Numerical Analysis

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Newton's Method

By $\nabla^2 f(x)$ we denote the Hessian matrix of f at x

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}.$$

If the second order partial derivatives of f are all continuous then $\nabla f^2(x)$ is a symmetric matrix.

Example

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x_1, x_2, x_3) = x_1^3 + 4x_2^2 x_1 + \sin(x_3)$.
Compute the Hessian matrix $\nabla^2 f(x_1, x_2, x_3)$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. Our aim is to find the minimizer of f.

Let $x^{(0)} \in \mathbb{R}^n$ be the starting point.

$$x^{(k+1)} = x^{(k)} - \left[\nabla^2 f\left(x^{(k)}\right)\right]^{-1} \nabla f\left(x^{(k)}\right), \quad k = 0, 1, \ldots.$$

Assume $x^{(0)} \in \mathbb{R}^n$ is the initial approximation and

$$x^{(k+1)} = x^{(k)} + d^{(k)}, \quad k = 0, 1, \ldots.$$

where $d^{(k)}$ is the solution of the following system

$$\left[\nabla^2 f\left(x^{(k)}\right)\right] d^{(k)} = -\nabla f\left(x^{(k)}\right).$$

Stopping conditions

- $||\nabla f\left(x^{(k)}\right)|| < \varepsilon$
- $||x^{(k+1)} - x^{(k)}|| < \varepsilon$ or $\frac{||x^{(k+1)} - x^{(k)}||}{||x^{(k)}||} < \varepsilon$ if $||x^{(k)}|| \neq 0$
- $\left|f\left(x^{(k+1)}\right) - f\left(x^{(k)}\right)\right| < \varepsilon$ or $\frac{\left|f\left(x^{(k+1)}\right) - f\left(x^{(k)}\right)\right|}{\left|f\left(x^{(k)}\right)\right|} < \varepsilon$ if $f\left(x^{(k)}\right) \neq 0$.

Example

Assume we want to use the Newton's Method to minimize

$$f(x_1, x_2) = 2x_1^2 + x_2^2 - 2x_1x_2.$$

We start with $x^{(0)} = (1, 1)^T$. Calculate $x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right)^T$ by using the Newton's Method. Show that $x^{(1)}$ is the global minimizer of $f(x)$.