Numerical Analysis

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Errors in Polynomial Interpolation

Theorem

If $P_n(x)$ is the polynomial of degree at most n that interpolates $f(x)$ at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f \in C^{n+1}[a, b]$, then for each $x \in [a, b]$ there is a $\xi \in (a, b)$ for which

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \ldots (x - x_n).$$

Theorem

If $P_n(x)$ is the polynomial of degree at most n that interpolates $f(x)$ at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$, then for any x that is not a node

$$f(x) - P_n(x) = f[x_0, x_1, \ldots, x_n, x](x - x_0)(x - x_1)\ldots(x - x_n).$$

Example

Assume $P_3(x)$ is the interpolating polynomial for the function $f(x) = \ln x$ with the nodes $x_0 = 100$, $x_1 = 101$, $x_2 = 102$ and $x_3 = 103$. Estimate the error

$$|f(100.5) - P_3(100.5)|.$$

Theorem

If $P_n(x)$ is the polynomial of degree at most n that interpolates $f(x)$ at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f \in C^{n+1}[a, b]$, then for any $x \in [a, b]$

$$|f(x) - P_n(x)| = \frac{\max\limits_{a \le \xi \le b} |f^{(n+1)}(\xi)|}{(n+1)!} |(x - x_0)(x - x_1) \ldots (x - x_n)|.$$

Theorem

If $P_n(x)$ is the polynomial of degree at most n that interpolates $f(x)$ at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f^{(n+1)}$ is continuous and $M_{n+1} = \max\limits_{a \le x \le b} |f^{(n+1)}(x)|$, then

$$\max_{a \le x \le b} |f(x) - P_n(x)| \le \frac{M_{n+1}}{(n+1)!} \max_{a \le x \le b} |(x - x_0)(x - x_1) \ldots (x - x_n)|.$$

Example

Assume $P_2(x)$ is the interpolating polynomial for the function $f(x) = \sin(x^2)$ with the nodes $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 2$. Estimate the global error

$$\max_{0 \le x \le 2} |f(x) - P_2(x)|.$$

Runge Phenomenon

Give an example of a function $f(x)$ such that if we consider its interpolation on $[-1, 1]$ with equidistant nodes, then

$$\lim_{n\to\infty} \max_{-1\le x\le 1} |f(x) - P_n(x)| \nrightarrow 0.$$

Let's consider Dirichlet function:

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{I} \end{cases} \quad .$$

Interpolating nodes:

$$x_i = -1 + \frac{2}{n} i, \quad i = 0, 1, \ldots, n.$$

Interpolating Polynomial:

$$P_n(x) \equiv 0.$$

The Global Error:

$$\max_{-1 \le x \le 1} |f(x) - P_n(x)| = 1.$$

Dirichlet function is discontinuous.

Let's consider Runge function:

$$f(x) = \frac{1}{1 + x^2}$$

on the interval $[-5, 5]$.

Runge function is infinitely differentiable.

Example

Write a MatLab program that divides the interval $[-5, 5]$ by equidistant points $-5 = x_0 < x_1 < ... < x_n = 5$, constructs the Interpolating Polynomial of degree $\leq n$ for the function $f(x) = \frac{1}{1+x^2}$ at the points x_i, $i = 0, 1, ..., n$, and then plots the function f, the Interpolating Polynomial P_n, the points $(x_i, f(x_i))$ on the same figure, when $n = 10$, $n = 20$, $n = 30$.

- Create an array of nodes $x_i = -5 + \frac{10}{n} i$, $i = 0, 1, \ldots, n$ (you can use MatLab function *linspace*).
- Calculate the values of f at the nodes.
- Use MatLab function *polyfit* to calculate the coefficients of Interpolating Polynomial.
- Create an array using which you are going to plot f and the Interpolating Polynomial.
- Use MatLab function *polyval* to calculate the values of the Interpolating Polynomial.
- Then plot the Interpolating Polynomial, the function, the points $(x_i, f(x_i))$.