Numerical Analysis

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Example

Construct a quadrature rule

$$\int_{-2}^{2} f(x)dx \approx Af(-1) + Bf(c) + Af(1)$$

which will have the maximal precision degree.

Numerical Optimization

We are going to consider the following problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, with $n \geq 1$.

The function f that we wish to minimize is called the **objective function** or **cost function.**
The set Ω is called the **constraint set** or **feasible set.**
If Ω is a proper subset of $\mathbb{R}^n$ then we have **constrained optimization problem.**
If $\Omega = \mathbb{R}^n$ then we have **unconstrained optimization problem.**

Definition

A point $x^* \in \Omega$ is a **local minimizer** of f over Ω if there exists $\varepsilon > 0$ such that $f(x) \geq f(x^*)$ for all $x \in \Omega \backslash \{x^*\}$ and $||x - x^*|| < \varepsilon$. A point $x^* \in \Omega$ is a **global minimizer** of f over Ω if $f(x) \geq f(x^*)$ for all $x \in \Omega \backslash \{x^*\}$.

If in the definitions above we replace "$\geq$" with "$>$" then we have a **strict local minimizer** and a **strict global minimizer**, respectively.

If x^* is a global minimizer of f over Ω, we write $f(x^*) = \min_{x \in \Omega} f(x)$and $x^* = \text{arg min}_{x \in \Omega} f(x)$. If the minimization is unconstrained, we simply write $x^* = \text{arg min}_x f(x)$ or $x^* = \text{arg min} f(x)$

One Dimensional Search Methods

Here we consider the minimization of univariate function $f : [a, b] \to \mathbb{R}$.

In an iterative algorithm we start with an initial candidate solution x_0 and generate sequence of points $x_1, x_2, \ldots$. Each x_{k+1} iteration depends on previous points $x_0, x_1, \ldots, x_k$. The algorithm may also use the value of f or f' or even f'' at some points:

- Golden section method (uses only f);
- Fibonacci method (uses only f);
- Bisection method (uses only f');
- Gradient descent (uses only f');
- Newton's method (uses f' and f'').

Definition

The function $f : [a, b] \to \mathbb{R}$ is called a unimodal function on interval $[a, b]$, if f has only one local minimizer in $[a, b]$.

Example

Check if the function f is unimodal on the given interval, if

a. $f(x) = \sin(x)$, $x \in [\pi/2, 2\pi]$;

b. $f(x) = \sin(x)$, $x \in [0, 2\pi]$;

c. $f(x) = \frac{x^5}{5} - x^3$, $x \in [-1, 2]$.

If $f : [a, b] \to \mathbb{R}$ is continuous and unimodal on $[a, b]$, then f is strictly decreasing up to the minimum point x^* and strictly increasing thereafter.

Golden Section Search

Assume $f : [a, b] \to \mathbb{R}$ is unimodal and continuous on interval $[a, b]$.

Let's denote

$$[a_0, b_0] = [a, b].$$

$$A = a_0 + \gamma(b_0 - a_0),$$

$$B = b_0 - \gamma(b_0 - a_0),$$

where $\gamma \in \left(0, \frac{1}{2}\right)$.

We define the new interval in the following way

$$[a_1, b_1] = \begin{cases} [a_0, B], & \text{if } f(A) < f(B), \\ [A, b_0], & \text{if } f(A) \geq f(B). \end{cases}$$

$$x^* \in [a_1, b_1]$$

$$b_1 - a_1 = (1 - \gamma)(b_0 - a_0).$$

The first approximation will be

$$x_1 = \frac{a_1 + b_1}{2}.$$

n-th step

$$[a_n, b_n] = \begin{cases} [a_{n-1}, B], & \text{if } f(A) < f(B), \\ [A, b_{n-1}], & \text{if } f(A) \geq f(B). \end{cases}$$

$$x^* \in [a_n, b_n]$$

$$b_n - a_n = (1-\gamma)(b_{n-1} - a_{n-1})$$

The n-th approximation will be

$$x_n = \frac{a_n + b_n}{2}.$$

Theorem

If f is a unimodal and continuous function on $[a, b]$ and $\gamma \in \left(0, \frac{1}{2}\right)$, then the golden section search approximation x_n converges to x^* and we have

$$|x_n - x^*| \leq 0.5(1-\gamma)^n(b-a).$$

Stopping conditions

- $|b_n - a_n| < \varepsilon$
- $|x_n - x_{n-1}| < \varepsilon$ or $\frac{|x_n - x_{n-1}|}{|x_{n-1}|} < \varepsilon$, if $x_{n-1} \neq 0$
- $|f(x_n) - f(x_{n-1})| < \varepsilon$ or $\frac{|f(x_n) - f(x_{n-1})|}{|f(x_{n-1})|} < \varepsilon$, if $f\left(x_{n-1}\right) \neq 0$

Example

Calculate the second approximation x_2 of the golden section search method with $\gamma = \frac{1}{3}$ for function $f(x) = \frac{x^3}{3} - 2x$ on the interval $[-3, 0]$.

In general case at each step we calculate the value of f at two points: A and B.

It is possible to find $\gamma \in (0, \frac{1}{2})$ such that we evaluate f at two points at first step but at each next step we evaluate f at one point.

Let's assume that at the step n $f(A) < f(B)$, which means that our new interval $[a_n, b_n]$ is going to be $[a_{n-1}, B]$.

$$\begin{aligned} a_{n-1} + \gamma(b_{n-1} - a_{n-1}) &= b_n - \gamma(b_n - a_n) \\ &= b_{n-1} - \gamma(2-\gamma)(b_{n-1} - a_{n-1}) \end{aligned}$$

This implies

$$\gamma^2 - 3\gamma + 1 = 0$$

and

$$\gamma = \frac{3-\sqrt{5}}{2} \approx 0.382.$$

$$|x_n - x^*| \leq 0.5 \left(\frac{\sqrt{5}-1}{2} \right)^n (b-a).$$

Example (MatLab)

Our aim is to find the minimum of $f(x) = x^4 + x^2 - 2x$ on the interval $[-4, 5]$ with an error of at most 10^{-3}. Write a MatLab program that calculates the approximation of the minimum point using the golden section search method with $\gamma = \frac{3-\sqrt{5}}{2}$.