Numerical Analysis

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Jacobi Method

Let

$$Ax = b$$

is a system of linear equations with an $n \times n$ square matrix A such that

$$a_{ii} \neq 0, \quad i = 1, 2, \ldots, n.$$

We are going to represent A in the following form

$$A = L + D + U,$$

where

$$L = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ a_{21} & 0 & \dots & 0 & 0 \\ a_{31} & a_{32} & \dots & 0 & 0 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn-1} & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n-1} & a_{1n} \\ 0 & 0 & \dots & a_{2n-1} & a_{2n} \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & a_{n-1n} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

As a result we can represent the system $Ax = b$ in the equivalent form

$$Dx = -(L+U)x + b.$$

from this we get the iterative method

$$Dx^{(k+1)} = -(L+U)x^{(k)} + b, , \quad k = 0, 1, \dots,$$

which is called Jacobi iterative method. As D is invertible we can write

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b, \quad k = 0, 1, \dots .$$

It is easy to see that

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(-\sum_{j=1, j\neq i}^{n} a_{ij}x_j^{(k)} + b_i\right), \quad i = 1, 2, \dots, n.$$

Definition

A square $n \times n$ matrix A is called diagonally dominant if $|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|$ for all i. A is called strictly diagonally dominant if $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$ for all i.

Theorem

If a square $n \times n$ matrix A is strictly diagonally dominant then the Jacobi method is convergent for any starting point $x^{(0)}$ and it converges to the unique solution of $Ax = b$.

Stopping conditions

- $||x^{(k)} - x^{(k-1)}|| < \varepsilon$
- $\frac{||x^{(k)} - x^{(k-1)}||}{||x^{(k)}||} < \varepsilon$
- $||Ax^{(k)} - b|| < \varepsilon$

Example

Our aim is to solve the following system of linear equations

$$\begin{cases} 4x_1 - x_2 = 3 \\ -x_1 + 4x_2 - x_3 = 2 \\ -2x_2 + 4x_3 = 2 \end{cases}$$

using Jacobi Methods.

a. show that Jacobi Method is convergent for any starting point $x^{(0)} = [x_1^{(0)}, x_2^{(0)}, x_3^{(0)}]$.

b. Carry out two iterations of the Jacobi Method starting with $x^{(0)} = [3, 2, 1]$.

Gauss-Seidel Method

Let

$$Ax = b$$

is a system of linear equations with an $n \times n$ square matrix. As in Jacobi's method we assume that

$$a_{ii} \neq 0, \quad i = 1, 2, \ldots, n.$$

By the same way we represent A in the following form

$$A = L + D + U,$$

where

$$L = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ a_{21} & 0 & \dots & 0 & 0 \\ a_{31} & a_{32} & \dots & 0 & 0 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn-1} & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n-1} & a_{1n} \\ 0 & 0 & \dots & a_{2n-1} & a_{2n} \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & a_{n-1n} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

As a result we can represent the system $Ax = b$ in the equivalent form

$$(L + D)x = -Ux + b.$$

from this we get the iterative method

$$(L + D)x^{(k+1)} = -Ux^{(k)} + b, \quad k = 0, 1, \dots,$$

which is called Gauss-Seidel iterative method. It is easy to see that

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} + b_i \right), \quad i = 1, 2, \dots, n.$$

Theorem

If a square $n \times n$ matrix A is strictly diagonally dominant then the Gauss-Seidel Method is convergent for any starting point $x^{(0)}$ and it converges to the unique solution of $Ax = b$.

Stopping conditions

- $||x^{(k)} - x^{(k-1)}|| < \varepsilon$
- $\frac{||x^{(k)} - x^{(k-1)}||}{||x^{(k)}||} < \varepsilon$
- $||Ax^{(k)} - b|| < \varepsilon$

Example

Our aim is to solve the following system of linear equations

$$\begin{cases} 4x_1 - x_2 + x_3 = 4 \\ -x_1 + 3x_2 + x_3 = 3 \\ -x_2 + 2x_3 = 1 \end{cases}$$

using Gauss-Seidel Method.

a. show that Gauss-Seidel Method is convergent for any starting point $x^{(0)} = [x_1^{(0)}, x_2^{(0)}, x_3^{(0)}]$.

b. Carry out the first iteration of the Gauss-Seidel Method starting with $x^{(0)} = [2, 3, 2]$.