Optimization

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Barrier Methods

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$.
Barrier Methods are procedures when constrained optimization problems are approximated by "unconstrained optimization" problem. The approximation is accomplished by adding to the objective function a term that favors points interior to the feasible set over those near boundary.

Here we assume that the feasible set is robust, i.e., each feasible point can be approached by a sequence of interior points.

Definition

A function $B(x)$ defined on the interior of Ω is called a barrier function for the constrained minimization problem above, if it satisfies the following conditions

1. B is continuous on the interior of Ω ($\text{int}(\Omega)$),
2. $B(x) \geq 0$, $\forall x \in \text{int}(\Omega)$,
3. $B(x) \to \infty$ as x approaches the boundary of Ω.

Assume $\{c_k\}$ is a strictly increasing sequence of positive numbers such that $\lim_{k\to\infty} c_k = \infty$. Now we consider the following problem

$$\begin{aligned} \text{minimize} \quad & f(x) + \tfrac{1}{c_k} B(x) \\ \text{subject to} \quad & \text{int}(\Omega). \end{aligned}$$

In fact we have a constrained optimization problem but we can apply the methods for unconstrained optimization problems. Take an initial approximation from $\text{int}(\Omega)$, use some method to calculate next approximation and it will be in $\text{int}(\Omega)$ (if carefully implemented) since $B(x) \to \infty$ as x approaches the boundary.

As equality constraints don't give robust sets we don't consider problems with equality constraints.

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \dots p. \end{aligned}$$

We assume that the objective function and constraint functions are continuous.

If $\text{int}(\Omega) = \{x : g_i(x) < 0, \forall i \in \overline{1, p}\}$, then the barrier function can be given by the following formula

$$B(x) = -\sum_{i=1}^{p} \frac{1}{g_i(x)}, \quad x \in \text{int}(\Omega).$$

Example

Consider the following constrained minimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) = -\frac{1}{x^2+1} \\ \text{subject to} \quad & x \le 2, \\ & x \ge -1. \end{aligned}$$

$$r(x, c) = -\frac{1}{x^2+1} - \frac{1}{c} \cdot \frac{1}{x-2} - \frac{1}{c} \cdot \frac{1}{-x-1}, \quad x \in (-1, 2)$$

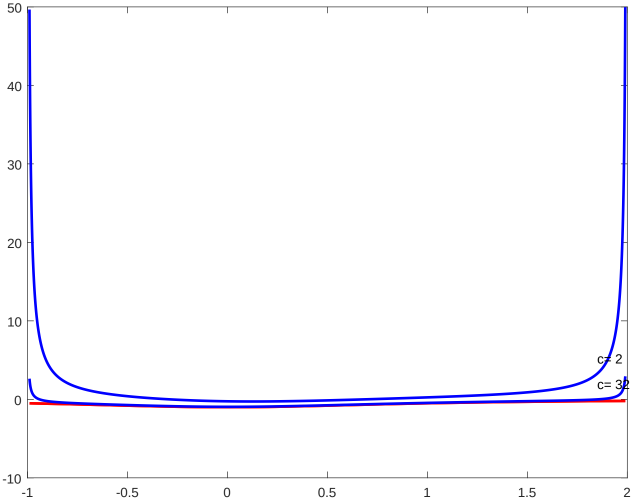

Figure: The red line is the graph of $f(x)$ and blue lines are the graphs of $r(x, c)$ for different parameters.

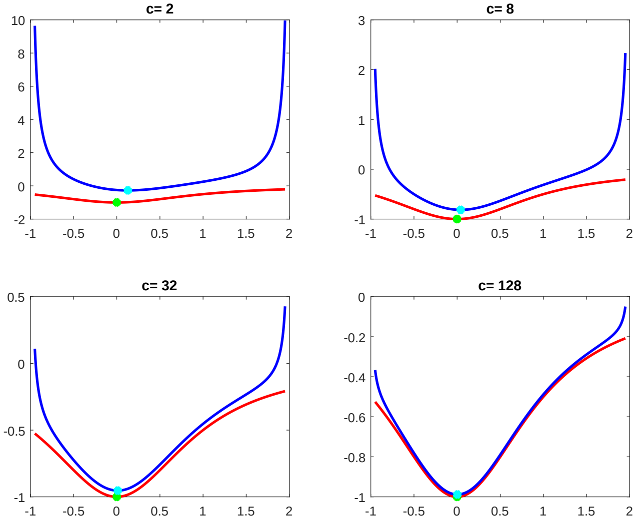

Figure: The green point is the solution of constrained minimization problem and the blue one is the minimizer of $r(x, c)$.