Numerical Analysis

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Spline Interpolation

A spline function is a function that consists of polynomial pieces joined together with certain smoothness conditions. A simple example is the spline of degree 1, whose pieces are linear polynomials joined together to achieve continuity.

Explicit definition of the first degree spline

$$S(x) = \begin{cases} s_0(x), & x \in [t_0, t_1], \\ s_1(x), & x \in [t_1, t_2], \\ \vdots & \\ s_{n-1}(x), & x \in [t_{n-1}, t_n], \end{cases}$$

where

$$S_i(x) = a_i + b_i x.$$

If $S(x)$ is continuous, we call it a first-degree spline.

As each piece of $S(x)$ is a linear function, then $S(x)$ is a piecewise linear function.

Definition

The points $t_0, t_1, \ldots$ are called knots in the theory of splines.

Definition

A function S is called a spline of degree 1 if:

1. The domain of S is an interval $[a, b]$.
2. S is continuous on $[a, b]$.
3. There is a partitioning of the interval $a = t_0 < t_1 < \cdots < t_n = b$ such that S is a linear function on each subinterval $[t_i, t_{i+1}]$.

Outside the interval $[a, b]$ S is usually defined in the following way

$$S(x) = \begin{cases} s_0(x), & x < a, \\ s_{n-1}(x), & x > b. \end{cases}$$

Example

Determine whether this function is a first-degree spline function:

$$S(x) = \begin{cases} x, & x \in [-1, 0], \\ 1 - x, & x \in (0, 1), \\ 2x - 2, & x \in [1, 2]. \end{cases}$$

The spline functions of degree 1 can be used for interpolation. Assume we have a table

x	t_0	t_1	t_2	$\dots$	t_n
$f(x)$	$f(t_0)$	$f(t_1)$	$f(t_2)$	$\dots$	$f(t_n)$

There is no loss of generality in supposing that $t_0 < t_1 < \cdots < t_n$.

$$S_i(x) = f(t_i) + m_i(x - t_i), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1 \dots, n-1,$$

where

$$m_i = \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}.$$

Theorem

Let $S(x)$ be a first-degree spline having knots $a = x_0 < x_1 < \cdots < x_n = b$. If S interpolates a function f at these knots, $f'(x)$ exists and continuous on $[a, b]$, then we have

$$|f(x) - S(x)| \leq M_1 \frac{h}{2}, \qquad \forall x \in [a, b],$$

where

$$M_1 = \max_{[a,b]} |f'(x)|,$$

$$h = \max_{1 \leq i \leq n} (x_i - x_{i-1}).$$

Theorem

Let $S(x)$ be a first-degree spline having knots $a = x_0 < x_1 < \cdots < x_n = b$. If S interpolates a function f at these knots, $f''(x)$ exists and continuous on $[a, b]$ then we have

$$|f(x) - S(x)| \leq M_2 \frac{h^2}{8}, \qquad \forall x \in [a, b],$$

where

$$M_2 = \max_{[a,b]} |f''(x)|,$$

$$h = \max_{1 \leq i \leq n} (x_i - x_{i-1}).$$

Example

Estimate the global error, when $f = \frac{1}{1+x^2}$ is interpolated with a first-degree spline at the knots $t_0 = -5$, $t_1 = -4.5$, $t_2 = -3.5$, $t_3 = -3$, $t_4 = -1$, $t_5 = 2.5$, $t_6 = 4$ and $t_7 = 5$.

Cubic Spline

Definition

A function S is called a spline of degree 3 or a cubic spline if:

1. The domain of S is an interval $[a, b]$.
2. S, S' S'' are continuous on $[a, b]$.
3. There is a partitioning of the interval $a = t_0 < t_1 < \cdots < t_n = b$ such that S is a polynomial of degree at most 3 on each subinterval $[t_i, t_{i+1}]$.

$$S(x) = \begin{cases} S_0(x), & x \in [t_0, t_1], \\ S_1(x), & x \in [t_1, t_2], \\ \vdots & \\ S_{n-1}(x), & x \in [t_{n-1}, t_n], \end{cases}$$

where

$$S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i.$$

Example

Determine whether this function is a cubic spline:

$$S(x) = \begin{cases} x^3 - x^2 + 3x - 1, & x \in [-5, 1], \\ 2x^3 - 4x^2 + 6x - 2, & x \in (1, 2], \end{cases}$$

Cubic spline can be used for interpolation. Assume we have a table

x	t_0	t_1	t_2	$\dots$	t_n
$f(x)$	$f(t_0)$	$f(t_1)$	$f(t_2)$	$\dots$	$f(t_n)$

.

We need to find a cubic spline such that

$$S(t_i) = f(t_i), \qquad i = 0, 1, \dots, n.$$

A cubic spline consists of n polynomials of order 3
$S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$, which means we have $4n$ unknowns.

Interpolation condition imposes on $S(x)$ $2n$ conditions.

The continuity of $S'(x)$ and $S''(x)$ adds $2n - 2$ more conditions.

We have $4n$ unknowns and $4n - 2$ conditions.

There are number of ways to impose these 2 extra conditions.

A cubic spline is called a **natural cubic spline** if

$$S''(t_0) = 0, \qquad S''(t_n) = 0.$$

A cubic spline is called a **not-a-knot cubic spline** if

$$S_0'''(t_1) = S_1'''(t_1) \qquad S_{n-2}'''(t_{n-1}) = S_{n-1}'''(t_{n-1}).$$

A cubic spline is called a **clamped cubic spline** if

$$S'(t_0) = d_0 \qquad S'(t_n) = d_n.$$

A cubic spline is called a **periodic cubic spline** if

$$S(t_0) = s(t_n) \qquad S'(t_0) = S'(t_n).$$

Example

Construct a natural cubic spline that passes through the points $(-1, 1)$, $(0, 0)$, and $(2, 1)$.

Example (MATLAB)

Write a MatLab program that divides the interval $[-5, 5]$ by equidistant points $-5 = t_0 < t_1 < ... < t_n = 5$, constructs the first-degree interpolating spline for the function $f(x) = \frac{1}{1+x^2}$ at the knots t_i, $i = 0, 1, ..., n$ in interval $[-5, 5]$, and then plots the function f, the Interpolating Spline, the points $(t_i, f(t_i))$ on the same figure, when $n = 5, 10, 15, 20$.

Example (MATLAB)

Write a MatLab program that divides the interval $[-5, 5]$ by equidistant points $-5 = t_0 < t_1 < ... < t_n = 5$, constructs not-a-knot cubic spline, which interpolates the function $f(x) = \frac{1}{1+x^2}$ at the knots t_i, $i = 0, 1, ..., n$ in interval $[-5, 5]$, and then plots the function f, the Interpolating Spline, the points $(t_i, f(t_i))$ on the same figure, when $n = 5, 10, 15, 20$.

Spline of degree k

Definition

A function S is called a spline of degree k:

1. The domain of S is an interval $[a, b]$.
2. $S, S', \dots, S^{(k-1)}$ are continuous on $[a, b]$.
3. There is a partitioning of the interval $a = t_0 < t_1 < \dots < t_n = b$ such that S is a polynomial of degree at most k on each subinterval $[t_i, t_{i+1}]$.

Assume we have a table

x	t_0	t_1	t_2	$\dots$	t_n
$f(x)$	$f(t_0)$	$f(t_1)$	$f(t_2)$	$\dots$	$f(t_n)$

.

A spline of degree k interpolates $f(x)$ if

$$S(t_i) = f(t_i), \qquad i = 0, 1, \dots, n.$$

A spline of degree k consists of n polynomials of order k, which means we have $(k+1)n$ unknowns.

Interpolation condition imposes on $S(x)$ $2n$ conditions.

The continuity of $S'(x)$, $S''(x)$ and $S^{(k-1)}$ adds $(k-1)(n-1)$ more conditions.

We have $(k+1)n$ unknowns and $(k+1)n-(k-1)$ conditions.

There are number of ways to impose these extra $k-1$ conditions.