Numerical Analysis

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Bisection method

Stopping conditions

- $|a_n - b_n| < \varepsilon$
- $|x_n - x_{n-1}| < \varepsilon$
- $\frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon, \quad |x_n| \neq 0$
- $|f(x_n)| < \varepsilon$

Example

Our aim is to solve the following equation

$$f(x) = x^3 - 3x - 10 = 0.$$

a. Isolate the roots;

b. Use the Bisection Method to find x_2 approximation of the root on $[1, 5]$;

c. Determine the number of iterations necessary to solve $f(x) = x^3 + 4x - 10 = 0$ within accuracy 10^{-3} using $a = 1$ and $b = 3$;

d. (MATLAB) Use Bisection method to determine an approximation of each root with error at most 10^{-4}.

Newton's (or Newton-Raphson) method

Let r be an isolated root of

$$f(x) = 0.$$

Suppose x_0 is an initial approximation to the zero of f such that $|r - x_0|$ is small. Let's assume that the second derivative of f exists and continuous in some neighborhood of r which also contains x_0.

$$0 = f(r) = f(x_0) + f'(x_0)(r - x_0) + f''(\alpha)\frac{(r - x_0)^2}{2},$$

where $\alpha \in (x_0, r)$.

As $r - x_0$ is small, the term involving $(r - x_0)^2$ is much smaller, so

$$f(x_0) + f'(x_0)(r - x_0) \approx 0.$$

Solving for r gives us

$$r \approx x_0 - \frac{f(x_0)}{f'(x_0)} = x_1.$$

This process can be repeated to produce a sequence of points:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1 \ldots .$$

Graphical interpretation

Let's assume that $f(x)$ is a differentiable function.

$$l(x) = f(x_0) + f'(x_0)(x - x_0)$$

We take the zero of linear function $l(x)$ as an approximation to the zero of f

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This process can be repeated to produce a sequence of points:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1 \ldots .$$

Stopping conditions

- $|x_n - x_{n-1}| < \varepsilon$
- $\frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon, \quad |x_n| \neq 0$
- $|f(x_n)| < \varepsilon$

None of these inequalities gives precise information about actual error $r - x_n$.

Example

Assume we want to approximate the root of the following equation

$$\ln x - 2 = 0.$$

Use Newton's method to calculate the third approximation x_2 with a starting point $x_0 = 1$.

Convergence Analysis

Theorem

If f, f' and f'' are continuous in a neighborhood of a zero r of f and $f'(r) \neq 0$, then there is a positive δ with the following property: If the initial point in Newton's method satisfies $|r - x_0| \leq \delta$, then all subsequent points x_n satisfy the same inequality, converge to r, and do so quadratically; that is

$$|r - x_{n+1}| \leq C|r - x_n|^2,$$

where $C = const > 0$.

Theorem

Let $f \in C^{(2)}(\mathbb{R})$ and satisfies following conditions: f has a zero, $f'(x) > 0$ and $f''(x) > 0, \forall x \in \mathbb{R}$, then for any initial point x_0 in Newton's method the sequence $\{x_n\}$ converges to r.

If the change of $f'(x)$ in the neighborhood of zero is little, then $f'(x_n)$ can be replaced by $f'(x_0)$, and we will have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}, \quad n = 0, 1 \ldots .$$