = y$. But x equals the circumference of the big wheel and y the circumference of the little wheel. Is it true that these wheels have the same perimeter? Try the experiment to see that it must be the case!

John Dockstader experiments with Aristotle's paradox.

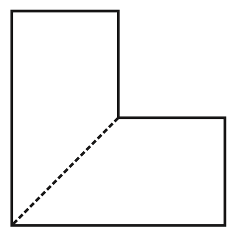
22.3 Breaking a Necklace



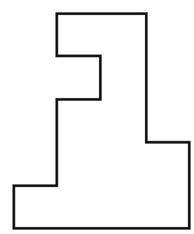
Here are 12 dots arranged in a circle. Color any six of them black. Two pirates have acquired a necklace containing six black pearls and six white pearls coincidentally arranged in the order you just colored! They would like to cut the necklace into as few pieces as possible so that, after divvying up the pieces, each pirate receives exactly three black and three white pearls. What is the minimal number of cuts they could make? Where should these cuts be placed?

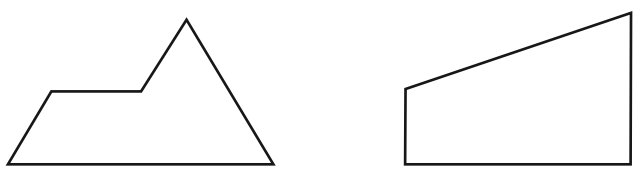
22.4 Congruent Halves

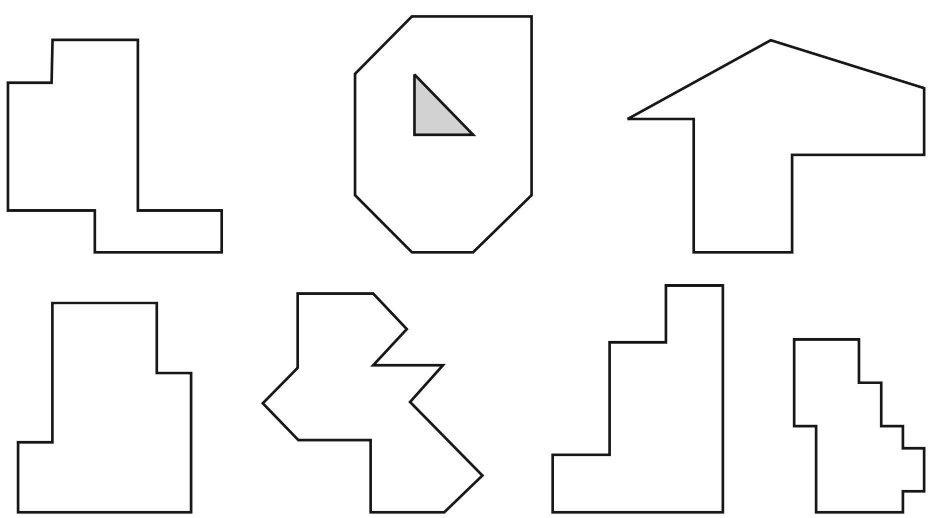
This L shape can be cut into two congruent halves. (Mirror images are considered congruent.)



Which of these shapes can also be so subdivided?







23

Playing with Playing Cards

23.1 A Pastiche of Card Surprises

Surprise 1. Shuffle a deck of cards and divide them into two piles of 26 cards each. Count the number of black cards in the first pile and the number of red cards in the second. What do you notice?

Surprise 2. Shuffle a deck of cards and divide them into two piles, one of 32 cards, the other of 20 cards. Count the number of black cards in the first pile and the number of red cards in the second. Subtract the smaller number from the larger. What do you get?

Surprise 3. Split a deck of cards into two piles according to color. Take ten cards from the red pile and shuffle them into the black pile. Without peeking, select ten cards from the (mostly) black pile and place them into the red pile. Count the number of "foreign" cards in each pile. What do you notice?

Surprise 4. Shuffle a deck of cards, take note of the top card, and place the deck face down on the table. Cut the deck, flip the top pile over, and place it back on the deck. Cut the deck again, deeper this time, and again flip the top pile over and place it back on the deck. Now remove all the cards that face up. What's the next card?

Surprise 5. Arrange a deck of cards so that all cards of the same suit appear in order from Ace, King, Queen, down to two. Each suit contains 13 cards. Cut the deck 13 times. Deal out, face down, a row of 13 cards from left to right. On top of them deal another row of 13 cards from left to right. Repeat this two more times until all the cards are dealt. Flip over each pile of four. What do you notice?

Surprise 6. Shuffle a deck of cards and note the bottom card. Call this the "magic card." Deal 12 cards face down and have someone turn over any four cards. Place the remaining eight cards on the bottom of the deck. Assigning the value 1 to an Ace, 10 to a Jack, Queen, or King, and the face value to a number card, sum the values of the four selected cards. Call this the "magic number."

On top of each selected card deal, in turn, the number of cards required to increase the face value of that card to 10. For example, on top of a 2 you would place eight cards, on top of a 7 three cards, and on top of a Jack zero cards. Collect all four piles of cards and place them on the bottom of the deck.

Now deal out the “magic number” of cards from the top of the deck. What’s the final card dealt?

Surprise 7. Lay out 21 cards face up in a grid of seven rows and three columns. Have a friend mentally select one card and indicate to you the column in which that card lies. Pick up the cards one column at a time, carefully preserving the order within the columns, and making sure to collect the indicated column second.

Lay out the 21 cards again, row by row, to obtain seven rows of three. Have your friend again indicate to you the column in which the selected card lies. Pick up the cards as above and repeat this process one more time.

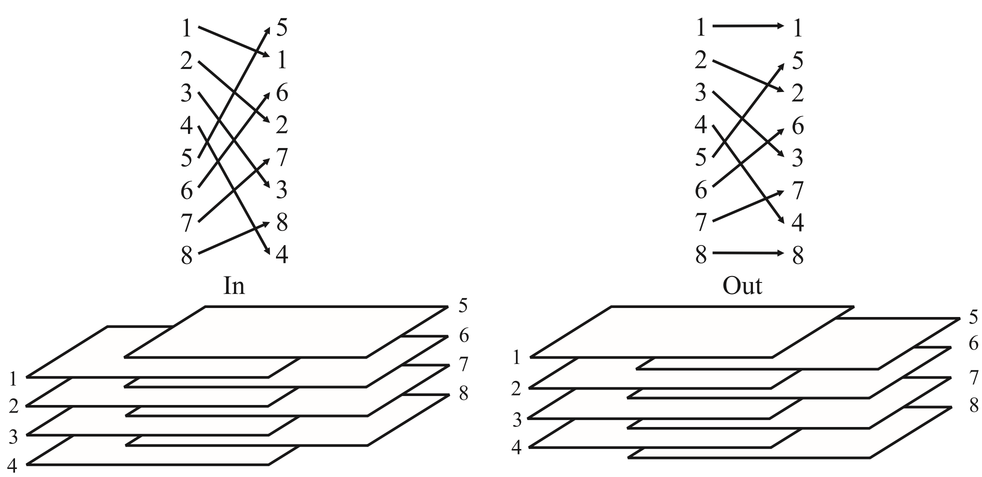
Pick up the cards again as above. Deal ten cards from the top of the pile and hand the 11th card to your friend!

23.2 Curious Piles

Shuffle a deck of cards and divide them into two equal piles of 26 cards. Clearly it is always possible to select a red card from one pile and a black card from the other. But suppose these cards were dealt instead into four piles of 13 cards. Would it be possible to select a spade from one pile, a club from another, a heart from a third, and a diamond from a fourth? (Try it!)
Is it always possible to accomplish this feat no matter how the cards are distributed?

23.3 On Perfect Shuffling

A *riffle shuffle*, also called a *Faro shuffle* or a *perfect shuffle*, begins by splitting a deck of cards precisely in half and then alternately interleaving the two halves back into a single pile. Thus, if one half of the deck were all the black cards, and the other half all the red, a riffle shuffle will result in a pile of alternating red and black. Dr. Brent Morris, a mathematician at the National Security Agency, is a master at perfect shuffling. He explains how you can perform it yourself on a full deck of cards in his wonderful book [Morr]. For ease, let’s just work with eight cards numbered 1 to 8. Splitting the deck in half yields two piles, 1 2 3 4 and 5 6 7 8.



There are two types of perfect shuffles one can be perform: an *in-shuffle*, which results in the ordering 5 1 6 2 7 3 8 4, and an *out-shuffle*, yielding the arrangement 1 5 2 6 3 7 4 8. Lay the cards in two rows of four and pick them up again by column to demonstrate these procedures.

What happens if you perform a perfect out-shuffle three times on a deck of eight cards? Using a combination of in- and out-shuffles, is it possible to move the top card of a deck of eight cards to an arbitrary position of the deck?

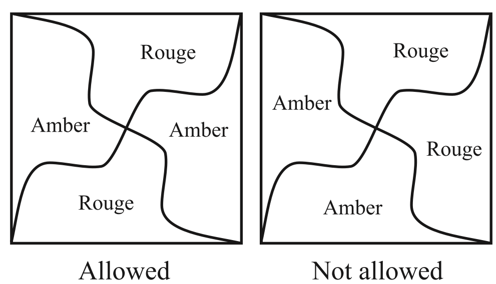
24

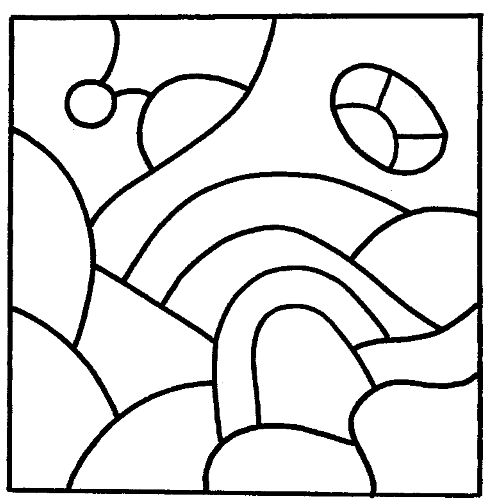
Map Mechanics

24.1 Cartographer's Wisdom

A map consists of *regions* bounded by *edges* that meet to form *vertices*. When painting a map, cartographers follow the convention that no two distinct regions sharing a common edge are assigned the same color (though two regions sharing a common vertex may be).

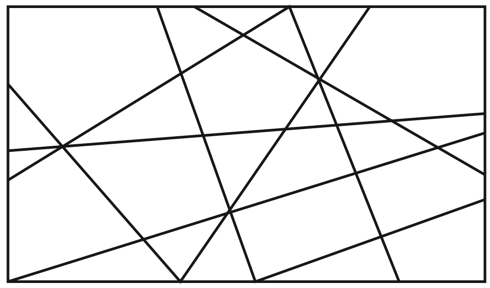
Cartographers have known for centuries (though see [May]) that just four colors are sufficient for coloring any map drawn on a plane. Some maps may be colored with less, but all maps can certainly be done with four. Try coloring this planar map with just four colors.



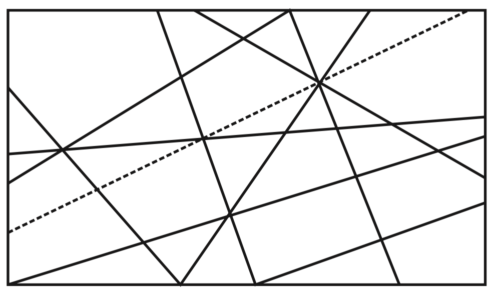


24.2 Simple Maps

This map is composed of regions arising from straight lines drawn across the entire page. It can be colored with fewer than four colors. How many fewer?

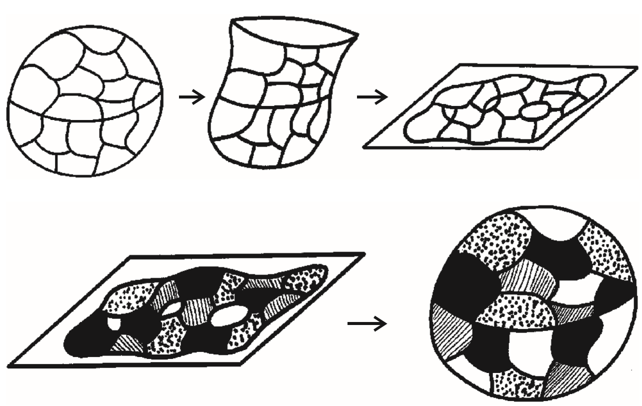


Taking It Further. How would your coloring scheme change if an extra line were added to the diagram?

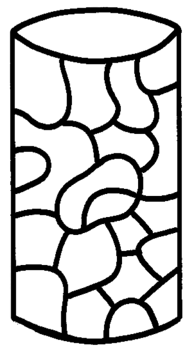


24.3 Toroidal Maps

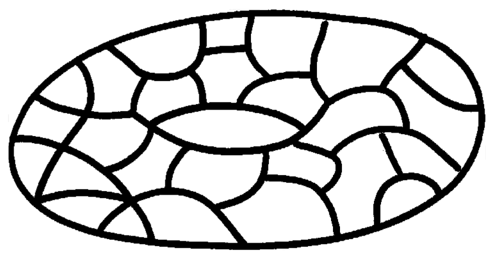
Four colors are sufficient to color any map drawn on the plane. This implies that four colors are sufficient for coloring any map on a sphere as well: Simply puncture the sphere at the interior of any region, flatten out the map, paint it as though it were a planar design, and then reform the sphere.



Four colors are also sufficient to paint any map on a cylinder. Can you see why?



But for maps drawn on a *torus* the situation is different. Can you design a map on a torus that requires a minimum of *five* colors to paint?



Try drawing and painting maps on real donuts or bagels!

25

Weird Lotteries

25.1 Winning Cake

A large group of people play an unusual lottery in the hopes of winning a cake. All write on a piece of paper their name and a positive integer greater than or equal to 1. All the entries are then collected and sorted through. If two or more people enter the same number, they are disqualified from the lottery. Only the unique numbers submitted are considered. The highest unique number wins, and the prize is that *fraction* of the cake! Thus if someone wins with the number 20 they win one twentieth of the cake and no one else will receive any cake. If you were to play this game, what strategy would you employ?

Comment. Try playing this game multiple times to experiment with alternative strategies. Cupcakes make good prizes.

25.2 Unexpected Winner

Some students write their names on individual ballots and place them into a hat. The professor selects one ballot at random to determine the winner of a fabulous chocolate cake. However, the professor suddenly makes this surprising announcement: “I am going to wait two minutes before announcing the name of the winner. No-one, except me of course, knows who has won the fabulous cake. You have no way of guessing who the winner could be, and it will remain that way for the next two minutes. The name of the winner will be a complete surprise to you all. John has won the cake.” The professor is then silent for two minutes. Has John won the cake?

25.3 Winning Tootsie Rolls®

Everyone in a room is to write on a card the word “Cooperate” or “Defect.” Tootsie Rolls® will be distributed according to the outcome of the following scheme. If everyone “cooperates” each person will receive ten Tootsie Rolls®. If a mixture of people cooperate and defect, or everyone defects, then the cooperators will each receive five tootsie rolls and the defectors none. But if there is just a single defector, he or she will receive 60 Tootsie Rolls® and the others none.

	C	D
All Cooperate	10	–
Two or more Defect	5	0
One Defects	0	60

Given the rules of this game, how would you vote?

25.4 Buying Tootsie Rolls

Clarence and Denise each receive a large pile of pennies with which to "buy" candy. The price, however, varies from turn to turn according to the roll of a die.

Die face	Candies per penny	Cost of six candies
1	1	6¢
2	2	3¢
3	2	3¢
4	3	2¢
5	3	2¢
6	6	1¢

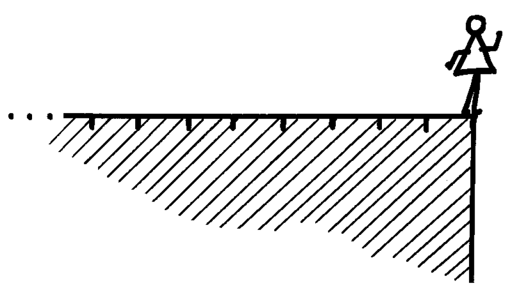
At every roll of the die, Clarence will buy one penny's worth of candy, Denise will buy six candies no matter the cost. Who in the long run gets the better deal? Try this experiment several times and see if you can detect who gets the most candies for the money.

26

Flipped Out

26.1 A Real Cliff-Hanger

Dorothy stands on the edge of a cliff; an infinite expanse of land is behind her. Taking one step forward would send her to her doom, whereas one step back would be a step toward safety. All of Dorothy's steps are precisely one foot long. Dorothy has gamely agreed to let her fate be determined by the flip of a coin. She will take one step forward if the result of a toss is heads, one step back if it is tails. If she survives the first toss, she is willing to do it again, and again, stepping forward and back one foot according to the toss of the coin. After an infinite number of tosses she hopes to be wandering off into the infinite expanse behind her. What are Dorothy's chances of survival?



Angela Dellano tests Dorothy's fate with several repeated experiments.

26.2 Too Big a Difference

Toss a coin ten times. Then do it again, and again, many times. What is the average difference between the number of heads and the number of tails that appear? Is it zero?

26.3 A Surprise

Delicately balance 20 pennies on edge on the surface of a table. Then bang the table so they all fall over. What do you notice?

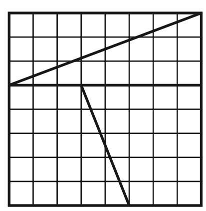
Now simultaneously spin 20 American pennies and let them come to rest. What do you notice?

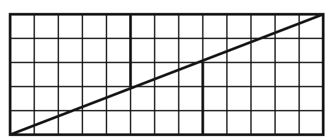
27

Parts That Do Not Add Up to Their Whole

27.1 A Fibonacci Mismatch

Who said area is always preserved? Take an 8×8 square inch piece of paper and subdivide it as shown. Now rearrange the pieces to form a 5×13 rectangle as shown. (Try it!) This transforms 64 square inches of paper into 65 square inches. What's going on?

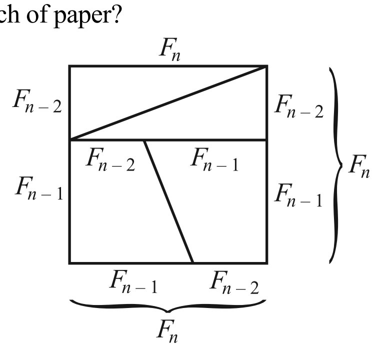


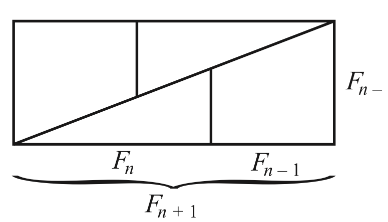


Taking It Further. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ is defined recursively by

$$F_1 = 1$$
$$F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad \text{if } n \geq 3.$$

Choose any Fibonacci number F_n equal to or larger than eight, and subdivide an $F_n \times F_n$ square as shown. Rearrange the pieces to produce an $F_{n-1} \times F_{n+1}$ rectangle. Have you again lost track of a square inch of paper?





27.2 Cake Please

Two brothers, Albert and Hubert, plan to share a last piece of cake. They could perform the familiar "you cut – I choose" division scheme, where one brother cuts the slice into what he believes to be two equal parts and the other then chooses a piece. The first is then guaranteed, in his estimation, precisely 50% of the cake, the other, in his measure, 50% and perhaps more if he has a different estimation of half. It seems, however, the second person has an advantage in this scheme. Is there a cake slicing scheme that will guarantee *both* brothers more than 50% of the cake in their own estimations?

27.3 Sharing Indivisible Goods

Bjorn and Elaina each have a large supply of Tootsie Rolls® but only one chocolate bar between them. For some reason they will not break the bar in two, but both are willing to trade their share of the bar for Tootsie Rolls®. Bjorn thinks the bar is worth 18 Tootsie Rolls®, Elaina 14. Who should keep the bar? How many Tootsie Rolls® should the other person receive as compensation? Devise a scheme so that each person gets more than his or her estimation of worth!

28

Making the Sacrifice

28.1 The Josephus Flavius Story

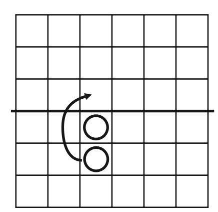
Let's begin on a positive note.

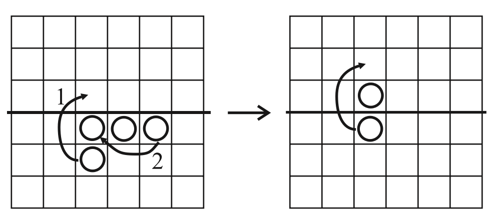
A group of people sit in a circle and begin to count off every third person. Whoever is selected third is called "out" and leaves the game. Counting continues until the last participant is declared the winner of the game. Given the size of a group and where the count begins, is it possible to predict beforehand who will be the winner? Try playing the game several times with different sized groups and see if you can detect any pattern to the location of the winner.

Taking It Further. What if every second person is counted? Every ninth person?

28.2 Soldiers in the Desert

A horizontal line is drawn on an arbitrarily large grid of squares. Behind this line stands an army of pennies, one penny per cell. The aim is to move a single penny into the desert by performing a series of checker jump moves within the troops. One penny can jump over any other penny in either a vertical or a horizontal direction to a vacant square. The penny jumped over is removed or "sacrificed". It is easy to design an army that could move a penny one line or two lines into the desert. Is it possible to move a penny three lines into the desert? How about four lines into the desert?





28.3 Democratic Pirates

Ten pirates ranging in rank from captain (whom we shall label pirate number 1) to cabin boy (pirate number 10) have come across ten gold coins. Being a democratic crew they decide upon the following process to distribute the coins. The cabin boy will first nominate a distribution pattern (two coins to pirates 1, 3, 4, 8, and himself, for example) and the pirates will take a vote. If 50% or more of the pirates agree with this distribution they will go with it, otherwise the cabin boy will be thrown overboard and the ninth ranking pirate will propose a new scheme and invoke a new vote. They will do this, up the rank, until they finally settle upon a favorable vote.

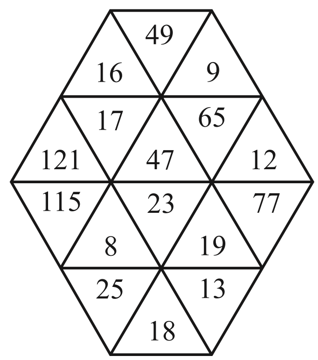
What distribution scheme should the cabin boy suggest? Assume the pirates are rational thinkers and none will vote against a proposed scheme if, as a consequence, he will be thrown overboard, or end up with fewer coins. If there is no personal ill effect, a pirate will otherwise be glad to see a fellow shipmate thrown overboard.

Comment. This game can be acted out using a supply of ten treats. Rather than ten pirates, consider first a game with just two or three pirates.

29

Problems in Parity

29.1 Magic Triangles



Consider the grid of triangles shown. Two triangles are said to be *neighbors* if they share a common edge. Thus 17 and 47 are neighbors, but 17 and 9 are not. A *path* in the grid is any sequence of neighboring cells. Apath might loop back on itself or even step back and forth repeatedly between a select few cells. For example, 17-47-65-9-65-47-23 is a valid path of six steps.

Place your finger on any even number in the grid and perform the following sequence of instructions.

1. Move your finger 11 steps along any meandering path of your choosing. Keep your finger in place at the end of your chosen path. You will start step 2 from here. (Despite the free will you possess I know you have not landed on a cell that is a multiple of 3!)

2. Avoiding all multiples of 3, move your finger seven steps along any path of your choosing. (I know you have not landed on a multiple of 5! I could make similar predictions as you continue.)

3. Avoiding all multiples of 3 and 5, move your finger six steps along any path.

4. Avoiding all multiples of 3, 5, and 7, move your finger five steps along any path.

5. Avoiding all multiples of 3, 5, 7, and 11, move your finger five steps along any path.

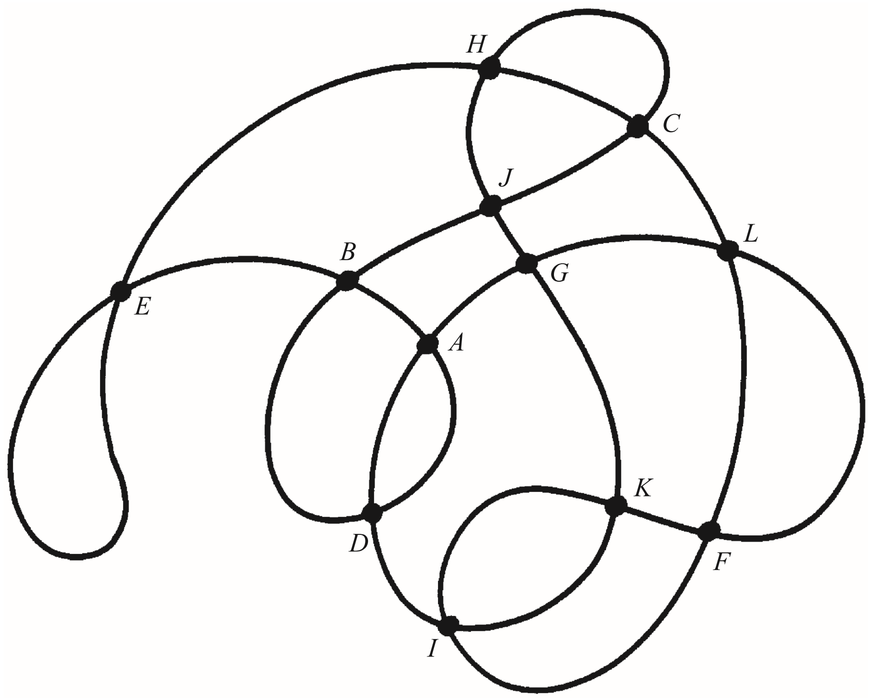
6. Avoiding all multiples of 3, 5, 7, 11, and 13, move your finger seven steps along any path.

7. Avoiding all multiples of 2, 3, 5, 7, 11, and 13, move your finger three steps along any path.

8. Avoiding all multiples of 2, 3, 5, 7, 11, 13, and 17, move one place over.

How do I know that you will never land on an inappropriate cell? How do I know that your finger is currently on cell number 23?

Comment. Make copies of this grid and perform the activity with a group. Everyone in the group will land on precisely the same cell in the end to their great surprise.



29.2 Walking a Loop

Here is a closed loop that crosses over itself several times. The curve has the property that it passes through each of its crossing points only twice. (One can draw curves that pass through the same point many more times, but we ignore curves of that type here.) I have labelled the crossing points randomly with the letters A through L. These letters can be used to record a journey that traverses the entire loop once:

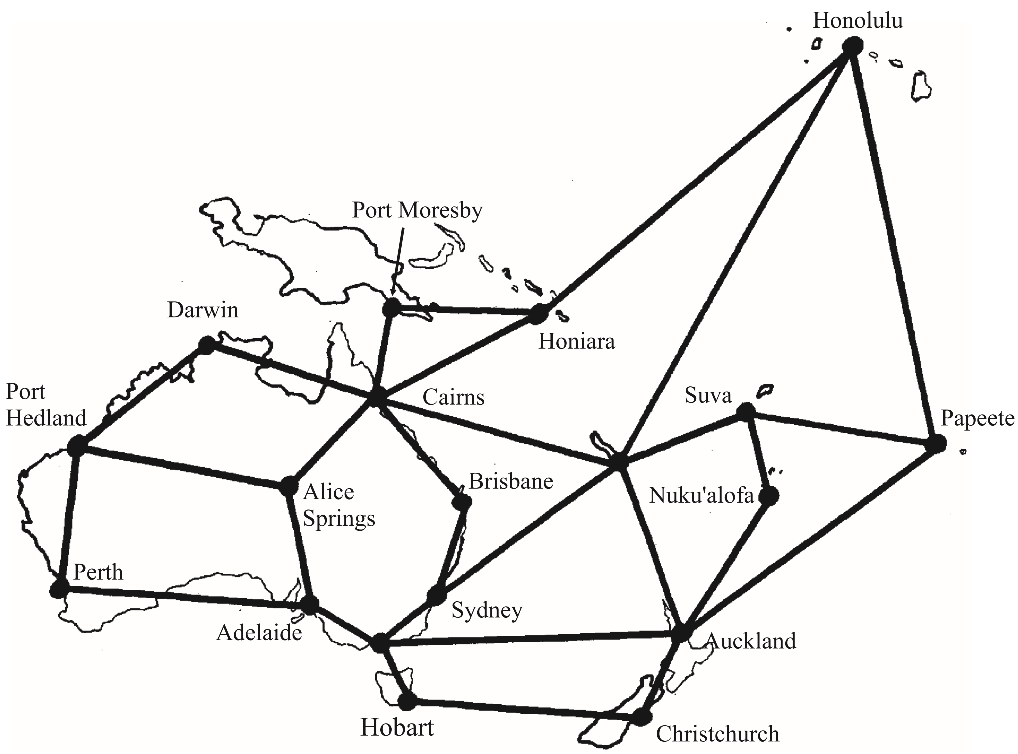
A-D-B-J-C-H-J-G-K-I-D-A-G-
L-F-K-I-F-L-C-H-E-E-B-A

Have a friend secretly draw a curve of this type on a piece of paper and label the crossing points with letters in some arbitrary fashion. Now have the person trace the curve with her finger and read out to you the names of the crossing points she encounters, but with one deliberate error: she is to transpose the names of two adjacent crossing points. (For example, in the curve above, when I am at crossing point G and about to move to L I could read out L G rather than G L.) Your friend should never reveal to you the picture, the labelling scheme, or the place of the error.

Relying only on the sequence of letters said out loud, how could you swiftly determine which two crossing points were switched? Once you have mastered this trick, use it to impress your friends!

29.3 Catch Me If You Can

Here is an international island-hopping game. Craig starts in Perth, and Joy in Papeete. Each takes turns hopping from one location to the next along an established air route as in the diagram.



Joy's mission is to land in the same location as Craig. Craig's mission is to avoid this. Joy goes first and has 12 turns in which to win this game. If she fails, Craig will be declared winner. What is Joy's best strategy in order to win this game?

29.4 A Game of Solitaire

This game of solitaire is played with a line of pennies placed heads or tails up in an arbitrary fashion. A move consists of removing any coin that is heads up and flipping over the coin in the space to its immediate left (if one is there to be flipped) and the coin in the space to its immediate right (if present). The goal is to remove all the pennies. What is the best strategy for winning this game of solitaire? When should you not even bother playing the game?

30

Chessboard Maneuvers

30.1 Grid Walking

Greta, Peter, Lashana, and Lopsided Charlie wander within a large grid of square paving stones, each following certain rules of motion.

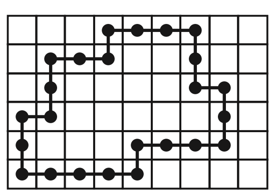
Greta moves only in horizontal and vertical directions, one square over at a time. Can anything be said about the total number of steps she must take to return to her initial square and thus form a loop of steps?

Peter moves about the grid like a knight on a chessboard. Each step takes him two squares over in one direction and one square in an orthogonal direction. Can anything be said about the total number of steps he must take to complete a loop?

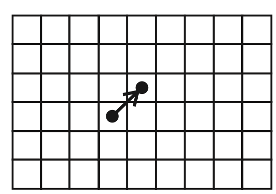
Lashana takes only single diagonal steps as she wanders about the grid. Can anything be said about the total number of steps she must take to complete a loop?

Lopsided Charlie follows slightly more complicated rules of motion. Any step he takes north or east moves him *two* squares, but any step south or west moves him only *one* square. Can anything be said about the total number of steps Charlie must take to complete a loop?

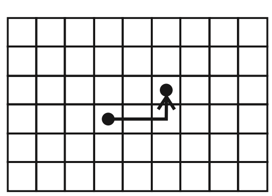
Greta



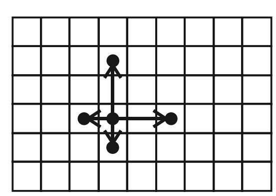
Lashana



Peter



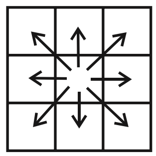
Lopsided Charlie

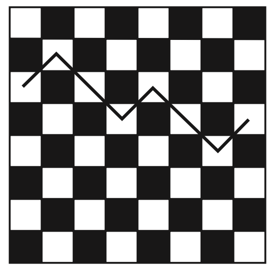


Comment. Use graph paper to experiment with any of the above motions.

30.2 Kingly Maneuvers

A king moves across a chessboard one square at a time in any one of eight directions: left, right, up, down, or diagonally. In this problem we want the king to move from the top row of an 8 × 8 chessboard to the bottom row solely on black squares, or from the leftmost column to the rightmost solely on white squares. Both are possible with the standard coloring scheme of a chessboard as shown.





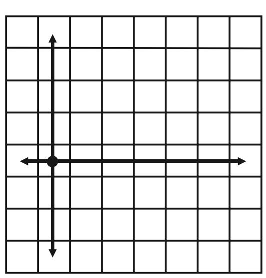
Make an 8 × 8 grid of squares and randomly color the cells black and white. (You need not have the same number of cells for each color). In your arbitrary coloring scheme, is there still a path of black squares from the top row to the bottom for a king to travel? If not, have you unwittingly created instead a path of white cells from left to right? Devise an 8 × 8 coloring scheme that creates no paths of either type.

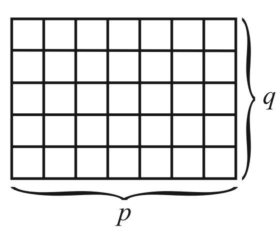
Comment. Turn this puzzle into a two-person game. Have two players take turns coloring the cells of an 8 × 8 grid, the first player coloring the cells black, the second white. The first player wins by creating a path of black cells connecting the top row to the bottom, before the second player creates a path of white cells linking the leftmost column to the rightmost (otherwise the second player wins.) Must there be a winner for every game?

30.3. Mutual Non-Attack: Rooks

A rook attacks other pieces on a chessboard by sliding along vertical or horizontal lines of squares and taking the place of any opponent it encounters. What is the maximal number of rooks that can be placed on a 4 × 4 chessboard so that no rook is in position to attack any other rook?

What is the maximal number of rooks that can be placed on an $n \times n$ board in a configuration of mutual non-attack? What about $p \times q$ rectangular boards with $q < p$? How many different ways are there to place a maximal number of rooks on an $n \times n$ board in mutual non-attack? On a $p \times q$ board?





30.4 Mutual Non-Attack: Queens

A queen attacks other pieces on a chessboard by sliding along vertical, horizontal, or diagonal lines of squares and taking the place of any opponent encountered. What is the maximal number of queens that can be placed on a 4×4 chessboard so that no queen is in position to attack another queen? What about 5×5, 6×6, and 8×8 chessboards?