Numerical Analysis

Lusine Poghosyan

AUA

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Example

Assume we are calculating the value of

$$f(x) = \sqrt{x} - \frac{1}{x^2 + 1}$$

at the point $a = 0.43156722$, i.e., we are calculating $f(a)$. Instead of this, we approximate a by a^* s.t. $abserr(a, a^*) \leq 0.01$ and calculate $f(a^*)$, that is, we use the approximation

$$f(a) \approx f(a^*).$$

Estimate the absolute and relative errors of this approximation. i.e., $abserr\,(f(a), f(a^*))$ and $relerr\,(f(a), f(a^*))$.

Example

Our aim is to find the zeros of the following function:

$$f(x) = x^3 - 12x - 20.$$

a. Show that $f(x)$ has exactly one root x^* and isolate it i.e. find an interval $[a, b]$ such that $x^* \in [a, b]$.

b. Assume the Bisection Method is applied on interval $[a, b]$. Find the second approximation of the root.

c. How many iterations are needed to calculate the root with an error at most 10^{-3} by using the Bisection Method?

d. Replace $f(x) = 0$ with an equivalent equation of the form $x = g(x)$ in such a way that $g : [2, 5] \to [2, 5]$ and $g(x)$ is a contraction mapping on $[2, 5]$.

e. Take $x_0 = 1$ as an initial approximation, and calculate x_2 using the Fixed Point Iteration Method.

Example

Let $g(x) = \frac{1}{2}\frac{x+6}{x+2}$. Prove that

a. g maps the interval $\left[\frac{1}{2}, 2\right]$ into itself, i.e., $g : \left[\frac{1}{2}, 2\right] \to \left[\frac{1}{2}, 2\right]$;

b. $g(x)$ is a contraction mapping on $\left[\frac{1}{2}, 2\right]$;

c. Prove that $g(x)$ has a unique fixed point x^* in $\left[\frac{1}{2}, 2\right]$;

e. Assume $x_0 = 1$, and let x_n be the sequence constructed by the Fixed Point Iteration Method applied to $g(x)$, starting from x_0. Show that x_n is a convergent sequence and estimate the absolute error

$$|x_{10} - x^*|.$$

Example

a. Construct the Interpolating Polynomial $P_2(x)$ for the function $f(x) = x(x-3) + \sin(\pi x)$ at the nodes $x_0 = 0$, $x_1 = 2$, $x_2 = 4$.

b. Estimate the local error of approximation at the points $x = 2$ and $x = 3$.

c. Estimate the global error

$$\max_{x \in [-1,5]} |f(x) - P_2(x)| \, .$$

Example

a. Calculate the 4-th order Chebyshev nodes.

b. Calculate $\max_{x \in [-1,1]} |T_4(x)|$, where $T_4(x)$ is the 4-th order Chebyshev Polynomial.

Example

We want to interpolate the function $f(x) = \frac{x}{1+x}$ on $[2, 5]$ using first degree splines.

a. Assume we divide our interval into 3 equal-length parts. Construct the interpolating spline of degree 1 for f at the obtained nodes.

b. In how many equal-length subintervals we need to divide our interval in order to have that the global error of approximation of f in $[2, 5]$ by the first degree interpolating spline at the obtained nodes is less than 0.001?

Example

Assume we have the following data

$$\begin{array}{lll} x_0 = -2 & x_1 = 0 & x_2 = 2 \\ y_0 = 12 & y_1 = 2 & y_2 = 24 \end{array}.$$

Our aim is to construct the polynomial of degree at most 2 that fits the data in the least square sense, i.e., find coefficients a_0^*, a_1^* and a_2^* that minimize the error function

$$\phi(a_0, a_1, a_2) = \sum_{i=0}^{2} \left(a_0 + a_1 x_i + a_2 x_i^2 - y_i \right)^2 .$$

Show that found (a_0^*, a_1^*, a_2^*) is the global minimizer of $\phi(a_0, a_1, a_2)$.

Example

Is the quadrature rule

$$\int_0^4 f(x)dx \approx 2 \cdot f(1) + 2 \cdot f(3)$$

Gaussian? Prove your statement.

Example

Assume the Quadrature Rule

$$\int_{-3}^{2} f(x)dx \approx A \cdot f(x_0) + B \cdot f(x_1)$$

$(x_0, x_1 \in [-3, 2])$ is Gaussian, i.e., it has the precision degree 3.

a. Show that $\int_{-3}^{2} (x - x_0)(x - x_1)dx = \int_{-3}^{2} (x - x_0)^2(x - x_1)dx = 0$;

b. Calculate $A + B$, $A \cdot x_0 + B \cdot x_1$ and $A \cdot x_0^2 + B \cdot x_1^2$;

c. Show that $A > 0$ and $B > 0$.

Example

Let $f(x) = e^{(x-2)^2} + 4x$. Our aim is to find the global minimizer x^* of f over $[0, 8]$.

- **a.** Show that $f(x)$ is a unimodal function in $[0, 8]$.
- **b.** Calculate x_2 approximation of the minimum point using the Golden Section (Ratio) Search Method with $\gamma = \frac{3-\sqrt{5}}{2}$.
- **c.** Calculate x_2 approximation of the minimum point using the Bisection Method.

Example

Assume we want to use the Steepest Descent Method to minimize

$$f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + x_3^2 - 2x_2x_3.$$

We start with $x^{(0)} = (1, 1, 0)^T$. Calculate $x^{(1)}$ by using the Steepest Descent Method.

Example

Assume we are solving the minimization problem

$$\begin{aligned} &\textit{minimize} \quad f(x_1, x_2) = \ln\left(x_1^2 + 1\right) + x_2^2 \\ &\textit{subject to} \quad x \in \mathbb{R}^2, \end{aligned}$$

by using line search methods. We start from the initial approximation $x^{(0)} = (1/2, 1)^T$. Calculate $x^{(1)}$ by using the Newton's Method.

Example

Assume we want to solve the following constrained minimization problem

$$\begin{aligned} \text{minimize} \quad & f(x_1, x_2) = 4x_1^2 + x_2^2 \\ \text{subject to} \quad & x_2 = x_1^2 + 1. \end{aligned}$$

a. Find all possible minimizers of this problem by using the Lagrange Multipliers Method;

b. Now let's use the Penalty Method and consider the following function for large $\gamma > 0$,

$$g(x_1, x_2) = f(x_1, x_2) + \gamma \cdot (x_2 - x_1^2 - 1)^2.$$

Assuming γ is fixed, find the minimum point $x^{(\gamma)}$ of g;

c. Prove that $x^{(\gamma)}$ tends to the minimum point, as $\gamma \to +\infty$.

Numerical Solution of Systems of Linear Equations:

Example

Solve the following system of linear equations

$$\begin{cases} 3x_1 + 3x_2 + 3x_3 = 6 \\ 2x_1 + 4x_2 + 4x_3 = 8 \\ x_1 + 2x_2 + 3x_3 = 5 \end{cases}$$

using LU Factorization Methods.

Example

Assume

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1.999 \\ -0.9 \end{bmatrix}$$

a. Calculate the norm $\|A\|_1$;

b. Calculate the condition number $cn_1(A)$;

c. Now assume we are solving the system $A\mathbf{x} = b$. During the calculations we replace b by $\tilde{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and we solve $A\tilde{\mathbf{x}} = \tilde{b}$ instead. Estimate the relative error of the approximate solution

$$\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|_1}{\|\mathbf{x}\|_1}.$$