Ma 236 Notes - Spring 2019

Contents

These notes are an outline of topics covered in the first part of Ma 236. For more information you should consult the text or Schaum's Outline of Logic.

1 Propositional Logic

References: Chapter 1 of the text and Chapter 3.1-3.6 of Schaum's Outline

1.1 Valid arguments

An argument is *valid* if its conclusion is true whenever all its premises are. The following argument is valid.

Premise 1:	Min is not both home and on board
Premise 2:	She's home
Conclusion:	She's not on board

We analyze this argument by first converting it to symbolic form. Let H stand for "Min is home" and B for "Min is on board". Our argument becomes

Premise 1:	Not (H and B)
Premise 2:	H
Conclusion:	Not B

Next use a truth table to check the truth values of the premises and the conclusion for all possible combinations of truth values for H and B.

H	B	Not	(H and B)	H	Not B
t	t	f	(t)	t	f
t	f	t	(f)	t	t
f	t	t	(f)	f	f
f	f	t	(f)	f	t

A truth table. Truth values of premises and conclusion are red.

There is one row for each choice of truth values for H and B. Since no row has $t\,t\,f$, the argument is valid.

Notice that whether or not the premises are true in real life does not affect whether or not the argument is valid.

1.1.1 Sound arguments

A valid argument in which all premises are true in real life is called *sound*. The conclusion of a sound argument is true in real life.

1.2 An invalid argument

The following argument is not valid.

Premise 1:	Min is home or on board
Premise 2:	She's home
Conclusion:	She's not on board

To test validity keep H and B as before and convert to symbolic form:

Premise 1:	$H \vee B$
Premise 2:	H
Conclusion:	$\neg B$

New notation: Write $\vee$ for "or" and $\neg$ for "not". Likewise $\wedge$ for "and".

"Or" is ambiguous. It can mean "one or the other or both", but it can also mean "one or the other but not both". The symbol $\vee$ carries the first meaning.

The argument above is not valid because there is a counterexample.

H	B	$H \vee B$	H	$\neg B$
t	t	t	t	f
t	f	t	t	t
f	t	t	f	f
f	f	f	f	t

The first row is a counterexample.

1.3 Sentences

1.3.1 Atomic sentences

Simple English sentences which are either true or false are represented symbolically by capital letters. For example

A : Frogs have feathers.

B : It will rain tomorrow.

C : He is a college student.

Some English sentences are neither true nor false:

- Please put away your cell phones.
- What is the point of this course?

1.3.2 Compound sentences

More complicated English sentences are represented symbolically by using $\neg, \vee, \wedge$. For example

> Either it will rain tomorrow and he is a college student, or it will not rain tomorrow and frogs have feathers.

may be translated as $(B \wedge C) \vee (\neg B \wedge A)$.

1.3.3 Logical equivalence

Every symbolic sentence has a truth table. Two sentences with the same truth table are *logically equivalent*.

1.3.4 Rules of formation

Symbolic sentences are often called propositions. We will call them just sentences for short.

1. Letters $A, B, C, \ldots$ with or without subscripts are sentences.
2. If $\square$ is any sentence, so is $\neg\square$.
3. If $\square$ and $\triangle$ are sentences, so are $(\square \vee \triangle)$ and $(\square \wedge \triangle)$. More generally if $\square_1, \ldots, \square_n$ are sentences, so are $(\square_1 \vee \cdots \vee \square_n)$ and $(\square_1 \wedge \cdots \wedge \square_n)$

Using these rules we can show formally that $(B \wedge C) \vee (\neg B \wedge A)$ is a sentence.

1.	A, B, and C are sentences	By Rule 1
2.	$\neg B$ is a sentence	From Step 1 and Rule 2
3.	$(B \wedge C)$ and $(\neg B \wedge A)$ are sentences	Steps 1,2 and Rule 3
4.	$((B \wedge C) \vee (\neg B \wedge A))$ is a sentences	Step 3 and Rule 3

Strictly speaking $(B \wedge C) \vee (\neg B \wedge A)$ is not correctly formed; it is missing its outer parentheses. For convenience we allow ourselves to leave off the outer parentheses when it does not cause confusion.

1.4 Conditionals

1.4.1 $\rightarrow$

We agree that the following argument is valid.

If Min works on board that tub, she is underpaid	If W, then U
She works on board that tub	W
She is underpaid	U

When W is true, and U is false, then "If W, then U" must be false or else the argument would not be valid. Thus we know Line 2 in the following truth table.

New notation. Write "If W, then U" symbolically as "$W \rightarrow U$".

W	U	$W \rightarrow U$
t	t	t
t	f	f
f	t	t
f	f	t

The other lines may seem strange, because in ordinary conversation we would say "If Min works on board that tub, she is underpaid" we know for a fact that she does not work on that tub or we know for a fact that she is underpaid.

$W \rightarrow U$ is logically equivalent to $\neg W \vee U$ (as you can easily check).

$\to$ has many translations into English: A:Min is home; C:Hen is home.

If Min is home, then Hen is	If A then C	$A \to C$
If Min is home, Hen is	If A, C	$A \to C$
Hen is home if Min is	C if A	$A \to C$
Min is home only if Hen is	A only if C	$A \to C$
Only if Hen is home is Min home	Only if C, A	$A \to C$

"Only if" can be confusing. Think of it this way: "Min is home only if Hen is" is equivalent to "Min cannot be home if Hen is not home", i.e. $\neg C \to \neg A$. But $\neg C \to \neg A$ is logically equivalent to $A \to C$ (as you can check).

Is the following argument valid? If not, find a counterexample.

> Look, we know that Min is on board if Henry is home. Then she has to be on board if she's home, because Henry is home if she is.

A:Henry is home B:Min is on board C:Min is home.

Min is on board if Henry is home	$A \to B$
Henry is home if she is	$C \to A$
She has to be on board if she's home.	$C \to B$

1.4.2 $\leftrightarrow$

A if and only if C is written symbolically as $A \leftrightarrow C$.

A	C	$A \leftrightarrow C$
t	t	t
t	f	f
f	t	f
f	f	t

$A \leftrightarrow C$ is logically equivalent to $(A \to C) \land (C \to A)$.

1.5 Translation

$A \to B$ is logically equivalent to $\neg(A \land \neg B)$ and also to $\neg A \lor B$, but these equivalences do not seem to hold up in everyday language.

Socrates is dead	B
If George sits down, then Socrates is dead	$A \to B$

It's not the case that if I break my leg today, I'll ski tomorrow	$\neg(A \to B)$
I'll break my leg today	A

If they withdraw if we advance, we'll win	$(A \to B) \to C$
We won't advance	$\neg A$
We'll win	C

1.6 Complete rules of formation

1. Capital letters are sentences: $A, B, \ldots, A_1, A_2, \ldots$.
2. If $\Box$ is a sentence, so is $\neg\Box$.
3. If $\Box_1, \ldots, \Box_n$, $n \geq 2$, are sentences, so are $(\Box_1 \wedge \Box_2 \wedge \cdots \wedge \Box_n)$ and $(\Box_1 \vee \Box_2 \vee \cdots \vee \Box_n)$
4. If $\Box$ and $\triangle$ are sentences, so are $(\Box \to \triangle)$ and $(\Box \leftrightarrow \triangle)$.
5. Nothing else is a sentence (until the next chapter).

1.7 Tautologies

A tautology is a sentence which is always true. For example $A \vee \neg A$.

The sentence $\Box$ is a tautology if and only if the argument (with no premises)

$\dfrac{}{\Box}$ is valid.

2 Refutation Trees

References: Chapter 2 of the text and Chapter 3.7 of Schaum's Outline

To test if a sentence α is a tautology, we can use truth tables to check that if α is always true. Or we can use truth tables to check if $\neg\alpha$ is never true. In this chapter we introduce a more efficient way to check if a sentence is never true.

We use refutation trees (called truth trees in the text) to decide whether or not a sentence is never true. In fact using refutation trees we can decide if a bunch of sentences are never all true at the same time.

Definition 1. A set of sentences *consistent* if the sentences can all be true at the same time. The set is *inconsistent* if the sentences are never all true at the same time.

2.1 A closed tree

Use the tree method to test the sentences $B \to \neg A$, $\neg B \to C$ and $\neg(A \to C)$ for consistency.

1.			$\surd_2$ $B \to \neg A$ Input	
2.			$\surd_3$ $\neg B \to C$ Input	
3.			$\surd_1$ $\neg(A \to C)$ Input	
4.			A 3.$\neg \to$	
5.			$\neg C$ 3.$\neg \to$	
6.		$\neg B$ 1.$\to$		$\neg A$ 1.$\to$ $\times$
7.	$\neg\neg B$ 2.$\to$ $\times$	C 2.$\to$ $\times$		

After each step put $\times$ at the bottom of every path which contains a sentence and its negation; there is no way all the sentences in such a path can be true at the same time.

The tree test is finished when there are no more rules to apply. In the present case all paths close, so the sentences are inconsistent. The subscripts on the $\surd$'s show the order in the rules of inference were applied. If the rules are applied in a different order, the resulting trees will be different, but all paths will still close.

2.2 A tree which does not close

Use the tree test to see if the sentences $A \to B$, $\neg A$ and $\neg B$ are consistent.

1.	$\surd\ A \to B$ Input	
2.	$\neg A$ Input	
3.	$\neg B$ Input	
4.	$\neg A$ 1.$\to$	B 1.$\to$
		$\times$

We are done after just one step. There is an open path, so the sentences $A \to B$, $\neg A$ and $\neg B$ are consistent. Truth values which make all the sentences true may be read off from any open path.

2.3 Rules of inference (preliminary version)

$\dfrac{\begin{array}{c}\square\\ \neg\square\end{array}}{\times}$	$\dfrac{\surd(\square \to \triangle)}{\neg\square \qquad \triangle}$	$\dfrac{\surd(\square \leftrightarrow \triangle)}{\begin{array}{cc}\square & \neg\square\\ \triangle & \neg\triangle\end{array}}$	$\dfrac{\surd(\square \vee \triangle)}{\square \qquad \triangle}$	$\dfrac{\surd(\square \wedge \triangle)}{\begin{array}{c}\square\\ \triangle\end{array}}$
$\dfrac{\surd\ \neg\neg\square}{\square}$	$\dfrac{\surd\neg(\square \to \triangle)}{\begin{array}{c}\square\\ \neg\triangle\end{array}}$	$\dfrac{\surd\neg(\square \leftrightarrow \triangle)}{\begin{array}{cc}\neg\square & \square\\ \triangle & \neg\triangle\end{array}}$	$\dfrac{\surd\neg(\square \vee \triangle)}{\begin{array}{c}\neg\square\\ \neg\triangle\end{array}}$	$\dfrac{\surd\neg(\square \wedge \triangle)}{\neg\square \qquad \neg\triangle}$

$\begin{array}{ccc} \multicolumn{3}{c}{\surd(\Box \lor \triangle \lor \bigcirc)} \\ \hline \Box & \triangle & \bigcirc \end{array}$	$\begin{array}{c} \surd(\Box \land \triangle \land \bigcirc) \\ \hline \Box \\ \triangle \\ \bigcirc \end{array}$
$\begin{array}{c} \surd\neg(\Box \lor \triangle \lor \bigcirc) \\ \hline \neg\Box \\ \neg\triangle \\ \neg\bigcirc \end{array}$	$\begin{array}{ccc} \multicolumn{3}{c}{\surd\neg(\Box \land \triangle \land \bigcirc)} \\ \hline \neg\Box & \neg\triangle & \neg\bigcirc \end{array}$

2.4 Exercises

1. Interpret "E", "J" and "M" as meaning that the earth is the third planet from the sun, that Jupiter is, and that Mars is. Work out the truth values (in real life) of the following sentences.

   (a) $(M \land J) \lor E$ (b) $M \land (J \lor E)$
   (c) $\neg(M \lor J \lor E)$ (d) $\neg M \lor J \lor E$
   (e) $\neg(\neg M \lor \neg J \lor \neg E)$ (f) $\neg(\neg M \land \neg J \land \neg E)$

2. "Thin is guilty," observed Watson,"because (i) either Holmes is right and the vile Moriarty is guilty or he is wrong and the scurrilous Thin did the job; but (ii) those scoundrels are either both guilty or both innocent; and, as usual, (iii) Holmes is right."

   (a) Write down the argument symbolically. Use "T, M" for "Thin is guilty, Moriarty is guilty", "$\neg T, \neg M$" for innocence, and "H" for "Holmes is right."

   (b) Is the argument valid?

   (c) Are the premises and the conclusion consistent, i.e., is there a case in which they are all true?

   Solve (b) and (c) efficiently by looking for counterexamples and then solve them by truth tables.

3. Use a truth tree to show that this argument is valid. $\begin{array}{l} A \lor B \\ A \to C \\ B \to C \\ \hline C \end{array}$

4. Convert the following arguments to symbolic notation, and then use truth trees to check if they are valid. Find a counterexample for each argument which is not valid.

   (a) Moriarty: “If Min is home, so is Henry.” Thin: “Indeed, and if Min is home, Henry isn’t.” Moriarty: “Ah, I see. Min’s not home.”

   (b) “Min’s home if Henry is, but he isn’t, so she isn’t.”

   (c) “It’s false that if Min is home, she’s on board. Then if she’s home, she’s not on board.”

   (d) “It’s false that if Min is home, she’s on board, because if she’s home, she’s not on board.”

   (e) “Look, we know that Min is on board if Henry is home. Then she has to be on board if she’s home, because Henry’s home if she is.”

3 Predicate Logic

References: Chapter 3 of the text and Chapter 6 of Schaum's Outline

3.1 The tree test for sentences with quantifiers

A valid argument: $\dfrac{\text{Everyone can trap Moriarty} \quad \forall x\, Txb}{\text{Watson can trap Moriarty} \quad Tab}$

An invalid argument: $\dfrac{\text{Someone can trap Moriarty} \quad \exists x\, Txb}{\text{Watson can trap Moriarty} \quad Tab}$

New notation:

$\forall x$	for all x
$\exists x$	for some x
a	Watson
b	Moriarty
Txy	x can trap y
x, y	variables

An argument is valid if and only if the premises and the negation of the conclusion are *inconsistent*. Thus we can use the tree test to check if the arguments above are valid, but we need some additional rules of inference.

3.1.1 Rules of inference for $\neg$

$\dfrac{\surd\neg\forall\square}{\exists\neg\square}$	$\dfrac{\surd\neg\exists\square}{\forall\neg\square}$	$\dfrac{\surd\neg\neg\square}{\square}$	$\dfrac{\begin{array}{c}\square\\ \neg\square\end{array}}{\times}$

3.1.2 Rules of inference for $\rightarrow, \leftrightarrow, \vee, \wedge$

$\dfrac{\surd(\square\rightarrow\triangle)}{\neg\square \qquad \triangle}$	$\dfrac{\surd(\square\leftrightarrow\triangle)}{\begin{array}{cc}\square & \neg\square\\ \triangle & \neg\triangle\end{array}}$	$\dfrac{\surd(\square\vee\triangle)}{\square \qquad \triangle}$	$\dfrac{\surd(\square\wedge\triangle)}{\begin{array}{c}\square\\ \triangle\end{array}}$
$\dfrac{\surd\neg(\square\rightarrow\triangle)}{\begin{array}{c}\square\\ \neg\triangle\end{array}}$	$\dfrac{\surd\neg(\square\leftrightarrow\triangle)}{\begin{array}{cc}\neg\square & \square\\ \triangle & \neg\triangle\end{array}}$	$\dfrac{\surd\neg(\square\vee\triangle)}{\begin{array}{c}\neg\square\\ \neg\triangle\end{array}}$	$\dfrac{\surd\neg(\square\wedge\triangle)}{\neg\square \qquad \neg\triangle}$

3.1.3 Rules of inference for $\forall$ and $\exists$

Universal Instantiation Rule. If an open path has a line $\forall x \ldots x \ldots$, do not check ($\surd$) this line, but add as many new lines as necessary so that for every name n in the path there is a line $\ldots n \ldots$. If there are no names in the path, choose a new name n and add a line $\ldots n \ldots$.

Everyone can trap Moriarty	$\forall x\, Txb$
Watson can trap Moriarty	Tab

1. $\forall x Txb$ Premise
2. $\neg Tab$ $\neg$ Conclusion
3. Tcb 1.$\forall$

Line 3 comes from applying the Universal Instantiation Rule to Line 1. Notice that Line 1 is not checked ($\surd$) because we may want to use it again. Strictly speaking we should use it to add lines with Taa, Tba and Tbb to the tree, but we don't need those lines because the tree closes with just Tab.

Existential Instantiation Rule. If an open path has a line $\exists x \ldots x \ldots$, check this line ($\surd$). If there are no lines $\ldots n \ldots$ in the path, choose a *new name* n, and add a line $\ldots n \ldots$.

Someone can trap Moriarty	$\exists x\, Txb$
Watson can trap Moriarty	Tab

1. $\surd$ $\exists x Txb$ Premise
2. $\neg Tab$ $\neg$ Conclusion
3. Tab 1.$\exists$

The argument is invalid because there is an unclosed path.

3.2 Examples

1. Alma paints so someone paints.

Alma paints	Pa
Someone paints	$\exists x\, Px$

1. Pa Premise
2. $\surd$ $\neg\exists x Px$ $\neg$ Conclusion
3. $\forall x \neg Px$ 2.$\neg\exists$
4. $\neg Pa$ 3.$\forall$

$\times$

2. Someone sings so someone paints, because all singers paint.

Someone sings	$\exists x Sx$
All singers paint	$\forall x(Sx \to Px)$
Someone paints	$\exists x\, Px$

1.	$\surd_2$ $\exists x Sx$ Premise		
2.	$\forall x(Sx \to Px)$ Premise		
3.	$\surd_1$ $\neg\exists x Px$ $\neg$ Conclusion		
4.	$\forall x \neg Px$ 3.$\neg\exists$		
5.	Sa 1.$\exists$		
6.	$Sa \to Pa$ 2.$\forall$		
7.		$\neg Sa$ 6.$\to$ $\times$	Pa 6.$\to$
8.			$\neg Pa$ 4.$\forall$ $\times$

3. Construct a refutation tree to decide if the following argument is valid. Justify each line in the tree.

$\forall x \forall y(Lxy \to Lyx)$
$\exists x Lax$
$\overline{\exists x Lxa}$

An answer is on the next page, but try the problem before you look. (There is more than one answer depending on the order in which you apply the rules of inference.)

1. $\forall x \forall y(Lxy \to Lyx)$ Premise

2. $\surd$ $\exists x Lax$ Premise

3. $\surd$ $\neg\exists x Lxa$ $\neg$ Conclusion

4. $\forall x \neg Lxa$ 3.$\neg\exists$

5. Lab 2.$\exists$

6. $\neg Laa$ 4.$\forall$

7. $\neg Lba$ 4.$\forall$

8. $\surd$ $Lab \to Lba$ 1.$\forall$

9. $\neg Lab$ 8.$\to$ $\qquad$ Lba 8.$\to$

$\times$ $\qquad$ $\times$

This argument is valid because all paths close. Since all paths close, it is not necessary to do all applications of the $\forall$ rule to line 1. In other words you do not need to add lines $Laa \to Laa$, $Lba \to Lab$ and $Lbb \to Lbb$ to the tree.

If there were an open path you would need all these lines. First because they might close the open path; and second because even if they do not, they ensure that the first premise is true in the counterexample corresponding to the open path.

In general as long as all paths close, the rules of inference may be applied in any order.

3.3 Identity

The identity predicate Ixy is always interpreted as equality. Ixy is true if and only if x and y are equal. That being the case, we write $x = y$ instead of Ixy and $x \neq y$ or $\neg x = y$ instead of $\neg Ixy$.

Example 2. "There are at most 2 things" can be translated into symbolic form as

$$\exists x \exists y \forall z\, z = x \lor z = y.$$

Exercise 3. Translate

1. “There are at most 3 things”.
2. “There are at least 2 things”
3. “There are exactly 2 things”

The rules of inference for $=$ are as follows.

$$\frac{a \neq a}{\times} \qquad \frac{\begin{array}{c} a = b \\ \ldots a \ldots \end{array}}{\ldots b \ldots} \quad \text{or} \quad \frac{\begin{array}{c} a = b \\ \ldots b \ldots \end{array}}{\ldots a \ldots}$$

See the flowchart on page 77 of the text.