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Euclidean preferences

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Abstract

This note is devoted to the question: how restrictive is the assumption that preferences be Euclidean in d dimensions. In particular it is proven that any preference profile with I individuals and A alternatives can be represented by Euclidean utilities with d dimensions if and only if $d \geq \min(I, A-1)$. The paper also describes the systems of A points which allow for the representation of any profile over A alternatives, and provides similar results when only strict preferences are considered. These findings contrast with the observation that if preferences are only required to be convex then two dimensions are always sufficient.

Keyword: Spatial preferences

1. Introduction

A popular model in Political Science is the "spatial model of preferences". It amounts to consider that the alternatives which are the objects of preferences are points in the Euclidean d-dimensional space $\mathbb{R}^d$, that an individual is characterized by his or her "ideal point" in that same space, and that alternatives are judged as good as they are close to the ideal point.

One-dimension Euclidean preference profiles are very specific, they show no Condorcet cycles (Hotelling, 1929; Arrow, 1951; Black, 1958). But this property is lost as soon as d is at least 2, and the chaotic behavior of majority rule can be seen in the planar Euclidean model (Davis et al., 1972; McKelvey, 1976). Multi-dimensional models are often used as an illustration of the theory (Stokes, 1963; Enelow and Hinich, 1990), applications to Public Economics and the theory of

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taxation are possible (for instance Gevers and Jacquemin, 1987, and De Donder, 2000) but not so common because of intrinsic limitations of the Euclidean model (Milyo, 2000). Empirical use of the spatial model in Politics are numerous, an important problem being to develop adequate statistical tools for estimation of voters ideal points or party positions (see Londregan, 2000; Bailey, 2001; Poole, 2005; Laslier, 2006).

This note is devoted to the following question: how restrictive is the assumption that a finite preference profile be Euclidean in d dimensions? We are not aware of any closely related studies. In Graph theory, Johnson and Slater (1988), and Gu et al. (1995),[1] considered the realization of asymmetric digraphs (also called "weak tournaments") as the majority relation associated to some preference profile, when preferences are supposed to be derived from distances on a graph. Working in a single dimension, Brams et al. (2002) introduced a distinction between *ordinally* and *cardinally* single-peaked preferences. Euclidean preferences (in one dimension) are a particular type of cardinally single-peaked preferences.

We investigate whether, given a preference profile, one can find an embedding of both the set of alternatives and the set of agents in some Euclidean space, so that this profile would be represented by Euclidean utilities in such embedding. We show that the number of dimensions needed to guarantee that any preference profile (or even just all strict preference profiles) can be represented by Euclidean utilities grows linearly with the number of alternatives, as well as with the number of agents. More precisely, the smallest number of dimensions necessary is approximately the minimum of the two (see below). We further compare this result with another, more general, situation. Assume that one seeks to represent a preference profile (both agents and alternatives) by points in some Euclidean space, but there is no requirement on preferences to be Euclidean. Agents are simply required to have convex preferences. This case is in sharp contrast with the preceding one. No matter how large are the set of alternatives and the set of agents, two dimensions are sufficient for the representation. Moreover, one can embed the set of alternatives in $\mathbb{R}^2$ in such a way, that any convex preference profile could be generated by an appropriate embedding of the agents.

We restrict our attention to the finite setting and use the following vocabulary. There is a finite number I of individuals $i \in \{1, \dots, I\}$ and a finite number A of alternatives $a \in \{1, \dots, A\}$. A *preference* R_i for individual i is a weak order on $\{1, \dots, A\}$; for alternatives a and b, aR_ib means that i prefers a to b, that is strictly prefers (denoted aP_ib) or is indifferent between a and b (denoted $a \sim_i b$). A preference is *strict* if $a \sim_i b$ implies $a = b$. A preference *profile* is a vector $R = (R_i)_{i=1}^{I}$. Let $\mathcal{R}_{A,I}$ be the set of preference profiles with I individuals and A alternatives. Let $\|\cdot\|$ denote the usual two-norm and $x^a \cdot v^i$ denote the usual scalar product in $\mathbb{R}^d$.

Definition 1. A profile $R \in \mathcal{R}_{A,I}$ is *Euclidean* of dimension d if there exist points $x^a, a = 1, \dots, A$ in $\mathbb{R}^d$ such that, for all a and b and for all individuals i, either there exists a point $\omega^i \in \mathbb{R}^d$ such that:

$$aR_ib \Longleftrightarrow \|x^a - \omega^i\| \leq \|x^b - \omega^i\|,$$

or there exists a direction $v^i \in \mathbb{R}^d$ such that:

$$aR_ib \Longleftrightarrow x^a \cdot v^i \geq x^b \cdot v^i.$$

If any profile in $\mathcal{R}_{A,I}$ is Euclidean of dimension d then we say that d is *sufficient* for I orders on A alternatives.

[1] Thanks to Michel Le Breton for indicating this reference to us.

Point x^a is called the location of alternative a, point ω^i is called the *ideal point* of individual i and v^i the *ideal direction* for individual i. Indifference curves are spheres in the first case and hyperplanes (or the whole space) in the second case. We refer to the first type of preferences as "quadratic", or "spheric". We could refer to the second type as "linear", but the term "linear preference" is usually used with another meaning, therefore we use the expression "directional" preference (Rabinowitz and MacDonald, 1989). The case of complete indifference corresponds to the degenerated ideal direction $v^i = 0$. Non degenerated directional preferences can be seen as limit case of quadratic preferences, when the ideal point ω^i goes to infinity in the direction v^i. Here are some obvious properties:

Proposition 2. *Suppose that d is sufficient for I orders on A alternatives, then*:

(i) *d is sufficient on A alternatives for all $I' \leq I$*;
(ii) *d is sufficient for I orders for all $A' \leq A$*;
(iii) *d' is sufficient for I orders on A alternatives for all $d' \geq d$.*

The following results will be proved. In Section 2.1 we determine when dimension d is sufficient; Theorem 8 states that d is sufficient for I orders on A alternatives if and only if $d \geq \min\{I, A-1\}$. In Section 2.2 we characterize the systems of locations which are able to represent all preferences; Theorem 11 states that a system of A points in $\mathbb{R}^d$ allows for the Euclidean representation of any preference profile over A alternatives if and only if it spans a space of dimension $A-1$.

Notice that we allow for indifferences, and that preferences with indifferences are used in some proofs. If one only considers profiles of strict preferences, then the smallest necessary number of dimensions is proven to be between $\min\{I-1, A-1\}$ and $\min\{I, A-1\}$. Section 2.3 is devoted to the strict preference case.

Finally, we compare our findings with much more general case, when one tries to represent a preference profile by a spatial model with convex (but not necessarily Euclidean or based on any other metric) preferences. Recall that a preference relation $\succcurlyeq_i$ on $\mathbb{R}^d$ is convex if its upper contour sets $U_{\succcurlyeq_i}(y) = \{x \in \mathbb{R}^d : x \succcurlyeq_i y\}$ are convex.

Definition 3. A profile $R \in \mathcal{R}_{A,I}$ is *convex spatial* of dimension d if there exist points x^a, $a = 1, \ldots, A$ in $\mathbb{R}^d$ such that for any agent i, there exists a convex preference relation $\succcurlyeq_i$ on $\mathbb{R}^d$ such that for all a and b:

$$aR_ib \Longleftrightarrow x^a \succcurlyeq_i x^b$$

We prove in Section 2.4 that any preference profile is convex spatial of dimention 2 (though not necessarily of dimension 1). Even stronger, for any set A of alternatives there is a set of points $\{x^a\}_{a\in A}$ in $\mathbb{R}^2$ such that any preference profile on A becomes convex spatial under an appropriate choice of convex $\succcurlyeq_i$, $i \in I$, on $\mathbb{R}^2$. Moreover, we can describe the sets $\{x^a\}_{a\in A}$ that allow for such representations.

2. Results

2.1. Determination of the sufficient dimension

The following result states that any profile is Euclidean provided one considers as many dimensions as there are individuals.

Proposition 4. *If $d \geq I$, d is sufficient for all A.*

Proof. It is enough to prove this for $d = I$. Define for each alternative a a point x^a in $\mathbb{R}^I$ by saying that its ith coordinate is:

$$x_i^a = -\#\{b : aR_ib\}.$$

For instance, on axis i, individual i's best preferred alternative has coordinate -1 and i's worst alternative has coordinate $-A$. Then, for some number M, define i's ideal point ω^i by saying that its coordinate on axis j is:

$$\omega_j^i = \begin{cases} M, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}.$$

Then it is easy to see that, for M large enough the points x^a and ω^i represent the profile R in $\mathbb{R}^I$. This proves the result using only spheric preferences. □

Proposition 5. *If $d \geq A - 1$ then d is sufficient for all I.*

Proof. It is enough to prove this for $d = A - 1$. Consider the A points x^a in $\mathbb{R}^A$, defined by the coordinate of x^a on axis $b \in \{1, \ldots, A\}$ being $x_b^a = 1$ if $a = b$ and $x_b^a = 0$ if $a \neq b$. Notice that the points x^a all belong to the linear space $\Delta_A = \left\{ y \in \mathbb{R}^A : \sum_{a=1}^{A} y_a = 1 \right\}$ of dimension $A - 1$. Let a and b be two alternatives The median to the segment $[x^a, x^b]$ is a hyperplane $H(a \sim b)$ which divides Δ_A in two-half spaces that can be denoted $H(a > b)$ and $H(b > a)$, $H(a > b)$ being the set of points in Δ_A which are closer to a than to b. In an Euclidean representation of her preference, an individual strictly prefers a to b if and only if her ideal point is in $H(a > b)$. Let R_i be a preference over the set $\{1, \ldots, A\}$, the condition for a point ω^i to serve as an ideal point for R_i is thus that ω^i belongs to $H(a > b)$ for all $a \neq b$ such that aR_ib, and we find that R_i can be represented if and only if:

$$\Omega(R_i) = \bigcap_{a \neq b : aR_ib} H(a > b) \neq \emptyset.$$

By symmetry, if $\Omega(R_i)$ is empty for some preference R_i, it is empty for all preferences, and this is obviously not the case. Therefore, for any preference R_i, $\Omega(R_i) \neq \emptyset$ and it follows that for any profile $R = (R_i)_{i=1}^I$ there exist points ω^i, $i = 1, \ldots, I$ that represent R in Δ_A with respect to the points x^a, $a = 1, \ldots, A$. This proves the result using only spheric preferences. □

The proof of the next result will rest on the following example, with slightly more alternatives than individuals.

Example 6. Consider $I = d + 1$ individuals and $A = d + 2$ alternatives. To avoid confusion, denote $a_0, a_1, \ldots, a_i, \ldots, a_{d+1}$ the alternatives. The individual $i \in \{1, \ldots, d + 1\}$ strictly prefers alternative a_i to any other and is indifferent between all the others:

$$\begin{aligned} &a_i P_i a_j, && \text{for } j \neq i \\ &a_j \sim_i a_k, && \text{for } j, k \neq i \end{aligned}.$$

Proposition 7. *Dimension d is not sufficient for $I = d + 1$ individuals and $A = d + 2$ alternatives.*

Proof. Consider the profile of Example 6.

For $d = 1$, consider three locations x_0, x_1 and x_2 on a line. These three locations must be different one from the other. Thus, for the indifferences to be possible, preferences cannot be directional. Considering the preference R_1, one can see that x^1 must be between x^0 and x^2, and for the preference R_2, x^2 must be between x^0 and x^1, impossible.

For $d = 2$, the profile is:

$i = 1$	$i = 2$	$i = 3$
a_1	a_2	a_3
$a_0 \sim a_2 \sim a_3$	$a_0 \sim a_1 \sim a_3$	$a_0 \sim a_1 \sim a_2$

The result will then be proven by induction on d, starting from $d = 2$. For a contradiction, consider an Euclidean representation of the profile in $\mathbb{R}^2$, with points $x^0, \ldots, x^3$ for the alternatives.

In a first part of the proof, suppose that some individual preference, for instance R_3 is directional. Then x_0, x_1 and x_2 are on a line. For the preference R_1, x^1 must be between x^0 and x^2, and for the preference R_2, x^2 must be between x^0 and x^1, impossible.

In a second part of the proof, suppose that all individual preferences are spheric. It is easy to see that the four locations $x^0, \ldots, x^3$ are distinct. For $i = 1$, the three points x^0, x^2, x^3 are on a circle S^1 centered at the ideal point ω^1 and the location x^1 is inside the disk, and similarly for $i = 2, 3$. Denote:

$$S^i = \{y \in \mathbb{R}^2 : \|y - \omega^i\| = \|x^0 - \omega^i\|\},$$
$$B^i = \{y \in \mathbb{R}^2 : \|y - \omega^i\| < \|x^0 - \omega^i\|\}.$$

the profile is such that, for all $i \neq j$:

$$x^j \in S^i$$
$$x^i \in B^i.$$

In particular, x^0 is on S^i for $i = 1, 2, 3$.

We now prove that this is impossible. Consider the inversion of center x^0 and ratio 1, that is the application ψ form $\mathbb{R}^2 \setminus \{x^0\}$ onto itself defined by:

$$\forall\, x \in \mathbb{R}^2 \setminus \{x^0\},\ \ \psi(x) - x^0 = \frac{x - x^0}{\|x - x^0\|^2}.$$

As is well-known, this application is involutive ($\psi(\psi(x)) = x$) and transforms the spheres that contain x^0 into hyperplanes that do not contain x^0.

For $i = 1, 2, 3$, denote $y^i = \psi(x^i)$. Suppose firstly that the points y^i are on a single hyperplane (a line) that does not contain x^0. Then, by ψ, the three circles S^i, $i \in \{1, \ldots, d + 1\}$ are identical, which is impossible.

Suppose secondly that the points y^i are on a line that contains x^0, then by ψ, the points x^i are on that same line. Then the three points x^0, x^1, x^2 being at the same distance from ω^3, two of them at least are equal, which is impossible.

Suppose now that the three points y^i span $\mathbb{R}^2$. Then there exists a unique vector $(\lambda_1, \lambda_2, \lambda_3)$ such that:

$$\sum_{i=1}^{3} \lambda_i y^i = x^0$$
$$\sum_{i=1}^{3} \lambda_i = 1 \quad .$$

For $i \geq 1$, the center of inversion is on the circle S^1 thus its image is a line that we denote by D^i. Moreover, if $x^i \in B^i$, its image y^i is one the side of D^i opposite to the center x^0, therefore $\lambda_i < 0$. Hence it cannot be the case that $x^i \in B^i$ for all i.

It remains to complete the induction. Suppose the result is true up to $d-1$ and consider an Euclidean representation of the profile in $\mathbb{R}^d$, with locations $x^0, \ldots, x^{d+1}$ for the alternatives.

If one preference, say R_{d+1} is directional, then the points $x^0, \ldots, x^d$ are on a hyperplane. Dropping individual $d+1$ and alternative $d+1$ yields the same profile at the previous order, by the induction hypothesis, it cannot be represented with $d-1$ dimensions.

Suppose now that all preferences are spheric. The argument is the same as for $d = 2$. The $d+1$ spheres $S^i = \{y \in \mathbb{R}^d : \|y - \omega^i\| = \|x^0 - \omega^i\|\}$ are different one from the other and intersect at x^0, and for $i = 1, \ldots, d+1$, x^i is inside S^i. By inversion, points x^i are transformed into $d+1$ points $y^1, \ldots, y^{d+1}$ that cannot be on a single hyperplane otherwise the points $x^0, x^1, \ldots, x^{d+1}$ would be either on the same hyperplane or on the same sphere, both situations being impossible. Thus, $y^1, \ldots, y^{d+1}$ span $\mathbb{R}^d$ and the conclusion follows. □

Theorem 8. *Dimension d is sufficient for I orders on A alternatives if and only if $d \geq \min\{I, A-1\}$.*

Proof. Propositions 4 and 5 prove that d is sufficient if $d \geq \min\{I, A-1\}$. Conversely, take $d < I$ and $d < A-1$, then $I \geq d+1$ and $A \geq d+2$ and we know from Proposition 7 that d is not sufficient for $I = d+1$ individuals on $A = d+2$ alternatives. □

2.2. *Systems of locations that represent all preferences*

Given a number A of alternatives, we identify the systems of points $(x^a)_{a=1}^A$ which are such that any preference over alternatives $1, \ldots, A$ can be represented with these points.

Lemma 9. *If $(x^a)_{a=1}^A$ is a system of A points in $\mathbb{R}^{A-1}$ that allows for the Euclidean representation of all preferences then the median hyperplanes $H(a \sim b)$, for $a, b \in \{1, \ldots, A\}$ have a non-empty intersection.*

Proof. If two such hyperplanes, say $H(a \sim b)$ and $H(c \sim d)$ have empty intersection, it must be the case that one-half space $H(a < b)$ or $H(a > b)$ is included in $H(c < d)$ or $H(c > d)$. If, for instance, $H(a < b) \subseteq H(c < d)$ the system is unable to represent a preference such that aR_ib and dR_ic. Thus, two hyperplanes intersect. Suppose, for a contradiction, that we can only finds k points, with $k < A$ whose median hyperplanes intersect. For instance:

$$\bigcap_{1 \leq a,b \leq k} H(a \sim b) \neq \emptyset$$

but:

$$\bigcap_{1 \leq a,b \leq k} H(a \sim b) \subseteq H(1 < k+1)$$

this implies that the system is unable to represent preferences such that aI_ib for all $a, b \leq k$ and $(k+1)P_i1$. □

Lemma 10.

(i) *If* $(x^a)_{a=1}^{A}$ *is a system of A points in* $\mathbb{R}^{A-1}$ *that allow for the Euclidean representation of all preferences on A alternatives, then the intersection of the median hyperplanes is a singleton.*
(ii) *If* $(x^a)_{a=1}^{A+1}$ *is a system of* $A+1$ *points in* $\mathbb{R}^d$ *that allow for the Euclidean representation of all preferences on* $A+1$ *alternatives then* $(x^a)_{a=1}^{A+1}$ *spans a space of dimension A.*

Proof. The lemma will be proved by induction on A.

For $A = 2$ point (i) is trivially true. For point (ii), consider three different points. If they are on a line then one is between the other two, then no preference can rank this point last.

For $A \geq 2$ suppose that both (i) and (ii) are true. To check (i) at the next order, consider $A+1$ points $\{x^1, \ldots, x^{A+1}\} \subseteq \mathbb{R}^A$ that allow for the Euclidean representation of any preference on $\{1, \ldots, A+1\}$. We know that the intersection of the median hyperplanes is nonempty. If it is not a singleton then it is some linear space of positive dimension. Let E be an affine subspace orthogonal to that intersection, the dimension of E is at most $A-1$. Let $\hat{x}^1, \ldots, \hat{x}^{A+1}$ be the projections of $x^1, \ldots, x^{A+1}$ on E. If $\omega^i \in \mathbb{R}^A$ is the ideal point for preference R_i with respect to $\{x^1, \ldots, x^{A+1}\}$, let $\hat{\omega}^i$ be the projection of ω^i on E. Then it is easy to check that:

$$\|x^a - \omega^i\| \leq \|x^b - \omega^i\| \iff \|\hat{x}^a - \hat{\omega}^i\| \leq \|\hat{x}^b - \hat{\omega}^i\|$$

so that R_i is well represented. We thus have found Euclidean representation of preferences over $A+1$ alternatives with at most $A-2$ dimensions. By the induction hypothesis (ii), this is impossible. This establish the induction step for point (i).

To check (ii) at the next order, consider a system $(x^a)_{a=1}^{A+2}$ of $A+2$ points in $\mathbb{R}^d$ that allows for the Euclidean representation of any preference on $A+2$ alternatives and suppose, for a contradiction, that these points do not span a space of dimension $A+1$, which means that they are included in a linear space of dimension A.

Each subset $\{x^1, \ldots, x^{A+2}\} \setminus \{x^a\}$ of $A+1$ of these points allows for the euclidean representation of any preference on $\{1, \ldots, A+2\} \setminus \{a\}$ and thus, by point (i), there exist a unique point, call it z^{A+2}, equidistant from $x^1, \ldots, x^{A+1}$. This point is such that an individual i is indifferent between alternatives $1, \ldots, A+1$ if and only if $\omega^i = z^{A+2}$. If $z^{A+2} \in H(1 < A+2)$, we find that an individual cannot have the preference $1 I_i 2 \ldots I_i (A+1) P_i (A+2)$, and $z^{A+2} \in H(1 \sim A+2)$ or $z^{A+2} \in H(A+2 < 1)$ would entail similar preference restrictions, in contradiction with the hypothesis. It thus must be the case that $(x^a)_{a=1}^{A+2}$ spans a space of dimension $A+1$. □

Theorem 11. *A system of A points* $(x^a)_{a=1}^{A}$ *in* $\mathbb{R}^d$ *allows for the Euclidean representation of all preferences over A alternatives if and only if* $d \geq A-1$ and $(x^a)_{a=1}^{A}$ *spans a space of dimension* $A-1$.

Proof. The "only if part" is point (ii) of the previous lemma. The converse will be proven by induction. For $A = 2$, it is easy. Take $A > 2$ and suppose that $(x^a)_{a=1}^{A}$ spans a space of dimension $A-1$, denote it $[x^1, \ldots, x^A]$. Consider a preference relation R_i.

If there exists an alternative (say alternative A) which is strictly preferred to all the other alternatives. The points $(x^a)_{a=1}^{A-1}$ span a space $[x^1, \ldots, x^{A-1}]$ of dimension $A-2$ therefore, by the induction hypothesis, there exists a point $\omega \in [x^1, \ldots, x^{A-1}]$ such that ω with respect to $x^1, \ldots, x^{A-1}$ represents the restriction of R_i to $\{1, \ldots, A-1\}$. Let n, with $\|n\| = 1$ be a vector in $[x^1, \ldots, x^A]$, orthogonal to $[x^1, \ldots, x^{A-1}]$, we can choose n such that $(x^A - \omega) \cdot n > 0$. Notice that for any λ, $\omega + \lambda n$ represents the restriction of R_i as well. Moreover, for any

$x^a \in [x^1, \dots, x^{A-1}]$:

$$\|\omega + \lambda n - x^a\|^2 = \lambda^2 + \|\omega - x^a\|^2$$

Write $x^A - \omega = x^A - y^A + y^A - \omega$ with y^A the projection of x^A on $[x^1, \dots, x^{A-1}]$, then:

$$\|\omega + \lambda n - x^A\|^2 = (\lambda - \|x^A - y^A\|)^2 + \|\omega - y^A\|^2.$$

It follows that, for λ large enough, $\omega + \lambda n$ is closer to x^A than to x^a. Thus, for λ large enough, $\omega^i = \omega + \lambda n$ represents R_i.

The reasoning is similar if there are several alternatives which are strictly preferred to the others and among which the individual is indifferent, say $x^{k+1} \sim_i x^{k+2} \sim_i \cdots \sim_i x^A$. Take $\tilde{\omega} \in [x^1, \dots, x^k]$ that represents the restriction of R_i to $\{1, \dots, k\}$. Any $\omega = \tilde{\omega} + y$ such that y is orthogonal to $[x^1, \dots, x^k]$ represents this restriction as well. Let E be the set of such ω, E is a linear space of dimension $(A-1)-(k-1) = A-k$. Let $\hat{\omega}$ be the center of the sphere in $[x^{k+1}, \dots, x^A]$ that contains points $x^{k+1}, \dots, x^A$, $\hat{\omega}$ represents the restriction of R_i to $\{k+1, \dots, A\}$, and any $\omega = \hat{\omega} + z$ such that z is orthogonal to $[x^{k+1}, \dots, x^A]$ represents this restriction as well. Let F be the set of such ω, F is a linear space of dimension $(A-1)-(A-k-1) = k$. Since the whole space has dimension $A-1$, $E \cap F$ contains a line L, which means that there exists a point t and a vector n with $\|n\| = 1$ which satisfies the following property: for all λ, the points in L, which we can denote $\omega(\lambda) = t + \lambda n$, are such that $\omega(\lambda) - \tilde{\omega}$ is orthogonal to $[x^1, \dots, x^k]$ and $\omega_\lambda - \hat{\omega}$ is orthogonal to $[x^{k+1}, \dots, x^A]$. Any such $\omega(\lambda)$ represents both restrictions.

Let $\omega(\tilde{\lambda}) = t + \tilde{\lambda} n$ be the projection of $\tilde{\omega}$ on L. Because L is orthogonal to $[x^1, \dots, x^k]\omega(\tilde{\lambda})$ is also the projection on L of $x^1, \dots, x^k$, and for all λ,

$$\|\omega(\lambda) - x^a\|^2 = (\lambda - \tilde{\lambda})^2 + \|\omega(\tilde{\lambda}) - x^a\|^2.$$

Similarly, let $\omega(\hat{\lambda}) = t + \hat{\lambda} n$ be the projection of $\hat{\omega}$ on L, for $k+1 \leq b \leq A$:

$$\|\omega(\lambda) - x^b\|^2 = (\lambda - \hat{\lambda})^2 + \|\omega(\hat{\lambda}) - x^b\|^2.$$

It follows that:

$$\|\omega(\lambda) - x^a\|^2 - \|\omega(\lambda) - x^b\|^2 = 2\lambda(\hat{\lambda} - \tilde{\lambda}) + \text{constant}.$$

If the $\omega(\tilde{\lambda}) = \omega(\hat{\lambda})$, then both $[x^1, \dots, x^k]$ and $[x^{k+1}, \dots, x^A]$ are included in the same hyperplane orthogonal to L, contradicting the hypothesis that $[x^1, \dots, x^A]$ is the whole space. Thus, $\hat{\lambda} \neq \lambda$ by taking λ large enough and with the sign of $\hat{\lambda} - \tilde{\lambda}$, the above difference will be positive, so that the ideal point $\omega(\lambda)$ will assure that alternatives $b > k$ are preferred to alternatives $a \leq k$.

Finally, if R_i is the complete indifference, the center of the sphere that contains all the points x^a can serve as the ideal point. □

2.3. Strict preferences

The previous proofs relied on indifferences in preferences. If we restrict our attention to strict preferences, things are different. Consider the case $d = 1$. Proposition 7 implies that there exists a profile of (non strict) preferences with two individuals and three alternatives, which is not Euclidean in one dimension. But, looking at all the possible cases, it is not difficult to check that any profile of strict preferences with two individuals and three alternatives is Euclidean in one dimension.

We know that a profile is Euclidean if $d \geq \min\{I, A-1\}$. For instance a profile of strict preferences with $I = 4$ individuals and $A = 4$ alternatives can always be represented in $d = 3$

dimensions, but it is not clear wether two dimensions are enough. An example will show that $d = 2$ is indeed not enough for four individuals and four alternatives. Notice that this leaves open the question "Is any profile of strict preferences with three individuals and four alternatives Euclidean of dimension 2?" The question for larger d is also left open.

Note that if a strict preference order of an agent i can be represented as a directional preference in some direction v^i, then it also can be represented as a spheric preference in the same direction by choosing the location ω^i for the agent i far enough in the direction v^i. We thus can exclude directional preferences, and only check whether it is possible to represent strict preference profiles by spheric preferences.

We will build on the following example, which exhibits a particular cyclic pattern.

Example 12. There are d alternatives $a_1, \ldots, a_d$, and d agents $1, \ldots, d$ with preferences

$$\begin{aligned}
1: &\quad a_1 P a_2 P a_3 P \ldots P a_{k-1} P a_k \\
2: &\quad a_2 P a_3 P \ldots P a_{k-1} P a_k P a_1 \\
3: &\quad a_3 P a_4 P \ldots P a_k P a_1 P a_2 \\
&\vdots \\
k: &\quad a_k P a_1 P a_2 P \ldots P a_{k-2} P a_{k-1}.
\end{aligned}$$

Proposition 13. *For all strict profiles to be Euclidean it is necessary that* $d \geq \min\{I - 1, A - 1\}$.

Proof. For any d, consider the profile of Example 12. It is enough to check that one cannot find d locations $x^1, \ldots, x^d$ for the alternatives and d locations $\omega^1, \ldots, \omega^d$ for the agents in $(d-2)$-dimensional Euclidean space, such that $||x^{j_1} - \omega^i|| < ||x^{j_2} - \omega^i||$ if and only if $a_{j_1} P_i a_{j_2}$.

Assume to the contrary that such locations can be found. Since preferences are strict, all points $x^1, \ldots, x^d$ must be all different.

First, note that any d points in $(d-2)$-dimensional Euclidean space are affinely dependent, i.e., there exist real numbers $\alpha_1, \ldots, \alpha_k$ such that $\sum_{i=1}^{d} \alpha_i x^i = \bar{0}$ and $\sum_{i=1}^{d} \alpha_i = 0$. We can rewrite this condition in the following way. Leave the members with $\alpha > 0$ on the left side of each of the two equations, and put the members with $\alpha \leq 0$ on the right side. Then rename variables, by calling $y^1, \ldots, y^n$ locations with $\alpha > 0$, and $z^1, \ldots, z^m$ locations with $\alpha > 0$ $(m + n = d)$; also rename positive α-s into β-s, and nonpositive α-s into $(-\gamma)$-s (thus, γ-s are nonnegative). We thus obtain for d points $y^1, \ldots, y^n, z^1, \ldots, z^m$ representing our d alternatives:

$$\sum_{i=1}^{n} \beta_i y^i = \sum_{j=1}^{m} \gamma_j z^j, \quad \text{where } \sum_{i=1}^{n} \beta_i = \sum_{j=1}^{m} \gamma_j,\ \beta_i > 0, \gamma_j \geq 0.$$

Next, an individual with Euclidean preferences, located at point ω, prefers b located at y to c located at z, if and only if $||\omega - y\|^2 < \|\omega - z\|^2$, i.e., if and only if $y \cdot y - 2\omega \cdot y < z \cdot z - 2\omega \cdot z$.

Think now about our alternatives as points $x^1, \ldots, x^d$, located on the circle (in that precise order clockwise), each also marked as y^i or z^j, and with an attached weight β_i or γ_j.

Start from some $y^{i_1} = x^t$ and go clockwise summing up separately all weights β_i, and separately all weights γ_j, until first sum becomes smaller then the second. That is, we start from $\sum_\beta = \beta_{i_1} \geq \sum_\gamma = 0$. If next alternative clockwise on the circle, x^{t+1}, is a y-alternative y^{i_1+1}, then $\sum_\beta = \beta_{i_1} + \beta_{i_1+1} \geq \sum_\gamma = 0$, and we continue. If next alternative x^{t+1} is a z-alternative z^{j_1}, then we continue if $\sum_\beta = \beta_{i_1} \geq \sum_\gamma = \gamma_{j_1}$, and stop if $\sum_\beta = \beta_{i_1} < \sum_\gamma = \gamma_{j_1}$. In general, we stop when for the first time we obtain $\sum_\beta = \beta_{i_1} + \beta_{i_1+1} + \beta_{i_1+2} + \cdots < \sum_\gamma = \gamma_{j_1} + \gamma_{j_1+1} + \gamma_{j_1+2} + \cdots$, or, if it never happens, we stop when we make the whole circle and return to the alternative $y^{i_1} = x^t$.

Assume that we were forced to stop before we made the whole circle. Then we attach the sum $\sum_{\beta} = \beta_{i_1} + \beta_{i_1+1} + \beta_{i_1+2} + \cdots$ we got so far to the alternative y^{i_1}, call the first y-alternative, clockwise after we stopped, y^{i_2} (note that we had to stop at some z-alternative), and repeat the same process, etc.

After no more then $n < d$ steps, we will be starting from some y-alternative, from which we already were starting before: assume without loss of generality that when we write down the y-alternatives we were choosing, $y^{i_1}, y^{i_2}, y^{i_3}, \ldots$, the first alternative which repeats itself is y^{i_1} (otherwise just through away first several alternatives), i.e., our sequence is

$$y^{i_1}, y^{i_2}, y^{i_3}, \ldots, y^{i_{q-1}}, y^{i_q}, y^{i_1}, \ldots$$

Consider the first q alternatives in this sequence (i.e., the longest sequence without repetition), $y^{i_1}, y^{i_2}, y^{i_3}, \ldots, y^{i_{q-1}}, y^{i_q}$ together with attached to them sums $\sum_{\beta}$. We remember that for each of these alternatives corresponding $\sum_{\beta} < \sum_{\gamma}$. Thus, the total sum of all their $\sum_{\beta}$ (we call it $\sum_{\beta\beta}$) is strictly smaller then total sum of all corresponding to them $\sum_{\gamma}$ (we call it $\sum_{\gamma\gamma}$).

But in constructing our sequence $y^{i_1}, y^{i_2}, y^{i_3}, \ldots, y^{i_{q-1}}, y^{i_q}$ we were moving clockwise along the circle, and since we stopped just before repeating y^{i_1} we made several (say, Q) whole circles. Our sums $\sum_{\beta}$ were calculated by summing up all coefficients β along the way, while our $\sum_{\gamma}$ were calculated by summing up coefficients γ along the way (probably skipping some γ-s—ones attached to z -alternatives between some stop and the next after it y-alternative). Hence, $\sum_{\beta\beta} = Q\sum_{i=1}^{n}\beta_i = Q\sum_{j=1}^{m}\gamma_j \geq Q\sum_{\gamma\gamma}$, which is a contradiction to $\sum_{\beta\beta} < \sum_{\gamma\gamma}$ we just proved.

It follows that, starting from at least some alternative y^{i_1}, we should be able to make the whole circle keeping $\sum_{\beta} \geq \sum_{\gamma}$ all the way.

Without loss of generality, assume $y^{i_1} = y^1 = x^t$.

Consider the agent t for whom the alternative a_t, located at $y^1 = x^t$, is the best one. Assume that this agent t is located at point ω. We know that she prefers an alternative b, located at y, to an alternative c, located at z, if and only if $y \cdot y - 2\omega \cdot y < z \cdot z - 2\omega \cdot z$.

Given our profile, we know that preferences of this agent t decrease when we go along our circle clockwise (starting from $y^1 = x^t$), and on the way the sum of weights β at y-alternatives is always at least as big as the sum of weights γ at z-alternatives, all weights β, γ being nonnegative. Thus, we obtain that

$$\sum_{i=1}^{n}\beta_i(y^i \cdot y^i - 2\omega \cdot y^i) < \sum_{j=1}^{m}\gamma_j(z^j \cdot z^j - 2\omega \cdot z^j)$$

or, given that $\sum_{i=1}^{n}\beta_i y^i = \sum_{j=1}^{m}\gamma_j z^j$, that $\sum_{i=1}^{n}\beta_i y^i \cdot y^i < \sum_{j=1}^{m}\gamma_j z^j \cdot z^j$.

Now, if we repeat the same circle argument from the beginning, but for z -alternatives, we obtain that $\sum_{i=1}^{n}\beta_i y^i \cdot y^i > \sum_{j=1}^{m}\gamma_j z^j \cdot z^j$, the desired contradiction. □

Proposition 13 tells us that for any strict profile to be representable in d dimensions it has to be true that $d \geq \min\{I-1, A-1\}$, while Propositions 5 and 4 tell that it is enough to have $d \geq \min\{I, A-1\}$. Thus, we know the minimal necessary number of dimensions needed to represent all strict profiles, for all cases with $\min\{I-1, A-1\} = \min\{I, A-1\}$.

Assume that $\min\{I-1, A-1\} \neq \min\{I, A-1\}$. Then $\min\{I-1, A-1\} = I-1 = \min\{I, A-1\} - 1 < A-1$, so this is the case of I agents and $A \geq I+1$ alternatives. For this case, our results give that the smallest necessary number of dimensions d is such that $I-1 \leq d \leq I$.

The next proposition tells us that, for any I, for A large enough it is necessary to use I dimensions. Its proof uses the following profiles, with very large numbers of alternatives.

Example 14. Consider a strict profile with I agents and $A = 2^I$ alternatives labelled a_S, for $S \subset \{1, \ldots, I\}$. Let, for any agent $i \in I$, all a_S such that $i \in S$ be above $a_\emptyset$, while all a_S such that $i \notin S$ be below $a_\emptyset$. Notice that, for such a profile, for any subset $S \subset \{1, \ldots, I\}$ with $S \neq \emptyset$, there is exactly one alternative $a_S \neq a_\emptyset$, such that all agents from S prefer a_S to $a_\emptyset$, while all agents from $\{1, \ldots, I\} \backslash S$ prefer $a_\emptyset$ to a_S.

Proposition 15.

(1) *There exists a strict profile with I agents and $A = 2^I$ alternatives, such that it cannot be represented with $I - 1$ dimensions.*
(2) *All strict profiles with I agents and $A = I + 1$ alternatives can be represented with $I - 1$ dimensions.*

Proof.

(1) Consider a strict profile as described in Example 14. We check that any such profile cannot be represented as Euclidean in $I - 1$ dimensions.
    Assume to the contrary that there is such a representation, and consider the inversion with the center at $a = a_\emptyset$ and ratio 1. Each sphere with the center at the location of an agent, containing $a = a_\emptyset$, transforms in a hyperplane. There are I such hyperplanes, and they divide the $(I - 1)$-dimensional Euclidean space in at most $2^I - 1$ different areas.
    Consider now the images under this inversion of the following 2^I points: the locations of $2^I - 1$ alternatives, namely all alternatives except the alternative $a = a_\emptyset$, and some additional point b which is further than $a = a_\emptyset$ from any agent. All these 2^I images must be in different areas, since for any two of our points there is at least one agent for whom one of these points is closer than $a = a_\emptyset$, while another one is further than $a = a_\emptyset$. This is the desired contradiction.
(2) Fix a strict profile with I agents and $A = I + 1$ alternatives. There is at least one alternative, say a, such that it is not the last in the preferences of any agent. Locate all remaining I alternatives in the vertices of the simplex in $\mathbb{R}^{I-1}$, and locate alternative a in the center of this simplex. It is easy to see that any strict preference order which does not have a as its last alternative can be represented by locating an agent with such order at some point in this $(I - 1)$-dimensional space. □

2.4. Arbitrary convex spatial preferences

The previous results essentially tell that many dimensions are needed to represent preference profiles with Euclidean representation. If we allow for general convex representations then the picture is totally different: A single dimension is not enough but two are.

Theorem 16. *A system of A points $(x^a)_{a=1}^A$ in $\mathbb{R}^2$ allows for the convex spatial representation of all preferences over A alternatives if and only if neither of these points is a convex combination of others* (i.e., *if and only if $(x^a)_{a=1}^A$ are the vertices of some convex polytope*).

Proof. "if": Assume that points $(x^{a_k})_{k=1}^A$ are the vertices of some convex polytope $K = \mathrm{Co}[x^{a_1}, \ldots, x^{a_A}]$. Consider an agent i with preference ordering $a_1 R_i a_2 R_i \ldots R_i a_A$. We will construct the convex preference $\succcurlyeq_i$ on $\mathbb{R}^2$, which respects this ordering. For any $y \in \mathbb{R}^2$ let $U_{\succcurlyeq_i}(y) = \{x \in \mathbb{R}^2 : x \succcurlyeq_i y\}$ be the upper contour set of y under $\succcurlyeq_i$. Choose $U(x^{a_k}) = \mathrm{Co}[x^{a_1}, \ldots, x^{a_k}]$, $k = 1, \ldots, A$. It is easy to see that all $U(x^{a_k})$ are convex, $U(x^{a_{k-1}}) \subset U(x^{a_k})$, $\{x^{a_1}, \ldots, x^{a_k}\} \subset$

$U(x^{a_k})$, and $x^{a_l} \notin U(x^{a_k})$ for any $l > k$. Now, one obviously can define a convex preference $\succcurlyeq_i$ on $\mathbb{R}^2$, so that $U_{\succcurlyeq_i}(x^{a_k}) = U(x^{a_k})$ for $k = 1, \ldots, A$.

"only if": Assume that x^{a_A} is a convex combination of other points: $x^{a_A} \in \mathrm{Co}[x^{a_1}, \ldots, x^{a_{k-1}}]$. Then any preference ordering which ranks a_A last is not representable by convex preferences. Indeed, assume that an agent i has preference ordering $a_1 R_i a_2 R_i \ldots R_i a_A$. For any convex $\succcurlyeq_i$, the upper counter set $U_{\succcurlyeq_i}(x^{a_{k-1}}) \supset \mathrm{Co}[x^{a_1}, \ldots, x^{a_{k-1}}]$, so $x^{a_k} \in U_{\succcurlyeq_i}(x^{a_{k-1}})$, a contradiction to $a_{A-1} R_i a_A$. □

Proposition 17. *If* $A, I \geq 3$, *then there exist preference profiles which are not convex spatial for dimension* 1.

Proof. This is obvious, since in the case of one dimension convex preference means single-peaked preference. Consider a profile where alternative a_i is the (unique) worst for agent i, for $i = 1, 2, 3$. Assume that there are $x^{a_1}, x^{a_2}, x^{a_3} \in \mathbb{R}$, which allow to represent this profile by convex preferences over $x^{a_1}, x^{a_2}, x^{a_3}$. Since we are in one-dimensional space, one of the points $x^{a_1}, x^{a_2}, x^{a_3}$, say x^{a_1}, is located between two others: $x^{a_1} \in \mathrm{Co}[x^{a_2}, x^{a_3}]$. But then agent 1 cannot rank a_1 strictly below a_2 and a_3, a contradiction. □

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