Linear Programming - Simplex

Operations research

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General concept

- Find any corner point.
- while (!optimal)
  - find better corner point
  - move to better corner point

Gausian elimination

$$\begin{cases} x_1 + x_2 + x_3 & = & 6 \\ 3x_1 + 2x_2 + 2x_3 & = & 13 \\ 2x_1 + x_2 + 2x_3 & = & 10 \end{cases}$$

The Standard Maximum Problem

Find

$$X = [x_1, x_2, ..., x_n]^T \tag{1}$$

Subject to

$$\begin{array}{ccccc} a_{11}x_1 + a_{12}x_2 + & \ldots & +a_{1n}x_n & \leq & a_{10} \\ a_{21}x_1 + a_{22}x_2 + & \ldots & +a_{2n}x_n & \leq & a_{20} \\ & \ldots & & & \\ a_{m1}x_1 + a_{m2}x_2 + & \ldots & +a_{mn}x_n & \leq & a_{m0} \end{array} \tag{2}$$

$$x_1 \geq 0, x_2 \geq 0 \ldots x_n \geq 0 \tag{3}$$

Maximize objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{4}$$

Properties

THEOREM 7 Point P is a corner point of LP feasible solution if and only if it sharply satisfies n linearly independent inequalities.

Standard Equation Problem

Determine vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$ that maximizes

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & a_{10} \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & a_{20} \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & a_{m0} \end{array}$$

$$x_1 \geq 0, \ldots, x_n \geq 0$$

Standard Matrix Representation

Find $X \in R^n$, subject to

$$A \cdot X = A_0$$

$$X \geq 0_n$$

where

$$z = C \cdot X$$

gets its maximum.
Where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ 0_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ C = [c_1, c_2, \ldots, c_n]$$

Standard Vector Representation

Find $X \in R^n$, that maximizes function

$$z = C \cdot X$$

subject to

$$A_1 x_1 + A_2 x_2 + \ldots + A_n x_n = A_0$$

$$X \geq 0_n$$

Where

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix},\ 0_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ C = [c_1, c_2, \ldots, c_n]$$

Properties

THEOREM 6. The most important theorem A vector $x = [x_1, x_2, ..., x_n]^T$ represents the coordinates of a feasible corner point if and only if in the linear combination

$$A_1x_1 + A_2x_2 + ... + A_nx_n = A_0$$

if there exists a subset of m (where m is the number of equations) linearly independent vectors A_j, such that decision variables associated with remaining vectors are 0.

Standard Simplex Representation

Find $X \in R^n$ that maximizes the function

$$z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to:

$$\begin{array}{lllllll}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n-m}x_{n-m} & + x_{n-m+1} & & & = a_{1,0} \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n-m}x_{n-m} & & + x_{n-m+2} & & = a_{2,0} \\
\vdots \qquad \vdots \qquad \ddots \qquad \vdots & & & \ddots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{m,n-m}x_{n-m} & & & + x_n & = a_{m,0}
\end{array}$$

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, a_{1,0} \geq 0, a_{2,0} \geq 0, \ldots, a_m, 0 \geq 0$$

Standard Simplex Representation - Example

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 4x_1 + 6x_2$$

subject to

$$\begin{aligned} 6x_1 + 8x_2 &\leq 48 \\ 10x_1 + 6x_2 &\leq 60 \\ 5x_1 + 15x_2 &\leq 75 \end{aligned}$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Standard Simplex Representation - Example

Find $X = [x_1, x_2, x_3, x_4, x_5]^T$ that maximizes

$$z = 4x_1 + 6x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to

$$\begin{array}{rcrcrcrcrcr} 6x_1 & + & 8x_2 & + & x_3 & & & & & = & 48 \\ 10x_1 & + & 6x_2 & & & + & x_4 & & & = & 60 \\ 5x_1 & + & 15x_2 & & & & & + & x_5 & = & 75 \end{array}$$

$$x_1 \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0, \;\; x_4 \geq 0, \;\; x_5 \geq 0$$

Standard Simplex (matrix) Representation

Find $X \in R^n$ that maximizes function

$$z = C \cdot X$$

subject to

$$A \cdot X = A_0$$

$$X \geq 0_n$$

$$A_0 \geq 0_n$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-m} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2,n-m} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{m,n-m} & 0 & 0 & \dots & 1 \end{bmatrix}, \; 0_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; C = [c_1, c_2, \dots, c_n]$$

Standard Simplex (vector) representation

Find $X \in R^n$
that maximizes function

$$z = C \cdot X$$

subject to

$$A_1 x_1 + A_2 x_2 + \ldots + A_n x_n = A_0$$

$$X \geq 0_n$$

$$A_0 \geq 0_n$$

where $A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, $0_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $C = [c_1, c_2, \ldots, c_n]$

Simplex tableau

			c_1	c_2	$\dots$	c_{n-m}	c_{n-m+1}	c_{n-m+2}	$\dots$	c_n	c_0
i	base	c_i	A_1	A_2	$\dots$	A_{n-m}	A_{n-m+1}	A_{n-m+2}	$\dots$	A_n	A_0
1	A_{n-m+1}	c_{n-m+1}	$x_{1,1}$	$x_{1,2}$	$\dots$	$x_{1,n-m}$	1	0	$\dots$	0	$x_{1,0}$
2	A_{n-m+2}	c_{n-m+2}	$x_{2,1}$	$x_{2,2}$	$\dots$	$x_{2,n-m}$	0	1	$\dots$	0	$x_{2,0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
m	A_n	c_n	$x_{m,1}$	$x_{m,2}$	$\dots$	$x_{m,n-m}$	0	0	$\dots$	1	$x_{m,0}$
	$z_k - c_k$		$z_1 - c_1$	$z_2 - c_2$	$\dots$	$z_{m,n-m} - c_{m,n-m}$	0	0	$\dots$	0	$z_0 - c_0$

Remark: Identify $x_{i,j}$ with $a_{i,j}$.

Simplex tableau - Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	6	8	1	0	0	48
2	A_4	0	10	6	0	1	0	60
3	A_5	0	5	15	0	0	1	75
	$z_j - c_j$		-4	-6	0	0	0	0

1. If $z_j - c_j \geq 0$ for all j $(1 \leq j \leq n)$ then the optimal (maximal) solution was found – stop the algorithm. Otherwise go to step 2.
2. Find that k which satisfies the condition

$$z_k - c_k = \min_{1 \leq j \leq n} (z_j - c_j)$$

3. Find that l which satiesfies the condition

$$\frac{a_{j0}}{a_{lk}} = \min_{a_{ik} > 0} \frac{a_{i0}}{a_{ik}}$$

If $a_{ik} \leq 0$ for all i $(1 \leq i \leq m)$ then the objective function reaches $+\infty$, so stop the algorithm. Otherwise go to step 4.

4. Calculate new coefficiants of the simplex tableou according to formulas (determine a new corner-point feasible solution)

$a'_{ij} = a_{ij} - \frac{a_{lj}}{a_{lk}} a_{ik}$ for $i = 1, 2, ..., l-1, l+1, ..., m+1$ and $j = 1, 2, ..., n, 0$
$a'_{lj} = \frac{a_{lj}}{a_{lk}}$ for $j =, 1, 2, ..., n, 0$
where $a'_{m+1,j} = z'_j - c_j$ for $j = 1, 2, ..., n, 0$.
Come back to step 1.

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	6	8	1	0	0	48
2	A_4	0	10	6	0	1	0	60
3	A_5	0	5	15	0	0	1	75
	$z_j - c_j$		-4	-6	0	0	0	0

$X^{(0)} = [0, 0, 48, 60, 75]^T$

$z(x^{(0)}) = 0$

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	6	8	1	0	0	48
2	A_4	0	10	6	0	1	0	60
3	A_5	0	5	15	0	0	1	75
	$z_j - c_j$		-4	-6	0	0	0	0

$X^{(0)} = [0, 0, 48, 60, 75]^T$

$z(x^{(0)}) = 0$

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	$\frac{10}{3}$	0	1	0	$-\frac{8}{15}$	8
2	A_4	0	8	0	0	1	$-\frac{6}{15}$	30
3	A_2	6	$\frac{1}{3}$	1	0	0	$\frac{1}{15}$	5
	$z_j - c_j$		-2	0	0	0	$\frac{2}{5}$	30

$X^{(1)} = [0, 5, 8, 30, 0]^T$

$z(x^{(1)}) = 30$

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	$\frac{10}{3}$	0	1	0	$-\frac{8}{15}$	8
2	A_4	0	8	0	0	1	$-\frac{6}{15}$	30
3	A_2	6	$\frac{1}{3}$	1	0	0	$\frac{1}{15}$	5
	$z_j - c_j$		-2	0	0	0	$\frac{2}{5}$	30

$X^{(1)} = [0, 5, 8, 30, 0]^T$

$z(x^{(1)}) = 30$

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_1	4	1	0	$\frac{3}{10}$	0	$-\frac{8}{50}$	$\frac{12}{5}$
2	A_4	0	0	0	$-\frac{24}{10}$	1	$\frac{44}{50}$	$\frac{52}{5}$
3	A_2	6	0	1	$-\frac{1}{10}$	0	$\frac{6}{50}$	$\frac{21}{5}$
	$z_j - c_j$		0	0	$\frac{3}{5}$	0	$\frac{2}{25}$	$34\frac{4}{5}$

$X^{(2)} = [\frac{12}{5}, \frac{21}{5}, 0, \frac{52}{5}, 0]^T$

$z(x^{(2)}) = 34\frac{4}{5}$

Explanation 1. How to get from one corner point to another?

Consider corner point $X^0 = [a^0_{1,0}, a^0_{2,0}, ...a^0_{m,0}, 0, ..., 0]^T$ represented by set of linearly independent $A_1, A_2, ..., A_m$ of vectors. (See Theorem 6.)

Vectors from the base are kept in form of versors.

Consider introducing vector A_k to the base. Before transforming (Gaussian elimination) it to versor we need to decide which vector is to be removed from base.

Explanation 1. How to get from one corner point to another?

Let us assume that vector associated with l-th row is deleted from base. We may not allow any negative value in A_0 so:

- $a^1_{1,0} = a^0_{1,0} - \frac{a^0_{1,k} a^0_{l,0}}{a^0_{l,k}} \geq 0$

- $a^1_{2,0} = a^0_{2,0} - \frac{a^0_{2,k} a^0_{l,0}}{a^0_{l,k}} \geq 0$

  $\vdots$

- $a^1_{m,0} = a^0_{m,0} - \frac{a^0_{m,k} a^0_{l,0}}{a^0_{l,k}} \geq 0$

Explanation 1. How to get from one corner point to another?

Let us assume that vector associated with l-th row is deleted. We may not allow any negative value in A_0 so:

- $\frac{a^0_{1,0}}{a^0_{1,k}} \geq \frac{a^0_{l,0}}{a^0_{l,k}}$
- $\frac{a^0_{2,0}}{a^0_{2,k}} \geq \frac{a^0_{l,0}}{a^0_{l,k}}$

  $\vdots$
- $\frac{a^0_{m,0}}{a^0_{m,k}} \geq \frac{a^0_{l,0}}{a^0_{l,k}}$

Explanation 1. How to get from one corner point to another?

- $\frac{a^0_{1,0}}{a^0_{1,k}} \geq \frac{a^0_{l,0}}{a^0_{l,k}}$

- $\frac{a^0_{2,0}}{a^0_{2,k}} \geq \frac{a^0_{l,0}}{a^0_{l,k}}$

  $\vdots$

- $\frac{a^0_{m,0}}{a^0_{m,k}} \geq \frac{a^0_{l,0}}{a^0_{l,k}}$

Take l such that $\min_{j \in \{1,\ldots,m\}} \left(\frac{a^0_{j,0}}{a^0_{j,k}} \right) = \frac{a^0_{l,0}}{a^0_{l,k}}$.

Explanation 2. How to get better corner point?

$$z(X^0) = c_1 a^0_{1,0} + c_2 a^0_{2,0} + ... + c_m a^0_{m,0}$$

$$z(X^1) = c_1 a^1_{1,0} + c_2 a^1_{2,0} + ... + c_{l-1} a^1_{l-1,0} + c_{l+1} a^1_{l+1,0} + ... + c_m a^1_{m,0} + c_k a^1_{k,0}$$

Explanation 2. How to get better corner point?

$$z(X^1) = c_1 a^1_{1,0} + c_2 a^1_{2,0} + ... + c_{l-1} a^1_{l-1,0} + c_{l+1} a^1_{l+1,0} + ... + c_m a^1_{m,0} + c_k a^1_k$$

Let $t = \frac{a^0_{l,0}}{a^0_{l,k}}$

$$a^1_{1,0} = a^0_{1,0} - \frac{a^0_{1,k} a^0_{l,0}}{a^0_{l,k}};\ a^1_{2,0} = a^0_{2,0} - \frac{a^0_{2,k} a^0_{l,0}}{a^0_{l,k}};\ \ldots x^1_{m,0} = x^0_{m,0} - \frac{a^0_{m,k} a^0_{l,0}}{a^0_{l,k}}$$

$$z(X^1) = c_1(a^0_{1,0} - t a^0_{1,k}) + c_2(a^0_{2,0} - t a^0_{2,k}) + ... + c_{l-1}(a^0_{l-1,0} - t a^0_{l-1,k}) + c_{l+1}(a^0_{l+1,0} - t a^0_{l+1,k}) + ... + c_m(a^0_{m,0} - t a^0_{m,k}) + c_k t$$

Explanation 2. How to get better corner point?

$$z(X^0) = c_1 a^0_{1,0} + c_2 a^0_{2,0} + ... + c_m a^0_{m,0}$$

$$z_k = c_1 a^0_{1,k} + c_2 a^0_{2,k} + ... + c_m a^0_{m,k}$$

$$z(X^1) = c_1(a^0_{1,0} - ta^0_{1,k}) + c_2(a^0_{2,0} - ta^0_{2,k}) + ... + c_{l-1}(a^0_{l-1,0} - ta^0_{l-1,k}) + c_{l+1}(a^0_{l+1,0} - ta^0_{l+1,k}) + ... + c_m(a^0_{m,0} - ta^0_{m,k}) + c_k t$$

$$z(X^1) = z(X^0) - c_l a^0_{l,0} - tz_k + tc_l a^0_{l,k} + c_k t$$

$$z(X^1) = z(X^0) - t(z_k - c_k) - c_l a^0_{l,0} + \frac{a^0_{l,0}}{a^0_{l,k}} c_l a^0_{l,k}$$

$$z(X^1) = z(X^0) - t(z_k - c_k)$$