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Local Models Semantics, or contextual reasoning = locality + compatibility ☆

Chiara Ghidini [a,*], Fausto Giunchiglia [b,c]

[a] Department of Computer Science, University of Liverpool, Liverpool L69 7ZF, UK
[b] DISA – Università degli Studi di Trento, Via Inama 5, 38100 Trento, Italy
[c] ITC – IRST, 38050 Povo, Trento, Italy

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Abstract

In this paper we present a new semantics, called *Local Models Semantics*, and use it to provide a foundation to reasoning with contexts. This semantics captures and makes precise the two main intuitions underlying contextual reasoning: (i) reasoning is mainly *local* and uses only part of what is potentially available (e.g., what is known, the available inference procedures), this part is what we call *context* (of reasoning); however (ii) there is *compatibility* among the reasoning performed in different contexts. We validate our semantics by formalizing two important forms of contextual reasoning: reasoning with viewpoints and reasoning about belief. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The notion of context is studied in many research areas, and it has been many years now. We only need to mention here that the notion of context is very important for disciplines such as philosophy of language [2], cognitive science [12,15,26], pragmatics [30], linguistics [15], and so on. In Artificial Intelligence, contexts were first introduced in Weyhrauch's work on mechanizing logical theories in the FOL system [43]. However contexts became a widely discussed issue in the late 1980s, when they were independently

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* Corresponding author.
E-mail addresses: chiara@csc.liv.ac.uk (C. Ghidini), fausto@irst.itc.it (F. Giunchiglia).

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proposed by Fausto Giunchiglia [25] and John McCarthy [35] as an important means for formalizing (certain forms of) reasoning. According to [25], contexts are a tool for formalizing the locality of reasoning, while in [35] contexts are introduced as a means of solving the problem of generality. Coherently with these two proposals, contexts have been used in various applications. Farquhar et al. [14], Ghidini and Serafini [22], Mylopoulos and Motschnig-Pitrik [37] and Theodorakis et al. [42] describe the use of contexts in dealing with issues concerning the integration of heterogeneous knowledge and data bases. In [27] contexts are used for formalizing meta reasoning and propositional attitudes. In [1] contexts are used in the formalization of reasoning with viewpoints. Bouquet and Giunchiglia [5] formalize context-based common-sense reasoning. In [3,16,19,24, 28] contexts are used to formalize theoretical issues concerning reasoning about beliefs, whereas in [4,9] contexts are used to model different aspects of agents and multi-agent systems. Ghidini and Serafini [21], Noriega and Sierra [38] and Parsons et al. [39] describe the use of contexts for the modeling of dialog, argumentation, and information integration in electronic commerce. Finally, the largest common-sense knowledge-base, CYC [33], implements and exploits an explicit notion of context [31].

Despite the plethora of different approaches, formalizations, and applications, two are the main intuitions underlying the use of context. We state these two intuitions as the following two principles:

Principle 1 (of Locality). Reasoning uses only part of what is potentially available (e.g., what is known, the available inference procedures). The part being used while reasoning is what we call *context* (of reasoning);

Principle 2 (of Compatibility). There is *compatibility* among the kinds of reasoning performed in different contexts.

The goal of this paper is to describe and motivate a new semantics, called *Local Models Semantics*, which formalizes the two principles listed above, and that we propose as a foundation for contextual reasoning. The core definitions are given in Section 3. To make the presentation clearer, but also to show the generality of the approach, we informally describe, and then formalize, using Local Models Semantics, two important examples of contextual reasoning, namely *reasoning with viewpoints*, and *reasoning about belief*. This material is covered in Sections 2 (informal presentation) and 4 (formalization using Local Models Semantics).

In previous papers, various proof-theoretic formalizations of contextual reasoning have been proposed (see [1,6,7,25,31,36]). One such axiomatization are Multi-Context Systems (also described as Multi-Language Systems, when there was a bigger interest in analyzing the structure of languages) [23,25,27]. To make the paper more self-contained, but also "to close the loop", in the second part of this paper, we analyze the relation existing between Local Models Semantics and Multi-Context Systems (MC systems from now on). In particular, in Section 5 we briefly overview the basic notion of MC systems and show how MC systems capture, at the proof-theoretic level, the notions of locality and compatibility. In Section 6 we give a formalization, in terms of MC systems, of reasoning with viewpoints and of reasoning about belief. The technical results are given in the appendices, which contain the proofs of correctness and completeness results between the MC systems defined in Sections 6.1 and 6.2 and the classes of models defined in Sections 4.1 and 4.2, respectively. We conclude with a short comparison with other frameworks for the formalization of reasoning with contexts.

2. Two examples

The examples introduced in this section are used throughout the paper to discuss and illustrate the ideas and the formalization of contextual reasoning we propose.

2.1. Reasoning with viewpoints

Consider the scenario of Fig. 1. There are two observers, *Mr*.1 and *Mr*.2, each having a partial viewpoint of a box. The box consists of six sectors, each sector possibly containing a ball. There cannot be balls hidden from the view of an observer. The box is "magic" and observers cannot distinguish the depth inside it. Fig. 2 shows what *Mr*.1 and *Mr*.2 can see in the scenario depicted in Fig. 1.

In this example we have two contexts, each context describing what an observer sees (its viewpoint) and the consequences that it is able to draw from it. The content of the two contexts is graphically represented in Fig. 2.

Locality. Both *Mr*.1 and *Mr*.2 have the notions of a ball being on the right or on the left. However these two notions are different and we may have a ball which is on the right for *Mr*.1 and not on the right for *Mr*.2. Furthermore *Mr*.2 has the notion of "a ball being in the center of the box" which is meaningless for *Mr*.1.

Compatibility. The contents of *Mr*.1 and *Mr*.2's contexts are obviously related. The relation is a consequence of the fact that *Mr*.1 and *Mr*.2 see the same box. Fig. 3 shows all the possible contexts for *Mr*.1 and *Mr*.2, and gives all their possible compatible combinations. Notice that we can describe this situation by listing all the possible

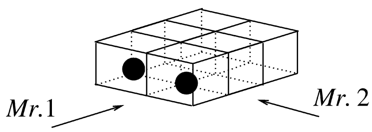

Fig. 1. The magic box.

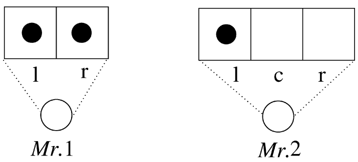

Fig. 2. *Mr*.1 and *Mr*.2's contexts.

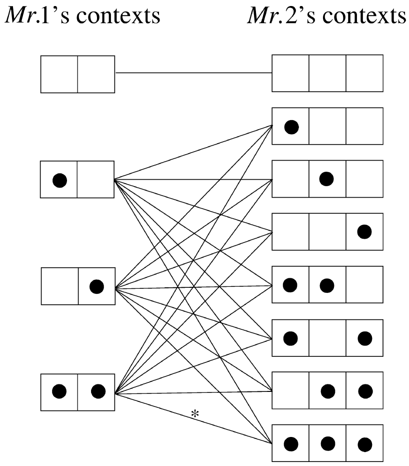

Fig. 3. Compatible contexts of $Mr.1$ and $Mr.2$.

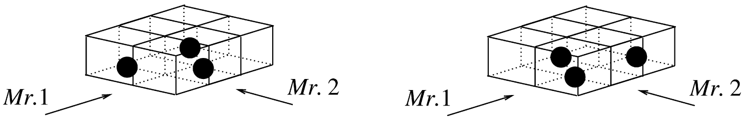

Fig. 4. Indistinguishable situations.

compatible pairs (as they are represented in Fig. 3), or we can describe it more synthetically using descriptions like: "if $Mr.1$ sees at least one ball then $Mr.2$ sees at least one ball".

Notice that the most straightforward formalization of this example would be a direct axiomatization of the box as a two-dimensional grid. $Mr.1$ and $Mr.2$'s views and contexts could then easily be constructed by projecting the grid in two one-dimensional views. Locality and compatibility would be guaranteed by construction. However this approach is based on the hypothesis that we have a complete description of the world (the box in this case), and that we can use it to build views of the world itself. This is not always the case. Quite often there are only partial views and only a partial or approximate view of the world can be reconstructed. This is, in fact, also the case for the magic box scenario depicted in Fig. 1. Consider, for instance, the situations depicted in Fig. 4. These two different situations cannot be distinguished by the two observers. The unique pair of compatible contexts associated to the two different situations in Fig. 4 is the one marked with "*" in Fig. 3. To obtain a complete description of the magic box, one also needs a third view from the top (as a matter of fact, the top view by itself provides a complete description of the balls contained into the magic box, as long as the box is only one cube deep).

An important application domain where we may or may not have a complete description of the world is the development and integration of data or knowledge bases. In a relational, possibly distributed, data base there is (assumed to be) a complete description of the world, and views are built by filtering out, and appropriately merging together, part of the available

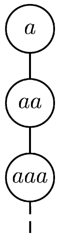

Fig. 5. The context structure of beliefs in a scenario with a single agent.

information. On the other hand, a federation of heterogeneous data or knowledge bases, possibly developed independently, can be seen as a set of views of an ideal data base which is often impossible or very complex to reconstruct completely. The work in [20,22] starts from this observation, further develops the semantics defined in this paper, and gives foundations to the various forms of federations described in, e.g., [14,37].

2.2. Reasoning about belief

Let us consider the situation of a single agent a (usually thought of as the computer itself or as an external observer) who is acting in a world, who has beliefs about this world and also beliefs about its own beliefs, and it is able to reason about them. We formalize beliefs about beliefs by exploiting the notion of *belief context*. The intuition is that a belief context formalizes the "mental image" that a has of itself, or the "mental image" that it has of the "mental image" of itself, or .... One more nesting of the belief operator corresponds to one more nesting in the structure of "mental images" (contexts).

Belief contexts are organized in a chain (see Fig. 5). We call a the root context; this context represents the beliefs of a. The context aa formalizes the beliefs that a ascribes to itself. Iterating the nesting, the context aaa formalizes the beliefs of a about the beliefs about its own beliefs, and so on. Let us consider only a and aa in Fig. 5, that is, the situation with an agent a having beliefs about its own beliefs.

Locality. The belief contexts tagged with a and aa are described using different languages. For instance a has a notion of "believing something" which aa doesn't have. The interpretation of a formula depends on the context we consider. For instance the sentence "it is raining" in the context a expresses the fact that, in the representation of the world made by the agent a, it is raining. The same sentence "it is raining" in the context aa expresses the fact that the agent a ascribes to itself the belief that it is raining. Notice also that, in general, a and aa may contain different beliefs about the world.

Compatibility. The contents of different contexts are obviously related. These relations, which in principle can be very different, express how a's beliefs and the beliefs that a ascribes to itself are connected. An obvious relation is the following: if a sentence of the form ϕ is in aa, then a sentence of the form "I believe that ϕ" is in a. In this case we say that a is a *correct* observer (with respect to the sentence "I believe that ϕ"). Another situation is when a sentence of the form ϕ is in aa, only if a sentence of the form

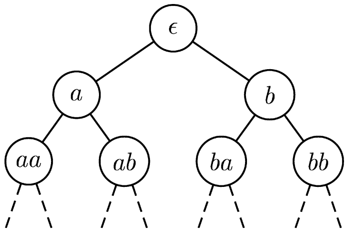

Fig. 6. The context structure of beliefs in a scenario with two agents.

"I believe that ϕ" is in a. In this case we say that a is a *complete* observer (with respect to the sentence "I believe that ϕ"). A taxonomy of the possible relations involving belief about belief is introduced in [28] and then refined in [24]. In these papers the authors show that, depending on the relations among different contexts, the agent a has different reasoning capabilities.

These observations about locality and compatibility can be easily generalized to consider a chain of any depth or to consider a multi-agent scenario, where each agent comes with its, usually different, language, knowledge base, and reasoning capabilities. Fig. 6 shows the structure of contexts in a multi-agent scenario where an external observer ε ascribes a collection of beliefs to two agents a and b. [1] The contexts tagged with a, and b, represent the beliefs that ε ascribes to a and b, respectively; the contexts tagged with aa, and ab, represent the mental images that a has of its own beliefs and of the beliefs of b, respectively (from the point of view of ε), and so on. For a more detailed description of this structure, a good reference is [9], where belief contexts are used to solve a well-known puzzle involving reasoning about belief and ignorance, namely the Three-Wise-Men problem.

An important application of the ideas and intuitions briefly illustrated in this section is the specification and development of complex agents platforms. The approach described above, first proposed in [24], is now current practice in much of the work in agent technology (see, e.g., [4,16,38,39,41]).

3. Local Models Semantics

We define in turn the notions of local model and model, context, local satisfiability and satisfiability, and logical consequence.

3.1. Local models and models

Let $\{L_i\}_{i\in I}$ be a family of languages defined over a set of indexes I (in the following we drop the index $i \in I$). Intuitively, each L_i is the (formal) language used to describe what is true in a context. For the purpose of our work we suppose that I is (at most) countable. Let us restrict ourselves to (classes of) first order languages. Let $\overline{M_i}$ be the class of all the models (interpretations) of L_i. We call $m \in \overline{M_i}$ a *local model* (of L_i).

[1] Taking a realistic attitude one might safely assume that ε describes what is actually true in the real world.

A *compatibility sequence* $\boldsymbol{c}$ (for $\{L_i\}$) is a sequence

$$\boldsymbol{c} = \langle \boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_i, \ldots \rangle,$$

where, for each $i \in I$, $\boldsymbol{c}_i$ is a subset of $\overline{M}_i$. We call $\boldsymbol{c}_i$ the ith element of $\boldsymbol{c}$. If $I = \{1, 2\}$, we call $\boldsymbol{c}$ a (*compatibility*) pair.

A *compatibility relation* $\boldsymbol{C}$ (for $\{L_i\}$) is a set $\boldsymbol{C} = \{\boldsymbol{c}\}$ of compatibility sequences $\boldsymbol{c}$. Formally, let $\prod_{i \in I} 2^{\overline{M}_i}$ be the Cartesian product of the collection $\{2^{\overline{M}_i}\colon i \in I\}$.[2] The compatibility relation $\boldsymbol{C}$ is a relation of type

$$\boldsymbol{C} \subseteq \prod_{i \in I} 2^{\overline{M}_i}.$$

A *model* is a compatibility relation which contains at least a sequence and does not contain the sequence of empty sets.

Definition 3.1 (*Model*). A *model* (for $\{L_i\}$) is a compatibility relation $\boldsymbol{C}$ such that:

(1) $\boldsymbol{C} \neq \emptyset$;
(2) $\langle \emptyset, \emptyset, \ldots, \emptyset, \ldots \rangle \notin \boldsymbol{C}$.

Conditions (1) and (2) eliminate meaningless compatibility relations and sequences, namely totally inconsistent context structures. In the following we write $\boldsymbol{C}$ to mean either a compatibility relation or a model, the context always makes clear what we mean. Fig. 7 gives a graphical representation of the construction we perform with $I = \{1, 2, 3\}$. We start from L_1, L_2, and L_3. Then, we associate each L_i with a set $M_i \subseteq \overline{M}_i$ of local models. Usually $M_i \subset \overline{M}_i$. Finally, we pair local models inside compatibility pairs and then compatibility sequences. The resulting compatibility relation is our model. Local models describe what is locally true. Compatibility sequences put together local models which are "mutually compatible", consistently with the situation we are describing (see Example 3.1 below). Compatibility relations and models are sets of "mutually compatible" sequences of local models.

Example 3.1. The construction described in Fig. 7 can be used to "build" the situation described in Fig. 3. First, we define the two languages L_1 and L_2 describing the views of *Mr*.1 and *Mr*.2, respectively. Both L_1 and L_2 are two propositional languages, L_1 describing that a ball can be on the left or on the right, and L_2 describing that a ball can be on the left, in the center, or on the right. Second, we construct all the possible situations (models) for L_1 and L_2. This leads to the definition of the four situations (models) for L_1 depicted on the lefthand side in Fig. 3, and of the eight possible situations (models) for L_2 depicted on the righthand side in Fig. 3. Finally, we construct all the compatibility pairs. Fig. 3 graphically represents all the possible pairs whose elements are singleton sets.

Notice that linking local models inside a compatibility relation may force us to throw away some of them. Consider, for instance, the case where we restrict the possible situations in Fig. 3 to the local models for *Mr*.1 which allow for exactly one ball. This

[2] Formally, the Cartesian product of a collection $\{X_i\colon i \in I\}$ of sets is denoted by $\prod_{i \in I} X_i$ and it is defined as the set of all functions f with domain I such that $f(i) \in X_i$ for all $i \in I$.

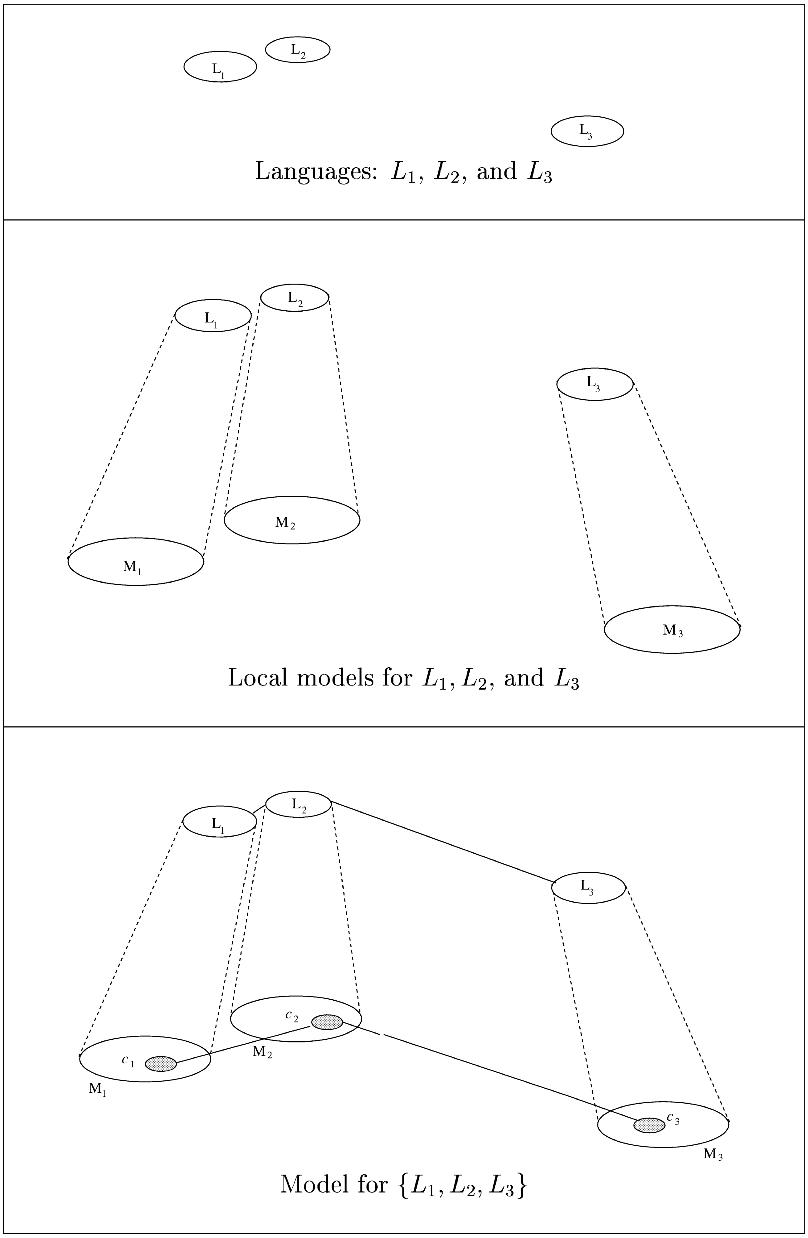

Fig. 7. The construction of a model.

fact, together with the definition of compatibility existing between the views of the two observers, forces us to throw away all the pairs, and corresponding local models for $Mr.2$, which allow for zero balls (see Fig. 3).

Given a family of languages $\{L_i\}$, different subclasses of models may be defined, depending on the definition of compatibility relation. Different compatibility relations model different situations. We introduce here two general classes of models which will be used throughout the paper.

Definition 3.2 (*Chain and chain model*). A compatibility sequence $\boldsymbol{c}$ is a *chain* if $|\boldsymbol{c}_i| = 1$ for each $i \in I$. A model $\boldsymbol{C}$ is a *chain model* if all the $\boldsymbol{c}$ in $\boldsymbol{C}$ are chains.

Definition 3.3 (*Weak chain and weak chain model*). A compatibility sequence $\boldsymbol{c}$ is a *weak chain* if $|\boldsymbol{c}_i| \leqslant 1$ for each $i \in I$. A model $\boldsymbol{C}$ is a *weak chain model* if all the $\boldsymbol{c}$ in $\boldsymbol{C}$ are weak chains.

3.2. Contexts

Given a model $\boldsymbol{C} = \{\langle \boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_i, \ldots\rangle\}$ we formally define a *context* to be any $\boldsymbol{c}_i$, namely the set of local models $m \in \overline{M_i}$ allowed by $\boldsymbol{C}$ within any particular compatibility sequence.

The intuition underlying the definition of context is that, semantically, a context consists of that set of models which captures exactly those facts which are locally true, given also the constraints posed by the local models of other contexts in the same compatibility sequence, as allowed by a given compatibility relation. Notice that this notion of context is the semantic formalization of the notion of context intuitively introduced in Principle 1 in Section 1. Notice also that defining a context as a set of models (instead of a single model) enables us to formalize it as a partial object, as explicitly required in, e.g., [25,34]. This is a key difference with possible worlds [32], which are complete objects (in the sense that a formula is either true or false in a world). We illustrate the advantage of having contexts as partial objects by using the following example.

Example 3.2. Consider the slightly modified magic box scenario depicted in Fig. 8, where *Mr*.2 is able to see only one box sector and knows that there are two sectors behind the wall. In this scenario *Mr*.2 is able to distinguish only two situations: there is a ball on the left, and there is no ball on the left. The fact that *Mr*.2 is uncommitted to whether there is a ball in a sector behind the wall is formalized by having the sentence *"there is a ball on the right"* true in some local models representing *Mr*.2's view and false in others. In the resulting context, describing *Mr*.2's viewpoint, "there is a ball on the right" will be neither true nor false because there will be models in $\boldsymbol{c}_2$ where the sentence is false and others where the sentence is true. Fig. 9 graphically describes the compatibility pairs involving

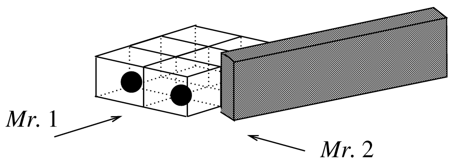

Fig. 8. A new magic box.

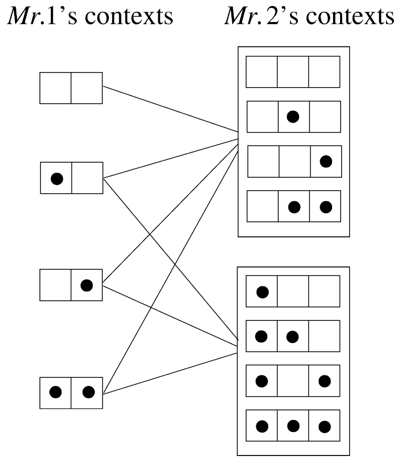

Fig. 9. Compatible contexts of $Mr.1$ and $Mr.2$ in the scenario of Fig. 8.

the four different possible situations for $Mr.1$ and the two different possible situations for $Mr.2$. Note that, in this case, contrarily to what happens in Fig. 3, compatibility sequences are not chains.

Given the above notion of context, we can now better understand the intuitions underlying the notion of compatibility sequence, and that of compatibility relation (model). A context is a partial description of the world. A compatibility sequence contains as many contexts as needed, one for each partial description of the world. Thus, in the magic box scenario we have compatibility sequences of length two, containing a context for the view of $Mr.1$ and a context for the view of $Mr.2$. Similarly, in the scenario concerning reasoning about belief we have two contexts, one each for the two mental images considered. In the more general scenario involving n belief contexts, we have to consider sequences of length n.

An interesting situation is the case of compatibility sequences in which all the contexts are singleton sets, that is, the case of chains as introduced in Definition 3.2. In this case, all the contexts are complete objects in the sense that each context, being a single model, assigns a truth value to all sentences in its language. A context which is a singleton set models the situation where a partial description of the world assigns a truth value to all the propositions it is able to express in its local (and limited) language. This is the case in Figs. 1, 2, and 3. Here, $Mr.1$ and $Mr.2$ have partial views of the world. However, within their partial views, they are able to "see everything".

A slightly different situation is the case of weak chains, introduced in Definition 3.3. In this case each context is either a singleton set ($|\boldsymbol{c}_i| = 1$) or an empty set ($|\boldsymbol{c}_i| < 1$). This means that a context is either a complete object, in the sense discussed above, or an inconsistent object. Indeed, in the latter case, being an empty set of models, a context assigns the truth value "true" to all sentences in its language, therefore describing an inconsistent situation.

3.3. Local satisfiability, satisfiability, and logical consequence

We can now say what it means for a model to *satisfy* a formula of a language L_i. Let $\models_{cl}$ be the (classical) satisfiability relation between local models and formulae of L_i. Let us call $\models_{cl}$ *local satisfiability*. Notationally, let us write $i:\phi$ to mean ϕ and that ϕ is a formula of L_i. We say that ϕ is an L_i-formula, and that $i:\phi$ is a formula or, also, a labelled L_i-formula. This notation and terminology allows us to keep track of the context we are talking about. Then we have the following:

Definition 3.4 (*Satisfiability*). Let $\boldsymbol{C} = \{\boldsymbol{c}\}$, with $\boldsymbol{c} = \langle \boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_i, \ldots \rangle$, be a model and $i:\phi$ a formula. $\boldsymbol{C}$ *satisfies* $i:\phi$, in symbols $\boldsymbol{C} \models i:\phi$, if for all $\boldsymbol{c} \in \boldsymbol{C}$

$$\boldsymbol{c}_i \models \phi,$$

where $\boldsymbol{c}_i \models \phi$ if, for all $m \in \boldsymbol{c}_i$, $m \models_{cl} \phi$.

Intuitively: an L_i-formula is satisfied by a model $\boldsymbol{C}$ if all the local models in each ith context satisfy it. A model $\boldsymbol{C}$ satisfies a set of formulae Γ, in symbols $\boldsymbol{C} \models \Gamma$, if $\boldsymbol{C}$ satisfies every formula $i:\phi$ in Γ.

The notion of *validity* is the obvious one.

Definition 3.5 (*Validity*). A formula $i:\phi$ is *valid*, in symbols $\models i:\phi$, if all models satisfy $i:\phi$.

What is more interesting is the notion of *logical consequence* which must take into account the fact that assumptions and conclusion may belong to distinct languages. Given a set of labelled formulae Γ, Γ_j denotes the set of formulae $\{\gamma \mid j:\gamma \in \Gamma\}$.

Definition 3.6 (*Logical consequence with respect to a model*). A formula $i:\phi$ is a logical consequence of a set of formulae Γ with respect to a model $\boldsymbol{C}$, in symbols $\Gamma \models_{\boldsymbol{C}} i:\phi$, if every sequence $\boldsymbol{c} \in \boldsymbol{C}$ satisfies:

$$\forall j \in I, j \neq i,\ \boldsymbol{c}_j \models \Gamma_j \quad \Rightarrow \quad (\forall m \in \boldsymbol{c}_i,\ m \models_{cl} \Gamma_i \quad \Rightarrow \quad m \models_{cl} \phi). \tag{1}$$

Intuitively: take a model $\boldsymbol{C}$ and a formula $i:\phi$. Take a set of assumptions Γ and, among them, isolate the set of assumptions Γ_j with $j \neq i$. Take all the sequences in $\boldsymbol{C}$ whose local models in $\boldsymbol{c}_j$ satisfy Γ_j (and throw away all the others). Consider now the local models in $\boldsymbol{c}_i$ of the remaining sequences. $\Gamma \models_{\boldsymbol{C}} i:\phi$ if in these remaining local models all the local models which satisfy Γ_i locally satisfy ϕ. Essentially, the intuition is that the formulae in Γ_j prune away compatibility sequences, while the formulae in Γ_i prune away local models in $\boldsymbol{c}_i$. This is due to the fact that the assumptions Γ_j ($j \neq i$) made in the context $\boldsymbol{c}_j$ induce "compatible" assumptions in other contexts, and in particular in the context $\boldsymbol{c}_i$. This, in turn, results in pruning away compatibility sequences. The role of the assumptions Γ_i is instead the usual one, that is, that of pruning away local models of L_i.

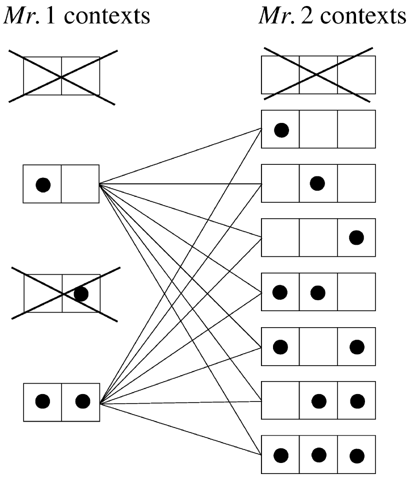

Fig. 10. Selecting compatibility sequences.

Example 3.3. Consider the model of the magic box informally depicted in Fig. 3, whose content has been informally described in Example 3.1. We want to verify that in this model

> if $Mr.1$ sees a ball on the left and $Mr.2$ doesn't see any ball on the right,
> then $Mr.2$ sees a ball on the left or in the center. (2)

Following Definition 3.6, the first step is to isolate all the pairs whose local models satisfy the property that $Mr.1$ sees a ball on the left, and throw away all the others. The remaining compatibility pairs are depicted in Fig. 10. The second step is to isolate all the $Mr.2$'s local models in the remaining pairs such that there are no balls on the right. The remaining $Mr.2$'s local models are depicted in Fig. 11. The last step is to check whether the remaining $Mr.2$'s local models represent the fact that $Mr.2$ sees a ball on the left or in the center. It is easy to see that all the remaining local models in Fig. 11 satisfy this property. Therefore the model depicted in Fig. 3 satisfies (2).

A formula $i:\phi$ is a logical consequence of a set of formulae Γ with respect to a class of models M, in symbols $\Gamma \models_{\mathsf{M}} i:\phi$, if $i:\phi$ is a logical consequence of Γ with respect to all the models in M. We say also that $i:\phi$ is an M-logical consequence of Γ. Finally, a formula $i:\phi$ is a *logical consequence* of Γ, in symbols $\Gamma \models i:\phi$, if $i:\phi$ is a logical consequence of Γ with respect to all the models $\boldsymbol{C}$.

The notion of logical consequence introduced in this section extends the notion of local logical consequence.

Theorem 3.1 (Extension with respect to local logical consequence). *Let Γ be a set of formulae. If $\Gamma_i \models_{cl} \phi$, then $\Gamma \models i:\phi$.*

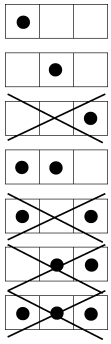

Fig. 11. Selecting local models.

Proof. $\Gamma_i \models_{cl} \phi$ implies that, for any local model m of L_i, if Γ_i holds, then ϕ holds as well. Therefore, the fact that for all $m \in \boldsymbol{c}_i$, if $m \models_{cl} \Gamma_i$ then $m \models_{cl} \phi$ is trivially true. This ends the proof. $\square$

The converse (i.e., if $\Gamma \models i:\phi$ then $\Gamma_i \models_{cl} \phi$) is not, in general, true. Trivially this is due to the possible existence of assumptions made in contexts with index $j \neq i$.

Notice that, if we restrict ourselves to consider classes of weak chain models, then Definition 3.6 can be simplified as follows: $\Gamma \models_{\boldsymbol{C}} i:\phi$ if

$$\forall j \in I,\ \boldsymbol{c}_j \models \Gamma_j \quad\Rightarrow\quad \boldsymbol{c}_i \models \phi. \tag{3}$$

The proof is straightforward. From the hypothesis that $|\boldsymbol{c}_i| \leqslant 1$, Eq. (1) can be rewritten as

$$\forall j \in I, j \neq i,\ \boldsymbol{c}_j \models \Gamma_j \quad\Rightarrow\quad (\boldsymbol{c}_i \models \Gamma_i \quad\Rightarrow\quad \boldsymbol{c}_i \models \phi)$$

which is, in turn, equivalent to Eq. (3). The notion of logical consequence given in Eq. (3) was first introduced in [29], where the authors define a semantics for an MC system formalizing meta-reasoning, called MK.

Notice also that the simplified notion of logical consequence given in Eq. (3) can be further simplified in the case of chain models. Indeed, from the fact that each $\boldsymbol{c}_i$ contains a single local model m_i, it follows that Eq. (3) can be rewritten as follows:

$$\forall j \in I,\ m_j \models_{cl} \Gamma_j \quad\Rightarrow\quad m_i \models_{cl} \phi. \tag{4}$$

As it will be clear in Section 4.1, this simplified notion of logical consequence applies to the magic box scenario graphically described in Figs. 1, 2, and 3.

3.4. The principles of locality and compatibility

The notions of model, context, satisfiability, and logical consequence given in this section formalize the principles of locality and compatibility in the following sense:

Locality. Everything is local. First of all, the language is local: not only do we have a language for each context, but, also, there is no notion of a not labelled L_i-formula ϕ

being satisfiable. We always talk of satisfiability of formulae in context, i.e., of labelled L_i-formulae. Second, the notion of satisfiability is local: the satisfiability of a (labelled) formula is given in terms of the local satisfiability of the formula with respect to its context. Third, the structures we consider to test local satisfiability are local: contexts have their own, generally different, domains of interpretation, sets of relations, and sets of functions.

Compatibility. Because of compatibility sequences, contexts mutually influence themselves. Compatibility has the structural effect of changing the set of local models defining each context. It forces local models to agree up to a certain extent. On the one extreme, any two contexts have two independent views of the world. In this case the compatibility relation allows for every pair of sets of local models and there is no relation between what holds in the distinct sets of local models. On the other extreme, any two contexts describe the same world from the same perspective. In this case all the languages are the same, for every local model in a context there is a corresponding compatible identical local model in the other context. In this case all the contexts are a replication of the same context, a compatibility relation is a set of sequences of identical contexts, and we are essentially in the classical situation of one language and one notion of satisfiability and truth.

4. The two examples—model theory

Let us see how the two examples introduced in Section 2 can be modeled by using Local Models Semantics.

4.1. Reasoning with viewpoints

Let us start by defining the propositional languages L_1 and L_2 used by $Mr.1$ and $Mr.2$, respectively, to describe their views. Let $P_1 = \{r, l\}$ and $P_2 = \{r, c, l\}$ be two sets of propositional constants (where intuitively, r, c, l stand for ball on the right, in the center and on the left, respectively). L_1 is formally defined as the smallest set containing P_1, the symbol for falsity $\bot$, and closed under implication; L_2 is formally defined as the smallest set containing P_2, the symbol for falsity $\bot$ and closed under implication. [3]

L_1 and L_2 have the usual propositional semantics. The local models of L_1 are (univocally defined by the following sets of formulae):

$$m_1 = \emptyset, \qquad m_2 = \{l\}, \qquad m_3 = \{r\}, \qquad m_4 = \{l, r\},$$

where we write $\emptyset$ to mean the local model describing the situation with no balls in the box, $\{l\}$ to mean the local model describing the situation with a ball on the left, and so on for the other cases. Analogously, the local models of L_2 are (univocally defined by the following sets of formulae):

$$\begin{array}{llll} m_1 = \emptyset, & m_2 = \{l\}, & m_3 = \{c\}, & m_4 = \{r\}, \\ m_5 = \{l, c\}, & m_6 = \{l, r\}, & m_7 = \{c, r\}, & m_8 = \{l, c, r\}. \end{array}$$

[3] In this paper we use the standard abbreviations from propositional logic, such as $\neg\phi$ for $\phi \supset \bot$, $\phi \vee \psi$ for $\neg\phi \supset \psi$, $\phi \wedge \psi$ for $\neg(\neg\phi \vee \neg\psi)$, $\top$ for $\bot \supset \bot$.

Following the definition given in Section 3, a generic compatibility relation $\boldsymbol{C}$ for the magic box is a relation

$$\boldsymbol{C} \subseteq 2^{\overline{M}_1} \times 2^{\overline{M}_2},$$

where $\overline{M}_1$ ($\overline{M}_2$) is the set of propositional models of L_1 (L_2). A compatibility pair $\langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle \in \boldsymbol{C}$ is a pair of sets of local models, being $\boldsymbol{c}_1$ a set of models of the view of $Mr.1$ and $\boldsymbol{c}_2$ a set of models of the view of $Mr.2$.

Let us construct a model for the scenario described in Fig. 3 (Section 2.1), by imposing the following compatibility constraints:

if $Mr.1$ sees at least one ball, then $Mr.2$ sees at least one ball, (5)

if $Mr.2$ sees at least one ball, then $Mr.1$ sees at least one ball, (6)

$Mr.1$ and $Mr.2$ are able to construct a complete description of their view. (7)

Definition 4.1 (*A model for the magic box*). A model $\boldsymbol{C}$ for the magic box is a compatibility relation such that, for all $\boldsymbol{c} \in \boldsymbol{C}$

$$\text{if for all } m \in \boldsymbol{c}_1,\ m \neq \emptyset, \text{ then for all } m \in \boldsymbol{c}_2,\ m \neq \emptyset, \tag{8}$$

$$\text{if for all } m \in \boldsymbol{c}_2,\ m \neq \emptyset, \text{ then for all } m \in \boldsymbol{c}_1,\ m \neq \emptyset, \tag{9}$$

$$|\boldsymbol{c}_1| = 1 \text{ and } |\boldsymbol{c}_2| = 1. \tag{10}$$

Eq. (8) models constraint (5). In fact, if $Mr.1$ sees a ball then this ball can be on the left or on the right and the local model $\emptyset$ cannot represent his view. Furthermore, in this case, $Mr.2$ sees a ball in one of the three possible positions, and, therefore the local model $\emptyset$ does not represent the view of $Mr.2$. A similar explanation can be given for Eq. (9), which models constraint (6). Eq. (10) is more interesting. It says that $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ contain a single local model, i.e., the magic box model is a chain model. This intuitively means that $Mr.1$ and $Mr.2$ have a complete model of their point of view about the box, namely, that both $Mr.1$ and $Mr.2$ see the box (from their point of view) and are able to construct a complete description of it. As a consequence of Eq. (10), a model $\boldsymbol{C}$ for the magic box example in Fig. 3 is a set of pairs $\langle \{m_1\}, \{m_2\} \rangle$ where m_1 and m_2 are local models of L_1 and L_2, respectively. Each pair corresponds to a possible combination of the observers' partial views.

Notice that Eq. (10) cannot be used in defining a model for the scenario depicted in Fig. 8. Indeed in that scenario $Mr.2$ is not able to construct a complete description of the box. Therefore the requirement $|\boldsymbol{c}_2| = 1$ must be removed from Eq. (10). Models for the scenario depicted in Fig. 8 are therefore sets of pairs $\langle \{m_1\}, \boldsymbol{c}_2 \rangle$ where $\boldsymbol{c}_2$ may contain different local models.

From now on, we call V the class of models introduced in Definition 4.1; we refer to a model in V as V-model for short, and to the logical consequence with respect to the class of V-models as V-logical consequence, in symbols $\models_{\mathsf{V}}$. The V-model containing *all* and *only* the chains depicted in Fig. 3 is the following:

$$\left\{\begin{array}{c}\langle\{\neg l,\neg r\},\{\neg l,\neg c,\neg r\}\rangle\\ \langle\{l,\neg r\},\{l,\neg c,\neg r\}\rangle\\ \langle\{l,\neg r\},\{\neg l,c,\neg r\}\rangle\\ \langle\{l,\neg r\},\{\neg l,c,r\}\rangle\\ \dots\\ \langle\{l,r\},\{l,c,\neg r\}\rangle\\ \langle\{l,r\},\{l,c,r\}\rangle\end{array}\right\}.$$

The models in V are all subsets of this model.

Example 4.1. It is easy to see that in all the V-models, if $Mr.1$ sees no balls then $Mr.2$ sees no balls (formally, $1:\neg l\wedge\neg r\models_{\mathsf{V}} 2:\neg l\wedge\neg c\wedge\neg r$).

To prove this, let us consider all the pairs $\langle \boldsymbol{c}_1,\boldsymbol{c}_2\rangle$ such that $\boldsymbol{c}_1$ satisfies $\neg l\wedge\neg r$. Suppose that there exists a $\boldsymbol{c}_2$ which does not satisfy $\neg l\wedge\neg c\wedge\neg r$. From Eq. (10) we know that $\boldsymbol{c}_2$ contains exactly a propositional local model. Therefore, $\boldsymbol{c}_2$ satisfies $l\vee c\vee r$ and the local model contained in $\boldsymbol{c}_2$ is not $\emptyset$. From Eq. (9) we obtain that, for all $m\in\boldsymbol{c}_1$, $m\neq\emptyset$. This is impossible because, from the hypothesis, we know that $\boldsymbol{c}_1$ satisfies $\neg l\wedge\neg r$. Therefore the hypothesis that $\boldsymbol{c}_2$ does not satisfy $\neg l\wedge\neg c\wedge\neg r$ must be false. Thus $1:\neg l\wedge\neg r\models_{\mathsf{V}} 2:\neg l\wedge\neg c\wedge\neg r$.

In a similar way, we can also prove the dual, that is, $2:\neg l\wedge\neg c\wedge\neg r\models_{\mathsf{V}} 1:\neg l\wedge\neg r$. These two logical consequences express the fact that

$$\text{for all } m\in\boldsymbol{c}_1,\ m=\emptyset \text{ if and only if for all } m\in\boldsymbol{c}_2,\ m=\emptyset \tag{11}$$

holds in Definition 4.1. It is easy to notice that Eqs. (8), (9), and (11) capture all the compatibility pairs represented in Fig. 3 (Eq. (11) capturing the one at the top). Eq. (10) in Definition 4.1 could therefore be substituted with Eq. (11).

4.2. Reasoning about belief

We consider a scenario involving an infinite chain of belief contexts, that is, an agent a able to express and reason about beliefs of arbitrary nesting. Let us start by defining the languages $L_0, L_1, L_2, \dots$ (over $I=\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers including 0), where the language L_0 is the language of context a, the language L_1 is the language of context aa, the language L_n is the language of context $aa\dots a$ ($n+1$ times), and so on. To express statements about the world, every L_n contains a set P of propositional constants. To express beliefs about beliefs described with L_{n+1}, L_n contains a predicate B, which intuitively stands for belief, and a name "ϕ" for each formula ϕ in L_{n+1}. Since each context is "above" an infinite chain and each level corresponds to a level of nesting of the belief predicate, all the languages L_i, with $i\in\mathbb{N}$, must have the same expressibility. Therefore, all languages are the same language $L(B)$ containing all the propositional formulae ϕ, $B(\text{“}\phi\text{”})$, $B(\text{“}B(\text{“}\phi\text{”})\text{”})$, $B(\text{“}B(\text{“}B(\text{“}\phi\text{”})\text{”})\text{”})$, and so on.

Formally, we define $L(B)$ as follows. Let L be a propositional language containing a set P of propositional letters, the symbol for falsity $\perp$, and closed under implication. Then for any natural number $i\in\mathbb{N}$, we define a language L_i as follows:

- if $\phi \in L$, then $\phi \in L_i$;
- $\bot \in L_i$;
- if $\phi \in L_i$ and $\psi \in L_i$, then $\phi \supset \psi \in L_i$;
- if $\phi \in L_i$, then $B(``\phi") \in L_{i+i}$;
- nothing else is in L_i.

$L(B)$ is defined as the union of all the L_n, i.e., $L(B) = \bigcup_{n\in\mathbb{N}} L_n$.

From now on we call HMB languages (where HMB stands for *Hierarchical Multilanguage Belief*) the family $\{L_i\}$ of languages over the set of indexes $\mathbb{N}$ such that for every $i \in \mathbb{N}$, $L_i = L(B)$.

An HMB language $\{L_i\}$ is a family of propositional languages containing the propositional letters in P, used to express statements about the world, and "special" propositional letters $B(``\phi")$, used to express beliefs about beliefs. Hence each $L_i$ has the usual propositional semantics. The local models of each $L_i$ are univocally defined by a subset of propositional letters in $P$ and a subset of "special" propositional letters of the form $B(``\phi")$. The satisfiability relation is the usual one between propositional models and propositional formulae.

Following the definition given in Section 3, a generic compatibility relation $\boldsymbol{C}$ for an HMB language is a relation

$$\boldsymbol{C} \subseteq \prod_{i\in\mathbb{N}} 2^{\overline{M}_i},$$

where each $\overline{M}_i$ is the set of propositional models of L_i. A sequence $\langle \boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_i, \ldots\rangle \in \boldsymbol{C}$ is a sequence of sets of local models, $\boldsymbol{c}_0$ being a set of models of a, $\boldsymbol{c}_1$ a set of models of aa, and so on. A set of sequences (i.e., a model of an HMB language) formalizes different sequences of mental images (contexts) that a has of itself, its own beliefs, its beliefs about beliefs, and so on, in possibly different situations.

Let us construct a model for a class of HMB languages by imposing the following compatibility constraints:

$$\text{whenever it believes } B(``\phi"), \text{ then } a \text{ believes that it believes } \phi; \tag{12}$$

$$\begin{aligned}&a \text{ believes } B(``\phi") \text{ only if}\\ &\quad\text{it believes that it believes } \phi \text{ in all the admissible situations.}\end{aligned} \tag{13}$$

Let us first consider constraint (12). Semantically, (12) imposes that, for all the compatibility sequences $\boldsymbol{c}$ in a model $\boldsymbol{C}$, if $\boldsymbol{c}_i$ satisfies $B(``\phi")$, then $\boldsymbol{c}_{i+1}$ satisfies ϕ. In order to define the structural relation formalizing (12) we introduce some extra notation.

- Let $\boldsymbol{c}_i$ be an element of a compatibility sequence $\boldsymbol{c}$. We write $\Theta(\boldsymbol{c}_i)$ to mean the set of L_i-formulae which are satisfied by all the local models in $\boldsymbol{c}_i$. Formally,

$$\Theta(\boldsymbol{c}_i) = \{\phi \mid \forall m \in \boldsymbol{c}_i \ m \models_{cl} \phi\}.$$

- Let Γ be a set of L_i-formulae. We write $B^{-1}(``\Gamma")$ to mean the set of $L_{i+1}$-formulae $\phi$ such that $B(``\phi")$ belongs to Γ.

$\Theta(\boldsymbol{c}_i)$ characterizes the formulae satisfied by the ith context in a sequence $\boldsymbol{c}$, while $B^{-1}(``\Gamma")$ characterizes a set of formulae obtained by "removing" the belief operator B to a set of formulae Γ. The structural constraint modeling (12) is obtained by imposing that all the sequences $\boldsymbol{c}$ in a model $\boldsymbol{C}$ satisfy the following property:

$$B^{-1}(\text{“}\Theta(\boldsymbol{c}_i)\text{”}) \subseteq \Theta(\boldsymbol{c}_{i+1}). \tag{14}$$

Eq. (14) imposes that all the L_{i+1}-formulae obtained by “removing” the belief operator B to the set of L_i-formulae satisfied by $\boldsymbol{c}_i$ are contained into the set $\Theta(\boldsymbol{c}_{i+1})$ of formulae satisfied by $\boldsymbol{c}_{i+1}$. This implies that for every sequence $\boldsymbol{c} \in \boldsymbol{C}$ if $\boldsymbol{c}_i$ satisfies $B(\text{“}\phi\text{”})$, then $\boldsymbol{c}_{i+1}$ satisfies ϕ.

Let us now turn to constraint (13). We start by noticing that different compatibility sequences may have common parts. For instance, given two sequences $\langle \boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_i, \ldots\rangle$ and $\langle \boldsymbol{c}'_0, \boldsymbol{c}'_1, \ldots, \boldsymbol{c}'_i, \ldots\rangle$ in $\boldsymbol{C}$, the two contexts $\boldsymbol{c}_i$ and $\boldsymbol{c}'_i$ may coincide (namely, $\boldsymbol{c}_i = \boldsymbol{c}'_i$), or partially coincide (namely, $\boldsymbol{c}_i \cap \boldsymbol{c}'_i \neq \emptyset$). Among partially coinciding contexts, an interesting case is given by $\boldsymbol{c}'_i \subseteq \boldsymbol{c}_i$. According to our interpretation of a belief context as a partial description of a mental image, $\boldsymbol{c}'_i \subseteq \boldsymbol{c}_i$ means that the description contained in the belief context $\boldsymbol{c}'_i$ is less partial (or more complete) than the one contained in $\boldsymbol{c}_i$. Notationally, if $\boldsymbol{c}'_i \subseteq \boldsymbol{c}_i$ we say that the sequence $\boldsymbol{c}'$ is *i-admissible* for the sequence $\boldsymbol{c}$. Analogously, we say that all the elements $\boldsymbol{c}'_j$ in $\boldsymbol{c}'$ are *i-admissible* for the sequence $\boldsymbol{c}$. Given a model $\boldsymbol{C}$ and a compatibility sequence $\boldsymbol{c}$, the notion of i-admissibility enables us to characterize the set of sequences $\boldsymbol{c}' \in \boldsymbol{C}$ whose belief contexts $\boldsymbol{c}'_i$ are less partial (or more complete) than the belief context $\boldsymbol{c}_i$ in the given $\boldsymbol{c}$. The notion of i-admissibility is important whenever we are interested in defining a compatibility relation $\boldsymbol{C}$ by imposing constraints on sets of belief contexts belonging to different compatibility sequences. For instance, we may define a compatibility relation $\boldsymbol{C}$ by imposing a certain relation between a belief context $\boldsymbol{c}_i$ and all its i-admissible sequences. Although this is slightly more complicated than defining a compatibility relation simply by imposing a certain constraint on two (or more) belief contexts $\boldsymbol{c}_i$, and $\boldsymbol{c}_j$ in the same sequence (as, e.g., in Eq. (14)), it enables us to express compatibility constraints involving more than one sequence at once. This is, in fact, also the case for the modeling of constraint (13).

Semantically, constraint (13) imposes that, for all the compatibility sequences $\boldsymbol{c}$ in a model $\boldsymbol{C}$, $\boldsymbol{c}_i$ satisfies $B(\text{“}\phi\text{”})$ only if all the $\boldsymbol{c}'_{i+1}$, that are i-admissible for $\boldsymbol{c}$, satisfy ϕ. Notice that the notion of i-admissibility has been used here in order to model the informal notion of admissibility in constraint (13). In order to formally define the structural relation formalizing (13) we introduce some extra notation.

- Let $\boldsymbol{C}$ be a compatibility relation and $\boldsymbol{c}$ a compatibility sequence in $\boldsymbol{C}$. We write $V^{\downarrow}(\boldsymbol{c}_i)$ to mean the set of L_{i+1}-formulae which are satisfied by every element $\boldsymbol{c}'_{i+1}$ which is i-admissible for $\boldsymbol{c}$. Formally,

$$V^{\downarrow}(\boldsymbol{c}_i) = \bigl\{\phi \in L_{i+1} \mid \forall \boldsymbol{c}' \in \boldsymbol{C},\ \boldsymbol{c}'_i \subseteq \boldsymbol{c}_i \ \Rightarrow\ \phi \in \Theta(\boldsymbol{c}'_{i+1})\bigr\}.$$

- Let Γ be a set of L_i-formulae, we write $B(\text{“}\Gamma\text{”})$ to mean the set of L_{i-1}-formulae $B(\text{“}\phi\text{”})$ such that ϕ belongs to Γ, $i > 0$;

$V^{\downarrow}(\boldsymbol{c}_i)$ characterizes the formulae satisfied by all the $(i+1)$th contexts within the i-admissible sequences of a given sequence $\boldsymbol{c}$. That is, given the set of sequences $\boldsymbol{c}' \in \boldsymbol{C}$ whose belief contexts $\boldsymbol{c}'_i$ are less partial than the belief context $\boldsymbol{c}_i$, $V^{\downarrow}(\boldsymbol{c}_i)$ characterizes the formulae satisfied by all the sequences $\boldsymbol{c}'$ at one more nesting in the structure of belief contexts with respect to $\boldsymbol{c}_i$. $B(\text{“}\Gamma\text{”})$ characterizes sets of formulae obtained by “applying” the belief operator B to a set of formulae Γ. Constraint (13) is obtained by imposing that all the sequences $\boldsymbol{c}$ in a model $\boldsymbol{C}$ satisfy the following property:

$$B(\text{“}V^{\downarrow}(\boldsymbol{c}_i)\text{”}) \subseteq \Theta(\boldsymbol{c}_i). \tag{15}$$

Eq. (15) imposes that all the L_i-formulae obtained by applying the belief operator B to the set of L_{i+1}-formulae satisfied by the i-admissible sequences for $\boldsymbol{c}$, are contained into the set $\Theta(\boldsymbol{c}_i)$ of formulae satisfied by $\boldsymbol{c}_i$. This implies that for every sequence $\boldsymbol{c} \in \boldsymbol{C}$, $\boldsymbol{c}_i$ satisfies $B(\text{“}\phi\text{”})$ only if all the i-admissible sequences $\boldsymbol{c}'$ of $\boldsymbol{c}$ are such that $\boldsymbol{c}'_{i+1}$ satisfies ϕ.

Definition 4.2 (*HMB models*). A model $\boldsymbol{C}$ for the belief example (HMB model) is a compatibility relation satisfying at least one among properties (14) and (15).

Models satisfying Eq. (14) are called *$\mathcal{R}$dw-models*, models satisfying Eq. (15) are called *$\mathcal{R}$upr-models*, and models satisfying both (14) and (15) are called *MBK-models*.

Example 4.2. For any MBK-model $\boldsymbol{C}$ and any $i \in \mathbb{N}$,

$$\boldsymbol{C} \models i : B(\text{“}\phi \supset \psi\text{”}) \supset \big(B(\text{“}\phi\text{”}) \supset B(\text{“}\psi\text{”})\big).$$

To prove this, we need to show that all the compatibility sequences in $\boldsymbol{C}$ satisfy $i : B(\text{“}\phi \supset \psi\text{”}) \supset (B(\text{“}\phi\text{”}) \supset B(\text{“}\psi\text{”}))$. Suppose that $\boldsymbol{c}_i$ satisfies both $B(\text{“}\phi \supset \psi\text{”})$ and $B(\text{“}\phi\text{”})$. From condition (14) in the definition of an MBK-model every $\boldsymbol{c}'_{i+1}$ i-admissible for $\boldsymbol{c}$ satisfies both $\phi \supset \psi$ and ϕ. Being all the local models in $\boldsymbol{c}'_{i+1}$ propositional models, they satisfy also ψ. Therefore, from condition (15) in the definition of MBK-model, $\boldsymbol{c}_i$ satisfies $B(\text{“}\psi\text{”})$.

5. The proof theory: MC systems

The goal of this section is to give a brief introduction to the notion of a formal system allowing multiple contexts, called *Multi-Context system (MC system)*, where contexts are formalized proof-theoretically. MC systems were first introduced in [25]. A more theoretical presentation is given in [27]. The formalization of MC systems used in this paper was first given in [23]. The novelty here is that we show how MC systems actually formalize the notions of locality and compatibility introduced in Section 1, that we use them to formalize the magic box scenario, and that we provide soundness and completeness results with respect to Local Models Semantics.

Definition 5.1 (*MC system*). Let I be a set of indexes. A *Multi-Context system* (*MC system*) *MS* is a pair

$$MS = \langle \{T_i\}, \Delta_{br} \rangle,$$

where:

- for each $i \in I$, $T_i = \langle L_i, \Omega_i, \Delta_i \rangle$ is an axiomatic formal system where L_i is the language, $\Omega_i \subseteq L_i$ is the set of axioms, and Δ_i is the set of inference rules;
- Δ_{br} is a set of inference rules with premises and conclusions in different languages.

An MC system is essentially a set of logical theories, plus a set of inference rules which allow for the propagation of consequences among theories. MC systems are a generalization of Natural Deduction (ND) systems [40]. The generalization amounts to use formulae tagged with the language they belong to. This allows for the effective use of the multiple languages. The deduction machinery of an MC system is composed of two kinds of inference rules: the inference rules in each Δ_i, called *internal rules,* and the inference rules in Δ_{br}, called *bridge rules*. Internal rules are inference rules with premises and conclusions in the same language, while bridge rules are inference rules with premises and conclusions belonging to different languages. Notationally, inference rules are written as follows:

$$\frac{i:\phi_1 \quad \ldots \quad i:\phi_n}{i:\psi}\,ir, \qquad \frac{i_1:\phi_1 \quad \ldots \quad i_n:\phi_n}{j:\psi}\,br,$$

where *ir* is an internal rule, while *br* is a bridge rule. Internal rules allow us to draw consequences inside a theory, while bridge rules allow us to export results from one theory to another. Indeed *ir* allows us to derive the formula ψ from the formulae $\phi_1, \ldots, \phi_n$ in the theory tagged with i, while *br* allows us to export the formula ψ to the theory tagged with j because of the fact that all the $\phi_1, \ldots, \phi_n$ are derivable in the theories tagged with $i_1, \ldots, i_n$, respectively. From now on, we write Δ to mean the deduction machinery of an MC system, i.e., $\Delta = \bigcup_{i\in I} \Delta_i \cup \Delta_{br}$. Using ND and following [40] in the notation and terminology, Δ contains also inference rules which discharge assumptions, written as:

$$\frac{i_1:\phi_1 \quad \ldots \quad i_n:\phi_n \quad \begin{array}{c}[k_1:\gamma_1]\\ \Pi_1\\ i_{i+1}:\phi_{n+1}\end{array} \quad \ldots \quad \begin{array}{c}[k_m:\gamma_m]\\ \Pi_m\\ i_{n+m}:\phi_{n+m}\end{array}}{j:\psi}\,dr$$

dr represents an inference rule which allow to infer $j:\psi$ from $i_1:\phi_1, \ldots, i_n:\phi_n$ discharging the assumptions $k_1:\gamma_1, \ldots, k_m:\gamma_m$.

Notationally, we use the Greek letter Π (possibly with subscripts) to denote deductions. For instance, in the inference rule *dr* above, Π_1 represents a deduction of $i_{i+1}:\phi_{n+1}$ from the assumption $k_1:\gamma_1$.

In Fig. 12 we show the construction of an MC system containing three logical theories and four bridge rules. We start from different languages, e.g., L_1, L_2, and L_3. Then, we associate each of them with a logical theory $T_i = \langle L_i, \Omega_i, \Delta_i\rangle$. Finally, we connect different logical theories with bridge rules, e.g., br_1, br_2, br_3, and br_4. The final result is an MC system.

Deductions in MC systems are trees of formulae built starting from a finite number of assumptions and axioms, possibly belonging to distinct languages, and by applying a finite number of inference rules. A formula $i:\phi$ is *derivable* from a set of formulae Γ in an MC system *MS*, in symbols $\Gamma \vdash_{MS} i:\phi$ if there is a deduction with bottom formula $i:\phi$ whose undischarged assumptions are in Γ. A formula $i:\phi$ is a *theorem* in *MS*, in symbols $\vdash_{MS} i:\phi$, if it is derivable from the empty set. The *deductive closure* of MS is denoted by $Th(MS)$ and is formally defined as $Th(MS) = \{i:\phi \mid \vdash_{MS} i:\phi\}$. A deduction in an MC system can be seen as composed of sub-deductions in distinct languages, obtained by

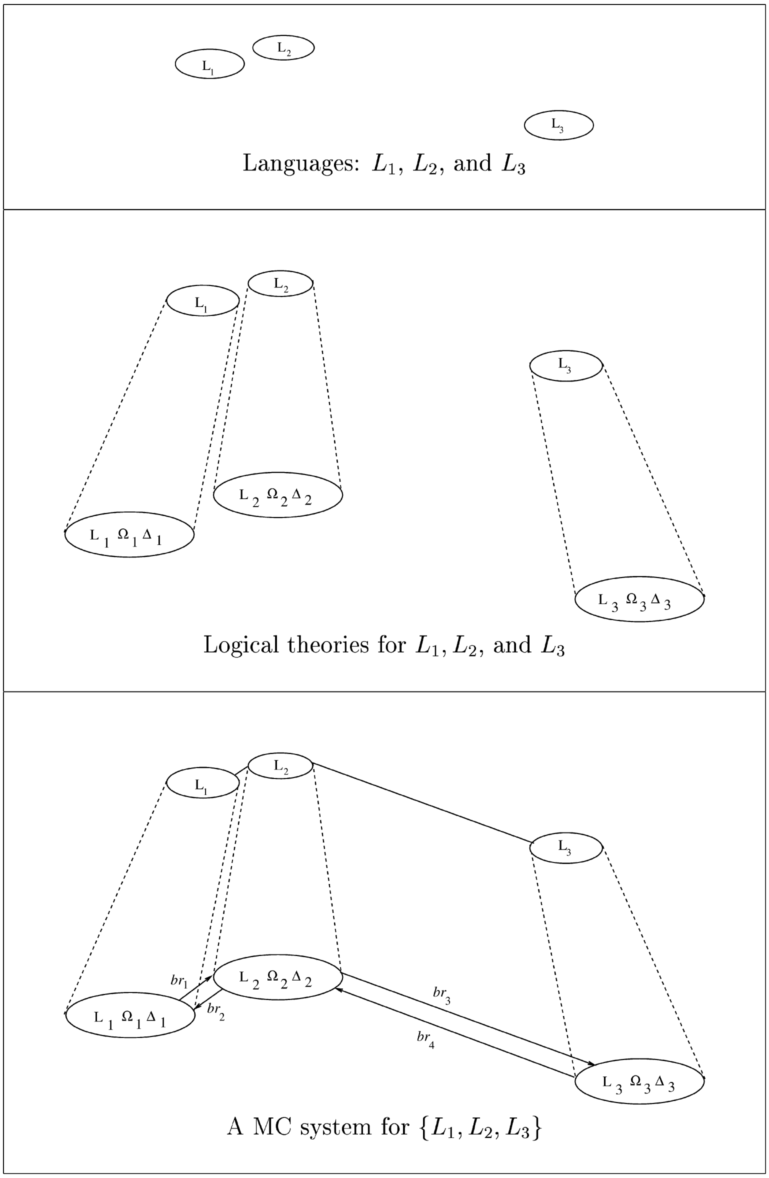

Fig. 12. The construction of a MC system.

repeated applications of internal rules, any two or more sub-deductions being concatenated by one or more applications of bridge rules. [4]

[4] MC systems can be thought of as particular Labelled Deductive Systems (LDSs) [17]. In particular MC systems are LDSs where labels are used only to keep track of the language formulae belong to, and where inference rules can be applied only to formulae belonging to the "appropriate" language.

Given an MC system $MS = \langle \{T_i\}, \Delta_{br} \rangle$ we formally define a *context* c_i to be the set of L_i-formulae belonging to the deductive closure $Th(MS)$ of MS. Formally, $c_i = Th(MS) \cap L_i$.

The intuition underlying the notion of context is that, proof-theoretically, a context consists of that set of formulae which are locally theorems, given also the theorems which can be derived (via applications of bridge rules) from theorems in other contexts. It can be noticed that the notion of context given above is the proof-theoretical counterpart of the notion of context introduced in Section 3.2.

An MC system formalizes the principles of locality and compatibility in the following sense:

Locality. First of all the signature and the notion of well formed formula is localized and distinct for each context c_i. This is achieved by providing a language L_i to each context c_i. Second, the set of facts (axioms) Ω_i which provides the context of reasoning (namely, describes what is true in a context) is local to c_i. Finally the inference engines Δ_i are distinct for each context. This allows us to localize the form of reasoning to each distinct context c_i and to define special inference engines which exploit the local form of formulae (e.g., we can use PROLOG on clausal languages) and capture different deduction capabilities.

Compatibility. Bridge rules in Δ_{br} formalize compatibility. Indeed via bridge rules, contexts mutually influence themselves. For instance, a bridge rule

$$\frac{j:\psi}{i:\phi}$$

has the effect of deriving ϕ in the context c_i because of the fact that another formula, ψ, has been derived in the context c_j. Bridge rules change the set of formulae derived in each context. Bridge rules force contexts to agree up to a certain extent. On one extreme the two contexts might have two independent views of the world. In this case we have a set of bridge rules which is the empty set and there is no relation between what is derivable in the distinct contexts. On the other extreme the two contexts describe the same world from the same perspective. This situation can be imposed by asking that all the languages, sets of axioms, and deduction rules are the same, and that the two contexts c_i and c_j are linked by the following bridge rules:

$$\frac{i:\phi}{j:\phi}, \qquad \frac{j:\phi}{i:\phi}.$$

In this case all the contexts consist of the same set of provable formulae.

6. The two examples—proof theory

Let us see how the two examples, described in Section 2, can be formalized using MC systems.

6.1. Reasoning with viewpoints

Let us start by defining the MC system $\mathsf{MV} = \langle \{T_1, T_2\}, \Delta_{br} \rangle$ modeling the magic box scenario depicted in Fig. 3. Let the two languages used by $Mr.1$ and $Mr.2$ be the two propositional languages defined in Section 4.1, that is L_1 (L_2) is the smallest set containing $\{r, l\}$ ($\{r, c, l\}$) and closed under the standard propositional connectives. To the purpose of this example we suppose that $\Omega_1 = \Omega_2 = \emptyset$. This formalizes the fact that we do not commit ourselves to any particular partial view among the ones depicted in Fig. 3. Since each partial view is modeled using propositional models, both Δ_1 and Δ_2 contain the following MC version of Natural Deduction rules for propositional calculus:

$$\frac{\begin{array}{c}[i:\phi]\\ \Pi\\ i:\psi\end{array}}{i:\phi\supset\psi}\supset\mathrm{I}_i,\qquad \frac{i:\phi\quad i:\phi\supset\psi}{i:\psi}\supset\mathrm{E}_i,\qquad \frac{\begin{array}{c}[i:\neg\phi]\\ \Pi\\ i:\bot\end{array}}{i:\phi}\bot_i.$$

The key part in the construction of the MC system MV is the formalization of compatibility constraints (5), (6), and (7) in Section 4.1. This is achieved by adding the following bridge rules to Δ_{br}

$$\frac{1:l\vee r}{2:l\vee c\vee r}br_{12},\qquad \frac{2:l\vee c\vee r}{1:l\vee r}br_{21},\qquad \frac{\begin{array}{c}[2:\neg\phi]\\ \Pi\\ 1:\bot\end{array}}{2:\phi}\bot_{12},\qquad \frac{\begin{array}{c}[1:\neg\phi]\\ \Pi\\ 2:\bot\end{array}}{1:\phi}\bot_{21}.$$

br_{12} formalizes constraint (5) in Section 4.1. In fact, if *Mr*.1 sees at least a ball in the box, then $1:l\vee r$ is derivable in his context. Furthermore, in this case *Mr*.2 sees a ball in one of the three possible positions, and therefore $2:r\vee c\vee l$ is derivable in his context. A similar explanation can be given for br_{21} which formalizes constraint (6) in Section 4.1. $\bot_{12}$ and $\bot_{21}$ formalize the fact that both *Mr*.1 and *Mr*.2 are able to construct a complete description of their view, and are the proof theoretical counterpart of constraint (7) in Section 4.1. Let us start by noticing that $\bot_{12}$ and $\bot_{21}$ are some kind of generalization of the classical law of reasoning by absurdum. $\bot_{12}$ and $\bot_{21}$ can be intuitively motivated as follows. Since we have contexts which are single models, then either ϕ or $\neg\phi$ holds. As a consequence, if assuming $\neg\phi$ in one context generates an inconsistency in another context, then it is possible to conclude that $\neg\phi$ doesn't hold in the first context, and therefore that ϕ holds.

Example 6.1. In MV, if *Mr*.1 sees no balls, then *Mr*.2 sees no balls. Formally:

$$1:\neg l\wedge\neg r\vdash_{\mathsf{MV}} 2:\neg l\wedge\neg c\wedge\neg r.$$

The proof in a Natural Deduction-like style is given in Fig. 13. Deductions local to the contexts describing *Mr*.1 (*Mr*.2) are surrounded by boxes labelled *Mr*.1 (*Mr*.2). This emphasizes the fact that a deduction in the MC system can be seen as composed of sub-deductions in distinct languages (L_1, and L_2), obtained by repeated applications of internal rules, these sub-deductions being concatenated by one or more applications of the bridge rules in Δ_{br}. Let us describe the deduction tree in detail. First we assume $2:l\vee c\vee r$ in the context of *Mr*.2. Applying br_{21} to this formula we deduce $1:l\vee r$ in the context of *Mr*.1. Then we assume $1:\neg l\wedge\neg r$ and applying ND rules of propositional calculus we obtain $1:\neg(l\vee r)$. From $1:l\vee r$ and $1:\neg(l\vee r)$ we obtain $1:\bot$. Applying the bridge rule $\bot_{12}$ we deduce $2:\neg(l\wedge c\wedge r)$ discharging the assumption $2:l\vee c\vee r$ in the context of

$$\cfrac{\boxed{\cfrac{\cfrac{\boxed{[2: l \vee c \vee r]}_{Mr.2}}{1: l \vee r}\,br_{21} \qquad \cfrac{1: \neg l \wedge \neg r}{1: \neg(l \vee r)}}{1: \bot}}_{Mr.1}}{\boxed{\cfrac{2: \neg(l \vee c \vee r)}{2: \neg l \wedge \neg c \wedge \neg r}}_{Mr.1}}\,\bot_{12}$$

Fig. 13. A deduction tree in MV.

$Mr.2$. Finally we obtain $2 : \neg l \wedge \neg c \wedge \neg r$ with a deduction involving rules of propositional calculus.

The MC system MV presented in this section can be proved to be a sound and complete axiomatization of the Local Models Semantics for the magic box scenario presented in Section 4.1, Fig. 3. This result is stated and proved in Appendix A.

6.2. *Reasoning about belief*

The idea underlying the formalization of the belief example using MC systems is straightforward. Every view is formalized by a theory T_i. To obtain the desired behavior, that is to make a able to reason about its own beliefs, it is sufficient to "link" deduction in the theory representing a's beliefs and deduction in the theory representing the mental images that a has of itself. "Links" are provided by bridge rules. Depending on the kind of bridge rule, a will have different reasoning capabilities.

Formally, an HMB system is an MC system $\langle \{T_i\}, \Delta_{br} \rangle$ defined over the index $I = \mathbb{N}$. For every $i \in \mathbb{N}$, the language L_i of the theory T_i is the language $L(B)$ defined in Section 4.2. For this example we assume $\Omega_i = \emptyset$. Since each view is modeled using propositional models, each Δ_i contains the MC version of Natural Deduction rules for propositional calculus described in Section 6.1. The key part in the construction of an HMB system is the formalization of compatibility constraints (12) and (13) in Section 4.2. This is achieved by adding the following bridge rules to Δ_{br}:

$$\frac{i : B(\text{“}\phi\text{”})}{i+1 : \phi}\mathcal{R}dw_i, \qquad \frac{i+1 : \phi}{i : B(\text{“}\phi\text{”})}\mathcal{R}upr_i.$$

RESTRICTIONS: $\mathcal{R}upr_i$ is applicable if and only if $i + 1 : \phi$ does not depend on any assumption $j : \psi$ with index $j \geqslant i + 1$.

$\mathcal{R}dw_i$ formalizes constraint (12) in Section 4.2. If $B(\text{“}\phi\text{”})$ is assumed in the context c_i, then a is able to conclude that ϕ holds in the context c_{i+1}. $\mathcal{R}upr_i$ formalizes constraint (13) in Section 4.2. If a is able to infer ϕ in the context c_{i+1} from a set of assumptions with index $j < i + 1$, then ϕ holds in all the contexts c'_{i+1} compatible with such a set of assumptions. In this case, and only in this case, a is able to infer $B(\text{“}\phi\text{”})$ in the context c_i. Intuitively,

$$\cfrac{\boxed{\cfrac{\cfrac{i:B(``\psi")}{i:B(``\phi") \supset B(``\psi")}\supset \mathrm{I}_i}{i:B(``\phi \supset \psi") \supset (B(``\phi") \supset B(``\psi"))}\supset \mathrm{I}_i}_{c_i}}{} \quad \text{from} \quad \cfrac{\boxed{\cfrac{\cfrac{\boxed{[i:B(``\phi")]}_{c_i}}{i+1:\phi}\mathcal{R}dw_i \qquad \cfrac{\boxed{[i:B(``\phi \supset \psi")]}_{c_i}}{i+1:\phi \supset \psi}\mathcal{R}dw_i}{i+1:\psi}\supset \mathrm{E}_{i+1}}_{c_{i+1}}}{i:B(``\psi")}\mathcal{R}upr_i$$

Fig. 14. A deduction tree in MBK.

the restriction on $\mathcal{R}upr_i$ prevents the case in which a consequence of an assumption in a belief context is treated, by the context above, as a theorem of that belief context. Notice that, the restriction on $\mathcal{R}upr_i$ corresponds to the fact that constraint (13) involves sets of i-admissible sequences.

The HMB system containing only bridge rules of the form $\mathcal{R}dw_i$ is called $\mathcal{R}dw$; the HMB system containing only bridge rules of the form $\mathcal{R}upr_i$ is called $\mathcal{R}upr$; the HMB system containing both $\mathcal{R}dw_i$ and $\mathcal{R}upr_i$ is called MBK. Giunchiglia and Serafini [27] show that, in MBK, the theory of each view is theorem equivalent with the minimal normal modal logic K. For a detailed investigation on MC systems obtained by imposing different combinations of bridge rules of the form $\mathcal{R}up$ and $\mathcal{R}dw$, called *reflection rules*, good references are [10,11], where different MC systems for the formalization of meta-reasoning are defined and studied. Another reference is [18] where Local Models Semantics is used to define classes of models for MC systems containing different reflection rules.

Example 6.2. It is easy to see that for any $i \in \mathbb{N}$,

$$\vdash_{\mathrm{MBK}} i : B(``\phi \supset \psi") \supset \big(B(``\phi") \supset B(``\psi")\big).$$

The proof in a Natural Deduction-like style is given in Fig. 14. Let us describe it in detail. First, we assume $i : B(``\phi")$ and $i : B(``\phi \supset \psi")$ in the context c_i. Applying $\mathcal{R}dw_i$ to these formulae we deduce $i + 1 : \phi$ and $i + 1 : \phi \supset \psi$ in c_{i+1} and we obtain $i + 1 : \psi$ in the same context by propositional reasoning. Applying the bridge rule $\mathcal{R}upr_i$ we deduce $i : B(``\psi")$. Then we obtain $i : B(``\phi \supset \psi") \supset (B(``\phi") \supset B(``\psi"))$ by applying the $\supset \mathrm{I}_i$ rule two times and discharging the assumptions $i : B(``\phi")$ and $i : B(``\phi \supset \psi")$.

The MC systems $\mathcal{R}dw$, $\mathcal{R}upr$ and MBK presented in this section can be proved to be sound and complete with respect to the class of $\mathcal{R}dw$-models, $\mathcal{R}upr$-models, MBK-models, respectively, defined in Section 4.2. This result is stated and proved in Appendix B. As a consequence of this result, the class of MBK-models formalizes an ideal agent a theorem equivalent to the minimal normal modal logics K. On the other hand, $\mathcal{R}dw$-models and $\mathcal{R}upr$-models formalize agents having extremely weak reasoning capabilities. Notice therefore that the representation of an agent's beliefs based on the notion of local semantics and compatibility relation provides enough modularity and flexibility to model agents with different reasoning capabilities in a uniform way. Ghidini [19] gives a more general definition of HMB model and shows how various forms of ideal and real agents (including agents with bounded reasoning capabilities) are modeled by using Local Models Semantics. Notice also how we construct the models of combinations of constraints (e.g., the MBK-model) simply by taking the intersection of the models of the constituent constraints (e.g., constraints modeled by Eqs. (14) and (15)).

7. Other frameworks—a comparison

The obvious, most studied, framework to start from is possible worlds semantics [32]. Both Local Models Semantics and possible worlds semantics allow for multiple objects (models or worlds) and have a notion of local satisfiability (to a local model, to a possible world). However there are also some important differences. First, in possible worlds there is a unique language which describes what is true in all the worlds and there is no notion of truth of a labelled formula. This is the case also for the extensions of possible worlds semantics aimed at formalizing local reasoning (see, e.g., [13]), where localization is achieved by adding a new modal operator to the language. Second, worlds are not (Tarskian) models, the key difference being that possible worlds allow for the use of modal operators. The satisfiability of a formula containing a modal operator is defined in terms of the accessibility relation, which must therefore be given while defining satisfiability in a world. The notion of satisfiability in a world is a function of the model of which the world is part. This is not the case for Local Models Semantics where each local model has its own notion of satisfiability. In Local Models Semantics, the model and its structure influence only the set of local models under consideration. The hypothesis of using a single unique global language and of being able to describe a priori the structure of the model under consideration is very useful and works in many situations. It does not seem to work in those cases where there is no global scheme describing the system, e.g., the federation of heterogeneous data or knowledge bases or multi-agent systems.

In the last few years various semantics for contextual reasoning have been proposed. Most of them are based on possible worlds semantics. As far as we know, the first attempt is described in [29]. In this work there is a notion of labelled formula and of (local) satisfiability to a set of (local) possible worlds. This semantics works well for contextual logics equivalent to modal K or stronger. Its main limitation is that it is not clear how to extend it to other logics, e.g., nonnormal modal logics or logics for reasoning with viewpoints.

Guha, in his Ph.D. Thesis [31], informally describes a semantics for reasoning with context. Understanding Guha's informal definitions is a non-trivial task. Some of the main ideas seem the following. There is a single global language from which it is possible to extract the (local) languages of all the contexts. There seems to be a notion of satisfaction of labelled formulae, and a notion of labelled formulae being meaningless in a context. There is distinguished symbol `ist`, whose intuitive meaning is "is true", which seems treated as a modal operator. Guha's semantics has been partially formalized in the work by Buvac and his co-authors (see for instance [7]). Buvac's semantics seems to have the same features and defects as the semantics in [29], with the further complication that, starting from a single language, there is a lot of work to do in order to achieve locality. In particular the formulae of the global language which are meaningless in a context must be treated as such (this is done using Bochvar three valued logic).

8. Conclusion

In this paper we have presented a new semantics, called Local Models Semantics, and proposed it as a foundation for reasoning with context. Local Models Semantics formalizes the two general principles underlying contextual reasoning, namely the principle of locality and the principle of compatibility. Finally, we have shown how Local Models Semantics can be used to model two important forms of contextual reasoning, namely reasoning with viewpoints and reasoning about belief.

Despite their (apparent) simplicity, the examples proposed in Section 2 show how the semantics and methodology developed in this paper can be applied, suitably modified, to the modeling of important problems. The work in [20] starts from the intuitions and the semantics presented in Sections 2.1 and 4.1 and defines a context-based logic for distributed representation and reasoning, called Distributed First Order Logics. Distributed First Order Logics has been successfully applied to model important theoretical aspects of federations of heterogeneous data or knowledge bases in [22]. The work in [4,16,19] suitably generalizes the intuitions and the formalization proposed in Sections 2.2 and 4.2 in order to model different aspects of agents and multi-agent systems.

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Appendix A. Viewpoints—soundness and completeness

The goal of this section is to show that the MC system for viewpoints MV defined in Section 6.1 is sound and complete with respect to the class of models V defined in Section 4.1. In Section A.1 we prove the Soundness theorem and in Section A.2 the Completeness theorem. The main body of this section concentrates on the proof of the Completeness theorem and on a method for constructing canonical models $\boldsymbol{C}^c$.

A.1. The proof of soundness

Theorem A.1 (Soundness theorem). *If $\Gamma \vdash_{\mathsf{MV}} k:\phi$, then $\Gamma \models_{\mathsf{V}} k:\phi$.*

This theorem states that the calculus provided using the MC system MV computes a derivability relation which is a subset of the consequence relation on models MV.

Proof of Theorem A.1. The proof is by induction on the structure of the derivation of $k:\phi$ from Γ.

Base case: If $\Gamma \vdash_{\mathsf{MV}} k:\phi$ with a zero steps derivation, then $k:\phi \in \Gamma$. Thus $\Gamma \models_{\mathsf{V}} k:\phi$ from the definition of consequence relation.

$\supset \mathrm{I}_k$: If $\Gamma \vdash_{\mathsf{MV}} k:\phi \supset \psi$ and the last rule used is $\supset \mathrm{I}_k$, then $\Gamma, k:\phi \models_{\mathsf{V}} k:\psi$ holds from the inductive hypothesis. Let $\boldsymbol{C}$ be a V-model and $\boldsymbol{c} \in \boldsymbol{C}$ be a sequence such that $\boldsymbol{c}_j$ satisfies the formulae in Γ_j, $j \neq k$. Let m be a model in $\boldsymbol{c}_k$ which satisfies all the formulae in Γ_k. From the inductive hypothesis $m \models_{cl} \phi$ implies $m \models_{cl} \psi$. Thus $m \models_{cl} \phi \supset \psi$ and $\Gamma \models_{\mathsf{V}} k:\phi \supset \psi$.

$\supset \mathrm{E}_k$: If $\Gamma \vdash_{\mathsf{MV}} k:\psi$ and the last rule used is $\supset \mathrm{E}_k$, then there are two formulae $k:\phi$ and $k:\phi \supset \psi$ such that both $\Gamma \models_{\mathsf{V}} k:\phi$ and $\Gamma \models_{\mathsf{V}} k:\phi \supset \psi$ hold from the inductive hypothesis. Let $\boldsymbol{C}$ be a V-model and $\boldsymbol{c} \in \boldsymbol{C}$ be a sequence such that $\boldsymbol{c}_j$ satisfies the formulae in Γ_j, $j \neq k$. Let m be a model in $\boldsymbol{c}_k$ which satisfies all the formulae in Γ_k. From the inductive hypothesis $m \models_{cl} \phi$ and $m \models_{cl} \phi \supset \psi$. Thus $m \models_{cl} \psi$ and $\Gamma \models_{\mathsf{V}} k:\psi$.

$\bot_k$: If $\Gamma \vdash_{\mathsf{MV}} k:\phi$ and the last rule used is $\bot_k$, then $\Gamma, k:\neg\phi \models_{\mathsf{V}} k:\bot$ holds from the inductive hypothesis. Let $\boldsymbol{C}$ be a V-model and $\boldsymbol{c} \in \boldsymbol{C}$ be a sequence such that $\boldsymbol{c}_j$ satisfies the formulae in Γ_j, $j \neq k$. Let m be a model in $\boldsymbol{c}_k$ which satisfies all the formulae in Γ_k. From the inductive hypothesis $m \models_{cl} \neg\phi$ implies $m \models_{cl} \bot$. From the definition of satisfiability in a propositional model it follows that $m \models_{cl} \phi$. Thus $\Gamma \models_{\mathsf{V}} k:\phi$.

br_{12}: If $\Gamma \vdash_{\mathsf{MV}} 2:l \vee c \vee r$ and the last rule used is br_{12}, then $\Gamma \models_{\mathsf{V}} 1:l \vee r$ holds from the inductive hypothesis. Let $\boldsymbol{C}$ be a V-model and $\boldsymbol{c} \in \boldsymbol{C}$ be a sequence such that $\boldsymbol{c}_1$ satisfies Γ_1. Let $m \in \boldsymbol{c}_2$ be a local model such that $m \models_{cl} \Gamma_2$. Both $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are singleton sets. Therefore $\boldsymbol{c}_2$ satisfies Γ_2 and for every $m \in \boldsymbol{c}_1$ $m \models_{cl} \Gamma_1$. From the inductive hypothesis, it follows that for every $m \in \boldsymbol{c}_1$ $m \models_{cl} l \vee r$, i.e., $m \neq \emptyset$. By Eq. (8) in Definition 4.1 every $m \in \boldsymbol{c}_2$ is different from $\emptyset$. Thus every $m \in \boldsymbol{c}_2$ satisfy $l \vee c \vee r$ and $\Gamma \models_{\mathsf{V}} 2:l \vee c \vee r$ holds.

br_{21}: Similar to br_{12}.

$\bot_{12}$: $\Gamma \vdash_{\mathsf{MV}} 2:\phi$ and the last rule used is $\bot_{12}$. From the inductive hypothesis $\Gamma, 2:\neg\phi \models_{\mathsf{V}} 1:\bot$ holds. Let $\boldsymbol{C}$ be a V-model and $\boldsymbol{c} \in \boldsymbol{C}$ be a sequence such that $\boldsymbol{c}_1$ satisfies the formulae in Γ_1. We must show that for every $m \in \boldsymbol{c}_2$, $m \models_{cl} \Gamma_2$ implies $m \models_{cl} \phi$. Let $m \in \boldsymbol{c}_2$ be a model satisfying Γ_2, and suppose that $m \models \neg\phi$. Both $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are singleton sets. Therefore, $\boldsymbol{c}_2 \models \neg\phi$ and, from the inductive hypothesis, every $m \in \boldsymbol{c}_1$ satisfies $\bot$. From the definition of satisfiability in propositional models it follows that $\boldsymbol{c}_2 \not\models \neg\phi$. Again, being $\boldsymbol{c}_2$ a singleton set, this implies $\boldsymbol{c}_2 \models \phi$, i.e., for every $m \in \boldsymbol{c}_2$, $m \models_{cl} \phi$. Thus $\Gamma \models_{\mathsf{V}} 2:\phi$ holds.

$\bot_{21}$: Similar to $\bot_{12}$. $\square$

A.2. The proof of completeness

Theorem A.2 (Completeness theorem). *If* $\Gamma \models_{\mathsf{V}} k:\phi$*, then* $\Gamma \vdash_{\mathsf{MV}} k:\phi$.

This theorem, together with the soundness theorem, states that the calculus provided using MV systems computes a derivability relation which coincides with the consequence relation on the set of V-models.

The contrapositive will be proved: it will be shown that if $\Gamma \nvdash_{\mathsf{MV}} k:\phi$, then there exists a V-model $\boldsymbol{C}^c$ containing a sequence $\boldsymbol{c}$ such that $\boldsymbol{c}_j$ satisfies Γ_j for every $j \neq k$, and $\boldsymbol{c}_k$ contains a model m satisfying Γ_k and not satisfying ϕ. The proof is via the construction of a "canonical model" in which the required sequence $\boldsymbol{c}$ can always be found. As with the canonical model proof of completeness for propositional logic the idea relies upon the being able to construct maximally consistent sets of formulae and being able to use them in defining canonical models. The situation in MC systems is slightly complicated by the division of the system into different languages. To make this possible, a form of consistency and maximal consistency, which generalize the analogous concepts given in [8], are defined.

Definition A.1 (*k-consistency*). Given an MC system *MS*, a set of indexed formulae $\Gamma \in \{L_i\}$ is *k-consistent* if $\Gamma \nvdash_{MS} k:\bot$.

Definition A.2 (*Maximal-k-consistency*). Given an MC system *MS*, a set of indexed formulae $\Gamma \in \{L_i\}$ is *maximal-k-consistent* if it is k-consistent and the only k-consistent set of formulae containing Γ is Γ itself.

In the following we first concentrate on a method for constructing constructing the canonical model $\boldsymbol{C}^c$. Once defined the canonical model $\boldsymbol{C}^c$, we will be able to prove the Completeness theorem at the end of the section. The definition of a canonical model for MV is composed by the following steps:

(1) We generalize the Lindenbaum's theorem [8] by showing that for any k-consistent set of formulae Γ there exists a maximal-k-consistent set Γ' with $\Gamma \subseteq \Gamma'$ (Lemma A.1).
(2) We show some relevant properties of Γ' (Corollary A.1).
(3) We define the canonical model $\boldsymbol{C}^c$ as a compatibility relation over sets of (local) models satisfying maximal-k-consistent sets of formulae (Definition A.4). We show that $\boldsymbol{C}^c$ is a V-model (Lemma A.4).

Lemma A.1. *For any k-consistent set of formulae Γ there exists a maximal-k-consistent set Γ' such that $\Gamma \subseteq \Gamma'$.*

Proof. Let $i_1:\phi_1, i_2:\phi_2, \ldots$ be any enumeration of all the formulae in $\{L_1, L_2\}$. Define $\Gamma^0, \Gamma^1, \ldots$ inductively as follows:

- $\Gamma^0 = \Gamma$;
- if $\Gamma^n \cup \{i_n:\phi_n\}$ is k-consistent then $\Gamma^{n+1} = \Gamma^n \cup \{i_n:\phi_n\}$, otherwise $\Gamma^{n+1} = \Gamma^n$.

$\Gamma' = \bigcup_{i\in\mathbb{N}} \Gamma^i$. Let us prove that Γ' is k-consistent. Suppose not. Then there is a deduction of $k:\bot$ from a finite set $\Gamma^f \subseteq \Gamma'$. Then there is an n such that $\Gamma^f \subseteq \Gamma^n$. But this means that Γ^n is not k-consistent which is a contradiction.

Having shown that Γ' is k-consistent, we next show that Γ' is maximal-k-consistent. Suppose that there exists a maximal-k-consistent set of formulae Δ with $\Gamma' \subseteq \Delta$. Let $i_n:\phi_n \in \Delta$, then $\Gamma^n \cup \{i_n:\phi_n\}$ is k-consistent and hence $i_n:\phi_n \in \Gamma'$. Thus $\Delta = \Gamma'$. □

Definition A.3 (*Maximal-L_k-consistent*). A set of formulae Γ is maximal-L_k-consistent if it is k-consistent and for all L_k-formulae ϕ either $k:\phi \in \Gamma$ or $k:\neg\phi \in \Gamma$.

Corollary A.1. *Let Γ' be maximal-k-consistent set of formulae.*

(i) *if $1:l \vee r \in \Gamma'$ then $2:l \vee c \vee r \in \Gamma'$;*
(ii) *if $2:l \vee c \vee r \in \Gamma'$ then $2:l \vee r \in \Gamma'$;*
(iii) *for each $i \in \{1,2\}$, Γ'_i is maximal-L_i-consistent.*

Proof.

(i) Suppose that $1:l \vee r \in \Gamma'$ and $2:l \vee c \vee r \notin \Gamma'$. Both $1:l \vee r$ and $2:l \vee c \vee r$ occur in some point of the enumeration $i_1:\phi_1, i_2:\phi_2, \ldots$. Then there are two sets $\Gamma^{j_1} \subseteq \Gamma'$ and $\Gamma^{j_2} \subseteq \Gamma'$ such that $\Gamma^{j_1} \cup 1:l \vee r$ is k-consistent and $\Gamma^{j_2} \cup 2:l \vee c \vee r$ is not. If $j_1 < j_2$ then $1:l \vee r \in \Gamma^{j_2}$. We know that $\Gamma^{j_2} \cup 2:l \vee c \vee r$ is not k-consistent, i.e., there exists a deduction Π of $k:\bot$ from $\Gamma^{j_2} \cup 2:l \vee c \vee r$. Being $1:l \vee r \in \Gamma^{j_2}$, the following deduction

$$\begin{array}{ll} \Gamma^{j_2} & \dfrac{1:l \vee r}{2:l \vee c \vee r}\,br_{12} \\ & \quad\Pi \\ & k:\bot \end{array}$$

is a deduction of $k:\bot$ from Γ^{j_2}. This is impossible because Γ^{j_2} is k-consistent. In a similar way we show that this holds even if $j_2 < j_1$. So if $1:l \vee r \in \Gamma'$ then $2:l \vee c \vee r \in \Gamma'$.

(ii) Similar to (i).

(iii) If $i = k$ then the proof follows from the fact that each theory in MV is closed under propositional logic. Let's consider the case $i \neq k$. First we have to prove that Γ'_i is i-consistent. Suppose not, then there exists a deduction Π of $i:\bot$ from Γ'. Applying the bridge rule $\bot_{ik}$ the following deduction

$$\dfrac{\dfrac{\begin{array}{c}[k:\neg A]\ \ \Gamma' \\ \Pi \\ i:\bot\end{array}}{k:A}\bot_{ik} \qquad \dfrac{\begin{array}{c}[k:\neg\neg A]\ \ \Gamma' \\ \Pi \\ i:\bot\end{array}}{k:\neg A}\bot_{ik}}{k:\bot}$$

is a deduction of $k:\bot$ from Γ'. This is impossible because Γ' is k-consistent. Therefore Γ'_i is i-consistent. Suppose now that that neither $i:\phi$, nor $i:\neg\phi$ belong to

Γ'. Both $i:\phi$ and $i:\neg\phi$ occur in some point of the enumeration $i_1:\phi_1, i_2:\phi_2, \ldots$. Then there are two sets $\Gamma^{j_1} \subseteq \Gamma'$ and $\Gamma^{j_2} \subseteq \Gamma'$ such that both $\Gamma^{j_1} \cup i:\phi$ and $\Gamma^{j_2} \cup i:\neg\phi$ are not k-consistent. Suppose $j_1 < j_2$ (the case $j_1 > j_2$ is similar). Then $\Gamma^{j_2} \cup i:\phi$ is not k-consistent. By Lemma A.2 it follows that Γ^{j_2} is not k-consistent as well. But this is impossible. Therefore the hypothesis that neither $i:\phi$, nor $\neg i:\phi$ belong to Γ' must be false. This allows us to conclude that each Γ'_i is maximal-L_i-consistent. $\square$

Lemma A.2. *If $\Gamma, i:\phi \vdash_{\mathsf{MV}} k:\bot$ and $\Gamma, i:\neg\phi \vdash_{\mathsf{MV}} k:\bot$, then $\Gamma \vdash_{\mathsf{MV}} k:\bot$.*

Proof. The case $i = k$ follows easily from the fact that each theory in MV is closed under classical logic. Suppose $i \neq k$. From the hypothesis there exist two deductions Π_1 and Π_2 of $k:\bot$ from $\Gamma, i:\phi$ and $\Gamma, i:\neg\phi$ respectively. Therefore the following deduction is a proof of $k:\bot$ from Γ.

$$\begin{array}{c} \Gamma, [i:\neg\phi] \\ \Pi_1 \\ \dfrac{k:\bot}{i:\phi}\,\bot_{ki} \\ \Pi_2 \\ k:\bot \end{array} \qquad \square$$

We can now define the canonical models starting from maximal-k-consistent sets of formulae Γ'. From the proof of completeness for propositional logic we know that every maximal-L_i-consistent set of formulae Γ'_i univocally defines a propositional model $m^{\Gamma'_i}$ such that $m^{\Gamma'_i} \models_{cl} \phi$ if and only if $\phi \in \Gamma'_i$.

Definition A.4 (*Canonical model*). Let Γ' be a maximal-k-consistent set of formulae. The canonical model $\boldsymbol{C}^c$ is a compatibility relation containing a single compatibility pair $\langle\{m^{\Gamma'_1}\}, \{m^{\Gamma'_2}\}\rangle$.

Lemma A.3. *For every L_i-formula ϕ, $m^{\Gamma'_i} \models_{cl} \phi$ if and only if $i:\phi \in \Gamma'_i$.*

The proof is similar to that for propositional logic.

Lemma A.4. *$\boldsymbol{C}^c$ is indeed a* V*-model.*

Proof. To show that $\boldsymbol{C}^c$ is a V-model it has to be shown that it is a compatibility relation over $2^{\overline{M}_1} \times 2^{\overline{M}_2}$, which satisfies both Definition 3.1 and Definition 4.1. It is clear, however, from the definition of $\boldsymbol{C}^c$, that $\boldsymbol{C}^c \neq \emptyset$. All that needs to be proved in order to satisfy Definition 3.1 is that $\langle\{m^{\Gamma'_1}\}, \{m^{\Gamma'_2}\}\rangle \neq \langle\emptyset, \emptyset\rangle$. This follows from item (iii) in Corollary A.1.

We show now that $\boldsymbol{C}^c$ satisfies Definition 4.1. We prove first that if $m^{\Gamma'_1} \neq \emptyset$ then $m^{\Gamma'_2} \neq \emptyset$. If $m^{\Gamma'_1} \neq \emptyset$, then $m^{\Gamma'_1}$ satisfies l or r (or both). Therefore $1:l \vee r \in \Gamma'_1$ by Lemma A.3. By Lemma A.1(i), $2:l \vee c \vee r \in \Gamma'_2$ and $m^{\Gamma'_2}$ satisfies $l \vee c \vee r$ again by

Lemma A.3. Thus $m^{\Gamma_2'} \neq \emptyset$. The proof that $m^{\Gamma_2'} \neq \emptyset$ implies $m^{\Gamma_1'} \neq \emptyset$ is similar. Finally, both $|\boldsymbol{c}_1| = 1$ and $|\boldsymbol{c}_2| = 1$ are easy consequences of the definition of $\boldsymbol{C}^c$. $\square$

It is now straightforward to complete the proof of completeness.

Proof of Theorem A.2. Recall that the contrapositive is to be proved: if $\Gamma \nvdash_{\mathsf{MV}} k : \phi$ then there exists a model $\boldsymbol{C}$ with a sequence $\boldsymbol{c}$ such that for all $j \neq k$, $\boldsymbol{c}_j \models \Gamma_j$, and there exists a $m \in \boldsymbol{c}_k$ such that $m \models_{cl} \Gamma_k$ but $m \nvDash_{cl} \phi$.

Assuming that $\Gamma \nvdash_{\mathsf{MV}} k : \phi$ holds, then $\Gamma \cup k : \neg\phi$ is k-consistent (if not then $\Gamma \cup k : \neg\phi \vdash_{\mathsf{MV}} k : \bot$ and so $\Gamma \vdash_{\mathsf{MV}} k : \phi$ would also hold by an application of the $\bot_k$ rule). By Lemma A.1 there is a maximal-k-consistent set of formulae Γ' containing $\Gamma \cup k : \neg\phi$. Consider the model $\boldsymbol{C}^c$ defined starting from Γ'. By Lemma A.3, $\boldsymbol{c}^c_j \models \Gamma'_j$, $j \neq k$. Similarly the unique local model $m^{\Gamma_k'}$ in $\boldsymbol{c}^c_k$ satisfies Γ'_k. From $k : \neg\phi \in \Gamma'_k$ and Γ'_k maximal-L_k-consistent, it follows that $\phi \notin \Gamma'_k$. Therefore $m^{\Gamma_k'} \nvDash \phi$ by Lemma A.3. This ends the proof of the completeness theorem. $\square$

Appendix B. Reasoning about belief—soundness and completeness

Let HMB $\subseteq \{\mathcal{R}dw, \mathcal{R}upr\}$. The goal of this section is to show that an HMB system is sound and complete with respect to the class of HMB models (where MBK $= \{\mathcal{R}dw, \mathcal{R}upr\}$). In Section B.1 we prove the Soundness theorem and in Section B.2 the Completeness theorem.

In order to prove the Soundness and Completeness theorems, we slightly modify the definition of HMB model (Definition 4.2), by introducing the following property.

Definition B.1 (*Pointwise property*). Let $\boldsymbol{C} = \{\boldsymbol{c}\}$ with $\boldsymbol{c} = \langle \boldsymbol{c}_0, \boldsymbol{c}_1, \ldots \boldsymbol{c}_k, \ldots \rangle$ be a model. $\boldsymbol{C}$ satisfies the *pointwise property* if, for all compatibility sequences $\boldsymbol{c} \in \boldsymbol{C}$, for all $i \in I$, for any local model $m \in \boldsymbol{c}_i$, there exists a sequence $\boldsymbol{c}' \in \boldsymbol{C}$ such that

(1) $\boldsymbol{c}'_i = \{m\}$;
(2) $\boldsymbol{c}'_j \subseteq \boldsymbol{c}_j$, with $j \neq i$.

Intuitively: take a model $\boldsymbol{C}$, a compatibility sequence $\boldsymbol{c}$ and a local model m belonging to the ith element $\boldsymbol{c}_i$ of $\boldsymbol{c}$. $\boldsymbol{C}$ satisfies the pointwise property if it contains another sequence $\boldsymbol{c}'$ such that

(i) the ith element of $\boldsymbol{c}'$ is exactly m, and
(ii) all the jth elements of $\boldsymbol{c}'$ are subsets of the corresponding jth elements of $\boldsymbol{c}$.

Fig. 15 graphically represents $\boldsymbol{c}$ and $\boldsymbol{c}'$. Notice that we have a different $\boldsymbol{c}'$ for any $m \in \boldsymbol{c}$.

From now on, an HMB model is a model as introduced in Definition 4.2, which satisfies also the pointwise property.

B.1. The proof of soundness

Theorem B.1 (Soundness theorem). *If $\Gamma \vdash_{\mathrm{HMB}} k : \phi$, then $\Gamma \models_{\mathrm{HMB}} k : \phi$.*

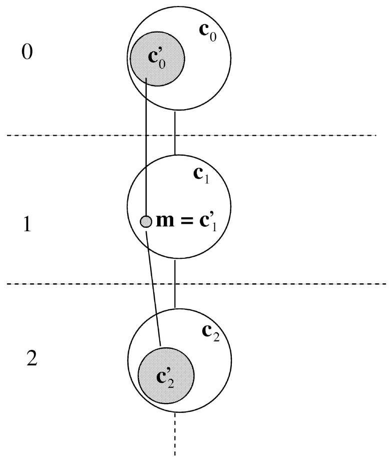

Fig. 15. The pointwise property.

Proof. The proof is by induction on the structure of the derivation of $k:\phi$ from Γ. The proof for the base case, $\supset \mathrm{I}_k$, $\supset \mathrm{E}_k$, and $\bot_k$ is equal to the one given in Section A.1. All that needs to be proven is soundness of bridge rules $\mathcal{R}dw_k$ and $\mathcal{R}upr_k$.

$\mathcal{R}dw_{k-1}$: If $\Gamma \vdash_{\mathrm{HMB}} k:\phi$ and the last rule used is $\mathcal{R}dw_{k-1}$, then $\Gamma \models_{\mathrm{HMB}} k-1 : B(\text{“}\phi\text{”})$ holds from the inductive hypothesis. Let $\boldsymbol{C}$ be an $\mathcal{R}dw$-model (MBK-model) and $\boldsymbol{c} \in \boldsymbol{C}$ be a sequence such that $\boldsymbol{c}_j$ satisfies Γ_j, $j \neq k$. We must show that for every $m \in \boldsymbol{c}_k$, $m \models_{cl} \Gamma_k$ implies $m \models_{cl} \phi$. Let $m \in \boldsymbol{c}_k$ be a local model such that $m \models_{cl} \Gamma_k$. From the pointwise property of HMB models there exists a sequence $\boldsymbol{c}'$ such that

(1) for $j \neq k$, $\boldsymbol{c}'_j \subseteq \boldsymbol{c}_j$;

(2) for $j = k$, $\boldsymbol{c}'_k = \{m\}$.

It is easy to see that this chain satisfies all the formulae in Γ. Thus, from the inductive hypothesis and from the fact that all the local models in $\boldsymbol{c}'_{k-1}$ satisfy Γ_{k-1}, it follows that all the local models in $\boldsymbol{c}'_{k-1}$ satisfy $B(\text{“}\phi\text{”})$, i.e., $B(\text{“}\phi\text{”}) \in \Theta(\boldsymbol{c}'_{k-1})$. From the definition of $\mathcal{R}dw$-model (MBK-model) $\phi \in \Theta(\boldsymbol{c}'_k)$. Thus $m \models_{cl} \phi$ and $\Gamma \models_{\mathrm{HMB}} k:\phi$ holds.

$\mathcal{R}upr_k$: If $\Gamma \vdash_{\mathrm{HMB}} k : B(\text{“}\phi\text{”})$ and the last rule used is $\mathcal{R}upr_k$, then $\Gamma \models_{\mathrm{HMB}} k+1:\phi$ from the inductive hypothesis. Let $\boldsymbol{C}$ be a $\mathcal{R}upr$-model (MBK-model) and $\boldsymbol{c} \in \boldsymbol{C}$ be a sequence such that $\boldsymbol{c}_j$ satisfies Γ_j, $j \neq k$. We must show that for every $m \in \boldsymbol{c}_k$, $m \models_{cl} \Gamma_k$ implies $m \models_{cl} B(\text{“}\phi\text{”})$. Let $m \in \boldsymbol{c}_k$ be a local model such that $m \models_{cl} \Gamma_k$. From the pointwise property of HMB models there exists a chain $\boldsymbol{c}'$ such that

(1) for every $j \neq k$, $\boldsymbol{c}'_j \subseteq \boldsymbol{c}_j$;

(2) for $j = k$, $\boldsymbol{c}'_k = \{m\}$;

$\boldsymbol{c}'$ satisfies all the formulae in Γ. Thus, from the inductive hypothesis it follows that all the local models in $\boldsymbol{c}'_{k+1}$ satisfy ϕ. Now, suppose that m does not satisfy $B(\text{“}\phi\text{”})$. From the definition of $\mathcal{R}upr$ model (MBK-model) there exists another sequence $\boldsymbol{c}''$ k-admissible for $\boldsymbol{c}'$ such that $\boldsymbol{c}''_{k+1}$ does not satisfy

ϕ. Consider the model containing all the sequences in $\boldsymbol{C}$ and the sequence $\langle \boldsymbol{c}'_0, \boldsymbol{c}'_1, \ldots, \boldsymbol{c}'_k, \boldsymbol{c}''_{k+1}, \boldsymbol{c}''_{k+2}, \ldots \rangle$. It is easy to see that this model is still an $\mathcal{R}upr$-model (MBK-model). From the fact that all the formulae in Γ have index $\leqslant k$ it follows that all the $\boldsymbol{c}'_j$ in this sequence satisfy Γ and $\boldsymbol{c}''_{k+1}$ does not satisfy $k+1:\phi$. This contradicts the inductive hypothesis. Therefore there is no m which does not satisfy $B(\text{“}\phi\text{”})$ and $\Gamma \models_{\mathrm{HMB}} k : B(\text{“}\phi\text{”})$ holds.

B.2. The proof of completeness

Theorem B.2 (Completeness theorem). *If $\Gamma \models_{\mathrm{HMB}} k:\phi$, then $\Gamma \vdash_{\mathrm{HMB}} k:\phi$.*

The proof is similar to that in Section A.2 and relies upon the being able to construct maximally consistent sets of formulae and being able to use them in the definition of the canonical model.

The definitions of k-consistency, maximal-k-consistency, and maximal-L_k-consistency, given in Appendix A, are used in the following.

Lemma B.1. *Let $\mathcal{R}dw \in HMB$. If Γ is k-consistent then Γ is j-consistent for all $j \leqslant k$.*

Proof. Suppose that $\Gamma \vdash_{\mathrm{HMB}} j:\bot$ holds for some $j \leqslant k$. Then $\Gamma \vdash_{\mathrm{HMB}} j:B(\text{“}\bot\text{”})$ holds from one assumption of $j:\neg B(\text{“}\bot\text{”})$ and one application of the $\bot_j$ rule. Therefore $\Gamma \vdash_{\mathrm{HMB}} j+i:\bot$. The same two steps can be repeated until $\Gamma \vdash_{\mathrm{HMB}} k:\bot$. But this is impossible because Γ is k-consistent. Thus $\Gamma \nvdash_{\mathrm{HMB}} j:\bot$ for all $j \leqslant k$. □

The steps towards the definition of canonical model for an HMB system are similar to the ones in Appendix A. It is easy to notice that the construction of the maximal-k-consistent set of formulae in Lemma A.1 does not depend upon any particular MC system. Therefore Lemma A.1 holds. What is different is the set of properties that the maximal-k-consistent set Γ' satisfies.

Corollary B.1.

(i) *Let $\mathcal{R}dw \in HMB$. If $i:B(\text{“}\phi\text{”}) \in \Gamma'$ then $i+1:\phi \in \Gamma'$.*
(ii) *Let $\mathcal{R}upr \in HMB$. If $i+1:\phi \in \Gamma'$ and $\vdash_{\mathrm{HMB}} i+1:\phi$ then $i:B(\text{“}\phi\text{”}) \in \Gamma'$.*
(iii) *Let $HMB = MBK$. For every $i \leqslant k$, Γ'_i is maximal-L_i-consistent.*

Proof.

(i) Suppose that $i:B(\text{“}\phi\text{”}) \in \Gamma'$ and $i+1:\phi \notin \Gamma'$. Both $i:B(\text{“}\phi\text{”})$ and $i+1:\phi$ occur in some point of the enumeration $i_1:\phi_1, i_2:\phi_2, \ldots$. Then there are two sets $\Gamma^{j_1} \subseteq \Gamma'$ and $\Gamma^{j_2} \subseteq \Gamma'$ such that $\Gamma^{j_1} \cup i:B(\text{“}\phi\text{”})$ is k-consistent and $\Gamma^{j_2} \cup i+1:\phi$ is not k-consistent. If $j_1 < j_2$ then $i:B(\text{“}\phi\text{”}) \in \Gamma^{j_2}$. We know that $\Gamma^{j_2} \cup i+1:\phi$

is not k-consistent, i.e., there exists a deduction Π of $k:\bot$ from $\Gamma^{j_2} \cup i+1:\phi$. Being $i:B(\text{“}\phi\text{”}) \in \Gamma^{j_2}$, the deduction

$$\begin{array}{cc} \Gamma^{j_2} & \dfrac{i+1:\phi}{i:B(\text{“}\phi\text{”})}\mathcal{R}dw_i \\ & \Pi \\ & k:\bot \end{array}$$

is a deduction of k:$\bot$ from Γ^{j_2}. This is impossible because Γ^{j_2} is k-consistent. In a similar way we show that this holds even if $j_2 < j_1$. So, if $i:B(\text{“}\phi\text{”}) \in \Gamma'$ then $i+1:\phi \in \Gamma'$.

(ii) Suppose that $i+1:\phi \in \Gamma'$, $i+1:\phi$ is provable (i.e. $\vdash_{\mathrm{HMB}} i+1:\phi$), and $i:B(\text{“}\phi\text{”}) \notin \Gamma'$. Both $i+1:\phi$ and $i:B(\text{“}\phi\text{”})$ occur in some point of the enumeration $i_1:\phi_1, i_2:\phi_2, \ldots$. Then there are two sets $\Gamma^{j_1} \subseteq \Gamma'$ and $\Gamma^{j_2} \subseteq \Gamma'$ such that $\Gamma^{j_1} \cup i+1:\phi$ is k-consistent and $\Gamma^{j_2} \cup i:B(\text{“}\phi\text{”})$ is not k-consistent. If $j_1 < j_2$ then $i+1:\phi \in \Gamma^{j_2}$. We know that $\Gamma^{j_2} \cup i:B(\text{“}\phi\text{”})$ is not k-consistent, i.e., there exists a deduction Π of $k:\bot$ from $\Gamma^{j_2} \cup i:B(\text{“}\phi\text{”})$. Being $i+1:\phi \in \Gamma^{j_2}$ and $i+1:\phi$ provable, the following deduction

$$\begin{array}{cc} \Gamma^{j_2} & \dfrac{i+1:\phi}{i:B(\text{“}\phi\text{”})}\mathcal{R}upr_i \\ & \Pi \\ & k:\bot \end{array}$$

is a deduction of $k:\bot$ from Γ^{j_2} (the hypothesis that $i+1:\phi$ is provable is crucial in order to satisfy the restriction of $\mathcal{R}upr_i$). This is impossible because Γ^{j_2} is k-consistent. In a similar way we show that this holds even if $j_2 < j_1$. So if $i+1:\phi \in \Gamma'$ then $i:B(\text{“}\phi\text{”}) \in \Gamma'$.

(iii) i-consistency of every Γ'_i with $i \leqslant k$ follows from Lemma B.1. Suppose that neither $i:\phi$ nor $i:\neg\phi$ are in Γ'_i. Both $i:\phi$ and $i:\neg\phi$ occur in some point of the enumeration $i_1:\phi_1, i_2:\phi_2, \ldots$. Then there are two sets of formulae $\Gamma^{j_1} \subseteq \Gamma'$ and $\Gamma^{j_1} \subseteq \Gamma'$ such that $\Gamma^{j_1} \cup i:\phi \vdash_{\mathrm{MBK}} k:\bot$ and $\Gamma^{j_2} \cup i:\neg\phi \vdash_{\mathrm{MBK}} k:\bot$. Suppose that $j_1 < j_2$. Then $\Gamma^{j_2} \cup i:\phi \vdash_{\mathrm{MBK}} k:\bot$ as well. From Lemma B.2 it follows that $\Gamma^{j_2} \vdash_{\mathrm{MBK}} k:\bot$, but this contradict the k-consistency of Γ^{j_2}. With a similar proof it can be shown that $j_2 < j_1$ implies $\Gamma^{j_1} \vdash_{\mathrm{MBK}} k:\bot$. But this contradict the k-consistency of Γ^{j_1}. So, for all L_i-formulae either $i:\phi \in \Gamma'_i$ or $i:\neg\phi \in \Gamma'_i$. □

Lemma B.2. *For all $i \leqslant j$, if $\Gamma, i:\phi \vdash_{\mathrm{MBK}} j:\psi$ and $\Gamma, i:\neg\phi \vdash_{\mathrm{MBK}} j:\psi$ then $\Gamma \vdash_{\mathrm{MBK}} j:\psi$.*

Proof. Being $i \leqslant j$, we can rewrite j as $i+n$ with $n \geqslant 0$. It will be shown that $\Gamma, i:\phi \vdash_{\mathrm{MBK}} i+n:\psi$ and $\Gamma, i:\neg\phi \vdash_{\mathrm{MBK}} i+n:\psi$ imply $\Gamma \vdash_{\mathrm{MBK}} i+n:\psi$ by induction on n. Assuming $n=0$, i.e., that both $\Gamma, i:\phi \vdash_{\mathrm{MBK}} i:\psi$ and $\Gamma, i:\neg\phi \vdash_{\mathrm{MBK}} i:\psi$ hold, it is easy to provide a derivation of $\Gamma \vdash_{\mathrm{MBK}} i:\psi$. This can be done because each MBK system is closed under propositional logic. The induction hypothesis is that $\Gamma', i:\phi \vdash_{\mathrm{MBK}} i+n:\psi'$ and $\Gamma', i:\neg\phi \vdash_{\mathrm{MBK}} i+n:\psi'$ imply $\Gamma' \vdash_{\mathrm{MBK}} i+n:\psi'$ for arbitrary $\Gamma', i+n:\psi'$.

It will be shown that $\Gamma, i:\phi \vdash_{\mathrm{MBK}} i+n+1:\psi$ and $\Gamma, i:\neg\phi \vdash_{\mathrm{MBK}} i+n+1:\psi$ imply $\Gamma \vdash_{\mathrm{MBK}} i+n+1:\psi$. On the assumption that $\Gamma, i:\phi \vdash_{\mathrm{MBK}} i+n+1:\psi$ and $\Gamma, i:\neg\phi \vdash_{\mathrm{MBK}} i+n+1:\psi$ hold, and from the finites of the derivation we know that there exists a $\Gamma_f \subseteq \Gamma$ for which $\Gamma_f, i:\phi \vdash_{\mathrm{MBK}} i+n+1:\psi$ and $\Gamma_f, i:\neg\phi \vdash_{\mathrm{MBK}} i+n+1:\psi$. Γ_f contains formulae with index $\leqslant i+n+1$ and can be rewritten as $\{i+n+1:\gamma_1, \ldots, i+n+1:\gamma_m\} \cup \Gamma'$ where all the indexes in Γ' are $\leqslant i+n$. By m applications of the $\supset\mathrm{I}_{i+n+1}$ rule followed by an application of the $\mathcal{R}upr$ rule, the following derivations hold:

$$\Gamma', i:\phi \vdash_{\mathrm{MBK}} i+n:B(\text{“}\gamma_1 \supset \cdots (\gamma_m \supset \psi)\ldots\text{”})$$

$$\Gamma, i:\neg\phi \vdash_{\mathrm{MBK}} i+n:B(\text{“}\gamma_1 \supset \cdots (\gamma_m \supset \psi)\ldots\text{”}).$$

The induction hypothesis is now applicable and so

$$\Gamma' \vdash_{\mathrm{MBK}} i+n:B(\text{“}\gamma_1 \supset \cdots (\gamma_m \supset \psi)\ldots\text{”})$$

holds. From this derivation, one application of $\mathcal{R}dw$ followed by the assumption of $i+n+1:\gamma_1, \ldots, i+n+1:\gamma_m$ and m applications of $\supset\mathrm{E}_{i+n+1}$ gives $\{i+n+1:\gamma_1, \ldots, i+n+1:\gamma_m\} \cup \Gamma' \vdash_{\mathrm{MBK}} i+n+1:\psi$. Being $\{i+n+1:\gamma_1, \ldots, i+n+1:\gamma_m\} \cup \Gamma'$ equal to Γ_f which is a subset of Γ, $\Gamma \vdash_{\mathrm{MBK}} i+n+1:\psi$ holds. □

We are now able to define canonical models starting from maximal-k-consistent sets of formulae Γ'. From the proof of completeness for propositional logics we know that every maximal-L_i-consistent set of formulae Γ'_i univocally defines a propositional model $m^{\Gamma'_i}$ such that $m^{\Gamma'_i} \models_{cl} \phi$ if and only if $\phi \in \Gamma'_i$. Let Γ' be a maximal-k-consistent set of formulae. A compatibility sequence $\boldsymbol{c}$ is defined *over* Γ' if:

(i) $\boldsymbol{c}_i = \{m_i^{\Gamma'}\}$, for Γ'_i maximal-i-consistent;
(ii) $\boldsymbol{c}_i = \{m \in \overline{M}_i \mid m \models_{cl} \Gamma'_i\}$, otherwise.

Lemma B.3. *For every L_i-formula ϕ, and every compatibility sequence $\boldsymbol{c}$ defined over Γ', $\boldsymbol{c}_i \models_{\mathrm{HMB}} \phi$ if and only if $i:\phi \in \Gamma'$.*

Proof. If $i:\phi \in \Gamma'$ then all the local models in $\boldsymbol{c}_i$ satisfy ϕ by construction. If $\boldsymbol{c}_i$ satisfies $i:\phi$, then all the local models in $\boldsymbol{c}_i$ satisfy ϕ. Being $\boldsymbol{c}_i$ the class containing all and only the models satisfying Γ'_i, then ϕ is a (propositional) logical consequence of the formulae in Γ'_i. From the completeness theorem for propositional logic there exists a deduction of $i:\phi$ from Γ'_i. Thus $\Gamma' \cup i:\phi$ is k-consistent (if not, there is a trivial deduction of $k:\bot$ from Γ' which contradicts the k-consistency of Γ') and $i:\phi \in \Gamma'$. □

Definition B.2 (*Canonical model*). Let $\overline{M}_0, \overline{M}_1, \ldots, \overline{M}_k, \ldots$ be the classes of models for the languages $L_0, L_1, \ldots, L_k, \ldots$ of an HMB system. The canonical model $\boldsymbol{C}^c$ is a compatibility relation of type $\boldsymbol{C} \subseteq \prod_{i\in I} 2^{\overline{M}_i}$ containing, for each maximal-k-consistent set of formulae Γ' for some index k, the compatibility sequence $\boldsymbol{c}$ defined over Γ'.

If HMB $= \mathcal{R}dw$, then $\boldsymbol{C}^c$ contains also a sequence $\boldsymbol{c}' = \langle \boldsymbol{c}_0, \ldots, \boldsymbol{c}_{i-1}, \{m\}, \emptyset, \ldots, \emptyset, \ldots\rangle$ for each local model $m \in \boldsymbol{c}_i$.

Lemma B.4. *$\boldsymbol{C}^c$ is indeed an HMB model.*

Proof. Let HMB $= \mathcal{R}dw$. $\boldsymbol{C}^c$ satisfies the pointwise property by definition. We have to show that for every $\boldsymbol{c} \in \boldsymbol{C}^c$, $B^{-1}(\text{“}\Theta(\boldsymbol{c}_i)\text{”}) \subseteq \Theta(\boldsymbol{c}_{i+1})$. The model contains compatibility sequences of two different forms, the ones defined over maximal-k-consistent sets of formulae and the ones added in order to satisfy the pointwise property. Consider the first ones. Suppose that $B(\text{“}\phi\text{”}) \in \Theta(\boldsymbol{c}_i)$. By Lemma B.3, $i : B(\text{“}\phi\text{”}) \in \Gamma'_i$, and by Corollary B.1(i), $i+1 : \phi \in \Gamma'_{i+1}$. Again by Lemma B.3, $i+1 : \phi \in \Theta(\boldsymbol{c}_{i+1})$. Consider now the second form of compatibility sequences. Suppose that $B(\text{“}\phi\text{”}) \in \Theta(\boldsymbol{c}'_i)$. Let j be greatest index such that $\boldsymbol{c}'_j \neq \emptyset$. For every $i \geqslant j$ the proof follows from the fact that $\boldsymbol{c}'_i = \emptyset$. For every $i < j-1$ the proof follows from the fact that every $\boldsymbol{c}'_i$ is equal to $\boldsymbol{c}_i$. If $i = j-1$, then the proof is a consequence of the fact that $\boldsymbol{c}'_j \subseteq \boldsymbol{c}_j$ and $\boldsymbol{c}'_{j-1} = \boldsymbol{c}_{j-1}$.

Let HMB $= \mathcal{R}upr$. First, we show that $\boldsymbol{C}^c$ satisfies the pointwise property. It is easy to observe that in a $\mathcal{R}upr$ system any assumption in L_j $(j \neq k)$ does not play any role in inferring $k : \bot$. Therefore, each Γ' maximal-k-consistent is such that $\Gamma'_j = L_j$ for each $j \neq k$. On the other hand, it is easy to show that Γ'_k is maximal-L_k-consistent. This is due to the fact that HMB systems are closed under propositional logic. Therefore, for each $j \neq k$, $\boldsymbol{c}_j = \emptyset$, and $\boldsymbol{c}_k = \{m^{\Gamma_k}\}$. As a consequence, $\boldsymbol{C}^c$ satisfies the pointwise property. Second, we show that for every $\boldsymbol{c} \in \boldsymbol{C}^c$ $B(\text{“}V^{\downarrow}(\boldsymbol{c}_i)\text{”}) \subseteq \Theta(\boldsymbol{c}_i)$. Suppose that there is a formula $i+1 : \phi$ such that $i : B(\text{“}\phi\text{”})$ is not in $\Theta(\boldsymbol{c}_i)$. We show that $i : B(\text{“}\phi\text{”}) \notin B(\text{“}V^{\downarrow}(\boldsymbol{c}_i)\text{”})$, i.e., there exists a sequence $\boldsymbol{c}' \in \boldsymbol{C}^c$ such that $\boldsymbol{c}'_i \subseteq \boldsymbol{c}_i$ and $\boldsymbol{c}'_{i+1} \nvDash \phi$. We know that $i+1 : \phi$ is not provable in $\mathcal{R}upr$ (otherwise $i : B(\text{“}\phi\text{”}) \in \Theta(\boldsymbol{c}_i)$ from Corollary B.1(ii) and Lemma B.3). Thus there exists an $i+1$-consistent set of formulae containing $i+1 : \neg\phi$ and, from Lemma A.1, a maximal-$i+1$-consistent set of formulae Γ' containing $i+1 : \neg\phi$. Consider the sequence $\boldsymbol{c}'$ defined over Γ'. From what we have said above the ith component of such sequence is the empty set. Being $\emptyset \subseteq \boldsymbol{c}_i$, $\boldsymbol{c}'$ is i-admissible for $\boldsymbol{c}$. From $\boldsymbol{c}'_{i+1} \nvDash_{\text{HMB}} \phi$ it follows that $i+1 : \phi \notin V^{\downarrow}(\boldsymbol{c}_i)$. So, $i : B(\text{“}\phi\text{”}) \notin B(\text{“}V^{\downarrow}(\boldsymbol{c}_i)\text{”})$ and the proof is done.

Let HMB $=$ MBK. First, we show that $\boldsymbol{C}^c$ satisfies the pointwise property. It is easy to observe that in this case each Γ' maximal-k-consistent is such that $\Gamma'_j = L_j$ for each $j > k$. This fact, together with Corollary B.1(iii) implies that for each $j > k$, $\boldsymbol{c}_j = \emptyset$, and for each $j \leqslant k$, $\boldsymbol{c}_j = \{m^{\Gamma'_i}\}$. As a consequence $\boldsymbol{C}^c$ satisfies the pointwise property. The proof that the model satisfies $B^{-1}(\text{“}\Theta(\boldsymbol{c}_i)\text{”}) \subseteq \Theta(\boldsymbol{c}_{i+1})$ and $B(\text{“}V^{\downarrow}(\boldsymbol{c}_i)\text{”}) \subseteq \Theta(\boldsymbol{c}_i)$ is similar to the ones for HMB $= \mathcal{R}dw$ and HMB $= \mathcal{R}upr$, respectively. □

It is now straightforward to complete the proof of completeness.

Proof of Theorem B.2. Recall that the contrapositive is to be proved: if $\Gamma \nvdash_{\text{HMB}} k : \phi$ then there exists a model $\boldsymbol{C}$ with a sequence $\boldsymbol{c}$ such that for all the $j \neq k$ $\boldsymbol{c}_j \vDash \Gamma_j$, and there exists a $m \in \boldsymbol{c}_k$ such that $m \vDash_{cl} \Gamma_i$ but $m \nvDash_{cl} \phi$.

Assuming that $\Gamma \nvdash_{\text{HMB}} k : \phi$ holds, then $\Gamma \cup k : \neg\phi$ is k-consistent (if not then $\Gamma \cup k : \neg\phi \vdash_{\text{HMB}} k : \bot$ and so $\Gamma \vdash_{\text{HMB}} k : \phi$ would also hold by an application of the $\bot_k$ rule). By Lemma A.1 there is a maximal-k-consistent set of formulae Γ' containing $\Gamma \cup k : \neg\phi$. Consider the model $\boldsymbol{C}^c$ defined in Definition B.2 and the sequence $\boldsymbol{c}$ defined over Γ'. From the definition of canonical model and Lemma B.3, for all $i \neq k$ $\boldsymbol{c}_i$ satisfies Γ_i. Moreover $\boldsymbol{c}_k = \{m^{\Gamma'_k}\}$ and it satisfies all the formulae in $\Gamma'_k \cup k : \neg\phi$. Being $m^{\Gamma'_k}$ a classical model it does not satisfy ϕ. Thus $\boldsymbol{c}$ is the sequence falsifying $\Gamma \models_{\mathrm{HMB}} k:\phi$. This ends the proof of the completeness theorem. □

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