Numerical Analysis

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Numerical Solution of Systems of Linear Equations

Our aim is to solve the following system of linear equations

$$Ax = b,$$

for the unknown vector x when the coefficient matrix A the right hand side vector b are known. We assume that A is $n \times n$ matrix and $b \in \mathbb{R}^n$.

There are two groups of methods for solving linear systems.
Direct methods:

- Gaussian Elimination
- LU Factorization

Iterative methods:

- Jacobi Method
- Gauss-Seidel Method

LU Factorization

Our aim is to replace the system $Ax = b$ with an equivalent system

$$LUx = b$$

where

$$L = \begin{pmatrix} \ell_{11} & 0 & \dots & 0 \\ \ell_{21} & \ell_{22} & \dots & 0 \\ \vdots & & & \\ \ell_{n1} & \ell_{n2} & \dots & \ell_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & u_{12} & \dots & u_{1n} \\ 0 & 1 & \dots & u_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

L is lower triangular matrix such that $\ell_{ii} \neq 0$, for $i = 1, \dots n$, and U is unit upper triangular matrix.

Example

Solve the following system of linear equations

$$\begin{cases} 6x_1 - 2x_2 + 2x_3 = 12 \\ 12x_1 - 8x_2 + 6x_3 = 16 \\ 3x_1 - 13x_2 + 9x_3 = -22 \end{cases}$$

using LU Factorization.

$$\ell_{ij} = a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} u_{kj}, \quad j = 1, 2, \ldots i,$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} \ell_{ik} u_{kj}}{\ell_{ii}}, \quad j = i+1, i+2, \ldots n.$$

After factorization let's denote

$$z = Ux.$$

Then we solve the system

$$Lz = b,$$

$$z_i = \frac{b_i - \sum_{k=1}^{i-1} \ell_{ik} z_k}{\ell_{ii}}, \quad i = 1, 2, \ldots n.$$

And the last step is

$$Ux = z,$$

$$x_i = z_i - \sum_{k=i+1}^{n} u_{ik} x_k, \quad i = n, n-1, \ldots 1.$$

Theorem

Let A be $n \times n$ matrix. If all leading principal minors of A are different from zero then one can represent A in the following form

$$A = LU,$$

where L is a lower triangular matrix with non zero diagonal elements and U is an unit upper triangular matrix.

Definition

A leading principal minor of a square $n \times n$ matrix A is the determinant of a submatrix obtained by deleting everything except the first m rows and columns, for $1 \leq m \leq n$.

Iterative Solutions of Linear Systems

We are going to consider the notion of norm in $\mathbb{R}^n$.

Definition

A vector norm $||\cdot||$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ that obeys the following three properties

- $||x|| \geq 0$, $||x|| = 0$ if and only if $x = 0$
- $||\alpha x|| = |\alpha|||x||$
- $||x + y|| \leq ||x|| + ||y||$

for vectors $x, y \in \mathbb{R}$ and scalars $\alpha \in \mathbb{R}$.

Examples of vector norms for $x = [x_1, x_2, \dots x_n]^T \in \mathbb{R}^n$ are

- $||x||_1 = \sum_{i=1}^{n} |x_i|$ ℓ_1-vector norm
- $||x||_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ Euclidean or ℓ_2-vector norm
- $||x||_\infty = \max\limits_{1 \le i \le n} |x_i|$ ℓ_∞-vector norm

Example

Determine the ℓ_1, ℓ_2 and ℓ_∞ vector norms of the vector $x = [-1, 1, -2, 0.5]^T$.

Definition

The sequence of vectors $\{x^{(k)}\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ is said to converge to $x \in \mathbb{R}^n$ with respect to the norm $||\cdot||$ if

$$||x^{(k)} - x|| \to 0, \quad \text{as} \quad k \to \infty.$$

Example

Show that

$$x^{(k)} = [x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)}]^T = \left[1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k} \sin k\right]^T$$

converges to $x = [1, 2, 0, 0]^T$ with respect to the norm ℓ_∞.

Definition

A matrix norm on the set of all $n \times n$ matrices is a real-valued function, $|| \cdot ||$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

- $||A|| \geq 0$, $||A|| = 0$ if and only if $A = 0$;
- $||\alpha A|| = |\alpha| ||A||$;
- $||A + B|| \leq ||A|| + ||B||$;
- $||AB|| \leq ||A|| \cdot ||B||$.

We will consider matrix norms that are related to a vector norm. The **natural**, or **induced**, matrix norm associated with the vector norm $||\cdot||$ is defined by

$$||A|| := \sup\{||Ax|| : \quad x \in \mathbb{R}^n \quad \text{and} \quad ||x|| = 1\}.$$

Here, A is $n \times n$ matrix.

It is easy to show that

$$||Ax|| \leq ||A|| \cdot ||x||, \quad \forall x \in \mathbb{R}^n.$$

Examples of natural matrix norms for an $n \times n$ matrix A are

- $||A||_1 = \max\limits_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$ ℓ_1-matrix norm
- $||A||_2 = \max\limits_{1 \le i \le n} \sqrt{|\sigma_i|}$ Spectral or ℓ_2-matrix norm
- $||A||_\infty = \max\limits_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$ ℓ_∞-matrix norm

Here, σ_i are the eigenvalues of $A^T A$.
If A is a symmetric matrix, then

$$||A||_2 = \rho(A),$$

where $\rho(A)$ is called the spectral radius of A. $\rho(A)$ is defined by the following

$$\rho(A) = \max_{1 \le i \le n} |\lambda_i|,$$

λ_i are the eigenvalues of A.

Example

Determine the ℓ_1, ℓ_2 and ℓ_∞ matrix norms of the following matrix

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$