Linear Programming - Simplex and Integer Programming

Operations research

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Linear Programming - Simplex and Integer Programming

- Adapting instances to standard simplex format.
  - equations
  - right-hand side inequalities
- Integer programming

Adapting to other model forms

Equality Constraints

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 7x_1 + 4x_2$$

subject to

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leq & 11 \\ & & x_2 & \leq & 4 \\ 2x_1 & + & 3x_2 & = & 24 \end{array}$$

$$x_1 \geq 0, \;\; x_2 \geq 0$$

Equality Constraints

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 7x_1 + 4x_2 + 0x_3 + 0x_4$$

subject to

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & & & = & 11 \\ & & x_2 & & & + & x_4 & = & 4 \\ 2x_1 & + & 3x_2 & & & & & = & 24 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

Equality Constraints

Big M metchod.

Numerical stability.

Equality Constraints

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 7x_1 + 4x_2 + 0x_3 + 0x_4 - M\bar{x}_5$$

subject to

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & & & & & = & 11 \\ & & x_2 & & & + & x_4 & & & = & 4 \\ 2x_1 & + & 3x_2 & & & & & + & \bar{x}_5 & = & 24 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ \bar{x}_5 \geq 0$$

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	$\bar{x}_5$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	1	1	0	0	11
2	A_4	0	0	1	0	1	0	4
3	A_5	-M	2	3	0	0	1	24
	$-c_j$		-7	-4	0	0	M	0

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	$\bar{x}_5$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	1	1	0	0	11
2	A_4	0	0	1	0	1	0	4
3	A_5	-M	2	3	0	0	1	24
	$z_j - c_j$		-2M-7	-3M-4	0	0	0	-24M

$X^0 = [0, 0, 11, 4, 24]^T$

$z(X^0) = -24M$

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	$\bar{x}_5$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	1	1	0	0	11
2	A_4	0	0	1	0	1	0	4
3	A_5	-M	2	3	0	0	1	24
	$z_j - c_j$		-2M-7	-3M-4	0	0	0	-24M

$X^0 = [0, 0, 11, 4, 24]^T$

$z(X^0) = -24M$

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	$\bar{x}_5$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	0	1	-1	0	7
2	A_2	4	0	1	0	1	0	4
3	A_5	-M	2	0	0	-3	1	12
	$z_j - c_j$		-2M-7	0	0	3M+6	0	-12M+16

$X^1 = [0, 4, 7, 0, 12]^T$

$z(X^1) = -12M + 16$

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	$\bar{x}_5$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	0	1	-1	0	7
2	A_2	4	0	1	0	1	0	4
3	A_5	-M	2	0	0	-3	1	12
	$z_j - c_j$		-2M-7	0	0	3M+6	0	-12M+16

$X^1 = [0, 4, 7, 0, 12]^T$

$z(X^1) = -12M + 16$

Equality Constraints - Example

		C	7	4	0	0	$-M$	0
			x_1	x_2	x_3	x_4	$\bar{x}_5$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1
2	A_2	4	0	1	0	1	0	4
3	A_1	7	1	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	6
	$z_j - c_j$		0	0	0	$-\frac{13}{2}$	$M + \frac{7}{2}$	58

$X^2 = [6, 4, 1, 0, 0]^T$

$z(X^2) = 58$

Equality Constraints - Example

		C	7	4	0	0	$-M$	0
			x_1	x_2	x_3	x_4	$\bar{x}_5$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1
2	A_2	4	0	1	0	1	0	4
3	A_1	7	1	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	6
	$z_j - c_j$		0	0	0	$-\frac{13}{2}$	$M + \frac{7}{2}$	58

$X^2 = [6, 4, 1, 0, 0]^T$

$z(X^2) = 58$

Equality Constraints - Example

		C	7	4	0	0	$-M$	0
			x_1	x_2	x_3	x_4	$\bar{x}_5$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_4	0	0	0	2	1	-1	2
2	A_2	4	0	1	-2	0	1	2
3	A_1	7	1	0	3	0	-1	9
	$z_j - c_j$		0	0	13	0	$M - \frac{5}{2}$	71

$X^3 = [9, 2, 0, 2, 0]^T$

$z(X^3) = 71$

Radiation therapy (Hilier)

In radiation therapy doses to which the tissues are exposed are crucial. Specialists are able to estimate the results of exposure to certain beams (intensity and direction) to both healthy body parts and cancer. Table shows the estimations absorption of radiation.

Area	Beam 1.	Beam 2.	restrictions
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	$=6$
Center of tumor	0.6	0.4	≥ 6

Adjust the strength of beams (in kilorads).

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0.4x_1 + 0.5x_2$$

subject to

$$\begin{array}{rcrcr} 0.3x_1 & + & 0.1x_2 & \leq & 2.7 \\ 0.5x_1 & + & 0.5x_2 & = & 6 \\ 0.6x_1 & + & 0.4x_2 & \geq & 6 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0.4x_1 + 0.5x_2 + 0x_3 + M\bar{x}_4$$

subject to

$$\begin{array}{rcrcrcrcr} 0.3x_1 & + & 0.1x_2 & + & x_3 & & & = & 2.7 \\ 0.5x_1 & + & 0.5x_2 & & & + & \bar{x}_4 & = & 6 \\ 0.6x_1 & + & 0.4x_2 & & & & & \geq & 6 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

Constraints in $\geq$ form

1. transform to equation by subtracting slack variable

2. add one another slack variable with big M coefficient in the objective function.

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0.4x_1 + 0.5x_2 + 0x_3 + M\bar{x}_4 + 0x_5$$

subject to

$$\begin{array}{rcrcrcrcrcr} 0.3x_1 & + & 0.1x_2 & + & x_3 & & & & & = & 2.7 \\ 0.5x_1 & + & 0.5x_2 & & & + & \bar{x}_4 & & & = & 6 \\ 0.6x_1 & + & 0.4x_2 & & & & & - & x_5 & = & 6 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ \bar{x}_4 \geq 0,\ x_5 \geq 0$$

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0.4x_1 + 0.5x_2 + 0x_3 + M\bar{x}_4 + 0x_5 + M\bar{x}_6$$

subject to

$$\begin{array}{ccccccccccccc}
0.3x_1 & + & 0.1x_2 & + & x_3 & & & & & & & = & 2.7 \\
0.5x_1 & + & 0.5x_2 & & & + & \bar{x}_4 & & & & & = & 6 \\
0.6x_1 & + & 0.4x_2 & & & & & - & x_5 & + & \bar{x}_6 & = & 6
\end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ \bar{x}_4 \geq 0,\ x_5 \geq 0,\ \bar{x}_6 \geq 0$$

Minimalization

$\min z = -\max(-z)$

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that maximizes

$$z' = -0.4x_1 - 0.5x_2 + 0x_3 - M\bar{x}_4 + 0x_5 - M\bar{x}_6$$

subject to

$$\begin{array}{rcrcrcrcrcrcl} 0.3x_1 & + & 0.1x_2 & + & x_3 & & & & & & & = & 2.7 \\ 0.5x_1 & + & 0.5x_2 & & & + & \bar{x}_4 & & & & & = & 6 \\ 0.6x_1 & + & 0.4x_2 & & & & & - & x_5 & + & \bar{x}_6 & = & 6 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ \bar{x}_4 \geq 0,\ x_5 \geq 0,\ \bar{x}_6 \geq 0$$

Radiation therapy (Hilier)

		C	-0.4	-0.5	0	-M	0	-M	0
			x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0.3	0.1	1	0	0	0	2.7
2	A_4	-M	0.5	0.5	0	1	0	0	6
3	A_6	-M	0.6	0.4	0	0	-1	1	6
	$-c_j$		0.4	0.5	0	M	0	M	0

Example - Radiation therapy (Hillier)

		C	-0.4	-0.5	0	-M	0	-M	0
			x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0.3	0.1	1	0	0	0	2.7
2	A_4	-M	0.5	0.5	0	1	0	0	6
3	A_6	-M	0.6	0.4	0	0	-1	1	6
	$z_j - c_j$		0.4 -1.1M	0.5-0.9M	0	0	M	0	-12M

$X^0 = [0, 0, 2.7, 6, 0, 6]^T$

$z'(X^0) = -12M$

Example - Radiation therapy (Hillier)

		C	-0.4	-0.5	0	-M	0	-M	0
			x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0.3	0.1	1	0	0	0	2.7
2	A_4	-M	0.5	0.5	0	1	0	0	6
3	A_6	-M	0.6	0.4	0	0	-1	1	6
	$z_j - c_j$		0.4 -1.1M	0.5-0.9M	0	0	M	0	-12M

$X^0 = [0, 0, 2.7, 6, 0, 6]^T$

$z'(X^0) = -12M$

Example - Radiation therapy (Hilier)

		C	-0.4	-0.5	0	-M	0	-M	0
			x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0.4	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
2	A_4	-M	0	$\frac{1}{3}$	$\frac{-5}{3}$	1	0	0	1.5
3	A_6	-M	0	0.2	-2	0	-1	1	0.6
	$z_j - c_j$		0	$\frac{11-16M}{30}$	$\frac{11M-4}{3}$	0	M	0	-2.1M- 3.6

$X^1 = [9, 0, 0, 1.5, 0, 0.6]^T$

$z'(X^1) = -2.1M - 3.6$

Example - Radiation therapy (Hillier)

		C	-0.4	-0.5	0	-M	0	-M	0
			x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0.4	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
2	A_4	-M	0	$\frac{1}{3}$	$\frac{-5}{3}$	1	0	0	1.5
3	A_6	-M	0	0.2	-2	0	-1	1	0.6
	$z_j - c_j$		0	$\frac{11-16M}{30}$	$\frac{11M-4}{3}$	0	M	0	-2.1M- 3.6

$X^1 = [9, 0, 0, 1.5, 0, 0.6]^T$

$z'(X^1) = -2.1M - 3.6$

Example - Radiation therapy (Hillier)

		C	-0.4	-0.5	0	-M	0	-M	0
			x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0.4	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	-$\frac{5}{3}$	8
2	A_4	-M	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	-$\frac{5}{3}$	0.5
3	A_2	-0.5	0	1	-10	0	-5	5	3
	$z_j - c_j$		0	0	$\frac{7-5M}{3}$	0	$\frac{11-10M}{6}$	$\frac{16M-11}{6}$	-0.5M- 4.7

$X^2 = [8, 3, 0, 1.5, 0, 0]^T$

$z'(X^2) = -0.5M - 4.7$

Example - Radiation therapy (Hillier)

		C	-0.4	-0.5	0	-M	0	-M	0
			x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0,4	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	-$\frac{5}{3}$	8
2	A_4	-M	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	-$\frac{5}{3}$	0.5
3	A_2	-0.5	0	1	-10	0	-5	5	3
	$z_j - c_j$		0	0	$\frac{7-5M}{3}$	0	$\frac{11-10M}{6}$	$\frac{16M-11}{6}$	-0.5M- 4.7

$X^2 = [8, 3, 0, 1.5, 0, 0]^T$

$z'(X^2) = -0.5M - 4.7$

Example - Radiation therapy (Hilier)

		C	-0.4	-0.5	0	-M	0	-M	0
			x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0.4	1	0	5	-1	0	0	7.5
2	A_5	0	0	0	1	0.6	1	-1	0.3
3	A_2	-0.5	0	1	-5	3	0	0	4.5
	$z_j - c_j$		0	0	0.5	$M - 1.1$	0	M	-5.25

$X^3 = [7.5, 4.5, 0, 0, 0.3, 0]^T$

$z'(X^3) = -5.25$

Negative variables

Replace variable x_i (possibly negative) with $x_i' - x_i''$
where $x_i' \geq 0$ and $x_i'' \geq 0$

Negative variables - Example Find $X = [x_1, \dot{x}_2]^T$ that maximizes

$$z = 2x_1 - \dot{x}_2$$

subject to

$$\begin{array}{rcrcl} x_1 & + & \dot{x}_2 & \leq & 2 \\ x_1 & - & \dot{x}_2 & \leq & 2 \\ x_1 & & & \leq & 1 \end{array}$$

$$x_1 \geq 0$$

Negative variables - Example

$$\dot{x}_2 = x_2 - x_3$$

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 2x_1 - x_2 + x_3$$

subject to

$$\begin{array}{ccccccc} x_1 & + & x_2 & - & x_3 & \leq & 2 \\ x_1 & - & x_2 & + & x_3 & \leq & 2 \\ x_1 & & & & & \leq & 1 \end{array}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Negative variables - Example

		C	2	-1	1	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	x_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_4	0	1	1	-1	1	0	0	2
2	A_5	0	1	-1	1	0	1	0	2
3	A_5	0	1	0	0	0	0	1	1
	$z_j - c_j$		-2	1	-1	0	0	0	0

Extremal cases and Simplex method

1. inconsistent system of equations

2. unbounded feasible set and infinite solution.

3. unbounded feasible set and unbounded set of solutions

Inconsistent system

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 2x_1 + x_2$$

subject to

$$\begin{array}{ccccc} x_1 & + & x_2 & \leq & 1 \\ x_1 & + & x_2 & \geq & 2 \end{array}$$

$$x_1 \geq 0, x_2 \geq 0$$

unbounded feasible set and infinite solution.

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 2x_1 + x_2$$

subject to

$$x_1 \ + \ x_2 \ \geq \ 1$$

$$x_1 \geq 0, x_2 \geq 0$$

unbounded feasible set and unbounded set of solutions

Find $X = [x_1, x_2]^T$ that minimizes

$$z = x_2$$

subject to

$$\begin{array}{rcrcl} & & x_2 & \geq & 1 \\ x_1 & + & x_2 & \geq & 2 \end{array}$$

$$x_1 \geq 0, x_2 \geq 0$$

Definition of Integer programming

Let us consider vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Which satisfies the constraints

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} a_{i0} \quad i = 1, 2, ..., m \tag{1}$$

$$x_1 \in \mathbb{N}, x_2 \in \mathbb{N}, \ldots, x_n \in \mathbb{N} \tag{2}$$

And maximized (minimized) objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{3}$$

Integer programming

- Rounding off
- Branch and bound
- Cutting plane method

01-programming

Cutting plane method (Ralph Gomory algorithm)

1. Find optimal solution of the integer programming problem in real numbers. Simplex algorithm may be used. If the solution is in integers finish.

2. Constitute the solving raw of the computational tableau as the raw which a_{i0} has minimal positive fraction.

3. Create a new inequality (right hand side) which coefficients are equal to the fractions (nonnegative) of the corresponding elements of the solving raw.

4. Append new-created constraint to the model and computational tableau and move back to step 1.

Cutting plane method (Ralph Gomory algorithm)

ad. 2. Sometimes all non zero fraction rows are taken, sometimes the first one.

ad. 3. Plane that cuts of current corner point but leaves all integer points.

$x_j + a_{1i}x_1 + ... + a_{ri}x_r = a_{0i}$

$x_j + \lfloor a_{1i} \rfloor x_1 + ... + \lfloor a_{ri} \rfloor x_r \leq a_{0i}$

$x_j + \lfloor a_{1i} \rfloor x_1 + ... + \lfloor a_{ri} \rfloor x_r \leq \lfloor a_{0i} \rfloor$

$x_j - x_j + (a_{1i} - \lfloor a_{1i} \rfloor)x_1 + ... + (a_{ri} - \lfloor a_{ri} \rfloor)x_r \geq a_{0i} - \lfloor a_{0i} \rfloor$

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 4x_1 + 1x_2$$

subject to

$$\begin{array}{lcl} 2x_1 + 3x_2 & \leq & 6 \\ x_1 & \leq & 2 \end{array}$$

$$x_1 \geq 0,\ \ x_2 \geq 0$$

x_1, x_2 are integers.

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 4x_1 + x_2 + 0x_3 + 0x_4$$

subject to

$$\begin{array}{ccccccccc} 2x_1 & + & 3x_2 & + & x_3 & & & = & 6 \\ x_1 & + & & + & & & x_4 & = & 2 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

x_1, x_2 are integers.

Relaxation of a problem

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_3	0	2	3	1	0	6
2	A_4	0	1	0	0	1	2
	$z_j - c_j$		-4	-1	0	0	0

Relaxation of a problem

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_3	0	0	3	1	-2	2
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	-1	0	4	8

Relaxation of a problem

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

Cutting plane

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

$0x_1 + x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = \frac{2}{3}$

$0x_1 + x_2 + 0x_3 - x_4 \leq 0$

Cutting plane

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

$$0x_1 + x_2 + \tfrac{1}{3}x_3 - \tfrac{2}{3}x_4 = \tfrac{2}{3}$$

$$0x_1 + x_2 + 0x_3 - x_4 \leq 0$$

$$0x_1 + 0x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 \leq -\tfrac{2}{3}$$

Cutting plane

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

$0x_1 + 0x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{2}{3}$

$0x_1 + 0x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{2}{3}$

Cutting plane

		C	4	1	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
2	A_1	4	1	0	0	1	2
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{26}{3}$

$$0x_1 + 0x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{2}{3}$$

$$0x_1 + 0x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{2}{3}$$

$$0x_1 + 0x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 - x_5 + \bar{x}_6 = \frac{2}{3}$$

Cutting plane

		C	4	1	0	0	0	$-M$	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_2	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	$\frac{2}{3}$
2	A_1	4	1	0	0	1	0	0	2
3	A_6	-M	0	0	$\frac{1}{3}$	$\frac{1}{3}$	-1	1	$\frac{2}{3}$
	$z_j - c_j$		0	0	$\frac{1}{3}$	$\frac{10}{3}$	0	0	$\frac{26}{3}$
			0	0	$-\frac{M}{3}$	$-\frac{M}{3}$	M	0	$-\frac{2M}{3}$

$$0x_1 + 0x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 - x_5 + \bar{x}_6 = \frac{2}{3}$$

Cutting plane

		C	4	1	0	0	0	$-M$	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0	3	1	-2	0	0	2
2	A_1	4	1	0	0	1	0	0	2
3	A_6	-M	0	-1	0	1	-1	1	0
	$z_j - c_j$		0 0	-1 M	0 0	4 $-M$	0 M	0 0	8 0

Cutting plane

		C	4	1	0	0	0	$-M$	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0	1	1	0	-2	2	2
2	A_1	4	1	1	0	0	1	-1	2
3	A_4	0	0	-1	0	1	-1	1	0
	$z_j - c_j$		0	3	0	0	4	-4	8
			0	0	0	0	0	M	0

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 2x_1 + 3x_2$$

subject to

$$\begin{aligned} -2x_1 + 4x_2 &\leq 9 \\ 14x_1 + 4x_2 &\leq 49 \end{aligned}$$

$$x_1 \geq 0, \;\; x_2 \geq 0$$

x_1, x_2 are integers.

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

subject to

$$\begin{array}{rcrcrcrcr} -2x_1 & + & 4x_2 & + & x_3 & & & = & 9 \\ 14x_1 & + & 4x_2 & + & & & x_4 & = & 49 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

x_1, x_2 are integers.

Relaxation of a problem

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_3	0	-2	4	1	0	9
2	A_4	0	14	4	0	1	49
	$z_j - c_j$		-2	-3	0	0	0

Relaxation of a problem

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	$-\frac{1}{2}$	1	$\frac{1}{4}$	0	$\frac{9}{4}$
2	A_4	0	16	0	1	1	40
	$z_j - c_j$		$-\frac{7}{2}$	0	$\frac{3}{4}$	0	$\frac{27}{4}$

Relaxation of a problem

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$x_2 + \frac{9}{32}x_3 + \frac{1}{32}x_4 = \frac{7}{2}$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$x_2 + \frac{9}{32}x_3 + \frac{1}{32}x_4 = \frac{7}{2}$

$x_2 + 0x_3 + 0x_4 \leq 3$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$0x_1 + 0x_2 - \frac{9}{32}x_3 - \frac{1}{32}x_4 \leq -\frac{1}{2}$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$0x_1 + 0x_2 + \frac{9}{32}x_3 + \frac{1}{32}x_4 \geq \frac{1}{2}$

Cutting plane

		C	2	3	0	0	0
			x_1	x_2	x_3	x_4	
i	Base		A_1	A_2	A_3	A_4	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{2}$
	$z_j - c_j$		0	0	$\frac{31}{32}$	$\frac{7}{32}$	$\frac{31}{2}$

$$0x_1 + 0x_2 + \frac{9}{32}x_3 + \frac{1}{32}x_4 - x_5 + \bar{x}_6 = \frac{1}{2}$$

Cutting plane

		C	2	3	0	0	0	-M	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	0	0	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{5}{2}$
3	A_6	-M	0	0	$\frac{9}{32}$	$\frac{1}{32}$	-1	1	$\frac{1}{2}$
	$z_j - c_j$		0	0	$\frac{31-9M}{32}$	$\frac{7-M}{32}$	M	0	$\frac{31-M}{2}$

Cutting plane

		C	2	3	0	0	0	-M	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_2	0	0	1	$\frac{9}{32}$	$\frac{1}{32}$	0	0	$\frac{7}{2}$
2	A_1	0	1	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{5}{2}$
3	A_6	-M	0	0	$\frac{9}{32}$	$\frac{1}{32}$	-1	1	$\frac{1}{2}$
	$z_j - c_j$		0 0	0 0	$\frac{31}{32}$ $\frac{-9M}{32}$	$\frac{7}{32}$ $\frac{-M}{32}$	0 M	0 0	$\frac{31}{2}$ $-\frac{M}{2}$

Cutting plane

		C	2	3	0	0	0	-M	0
			x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_2	0	0	1	0	$\frac{1}{32}$	0	0	$\frac{7}{2}$
2	A_1	0	1	0	0	$\frac{1}{16}$	0	0	$\frac{5}{2}$
3	A_3	-M	0	0	1	$\frac{1}{9}$	$-\frac{32}{9}$	$\frac{32}{9}$	$\frac{16}{9}$
	$z_j - c_j$		0	0	0	$\frac{7-M}{32}$	M	0	$\frac{31}{2}$

Practical aspects of integer programming

- NP-complete
- Gomory algorithm has halting property
- 01-programming
- Multiple cutting plane