Numerical Analysis

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Interpolatory quadratures

$$a \leq x_0 < x_1 < \cdots < x_n \leq b$$

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} A_k f(x_k),$$

where

$$A_k = \int_a^b \ell_n^{(k)}(x)dx,$$

$$\ell_n^{(k)}(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}.$$

Theorem

The quadrature rule

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} A_k f(x_k),$$

is interpolatory quadrature rule if and only if it is exact for the class of polynomials of order n.

Example

Determine the interpolatory quadrature rule when the interval is $[-2, 2]$ and the nodes are -1, 0 and 1. What is the precision degree of the quadrature rule?

Gaussian quadrature

For given n construct a quadrature rule

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} A_k f(x_k)$$

that will have the highest precision degree.

Example

Obtain the Gaussian Quadrature Formula for $n = 0$ for the interval $[-1, 1]$, i.e., find the Quadrature Rule of the form

$$\int_{-1}^{1} f(x)dx \approx A_0 f(x_0)$$

which has the maximum degree of precision.

Let's take $f(x) = (x - x_0)^2(x - x_1)^2 \dots (x - x_n)^2$.

$$\int_a^b f(x)dx > 0,$$

$$\sum_{k=0}^{n} A_k f(x_k) = 0,$$

hence the quadrature rule is not exact for the class of polynomials of order $2n + 2$.

It is enough to consider the interval $[-1,1]$, as after $x = \frac{b-a}{2}t + \frac{a+b}{2}$ substitution we will have

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{a+b}{2}\right) dt.$$

If t_0, t_1, ..., t_n are Gaussian quadrature nodes and A'_0, A'_1, ..., A'_n are Gaussian quadrature weights on the interval $[-1,1]$, then

$$x_k = \frac{b-a}{2}t_k + \frac{a+b}{2}, \quad k = 0, 1, \ldots, n$$

and

$$A_k = \frac{b-a}{2}A'_k, \quad k = 0, 1, \ldots, n.$$

Let's consider Legendre Polynomials

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n, \quad , n = 0, 1, \ldots .$$

The Gaussian quadrature nodes on interval $[-1, 1]$ are the zeros of $L_{n+1}(x)$.

Example

Obtain the Gaussian Quadrature Formula for $n = 1$ for the interval $[-2, 2]$, i.e., find the Quadrature Rule of the form

$$\int_{-2}^{2} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1)$$

which has the maximum degree of precision.