Numerical Analysis

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Newton's method for systems of nonlinear equations

Assume we have a system of N nonlinear equations

$$\begin{cases} f_1(x_1, x_2, \dots, x_N) = 0 \\ f_2(x_1, x_2, \dots, x_N) = 0 \\ \vdots \\ f_N(x_1, x_2, \dots, x_N) = 0 \end{cases}$$

where $f_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, $i = \overline{1, N}$.

Using vector notation, we can write this system in a more elegant form:

$$F(X) = 0$$

by defining column vectors as

$$X = [x_1, x_2, \ldots, x_N]^T$$

$$F = [f_1, f_2, \ldots, f_N]^T$$

Let's recollect that in case of $N = 1$ Newton's method with a starting point x_0 is given by the following formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots .$$

The extension of Newton's method for nonlinear systems is

$$X^{(n+1)} = X^{(n)} - [F'(X^{(n)})]^{-1} F(X^{(n)}), \quad n = 0, 1 \dots .$$

with an initial approximation vector $X^{(0)} = [x_1^{(0)}, x_2^{(0)}, \dots, x_N^{(0)}]^T$, taken to be close to the solution of the nonlinear system.

$F'\left(X^{(n)}\right)$ is the Jacobian matrix defined by

$$F'\left(X^{(n)}\right)=\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}.$$

Here all partial derivatives are evaluated at $X^{(n)}$.

In practice, the computational form of Newton's method does not involve inverting the Jacobian matrix but rather solves the Jacobian linear systems

$$F'(X^{(n)})Z^{(n)} = -F(X^{(n)}).$$

The next iteration of Newton's method is then

$$X^{(n+1)} = X^{(n)} + Z^{(n)}.$$

Example

For the following system

$$\begin{cases} 4x_1^2 - x_2^2 = 0 \\ 4x_1 x_2^2 - x_1 = 1 \end{cases}$$

carry out one step of Newton's method, as initial approximation take $X^{(0)} = [0, 1]^T$.

Fixed Point Iteration Method for systems of nonlinear equations

$$F(X) = 0$$

where

$$F : \mathbb{R}^N \to \mathbb{R}^N$$

and

$$F = [f_1, f_2, \ldots, f_N]^T.$$

Let's recollect that in case of $N = 1$ in fixed point iteration method $f(x) = 0$ is replaced with an equivalent equation $x = g(x)$ and if x_0 is the staring point then

$$x_{n+1} = g(x_n), \quad n = 0, 1, \dots .$$

Now this idea is generalized and the system $F(X) = 0$ is replaced with with an equivalent system

$$X = G(X),$$

where $G : \mathbb{R}^N \to \mathbb{R}^N$.

Definition

$X^* \in \mathbb{R}^N$ is called a fixed point for a given function G if $G(X^*) = X^*$.

If $X^{(0)} \in \mathbb{R}^N$ is an initial approximation of X^* then Fixed Point Method is

$$X^{(n+1)} = G\left(X^{(n)}\right) \qquad n = 0, 1, \dots .$$

which is the same as

$$\begin{cases} x_1^{(n+1)} = g_1\left(x_1^{(n)}, x_2^{(n)}, \dots, x_N^{(n)}\right) \\ x_2^{(n+1)} = g_2\left(x_1^{(n)}, x_2^{(n)}, \dots, x_N^{(n)}\right) \\ \vdots \\ x_N^{(n+1)} = g_N\left(x_1^{(n)}, x_2^{(n)}, \dots, x_N^{(n)}\right) \end{cases},$$

Example

For the following system

$$\begin{cases} 4x_1 x_2^3 - x_1 = 1 \\ 4x_1 - x_2 = 0 \end{cases}$$

carry out two steps of Fixed Point Iteration method, as initial approximation take $X^{(0)} = [1, 1]^T$.