Numerical Analysis

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Interpolating Polynomial: Newton Form

Suppose that we have succeeded in finding a polynomial $P(x)$ that $P(x_i) = f(x_i)$ for $0 \leq i \leq k$.
We shall attempt to add to $P(x)$ another term that will enable the new polynomial to take value $f(x_{k+1})$ at x_{k+1}.

$$P(x) + c(x - x_0)(x - x_1)\dots(x - x_k)$$

We will find c from the condition that

$$P(x_{k+1}) + c(x_{k+1} - x_0)(x_{k+1} - x_1)\dots(x_{k+1} - x_k) = f(x_{k+1}).$$

Example

Using the Newton algorithm, find the interpolating polynomial for this table:

x	0	1	-1	2	-2
y	-5	-3	-15	39	-9

.

The Interpolating Polynomial will be

$$
\begin{aligned}
P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots \\
+ a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}),
\end{aligned}
$$

where

$$
\begin{cases}
f(x_0) = a_0 \\
f(x_1) = a_0 + a_1(x_1 - x_0) \\
f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\
etc.
\end{cases}.
$$

In general, a_n depends on $f(x_0)$, $f(x_1), \ldots f(x_n)$. In other words, a_n depends on the values of f at the nodes x_0, x_1, $\ldots$, x_n.
The traditional notation is

$$a_n = f[x_0, x_1, \ldots, x_n].$$

The quantity $f[x_0, x_1, \ldots, x_n]$ is called the divided difference of order n for f.

Example

For the table

x	1	-4	0
$f(x)$	3	13	-23

.

determine the quantities $f[x_0]$, $f[x_0, x_1]$, and $f[x_0, x_1, x_2]$.

Newton form of the interpolating polynomial

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots$$
$$+ f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1)\dots(x - x_{n-1}).$$

Theorem

The divided differences obey the formula

$$f[x_0] = f(x_0)$$

and

$$f[x_0, x_1, \dots x_k] = \frac{f[x_1, x_1, \dots x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_0}, \quad k > 0.$$

Theorem

The divided differences obey the formula

$$f[x_0, x_1, \dots x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{(x_k - x_0)(x_k - x_1)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)}.$$

Theorem

The divided difference $f[x_0, x_1, \ldots, x_n]$ *is invariant under all permutations of the arguments* $x_0, x_1, \ldots, x_n$.

Example

Construct the interpolating polynomial of corresponding order for this table

x	0	1	-1
y	-5	-3	-15

by using Newton's form.

Example

Calculate the divided difference $f[1, 3, -5, 0]$ for the function $f(x) = (x - 1)(x - 3)(x + 5)$.