Optimization

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Definition

Assume $A = \left[a_{ij}\right]_{i,j=1}^{n}$ is a $n \times n$ symmetric matrix, i.e. $a_{ij} = a_{ji}$. A function $QF_A : \mathbb{R}^n \to \mathbb{R}$ is called a **quadratic form** associated to the matrix A if

$$QF_A(y) = y^T A y = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} y_i y_j.$$

Example

Construct the quadratic form associated to the matrix A if

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 3 & 5 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Definition

We will say that the symmetric $n \times n$ matrix A or the quadratic form QF_A is

- **positive definite** if $QF_A(y) > 0$, $\forall y \in \mathbb{R}^n$ and $y \neq 0$;
- **positive semidefinite** if $QF_A(y) \geq 0$, $\forall y \in \mathbb{R}^n$;
- **negative definite** if $QF_A(y) < 0$, $\forall y \in \mathbb{R}^n$ and $y \neq 0$;
- **negative semidefinite** if $QF_A(y) \leq 0$, $\forall y \in \mathbb{R}^n$;
- **indefinite** if there exist $y_1, y_2 \in \mathbb{R}^n$ such that $QF_A(y_1) > 0$ and $QF_A(y_2) < 0$.

Example

Determine whether the matrix A is positive definite (semidefinite), negative definite (semidefinite) or indefinite if

a.

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

b.

$$A = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix};$$

c.

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 5 \end{pmatrix}.$$

Definition

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ matrix. The leading principal minors are $\det A$ and the minors obtained by successively removing the last row and the last column. That is, the leading principal minors are

$$\Delta_1 = a_{11},\ \Delta_2 = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ \Delta_3 = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},\ \ldots,$$

$\Delta_n = \det A.$

Theorem

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ symmetric matrix. The following three statements are equivalent

- *A is positive definite;*
- *All eigenvalues of A are positive;*
- *All leading principal minors of A are positive (Sylvester's criterion).*

Definition

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ matrix. The principal minors are $\det A$ itself and the determinants of matrices obtained by successively removing an i-th row and and i-th column. That is, the principal minors are

$$\det \begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \dots & a_{i_1 i_p} \\ a_{i_2 i_1} & a_{i_2 i_2} & \dots & a_{i_2 i_p} \\ \vdots & & & \\ a_{i_p i_1} & a_{i_p i_2} & \dots & a_{i_p i_p} \end{pmatrix}, 1 \leq i_1 < i_2 < \dots < i_p \leq n, p = 1, \dots, n.$$

Theorem

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ *be* $n \times n$ *symmetric matrix. The following three statements are equivalent*

- *A is positive semidefinite;*
- *All eigenvalues of A are nonnegative;*
- *All principal minors of A are nonnegative.*

Example

Determine whether the matrix A is positive definite (semidefinite) if

a.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

b.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Theorem

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ symmetric matrix. The following three statements are equivalent

- *A is negative definite;*
- *All eigenvalues of A are negative;*
- *All leading principal minors of even order are positive and of odd order negative (Sylvester's criterion).*

Theorem

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ symmetric matrix. The following three statements are equivalent

- *A is negative semidefinite;*
- *All eigenvalues of A are nonpositive;*
- *All principal minors of even order are nonnegative and of odd order nonpositive.*

Example

Determine whether the matrix A is positive definite (semidefinite), negative definite (semidefinite) or indefinite if

a.

$$A = \begin{pmatrix} -1 & -2 & 0 \\ -2 & -5 & 1 \\ 0 & 1 & -4 \end{pmatrix};$$

b.

$$A = \begin{pmatrix} -1 & -3 & 0 \\ -3 & -9 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

c.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$