Numerical Analysis

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Characterizing Algorithms

If small changes in initial data produce correspondingly small changes in the final result then the algorithm is stable; otherwise it is unstable. Some algorithms are stable only for certain choices of initial data, and are called conditionally stable.

Example

Calculate the terms of sequence

$$E_n = \int_0^1 x^n e^{x-1} dx, \quad n = 1, 2, \ldots.$$

E_n is a decreasing sequence and tends to 0 as n goes to infinity.

$$E_1 > E_2 > \cdots > 0, \quad \lim_{n \to \infty} E_n = 0.$$

After integration by parts

$$\int_0^1 x^n e^{x-1} dx = x^n e^{x-1}\Big|_0^1 - n\int_0^1 x^{n-1} e^{x-1} dx$$

or

$$E_n = 1 - nE_{n-1}, \quad n = 2, 3 \ldots, \quad E_1 = \frac{1}{e}.$$

$$E_{n-1} = \frac{1 - E_n}{n}, \quad n = \dots 3, 2.$$

$$E_n = \int_0^1 x^n e^{x-1} dx \le \int_0^1 x^n dx \le \frac{1}{n+1}.$$

Convergence of Numerical Methods

Definition

A sequence x_n exhibits **linear** convergence to a limit x if there is a constant C in the interval $[0\ ,\ 1)$ and an integer N such that

$$|x_{n+1} - x| \leq C|x_n - x|, \quad \forall n \geq N.$$

Example

$$x_n = \frac{1}{2^n}.$$

Definition

A sequence x_n exhibits **superlinear** convergence to a limit x if there is a sequence β_n, which converges to 0, and an integer N such that

$$|x_{n+1} - x| \leq \beta_n |x_n - x|, \quad \forall n \geq N.$$

Example

$$x_n = \frac{n}{2^{n^2}} + 1.$$

Definition

We will say that $\alpha \geq 1$ is the rate of convergence of sequence x_n if there is a constant $C > 0$ (if $\alpha = 1$ then $0 < C < 1$) and an integer N such that

$$|x_{n+1} - x| \leq C|x_n - x|^{\alpha}, \quad \forall n \geq N.$$

Example

Find the limit and the rate of convergence to that limit for the following sequences:

a. $x_n = \frac{1}{2^{2^n}}$;

b. $x_n = \frac{1}{2^{3^n}+n}$;

Locating Roots of Nonlinear equations

Why do we need numerical methods to solve equations?

- We can't find precise solutions of the equation.
- Sometimes equations contain some parameters, which have been approximated and already there is no need solve that equation precisely.

Assume f is defined and continuous on some interval and we need to solve the following equation

$$f(x) = 0.$$

Definition

A number r for which $f(r) = 0$ is called a **root** of that equation or a **zero** of f.

Assume $f(x) = 0$ has only isolated roots, which means for each root we can find an interval, where the equation doesn't have any other roots.

- Isolate the roots.
- Calculate the roots within given accuracy.

Isolating the roots

Assume we can solve equation $f'(x) = 0$ and $x_1 < x_2 < \ldots x_n$ are the roots, then isolation of zeros of $f(x)$ can be easily done.

If $\lim_{x \to a} f(x) f(x_1) < 0$, then $f(x)$ has a zero in (a, x_1).

If $f(x_i) f(x_{i+1}) < 0$, $i = 1, \ldots n - 1$, then $f(x)$ has a zero 1n (x_i, x_{i+1}).

If $f(x_n) \lim_{x \to b} f(x) < 0$, then $f(x)$ has a zero in (x_n, b).

$f(x)$ has at most $n + 1$ zeros.

Example

Isolate the roots of the following equation

$$x^4 - 4x - 1 = 0.$$

If the equation $f(x) = 0$ doesn't have close roots then we can isolate the roots graphically.

Example

Isolate the roots of the following equation

$$x \ln x - 1 = 0.$$

Bisection method

Assume f is continuous on the closed interval $[a, b]$ and $f(a)f(b) < 0$. The Intermediate Vale theorem implies that a number r exists in (a, b) with $f(r) = 0$.

Let's set $a_0 = a$, $b_0 = b$ and x_0 is the midpoint of $[a, b]$; that is

$$x_0 = a_0 + \frac{b_0 - a_0}{2} = \frac{b_0 + a_0}{2}.$$

- If $f(x_0) = 0$, then $r = x_0$, and we are done.
- If $f(x_0) \neq 0$, then $f(x_0)$ has the same sign as either $f(a_0)$ or $f(b_0)$.
  - If $f(x_0)$ and $f(a_0)$ have the same sign, then $r \in (x_0, b_0)$. Set $a_1 = x_0$, $b_1 = b_0$.
  - If $f(x_0)$ and $f(b_0)$ have the same sign, then $r \in (a_0, x_0)$. Set $a_1 = a_0$, $b_1 = x_0$.

Then reapply the process to the interval $[a_1, b_1]$.

As a result we will have a sequence of nested intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \cdots \supset [a_n, b_n] \supset \ldots .$$

Moreover

$$f(a_n)f(b_n) < 0, \quad n = 0, 1, \ldots$$

and

$$b_n - a_n = \frac{b - a}{2^n}.$$

As $\{a_n\}$, $\{b_n\}$ are monotonic and bounded sequences and $\lim_{n\to\infty} b_n - a_n = 0$, then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = r$$

and

$$f(r) = 0.$$

x_n approximates the root r with an error at most

$$|r - x_n| \leq \frac{b-a}{2^{n+1}}.$$

Stopping conditions

- $|a_n - b_n| < \varepsilon$
- $|x_n - x_{n-1}| < \varepsilon$
- $\frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon, \quad |x_n| \neq 0$
- $|f(x_n)| < \varepsilon$