Numerical Analysis

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Condition Number and Ill-Conditioning

Suppose we want to solve an invertible linear system of equations $Ax = b$ for a given coefficient matrix A and right-hand side b but there may have been perturbations of the data owing to uncertainty in the measurements and roundoff errors in the calculations. As a result the solution is changed, but how far is the resulting solution from the actual solution?

Example

Consider the following systems of linear equations

$$\begin{cases} x_1 + 1.01x_2 = 2.01 \\ 0.99x_1 + x_2 = 1.99 \end{cases},$$

$$\begin{cases} x_1 + 1.01x_2 = 2.0099 \\ 0.99x_1 + x_2 = 1.9901 \end{cases}.$$

Definition

The following quantity

$$\mathcal{K}(A) = ||A|| \cdot ||A^{-1}||$$

is called the **condition number** of the matrix A.

If the linear system is sensitive to perturbations in the elements of A, or to perturbations of the components of b, then this fact is reflected in A having a large condition number. In such a case, the matrix A is said to be ill-conditioned.

Some properties of conditional number

- $\mathcal{K}(A) \geq 1$
- $\mathcal{K}(A^{-1}) = \mathcal{K}(A)$
- $\mathcal{K}(AB) \leq \mathcal{K}(A)\mathcal{K}(B)$

Let's consider a system of equations

$$Ax = b,$$

where $\det A \neq 0$ and $b \neq 0$.
Suppose that the right-hand side is perturbed by an amount Δb and the corresponding solution is perturbed an amount Δx i.e.

$$A(x + \Delta x) = b + \Delta b.$$

We can show that

$$\frac{||\Delta x||}{||x||} \leq \mathcal{K}(A) \frac{||\Delta b||}{||b||}.$$

Assume x^* is the solution that we obtained for the system $Ax = b$. Let

$$\Delta x = x - x^*.$$

Using the quantity

$$r = b - Ax^*$$

it's possible to estimate the relative error $\frac{||\Delta x||}{||x||}$

$$\frac{1}{\mathcal{K}(A)} \frac{||r||}{||b||} \leq \frac{||\Delta x||}{||x||} \leq \mathcal{K}(A) \frac{||r||}{||b||}.$$