# Optimization

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## Conditions for Local Minimizers

$$\begin{aligned} &\textit{minimize} \quad f(x) \\ &\textit{subject to} \quad x \in \Omega, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, with $n \geq 1$.

## Definition

A vector $d \in \mathbb{R}^n$, $d \neq 0$ is a feasible direction at $x^* \in \Omega$ if there exists $\alpha_0 > 0$ such that $x^* + \alpha d \in \Omega$ for all $\alpha \in [0, \alpha_0]$.

## Example

If it is possible, find two vectors $d_1$, $d_2 \in \mathbb{R}^n$ different from 0 vector such that $d_1$ is a feasible direction at $x^* \in \Omega$ and $d_2$ is not a feasible direction at $x^* \in \Omega$, if

**a.** $\Omega = [1, 3]$, $x^* = 1$;

**b.** $\Omega = [1, 3]$, $x^* = 3$;

**c.** $\Omega = [1, 3]$, $x^* = 2$;

**d.** $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$, $x^* = [0, 0]^T$;

**e.** $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$, $x^* = [1, 0]^T$.

### Definition

A point $x^* \in \Omega$ is said to be an interior point of the set $\Omega$ if $\Omega$ contains some neighborhood of $x^*$.

### Example

Determine whether $x^* \in \Omega$ is an interior point of $\Omega$, if

**a.** $\Omega = [1, 3]$, $x^* = 1$;

**b.** $\Omega = [1, 3]$, $x^* = 2$;

**c.** $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$, $x^* = [0, 0]^T$;

**d.** $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$, $x^* = [1, 0]^T$.

**Proposition.** Assume $x^* \in \Omega$ is an interior point of $\Omega$, then any vector $d \in \mathbb{R}^n$ , $d \neq 0$ is a feasible direction at $x^*$.

**Theorem (First-Order Necessary Conditions (FONC))**

*Assume $f$ is a continuously differentiable function in $\Omega$. If $x^*$ is a local minimizer of $f$ over $\Omega$, then for any feasible direction $d$ at $x^*$, we have*

$$d^T \nabla f(x^*) \geq 0.$$

**Theorem (First-Order Necessary Conditions (FONC))**

*Assume $f$ is a continuously differentiable function in $\Omega$. If $x^*$ is a local minimizer of $f$ over $\Omega$ and $x^*$ is an interior point of $\Omega$, then*

$$\nabla f(x^*) = 0.$$

**Definition**

We call $x^*$ a **stationary point** if $\nabla f(x^*) = 0$.

**Example**

Consider the problem

$$\begin{aligned} \text{minimize} \quad & x_1^2 + 0.5x_2^2 + 3x_2 + 4.5 \\ \text{subject to} \quad & x_1, x_2 \geq 0. \end{aligned}$$

Is the first-order necessary condition (FONC) for a local minimizer satisfied at:

**a.** $x^* = [1, 3]^T$;

**b.** $x^* = [0, 3]^T$;

**c.** $x^* = [1, 0]^T$;

**d.** $x^* = [0, 0]^T$.

**Example**

Consider the problem

$$\begin{aligned} \textit{minimize} \quad & x_1^2 + 0.5x_2^2 + 3x_2 + 4.5 \\ \textit{subject to} \quad & x \in \mathbb{R}^2. \end{aligned}$$

Find the points which satisfy the first-order necessary condition (FONC) for a local minimizer.

**Theorem (Second-Order Necessary Conditions (SONC))**

*Assume $f$ is twice continuously differentiable in $\Omega$. If $x^*$ is a local minimizer (maximizer) of $f$ over $\Omega$ and $x^*$ is an interior point of $\Omega$, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive (negative) semidefinite.*

**Definition**

A **saddle point** is a stationary point which is not a local extremum.

**Example**

Show that $x^* = (0,0)^T$ is a saddle point for the function $f(x_1, x_2) = x_1^2 + 8x_1x_2 + x_2^2$.

**Theorem (Second-Order Sufficient Conditions (SOSC))**

*Assume $f$ is twice continuously differentiable in $\Omega$. If $x^*$ is an interior point of $\Omega$ such that $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive (negative) definite, then $x^*$ is a strict local minimizer (maximizer) of $f$.*

**Example**

Find all stationary points of $f$ and check if these points are local maximum, minimum or saddle points for that function if

**a.** $f(x_1, x_2) = 4x_1^4 + x_2^4 + 4x_1x_2;$

**b.** $f(x_1, x_2) = \frac{1}{3}x_1^3 - 4x_1 + \frac{1}{3}x_2^3 - 16x_2;$

**c.** $f(x_1, x_2, x_3) = 3x_1^3 - 9x_1 + x_2^3 + x_3^3 - 6x_3^2 - 10.$

**Theorem**

*When $f$ is convex, any local minimizer $x^*$ is a global minimizer of $f$. If in addition $f$ is differentiable, then any stationary point $x^*$ is a global minimizer of $f$.*

## Example

Find the global minimizer of $f$ on $\Omega$ if

**a.** $f(x_1, x_2, x_3) = x_1^4 + x_2^4 + x_1^2 x_2^2 + x_3^2$, $\Omega = \mathbb{R}^3$;

**b.** $f(x) = x^T A x$, where $A$ is symmetric and positive definite matrix and $\Omega = \mathbb{R}^n$.