Linear Programming - Applications

Operations research

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Applications

Profit maximization

An enterprise has m factors of production (ex. machines, resources, workers) each in limited amount a_{i0}.

Enterprise produces n different product. There should be produced at least p_i and at most q_i peaces of i-th product $(i = 1, 2, \ldots, n)$.

The matrix bellow expresses the requirements a_{ij} of i-th factor of production to produce j-th product.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Each product brings profit c_i $(i = 1, 2, \ldots, n)$.

How should we distribute factors of production to reach the highest total profit.

Economic interpretation, Production optimization Consider an unified product. It has m different raw materials $S_1, S_2, \ldots, S_m$ which are available in monthly (daily, annula) reserves, respectively a_{i0}. Product can by produced in n different manners, $x_1, x_2, \ldots, x_n$ express the amount of time spent on the certain production manner. The parameter a_{ij} in the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

says how much of i-th material is used for j-th production manner in an unit of time. Finally parameters $c_1, c_2, \ldots, c_n$ represent the number of goods produced in the $i-th$ manner in one unit of time. Hom to maximize the production? What is the optimal distribution of materials?

Feed mix (diet) problem (Blending aviation gasoline 1952)

Given n ingredients (I_i for $i = 1, 2, ..., n$) with certain properties and prises, compose the cheapest mixture satisfying prespecified standards. Assume that ingredients contain both desirable and undesirable substances. Components of each ingredient and desired parameters of mixture are known and collected in the table.

	P_1	P_2	$\ldots$	P_m	Price
I_1	$a_{1,1}$	$a_{1,2}$	$\ldots$	$a_{1,m}$	c_1
I_2	$a_{2,1}$	$a_{2,2}$	$\ldots$	$a_{2,m}$	c_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
I_n	$a_{n,1}$	$a_{n,2}$	$\ldots$	$a_{n,m}$	c_n
Limits	$\{\geq, \leq, =\}$	$\{\geq, \leq, =\}$	$\ldots$	$\{\geq, \leq, =\}$	
	$a_{0,1}$	$a_{0,2}$	$\ldots$	$a_{0,m}$	

Example

Let us consider two coal types: A and B. Both are polluted with ash and phosphorus.

We need 90 t of fuel containing at most 0.03% of phosphorus and 4% of ash.

The table shows the prices of different types of coal and the percentage rate of pollution.

	Pollution		price
	phosphorus	ash	
A	0.02	3	100
B	0.05	5	80

How much coal of different types should be mixed to minimize the cost and satisfy the restrictions?

Example

Stock-farm needs feed for the cattle in prespecified amounts. Farm needs to determine the proportions of two ingredients in the mix. Ingredients are limited. They have 16 kg of first ingredient per day and 10 kg of second. There are two nutrients S_1 i S_2, mix should contain at least of 3.3 kg of S_1 and at most 2.6 kg of S_2 per day. Table shows the percentage amounts of nutrients in ingredient.

	Percentage amounts of nutrients	
	S_1	S_2
Ingredient I	30	20
Ingredient II	10	20

Compose the mixture to minimize cost, sine I ingredient costs 2 zł per kg and II ingredient costs 5 zł per kg.

Rucksack problem (knapsack problem)

There are n different commodities $r_1, r_2, \ldots, r_n$ each in unrestricted number. Peace of r_i weights w_i kg and is worth p_i units of money. The capacity (strength) W of the rucksack is known in advance. Determine the numbers of each commodity that fit the rucksack and maximize walue. In general rucksack problem can by applied in many practical cases, ex. loading of train, container, car, case.

Example

Alibaba found himself in the cave full of treasures. Unfortunately his camel is able to carry only 40 kg of additional baggage. Alibaba is able to estimate quite precisely both the price and the weight of the commodity. Help him taking the most valuable treasures.

i	1	2	3	W
p_i	9	6	4	
w_i	7	5	3	40

Transportation problem

There are m ports or centres of supply of a certain commodity, and n destinations or markets to which this commodity must be shipped. The i-th port ($i = 1, 2, \ldots, m$) possesses an amount a_i of the commodity, and the requirements are such that the j-th destination is to receive the amount b_j of the commodity. The cost of shipping one unit of the commodity from port i to destination j is equal to c_{ij} ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$). The task is to determine x_{ij} ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$) the quantity of the commodity which must be shipped from port i to destination j to minimize the total cost of shipping.

Example

A transportation company has two bases B_1 and B_2 from which it sends buses to tree bus stations D_1, D_2, D_3. Transit between base and bus station is considered as a wast. In the table you can see the distances, numbers of buses in bases and requirements.

Bases	Bus stations			Number of buses
	D_1	D_2	D_3	
B_1	15	12	10	80
B_2	5	18	24	120
requirement	40	65	45	

Find the optimal buses division.

Assignment problem

n assignees (employees, machines, vehicles, plants, time slots) are being assigned to perform n tasks. The assignment has to satisfy the following assumptions:

- each assignee is to be assigned to exactly one task,
- each task is to be performed by exactly one assignee,
- there is a cost c_{ij} associated with assignee i ($i = 1, 2, \ldots, n$) performing task j ($j = 1, 2, \ldots, n$).

The objective is to determine how all n assignments should be made to minimize the total cost.

1931 Egervary

1955 Khun Hungarian algorithm

1890 Carl Gustav Jacobi discovered again in 2006

Example

There are four workers and four tasks to be performed. The time in which each worker would completes each task is given in the table. Assign bijectively workers to the tasks in the way that minimizes the total time of work.

	W_1	W_2	W_3	W_4
T_1	210	450	390	330
T_2	270	390	450	330
T_3	270	510	390	390
T_4	330	450	330	450

Hungarian algorithm

1. Subtract the smallest number in each row from every number in the row.

2. Subtract the smallest number in each column from every number in the column.

3. Test whether an optimal assignment can be made. You do this by determining the minimum number of vertical and horizontal lines needed to cross out all zeros. If the minimum number of lines equals the number of rows, an optimal set of assignments is possible. In that case, go to step 6. Otherwise go on to step 4.

4. If the minimum number of lines is less than the number of rows, modify the the table in the following way:

Hungarian algorithm (continued)

a) Subtract the smallest uncrossed number from every uncrossed number in the table.
b) Add the smallest uncrossed number to the numbers at the intersections of crossing lines.
c) Numbers cross out but not at the intersections of cross-out lines carry over unchanged to the table.
5. Repeat steps 3 and 4 until an optimal set of assignments is possible.
6. Make the assignments one at a time in positions that have zero elements. Begin with rows and columns that have only one zero. Since each row and each column needs to receive exactly one assignment, cross out both the row and the column involved after each assignment is made. Continue step 6 until every row and every column has exactly one assignment.

Example

Find minimal matching.
Initial table:

	W_1	W_2	W_3	W_4
T_1	210	450	390	330
T_2	270	390	450	330
T_3	270	510	390	390
T_4	330	450	330	450

Example

Subtract the smallest number in each row from every number in the row.

	W_1	W_2	W_3	W_4
T_1	0	240	180	120
T_2	0	120	180	60
T_3	0	240	120	120
T_4	0	120	0	120

Example

Subtract the smallest number in each column from every number in the column.

	W_1	W_2	W_3	W_4
T_1	0	120	180	60
T_2	0	0	180	0
T_3	0	120	120	60
T_4	0	0	0	60

Example

Determine the minimum number of vertical and horizontal lines needed to cross out all zeros.

	W_1	W_2	W_3	W_4
T_1	0	120	180	60
T_2	0	0	180	0
T_3	0	120	120	60
T_4	0	0	0	60

Example

Only three lines are required. Find minimal uncrossed element.

	W_1	W_2	W_3	W_4
T_1	0	120	180	60
T_2	0	0	180	0
T_3	0	120	120	60
T_4	0	0	0	60

Example

The minimal uncrossed element should be:

- subtracted form all uncrossed elements,
- added to all crossed by two lines,
- elements crossed only once remain unchanged.

	W_1	W_2	W_3	W_4
T_1	0	60	120	0
T_2	60	0	180	0
T_3	0	60	60	0
T_4	60	0	0	60

Example

Now you need at least four lines to cross all the zeros.

	W_1	W_2	W_3	W_4
T_1	0	60	120	0
T_2	60	0	180	0
T_3	0	60	60	0
T_4	60	0	0	60

Example

Let us construct the optimal solution:

- one can be placed only on position where there were zeros ,
- in each raw and each column there should be exactly one one.

	W_1	W_2	W_3	W_4
T_1	0			0
T_2		0		0
T_3	0			0
T_4		0	0	

Example

There are two optimal solutions:

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

	W_1	W_2	W_3	W_4
T_1	0			0
T_2		0		0
T_3	0			0
T_4		0	0	

Example

There are two optimal solutions:

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

	W_1	W_2	W_3	W_4
T_1	210	450	390	330
T_2	270	390	450	330
T_3	270	510	390	390
T_4	330	450	330	450

Example

There are two optimal solutions:

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$z = 210 + 390 + 330 + 390 = 1320$ $\qquad z = 330 + 390 + 270 + 330 = 1320$

	W_1	W_2	W_3	W_4
T_1	210	450	390	330
T_2	270	390	450	330
T_3	270	510	390	390
T_4	330	450	330	450

How can we check how many lines are necessary to cross all zeros in matrix?

Cutting Stock Problem (Kantorovich 1939)

Let us consider a manufacturer who produces sheet of material (ex. steel, wood) of standard size. An order is placed by a customer who needs sheets of different sizes. Assume that m types of sheets are to be produced. In particular, we need b_i sheets of i-th type. A standard sheet can by cut in n different ways. j cutting pattern gives a_{ij} peaces of sheet of type i. Scraps can not be recycled so we want to minimize the amount of wasts or we simply want to minimize the number of used standard sheets.

Example

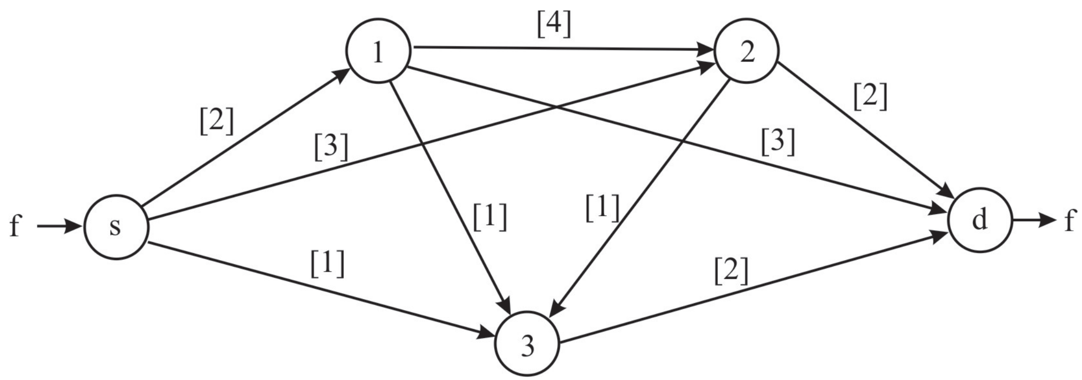
Sawmill was placed an order for 300 sets of desks (for desks for example). Each set consists of 7 desks of length 0.7 m and 4 of length 2.5 m. How should the standard desks be cut to minimize waste?

Maximum flow in network

Consider a directed weighted graph without cycles. If there is one staring and one finishing vertex what is the maximal flow, between them.

Example

Let us consider a network of oil pipelines modeled by the digraph bellow D. Weights $c(x, y)$ on arcs stand for the capacity of pipeline. Oil can be sent in only one direction. Hom much oil can be sent from point s to d?



Algorithm Edmonds-Karp

1. Construct a digraph D' with the same vertex set as D and edge set $E(D') = \{(x,y) : ((x,y) \in E(D)) \wedge c(x,y) > f(x,y)) \vee ((y,x) \in E(D) \wedge f(y,x) > 0)\}$
2. Using BFS (Breadth-First Search Algorithm) find shortest path from s to d if not possible go to step 5 else let $Q : s = u_0, u_1, u_2, ..., u_r = d$
3. Assign to $(u_{i-1}, u_i) \in Q$ weight $\Delta_i = c(u_{i-1}, u_i) - f(u_{i-1}, u_i)$ if $(u_{i-1}, u_i) \in D$ or $\Delta_i = f(u_i, u_{i-1})$ if $(u_i, u_{i-1}) \in D$. Let $\Delta = \min(\Delta_i)$. Adjust $f(u_{i-1}, u_i) = f(u_{i-1}, u_i) + \Delta$ if $(u_{i-1}, u_i) \in E(D)$ or $f(u_i, u_{i-1}) = f(u_i, u_{i-1}) - \Delta$ if $(u_i, u_{i-1}) \in E(D)$.
4. Go to the step 1.
5. Output f as an optimal flow.

Maximum flow in network

Multi starting points or destinations.

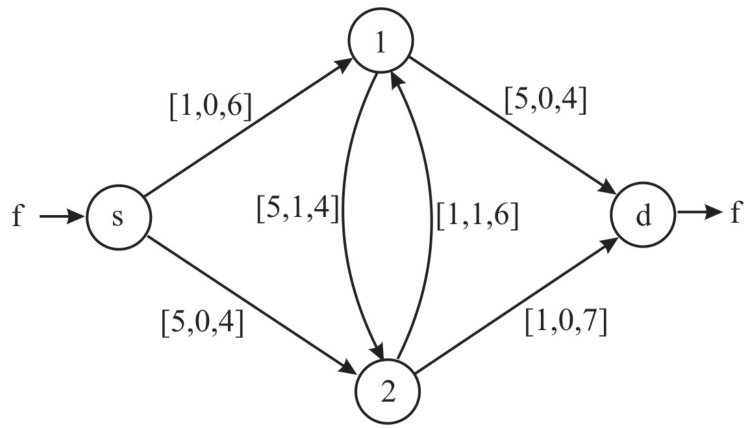
Minimal cost of flow in network

Similarly as on the previous slide, we have a directed network. Here each arc (i, j) is assigned a triple $[c_{ij}, d_{ij}, g_{ij}]$, where c_{ij} stands for the cost of transporting one entity along the arc (i, j), d_{ij} is the minimal and g_{ij} is the maximal amount of resource.

The aim is to minimize the cost of transporting f units of resources from s to d.

Example

Calculate the minimal cost of flow f from point s to d.



Graph coloring

Consider a graph $G = (V, E)$ and the problem of k-colorability of G.

We assume that $V = \{v_1, v_2, ..., v_n\}$ and G is given by maximum clique representation. So we know array

$$Q = \begin{bmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,n} \\ \vdots & \vdots & & \vdots \\ q_{r,1} & q_{r,2} & \cdots & q_{r,n} \end{bmatrix}$$

Where $q_{i,j} = 1$ if v_i belongs to Q_j clique in other cases $g_{i,j} = 0$.

Graph coloring - integer programming

Collect decision variables in matrix

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,k} \\ x_{2,1} & x_{2,2} & \dots & x_{2,k} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,k} \end{bmatrix}$$

where $x_{i,j} = 1$ if v_i is colored with color j.

Minimize $z = \sum_{i=1}^{n} \sum_{j=1}^{k} x_{i,j}$ subject to:

$$QX \leq \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$x_{i,j} \in \{0, 1\}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.

$\sum_{j=0}^{k} x_{i,j} = 1$ for $i = 1, 2, ..., n$

Graph coloring - related problems

- Sum-coloring
- list coloring
- restricted coloring
- edge coloring

Prymal representation.

Find vector $X = [x_1, x_2, \ldots, x_n]^T$, subjected to conditions

$$
\begin{array}{ccccc}
a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n \leq a_{10} \\
a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n \leq a_{20} \\
\vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n \leq a_{m0}
\end{array}
$$

$$x_1 \geq 0,\ x_2 \geq 0, \ldots, x_n \geq 0$$

that maximizes the cost function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Dual representation.

Find vector $Y = [y_1, y_2, \ldots, y_m]^T$, subject to conditions

$$\begin{array}{ccccc} a_{11}y_1 + a_{21}y_2 + \ldots a_{m1}y_n \geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots a_{m2}y_n \geq c_2 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m1}y_1 + a_{m2}y_2 + \ldots a_{mn}y_n \geq c_m \end{array}$$

$$y_1 \geq 0,\ y_2 \geq 0, \ldots, y_n \geq 0$$

that minimizes the cost function

$$z = a_{10}y_1 + a_{20}y_2 + \ldots + a_{m0}y_m$$

Game theory

Interactive decision theory.

Elements:

- players
- moves (strategies)
- payoff matrix

Game theory - game representation

- Extensive representation. (decision tree)

- Normal form. (matrix)

Game theory - classification

- Cooperative or non-cooperative
- Symmetric and asymmetric
- Zero-sum and non-zero-sum
- Simultaneous and sequential
- Full information imperfect information (probabilistic)

Zero sum matrix games

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

where ai, j is the won value of P_1 if he chooses i strategy on condition that P_2 chooses j strategy.

Zero sum matrix games

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Clean strategy for P_1 $\max_{1\le i\le m}\min_{1\le j\le n} a_{i,j}$.

Zero sum matrix games

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

Clean strategy for P_1 $\max_{1 \le i \le m} \min_{1 \le j \le n} a_{i,j}$.
Clean strategy for P_2 $\min_{1 \le j \le n} \max_{1 \le i \le m} a_{i,j}$.

Zero sum matrix games

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Clean strategy for P_1 $\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{i,j}$.

Clean strategy for P_2 $\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{i,j}$.

$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{i,j} \leq \min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{i,j}$

Zero sum matrix games, example

$$A = \begin{bmatrix} 1 & 5 & 0 & 3 \\ 2 & 1 & 4 & 2 \\ 4 & 2 & -1 & 0 \end{bmatrix}$$

Clean strategy for P_1 $\max_{1\leq i\leq m} \min_{1\leq j\leq n} a_{i,j} = 1$.
Clean strategy for P_2 $\min_{1\leq j\leq n} \max_{1\leq i\leq m} a_{i,j} = 3$.

Zero sum matrix games, mixed strategies

Player P_1 strategy.

$$X = \begin{bmatrix} x_1 & x_2 & \dots & x_m \end{bmatrix}$$

$x_1, x_2, \dots, x_m \geq 0$ and $x_1 + x_2 + \dots + x_m = 1$

Player P_2 strategy.

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$y_1, y_2, \dots, y_m \geq 0$ and $y_1 + y_2 + \dots + y_m = 1$

Zero sum matrix games, mixed strategies