



Linear Programming - Simplex and applications

Operations research

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Adapting to other model forms

Equality Constraints

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 7x_1 + 4x_2$$

subject to

$$\begin{array}{rclcl} x_1 & + & x_2 & \leq & 11 \\ & & x_2 & \leq & 4 \\ 2x_1 & + & 3x_2 & = & 24 \end{array}$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Equality Constraints

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 7x_1 + 4x_2 + 0x_3 + 0x_4$$

subject to

$$\begin{array}{rcccccl} x_1 & + & x_2 & + & x_3 & & = & 11 \\ & & x_2 & & & + & x_4 & = & 4 \\ 2x_1 & + & 3x_2 & & & & & = & 24 \end{array}$$

$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0$$

Equality Constraints

Big M method.

Numerical stability.

Equality Constraints

Find $X = [x_1, x_2]^T$ that maximizes

$$z = 7x_1 + 4x_2 + 0x_3 + 0x_4 - M\bar{x}_5$$

subject to

$$\begin{array}{rcccccccl} x_1 & + & x_2 & + & x_3 & & & = & 11 \\ & & x_2 & & & + & x_4 & = & 4 \\ 2x_1 & + & 3x_2 & & & & + & \bar{x}_5 & = & 24 \end{array}$$

$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0, \ \bar{x}_5 \geq 0$$

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	\bar{x}_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	1	1	0	0	11
2	A_4	0	0	1	0	1	0	4
3	A_5	-M	2	3	0	0	1	24
	$-c_j$		-7	-4	0	0	M	0

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	\bar{x}_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	1	1	0	0	11
2	A_4	0	0	1	0	1	0	4
3	A_5	-M	2	3	0	0	1	24
	$z_j - c_j$		-2M-7	-3M-4	0	0	0	-24M

$$X^0 = [0, 0, 11, 4, 24]^T$$

$$z(X^0) = -24M$$

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	\bar{x}_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	1	1	0	0	11
2	A_4	0	0	1	0	1	0	4
3	A_5	-M	2	3	0	0	1	24
	$z_j - c_j$		-2M-7	-3M-4	0	0	0	-24M

$$X^0 = [0, 0, 11, 4, 24]^T$$

$$z(X^0) = -24M$$

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	\bar{x}_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	0	1	-1	0	7
2	A_2	4	0	1	0	1	0	4
3	A_5	-M	2	0	0	-3	1	12
	$z_j - c_j$		-2M-7	0	0	3M+6	0	-12M+16

$$X^0 = [0, 4, 7, 0, 12]^T$$

$$z(X^0) = -12M + 16$$

Equality Constraints - Example

		C	7	4	0	0	-M	0
			x_1	x_2	x_3	x_4	\bar{x}_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	1	0	1	-1	0	7
2	A_2	4	0	1	0	1	0	4
3	A_5	-M	2	0	0	-3	1	12
	$z_j - c_j$		-2M-7	0	0	3M+6	0	-12M+16

$$X^0 = [0, 4, 7, 0, 12]^T$$

$$z(X^0) = -12M + 16$$

Equality Constraints - Example

		C	7	4	0	0	$-M$	0
			x_1	x_2	x_3	x_4	\bar{x}_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1
2	A_2	4	0	1	0	1	0	4
3	A_1	7	1	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	6
	$z_j - c_j$		0	0	0	$-\frac{13}{2}$	$M + \frac{7}{2}$	58

$$X^0 = [6, 4, 1, 0, 0]^T$$

$$z(X^0) = 58$$

Equality Constraints - Example

		C	7	4	0	0	$-M$	0
			x_1	x_2	x_3	x_4	\bar{x}_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1
2	A_2	4	0	1	0	1	0	4
3	A_1	7	1	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	6
	$z_j - c_j$		0	0	0	$-\frac{13}{2}$	$M + \frac{7}{2}$	58

$$X^0 = [6, 4, 1, 0, 0]^T$$

$$z(X^0) = 58$$

Equality Constraints - Example

		C	7	4	0	0	$-M$	0
			x_1	x_2	x_3	x_4	\bar{x}_5	
i	Base		A_1	A_2	A_3	A_4	A_5	A_0
1	A_4	0	0	0	2	1	-1	2
2	A_2	4	0	1	-2	0	1	2
3	A_1	7	1	0	3	0	-1	9
	$z_j - c_j$		0	0	13	0	$M - \frac{5}{2}$	71

$$X^0 = [9, 2, 0, 2, 0]^T$$

$$z(X^0) = 71$$

Radiation therapy (Hilier)

In radiation therapy doses to which the tissues are exposed are crucial. Specialists are able to estimate the results of exposure to certain beams (intensity and direction) to both healthy body parts and cancer. Table shows the estimations absorption of radiation.

Area	Beam 1.	Beam 2.	restrictions
Healthy anatomy	0.4	0.5	Minimize
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	$=6$
Center of tumor	0.6	0.4	≥ 6

Adjust the strength of beams (in kilorads).

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0,4x_1 + 0,5x_2$$

subject to

$$0,3x_1 + 0,1x_2 \leq 2,7$$

$$0,5x_1 + 0,5x_2 = 6$$

$$0,6x_1 + 0,4x_2 \geq 6$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0,4x_1 + 0,5x_2$$

subject to

$$0,3x_1 + 0,1x_2 \leq 2,7$$

$$0,5x_1 + 0,5x_2 = 6$$

$$0,6x_1 + 0,4x_2 \geq 6$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0,4x_1 + 0,5x_2 + 0x_3 + M\bar{x}_4$$

subject to

$$\begin{array}{rclcl} 0,3x_1 & + & 0,1x_2 & + & x_3 & = & 2,7 \\ 0,5x_1 & + & 0,5x_2 & & + \bar{x}_4 & = & 6 \\ 0,6x_1 & + & 0,4x_2 & & & \geq & 6 \end{array}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0$$

Constraints in \geq form

-> equation -> \leq
negative right hand side

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0,4x_1 + 0,5x_2 + 0x_3 + M\bar{x}_4 + 0x_5$$

subject to

$$\begin{array}{rclclcl} 0,3x_1 & + & 0,1x_2 & + & x_3 & & = & 2,7 \\ 0,5x_1 & + & 0,5x_2 & & & + & \bar{x}_4 & = & 6 \\ 0,6x_1 & + & 0,4x_2 & & & & - & x_5 & = & 6 \end{array}$$

$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ \bar{x}_4 \geq 0, \ x_5 \geq 0$$

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that minimizes

$$z = 0,4x_1 + 0,5x_2 + 0x_3 + M\bar{x}_4 + 0x_5 + M\bar{x}_6$$

subject to

$$\begin{array}{rclclcl} 0,3x_1 & + & 0,1x_2 & + & x_3 & & = & 2,7 \\ 0,5x_1 & + & 0,5x_2 & & & + & \bar{x}_4 & = & 6 \\ 0,6x_1 & + & 0,4x_2 & & & & - & x_5 & + & \bar{x}_6 & = & 6 \end{array}$$

$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ \bar{x}_4 \geq 0, \ x_5 \geq 0, \ \bar{x}_6 \geq 0$$

Minimalization

$$\min z = - \max -z$$

Radiation therapy (Hilier)

Find $X = [x_1, x_2]^T$ that maximizes

$$z' = -0,4x_1 - 0,5x_2 + 0x_3 - M\bar{x}_4 + 0x_5 - M\bar{x}_6$$

subject to

$$\begin{array}{rclclclcl} 0,3x_1 & + & 0,1x_2 & + & x_3 & & & = & 2,7 \\ 0,5x_1 & + & 0,5x_2 & & & + & \bar{x}_4 & = & 6 \\ 0,6x_1 & + & 0,4x_2 & & & & - & x_5 & + & \bar{x}_6 & = & 6 \end{array}$$

$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ \bar{x}_4 \geq 0, \ x_5 \geq 0, \ \bar{x}_6 \geq 0$$

Radiation therapy (Hilier)

		C	-0,4	-0,5	0	-M	0	-M	0
			x_1	x_2	x_3	\bar{x}_4	x_5	\bar{x}_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0,3	0,1	1	0	0	0	2,7
2	A_4	-M	0,5	0,5	0	1	0	0	6
3	A_6	-M	0,6	0,4	0	0	-1	1	6
	$-c_j$		0,4	0,5	0	M	0	M	0

Example - Radiation therapy (Hillier)

		C	-0,4	-0,5	0	-M	0	-M	0
			x_1	x_2	x_3	\bar{x}_4	x_5	\bar{x}_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0,3	0,1	1	0	0	0	2,7
2	A_4	-M	0,5	0,5	0	1	0	0	6
3	A_6	-M	0,6	0,4	0	0	-1	1	6
	$z_j - c_j$		0,4 -1,1M	0,5-0,9M	0	0	M	0	-12M

$$X^0 = [0, 0, 2.7, 6, 0, 6]^T$$

$$z'(X^0) = -12M$$

Example - Radiation therapy (Hillier)

		C	-0,4	-0,5	0	-M	0	-M	0
			x_1	x_2	x_3	\bar{x}_4	x_5	\bar{x}_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_3	0	0,3	0,1	1	0	0	0	2,7
2	A_4	-M	0,5	0,5	0	1	0	0	6
3	A_6	-M	0,6	0,4	0	0	-1	1	6
	$z_j - c_j$		0,4 -1,1M	0,5-0,9M	0	0	M	0	-12M

$$X^0 = [0, 0, 2.7, 6, 0, 6]^T$$

$$z'(X^0) = -12M$$

Example - Radiation therapy (Hilier)

		C	-0,4	-0,5	0	-M	0	-M	0
			x_1	x_2	x_3	\bar{x}_4	x_5	\bar{x}_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0,4	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
2	A_4	-M	0	$\frac{1}{3}$	$\frac{-5}{3}$	1	0	0	1,5
3	A_6	-M	0	0,2	-2	0	-1	1	0,6
	$z_j - c_j$		0	$\frac{11-16M}{30}$	$\frac{11M-4}{3}$	0	M	0	-2,1M- 3,6

$$X^0 = [9, 0, 0, 1.5, 0, 0.6]^T$$

$$z'(X^0) = -2,1M - 3,6$$

Example - Radiation therapy (Hillier)

		C	-0,4	-0,5	0	-M	0	-M	0
			x_1	x_2	x_3	\bar{x}_4	x_5	\bar{x}_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0,4	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
2	A_4	-M	0	$\frac{1}{3}$	$-\frac{5}{3}$	1	0	0	1,5
3	A_6	-M	0	0,2	-2	0	-1	1	0,6
	$z_j - c_j$		0	$\frac{11-16M}{30}$	$\frac{11M-4}{3}$	0	M	0	-2,1M- 3,6

$$X^0 = [9, 0, 0, 1.5, 0, 0.6]^T$$

$$z'(X^0) = -2,1M - 3,6$$

Example - Radiation therapy (Hillier)

		C	-0,4	-0,5	0	-M	0	-M	0
			x_1	x_2	x_3	\bar{x}_4	x_5	\bar{x}_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0,4	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	8
2	A_4	-M	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	$-\frac{5}{3}$	0,5
3	A_2	-0,5	0	1	-10	0	-5	5	3
	$z_j - c_j$		0	0	$\frac{7-5M}{3}$	0	$\frac{11-10M}{6}$	$\frac{16M-11}{6}$	-0,5M- 4,7

$$X^0 = [8, 3, 0, 1.5, 0, 0]^T$$

$$z'(X^0) = -0.5M - 4.7$$

Example - Radiation therapy (Hillier)

		C	-0,4	-0,5	0	-M	0	-M	0
			x_1	x_2	x_3	\bar{x}_4	x_5	\bar{x}_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0,4	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	8
2	A_4	-M	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	$-\frac{5}{3}$	0,5
3	A_2	-0,5	0	1	-10	0	-5	5	3
	$z_j - c_j$		0	0	$\frac{7-5M}{3}$	0	$\frac{11-10M}{6}$	$\frac{16M-11}{6}$	-0,5M- 4,7

$$X^0 = [8, 3, 0, 1.5, 0, 0]^T$$

$$z'(X^0) = -0.5M - 4.7$$

Example - Radiation therapy (Hilier)

		C	-0,4	-0,5	0	-M	0	-M	0
			x_1	x_2	x_3	\bar{x}_4	x_5	\bar{x}_6	
i	Base		A_1	A_2	A_3	A_4	A_5	A_6	A_0
1	A_1	-0,4	1	0	5	-1	0	0	7,5
2	A_5	0	0	0	1	0,6	1	-1	0,3
3	A_2	-0,5	0	1	-5	3	0	0	4,5
	$z_j - c_j$		0	0	0,5	$M - 1, 1$	0	M	-5,25

$$X^0 = [7.5, 4.5, 0, 0, 0.3, 0]^T$$

$$z'(X^0) = -5.25$$

Negative variables

$$x_i = x'_i - x''_i$$

for $x'_j \geq 0$ and $x''_j \geq 0$

Applications

Profit maximization

An enterprise has m factors of production (ex. machines, resources, workers) each in limited amount a_{i0} .

Enterprise produces n different product. There should be produced at least p_i and at most q_i peaces of i -th product ($i = 1, 2, \dots, n$).

The matrix bellow expresses the requirements a_{ij} of i -th factor of production to produce j -th product.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Each product brings profit c_i ($i = 1, 2, \dots, n$).

How should we distribute factors of production to reach the highest total profit.

Economic interpretation, Production optimization Consider an unified product. It has m different raw materials S_1, S_2, \dots, S_m which are available in monthly (daily, annula) reserves, respectively a_{i0} . Product can be produced in n different manners, x_1, x_2, \dots, x_n express the amount of time spent on the certain production manner. The parameter a_{ij} in the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

says how much of i -th material is used for j -th production manner in an unit of time.

Finally parameters c_1, c_2, \dots, c_n represent the number of goods produced in the i -th manner in one unit of time.

How to maximize the production? What is the maximal value of it?

Feed mix (die) problem (Blending aviation gasoline 1952)

Given n ingredients (I_i for $i = 1, 2, \dots, n$) with certain properties and prices compose the cheapest mixture satisfying prespecified standards. Assume that ingredients contain both desirable and undesirable particles. Amount of a certain particle in the unit of ingredients are given in the table. Minimal and maximal

	P_1	P_2	\dots	P_m	Price
I_1	$a_{1,1}$	$a_{1,2}$	\dots	$a_{1,m}$	c_1
I_2	$a_{2,1}$	$a_{2,2}$	\dots	$a_{2,m}$	c_2
\vdots	\vdots	\vdots		\vdots	\vdots
I_n	$a_{n,1}$	$a_{n,2}$	\dots	$a_{n,m}$	c_n
Limits	$\{\geq, \leq, =\}$	$\{\geq, \leq, =\}$	\dots	$\{\geq, \leq, =\}$	
	$a_{0,1}$	$a_{0,2}$	\dots	$a_{0,m}$	

Example

Let us consider two coal types: A and B. Both are polluted with ash and phosphorus.

We need 90 t of fuel containing at most 0.03% of phosphorus and 4% of ash.

The table shows the prices of different types of coal and the percentage rate of pollution.

	Pollution		price
	phosphorus	ash	
A	0.02	3	100
B	0.05	5	80

How much coal of different types should be mixed to minimize the cost and satisfy the restrictions?

Example

Stock-farm needs feed for the cattle in prespecified amounts. Farm needs to determine the proportions of two ingredients in the mix. Ingredients are limited. They have 16 kg of first ingredient per day and 10 kg of second. There are two nutrients S_1 i S_2 , mix should contain at least of 3.3 kg of S_1 and at most 2.6 kg of S_2 per day. Table shows the percentage amounts of nutrients in ingredient.

	Percentage amounts of nutrients	
	S_1	S_2
Ingredient I	30	20
Ingredient II	10	20

Compose the mixture to minimize cost, sine I ingredient costs 2 zł per kg and II ingredient costs 5 zł per kg.

Rucksack problem (knapsack problem)

There are n different commodities r_1, r_2, \dots, r_n each in unrestricted number. Piece of r_i weights w_i kg and is worth p_i units of money. The capacity (strength) W of the rucksack is known in advance. Determine the numbers of each commodity that fit the rucksack and maximize value. In general rucksack problem can be applied in many practical cases, ex. loading of train, container, car, case.

Example

Alibaba found himself in the cave full of treasures. Unfortunately his camel is able to carry only 40 kg of additional baggage. Alibaba is able to estimate quite precisely both the price and the weight of the commodity. Help him taking the most valuable treasures.

i	1	2	3	W
p_i	9	6	4	
w_i	7	5	3	40

Transportation problem

There are m ports or centres of supply of a certain commodity, and n destinations or markets to which this commodity must be shipped. The i -th port ($i = 1, 2, \dots, m$) possesses an amount a_i of the commodity, and the requirements are such that the j -th destination is to receive the amount b_j of the commodity. The cost of shipping one unit of the commodity from port i to destination j is equal to c_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$). The task is to determine x_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) the quantity of the commodity which must be shipped from port i to destination j to minimize the total cost of shipping.

Example

A transportation company has two bases B_1 and B_2 from which it sends buses to three bus stations D_1, D_2, D_3 . Transit between base and bus station is considered as a waste. In the table you can see the distances, numbers of buses in bases and requirements.

Bases	Bus stations			Number of buses
	D_1	D_2	D_3	
B_1	15	12	10	80
B_2	5	18	24	120
requirement	40	65	45	

Find the optimal buses division.

Assignment problem

n assignees (employees, machines, vehicles, plants, time slots) are being assigned to perform n tasks. The assignment has to satisfy the following assumptions:

- each assignee is to be assigned to exactly one task,
- each task is to be performed by exactly one assignee,
- there is a cost c_{ij} associated with assignee i ($i = 1, 2, \dots, n$) performing task j ($j = 1, 2, \dots, n$).

The objective is to determine how all n assignments should be made to minimize the total cost.

1931 Egervary

1955 Khun Hungarian algorithm

1890 Carl Gustav Jacobi discovered again in 2006

Example

There are four workers and four tasks to be performed. The time in which each worker would complete each task is given in the table. Assign bijectively workers to the tasks in the way that minimizes the total time of work.

	W_1	W_2	W_3	W_4
T_1	210	450	390	330
T_2	270	390	450	330
T_3	270	510	390	390
T_4	330	450	330	450

Hungarian algorithm

1. Subtract the smallest number in each row from every number in the row.
2. Subtract the smallest number in each column from every number in the column.
3. Test whether an optimal assignment can be made. You do this by determining the minimum number of vertical and horizontal lines needed to cross out all zeros. If the minimum number of lines equals the number of rows, an optimal set of assignments is possible. In that case, go to step 6. Otherwise go on to step 4.
4. If the minimum number of lines is less than the number of rows, modify the the table in the following way:

Hungarian algorithm (continued)

- a) Subtract the smallest uncrossed number from every uncrossed number in the table.
 - b) Add the smallest uncrossed number to the numbers at the intersections of crossing lines.
 - c) Numbers cross out but not at the intersections of cross-out lines carry over unchanged to the table.
5. Repeat steps 3 and 4 until an optimal set of assignments is possible.
6. Make the assignments one at a time in positions that have zero elements. Begin with rows and columns that have only one zero. Since each row and each column needs to receive exactly one assignment, cross out both the row and the column involved after each assignment is made. Continue step 6 until every row and every column has exactly one assignment.

Example

Find minimal matching.

Initial table:

	W_1	W_2	W_3	W_4
T_1	210	450	390	330
T_2	270	390	450	330
T_3	270	510	390	390
T_4	330	450	330	450

Example

Subtract the smallest number in each row from every number in the row.

	P_1	P_2	P_3	P_4
K_1	0	240	180	120
K_2	0	120	180	60
K_3	0	240	120	120
K_4	0	120	0	120

Example

Subtract the smallest number in each column from every number in the column.

	P_1	P_2	P_3	P_4
K_1	0	120	180	60
K_2	0	0	180	0
K_3	0	120	120	60
K_4	0	0	0	60

Example

Determine the minimum number of vertical and horizontal lines needed to cross out all zeros.

	P_1	P_2	P_3	P_4
K_1	0	120	180	60
K_2	0	0	180	0
K_3	0	120	120	60
K_4	0	0	0	60

Example

Only three lines are required. Find minimal uncrossed element.

	P_1	P_2	P_3	P_4
K_1	0	120	180	60
K_2	0	0	180	0
K_3	0	120	120	60
K_4	0	0	0	60

Example

The minimal uncrossed element should be:

- subtracted from all uncrossed elements,
- added to all crossed by two lines,
- elements crossed only once remain unchanged.

	P_1	P_2	P_3	P_4
K_1	0	60	120	0
K_2	60	0	180	0
K_3	0	60	60	0
K_4	60	0	0	60

Example

Now you need at least four lines to cross all the zeros.

	P_1	P_2	P_3	P_4
K_1	0	60	120	0
K_2	60	0	180	0
K_3	0	60	60	0
K_4	60	0	0	60

Example

Let us construct the optimal solution:

- one can be placed only on position where there were zeros ,
- in each raw and each column there should be exactly one one.

	P_1	P_2	P_3	P_4
K_1	0			0
K_2		0		0
K_3	0			0
K_4		0	0	

Example

There are two optimal solutions:

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

	P_1	P_2	P_3	P_4
K_1	0			0
K_2		0		0
K_3	0			0
K_4		0	0	

Example

There are two optimal solutions:

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

	P_1	P_2	P_3	P_4
K_1	210	450	390	330
K_2	270	390	450	330
K_3	270	510	390	390
K_4	330	450	330	450

Example

There are two optimal solutions:

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$z = 210 + 390 + 330 + 390 = 1320$$

$$z = 330 + 390 + 270 + 330 = 1320$$

	P_1	P_2	P_3	P_4
K_1	210	450	390	330
K_2	270	390	450	330
K_3	270	510	390	390
K_4	330	450	330	450

How can we check how many lines are necessary to cross all zeros in matrix?

Cutting Stock Problem (Kantorovich 1939)

Let us consider a manufacturer who produces sheet of material (ex. steel, wood) of standard size. An order is placed by a customer who needs sheets of different sizes. Assume that m types of sheets are to be produced. In particular, we need b_i sheets of i -th type. A standard sheet can be cut in n different ways. j cutting pattern gives a_{ij} pieces of sheet of type i . Scraps can not be recycled so we want to minimize the amount of waste or we simply want to minimize the number of used standard sheets.

Example

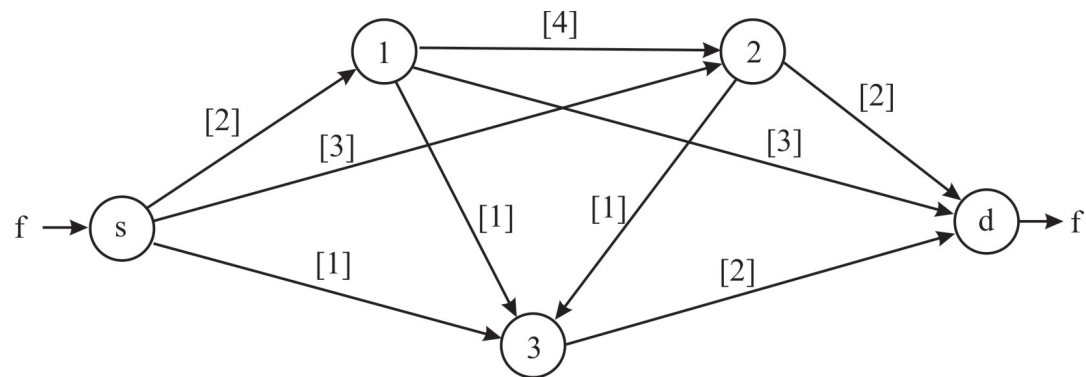
Sawmill was placed an order for 300 sets of desks (for desks for example). Each set consists of 7 desks of length 0.7 m and 4 of length 2.5 m. How should the standard desks be cut to minimize waste?

Maximum flow in network

Consider a directed weighted graph without cycles. If there is one starting and one finishing vertex what is the maximal flow, between them.

Example

Let us consider a network of oil pipelines modeled by the digraph bellow. Weights on arcs stand for the capacity of pipeline. Oil can be sent in only one direction. How much oil can be sent from point s to d ?



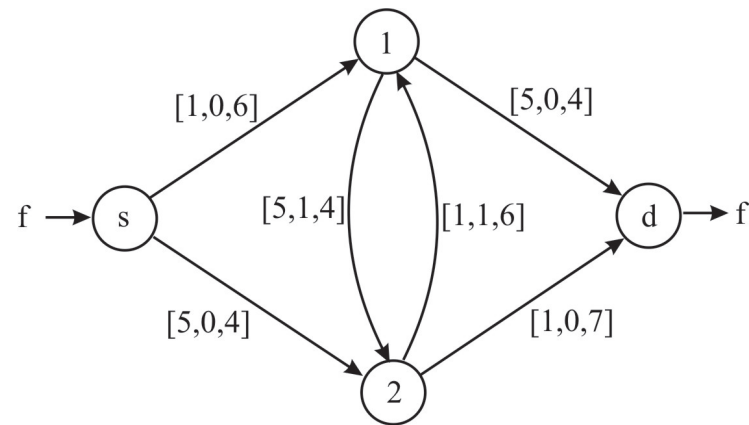
Minimal cost of flow in network

Similarly as on the previous slide, we have a directed network. Here each arc (i, j) is assigned a triple $[c_{ij}, d_{ij}, g_{ij}]$, where c_{ij} stands for the cost of transporting one entity along the arc (i, j) , d_{ij} is the minimal and g_{ij} is the maximal amount of resource.

The aim is to minimize the cost of transporting f units of resources from s to d .

Example

Calculate the minimal cost of flow f from point s to d .



Aircraft rooting (Ferguson 1955)