Numerical Analysis

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Interpolation and Approximation of functions and data

- Suppose we have values of a function f at some points x_0, x_1, ..., x_n. We need to calculate at least approximately the values of f at other points.
- A function f is given, perhaps in the form of a computer procedure, but it is an expensive function to evaluate and we need to calculate its values at many points.

In both of this cases a simple function can be obtained which approximates f.

- Polynomial Interpolation
- Spline Interpolation
- The Least Squares Method

Interpolation

Assume

$$x_0,\ x_1,\ \dots,x_n$$

are distinct points and we know the values of a function f at those points

$$f(x_0),\ f(x_1),\ \dots,f(x_n).$$

We need to find a function F such that

$$F(x_i) = f(x_i), \quad i = 0,\ 1,\ \dots,\ n.$$

Formulated problem is called **interpolation problem.**
$F(x)$ is called **interpolating function.**
x_0, x_1, ..., x_n are called **interpolation nodes.**

Polynomial Interpolation

Often interpolating function F is sought among polynomials.

Example

Find a polynomial of degree 0 that passes through the points $(1, 1)$ and $(2, 2)$.

Example

Find a polynomial of degree 0 that passes through the points $(1, 1)$ and $(2, 1)$.

Example

Find a polynomial of degree 1 that passes through the point $(1, 5)$.

Assume

$$x_0,\ x_1,\ \ldots, x_n$$

are distinct points and we know the values of a function f at those points

$$f(x_0),\ f(x_1),\ \ldots, f(x_n).$$

We need to find a polynomial $P(x)$ of degree at most n such that

$$P(x_i) = f(x_i), \quad i = 0,\ 1,\ \ldots,\ n.$$

Interpolating Polynomial: Lagrange Form

Let's consider following polynomials of degree n

$$\ell_n^{(k)}(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_{k-1})(x - x_{k+1})\dots(x - x_n)}{(x_k - x_0)(x_k - x_1)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)}.$$

This polynomials are known as cardinal polynomials.

$$\ell_n^{(k)}(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases}$$

The interpolating polynomial will be

$$L_n(x) = f(x_0)\ell_n^{(0)}(x) + f(x_1)\ell_n^{(1)}(x) + \cdots + f(x_n)\ell_n^{(n)}(x).$$

This form is called Lagrange form of the n-th order interpolating polynomial.

Theorem

If points x_0, x_1, ..., x_n are distinct, then for arbitrary real values y_0, y_1, ..., y_n, there is a unique polynomial $P(x)$ of degree at most n such that $P(x_i) = y_i$, $i = 0, 1, \ldots, n$.

Example

a. Find the interpolating polynomial of Lagrange form for the function $f(x) = \sin(\pi x)$ withe the nodes $x_0 = 0$, $x_1 = \frac{1}{6}$ and $x_2 = \frac{1}{2}$.

b. Use the polynomial to approximate $f\left(\frac{1}{4}\right) = \frac{\sqrt{2}}{2}$.

c. (MATLAB) Plot the interpolating polynomial and $f(x)$ on the same figure.