Numerical Analysis

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Fixed-Point Iteration

To apply Fixed-Point Iteration Method we need to replace $f(x) = 0$ by an equivalent equation

$$x = g(x),$$

where $g(x)$ can be defined in a number of ways, for example, as

$$g(x) = x + f(x), \qquad g(x) = x - 3f(x).$$

Definition

The number r is called a fixed point for a given function g if $g(r) = r$.

Fixed-Point Iteration Method

Let x_0 be the initial approximation, then

$$x_{n+1} = g(x_n), \quad n = 0, 1, \ldots.$$

Definition

A function $g : [a, b] \to \mathbb{R}$ is a contraction mapping if there is a constant $0 \leq \alpha < 1$, such that

$$|g(x) - g(y)| \leq \alpha|x - y|, \quad \forall x, y \in [a, b].$$

Example

$\sqrt{x} : [1, 5] \to \infty$ is a contraction mapping.

Example

$\sqrt{x} : [0, 1] \to \infty$ is not a contraction mapping.

If $g \in \mathbb{C}^1[a, b]$ and $\max\limits_{x \in [a,b]} |g'(x)| \leq \alpha < 1$, then g is a contraction mapping.

Theorem

Let $g : [a, b] \to [a, b]$ is a contraction mapping, then

a. *there is exactly one fixed point $r \in [a, b]$;*

b. *for any number $x_0 \in [a, b]$, the sequence defined by Fixed Point Iteration Method converges to the fixed point in $[a, b]$ and*

$$|x_n - r| \leq \frac{\alpha^n}{1-\alpha}|x_1 - x_0|, \quad n \geq 1.$$

Stopping conditions

- $|r - x_n| < \varepsilon$
- $|x_n - x_{n-1}| < \varepsilon$
- $\frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon, \quad |x_n| \neq 0$
- $|f(x_n)| < \varepsilon$

Example

Our aim is to solve the equation

$$x^3 + 4x^2 - 10 = 0$$

numerically using Fixed-Point Iterative method.

a. Show that the equation has a unique solution.

b. Write the equation $f(x) = 0$ in an equivalent form $x = g(x)$ in such a way, that $g(x)$ will be a contraction mapping in some interval $[a, b]$, containing the solution, and $g : [a, b] \to [a, b]$.

c. (MATLAB) Write a MatLab program which calculates the solution with an initial guess $x_0 \in [a, b]$, use the following stopping condition $|x_{n+1} - x_n| < 10^{-4}$.