Utility representations of risk neutral preferences in multiple dimensions*

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Abstract

I show that in a multidimensional spatial model, if an agent is risk neutral on each side of the policy space away from her ideal point, then her utility function is linearly decreasing in the city block distance to the ideal policy of the agent.

Keywords: Utility representation, spatial models, multidimensional preferences, risk neutrality, city block preferences.

A multidimensional spatial model represents preferences and choices over multiple policy issues. Standard assumptions are that the utility of an agent decreases in the Euclidean distance to the ideal point of the agent and that the loss function is either linear or quadratic.[1]

The standard interpretation of the loss function is that it captures the risk attitude of the agents: If the loss function is concave, the agent is risk averse, whereas if it is linear,

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[1] See the recent textbook by McCarty and Meirowitz [9], sections 2.5 and 4.3.

the agent is said to be risk neutral over policies that all lie to the same side of her ideal point (see Berinsky and Lewis [2]). However, this interpretation is correct only if the space is unidimensional. With multiple dimensions, the shape of the indifference curves affects the risk attitude of the agent.

I adapt the definition of risk neutrality to political applications with satiated preferences over multiple issues. I show that the utility function of risk neutral agents must be linearly decreasing in the city block distance to the ideal point of the agent.[2]

I first illustrate the result by example.

Assume that there are two policy issues, and on each issue, policies can take values from -2 to $+2$, so the policy space is the square $[-2, 2] \times [-2, 2]$. Assume that the ideal policy of an agent is $(0, 0)$ and consider the non-negative quadrant. On the left side of figure 1, the utility function $u(x, y) = -\sqrt{x^2 + y^2}$ represents linear Euclidean preferences, and on the right the utility function $v(x, y) = -|x| - |y|$ represents linear city block preferences. Along the horizontal axis, Euclidean and city block utilities coincide: $u(x, 0) = v(x, 0) = -x$. Consider a fair lottery between $(0, 0)$ and $(2, 0)$. Its expected value is $(1, 0)$. The expected utility of the lottery is $\frac{1}{2}(0) + \frac{1}{2}(-2) = -1$ which is equal to the utility of the expected value, so the agent is risk neutral for this particular lottery according to either utility function.

Suppose instead that $y = 1$, and consider the fair lottery between $(0, 1)$ and $(2, 1)$, with expected value $(1, 1)$. According to the linear Euclidean utility, $u(1, 1) = -\sqrt{2} \approx -1.41$, and the expected utility of the lottery is $\frac{1}{2}u(0, 1) + \frac{1}{2}u(2, 1) = -1.62$, so the agent is risk averse. This result generalizes: Fixing any value $y \neq 0$, for any lottery over x, an agent with linear

[2] I assume that the spatial representation of the set of feasible policies is exogenously given. If alternatives are not quantifiable, the mapping from the set of alternatives to $\mathbb{R}^K$ is endogenous and any notion of risk attitudes based on the shape of the utility functions becomes problematic. See Kalandrakis [8], Bogomolnaia and Laslier [3] and Eguia [4] for spatial models with an endogenous representation of the set of alternatives.

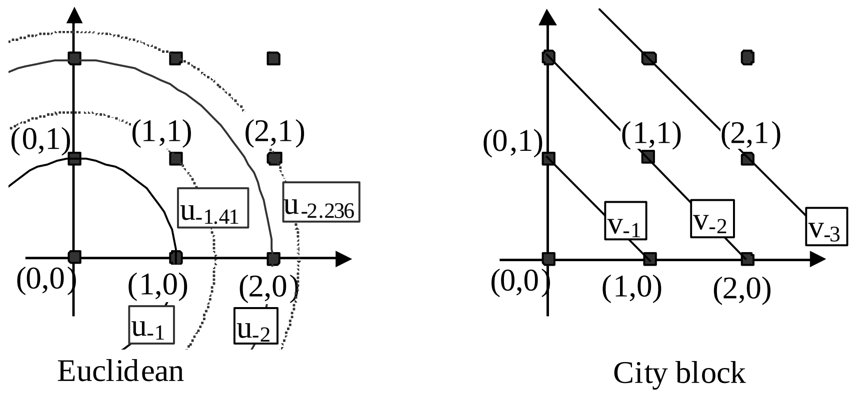

Figure 1: Indifference curves and utilities in two dimensions.

Euclidean utilities prefers the expected value of the lottery over the lottery. On the other hand, with linear city block utilities, $v(0,1) = -1$, $v(1,1) = -2$ and $v(2,1) = -3$ so the agent is indifferent about a fair lottery between $(0,1)$ and $(2,1)$ with an expected utility of -2, or the expected value of the lottery, which yields a utility of -2 for certain. For any lottery over points in the first quadrant, an agent with linear city block preferences is risk neutral.

Theory

Let $X = \mathbb{R}^K$ be a set of alternatives.[3] Each of the K dimension corresponds to a policy issue, and an alternative is a policy bundle that specifies a policy for each issue. Let ΔX be the set of all simple lotteries defined over X.[4] For any given lottery $p \in \Delta X$, let $p(x)$ denote the probability that p assigns to $x \in X$. Slightly abusing notation, let $x, y, z, w \in X$ also denote degenerate lotteries,[5] so they belong to ΔX. Let x_k denote the *k-th* coordinate of

[3] The result generalizes to $X \subset \mathbb{R}^K$ with an open interior. The proof is available from the author.
[4] A simple lottery is one that assigns positive probability only to a finite number of outcomes.
[5] Degenerate lotteries are those that generate a specific outcome with probability 1.

x and let x_{-k} denote the vector of $K-1$ dimensions that contains all the coordinates of x except x_k. Then one can write x as $x = (x_k, x_{-k})$.

Let $\succsim$ be a complete and transitive binary relation on ΔX representing the weak preferences of agent i over lotteries on X. Let $x \succ y$ denote ($x \succsim y$, not $y \succsim x$) and let $x \sim y$ denote ($x \succsim y$, $y \succsim x$). Let $\succsim$ be representable by the expected utility of a Von Neuman and Morgenstern [13] utility function $u : X \longrightarrow \mathbb{R}$ such that for any $p, q \in \Delta X$, $p \succsim q$ if and only if $\sum_X p(x)u(x) \geq \sum_X q(x)u(x)$.

Assume that $\succsim$ has a unique maximal element $x^* \in X$ such that $x^* \succ p$ for any $p \in \Delta X$, $p \neq x^*$. The degenerate lottery x^* is the most preferred alternative of the agent. Without loss of generality, let $x^* = (0, ..., 0)$. The set X is divided into 2^K orthants. Each orthant is one of the subsets composed of the points that do not contain both points that are strictly positive and points that are strictly negative in any given dimension; the analog of a quadrant on $\mathbb{R}^2$ or an octant on $\mathbb{R}^3$. Let O^j denote an arbitrary one of them, for $j \in \{1, 2, 3, ..., 2^K\}$

An agent is risk neutral if for any $p \in \Delta X$, $p \sim \sum_{x \in X} p(x)x$ that is, if she is indifferent between the lottery p and its expected value $\sum_{x \in X} p(x)x \in X$. This is the standard condition with non-satiated agents. With ideological preferences satiated at x^*, this condition is violated by all agents, who prefer their ideal point for sure over any lottery. A weaker notion of risk neutrality is appropriate. Authors as early as Shepsle [11] noticed this problem. Bendor and Meirowitz [1] suggest in a footnote that "one could define a notion of risk-aversion relative to certain lotteries." This is exactly what I do. I propose a weaker notion of risk neutrality for agents who prefer their ideal point to any lottery. I call it *orthant risk aversion*. The intuition pins back the notion of risk in an environment with satiated preferences to the standard definition of risk with monotonic preferences.

Consider the orthant that is non-positive in all dimensions. Within that orthant, the agent wants more of everything. This is the standard economic environment. Hence the standard risk neutrality concept applies, *within this orthant.* A risk neutral agent is indifferent between a lottery that assigns positive probability to outcomes in this orthant and its expected value. I impose the same condition within any other orthant: A risk neutral agent is indifferent between any lottery that assigns positive probability to outcomes in only one orthant and the expected value of this lottery.

Axiom 1 *(Orthant risk neutrality)* $p \sim \sum_{x \in X} p(x)x$ *for any* p *whose support is contained in a single orthant.*[6]

Similarly, $\succsim$ is orthant risk averse if $\sum_{x \in X} p(x)x \succ p$ for any p whose support is contained in a single orthant and contains x, y such that $x \succ y$.

If preferences over lotteries satisfy orthant risk neutrality, then the preferences over sure outcomes can be represented by a utility function that is linear in the distance measured by a generalization of the city block norm. To state the result formally, I first define the generalized weighted city block distance.[7]

Definition 1 *For any* $\lambda \in \mathbb{R}_+^{2K}$, *let* $d_\lambda : \mathbb{R}^K \longrightarrow \mathbb{R}$ *be the generalized weighted city block distance:*

$$d_\lambda(x) = \sum_{k=1}^{K} \lambda_k(x_k)|x_k|, \textit{ with } \lambda_k(x_k) = \left\{ \begin{array}{c} \lambda_{k+} > 0 \textit{ if } x_k > 0 \\ \\ \lambda_{k-} > 0 \textit{ if } x_k < 0 \end{array} \right\}, \; k \in \{1, ..., K\}.$$

The standard city block norm has equal weights in every dimension, and on each side of the origin, that is, $\lambda_{k+} = \lambda_{k-} = \lambda_{j+} = \lambda_{j-}$ for any $j, k \in \{1, ..., K\}$. A weighted city block

[6] The support of p is the set of alternatives to which lottery p assigns positive probability.

[7] I use the term *distance* informally, to refer to a function that quantifies proximity to the origin. Since this function is not symmetric, it is not a mathematical *norm.*

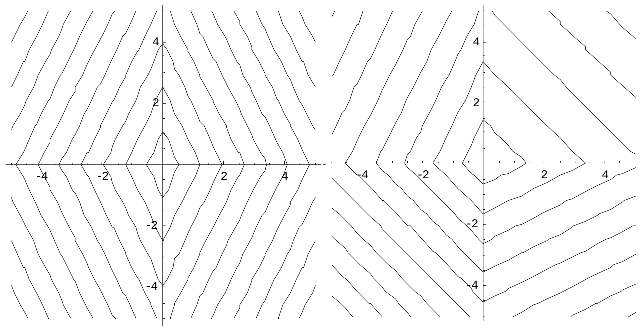

Figure 2: Generalized city block indifference curves in two dimensions.

norm has different weights for each dimension, but equal weights along each dimension, $\lambda_{k+} = \lambda_{k-}$ for any $k \in \{1, ..., K\}$. The generalized weighted city bloc distance allows for weights to be different to each side of the origin along the same dimension, as depicted in figure 2. If an agent with satiated preferences is risk neutral in each orthant, then only a generalization of the city block norm represents her preferences.

Theorem 1 *Let $\succsim$ have a unique maximal element x^*, be orthant risk neutral and be representable by the expected utility of a function $u : X \longrightarrow \mathbb{R}$. Then $\exists \lambda \in \mathbb{R}^{2K}_{+}$ such that u is linearly decreasing in the distance $d_\lambda(x)$.*

Proof. Recall $x^* = \{0\}^K$ and without loss of generality let $u(0, ..., 0) = 0$ and $u(1, 0, ..., 0) = -1$. For each $k \in \{2, ..., K\}$, let $x^k \in X$ be such that $x^k_k = 1$ and $x^k_j = 0$ for any $j \neq k$, and let $u(x^k) = \alpha^k$, where $\alpha^k < 0$. Let $\lambda_{k+} = -\alpha^k$. For each k, let $\frac{1}{\lambda_{k+}} x^k = z^k$. By orthant risk neutrality, $u(z^k) = -1$. Let $Z = \{x \in X : x = \sum_{k=1}^{K} p_k z^k$ for some $(p_1, ..., p_K)$ such that $p_k \geq 0$ for all k and $\sum_{k=1}^{K} p_k = 1\}$. The set Z is the set of points in the non-negative orthant that are a convex combination of points along each of the axes that yield utility -1. By orthant risk neutrality, $u(z) = -1$ for any $z \in Z$.

Let O^1 denote the non-negative orthant. Note that any $y \in O^1$ can be expressed as a linear transformation $y = \alpha z$ of some $z \in Z$ for some $\alpha > 0$. By orthant risk neutrality, it follows $u(y) = \alpha u(z) = -\alpha$.

$$d_\lambda(y) = \sum_{k=1}^{K} \lambda_{k+} y_k = \alpha \sum_{k=1}^{K} \lambda_{k+} z_k = \alpha \sum_{k=1}^{K} \lambda_{k+} p_k z_k^k = \alpha \sum_{k=1}^{K} p_k x_k^k = \alpha;$$

therefore, $u(y) = -d_\lambda(y)$. The second equality follows from $y = \alpha z$; the third from $z \in Z$; the fourth from $\frac{1}{\lambda_{k+}} x^k = z^k$, and the last from $x_k^k = 1$ and $\sum_{k=1}^{K} p_k = 1$.

The non-negative orthant was arbitrarily chosen. Repeating the above procedure on each orthant j, for each dimension k we find weights λ_k^j that correspond to the weight on dimension k found in the proof constructed for orthant j. Consider any two orthants O^2 and O^3 with positive sign on dimension i. Take $x \in O^2 \cap O^3$ such that $x_i > 0$ and $x_j = 0$ for all $j \neq i$. Since x belongs to the same indifference curve when measured as a point in orthant O^2 or orthant O^3, it must be $\lambda_{i+}^2 = \lambda_{i+}^3$, and thus the weight on dimension i is the same for all orthants that are positive on dimension i. Choosing x_i positive was arbitrary; similarly, for any $i \in \{1, ..., K\}$, the weight on dimension i is constant for all orthants that are negative on dimension i. ■

As a corollary, if preferences are symmetric so that along each dimension utility loses are equal in each direction away from the origin, then the generalized weighted city block distance reduces to a standard weighted city block norm.

Discussion

In spatial models, the two most popular utility functions are linear and quadratic Euclidean (McCarty and Meirowitz [9], p.22-24, p.80). The quadratic Euclidean utility function in multiple dimensions is the natural extension of the quadratic utility function in one dimension. However, the linear Euclidean utility function in multiple dimensions is *not* the natural extension of the linear utility in one dimension. In two dimensions, a linear Euclidean utility function is a cone. If we fix a non-zero value in the second dimension, and we cut the cone with a plane at that value, the resulting utility function in one dimension is not a linear function, but a hyperbola. The extension of linear utility to multiple dimensions is a linear city block utility function, whose shape in two dimensions is a pyramid, which, when cut by a plane, generates the linear utility function in one dimension.

An implication is that the popularity of linear Euclidean functions is inconsistent with the inattention to city block functions. If the appropriate assumption in one dimension is linearity, then the consistent choice is to use linear city block utilities in multiple dimensions. Whether a linear city block or a quadratic Euclidean utility function is more appropriate depends on the risk attitude of the agent. If agents are risk neutral, then the linear city block utility is appropriate, and if so, the theoretical implications for the stability of majority rule in multiple dimensions are huge: While the core is generically empty if indifference curves are Euclidean (or more generally, smooth; Plott [10]), core outcomes exist more generally if utilities are city block (Humphreys and Laver [7]).

A recent article by Berinsky and Lewis [2] attempts to estimate the risk attitude of US voters. They assume that the utility function is decreasing in some power of the Euclidean distance in three dimensions, and they estimate the parameter α in the utility function $\left(\left(\sum_{k=1}^{3} w_k |x_{ki} - x_{ki}^*|^2\right)^{1/2}\right)^{\alpha}$, where w_k is the weight of each issue. They estimate that $\alpha = 0.98$, and they interpret this finding as risk neutrality. However, this interpretation is not correct. As I have shown, the risk attitude is not only determined by the loss function, it also depends on the shape of the indifference curves. Their estimation that $\alpha \approx 1$ implies that starting at their multidimensional ideal point, agents are risk neutral on any direction, but starting away from their ideal point, agents are risk averse along any direction.

To improve upon their results and obtain an estimate of risk aversion that is robust over the whole policy space, I suggest estimating α using the utility function $\sum_{k=1}^{K} w_k |x_{ki} - x_{ki}^*|^{\alpha}$. If agents are not very risk averse, then α is close to one, the indifference curves are approximately linear, and a linear city block utility function is a better assumption than quadratic Euclidean utilities.

Tomz and Van Houweling [12] conduct laboratory experiments to test the effect on voters of uncertainty about the policy platform espoused by a candidate. They find that "on average, ambiguity does not repel and may, in fact, attract voters." Their finding is consistent with risk neutrality. Empirical studies on voters' choices in presidential and legislative elections by Enelow, Mendell and Ramesh [5], Westholm [14] and Grynaviski and Corrigan [6] obtain better results using city block utilities than using Euclidean utilities. These results provide more indirect evidence that a linear city block function may better represent the preferences of voters.

These findings are relevant for models of ideal point estimation. While they all use the Euclidean distance, using a linear city block utility may improve their results. Better yet, once we have a reliable estimate $\widehat{\alpha}$ of risk aversion for the whole policy space, ideal point estimation methods will achieve the best results using the utility function $u_i(x) = \sum_{k=1}^{K} |x_k - x_k^i|^{\widehat{\alpha}}$.

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