Numerical Analysis

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The Least-Squares Method

Example (Finance)

In The Figure below we report the price of a stock at the Zurich stock exchange over two years. The curve was obtained by joining with a straight line the prices reported at every day's closure. We ask whether from this graph one could predict the stock price for a short time interval beyond the time of the last quotation.

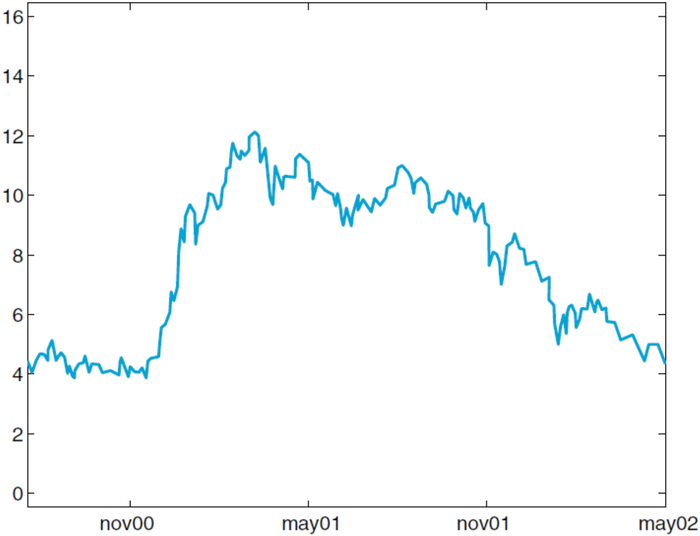

Figure: Price variation of a stock over two years

Linear Least-Squares

Assume we have a data

x	x_0	x_1	x_2	$\dots$	x_n
y	y_0	y_1	y_2	$\dots$	y_n

.

We want to find a linear function $y = ax + b$ that fits this data the best.

If the point (x_i, y_i) falls on the line, then

$$ax_i + b - y_i = 0.$$

If it doesn't, then there is an error of magnitude

$$|ax_i + b - y_i|$$

The total error for all points will be

$$\sum_{i=0}^{n} |ax_i + b - y_i|$$

This is a function of two variables a and b. Our aim is to minimize this function.

It is equivalent to the minimization of the following function

$$\varphi(a,b)=\sum_{i=0}^{m}(ax_i+b-y_i)^2\,.$$

If (a^*,b^*) is the minimum point of $\varphi(a^*,b^*)$, then (a^*,b^*) is the solution of the following system

$$\begin{cases}\frac{\partial\varphi}{\partial a}(a,b)=0,\\ \frac{\partial\varphi}{\partial b}(a,b)=0.\end{cases}$$

The resulting function $y=a^*x+b^*$ is known as the least-squares straight line, or regression line.

Example

Using the Least-Squares Method, find the linear function that best fits with the following

x	1	2	4	8
$f(x)$	2	0	2	6

.

In general case want to find a polynomial of degree m $P_n(x) = a_0 + a_1 x + \cdots + a_m x^m$ that fits this data the best.

x	x_0	x_1	x_2	$\dots$	x_n
y	y_0	y_1	y_2	$\dots$	y_n

.

As a result we need to minimize the following function

$$\varphi(a_0, a_1, \dots, a_m) = \sum_{i=0}^{n} \left(a_0 + a_1 x_i + \cdots + a_m x_i^m - y_i\right)^2 .$$

The minimum point of $\varphi(a_0, a_1, \ldots, a_m)$ is the solution of the following system

$$\begin{cases} \frac{\partial \varphi}{\partial a_0}(a_0, a_1, \ldots, a_m) = 0, \\ \frac{\partial \varphi}{\partial a_1}(a_0, a_1, \ldots, a_m) = 0, \\ \vdots \\ \frac{\partial \varphi}{\partial a_m}(a_0, a_1, \ldots, a_m) = 0. \end{cases}$$

What can we say about the case when $m = n$?

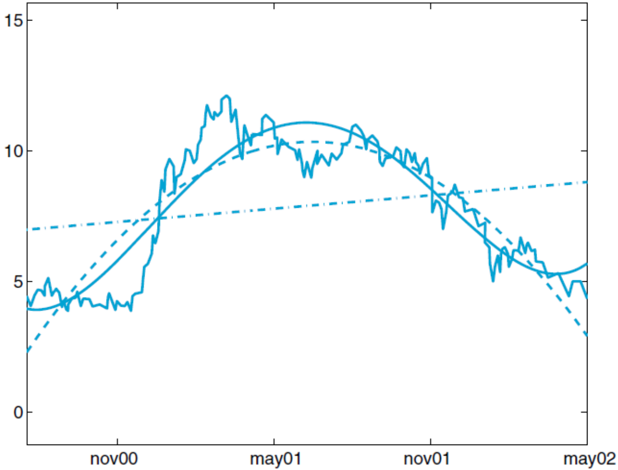

Figure: Least-squares approximation of the data with polynomials of degree 1 (dashed-dotted line), degree 2 (dashed line) and degree 4 (thick solid line). The exact data are represented by the thin solid line.

Example

Write a MatLab program that divides the interval $[-1, 1]$ by equidistant points $-1 = t_0 < t_1 < ... < t_{40} = 1$, then calculates the values of function $f(x) = |\sin(4x)|$ at the points t_i, $i = 0, 1, ..., 40$, and then calculates the approximating polynomial of degree n for $n = 1, 2, \ldots, 8$. Plots the function f, the polynomial $P_n(x)$ and the points $(t_i, f(t_i))$ on the same figure.