Linear Programming - Applications and Integer Programming

Operations research

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Cutting Stock Problem (Kantorovich 1939)
Let us consider a manufacturer who produces sheet of material (ex. steel, wood) of standard size. An order is placed by a customer who needs sheets of different sizes. Assume that m types of sheets are to be produced. In particular, we need b_i sheets of i-th type. A standard sheet can by cut in n different ways. j cutting pattern gives a_{ij} peaces of sheet of type i. Scraps can not be recycled so we want to minimize the amount of wasts or we simply want to minimize the number of used standard sheets.

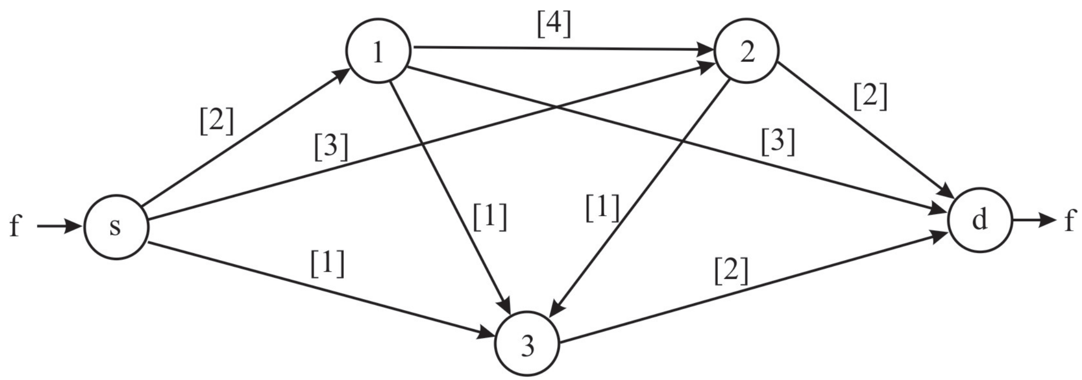
Example
Sawmill was placed an order for 300 sets of desks (for desks for example). Each set consists of 7 desks of length 0.7 m and 4 of length 2.5 m. How should the standard desks (5,2 m) be cut to minimize waste?

Maximum flow in network

Consider a directed weighted graph without cycles. If there is one staring and one finishing vertex what is the maximal flow, between them.

Example

Let us consider a network of oil pipelines modeled by the digraph bellow D. Weights $c(x, y)$ on arcs stand for the capacity of pipeline. Oil can be sent in only one direction. Hom much oil can be sent from point s to d?



Algorithm Edmonds-Karp

1. Construct a digraph D' with the same vertex set as D and edge set $E(D') = \{(x, y) : ((x, y) \in E(D)) \wedge c(x, y) > f(x, y)) \vee ((y, x) \in E(D) \wedge f(y, x) > 0)\}$
2. Using BFS (Breadth-First Search Algorithm) find shortest path from s to d if not possible go to step 5 else let $Q : s = u_0, u_1, u_2, ..., u_r = d$
3. Assign to $(u_{i-1}, u_i) \in Q$ weight $\Delta_i = c(u_{i-1}, u_i) - f(u_{i-1}, u_i)$ if $(u_{i-1}, u_i) \in D$ or $\Delta_i = f(u_i, u_{i-1})$ if $(u_i, u_{i-1}) \in D$. Let $\Delta = \min(\Delta_i)$. Adjust $f(u_{i-1}, u_i) = f(u_{i-1}, u_i) + \Delta$ if $(u_{i-1}, u_i) \in E(D)$ or $f(u_i, u_{i-1}) = f(u_i, u_{i-1}) - \Delta$ if $(u_i, u_{i-1}) \in E(D)$.
4. Go to the step 1.
5. Output f as an optimal flow.

Maximum flow in network

Multi starting points or destinations.

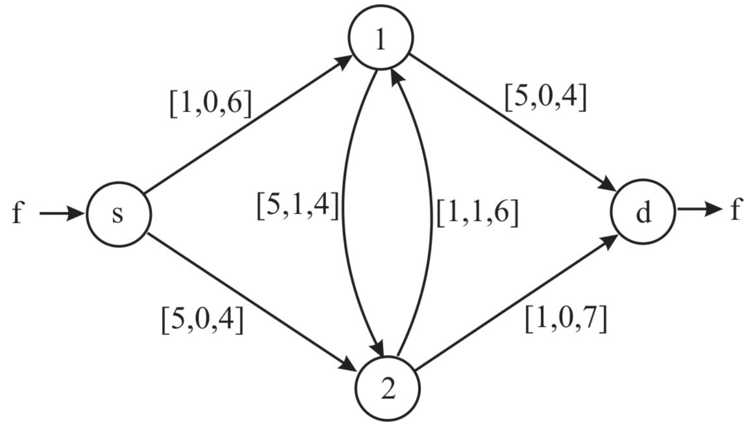
Minimal cost of flow in network

Similarly as on the previous slide, we have a directed network. Here each arc (i, j) is assigned a triple $[c_{ij}, d_{ij}, g_{ij}]$, where c_{ij} stands for the cost of transporting one entity along the arc (i, j), d_{ij} is the minimal and g_{ij} is the maximal amount of resource.

The aim is to minimize the cost of transporting f units of resources from s to d.

Example

Calculate the minimal cost of flow f from point s to d.



Integer programming

Integer programming

Rounding off

01-programming

Cutting plane method (Ralph Gomory algorithm)

1. Find real optimal solution (relaxation) of the integer programming problem (using, for example, the simplex method).
2. Constitute the solving raw of the computational tableau as the raw which a_{i0} has minimal positive fraction.
3. Create a new inequality (right hand side) which elements are equal to the fractions of the suitable elements of the solving raw.
4. Append the created inequality to the computational tableau and come back to step 1.

Cutting plane method (Ralph Gomory algorithm)

ad. 2. Sometimes all non zero fraction rows are taken, sometimes the first one.

ad. 3. Plane that cuts of current corner point but leaves all integer points.

$x_j + a_{1i}x_1 + ... + a_{ri}x_r = a_{0i}$

$x_j + \lfloor a_{1i} \rfloor x_1 + ... + \lfloor a_{ri} \rfloor x_r \leq a_{0i}$

$x_j + \lfloor a_{1i} \rfloor x_1 + ... + \lfloor a_{ri} \rfloor x_r \leq \lfloor a_{0i} \rfloor$

$x_j - x_j + (a_{1i} - \lfloor a_{1i} \rfloor)x_1 + ... + (a_{ri} - \lfloor a_{ri} \rfloor)x_r \geq a_{0i} - \lfloor a_{0i} \rfloor$

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 4x_1 + 1x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 6 \\ x_1 &\leq 2 \end{aligned}$$

$$x_1 \geq 0, \ x_2 \geq 0$$

x_1, x_2 are integers.

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 4x_1 + x_2 + 0x_3 + 0x_4$$

subject to

$$\begin{array}{ccccccccc} 2x_1 & + & 3x_2 & + & x_3 & & & = & 6 \\ x_1 & + & & + & & & x_4 & = & 2 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

x_1, x_2 are integers.

Relaxation of a problem

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_3	0	2	3	1	0	6
2	A_4	0	1	0	0	1	2
	$z_j - c_j$		-4	-1	0	0	0

Relaxation of a problem

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_3	0	0	3	1	-2	2
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	-1	0	4	8

Relaxation of a problem

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

Cutting plane

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

$0x_1 + x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = \frac{2}{3}$

$0x_1 + x_2 + 0x_3 - x_4 \leq 0$

Cutting plane

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

$0x_1 + x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = \frac{2}{3}$

$0x_1 + x_2 + 0x_3 - x_4 \leq 0$

$0x_1 + 0x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{2}{3}$

Cutting plane

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

$0x_1 + 0x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{2}{3}$

$0x_1 + 0x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{2}{3}$

Cutting plane

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

$$0x_1 + 0x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{2}{3}$$
$$0x_1 + 0x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{2}{3}$$
$$0x_1 + 0x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 - x_5 + \bar{x}_6 = \frac{2}{3}$$

Cutting plane

		C	4	1	0	0	0	$-M$	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	$\frac{2}{3}$
2	A_1	4	1	0	0	1	0	0	2
3	A_6	-M	0	0	$\frac{1}{3}$	$\frac{1}{3}$	-1	1	$\frac{2}{3}$
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	0	0	$\frac{26}{3}$
			0	0	$-\frac{M}{3}$	$-\frac{M}{3}$	M	0	$-\frac{2M}{3}$

$$0x_1 + 0x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 - x_5 + \bar{x}_6 = \frac{2}{3}$$

Cutting plane

		C	4	1	0	0	0	$-M$	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0	3	1	-2	0	0	2
2	A_1	4	1	0	0	1	0	0	2
3	A_6	-M	0	-1	0	1	-1	1	0
	$z_j - c_j$		0	-1	0	4	0	0	8
			0	M	0	$-M$	M	0	0

Cutting plane

		C	4	1	0	0	0	$-M$	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0	1	1	0	-2	2	2
2	A_1	4	1	1	0	0	1	-1	2
3	A_4	0	0	-1	0	1	-1	1	0
	$z_j - c_j$		0	3	0	0	4	-4	8
			0	0	0	0	0	M	0

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 2x_1 + 3x_2$$

subject to

$$\begin{aligned} -2x_1 + 4x_2 &\leq 9 \\ 14x_1 + 4x_2 &\leq 49 \end{aligned}$$

$$x_1 \geq 0,\ x_2 \geq 0$$

x_1, x_2 are integers.

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

subject to

$$\begin{array}{rcrcrcrcr} -2x_1 & + & 4x_2 & + & x_3 & & & = & 9 \\ 14x_1 & + & 4x_2 & + & & & x_4 & = & 49 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

x_1, x_2 are integers.

Relaxation of a problem

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_3	0	-2	4	1	0	9
2	A_4	0	14	4	0	1	49
	$z_j - c_j$		-2	-3	0	0	0

Relaxation of a problem

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	$-\frac{1}{2}$	1	$\frac{1}{4}$	0	$\frac{9}{4}$
2	A_4	0	16	0	1	1	40
	$z_j - c_j$		$-\frac{7}{2}$	0	$\frac{3}{4}$	0	$\frac{27}{4}$

Relaxation of a problem

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$x_2 + \frac{9}{32}x_3 + \frac{1}{32}x_4 = \frac{7}{2}$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$x_2 + \frac{9}{32}x_3 + \frac{1}{32}x_4 = \frac{7}{2}$

$x_2 + 0x_3 + 0x_4 \leq 3$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$$0x_1 + 0x_2 - \frac{9}{32}x_3 - \frac{1}{32}x_4 \leq -\frac{1}{2}$$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$0x_1 + 0x_2 + \frac{9}{32}x_3 + \frac{1}{32}x_4 \geq \frac{1}{2}$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$$0x_1 + 0x_2 + \frac{9}{32}x_3 + \frac{1}{32}x_4 - x_5 + \bar{x}_6 = \frac{1}{2}$$

Cutting plane

		C	2	3	0	0	0	-M	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	0	0	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{5}{2}$
3	A_6	-M	0	0	$\frac{9}{32}$	$\frac{1}{32}$	-1	1	$\frac{1}{2}$
	$z_j - c_j$		0	0	$\frac{31-9M}{32}$	$\frac{7-M}{32}$	M	0	$\frac{31-M}{2}$

Cutting plane

		C	2	3	0	0	0	-M	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	0	0	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{5}{2}$
3	A_6	-M	0	0	$\frac{9}{32}$	$\frac{1}{32}$	-1	1	$\frac{1}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	0	0	$\frac{31}{2}$
			0	0	$\frac{-9M}{32}$	$\frac{-M}{32}$	M	0	$-\frac{M}{2}$

Cutting plane

		C	2	3	0	0	0	-M	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_2	0	0	1	0	$\frac{1}{32}$	0	0	$\frac{7}{2}$
2	A_1	0	1	0	0	$\frac{1}{16}$	0	0	$\frac{5}{2}$
3	A_3	-M	0	0	1	$\frac{1}{9}$	$-\frac{32}{9}$	$\frac{32}{9}$	$\frac{16}{9}$
	$z_j - c_j$		0	0	0	$\frac{7-M}{32}$	M	0	$\frac{31}{2}$

Prymal reprezentation.

Find vector $X = [x_1, x_2, \ldots, x_n]^T$, subjected to conditions

$$\begin{array}{ccccc} a_{11}x_1 + & a_{12}x_2 + & \ldots & a_{1n}x_n \leq & a_{10} \\ a_{21}x_1 + & a_{22}x_2 + & \ldots & a_{2n}x_n \leq & a_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \ldots & a_{mn}x_n \leq & a_{m0} \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0, \ldots, x_n \geq 0$$

that maximizes the cost function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Dual representatiion.

Find vector $Y = [y_1, y_2, \ldots, y_m]^T$, subject to conditions

$$\begin{array}{ccccc} a_{11}y_1 + a_{21}y_2 + \ldots a_{m1}y_n \geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots a_{m2}y_n \geq c_2 \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \quad\quad \vdots \\ a_{m1}y_1 + a_{m2}y_2 + \ldots a_{mn}y_n \geq c_m \end{array}$$

$$y_1 \geq 0,\ y_2 \geq 0, \ldots, y_n \geq 0$$

that minimizes the cost function

$$z = a_{10}y_1 + a_{20}y_2 + \ldots + a_{m0}y_m$$