Linear Programming - Simplex

Operations research

Paweł Obszarski

Gausian eliminiation

$$\begin{cases} x_1 + x_2 + x_3 & = & 6 \\ 3x_1 + 2x_2 + 2x_3 & = & 13 \\ 2x_1 + x_2 + 2x_3 & = & 10 \end{cases}$$

The Standard Maximum Problem

Find

$$X = [x_1, x_2, ..., x_n]^T \tag{1}$$

Subject to

$$\begin{array}{ccccc} a_{11}x_1 + a_{12}x_2 + & \ldots & + a_{1n}x_n & \leq & a_{10} \\ a_{21}x_1 + a_{22}x_2 + & \ldots & + a_{2n}x_n & \leq & a_{20} \\ & \ldots & & & \\ a_{m1}x_1 + a_{m2}x_2 + & \ldots & + a_{mn}x_n & \leq & a_{m0} \end{array} \tag{2}$$

$$x_1 \geq 0, x_2 \geq 0 \ldots x_n \geq 0 \tag{3}$$

Maximize objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{4}$$

Properties

THEOREM 7 Point P is a corner point of LP feasible solution if and only if it sharply satisfies n linearly independent inequalities.

Standard Equation Problem

Determine vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$ that maximizes

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & a_{10} \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & a_{20} \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & a_{m0} \end{array}$$

$$x_1 \geq 0, \ldots, x_n \geq 0$$

Standard Matrix Representation

Find $X \in R^n$, subject to

$$A \cdot X = A_0$$

$$X \geq 0_n$$

where

$$z = C \cdot X$$

gets its maximum.
Where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ 0_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ C = [c_1, c_2, \ldots, c_n]$$

Standard Vector Representation

Find $X \in R^n$, that maximizes function

$$z = C \cdot X$$

subject to

$$A_1 x_1 + A_2 x_2 + \ldots + A_n x_n = A_0$$

$$X \geq 0_n$$

Where

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix},\ 0_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ C = [c_1, c_2, \ldots, c_n]$$

Properties

THEOREM 6. The most important theorem A vector $x = [x_1, x_2, ..., x_n]^T$ represents the coordinates of a feasible corner point if and only if in the linear combination

$$A_1x_1 + A_2x_2 + ... + A_nx_n = A_0$$

if A_j is an independent vector then $x_j > 0$ (so if A_j is a dependent vector then $x_j = 0$).

Standard Simplex Representation

Find $X \in R^n$ that maximizes the function

$$z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to:

$$\begin{array}{ccccccccc}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n-m}x_{n-m} & + x_{n-m+1} & & & = a_{1,0} \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n-m}x_{n-m} & & + x_{n-m+2} & & = a_{2,0} \\
\vdots \quad \vdots \quad \ddots \quad \vdots & & \ddots & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{m,n-m}x_{n-m} & & & + x_n & = a_{m,0}
\end{array}$$

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, a_{1,0} \geq 0, a_{2,0} \geq 0, \ldots, a_m, 0 \geq 0$$

Standard Simplex Representation - Example

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 4x_1 + 6x_2$$

subject to

$$\begin{aligned} 6x_1 + 8x_2 &\leq 48 \\ 10x_1 + 6x_2 &\leq 60 \\ 5x_1 + 15x_2 &\leq 75 \end{aligned}$$

$$x_1 \geq 0, \ x_2 \geq 0$$

Standard Simplex Representation - Example

Find $X = [x_1, x_2, x_3, x_4, x_5]^T$ that maximizes

$$z = 4x_1 + 6x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to

$$\begin{array}{rcrcrcrcrcl} 6x_1 & + & 8x_2 & + & x_3 & & & & & = & 48 \\ 10x_1 & + & 6x_2 & & & + & x_4 & & & = & 60 \\ 5x_1 & + & 15x_2 & & & & & + & x_5 & = & 75 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

Standard Simplex (matrix) Representation

Find $X \in R^n$ that maximizes function

$$z = C \cdot X$$

subject to

$$A \cdot X = A_0$$

$$X \geq 0_n$$

$$A_0 \geq 0_n$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-m} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2,n-m} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{m,n-m} & 0 & 0 & \dots & 1 \end{bmatrix}, \; 0_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; C = [c_1, c_2, \dots, c_n]$$

Standard Simplex (vector) representation

Find $X \in R^n$
that maximizes function

$$z = C \cdot X$$

subject to

$$A_1 x_1 + A_2 x_2 + \ldots + A_n x_n = A_0$$

$$X \geq 0_n$$

$$A_0 \geq 0_n$$

where $A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, $0_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $C = [c_1, c_2, \ldots, c_n]$

Simplex tableau

			c_1	c_2	$\dots$	c_{n-m}	c_{n-m+1}	c_{n-m+2}	$\dots$	c_n	c_0
i	base	c_i	A_1	A_2	$\dots$	A_{n-m}	A_{n-m+1}	A_{n-m+2}	$\dots$	A_n	A_0
1	A_{n-m+1}	c_{n-m+1}	$x_{1,1}$	$x_{1,2}$	$\dots$	$x_{1,n-m}$	1	0	$\dots$	0	$x_{1,0}$
2	A_{n-m+2}	c_{n-m+2}	$x_{2,1}$	$x_{2,2}$	$\dots$	$x_{2,n-m}$	0	1	$\dots$	0	$x_{2,0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
m	A_n	c_n	$x_{m,1}$	$x_{m,2}$	$\dots$	$x_{m,n-m}$	0	0	$\dots$	1	$x_{m,0}$
	$z_k - c_k$		$z_1 - c_1$	$z_2 - c_2$	$\dots$	$z_{m,n-m} - c_{m,n-m}$	0	0	$\dots$	0	$z_0 - c_0$

Simplex tableau - Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	6	8	1	0	0	48
2	A_4	0	10	6	0	1	0	60
3	A_5	0	5	15	0	0	1	75
	$z_j - c_j$		-4	-6	0	0	0	0

1. If $z_j - c_j \geq 0$ for all j $(1 \leq j \leq n)$ then the optimal (maximal) solution was found – stop the algorithm. Otherwise go to step 2.
2. Find that k which satisfies the condition

$$z_k - c_k = \min_{1 \leq j \leq n} (z_j - c_j)$$

3. Find that l which satiesfies the condition

$$\frac{x_{j0}}{x_{lk}} = \min_{x_{ik} > 0} \frac{x_{i0}}{x_{ik}}$$

If $x_{ik} \leq 0$ for all i $(1 \leq i \leq m)$ then the objective function reaches $+\infty$, so stop the algorithm. Otherwise go to step 4.

4. Calculate new coefficiants of the simplex tableou according to formulas (determine a new corner-point feasible solution)

$x'_{ij} = x_{ij} - \frac{x_{lj}}{x_{lk}} x_{ik}$ for $i = 1, 2, ..., l-1, l+1, ..., m+1$ and $j = 1, 2, ..., n, 0$
$x'_{lj} = \frac{x_{lj}}{x_{lk}}$ for $j =, 1, 2, ..., n, 0$
where $x'_{m+1,j} = z'_j - c_j$ for $j = 1, 2, ..., n, 0$.
Come back to step 1.

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	6	8	1	0	0	48
2	A_4	0	10	6	0	1	0	60
3	A_5	0	5	15	0	0	1	75
	$z_j - c_j$		-4	-6	0	0	0	0

$X^{(0)} = [0, 0, 48, 60, 75]$

$z(x^{(0)}) = 0$

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	6	8	1	0	0	48
2	A_4	0	10	6	0	1	0	60
3	A_5	0	5	15	0	0	1	75
	$z_j - c_j$		-4	-6	0	0	0	0

$X^{(0)} = [0, 0, 48, 60, 75]$

$z(x^{(0)}) = 0$

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	$\frac{10}{3}$	0	1	0	$-\frac{8}{15}$	8
2	A_4	0	8	0	0	1	$-\frac{6}{15}$	30
3	A_2	6	$\frac{1}{3}$	1	0	0	$\frac{1}{15}$	5
	$z_j - c_j$		-2	0	0	0	$\frac{2}{5}$	30

$X^{(1)} = [0, 5, 8, 30, 0]$

$z(x^{(1)}) = 30$

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	$\frac{10}{3}$	0	1	0	$-\frac{8}{15}$	8
2	A_4	0	8	0	0	1	$-\frac{6}{15}$	30
3	A_2	6	$\frac{1}{3}$	1	0	0	$\frac{1}{15}$	5
	$z_j - c_j$		-2	0	0	0	$\frac{2}{5}$	30

$X^{(1)} = [0, 5, 8, 30, 0]$

$z(x^{(1)}) = 30$

Example

		C	4	6	0	0	0	0
			x_1	x_2	x_3	x_4	x_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_1	4	1	0	$\frac{3}{10}$	0	$-\frac{8}{50}$	$\frac{12}{5}$
2	A_4	0	0	0	$-\frac{24}{10}$	1	$\frac{44}{50}$	$\frac{4}{5}$
3	A_2	6	0	1	$-\frac{1}{10}$	0	$\frac{6}{50}$	$\frac{21}{5}$
	$z_j - c_j$		0	0	$\frac{3}{5}$	0	$\frac{2}{25}$	$34\frac{4}{5}$

$X^{(2)} = [\frac{12}{5}, \frac{21}{5}, 0, \frac{4}{5}, 5]$

$z(x^{(2)}) = 34\frac{4}{5}$

Explanation 1. How to get from one corner point to another?

Explanation 1. How to get better corner point?