Optimization

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Example

Solve the problem

$$\begin{aligned} \textit{minimize} \quad & f(x) \\ \textit{subject to} \quad & x \in \Omega, \end{aligned}$$

i.e., find the global minimum points of $f(x)$ on Ω, if

d. $f(x) = \frac{x+1}{x^2+3}$, $\Omega = [0, +\infty)$.

Finite-Dimensional Optimization

We are going to consider the following problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, with $n \geq 1$.

Definition

A point $x^* \in \Omega$ is a **local minimizer** of f over Ω if there exists $\varepsilon > 0$ such that $f(x) \geq f(x^*)$ for all $x \in \Omega \backslash \{x^*\}$ and $||x - x^*|| < \varepsilon$. A point $x^* \in \Omega$ is a **global minimizer** of f over Ω if $f(x) \geq f(x^*)$ for all $x \in \Omega \backslash \{x^*\}$.

If in the definitions above we replace "$\geq$" with "$>$" then we have a **strict local minimizer** and a **strict global minimizer**, respectively.

If x^* is a global minimizer of f over Ω, we write $f(x^*) = \min_{x \in \Omega} f(x)$ and $x^* = \arg\min_{x \in \Omega} f(x)$. If the minimization is unconstrained, we simply write $x^* = \arg\min_x f(x)$ or $x^* = \arg\min f(x)$.

Existence of solution

Weierstrass Extreme Value Theorem

If $f \in \mathbb{C}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is compact, then the problem (1) has a solution.

Definition

A point $x \in \mathbb{R}^n$ is said to be a **limit point** of $\Omega \subset \mathbb{R}^n$, if each neighborhood of x contains a point of Ω other than x.

Example

Let $\Omega = [0,3) \cup \{4\}$. Is x a limit point of Ω?

a. $x = 0$

b. $x = 3$

c. $x = 2$

d. $x = 4$

Example

Let $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1 \quad \text{and} \quad x_1 > 0\}$. Is x a limit point of Ω, if

a. $x = [0,0]^T$;

b. $x = [1,0]^T$.

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **closed set** if it contains all its limit points.

Example

Check if the set Ω is a closed set, if

a. $\Omega = [0, 3)$;

b. $\Omega = [0, 3]$;

c. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$;

d. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1 \quad \text{and} \quad x_1 > 0\}$.

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **bounded** if there exists $M \in \mathbb{R}$ such that $||x|| \leq M$, for all $x \in \Omega$.

Example

Check if the set Ω is bounded, if

a. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$;

b. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \geq 1 \quad \text{and} \quad x_1 > 0\}$.

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **compact** if Ω is closed and bounded.

Example

Check if the set Ω is compact, if

a. $\Omega = [0, 3)$;

b. $\Omega = [0, 3]$;

c. $\Omega = \{x = [x_1, x_2, x_3]^T : x_1^2 + x_2^2 + x_3^2 \leq 1\}$;

d. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \geq 1 \quad \text{and} \quad x_1 > 0\}$.