Numerical Analysis

Lusine Poghosyan

AUA

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Numerical Integration

In this section we introduce numerical methods suitable for approximating the integral

$$\int_a^b f(x)dx,$$

where $f : [a, b] \to \mathbb{R}$ is a continuous function.

If $F(x)$ is the antiderivative of $f(x)$, then using The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a).$$

It's possible that there is no elementary function $F(x)$ such that $F'(x) = f(x)$, e.g.

$$\int_a^b e^{x^2} dx,$$

$$\int_a^b \frac{\sin x}{x} dx,$$

$$\int_a^b \frac{dx}{\ln x}.$$

When the following formula

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} A_k f(x_k)$$

is used to approximate the definite integral we say that we have a quadrature rule, where the coefficients $A_0, A_1, \ldots, A_n$ are called the weights of quadrature rule and $x_0, x_1, \ldots, x_n$ are called the nodes of quadrature rule.

Let's denote by $R(f)$ the error of quadrature rule

$$R(f) = \int_a^b f(x)dx - \sum_{k=0}^{n} A_k f(x_k).$$

Definition

We will say that the quadrature rule is exact for some class of functions $\mathcal{F}$, if

$$R(f) = 0, \qquad \forall f \in \mathcal{F}.$$

Definition

The precision degree of a quadrature formula is n if and only if the error is zero for all polynomials of degree $k = 0, 1, \ldots, n$, but is not zero for some polynomial of degree $n + 1$.

Example

Consider the quadrature rule

$$\int_{-\pi}^{\pi} f(x)dx \approx \frac{\pi}{2} \cdot f(-\pi) + \frac{3}{2}\pi \cdot f\left(\frac{\pi}{2}\right).$$

Is this quadrature rule exact for the class

a. of all polynomials of degree ≤ 1?

b. $\mathcal{F} = \{a \sin x + b \cos x : a, b \in \mathbb{R}\}$?

What is the precision degree of this QR (quadrature rule)?

Left endpoint rule

$$\int_a^b f(x)dx \approx f(a)(b-a).$$

Theorem

If $f \in C^1[a,b]$ and $M_1 = \max\limits_{a \le x \le b} |f'(x)|$, then

$$\left| \int_a^b f(x)dx - f(a)(b-a) \right| \le \frac{M_1}{2}(b-a)^2.$$

The left endpoint rule is exact for constant functions.

Right endpoint rule

$$\int_a^b f(x)dx \approx f(b)(b-a).$$

Theorem

If $f \in C^1[a,b]$ and $M_1 = \max_{a \le x \le b} |f'(x)|$, then

$$\left| \int_a^b f(x)dx - f(b)(b-a) \right| \le \frac{M_1}{2}(b-a)^2.$$

The right endpoint rule is exact for constant functions.

Midpoint rule

$$\int_a^b f(x)dx \approx f\left(\frac{a+b}{2}\right)(b-a).$$

Theorem

If $f \in C^2[a,b]$ *and* $M_2 = \max\limits_{a\le x\le b} |f''(x)|$, *then*

$$\left|\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a)\right| \le \frac{M_2}{24}(b-a)^3.$$

The midpoint rule is exact for linear functions.

Let's divide the interval $[a, b]$ into n equal-length parts by the following points

$$x_k = a + kh, \qquad k = 0, 1, \ldots, n \qquad h = \frac{b-a}{n}$$

and apply the left endpoint rule to each subinterval $[x_k, x_{k+1}]$.
Composite left endpoint rule

$$\int_a^b f(x)dx \approx h \sum_{k=0}^{n-1} f(x_k).$$

Composite right endpoint rule

$$\int_a^b f(x)dx \approx h \sum_{k=0}^{n-1} f(x_{k+1}).$$

Composite midpoint rule

$$\int_a^b f(x)dx \approx h \sum_{k=0}^{n-1} f\left(\frac{x_k + x_{k+1}}{2}\right).$$

Error estimate for the composite left endpoint rule.

Theorem

If $f \in C^1[a,b]$ *and* $M_1 = \max_{a \le x \le b} |f'(x)|$*, then*

$$\left| \int_a^b f(x)dx - h \sum_{k=0}^{n-1} f(x_k) \right| \le \frac{M_1}{2}(b-a)h.$$

Error estimate for the composite right endpoint rule.

Theorem

If $f \in C^1[a,b]$ and $M_1 = \max\limits_{a \le x \le b} |f'(x)|$, then

$$\left| \int_a^b f(x)dx - h \sum_{k=0}^{n-1} f(x_{k+1}) \right| \le \frac{M_1}{2}(b-a)h.$$

Error estimate for the composite midpoint rule.

Theorem

If $f \in C^2[a,b]$ *and* $M_2 = \max_{a \le x \le b} |f''(x)|$, *then*

$$\left| \int_a^b f(x)dx - h \sum_{k=0}^{n-1} f\left(\frac{x_k + x_{k+1}}{2} \right) \right| \le \frac{M_2}{24}(b-a)h^2.$$

Example

a. In how many equal-length parts one needs to divide the interval $[0, 1]$ to calculate the integral

$$\int_0^1 e^{x^2} dx$$

within the precision 10^{-4} by using the composite midpoint rule?

b. Write a MatLab program that calculates the approximate value of the integral

$$\int_0^1 e^{x^2} dx$$

by using the composite midpoint rule with n found above.

Trapezoid rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b)).$$

Theorem

If $f \in C^2[a,b]$ and $M_2 = \max_{a\le x\le b} |f''(x)|$, then

$$\left| \int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b)) \right| \le \frac{M_2}{12}(b-a)^3.$$

Let's divide the interval $[a, b]$ into n equal-length parts by the following points

$$x_k = a + kh, \qquad k = 0, 1, \dots, n \qquad h = \frac{b-a}{n}$$

and apply the trapezoid rule to each subinterval $[x_k, x_{k+1}]$.

Composite trapezoid rule

$$\int_a^b f(x)dx \approx \frac{h}{2}\sum_{k=0}^{n-1}(f(x_k) + f(x_{k+1})) = \frac{h}{2}\left(f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right).$$

Error estimate for the composite trapezoid rule.

Theorem

If $f \in C^2[a,b]$ and $M_2 = \max\limits_{a \le x \le b} |f''(x)|$, then for the error of the composite trapezoid rule we have

$$|R(f)| \le \frac{M_2}{12}(b-a)h^2.$$

Example

Consider the integral

$$\int_{-1}^{0} \sin\left(\frac{\pi x^2}{2}\right) dx.$$

Suppose that we wish to integrate numerically, with an error of magnitude less than 10^{-3}. What width h is needed if we wish to use the composite Trapezoid Rule?

Simpson's rule

Find a quadrature rule of the form

$$\int_a^b f(x)dx \approx A \cdot f(a) + B \cdot f\left(\frac{a+b}{2}\right) + C \cdot f(b),$$

that will be exact for the class of second order polynomials.

Simpson's rule

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Error estimate for the Simpson's rule

Theorem

If $f \in C^4[a,b]$ and $M_4 = \max\limits_{a \le x \le b} |f^{(4)}(x)|$, then for the error of the Simpson's rule we have

$$|R(f)| \le \frac{M_4}{90}\left(\frac{b-a}{2}\right)^5.$$

What is the precision degree of the Simpson’s rule?

Let's divide the interval $[a, b]$ into $2n$ equal-length parts by the following points

$$x_k = a + kh, \qquad k = 0, 1, \ldots, 2n \qquad h = \frac{b-a}{2n}$$

and apply the Simpson's rule to each subinterval $[x_{2k}, x_{2k+2}]$.

Composite Simpson's rule

$$\int_a^b f(x)dx \approx \frac{h}{3}\sum_{k=0}^{n-1}\left(f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\right).$$

Error estimate for the Composite Simpson's rule

Theorem

If $f \in C^4[a,b]$ and $M_4 = \max\limits_{a \le x \le b} |f^{(4)}(x)|$, then for the error of the Composite Simpson's rule we have

$$|R(f)| \le \frac{M_4}{180}(b-a)h^4.$$

Interpolatory quadratures

$$a \le x_0 < x_1 < \cdots < x_n \le b$$

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} A_k f(x_k),$$

where

$$A_k = \int_a^b \ell_n^{(k)}(x)dx,$$

$$\ell_n^{(k)}(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)}.$$

Theorem

The quadrature rule

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} A_k f(x_k),$$

is interpolatory quadrature rule if and only if it is exact for the class of polynomials of order n.