Linear Programming - Introduction and Applications

Operations research

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Bibliography

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1. History

2. Definition

3. Applications and examples

    - Profit maximization
    - Diet problem
    - Transportation problem
    - Assignment problem
    - Problem optymalnego wykroju
    - Maximal flow problem

History

- 1823 Fourier - the unpublished foundations
- 1911 Poussin - article, same problem as with Fourier
- 1939 Leonid Kantorovich (Russia) - Worked in USSR and discovered the model, again unpublished in the western countries
- II World War Dantzig works for US Air Force, logistics, scheduling
- 1947 George B. Dantzig - simplex method published

- 1947 John von Neumann - duality

- 1975 Leonid Kantorovich (USSR) Tjalling Koopmans (USA) Nobel price in economics, for applications of LP

- 1979 Leonid Khachiyan (Chaczijan) - polynomial algorithm, better then simplex from 1000 constraints and 50 000 variables.

- 1984 Narendra Karmarkar - interior point method, polynomial and more efficient similar to simplex

Definition of linear programing

Let us consider vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Which satisfies the conditions

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} a_{i0} \quad i = 1, 2, ..., m \tag{1}$$

$$x_1 \geq 0, x_2 \geq 0 \ldots x_n \geq 0 \tag{2}$$

And maximized (minimized) objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{3}$$

Terminology

- Objective function
- Decision variables
- Coefficients (cost, technological) - parameters
- Constraints
- Feasible solution

- Feasible region

- Optimal solution

Example 0 Jasiu is obligated to buy at least 10 kg of potato. Potato costs 2 PLN per kg. Jasiu wants to save money. How much will he buy and spend?

Example 1

Let us consider an enterprise which produces two different goods W_1 and W_2 (for example carriages and boats). Only three different resources are used S_1, S_2 and S_3 (for example: Labor force, wood, ropes).

	Resources		
	S_1	S_2	S_3
Product W_1	6	10	5
Product W_2	8	6	15
Amount of resource	48	60	75

The requirements are stated in the table above. The profit from selling one peace of first product is 4 units of money, and profit from second good W_2 is 6.

Assumptions

- Proportionality
- Additivity
- Divisibility
- Determinism

The Standard Maximum Problem

Find

$$X = [x_1, x_2, ..., x_n]^T \tag{4}$$

Subject to

$$\begin{array}{cccccc} a_{11}x_1 + a_{12}x_2 + & \dots & +a_{1n}x_n & \leq & a_{10} \\ a_{21}x_1 + a_{22}x_2 + & \dots & +a_{2n}x_n & \leq & a_{20} \\ & \dots & & & \\ a_{m1}x_1 + a_{m2}x_2 + & \dots & +a_{mn}x_n & \leq & a_{m0} \end{array} \tag{5}$$

$$x_1 \geq 0, x_2 \geq 0 \dots x_n \geq 0 \tag{6}$$

Maximize objective function

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \tag{7}$$

The Standard Minimum Problem

Find

$$X = [x_1, x_2, ..., x_n]^T \tag{8}$$

Subject to

$$\begin{array}{ccccc} a_{11}x_1 + a_{12}x_2 + & \dots & +a_{1n}x_n & \geq & a_{10} \\ a_{21}x_1 + a_{22}x_2 + & \dots & +a_{2n}x_n & \geq & a_{20} \\ & \dots & & & \\ a_{m1}x_1 + a_{m2}x_2 + & \dots & +a_{mn}x_n & \geq & a_{m0} \end{array} \tag{9}$$

$$x_1 \geq 0, x_2 \geq 0 \dots x_n \geq 0 \tag{10}$$

Minimize objective function

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \tag{11}$$

The Standard Equality Problem

Find

$$X = [x_1, x_2, ..., x_n]^T \tag{12}$$

Subject to

$$\begin{array}{rcrcl} a_{11}x_1 + a_{12}x_2+ & \dots & +a_{1n}x_n & = & a_{10} \\ a_{21}x_1 + a_{22}x_2+ & \dots & +a_{2n}x_n & = & a_{20} \\ & \dots & & & \\ a_{m1}x_1 + a_{m2}x_2+ & \dots & +a_{mn}x_n & = & a_{m0} \end{array} \tag{13}$$

$$x_1 \geq 0, x_2 \geq 0 \dots x_n \geq 0 \tag{14}$$

Maximize (minimize) objective function

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \tag{15}$$

Properties

THEOREM 1. Objective function of the linear programming problem is a convex function (at the same time is a concave function too).

Properties

THEOREM 2. Feasible region of the linear programming problem is a convex set.

Properties

THEOREM 3. Every point P of the feasible region of the linear programming problem may be expressed as convex linear combination of feasible corner points P_1, P_2, ... , P_s, that is $P = \lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_s P_s$, where $\lambda_1 \geq 0, \lambda_2 \geq 0, ..., \lambda_s \geq 0$ and $\lambda_1 + \lambda_2 + ... + \lambda_s = 1$.

Properties

THEOREM 4. The objective function of a linear programming problem achieves the maximum value at a feasible corner point.

Properties

THEOREM 5. If the objective function of a linear programming problem achieves the maximum value at s ($s \geq 2$) feasible corner points $P_1, P_2, ..., P_s$ then it achieves the maximum value at every point that is a convex linear combination of $P_1, P_2, ..., P_s$.

Example 3

Find $x = [x_1, x_2]^T$ subject to:

$x_1 + x_2 \leq 1$

$x_1 \geq 0$ $x_2 \geq 0$

That mimimizes:

a)$z = -x_1 - 2x_2 + 3$

b)$z = x_2 + 1$

c)$z = 2$

Properties

THEOREM 6. The most important theorem A vector $x = [x_1, x_2, ..., x_n]^T$ represents the coordinates of a feasible corner point if and only if in the linear combination

$$A_1 x_1 + A_2 x_2 + ... + A_n x_n = A_0$$

if A_j is an independent vector then $x_j > 0$ (so if A_j is a dependent vector then $x_j = 0$).

Definition of mathematical programing

For vector $X = [x_1, x_2, \ldots, x_n]^T$ maximize (minimize) objective function

$$z = f(x_1, x_2, \ldots, x_n) \tag{16}$$

Subject to

$$g_i(x_1, x_2, \ldots, x_n) \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} 0 \quad i = 1, 2, ..., m \tag{17}$$

Feasible region set $D = \left\{ X \in R^2 : g_i(x_1, \ldots, x_n) \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} 0, i = 1, ..., m, \right\}$.

Gradient vector

$$\nabla f(x_1, x_2, \ldots, x_n) = \left[\frac{\delta f(X)}{\delta x_1}, \frac{\delta f(X)}{\delta x_2}, \ldots, \frac{\delta f(X)}{\delta x_n}\right]$$

Fermat theorem - necessary condition for extreme function values If function $f(x_1, x_2, \ldots, x_n)$ reaches local maximum (minimum) in the interior point $\hat{X} = [\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_n}] \in D \subset R^n$ $\hat{X}$, where D is a domain of function f, then $\nabla f(\hat{X}) = [0, 0, \ldots, 0]$. In other words $\frac{\delta f(\hat{X})}{\delta x_1} = 0$, $\frac{\delta f(\hat{X})}{\delta x_2} = 0$, ... , $\frac{\delta f(\hat{X})}{\delta x_n} = 0$.

Applications

Profit maximization

An enterprise has m factors of production (ex. machines, resources, workers) each in limited amount a_{i0}.

Enterprise produces n different product. There should be produced at least p_i and at most q_i peaces of i-th product ($i = 1, 2, \ldots, n$).

The matrix bellow expresses the requirements a_{ij} of i-th factor of production to produce j-th product.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Each product brings profit c_i ($i = 1, 2, \ldots, n$).

How should we distribute factors of production to reach the highest total profit.

Economic interpretation, Production optimization Consider an enterprise that produces an unified product. It has m different raw materials $S_1, S_2, \ldots, S_m$ which are available in monthly (daily, annula) reserves, respectively a_{i0}. Product can by produced in n different manners, $x_1, x_2, \ldots, x_n$ express the numbers (quantities) time units spent on the certain production manner. The parameter a_{ij} in the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

says how much of i-th material is used for j-th production manner in an unit of time.

Finally parameters $c_1, c_2, \ldots, c_n$ represent the number of goods pro-

duced in the $i-th$ manner in one unit of time.
Hom to maximize the production? What is the maximal value of it?

Feed mix (die) problem (Blending aviation gasoline 1952)
A mill produces feed (for chickens for example). This requires mixing several ingredients (ex. corn, hay, wheat etc.). The final mixture should consist minimal quantities of some nutrients (ex. protein, fat, calcium, carbohydrates, vitamins etc.). We know those amounts a_{i0} (maximal, minimal or exact) of nutrients, that feed should consist. Let's assume that we have n ingredients and m nutrients. Matrix A determines the amounts a_{ij} of i-th nutrient in j-th ingredient. We also know the unit cost c_i of i-th ingredient. How to minimize the total cost?

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Example

Let us consider two coal types: A and B. Both are polluted with ash and phosphorus.

We need 90 t of fuel containing at most 0.03% of phosphorus and 4% of ash.

The table shows the prices of different types of coal and the percentage rate of pollution.

	Pollution		
	phosphorus	ash	price
A	0.02	3	100
B	0.05	5	80

How much coal of different types should be mixed to minimize the cost and satisfy the restrictions?

Example

Stock-farm needs feed for the cattle in prespecified amounts. Farm needs to determine the proportions of two ingredients in the mix. Ingredients are limited. They have 16 kg of first ingredient per day and 10 kg of second. There are two nutrients S_1 i S_2, mix should consist at least of 3.3 kg of S_1 and at most 2.6 kg of S_2 per day. Table shows the percentage amounts of nutrients in ingredient.

	Percentage amounts of nutrients	
	S_1	S_2
Ingredient I	30	20
Ingredient II	10	20

Compose the mixture to minimize cost, sine I ingredient costs 2 zł per kg and II ingredient costs 5 zł per kg.

Rucksack problem (knapsack problem)
There are n different commodities $r_1, r_2, \ldots, r_n$ each in unrestricted number. Peace of r_i weights w_i kg and is worth p_i units of money. The capacity (strength) W of the rucksack is known in advance. Determine the numbers of each commodity that fit the rucksack and maximize walue. In general rucksack problem can by applied in many practical cases, ex. loading of train, container, car, case.

Example

Solve the rucksack problem for the date in table bellow.

i	1	2	3	W
p_i	9	6	4	
w_i	7	5	3	20

Transportation problem

m dostawców posiada jednorodny towar odpowiednio w ilościach $a_1, a_2, \ldots, a_m$ jednostek. Całą masę towarową należy przetransportować do n odbiorców, których zapotrzebowanie odpowiednio wynosi $b_1, b_2, \ldots, b_n$ jednostek. Znana jest macierz C, w której element c_{ij} oznacza koszt transportu jednostki towaru od i-tego dostawcy do j-tego odbiorcy.

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

W jaki sposób należy zorganizować transport by całkowity koszt był minimalny?

Example

A transportation company has two bases B_1 and B_2 from which it sends buses to tree bus stations D_1, D_2, D_3. Transit between base and bus station is considered as a wast. In the table you can see the distances, numbers of buses in bases and requirements.

Bases	Bus stations D_1	D_2	D_3	Number of buses
B_1	15	12	10	80
B_2	5	18	24	120
requirement	40	65	45	

Find the optimal buses division.

Assignemstn problem

1955 Khun Hungarian algorithm

1890 Carl Gustav Jacobi discovered again in 2006

Example

	P_1	P_2	P_3	P_4
K_1	210	450	390	330
K_2	270	390	450	330
K_3	270	510	390	390
K_4	330	450	330	450

Cutting Stock Problem (Kantorovich 1939)
Let us consider a manufacturer who produces sheet of material (ex. steel, wood) of standard size. An order is placed by a customer who needs sheets of different sizes. Assume that m types of sheets are to be produced. In particular, we need b_i sheets of i-th type. A standard sheet can by cut in n different ways. j cutting pattern gives a_{ij} peaces of sheet of type i. Scraps can not be recycled so we want to minimize the amount of wasts or we simply want to minimize the number of used standard sheets.

Example

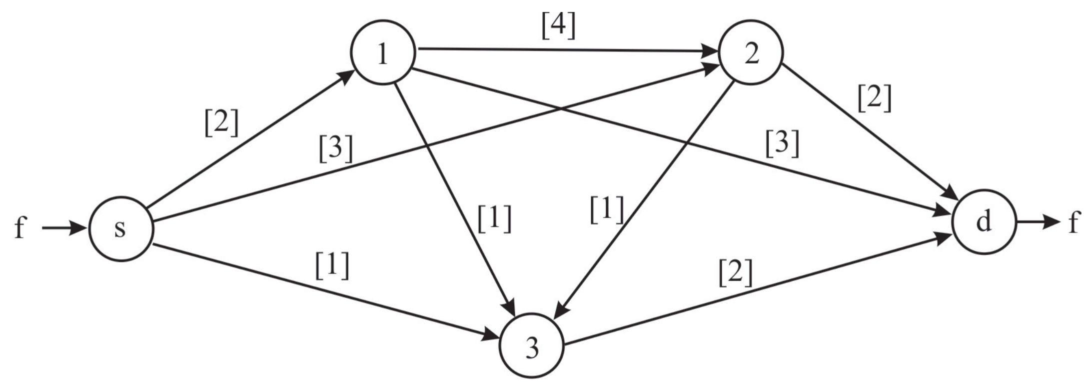
Sawmill was placed an order for 300 sets of desks (for desks for example). Each set consists of 7 desks of length 0.7 m and 4 of length 2.5 m. How should the standard desks be cut to minimize waste?

Maximum flow in network

Consider a directed weighted graph without cycles. If there is one staring and one finishing vertex what is the maximal flow, between them.

Example

Let us consider a network of oil pipelines modeled by the digraph bellow. Weights on arcs stand for the capacity of pipeline. Oil can be sent in only one direction. Hom much oil can be sent from point s to d?



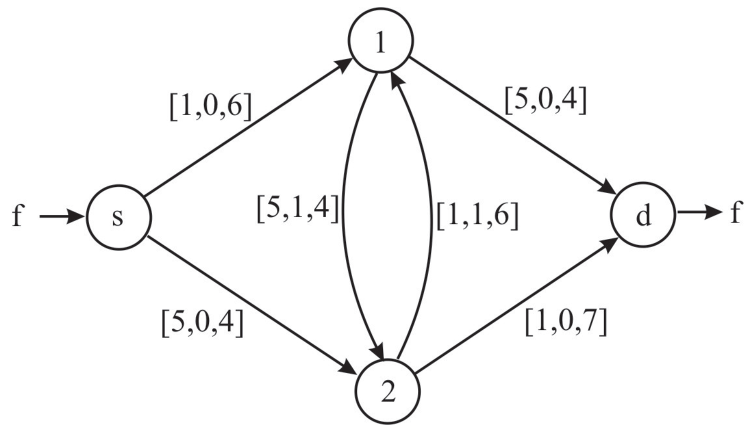
Minimal cost of flow in network

Similarly as on the previous slide, we have a directed network. Here each arc (i, j) is assigned a triple $[c_{ij}, d_{ij}, g_{ij}]$, where c_{ij} stands for the cost of transporting one entity along the arc (i, j), d_{ij} is the minimal and g_{ij} is the maximal amount of resource.

The aim is to minimize the cost of transporting f units of resources from s to d.

Example

Calculate the minimal cost of flow f from point s to d.



Aircraft rooting (Ferguson 1955)