Linear Programming - Definition and properties

Operations research

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Bibliography

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- D.B. Judin, E. G. Golsztejn, *Metody programowania liniowego*

1. History

2. Definition

3. Applications and examples

   - Profit maximization
   - Diet problem
   - Transportation problem
   - Assignment problem
   - Problem optymalnego wykroju
   - Maximal flow problem

History

- 1823 Fourier - the unpublished foundations
- 1911 Poussin - article, same problem as with Fourier
- 1939 Leonid Kantorovich (Russia) - Worked in USSR and discovered the model, again unpublished in the western countries
- II World War Dantzig works for US Air Force, logistics, scheduling
- 1947 George B. Dantzig - simplex method published

- 1947 John von Neumann - duality
- 1975 Leonid Kantorovich (USSR) Tjalling Koopmans (USA) Nobel price in economics, for applications of LP
- 1979 Leonid Khachiyan (Chaczijan) - polynomial algorithm, better then simplex from 1000 constraints and 50 000 variables.
- 1984 Narendra Karmarkar - interior point method, polynomial and more efficient similar to simplex

Definition of linear programing

Let us consider vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Which satisfies the conditions

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} a_{i0} \quad i = 1, 2, ..., m \tag{1}$$

$$x_1 \geq 0, x_2 \geq 0 \ldots x_n \geq 0 \tag{2}$$

And maximized (minimized) objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{3}$$

Terminology

- Objective function
- Decision variables
- Coefficients (cost, technological) - parameters
- Constraints
- Feasible solution

- Feasible region

- Optimal solution

Example 0 Jasiu is obligated to buy at least 10 kg of potato. Potato costs 2 PLN per kg. Jasiu wants to save money. How much will he buy and spend?

Example 1

Let us consider an enterprise which produces two different goods W_1 and W_2 (for example carriages and boats). Only three different resources are used S_1, S_2 and S_3 (for example: Labor force, wood, ropes).

	Resources		
	S_1	S_2	S_3
Product W_1	6	10	5
Product W_2	8	6	15
Amount of resource	48	60	75

The requirements are stated in the table above. The profit from selling one peace of first product is 4 units of money, and profit from second good W_2 is 6.

Assumptions

- Proportionality
- Additivity
- Divisibility
- Determinism

The Standard Maximum Problem

Find

$$X = [x_1, x_2, ..., x_n]^T \tag{4}$$

Subject to

$$\begin{array}{ccccc} a_{11}x_1 + a_{12}x_2 + & \dots & + a_{1n}x_n & \leq & a_{10} \\ a_{21}x_1 + a_{22}x_2 + & \dots & + a_{2n}x_n & \leq & a_{20} \\ & \dots & & & \\ a_{m1}x_1 + a_{m2}x_2 + & \dots & + a_{mn}x_n & \leq & a_{m0} \end{array} \tag{5}$$

$$x_1 \geq 0, x_2 \geq 0 \dots x_n \geq 0 \tag{6}$$

Maximize objective function

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \tag{7}$$

The Standard Minimum Problem

Find

$$X = [x_1, x_2, ..., x_n]^T \tag{8}$$

Subject to

$$\begin{array}{ccccc} a_{11}x_1 + a_{12}x_2 + & \dots & + a_{1n}x_n & \geq & a_{10} \\ a_{21}x_1 + a_{22}x_2 + & \dots & + a_{2n}x_n & \geq & a_{20} \\ & \dots & & & \\ a_{m1}x_1 + a_{m2}x_2 + & \dots & + a_{mn}x_n & \geq & a_{m0} \end{array} \tag{9}$$

$$x_1 \geq 0, x_2 \geq 0 \dots x_n \geq 0 \tag{10}$$

Minimize objective function

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \tag{11}$$

The Standard Equality Problem

Find

$$X = [x_1, x_2, ..., x_n]^T \tag{12}$$

Subject to

$$\begin{array}{rcrcl} a_{11}x_1 + a_{12}x_2 + & \dots & +a_{1n}x_n & = & a_{10} \\ a_{21}x_1 + a_{22}x_2 + & \dots & +a_{2n}x_n & = & a_{20} \\ & \dots & & & \\ a_{m1}x_1 + a_{m2}x_2 + & \dots & +a_{mn}x_n & = & a_{m0} \end{array} \tag{13}$$

$$x_1 \geq 0, x_2 \geq 0 \dots x_n \geq 0 \tag{14}$$

Maximize (minimize) objective function

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \tag{15}$$

Properties

THEOREM 1. Objective function of the linear programming problem is a convex function (at the same time is a concave function too).

Properties

THEOREM 2. Feasible region of the linear programming problem is a convex set.

Properties

THEOREM 3. Every point P of the feasible region of the linear programming problem may be expressed as convex linear combination of feasible corner points P_1, P_2, ... , P_s, that is $P = \lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_s P_s$, where $\lambda_1 \geq 0, \lambda_2 \geq 0, ..., \lambda_s \geq 0$ and $\lambda_1 + \lambda_2 + ... + \lambda_s = 1$.

Properties

THEOREM 4. The objective function of a linear programming problem achieves the maximum value at a feasible corner point.

Properties

THEOREM 5. If the objective function of a linear programming problem achieves the maximum value at s ($s \geq 2$) feasible corner points $P_1, P_2, ..., P_s$ then it achieves the maximum value at every point that is a convex linear combination of $P_1, P_2, ..., P_s$.

Example 3

Find $x = [x_1, x_2]^T$ subject to:

$x_1 + x_2 \leq 1$

$x_1 \geq 0$ $x_2 \geq 0$

That mimimizes:

a)$z = -x_1 - 2x_2 + 3$

b)$z = x_2 + 1$

c)$z = 2$

Definition of mathematical programing

For vector $X = [x_1, x_2, \ldots, x_n]^T$ maximize (minimize) objective function

$$z = f(x_1, x_2, \ldots, x_n) \tag{16}$$

Subject to

$$g_i(x_1, x_2, \ldots, x_n) \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} 0 \quad i = 1, 2, ..., m \tag{17}$$

Feasible region set $D = \left\{ X \in R^2 : g_i(x_1, \ldots, x_n) \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} 0, i = 1, ..., m, \right\}.$

Gradient vector

$$\nabla f(x_1, x_2, \ldots, x_n) = \left[\frac{\delta f(X)}{\delta x_1}, \frac{\delta f(X)}{\delta x_2}, \ldots, \frac{\delta f(X)}{\delta x_n}\right]$$

Fermat theorem - necessary condition for extreme function values If function $f(x_1, x_2, \ldots, x_n)$ reaches local maximum (minimum) in the interior point $\hat{X} = [\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_n}] \in D \subset R^n$ $\hat{X}$, where D is a domain of function f, then $\nabla f(\hat{X}) = [0, 0, \ldots, 0]$. In other words $\frac{\delta f(\hat{X})}{\delta x_1} = 0$, $\frac{\delta f(\hat{X})}{\delta x_2} = 0$, ... , $\frac{\delta f(\hat{X})}{\delta x_n} = 0$.