Numerical Analysis

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How to choose the nodes $x_0, x_1, \dots, x_n$ belonging to the interval $[a, b]$ in a way that

$$\max_{a \le x \le b} |(x - x_0)(x - x_1)\dots(x - x_n)|$$

will be minimized?

To find those nodes we need to consider Chebyshev polynomials of the first kind.

Chebyshev Polynomials

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, \cdots :$$

Example

Find the algebraic representation of the Chebyshev Polynomial $T_3(x)$.

Prop. Chebyshev Polynomials obey the following formula

$$T_n(x) = \cos(n \arccos x), \quad \forall x \in [-1, 1], \quad n = 0, 1, \ldots.$$

Example

Express $\cos(3\alpha)$ in terms of $\cos\alpha$.

Example

Calculate the roots of the third order Chebyshev polynomial.

Prop. The Chebyshev polynomial $T_n(x)$ of degree $n \geq 1$ has n real roots belonging to the interval $(-1, 1)$ and those roots are given by the following formula

$$x_k = \cos \frac{\pi(2k+1)}{2n}, \quad n = 0, 1, \dots, n-1.$$

Interpolation on Chebyshev nodes

If nodes are placed at the roots of $n+1$ order Chebyshev polynomial $T_{n+1}(x)$, i.e.,

$$x_i = \cos \frac{\pi(2i+1)}{2(n+1)}, \quad n = 0, 1, \ldots n.$$

then we will have the smallest possible value for

$$\max_{-1 \le x \le 1} |(x - x_0)(x - x_1) \ldots (x - x_n)|.$$

The corresponding set of nodes on an arbitrary interval $[a, b]$:

$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{2i+1}{2n+2}\pi\right), \quad i = 0, 1, \ldots, n.$$

Example

Write a MatLab program that constructs the Interpolating Polynomial of degree $\leq n$ for the function $f(x) = \frac{1}{1+x^2}$ at Chebyshev nodes x_i, $i = 0, 1, ..., n$ in interval $[-5, 5]$, and then plots the function f, the Interpolating Polynomial P_n, the points $(x_i, f(x_i))$ on the same figure, when $n = 10$, $n = 20$, $n = 30$.