Structured Contexts with Fibred Semantics

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Abstract

Fibred Semantics is used as a formal basis for contextual reasoning with arbitrary structure in the terms that describe context. Fibring is used to construct a class of graph-structured models for formulas $\mathbf{ist}(\phi, \psi)$, with the informal reading "ψ is true in the context described by ϕ". A corresponding axiomatic presentation is given, and proven to be sound and complete for this class of models.

A metatheory of fibred models of context is developed through self fibring of predicate logics. This covers both propositional and quantificational logics.

Some previously proposed logics of context occur naturally in this framework. We exemplify this by obtaining, as a special case, a close variant of Buvač's quantificational logic of context.

1 Introduction

The logical treatment of contexts in AI was suggested by John McCarthy in his Turing Award lecture [McCarthy, 1987], as a means to overcome the apparent lack of generality in AI systems. The notion was developed further in a series of papers, see [McCarthy and Buvač, 1994; Buvač *et al.*, 1995; Buvač, 1996] for some recent developments.

The notation $\mathbf{ist}(c, \psi)$, with the reading that ψ is true in the context c, was introduced in [Guha, 1991], where the application is to localized contexts in the CYC knowledge base. We augment this notation by allowing formulas to appear in the first argument of $\mathbf{ist}(\cdots, \cdots)$, and give formal semantics corresponding to the informal reading of $\mathbf{ist}(\phi, \psi)$ as "ψ is true in the context **described by** ϕ".

What is a context? The term is used in a variety of senses, and it is doubtful whether any single conception unifies them all. The range of phenomena found under the heading of 'context', includes:

- Mathematical theories, e.g. vector spaces. Mathematicians are in the habit of developing highly abstract, self-contained context descriptions, with precise conditions on when and how they are applicable.
- Conversational settings, that determine the interpretation of indexicals like 'we', 'here', and 'this'. Context accumulates with discourse.
- University politics during the annual budget discussions. Standards and patterns of interaction between colleagues tend to be influenced by this context.

In this paper we present logical tools for reasoning about contexts. We define a formula language with names for contexts, give a fibred model structure and semantics, devise a sound and complete axiomatic system for it, and investigate properties of the resulting logic. Furthermore, we present a general method for deriving logics of context by self fibring.

2 A logic of implicit and explicit context

We define a language of formulas inductively, starting from a set A of atomic formulas, and a nonempty set C of context names. We take both of these to be countably infinite in number, although in any given discourse only a finite number of each will actually be used.

Definition 2.1 (The language L) *The set L of formulas is defined as:*

$$L ::= A|\neg L|L \rightarrow L|\mathbf{ist}(C, L)|\mathbf{ist}(L, L)$$

The symbols $\neg$ and $\rightarrow$ stand for negation and material implication, respectively. The other classical connectives $\wedge, \vee, \leftrightarrow$ etc., and the constant atoms $\top$ and $\perp$ can be added to the language in the usual way.

For example, C might contain contexts like

- ctp "the College end-of-term party"
- abr "the annual budget round"
- wdd "the wedding of the Dean's daughter"

and A might contain the propositions

- sn "it is snowing outside"
- Dh "the Dean of the College is happy"
- rah "road accident rate is high"

Then the following are examples of well-formed formulas, and our logic will allow all of these to be simultaneously satisfiable. We annotate them by the story of some fictitious College and its Dean, whose mood is known to vary with context. It is wintertime:

$$sn$$

$$\mathbf{ist}(sn, rah)$$

The annual budget negotiations take place at this time of year, and the Dean is usually in a sombre and pessimistic mood:

$$\mathbf{ist}(abr, \neg Dh)$$

During a break in the negotiations, invitations to the College end-of-term party are handed out, and the Dean's mood lifts temporarily:

$$\mathbf{ist}(abr, \mathbf{ist}(ctp, Dh))$$

At the party, there is general merriment:

$$\mathbf{ist}(ctp, Dh)$$

Between the tango and the rumba, a young lecturer draws the Dean aside to argue the financial needs of a particular research project. The Dean's reply is non-committal, with admonitions of austerity in these troubled times for the College:

$$\mathbf{ist}(ctp, \mathbf{ist}(abr, \neg Dh))$$

A few days later the Dean's daughter gets married. When somebody makes a jocular comparison between the lavishness of the wedding reception and the frugality of College budgets, the Dean dismisses this with a laugh and proposes a toast to happiness and prosperity:

$$\mathbf{ist}(wdd, \mathbf{ist}(abr, Dh))$$

The last two examples illustrate nested contexts, and show that the truth of a formula depends on the sequence of surrounding contexts.

2.1 Fibred models of context formulas

To make precise the meaning of formulas, we give a model framework within which to interpret elements of the language L. Our models have a "possible-worlds" structure, that is to say the truth value of a formula is evaluated at each of a set of points in each model.

Each point, or possible world, represents a coherent interpretation of the propositional language fragment, and the points are connected in a way that facilitates interpretation of $\mathbf{ist}(\ldots, \ldots)$ formulas.

The interpretation of a formula $\mathbf{ist}(c, \phi)$ at a point w depends on the interpretation of ϕ at another point, which is connected to w by a fibre labelled with the context c.

For each point, there is a bundle of fibres connecting it with other pints, one fibre per context. Formulas of the form $\mathbf{ist}(\phi, \psi)$ are interpreted at a point by looking at all other points connected to it by a fibre in the bundle.

Formally, a model in this framework is a triple $M = \langle W, F, V \rangle$, where

- $W \neq \emptyset$,
- $F : W \times C \rightarrow W$, and
- $V : W \times A \rightarrow 2$.

The F component imposes a graph structure on W, where the edges are labeled by contexts. Paths in the graph correspond to sequences of nested contexts.

If $\vec{\sigma}$ is a finite sequence $\langle c_1, \ldots, c_k \rangle$ of contexts, we shall write $\mathbf{ist}(\vec{\sigma}, \phi)$ as an abbreviation for $\mathbf{ist}(c_1, \ldots \mathbf{ist}(c_k, \phi) \ldots)$, and $F(w, \vec{\sigma})$ as an abbreviation for $F \ldots F(w, c_1) \ldots, c_k)$. When $\vec{\sigma}$ is empty, $\mathbf{ist}(\vec{\sigma}, \phi)$ just ϕ, and $F(w, \vec{\sigma})$ is just w.

Definition 2.2 ($\models$) *The truth value of a formula ϕ at a point $w \in W$ in a model M is denoted $M, w \models \phi$, and defined by the following clauses:*

$$M, w \models a \text{ iff } V(w, a)$$

$$M, w \models \neg\psi \text{ iff } M, w \not\models \psi$$

$$M, w \models \phi \rightarrow \psi \text{ iff } M, w \models \phi \text{ implies } M, w \models \psi$$

$$M, w \models \mathbf{ist}(c, \psi) \text{ iff } M, F(w, c) \models \psi$$

$$M, w \models \mathbf{ist}(\phi, \psi) \text{ iff } M, w \models \mathbf{ist}(c, \phi \rightarrow \psi)$$

$$\text{for all } c \in C$$

Definition 2.3 (Satisfaction.) *Truth of a formula ϕ at all points of a model M is denoted $M \models \phi$, and we then say that M* satisfies *ϕ.*

Definition 2.4 (Validity.) *Truth of a formula ϕ at all points in all models, is denoted $\models \phi$, and we then say that ϕ is* valid.

Every atomic formula $a \in A$ is interpreted as a proposition which is either true or false, and the connectives $\neg$ and $\rightarrow$ are interpreted classically. Therefore, all substitution instances of propositional tautologies are valid. A contextualised formula $\mathbf{ist}(c, \psi)$ is interpreted as truth of ψ in context c, and $\mathbf{ist}(\phi, \psi)$ is interpreted as truth of ψ in the context described by ϕ.

Strictly speaking, the symbol **ist** is standing in for a pair of distinct symbols corresponding to its two different interpretations, but it will always be clear from syntax which one is intended.

2.2 Deductive system

We shall define a deductive system which will turn out to determine exactly the set of valid formulas in the model framework we just presented. This is useful, since it gives us two alternative perspectives on the set of valid formulas, a semantic view and a syntactic one.

The axiomatic system consists of axiom schemas and rules of inference.

Definition 2.5 ($\vdash$) *We say that ϕ is a* theorem, *and write $\vdash \phi$, if ϕ is an instance of an axiom schema, or follows from other theorems by application of a rule of inference.*

Definition 2.6 (Soundness.) *An axiom is said to be* sound *with respect to a model framework if it is valid, and a rule of inference is said to be sound if it takes valid premises to valid conclusions.*

If all axioms and rules are sound, we say the whole system is sound, and we then have

$$\vdash \phi \text{ implies } \models \phi$$

for all formulas ϕ.

Definition 2.7 (Completeness.) *If every valid formula is a theorem;*

$$\models \phi \textit{ implies } \vdash \phi$$

then the system is said to be complete.

Completeness is usually trickier to establish than soundness, and in the proof of completeness below we shall need some terminology concerning theoremhood of formulas, their negations, and sets of formulas:

Definition 2.8 (Consistency.) *We say a formula ϕ is* consistent *iff $\not\vdash \neg\phi$, thus* inconsistent *iff $\vdash \neg\phi$. A finite set of formulas is said to be consistent iff the conjunction of its members is consistent, and an infinite set is consistent iff all its finite subsets are consistent. A formula ϕ is* consistent with *a set Γ according to the consistency of $\Gamma \cup \{\phi\}$.*

Clearly, when Γ is a consistent set, ϕ is consistent with Γ iff $\neg\phi$ is inconsistent with Γ.

Note that by comparison, we did *not* extend the notion of truth, satisfaction and validity to sets of formulas, in particular not to infinite sets. Thus our notion of completeness can be rephrased as "every consistent formula is true at some point in some model", but this does not carry over to infinite sets of formulas. There exists a stronger notion of completeness, which does not hold in this logic, the technical reason being that it is not compact. Our only use of infinite sets of formulas is in maximal consistent sets, to be defined next. These will in turn be used to construct a particular model during the proof of completeness.

Definition 2.9 (Maximality.) *A consistent set* Γ *is* maximal *iff, for all formulas* ϕ*, consistency of* $\Gamma \cup \{\phi\}$ *implies* $\phi \in \Gamma$*.*

In the proof below, we use the following standard properties of maximal consistent sets of formulas. For a full treatment of maximal consistent sets and their properties, consult e.g. [Chellas, 1980].

- Γ is maximal consistent if it satisfies these three conditions:
  - Contains all theorems: $\vdash \phi$ implies $\phi \in \Gamma$
  - Separates formulas from their negations: $\neg\phi \in \Gamma$ iff $\phi \notin \Gamma$
  - Is propositionally closed: $\phi \in \Gamma$ and $\phi \rightarrow \psi \in \Gamma$ implies $\psi \in \Gamma$
- if Γ is maximal consistent, then
  - $\neg\phi \in \Gamma$ iff $\phi \notin \Gamma$
  - $\phi \rightarrow \psi \in \Gamma$ iff $\phi \in \Gamma$ implies $\psi \in \Gamma$
  - If $\vdash \phi \rightarrow \psi$ then $\phi \in \Gamma$ implies $\psi \in \Gamma$

Axiom schemas

$\mathcal{PT}$ All instances of propositional tautologies.

$\mathcal{DD}$ $\mathbf{ist}(c, \neg\phi) \leftrightarrow \neg\mathbf{ist}(c, \phi)$

$\mathcal{K}$ $\mathbf{ist}(c, \phi \rightarrow \psi) \rightarrow (\mathbf{ist}(c, \phi) \rightarrow \mathbf{ist}(c, \psi))$

$\mathcal{SP}$ $\mathbf{ist}(\phi, \psi) \rightarrow \mathbf{ist}(c, \phi \rightarrow \psi)$

Inference rules

$\mathcal{MP}$ $\dfrac{\vdash \phi, \vdash \phi \rightarrow \psi}{\vdash \psi}$

$\mathcal{RN}$ $\dfrac{\vdash \phi}{\vdash \mathbf{ist}(c, \phi)}$

$\mathcal{RG}$ $\dfrac{\vdash \phi \rightarrow \mathbf{ist}(\vec{\sigma}, \mathbf{ist}(c, \psi \rightarrow \chi))}{\vdash \phi \rightarrow \mathbf{ist}(\vec{\sigma}, \mathbf{ist}(\psi, \chi))}$ where c does not occur in ϕ.

We now proceed to prove soundness and completeness of this axiomatic presentation of the logic.

Soundness proof

We verify the soundness of $\mathcal{SP}$ and $\mathcal{RG}$ as examples. The other axiom schemas and rules are less complicated, and use the same patterns of reasoning.

$\mathcal{SP}$ We fix some arbitrary model $M = \langle W, F, V\rangle$, and show that $\mathcal{SP}$ is true at all $w \in W$: assuming the antecedent of the implication true: $M, w \models \mathbf{ist}(\phi, \psi)$, we must prove its consequent true for the same M and w: $M, w \models \mathbf{ist}(c, \phi \rightarrow \psi)$. But from the assumption we get: $M, w \models \mathbf{ist}(c, \phi \rightarrow \psi)$ for every $c \in C$, which is sufficient.

$\mathcal{RG}$ Assuming that the premise of the rule is valid in all models: $M, w \models \phi \rightarrow \phi \rightarrow \mathbf{ist}(\vec{\sigma}, \mathbf{ist}(c, \psi \rightarrow \chi))$ for every model $M = \langle W_M, F_M, V_M\rangle$ and every $w \in W_M$, we must prove that the conclusion of the rule valid in any arbitrary model $N = \langle W_N, F_N, V_N\rangle$, in other words $N, u \models \phi \rightarrow \mathbf{ist}(\vec{\sigma}, \mathbf{ist}(\psi, \chi))$ for every $u \in W_N$. So let us fix some such N and a $u \in W_N$, and assume $N, u \models \phi$, to prove $N, u \models \mathbf{ist}(\vec{\sigma}, \mathbf{ist}(\psi, \chi))$. Now, following the semantical definition, we take an arbitrary context $d \in C$, and prove $N, F_N(t, d) \models \phi \rightarrow \psi$ with $t = F_N(u, \vec{\sigma})$. To this end, we construct a special model M from N as follows:

- $W_M = W_N$
- $F_M(t, c) = F_N(t, d)$
- $F_M(w, e) = F_N(w, e)$ when $(w, e) \neq (t, c)$
- $V_M = V_N$

Now since $N, u \models \phi$ we have $M, u \models \phi$. Therefore by assumption $M, F_M(u, c) \models \phi \rightarrow \psi$, and by the construction of M it follows that $N, F_N(t, d) \models \phi \rightarrow \psi$, as required.

Completeness proof

To establish the completeness of the axiomatic presentation relative to the class of models, we start with a consistent formula $\delta \in L$, and construct a model $M = \langle W, F, V\rangle$ such that $M, w \models \delta$ for a particular $w \in W$.

The structure of the verifying model will be:

- $W = \{w_0\} \cup \{F(w, c) | w \in W, c \in C\}$, where w_0 is a certain set of formulas, containing δ and constructed as described below,
- $F(w, c) = \{\sigma | \mathbf{ist}(c, \sigma) \in w\}$,
- and $V(w, a)$ iff $a \in w$.

The construction of w_0 proceeds in steps as follows. We start with the set $w_0 := \{\delta\}$, and traverse the whole of L, including more formulas as we go: L is clearly enumerable since A and C are, so we fix some enumeration $L = \langle \lambda_1, \lambda_2, \ldots \rangle$. Now we consider each λ_i in turn, and if λ_i is consistent with w_0, then we add λ_i to w_0. If furthermore λ_i is of the form $\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(\psi, \chi))$, then we also add $\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(c_*, \psi \to \chi))$ to w_0, where c_* is chosen as a member of C that does not occur in any member of w_0. Since at each stage of the process w_0 is finite, while C is countably infinite, this is always feasible.

In order to establish completeness we need some lemmas:

Lemma 2.1 *w_0 is a maximal consistent set.*

Proof: by induction on the number of steps in the construction process. w_0 is consistent to begin with, and we show that each addition to it preserves consistency. Then, in the limit, every finite subset of w_0 will be consistent, therefore w_0 itself will be consistent too. Also it will be maximal, for suppose that $w_0 \cup \{\lambda_i\}$ is consistent for some $\lambda_i \in L$, then $\lambda_i \in w_0$, since it was added in step i of the process.

Addition of λ_i in the i'th step is only done if it preserves consistency, so it remains to show that, after adding $\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(\psi, \chi))$ consistently, adding $\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(c_*, \psi \to \chi))$ to w_0 also preserves consistency.

To see this, suppose for contradiction that $\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(c_*, \psi \to \chi))$ is inconsistent with w_0, in other words,

$$\vdash \neg(\phi \land \mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(c_*, \psi \to \chi)))$$

where ϕ is the (finite) conjunction of members of w_0 after adding λ_i consistently. Equivalently

$$\vdash \phi \to \neg\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(c_*, \psi \to \chi))$$

or equivalently, by repeated application of $\mathcal{DD}$,

$$\vdash \phi \to \mathbf{ist}(\vec{\sigma}, \mathbf{ist}(c_*, \psi \to \chi))$$

But then by $\mathcal{RG}$:

$$\vdash \phi \to \mathbf{ist}(\vec{\sigma}, \mathbf{ist}(\psi, \chi))$$

since c_* is chosen so as to not occur in ϕ. By repeatedly applying $\mathcal{DD}$ again, we get

$$\vdash \phi \to \neg\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(\psi, \chi))$$

or equivalently

$$\vdash \neg(\phi \land \mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(\psi, \chi)))$$

But this contradicts the consistency of λ_i with w_0, so it follows that consistency of w_0 is preserved at every step. This proves that w_0 as constructed above is a maximal consistent set.

Next we establish that

Lemma 2.2 *$F(w, c)$ is a maximal consistent set whenever w is.*

Proof: By the properties of maximal consistent sets, this follows from these three items:

- $F(w, c)$ contains all theorems: Suppose $\vdash \phi$. Then $\vdash \mathbf{ist}(c, \phi)$ by $\mathcal{RN}$, so $\mathbf{ist}(c, \phi) \in w$ since w is a maximal consistent set. Then it follows that $\phi \in F(w, c)$ by the definition of F.
- $F(w, c)$ separates formulas from their negations, i.e. $\phi \notin F(w, c)$ iff $\neg\phi \in F(w, c)$:
  Expanding the definition of the former we obtain: $\mathbf{ist}(c, \phi) \notin w$ which is equivalent to $\neg\mathbf{ist}(c, \phi) \in w$ by the fact that w is maximal and consistent. By $\mathcal{DD}$ this is equivalent to $\mathbf{ist}(c, \neg\phi) \in w$, which again by definition of F is equivalent to $\neg\phi \in F(w, c)$.
- $F(w, c)$ is propositionally closed, i.e. if $\phi \in F(w, c)$ and $\phi \to \psi \in F(w, c)$ then $\psi \in F(w, c)$:
  Suppose $\phi \in F(w, c)$, i.e. by definition $\mathbf{ist}(c, \phi) \in w$, and

suppose also $\phi \to \psi \in F(w,c)$, which develops into $\mathbf{ist}(c, \phi \to \psi) \in w$. We must show $\psi \in F(w,c)$, which means $\mathbf{ist}(c,\psi) \in w$. But this follows from $\mathcal{K}$ and the fact that w is a maximal consistent set.

This proves that the set $F(w,c)$ is maximal and consistent whenever w is, and by induction the previous two lemmas prove that all $w \in W$ are maximal consistent sets.

Lemma 2.3 *$M, w \models \phi$ iff $\phi \in w$*

Proof: This is proved by induction on the structure of the formulas.

- The atomic case follows directly from the definition of V
- $\neg\phi$: $M, w \models \neg\phi$ iff (by definition) $M, w \not\models \phi$ iff (by induction) $\phi \notin w$ iff (since w is maximal consistent) $\neg\phi \in w$
- $\phi \to \psi$: $M, w \models \phi \to \psi$ iff (by definition) $M, w \models \phi$ implies $M, w \models \psi$ iff (by induction) $\phi \in w$ implies $\psi \in w$ iff (since w is maximal consistent) $\phi \to \psi \in w$
- $\mathbf{ist}(c,\chi)$: By definition, $M, w \models \mathbf{ist}(c,\chi)$ iff $M, F(w,c) \models \chi$, equivalent by induction to $\chi \in F(w,c)$, which by definition of F is equivalent to: $\mathbf{ist}(c,\chi) \in w$.
- $\mathbf{ist}(\phi,\psi)$: By definition, $M, w \models \mathbf{ist}(\phi,\psi)$ iff for every $c \in C$, $M, w \models \mathbf{ist}(c, \phi \to \psi)$. By induction, this is equivalent to: for every $c \in C$, $\mathbf{ist}(c, \phi \to \psi) \in w$.

  Now suppose that $\mathbf{ist}(\phi,\psi) \in w$. Then, by $\mathcal{SP}$ and the fact that w is a maximal consistent set, $\mathbf{ist}(c, \phi \to \psi) \in w$ for any c, which as seen above is equivalent to $M, w \models \mathbf{ist}(\phi,\psi)$.

  Suppose on the contrary that $\mathbf{ist}(\phi,\psi) \notin w$. Then, since w is a maximal consistent set, $\neg\mathbf{ist}(\phi,\psi) \in w$. By construction of the model,
  - $w = F(w_0, \vec{\sigma})$ for some (possibly empty) sequence $\vec{\sigma}$ of contexts
  - $\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(\phi,\psi)) \in w_0$
  - $\mathbf{ist}(\vec{\sigma}, \neg\mathbf{ist}(c_*, \phi \to \psi)) \in w_0$ for some select c_*

  It follows that $\neg\mathbf{ist}(c_*, \phi \to \psi) \in w$, which as seen above is equivalent to $M, w \not\models \mathbf{ist}(\phi,\psi)$.

This completes the proof of the lemma.

Theorem 2.1 *$M, w_0 \models \delta$*

Proof: Follows from the previous lemma, since $\delta \in w_0$.

We have shown that every consistent formula is true somewhere, so we have completeness.

2.3 Further properties of the logic

Now that we have established a sound and complete axiomatic basis for the logic, we comment on a few additional properties of it, in the form of sound axiom schemas and rules of inference.

The soundness of each schema and rule is sufficient to ensure their dependence on the complete axiomatic basis. Soundness is straightforward to verify every case, and we omit the proofs.

Axiom schemas

$\mathcal{TC}$ $\mathbf{ist}(\top,\phi) \to \mathbf{ist}(c,\phi)$

$\mathcal{DD'C}$ $\neg(\mathbf{ist}(\top,\neg\phi)) \to (\mathbf{ist}(\phi,\neg\psi) \leftrightarrow \neg\mathbf{ist}(\phi,\psi))$

The least specific description of a context is $\top$, which describes everything by virtue of being true no matter what. $\mathcal{TC}$ expresses this, and $\mathcal{DD'C}$ says that whenever ϕ is not false in every context, then whatever is false in the contexts described by ϕ, is not true there. The restriction on ϕ is to avoid empty quantification.

$\mathcal{REF}$ $\mathbf{ist}(\phi,\phi)$

$\mathcal{TRA'}$ $\mathbf{ist}(x,\phi) \to (\mathbf{ist}(\phi,\psi) \to \mathbf{ist}(x,\psi))$, for $x \in C$ or $x \in L$.

This shows that $\mathbf{ist}$ is a partial preorder on L.

$\mathcal{I}'\mathcal{A}$ $(\phi \to \psi) \to (\mathbf{ist}(\psi,\chi) \to \mathbf{ist}(\phi,\chi))$

$\mathcal{IC}$ $(\phi \to \psi) \to (\mathbf{ist}(x,\phi) \to \mathbf{ist}(x,\psi))$ for $x \in C$ or $x \in L$.

These axioms express that **ist** is antitone in its first coordinate and monotone in its second one.

Rules of inference

$$\mathcal{REA}\ \frac{\vdash \phi \leftrightarrow \psi}{\vdash \mathbf{ist}(\phi,\chi) \leftrightarrow \mathbf{ist}(\psi,\chi)}$$

$$\mathcal{RKC}\ \frac{\vdash \wedge\phi_i \to \psi}{\vdash \wedge\mathbf{ist}(x,\phi_i) \to \mathbf{ist}(x,\psi)}$$ for $x \in C$ or $x \in L$.

These two rules show that, in the terminology of [Chellas, 1980], if we look at **ist** as a binary modality, it is *classical* in its first coordinate and *normal* in its second one. It shares these properties with the class of conditional logics investigated there, and for which the model framework was minimal models. We feel that the class of models we have devised here is simpler and more intuitive.

3 Context systems through self fibring of predicate logics

We now proceed to give a more general treatment of the $\mathbf{ist}(\cdots,\cdots)$ predicate. This section will show how various formal systems of context can be naturally presented as self fibring of predicate logics. The term self fibring refers to fibres that connect models of the same logic.

There is a strong similarity of context formalism with that of Labelled Deductive Systems (*LDS*) [Gabbay, 1996b][1]. In fact, the machinery of *LDS* can provide the formal logical framework for theories of context.

An algebraic LDS is a triple $\Lambda = (\Delta, L, R)$ where Δ is a set of formulas and L is an algebra of labels. These give rise to so-called *declarative units*, which are pairs $\lambda : F$ consisting of a label and a formula. R is a deductive discipline for deriving declarative units.

In [McCarthy and Buvač, 1994] and elsewhere, asserting $\mathbf{ist}(\phi,\psi)$ in context c is denoted by

$$c : \mathbf{ist}(\phi,\psi)$$

which is readily identified with a declarative unit in an LDS with labels as a part of the formula language.

The notion of fibred semantics was introduced in 1991 to give semantics for *LDS* and from it the notions of fibring logics and self fibring emerged, grew and developed into [Gabbay, 1996a]. It gives a fuller account of fibred semantics than is possible here.

3.1 The system B of context

We begin by defining a certain self fibred predicate logic which will naturally specialise to the system of [Buvač, 1996].

Let $\mathbf{L}_1$ be a copy of predicate logic and let $\mathbf{ist}(x,y)$ be a binary predicate of $\mathbf{L}_1$. Let $\mathbf{L}_0$ be the language of the predicate logic with the parameterised unary predicate $\lambda y \mathbf{ist}(x,y)$.

Consider the self fibred language $\mathbf{L}_0[\mathbf{L}_1]$. This means we can repeatedly substitute fibred formulas for the y coordinate of $\mathbf{ist}(x,y)$.

Consider a fibred model for this language. As is shown in [Gabbay, 1996a], we can take as models for this language structures of the form $\mathbf{n} = (S, D, a, \mathbf{F}, h, g)$, satisfying certain conditions.

In such models,

- S is the set of labels naming classical models.
- D is the domain which we assume for the purpose of this section to be constant (the same) for all labels (classical models)
- $a \in S$ is the actual label
- $\mathbf{F}$ is the fibring function associating with each $X \subseteq D$ and $t \in S$ a set of labels $\mathbf{F}(X,t) \subseteq S$
- h is the interpretation associating for each $t \in S$ and each $\mathbf{m}$-place predicate P a subset $h(t,P) \subseteq D^m$
- g is a rigid assigment giving for each variable or constant x of the language an element $g(x) \in D$.

Satisfaction is defined as in [Gabbay, 1996a] with the clause for $\mathbf{ist}(x,\phi)$ being:

- $t \vDash \mathbf{ist}(x,\phi)$ iff for all $s \in \mathbf{F}(X_{t,x},t)$, we have $s \vDash \phi$, where $X_{t,x} = \{y \mid t \vDash \mathbf{ist}(x,y)\}$.

Let $\mathfrak{M}^t_x = \mathbf{F}(X_{t,x},t)$. We have

[1] Gabbay's *LDS* position paper was presented in Logic Colloquium 90 [Gabbay, 1993]. A first draft of the *LDS* book was available in 1989 [Gabbay, 1989].

- $t \vDash \mathbf{ist}(X, \phi)$ iff $s \vDash \phi$, for all $s \in \mathfrak{M}^t_x$.

Remark: The system $\mathbf{B}_0$. Let $\mathbf{B}_0$ be the system semantically defined above. $\mathbf{B}_0$ can be axiomatised as shown in [Gabbay, 1996a]. This gives us a straightforward basic system of context. The system of [Buvač, 1996], which we shall call $\mathbf{B}_u$, is obtained by further simplifying assumptions on the semantics. This we discuss below.

We now examine what kind of simplifying assumptions we can have on $\mathbf{F}$ and on $h(t, \mathbf{ist})$.

Option 1

We can assume that $\mathfrak{M}^t_x$ is independent of t. This can be achieved by assuming that $X_{t,x}$ does not depend on t and further that $\mathbf{F}$ does not depend on t. The first assumption means that $\mathbf{ist}$ is a rigid predicate, i.e. there exists a relation $R_{\mathbf{ist}} \subseteq D^2$ such that for all t $h(t, \mathbf{ist}) = R_{\mathbf{ist}}$. The second assumption means that $\mathbf{F}(X, t)$ is rigid, i.e. does not depend on t. To appreciate the implications of this option, consider now the evaluationof $t \vDash \mathbf{ist}(x_1, \mathbf{ist}(x_2, \phi))$. Expanding the definition we get iff $(\forall s \in \mathfrak{M}^t_{x_1})(\forall r \in \mathfrak{M}^s_{x_2})(r \vDash \phi)$, which we can write compactly as $(\forall r \in \mathfrak{M}^t_{(x_1,x_2)})(r \vDash \phi)$, where $r \in \mathfrak{M}^t_{(x_1,x_2)}$ iff (definition) there exists an s such that $(s \in \mathfrak{M}^t_{x_1} \wedge r \in \mathfrak{M}^s_{x_2})$.

Since we assumed that $\mathfrak{M}^t_x$ does not depend on t at all, we get

- $t \vDash \mathbf{ist}(x_1, \mathbf{ist}(x_2, \phi))$ iff for all $r \in \mathfrak{M}_{(x_1,x_2)}, r \vDash \phi$.

Let us take a closer look at $\mathfrak{M}_{(x_1,x_2)}$ in case $\mathfrak{M}$ does not depend on h. $r \in \mathfrak{M}_{(x_1,x_2)}$ iff $\exists s(s \in \mathfrak{M}_{x_1} \wedge r \in \mathfrak{M}_{x_2})$. Assuming that $\mathfrak{M}_{x_1}$ is not empty we get $\mathfrak{M}_{(x_1,x_2)} = \mathfrak{M}_{x_2}$.

This immediately entails that $\mathbf{ist}(x_1, \mathbf{ist}(x_2, \phi)) = \mathbf{ist}(x_2, \phi)$, which is the flatness property found in e.g. [Buvač, 1996].

Option 2

We note the expression $t \vDash \mathbf{ist}(x, \phi)$. x ranges over D and t ranges over S. We can opt to make $\mathbf{ist}$ a two sorted predicate and let x range over S. This assumption does not imply any loss of generality since there is no other 'serious' substitution in the x-coordinate of $\mathbf{ist}$ and the second coordinate of $\mathbf{ist}$ is not affected. Choosing this option brings us closer to the system of [Buvač, 1996].

We can simplify further by letting $S = D$, and we need not have a two sorted system. This can also be done without loss of generality. The two choices give the same system in essence.

If we adopt the above two options, and further assume no function symbols, what we get is the self fibring of predicate logic into a family of unary predicates $\mathbf{ist}_t(y)$ indexed by the labels $t \in S$, whose models have the form $(S, D, a, \mathbf{F}, h, g)$ where both $\mathbf{F}$ and $h(t, \mathbf{ist}_s)$ are *rigid*.

Using results from [Gabbay, 1996a], we immediately get axiomatisability and decidability, summarised in the definition and theorem below.

Definition 3.1 *The context system $\mathbf{B}_u$. Let $\mathbf{L}_1$ be the monadic language with a family of unary predicates $\mathbf{ist}_t$, $t \in C$ and $\mathbf{L}_2$ be the classical predicate language. Let $\mathbf{L}_1[\mathbf{L}_2]$ be the self fibred language obtained by allowing repeated substitution from $\mathbf{L}_2$ into $\mathbf{L}_1$, as defined in [Gabbay, 1996a]. Let $\mathbf{B}$ be the Hilbert system for this language with the following axioms and rules.*

1. *All axioms and rules of free predicate logic with non-denoting constants $\mathfrak{T}$. These include*
   - *All substitution instances of truth functional tautologies*
   - $\dfrac{\vdash \phi, \vdash \phi \to \psi}{\vdash \psi}$
   - $\forall x(\phi(x) \wedge \psi(x)) \to \forall x \phi(x) \wedge \forall x \psi(x)$
   - $\forall x \phi(x) \wedge \forall x(\phi(x) \to \psi(x)) \to \forall x \psi(x)$
   - $\dfrac{\vdash \phi \to \psi(x)}{\vdash \phi \to (x)\psi(x)}$ *where x is not free in ϕ*
   - $\forall x \forall y \phi \to \forall y \forall x \phi$
   - $\forall u(\psi \wedge \forall x \phi(x) \to \phi(u))$.
2. *Modal rules $\mathfrak{K}$*
   - $\dfrac{\vdash \wedge \phi_i \to \phi}{\vdash \mathbf{ist}(t, \wedge \phi_i) \to \mathbf{ist}(t, \phi)}$
   - $\forall x\ \mathbf{ist}(t, \phi(x)) \to \mathbf{ist}(t, \forall x\ \phi(x))$.
3. *Axiom for the rigidity of $\mathbf{F}$.*

- $\forall x[\mathbf{ist}(t_1, x) \leftrightarrow \mathbf{ist}(t, \mathbf{ist}(t_2, x))] \rightarrow [\mathbf{ist}(t_1, \phi) \leftrightarrow \mathbf{ist}(t, \mathbf{ist}(t_2, \phi))]$.

4. Axiom for the rigidity of **ist**

- $\mathbf{ist}(t, x) \rightarrow \mathbf{ist}(s, \mathbf{ist}(t, x))$
- $\neg\mathbf{ist}(t, x) \rightarrow \mathbf{ist}(s, \neg\mathbf{ist}(t, x))$

Remark: Here we are applying the ready-made machinery of [Gabbay, 1996a] to the particular case of the context langauge.

Theorem 3.1 Completeness theorem for $\mathbf{B}_u$ *$\mathbf{B}_u$ is complete for the class of fibred models with* **F** *and* **ist** *rigid.*

Proof: see [Gabbay, 1996a].

Theorem 3.2 Decidability theorem for $\mathbf{B}_u$ *The tight fragment of monadic $\mathbf{B}_u$ is decidable.*

Proof: see [Gabbay, 1996a].

Note that by monadic $\mathbf{B}_u$, we mean we have only monadic predicates in the classical language and by *tight* fragment we mean that we substitute in $\mathbf{ist}(x, y)$ in the y position only wffs with at most y free. Otherwise we get undecidability!

We compare with [Buvač, 1996]. Our models for $\mathbf{B}_u$ are practically identical but for equality, but that can be added without any problem. Other minor differences: we use letters t for models instead of $\mathfrak{A}$, we write $t \vDash \phi$ for evaluation, as compared with $t \vDash s : \phi$, where the s is redundant as can be seen from [Buvač, 1996] by inspection.

The following axioms are given in [Buvač, 1996]:

1. Classical predicate calculus axioms and rules
2. Propositional properties of contexts:

   $\mathcal{K}$ $k : \mathbf{ist}(k', \phi \rightarrow \psi) \rightarrow (\mathbf{ist}(k', \phi) \rightarrow \mathbf{ist}(k', \psi))$

   (Δ) $k : \mathbf{ist}(k_1, \mathbf{ist}(k_2, \phi)) \vee \mathbf{ist}(k_1, \neg\mathbf{ist}(k_2, \phi))$

   (Flat) $k : \mathbf{ist}(k_1, \mathbf{ist}(k_2, \phi)) \leftrightarrow \mathbf{ist}(k_2, \phi)$

   $$\text{(Enter)}\ \frac{s : \mathbf{ist}(t, \phi)}{t : \phi}$$

   $$\text{(Exit)}\ \frac{t : \phi}{s : \mathbf{ist}(t, \phi)}$$

   The last two rules correspond to saying

   $t : \phi$ is the same as $\mathbf{ist}(t, \phi)$

   $t : (s : \phi)$ equals $s : \phi$
3. Quantificational properties of contexts:

   (BF) $k : \forall x(\mathbf{ist}(k', \phi) \rightarrow \mathbf{ist}(k', \forall x \phi)$

   ($\mathbf{ist}$ =) $k : t_1 = t_2 \leftrightarrow \mathbf{ist}(k', t_1 = t_2)$

Let us compare this axiom system with our own system defined above: Group axioms (1) of our system yields all the classical axioms. The $\mathfrak{K}$ group of rules yield context axioms $\mathcal{K}$, (Δ) and (BF). The axiom (Flat) is obtained from our rigidity axioms. The rules (Enter) and (Exit) are obviously conservative additional axioms, and we can add them if desired. The reader should bear in mind that we did not invent our axioms for $\mathbf{B}_u$ as a specific system. We have a general self fibring mechanism that is axiomatised and it specialises to certain axioms for the case of $\mathbf{B}_u$. It will specialise to other axioms should the reader choose a different context logic.

Let us list what we believe self fibring methodology can offer to context theories.

1. We provide a wide framework for general self fibring. This is useful since in AI logics and applications there is a lot of use of substitutions of formulas inside formulas.
2. More specifically for the McCarthy–Buvač systems with the $\mathbf{ist}(x, y)$ predicate, we can use the methodology to allow the use of expressions of the form $\mathbf{ist}(\phi, \psi)$, where the context is given by wffs of possibly another logic and language. We can describe contexts in a logic (not only name them) and reason logically about context. We can even allow statements on the context which may involve considerations of what happens in them, i.e. what kinds of wffs ψ hold in them (i.e. $\mathbf{ist}(\phi, \psi)$ holds).
3. Given two known logics, one for L_1 wffs and one for L_2 wffs, we can let the L_1-language serve as context for the L_2-language, by using the $\mathbf{ist}(\phi, \psi)$ predicate. In this way, we have a way to express under what conditions known properties of the components systems (e.g. decision procedure, proof theory) transfer to the combined context system with $\mathbf{ist}(\phi, \psi)$.

4 Comparison with the literature

In comparison with the propositional $\mathbf{ist}(c, \psi)$ of [Guha, 1991] and the two-sorted quantified version in [Buvač, 1996], our logic of $\mathbf{ist}(\phi, \psi)$ introduces new notation and develops a new logic for it. Where [Buvač, 1996] admits arbitrary quantification over context variables, our system of section 2 harnesses generalization over contexts in a two-layered multimodal system, the semantics of one modality quantifying over a set of other modalities. In spite of its richness of expression, we obtained a simple axiomatic characterization of this system.

van Benthem [Benthem, 1996] reviews diverse uses of context, and suggests that the term denotes a convenient methodological fiction, rather than a well-defined ontological category. He finds that admitting full quantification over context variables swamps the language with extraneous elements, and trades them in for greater technical economy. This recalls our position in section 2. His proposal is for an indexing scheme, where each language element can be decorated with an index specifying an intended context for evaluation. This results in a scheme where transition between contexts has a natural expression.

Our system $\mathbf{B}_u$, which we obtained above by self fibring, subsumes the system of [Buvač, 1996], except for minor differences discussed in the previous section.

In [Besnard and Tan, 1995], contextual reasoning is formalized by indexing formulas with sets of formulas, the index set specifying the context. A natural deduction system is given, and a connection with default logic is made via an operator on assumptions. The sets denoting contexts are arbitrary, whereas in our $\mathbf{ist}(\phi, \psi)$, the context is given indirectly by ϕ.

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