Numerical Analysis

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About CS 112 Numerical Analysis

- **Number of Credits:** 3

Section A

- **Class Schedule:** Mondays, Wednesdays and Fridays 9:30 - 10:20 in Room 208E
- **Office Hours:** Thursdays 12:00-14:00 in Room 335W or by an appointment (lpoghosyan@aua.am)
- **TA:** TBD
- **Problem Solving Sessions:** TBD
- **Moodle Enrollment Key:** NumAn18-A

Section B

- **Class Schedule:** Mondays, Wednesdays and Fridays 10:30 - 11:20 in Room 208E
- **Office Hours:** Thursdays 14:00-16:00 in Room 335W or by an appointment (lpoghosyan@aua.am)
- **TA:** TBD
- **Problem Solving Sessions:** TBD
- **Moodle Enrollment Key:** NumAn18-B

- **Main Textbooks:**
  1. Ward Cheney, David Kincaid, *Numerical Mathematics and Computing*, 6th Edition, Cengage Learning, 2007
  2. Alfio Quarteroni, Fausto Saleri, Paola Gervasio, *Scientific Computing with MATLAB and Octave, 4th Edition, Texts in Computational Science and Engineering*, Springer Science & Business Media, 2014

- **Additional Textbooks:**
  1. Burden, R.L.; Faires, J.D.; Reynolds, A.C. *Numerical analysis*. Prindle, Weber & Schmidt, Boston, Mass., 1978. ix+579 pp. ISBN: 0-87150-243-7
  2. Josef Stoer, R. Bulirsch, *Introduction to Numerical Analysis*, 3rd Edition, Springer Science & Business Media, 2013
  3. Cleve Moler, Numerical Computing with MATLAB : Revised Reprint, SIAM, 2010

- **Software:**
  MATLAB, http://www.mathworks.com/products/matlab/ (shareware)
  GNU Octave, https://www.gnu.org/software/octave/ (freeware)
- **Homeworks:** Every week, all assignments will be posted on Moodle. No late homeworks will be accepted.
- **Quizzes:** The course will include 4 pop-up quizzes.
- **Method of Evaluation**

Midterm1 20%
Midterm2 20%
Final Exam 40%
Homeworks 10%
Quizzes 10%
Total=0.1* (HW+Q)+0.2* (M1+M2) +0.4*F

Topics

- Error Analysis
- Locating Roots of Nonlinear equations and systems
- Interpolation and Approximation of functions and data
- Numerical Integration
- Numerical Optimization
- Numerical Solution of Systems of Linear Equations

Introduction

1. Formulation of the problem
2. Mathematical model of the problem
3. Development of the numerical method
4. Construction of the algorithm
5. Write a program based on the algorithm
6. Implementation of the program

V is the value we need to find

v is the value corresponding to mathematical model

$\bar{v}$ is the value corresponding to numerical method

v^* is the result we get after calculations by computer

$\epsilon_1 = V - v$

$\epsilon_2 = v - \bar{v}$ method error

$\epsilon_3 = \bar{v} - v^*$ calculations error

Round-off errors

We assume that machine numbers are represented in the normalized decimal floating-point form

$$\pm 0.d_1 d_2 ... d_k \times 10^n, \quad 1 \leq d_1 \leq 9, \quad \textit{and} \quad 0 \leq d_i \leq 9, \quad i = 2, ..., k.$$

Numbers of this form are called k- digit decimal machine numbers

Normalized form of number

$$y = \pm 0.d_1 d_2 ... d_k d_{k+1} d_{k+2} ... \times 10^n, \quad 1 \le d_1 \le 9, \text{ and } 0 \le d_i \le 9, \quad i \ge 2.$$

Example

Write $\frac{2}{30}$, $\frac{100}{3}$ and $\frac{1}{2}$ in normalized decimal form.

How do we obtain the floating-point form of y?

- **Chopping**

$$fl(y) = \pm 0.d_1 d_2 ... d_k \times 10^n, \quad 1 \le d_1 \le 9, \text{ and } 0 \le d_i \le 9, \quad 2 \le i \le k$$

- **Rounding**

$$fl(y) = \pm 0.\delta_1 \delta_2 ... \delta_k \times 10^n, \quad 1 \le \delta_1 \le 9, \text{ and } 0 \le \delta_i \le 9, \quad 2 \le i \le k.$$

When $d_{k+1} \ge 5$ we add 1 to d_k to obtain $fl(y)$

$$\delta_k = 1 + d_k.$$

When $d_{k+1} < 5$ we simply chop off all but the first k digits

$$\delta_k = d_k.$$

Example

Determine the five-digit (**a**)chopping and (**b**)rounding values of the irrational number $\pi = 3.14159265...$.

The error that results from replacing a number with its floating-point form is called round-off error regardless of whether the rounding or chopping method is used.

Definition

Suppose p^* is an approximation to p. The absolute error is $|p - p^*|$ and relative error is $\frac{|p-p^*|}{|p|}$ provided that $p \neq 0$.

We will write

$$abserr(p, p^*) = |p - p^*|,$$

$$relerr(p, p^*) = \frac{|p - p^*|}{|p|}.$$

Example

Determine the absolute and relative errors when approximating p by p^* when

- **a.** $p = 0.3 \times 10^1$ and $p^* = 0.31 \times 10^1$;
- **b.** $p = 0.3 \times 10^{-3}$ and $p^* = 0.31 \times 10^{-3}$.

Example

Suppose k decimal digits and chopping (rounding) are used for the machine representation of

$$y = 0.d_1 d_2 ... d_k d_{k+1} d_{k+2} ... \times 10^n.$$

Estimate $abserr(y, fl(y))$ and $relerr(y, fl(y))$.

Finite-Digit Arithmetic

Symbols $\oplus$, $\ominus$, $\otimes$ and $\oslash$ represent machine addition, subtraction, multiplication and division respectively.

$$x \oplus y = fl(fl(x) + fl(y))$$

$$x \ominus y = fl(fl(x) - fl(y))$$

$$x \otimes y = fl(fl(x) \times fl(y))$$

$$x \oslash y = fl(fl(x)/fl(y))$$

Example

Suppose $u = 0.714251$ and $v = 98765.9$. Use five-digit chopping to calculate $u + v$.

Operation	Result	Actual Value	Abs. Error	Rel. Error
$u \oplus v$	0.98765×10^5	0.98766×10^5	0.161×10^1	0.63×10^{-4}.

Example

Suppose $x = \frac{5}{7}$ and $u = 0.714251$. Use five-digit chopping to calculate $x - u$.

$$x = \frac{5}{7} = 0.\overline{714285}$$

Operation	Result	Actual Value	Abs. Error	Rel. Error
$x \ominus u$	0.30000×10^{-4}	0.34714×10^{-4}	0.471×10^{-5}	0.136

Suppose two nearly equal numbers x and y, with $x > y$ have k digit representations

$$fl(x) = 0.d_1 d_2 \ldots d_p \alpha_{p+1} \alpha_{p+2} \ldots \alpha_k \times 10^n,$$

$$fl(y) = 0.d_1 d_2 \ldots d_p \beta_{p+1} \beta_{p+2} \ldots \beta_k \times 10^n,$$

$$fl(fl(x) - fl(y)) = 0.\sigma_{p+1} \sigma_{p+2} \ldots \sigma_k \times 10^{n-p},$$

where

$$0.\sigma_{p+1} \sigma_{p+2} \ldots \sigma_k = 0.\alpha_{p+1} \alpha_{p+2} \ldots \alpha_k - 0.\beta_{p+1} \beta_{p+2} \ldots \beta_k.$$

Example

Suppose $x = \frac{5}{7}$, $u = 0.714251$ and $v = 98765.9$. Use five-digit chopping to calculate $(x - u) \times v$.

$$x = \frac{5}{7} = 0.\overline{714285}$$

Operation	Result	Actual Value	Abs. Error	Rel. Error
$x \ominus u$	0.30000×10^{-4}	0.34714×10^{-4}	0.471×10^{-5}	0.136
$(x \ominus u) \otimes v$	0.29629×10^{1}	0.34285×10^{1}	0.465	0.136