CHAPTER THREE

A Baseline ABM of Party Competition

WE JUST SAW THAT PARTY leaders involved in the complex dynamics of multiparty competition face analytically intractable decision problems. This implies that real party leaders use informal rules of thumb rather than formally provable best response strategies. This *behavioral* assumption about party leaders motivates using agent-based models (ABMs) of decision making in complex dynamic settings. In the rest of this book, we develop and analyze an ABM of multiparty competition in multidimensional policy spaces.

Recent examples of such models include widely cited and seminal work by Kollman, Miller, and Page (1992, 1998, 2003). Building on this, Laver developed an ABM of party competition (Laver 2005). He assumed a two-dimensional policy space and two sets of agents: voters and party leaders. At the outset of the simulation, voters are given random ideal policy points drawn from a normal density function over the two-dimensional policy space. Voters do not vote strategically but choose the party advocating the policy position that is closest to their ideal point. Party leaders choose policy positions in this two-dimensional policy space using simple decision rules.

The complex system described by the Laver (2005) model is summarized in Figure 3.1. Following a random start that is essentially a model artifact, political competition cycles endlessly around the loop at the bottom of Figure 3.1. Voters adapt their party support to the current configuration of party policy positions. Leaders then adapt their party policy positions to the current configuration of voter support levels. Voters then readapt, and this process continues ad infinitum.

Laver specified four different decision rules for party leaders, each distilled from the substantive literature on intraparty decision making:

Sticker: never change position (an "ideological" leader)

Aggregator: set party policy on each dimension at the mean ideal point coordinates of current party supporters (a leader who responds perfectly to supporters' preferences)

Hunter: if the last move increased support, make the same move; else, reverse heading and make a unit move in a heading chosen randomly from the 180° arc centered on the direction now being faced (a leader who is a Pavlovian vote forager)

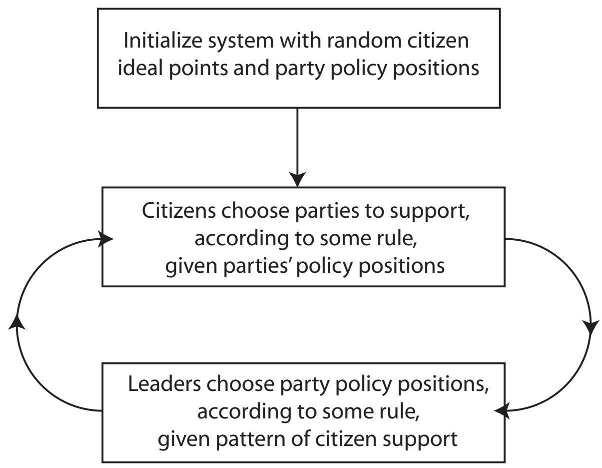

Figure 3.1. Complex dynamics of multiparty competition.

Predator: identify the largest party; if this is you, stand still; else, make a unit move toward the largest party (a leader seeking votes by attacking larger parties)

The big news in Laver's 2005 paper was that the adaptive Hunter rule, despite its simplicity, was good at finding high concentrations of voter support. Furthermore, when several different party leaders use the same vote-seeking Hunter rule, Laver found that *party positions tend to stay away from the dead center of the policy space*, despite the fact that this is the place with the highest voter densities. Another important finding was that party systems in which all party leaders use the Aggregator rule quickly reach a deterministic steady state with no party movement at all. In competition for votes with other rules, however, parties using the Aggregator rule tend to fare poorly, occupying systematically more off-center locations and winning fewer votes.

In the rest of this chapter, we rebuild this ABM of multiparty competition from the bottom up. We go on in later chapters to add features to this baseline model that allow us to explore processes of substantive interest to us, endogenizing the birth and death of political parties, adding new decision rules and rule features, and investigating the effects of candidates' nonpolicy attractions for voters and the impact of having politicians who may care as much about their own policy preferences as they do about winning votes. Before we do any of this, we must be more systematic about how we characterize the preferences of voters.

Characterizing the Preferences of Voters

Spatial models of party competition, as we have seen, are "spatial" in the sense that they describe voters' preferences on a wide range of issues using a small number of policy dimensions. This is substantively plausible because, *as a matter of empirical fact*, real voters' views on many different issues tend to be interrelated. Preferences on each of a substantial "bundle" of issues can be treated *as if* they were all correlated to a single underlying dimension. This "latent" dimension can be used in a parsimonious way to describe the empirical structure of voters' preferences on all issues in the bundle. For example, it is often possible to summarize voters' preferences on abortion, gay rights, stem cell research, capital punishment, and gun control in terms of a single "liberal-conservative" dimension of social policy. Once we know a voter's position on one of these issues, we can make a good prediction of her position on other issues in the same bundle. A set of latent dimensions such as this, together with information or assumptions about the distribution of voters' preferences on these, are fundamental primitives in all spatial models of political competition.

For most of this book we assume voters care only about the policy positions that are on offer at election time. (In chapter 9 we modify that assumption to model voters' evaluations of candidates' nonpolicy attributes.) We assume that each voter's preferences can be characterized by an ideal policy position in some *n*-dimensional policy space and a "loss function" that describes how different policy positions are evaluated, given how far these positions differ from the voter's ideal point.[1] This means that a crucial feature of our model is a description of the distribution of voters' ideal points. We assume that ideal points, whether in the overall voting population *or in well-specified subpopulations*, are normally distributed on policy dimensions of interest. This is in line with common practice. For example, Downsian one-dimensional models of party competition typically assume ideal point distributions that are bell shaped (Downs 1957). The normal distribution is bell shaped and occurs naturally in a huge variety of settings. Another reason for assuming a normal or bell-shaped distribution is that an *n*-dimensional Euclidean real space extends to infinity in all directions, but we typically do not think that some real voters have "infinitely extreme" ideal policy positions. This is an important restriction on possible distributions of voter ideal points. For example, it excludes the possibility that ideal points are *uniformly* or *arbitrarily* distributed, since these distributions imply we are just as likely to find someone with an infinitely extreme ideal point as

[1] In the language of Humphreys and Laver (2010), ours is a "strongly spatial" model—one that makes spatial assumptions about the *cognitive processes of real agents*.

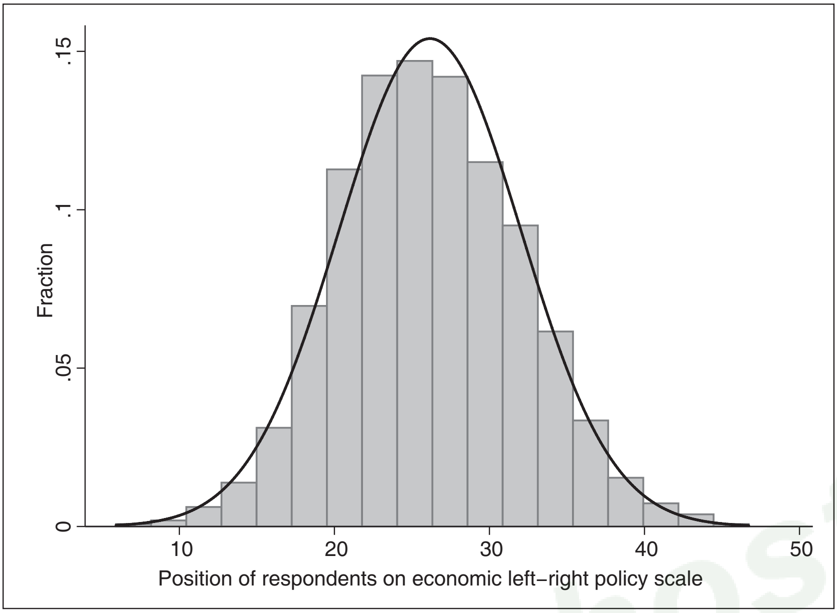

Figure 3.2. Density of Irish voters' attitudes on economic policy in 2002, with normal curve superimposed.

someone closer to the center.[2] Precluding infinitely extreme voters thus implies that densities of voter ideal points asymptotically approach zero in every direction as policy positions become more extreme. This implies distributions that are in this sense "bell shaped," at least at the extremes. We show in the appendix to this chapter that normal distributions of voter ideal points do not have to be imposed as "brute force" assumptions but may evolve endogenously on the basis of standard models of social interaction.

The justification for assuming normal distributions of voters' ideal points is not purely theoretical. It is also based on findings about distributions of policy preferences in real voting populations. Figure 3.2 shows the density of preferences, on a left-right dimension of economic policy, of 2,605 Irish survey respondents in 2002.[3] A theoretical normal curve is su-

[2] We return shortly to define the origin or "center" of a policy space.

[3] This scale was built, using data from the Irish National Election Study (INES), by aggregating each respondent's answers on seven different questions that probed attitudes on economic policy. These included agreement or disagreement with statements such as "income tax should be increased for people on higher than average incomes" and "business

perimposed and matches the empirically observed distribution of public opinion very closely. We can plot similar bell-shaped distributions of Irish survey respondents' positions on two other policy scales—one measuring liberal versus conservative attitudes on social policy, the other measuring attitudes on environmental policy. In each case, the empirically observed distribution of survey respondents' preferences closely resembles a normal curve.[4] More generally, when we measure the policy positions of populations of real humans on particular policy dimensions, we often encounter ideal point distributions that are essentially bell shaped. In short, the assumption of normal distributions of voters' ideal points is, for both theoretical and empirical reasons, nothing like as "special" as it might seem at first sight; normal distributions of public opinion can, and do, evolve in a range of plausible ways.

As we noted above, the models in this book are implemented for two-dimensional policy spaces. We assume, when specifying distributions of voter ideal points in particular populations or subpopulations, that the spatial distribution of ideal points has a bivariate normal density. For the sake of simplicity, we assume that standard deviations of ideal point locations are the same on both dimensions, with ideal points uncorrelated between dimensions; policy dimensions are taken to be "orthogonal" in this sense. This is a standard assumption in most implementations of the classical spatial model, giving us an assumed distribution of voter ideal points that is perfectly symmetric about its center.

A Coordinate System for Describing Ideal Points and Policy Positions

Any particular coordinate system for describing positions in a real Euclidean space is essentially arbitrary. A particular realization of any spatial model, whether computational or empirical, nonetheless requires some scale for specifying policy positions. In what follows, we use the distribution of ideal points as the basis of such a scale, describing party policy positions in terms of standard deviations, σ, of this distribution. Without loss of generality, we set the mean of the ideal point distribution at zero and the standard deviation at one. We thereby define the origin of the coordinate system of our policy space as the centroid of the set of population ideal points. We describe a policy position as being at (x, y) if it is at the position of an ideal point that is x standard deviations (of the distribution of population ideal points) from the population centroid

and industry should be strictly regulated by the State." A full description of this scale can be found in Appendix E3.

[4] See Appendix E3.

on the x dimension and y standard deviations from this on the y dimension. Any other coordinate system for describing party positions can be transformed into our system, but ours has the advantage of describing party positions explicitly in terms of the distribution of voters' ideal points. *Every party policy position can thereby be described in terms of the ideal point of some actual or possible voter*. Policy "positions" have no absolute meaning and must be described in relation to something; the distribution of voter ideal points is a natural benchmark.

Asymmetric Ideal Point Distributions Arising from More Than One Subpopulation

While we find bell-shaped distributions of public opinion in many different settings this is not always the case empirically, and it is certainly not the general case. An important issue for any decision maker, furthermore, is the possibility of getting stuck at some local maximum. In seminal work on ABMs of party competition by Kollman, Miller, and Page, some party leaders use "hill-climbing" rules, for example, polling voters in the local environment of their ideal point and moving in the policy direction that yields the greatest local improvement in voter support (Kollman et al. 1992, 1998).[5] Having climbed to the top of some local voter density hill, however, a party leader using a hill-climbing rule is unlikely to climb down the other side. This is true even when we know, from an all-knowing external vantage point, that this local hill is but a tiny bump on the population landscape. This decision rule may therefore perform poorly at maximizing party support. The relative effectiveness of different decision rules may be very different when spatial distributions of ideal points are asymmetric and multimodal.

For this reason, we generalize our description of voter preferences and assume the overall voter population to be the sum of a set of well-defined subpopulations. Think substantively of these subpopulations as, for example, ethnic, religious, economic, linguistic, or regional groups. We now modify our description of the distribution of voter ideal points to allow for populations that are aggregations of two subpopulations. While this is far from the most general possible case, it does generalize our description of the voting population in two important ways. First, it accommodates the possibility of *local maxima* in ideal point distributions. Second, generalizing the unrealistic assumption that the overall distribution of voters' ideal points is perfectly symmetrical about the origin, ideal point

[5] The catch-22 of hill climbing in the context of multiparty competition, of course, is how parties would know which voters to poll, as being in their ideological neighborhood, without first polling them. We do not torment ourselves with this conundrum here.

densities are typically *asymmetrical* about any given point in the policy space when the overall population is an aggregation of two distinct subpopulations. As we shall see, such asymmetries are important features of the environment for party competition and make a substantial difference to the outcomes we predict.

Even confining attention to two subpopulations, each with a bivariate normal distribution of members' ideal points, there may be an infinite number of possible ideal point distributions. There are three basic ways in which two distinct normally distributed subpopulations can differ: they can differ in their relative sizes, the means of their ideal point distributions, and the variances of these distributions. Since we confine ourselves to pairs of subpopulations with ideal points distributed in a Euclidean policy space, matters are simplified by the fact that all policy distances are invariant to rotations of the space. This means that we can, without loss of generality, take the line joining the centroids of two subpopulation ideal point distributions and rotate this to define it as the x-axis. We interpret this substantively as the *main axis of policy disagreement* between the two subpopulations of voters—the main axis of political competition in the political system under investigation.

In what follows, therefore, we consider two subpopulations with ideal point centroids that are separated on this axis of disagreement, which we specify as the x-axis. In addition, we set subpopulation ideal point variances equal. This enables research designs that explore the political effects of having increasingly "polarized" subpopulations, with ideal point centroids that are increasingly far apart on the x-axis. We do this, in the computational work we specify in chapters 5 and 6 below, by investigating the effects of a range of parameter settings for aggregated voter subpopulations. Using our baseline symmetrical bivariate normal distribution with means (0, 0) and standard deviations (1, 1) as a point of reference, we set standard deviations of each of the two subpopulations, l and r, at (0.5, 0.5). We investigate subpopulations with relative sizes, n_l/n_r, that are sampled from a uniform distribution on the interval [2.0, 1.0] and with ideal point means, $\mu_r = -\mu_l$, sampled from a uniform distribution on the interval [0.0, 1.5].

Table 3.1 describes aggregate populations generated by examples of two distinct subpopulation ideal point distributions parameterized within these ranges. The top three rows describe ideal point distributions arising from the aggregation of two equal-sized subpopulations ($n_l/n_r = 1$); the lower three describe aggregate populations in which the left subpopulation is twice the size of the right ($n_l/n_r = 2$). If these distributions are plotted,[6] it transpires that, with relatively unpolarized subpopulations

[6] These plots can be found in Table E3.4 in the electronic appendix.

Table 3.1
Summary Measures for Ideal Point Densities in Six Sample Populations Aggregated from Two Subpopulations ($\sigma_l = \sigma_r = 0.50$)

n_1/n_2	μ_r ($= -\mu_l$)	$\mu_{\text{x-pop}}$	*Median* $_{x\text{-}pop}$	$\sigma_{\text{x-pop}}$
1.0	0.5	0.0	0.0	0.707
1.0	1.0	0.0	0.0	1.118
1.0	1.5	0.0	0.0	1.582
2.0	0.5	– 0.166	– 0.218	0.687
2.0	1.0	– 0.334	– 0.662	1.067
2.0	1.5	– 0.500	– 1.162	1.500

($\mu_r = 0.5$), the aggregate population has a *unimodal* distribution of ideal points, though this distribution is asymmetrical, being "stretched" along the x-axis of primary policy disagreement. In sharp contrast, for the most polarized pairs of subpopulations we investigate ($\mu_r = 1.5$), subpopulation ideal point centroids are now so far apart that there is almost no overlap between subpopulations. Plots of aggregate population ideal point densities are, on these parameterizations, in effect plots of two distinct subpopulations. Intermediate examples ($\mu_r = 1.0$) generate two distinct but overlapping subpopulations that combine to form bimodal distributions for the population as a whole.

When we randomly sample parameters for the distribution of voter ideal points in this way, we not only generalize the model to comprehend asymmetric and potentially multimodal distributions but also introduce a very important new challenge for party decision rules. These now encounter many different ideal point distributions with unknown parameter settings rather than one single distribution to which they can adapt. Robust decision rules will perform well in the range of different environments we now specify. In contrast, more "brittle" rules may perform better than robust rules in certain specific settings, but not in any possible setting that might be encountered. This is perhaps the most important reason to model party competition using a large variety of different asymmetric distributions of voter ideal points.

Using asymmetric distributions of voter ideal points that aggregate two distinct subpopulations has the *theoretical* advantage of specifying more general ideal point distributions that are nonetheless well parameterized. It has the great *substantive* advantage that, anticipating empirical work we report in chapter 11, we find it relatively easy to calibrate observed party support shares to observed party policy positions in real party systems using parameterizations of the two-subpopulation model of voter ideal points that we specify above. While this does not mean that our two-subpopulation model of voter ideal point densities is "correct" in some sense, it does at least mean that it is consistent with what we observe.

Smooth *Densities of Voter Ideal Points*

Rigorous and exhaustive computational interrogation of dynamic models requires that these models be *parsimonious*, with as few parameters as is consistent with substantive realism. We therefore make an important modification to the Laver (2005) ABM and its extensions,[7] which treat each *voter* as an independent decision-making agent. Reported simulations typically involve one thousand voters; each voter in each simulation is given an ideal point randomly drawn from the symmetric bivariate normal density distribution specified above. Different repetitions of the same model run, even with identical parameter settings, generate different results because each repetition uses a different random draw of ideal points.[8]Given the stochastic components in the model, using many different random draws of ideal points for every vector of parameter settings means we must repeat simulation runs many times, *for each random draw*, if we want to ensure that run results are not a product of some particular random ideal point draw. This increases our computational budget by an order of magnitude.

We address this by replacing a *finite* population of voters, with ideal points drawn randomly from an underlying density function, with an effectively infinite population of voters characterized by the underlying density function itself. Since party support shares no longer depend on details of a discrete random draw of ideal points, the same configuration of party policy positions always generates the same configuration of party support in a given type of population. Our approach thus characterizes the population as a density function, not as a set of autonomous agents. This is directly analogous to the "electoral landscapes" used by Kollman, Miller, and Page and by de Marchi (Kollman et al. 1998; De Marchi 1999).

[7] Laver and Schilperoord (2007) extend the model to include endogenous political parties, while Fowler and Laver (2008) add a more rigorous examination of competition between different decision rules.

[8] Another way of looking at this is to see the seed of the pseudo-random number generator used for each run as a model parameter.

Voters' Evaluations of Different Policy Positions

Metrics for Measuring Policy Distances

Given a "common" policy space that can be used to describe voter ideal points, party policy positions, and the relationships between these, we must now specify how we assume voters to evaluate different policy positions when deciding which of these they prefer. If voters care only about policy, then the "policy distances" between their ideal points and the party policy positions on offer are crucial quantities in any spatial model. We are accustomed to living our daily lives in local regions of natural spaces that we treat as Euclidean, thinking of a straight line as the shortest distance between two points, expressing our ideas on Euclidean planes such as sheets of paper or computer monitors. It is hardly surprising, therefore, that human theorists tend to have in mind, albeit often implicitly, the notion that real people use a Euclidean metric when they measure *cognitive* distances in a *policy*, as opposed to a *physical*, space.

The Euclidean metric defines the distance between two points in a two-dimensional space using the well-known Pythagorean formula.[9] However, there is no good reason, theoretical or empirical, to believe this is an accurate description of how real people think about policy, given the infinite number of possible metrics (including the plausible "infinity metric")[10] that they could use when measuring policy distances. One cognitive metric that has been subject to empirical investigation over a long period by psychologists, and has explicit behavioral justification in certain real settings, is the "city block" or "Manhattan" metric (Attneave 1950; Shepard 1991; Gärdenfors 2000). The cognitive justification for using a city block metric in the context of political competition is that, when considering two voters with ideal points on two unrelated policy dimensions, one voter's perception of her difference from the other might, *as an empirical generalization about how real people actually think about politics*, be better described as the sum of their policy differences on each dimension,[11] not the square root of the sum of the squares of these differences. Humphreys and Laver show that it can make a big difference to

[9]
$$d_2\big((x_1, x_2), (y_1, y_2)\big) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$$

[10]
$$d_\infty\big((x_1, x_2), (y_1, y_2)\big) = \sqrt[\infty]{|x_1 - x_2|^\infty + |y_1 - y_2|^\infty}$$

According to the infinity metric, the distance between two policy positions is their distance on the dimension on which they most differ.

[11]
$$d_1\big((x_1, x_2), (y_1, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$

key theoretical results to assume voters perceive policy distances in terms of the city block metric, as opposed to making the orthodox Euclidean assumption (Humphreys and Laver 2010). Moreover, Eguia argues axiomatically that assuming Euclidean as opposed to city block preferences implies making some odd assumptions about how real people think about risk (Eguia 2009).

Having switched from formal analysis to computation, we are not constrained to make behavioral assumptions simply because these generate analytically tractable models. Liberation from the need to choose unrealistic but analytically tractable assumptions is, indeed, a signal virtue of the computational approach. For the purposes of the present book, however, we maintain theoretical continuity with previous scholars and use a Euclidean metric for measuring policy distances. We do not want to distract attention from our key results by raising the possibility these are driven by what would currently be the unorthodox choice of a city block metric to measure policy distances.[12]

Loss Functions for Measuring Individual Voter Utility

Having considered how real people *perceive* the "distance" between two different policy positions, we turn now to the closely related matter of how they *feel* about this perceived distance. When voters choose to evaluate the positions of two political parties competing for their support, for example, we could assume they favor the closest party, in the sense we just defined this, and that their preferences are single peaked. This amounts to a fairly weak assumption about how real people think about politics—that each voter has some "ideal" policy position and that the utility she anticipates from some putative policy outcome is monotonically decreasing in the distance between this outcome and the ideal point. If we model voters' decisions as being based solely on policy, there is often no need to be more specific about which precise function describes how voters' utility declines as policy distance increases—we need to know just that the closer alternative is seen as better. There are, however, circumstances in which we do need to be more precise. We may want, for example, to model decision making that combines evaluations of policy positions with other, nonpolicy, elements in the value of some particular

[12] It is also important not to underestimate the fundamental reconsideration of party competition that is implied by shifting assumptions to the city block metric. For example, moving "toward" some other position takes on a whole new meaning in a city block space. There are arguments in the philosophy of geometry, furthermore, that real humans find it difficult to visualize non-Euclidean spaces. This is an issue for *human analysts* who plot and interpret spatial maps of policy positions, though it is never argued that *real voters* actually visualize policy distances when making their decisions.

outcome or to model how voters choose between risky alternatives. There are infinitely many mathematical functions that decrease monotonically in the distance between two points—"loss functions" in common scholarly usage. The assumption of a particular loss function is, once again, *a behavioral assumption about how real people think*. Once again, this choice is in practice often motivated more by analytical convenience than behavioral realism.

The most common assumption we find, in a wide variety of published work on many different matters, is of *quadratic* policy loss (Adams et al. 2005). On this assumption, voter utility declines as the square of the distance between the policy position being evaluated and the voter's ideal point. The assumption of quadratic loss is common whether the model is analytical (Schofield and Sened 2006; Ansolabehere and Snyder 2000; Groseclose 2001; Adams et al. 2005; Adams 2001; Hinich and Munger 1994) or computational (De Marchi 2003; Jackson 2003). As Adams, Merrill, and Grofman point out (Adams et al. 2005: 17) an alternative plausible assumption about voter utility involves *linear* policy loss. Setting analytical tractability aside, there is at least one *substantive* argument for quadratic loss, based on the assumption that real voters tend to be risk averse. Faced with a choice between the certainty of a middling outcome *b* and a fair lottery between a good outcome *a* and a bad outcome *c*, the empirical claim is that most people are risk averse in the sense they prefer the certain alternative to the fair lottery, despite the fact that both appear on the face of things to have the same expected payoff. For voters who are risk averse, bearing risk has a cost in itself. If voters are assumed to be risk neutral *about policy choices*, then this implies a linear loss function. If they are assumed to be risk averse, and empirical research on both referendum dynamics and "economic" voting seems to imply this assumption may be behaviorally realistic, this implies a "concave" loss function, of which simple quadratic loss is a tractable example. Given what we take to be a consensus that real voters tend to be risk averse when they make important political choices, as opposed to being risk neutral or risk seeking, we assume a quadratic loss function when defining the utility $U(i, j)$ of a policy at j for a voter with an ideal point at i:

$$U(i, j) = -d(i, j)^2 \qquad (3.1)$$

Decision Rules for Voters

We now turn to the decisions made by voters when choosing which party to support. This is an important matter on which, given fundamental intractability of dynamic multiparty competition, we depart substantively from many classical analytical models of party competition. In these classical models, especially those that assume a small number of parties and voters' preferences well described using a single policy dimension, it is natural to consider the possibility that voters might make *strategic* calculations. Such calculations could imply supporting not the party with the policy position closest to the voter's ideal point but the party for which an added vote would maximize the voter's expected utility—given its policy position, given the policy position of all other parties, given a forecast of the behavior of all other voters, and given a forecast of the eventual policy outcome, conditional on different election results. These severe conditions show us why the analysis of strategic voting is most developed with reference to party systems with first-past-the-post elections, a small number of parties, and a single salient policy dimension. In such settings, the assumed rules of the game are that the party with more votes than any other "wins" the election and then instantly implements all of its policy promises. Given these assumptions and some estimate of the ideal positions of all other voters on the one salient policy dimension, it becomes feasible to consider the possibility of strategic voting. This remains far from straightforward if all voters are assumed to make the same strategic calculations, possibly dissembling about their "true" preferences as part of this process, but it is a feasible analytical objective. However we have seen that, when we add more parties and more salient policy dimensions and consider the dynamics of electoral competition, the resulting models of dynamic party competition are analytically intractable. Crucially, they are intractable for real voters since it is not possible formally to "solve" for party policy positions and consequent election results, conditional on a profile of voter preferences. This makes strategic voting, in the classical sense, infeasible.

The problem of designing an effective strategic voting rule runs much deeper than analytical intractability of dynamic multiparty competition. The norm in multiparty systems with proportional representation, the typical case, is for no party to win a legislative majority. This means that coalitions of parties, at either the legislative or executive level, are needed to sustain a government in office and implement any policy position. Axiomatically, more than one government may emerge from a given election result in which no single party wins a majority. This in turn means that every strategic voter needs to anticipate the government formation process before she can forecast how an election result maps onto a government and hence a policy outcome (Kedar 2009). This process is in itself quite complex, and has been the subject of considerable theorizing and empirical research (Laver and Schofield 1998; Laver and Shepsle 1996). Any human voter considering strategic voting must thus append a (inevitably complicated) model of government formation, not to mention a model of downstream policy implementation by coalition cabinets, to an already intractable model of dynamic multiparty competition. This seems to us to stretch the notion of "rational" strategic voting in national elections so far beyond the limits of credibility—in the context of dynamic multiparty competition—that it implies preposterous assumptions about how real voters actually think. For all of these reasons we assume, in the tradition of the classical Downsian model, that voters use a nonstrategic "proximity" voting rule when deciding which party to support.

There is more to the assumption of proximity voting than complexity of any strategic decision facing voters. Another important argument treats voting as a form of "expressive" behavior. Expressive voters *get more utility from the very act of voting for things they like* than from voting for things they do not like, regardless of the downstream strategic implications of this (Brennan and Hamlin 2000; Brennan and Lomasky 1993; Schuessler 2000). The argument for expressive voting is that rational voters in large populations know with near certainty that their individual votes make no difference whatsoever to the eventual result. Strategically instrumental voters in large electorates will thus typically not vote at all if there is any cost whatsoever associated with voting. On this argument voters who vote, and we observe that many voters who we think of as rational do indeed vote, must do so *for reasons that are not strategically instrumental*. The expressive voter, in these terms, derives *intrinsic* satisfaction from supporting particular political parties. Looked at in this way, a Downsian "proximity" voter gets most satisfaction from supporting the party policy position closest to her ideal point and feels increasing pain if she votes for parties whose positions are further away from this.[13]

Our baseline decision rule for voters is thus that they support the party with the closest policy position—the party whose position generates the Voronoi region in which their ideal point is located.

Characterizing the Preferences of Party Leaders

Having characterized the preferences of voters and specified a decision rule for voting in light of these, we turn our attention to party leaders.

[13] We have come across no theoretical account of "expressive" strategic voting, whereby the voter derives pleasure from the very act of making complex strategic calculations, over and above *or even despite* the downstream consequences of strategic voting. In our experience, however, there are economists and political scientists who actually think like this. Thus the core distinction is between voters who get some value out of voting as an act of self-expression and those who see voting as an instrumental way to change the odds of receiving some downstream benefit.

We first draw attention to an important question we beg by defining the problem in terms of "party leaders." In the spirit of both game theory, strictly applied, and agent-based modeling, we do not treat any self-evidently collective entity such as a "political party" in anthropomorphic terms, as if this were a discrete agent with a single collective brain. Our prime concern is with decision makers, and we thus focus our attention on party leaders. In effect we assume *either* that decisions for each party we model are taken by a single leader *or* that there is some leadership cohort of individuals with identical preferences who can be modeled *as if* they were a single leader. This sets aside the interesting and important matter of *intraparty* politics between politicians with conflicting preferences. We relish the prospect of returning to this in future work.

Earlier, when we specified the decision rules used in the Laver (2005) model and its extensions, we conflated two distinct things. The first concerns the *preferences* of party leaders; the second concerns the *decision rules* these leaders use when trying to realize these preferences. Stickers were presented as ideological party leaders who care only about their party policy position and not at all about their party's vote share. Aggregators were presented as "democratic" party leaders who care only about representing the preferences of current party supporters and not at all about either vote share or any particular policy position. Hunters and Predators were presented as party leaders who care only about vote share. The distinction between Hunters and Predators is important in this regard. Both types of party leaders care only about party vote share, but they use different decision rules when trying to increase this.

We now clarify the distinction between the preferences of the party leader and the decision rule she uses. Consider the vote share of party p (labeled v_p) and three policy positions of possible interest to the leader of party p. These are the ideal point of the party leader (labeled i_p), the current party position (labeled j_p), and the optimal representation of the ideal points of party supporters (labeled k_p). Having assumed a Euclidean distance metric for agents' perceptions of policy differences, we specify k_p as the centroid of the ideal points of current party supporters. A substantive interpretation of the rationale for the Sticker rule is that a leader who is a Sticker cares only about i_p, without regard for v_p or k_p and uses the rule of thumb of setting j_p at i_p. A leader who uses a Hunter or Predator rule may care only about v_p; she uses some rule to set j_p, seeking to maximize v_p without regard for i_p or k_p. A leader using an Aggregator rule may care only about moving to k_p, without regard for v_p or i_p. In light of this we specify a general utility function for party leaders,

$$U(v_p, i_p, j_p, k_p) = (1 - \varphi_1 - \varphi_2) \cdot v_p - \varphi_1 \cdot d(i_p, j_p)^2 - \varphi_2 \cdot d(j_p, k_p)^2 \qquad (3.2)$$

where φ_1 and φ_2 are trade-off parameters specifying the relative contributions of the party leader's ideal point, and the centroid of supporter ideal points, to the utility of the party leader, and $\varphi_1 + \varphi_2 \leq 1$.[14] One possible implication of this utility function is that party leaders using the Sticker rule do so because, for them, $\varphi_1 = 1$ and they are concerned only with having party policy at their ideal points. They achieve this by setting $j_p = i_p$. Similarly, party leaders using the Aggregator rule may do so because, for them, $\varphi_2 = 1$ and they are concerned only with setting a party policy that is an optimal representation of current supporters' ideal points. They achieve this by setting $j_p = k_p$. Finally, leaders using the vote-seeking Hunter or Predator rules may do so because, for them, $\varphi_1 = \varphi_2 = 0$; their utility derives solely from their party's vote share. There are many alternative parameterizations of this general party leader utility function, but in what follows we concentrate on these "ideal types" of party leader. We return in chapter 9 to consider party leaders who are also concerned both with their party's vote share and with their own ideal policy position, investigating the behavior of leaders who have utility functions such that $0 < \varphi_1 < 1$.

Decision Rules for Party Leaders

In light of the utility function for party leaders specified above, our baseline model of party competition involves three decision rules, defined and investigated by Laver (2005) and discussed briefly above.[15] These are Sticker, Aggregator, and Hunter.

Sticker

The simplest rule is Sticker: publish a policy position and never change this.[16] As we have just seen, the Sticker rule is an obvious choice for a party leader for whom $\varphi_1 = 1$, who maximizes utility when party policy

[14] Note that, while the specified utility function for party leaders is conceptually accurate, any realization of this in a model generating numerical values for $U(\cdot)$ requires an additional "scale" parameter, δ, applied to v_p . This is in effect an exchange rate that gives a common *numeràire* allowing expected utility derived from policy loss to be compared with expected utility deriving from vote share. This is not a model artifact but a real substantive matter, since these sources of utility are denominated in different "currencies"—policy distances and votes. At no point in this book will we make interpersonal comparisons of politicians' utility that, for obvious reasons given the above, and more generally, are deeply problematic.

[15] We return in chapter 6 to consider the Predator rule, which we redefine in the light of pathological behavior by agents using the rule as defined in Laver (2005).

[16] NetLogo code to the Sticker rule is thus:

```
to stick
end
```

is set at her ideal point. On March 4, 2008, for example, in a speech announcing his withdrawal from the U.S. presidential primary, Mike Huckabee said: "[W]e've kept the faith. And that for me has been the most important goal of all. I'd rather lose an election than lose the principles that got me into politics in the first place." Casual observation suggests that Huckabee is not unique in being doggedly unwilling to adapt stated policy positions for reasons of political expediency, and thus that we do indeed observe Stickers in the real political world. Over and above this substantive justification, Sticker is a theoretically interesting component of any portfolio of decision rules under investigation in a dynamic setting precisely because it is fundamentally static. The performance of any dynamically responsive decision rule can be measured relative to the performance of a static Sticker benchmark. Dynamically responsive decision rules should, if they are to be seen as effective, perform at least as well as the static Sticker rule.

Aggregator

The Aggregator rule sets party policy at the centroid of current supporters' ideal points.[17] This is a heuristic that is suitable for use, as we have seen, by a party leader who is concerned above all else to represent the views of current party supporters—for whom $\varphi_2 = 1$. We can also think of such a leader as a "benevolent social planner." This does not require knowledge of any particular citizen's ideal point but in effect assumes the party leader has some internal system (a general meeting of supporters, or an unbiased representative sample of these, or some other internal polling process) that allows her to estimate the aggregated preferences of party supporters. Alternatively, we could assume party rules to mandate that party policy positions are set at the centroid of current supporter ideal points by some unmodeled but binding "democratic" internal procedure. Empirically, furthermore, we do find real political parties (many Green parties, for example) with internal procedures designed to represent the wishes of supporters in party policy platforms.

[17] Note that, when all points in a set are weighted equally, the centroid of the set is the average of these points. NetLogo code for the Aggregator rule is thus:

```
to aggregate
    let xbar (sum [votes * pxcor] of patches with [closest-party = myself] / mysize)
    let ybar (sum [votes * pycor] of patches with [closest-party = myself] / mysize)
    setxy xbar ybar
end
```

Hunter

Most traditional spatial models of party competition assume that party leaders are vote seekers (for whom $\varphi_1 = \varphi_2 = 0$). Hunter is a decision rule based on Pavlovian learning that is designed for vote seeking in the analytically intractable context in which we are interested. If a Hunter's move at time t-1 was rewarded by an increase in vote share, then it makes a unit move at time t in the same direction as the move at t-1.[18] If not, a Hunter reverses direction and makes a unit move on a heading randomly selected within the half-space now being faced.[19] In a nutshell, a party leader using a Hunter rule keeps moving party policy in the same policy direction as long as party vote share keeps increasing; otherwise she makes a random policy move in the opposite direction. Hunter is a straightforward "win-stay, lose-shift" algorithm of the type investigated by scholars interested in adaptive learning (Nowak and Sigmund 1993). Substantively, Hunter encodes the behavior of a party leader who relentlessly forages in the policy space, always searching for more votes and never being satisfied—changing policy in the same direction as long as this is rewarded with more votes, but casting around for a new policy direction when the previous policy move was punished with falling or static support.

Characterizing Key Outcomes of Political Competition

There are many "outcomes" of any evolving system of multiparty competition, most of which are difficult to characterize in a systematic way. At any given time, for example, there is a spatial "constellation" of party policy positions and sizes. As with the constellations we see in the night sky, we can try to describe these, but the shapes we see may be in the eye of the beholder and systematic characterization is problematic. Nonetheless, the whole point of modeling party competition is to generate outputs that can indeed be described and analyzed in a systematic way. In what

[18] For stylistic reasons we often refer in what follows to, for example, a "Hunter Party" or a "Hunter," by which we mean a "party with a party leader who uses the Hunter rule."

[19] We specify this unit move, given our coordinate system, as 0.1 standard deviations of the baseline distribution of voter ideal points. NetLogo code for the Hunter rule is thus:

```
to hunt
    ifelse (mysize > old-size) [jump 1]
    [set heading heading + 90 + random-float 180 jump 1]
    set old-size mysize
end
```

follows, therefore, we specify four types of metric that characterize substantively important outputs of multiparty competition. These deal with the number of surviving political parties, the typical policy positions of surviving parties, the extent to which the current configuration of party positions represents the preferences of voters, and the number of different decision rules in use by surviving party leaders.

Effective Number of Parties

It is easy to count *N*, the absolute number of surviving political parties. This is clearly an important output of multiparty competition, especially if we endogenize the birth and death of parties. However, *N*-party systems can differ in important ways. For example the parties may have roughly equal levels of support, or support may be concentrated on a single political party. An index of the "effective" number of parties, *ENP*, defined by Laakso and Taagepera, compares the extent to which party support is concentrated on a small subset of parties in a system, or is distributed evenly between them (Laakso and Taagepera 1979). This index is substantively intuitive because, when all parties are of perfectly equal size in an *N*-party system, then $ENP = N$. When all votes are concentrated on a single party, then $ENP = 1$. As support becomes more evenly distributed among parties, *ENP* approaches *N*.

Eccentricity of Party Policy Positions

It is difficult to be systematic about the "shape" of different party policy configurations. We can however measure the *policy eccentricity* of any given party as the distance of its policy position from the centroid of voter ideal points.[20] This gives us a simple summary measure for any configuration *P* of political parties, as the mean policy eccentricity of the parties (labeled E_p). When interpreting the mean policy eccentricity of any configuration of parties, we need a sense of the eccentricities of voter ideal points underlying this. Figure 3.3 plots eccentricities of one hundred thousand voter ideal points, drawn at random from our baseline standardized bivariate normal distribution. Mean voter ideal point eccentricity is 1.25 and the median is 1.18. Contrary to casual intuition, few voters are at the precise policy centroid, since this is a single point location. Considering voter density *contours* that describe concentric circles of increasing circumference around this centroid, the number of voters with a given eccentricity increases as we move away from the center,

[20] In a Euclidean space such as we use here, we can rotate any configuration of points without changing interpoint distances, so policy eccentricity is invariant to the rotation of party positions. This would not be true, for example, in a city block space.

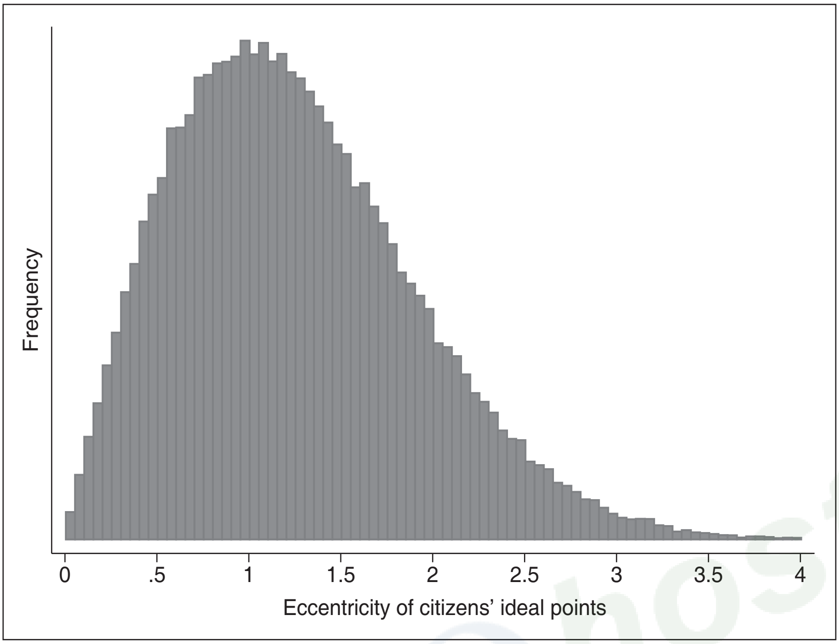

Figure 3.3. Eccentricities of one hundred thousand simulated ideal points drawn randomly from a bivariate normal distribution with mean 0 and standard deviation 1.

peaking at an eccentricity of about 1.00 standard deviations away, after which the density of voters declines as ideal point eccentricity increases. Looking at things another way, median voter eccentricity in this baseline distribution of ideal points tells us that half of all voters have ideal points less than 1.18 standard deviations from the voter centroid, while half have ideal points further away than this from the centroid. This gives a sense of scale to our measure of party policy eccentricity.

"Representativeness" of a Given Party Configuration

It is self-evidently important to measure how well any evolved configuration of party positions represents the policy preferences of voters. There are several ways to think about what we call *representativeness*, and what Golder and Stramski, in their extensive review of this matter, call "congruence" between ideal points and policy outputs (Golder and Stramski 2010). First, there is a "one-to-one" congruence between some summary of voter ideal points (for example, the ideal point of the median voter on some policy dimension) and some summary of policy positions on offer (the position of the eventual government on this dimension, for example). Second there is a "many-to-one" congruence between the full set of voter ideal points and some summary of policy positions on offer. Finally, there is a "many-to-many" congruence between the full set of ideal points and the full set of policy positions arising from party competition (the set of elected legislators' ideal points, for example).[21]

Here, we use a many-to-many measure of the extent to which the configuration of party positions represents the set of voter ideal points. We see each voter ideal point as "represented" at the time of the election by the ideal point of her closest party; we specified how voters feel about this in expression 3.1 above. This gives us a natural measure of representation, R_p, as the mean quadratic distance of all voters from their closest party. If n is the number of voters and $d(i_v, j_{vp})$ is the Euclidean distance between the ideal point of voter v (labeled i_v) and the policy position of her closest party in configuration P (labeled j_{vp}), then:

$$R_p = -\frac{\sum_v d\left(i_v, j_{vp}\right)^2}{n}$$

Higher values of R_p imply a more representative configuration of party policy positions. This measure has a theoretical maximum of zero, in the unlikely event there is a party policy position at the ideal point of every voter. In the general case with more unique voter ideal points than distinct party policy positions, R_p is constrained to be negative.

This measure fits perfectly with the notion of "optimal representation" in Voronoi geometry, which as we have seen encompasses an account of competitive spatial location. Given a set of p population points (voter ideal points) defined in a space and a set of n generating points (party positions), the set of generating points is an optimal n-point representation of the population points if this minimizes the sum of the squared distances between population points and generating points. The allows us to exploit the analytical result that a set of n points is an optimal n-point representation of a p-point population if it generates a centroidal Voronoi tessellation (CVT) of the population points, in which each generating point is at the centroid of its Voronoi region (Du et al. 1999). This tells us that *representativeness of any party system is maximized when party policy positions are configured in a CVT*, a very useful result indeed.

This insight gives us access to another result of substantive and normative significance in the context of dynamic multiparty competition.

[21] Any measure defined to characterize these different types of congruence is subject to the assumptions about cognitive metrics and loss functions that we just discussed.

A widely used computational procedure for *finding* a CVT from an arbitrary starting configuration of generating points is Lloyd's Algorithm (Lloyd 1982). This recursive algorithm is simple: (1) generate a Voronoi tessellation of the space; (2) move each generating point to the centroid of its Voronoi region; (3) go to (1). There are formal proofs in the computer science literature that a CVT can be found using Lloyd's Algorithm for any arbitrary *one-dimensional* space and starting configuration of generating points (Du et al. 1999). In multidimensional spaces, it has been found that Lloyd's Algorithm does converge on a CVT, in finite time and for any arbitrary level of precision. This has been used in a wide variety of heavy-duty computational applications, never failing but never being proved formally (Okabe et al. 2000).[22]

Lloyd's Algorithm is important for students of dynamic multiparty competition because a setting in which all party leaders use the Aggregator rule is, precisely, a party system implementing Lloyd's Algorithm. Consequently, we know that party positions in an all-Aggregator system converge on a CVT of the policy space that maximizes representativeness of the configuration of party policy positions as we have defined this.[23] Intriguingly, this emergent phenomenon arises despite the fact that leaders of individual Aggregator parties never try to maximize overall representativeness of the party system, but simply try to represent the ideal points of their own current supporters.

Effective Number of Decision Rules

In most of the models of party competition we investigate in this book, the set of surviving parties is endogenous and the set of rules used by party leaders is diverse. This means that the diversity of the set of decision rules in use by the leaders of *surviving* parties is an important output of party competition.[24] In some settings, each surviving party leader may use a different decision rule; in others, all leaders may use the same rule. We therefore define a measure of the *effective number of rules* (ENR) that is directly analogous to the effective number of parties (ENP). If the aggregate vote share of all parties with leaders using rule r is v_r, then the ENR is:

[22] We can think of this as a "feral" de facto proof or, in the language of political science, a "folk theorem."

[23] In terms of computational method we also know from Lloyd's Algorithm that an all-Aggregator system reaches steady state, in finite time, for an arbitrary level of precision determined by the floating point precision of the computation. We make use of this helpful result when specifying run designs in chapter 4.

[24] For stylistic reasons we often refer to *rules* in what follows, rather than to parameterized rules or rule-agent pairings.

$$ENR = \frac{1}{\sum_r (v_r)^2}$$

If all party leaders use the same decision rule, then $ENR = 1$; if every leader uses a different parameterized rule, then $ENR = ENP$. Thus, for a given number of parties, the closer ENR is to 1, the closer is some single decision rule to dominating the party system.

Moving Forward

We used this chapter to specify our baseline model of multiparty competition and define a set of measures that characterize evolving dynamic systems of competing political parties. We move on in the next chapter to develop the methodological tools we need to investigate this model in a rigorous way. While this next chapter may seem something of a methodological interlude, we urge readers not to skip ahead to the substantive findings that emerge from our model later in the book. As we have seen, the complexity of our model mandates the use of computational methods. If we are to feel confident in the inferences we draw from our computational work, we must feel confident that these inferences are as rigorous, to all intents and purposes, as the inferences drawn from rigorous classical analysis. Only by giving careful consideration to the methodological matters we discuss in the next chapter can we feel confident that our results are just as solid, both logically and statistically, as conclusions drawn rigorously from traditional analytical models.

Appendix to Chapter 3: Endogenous Evolution of Normal Distributions of Voter Ideal Points

Although not a central feature of our core argument, we now show that our specification of normally distributed ideal points in voter populations and subpopulations is by no means the brute force assumption it might seem at first sight. It is easy to show that such distributions of voters *tend to evolve endogenously* from simple but plausible models of social interaction between voters. Imagine, for example, a *uniform* random distribution of ideal points in some policy space, such as the distribution shown in the left panel of Figure 3.4. Sketch a simple model of social interaction as follows. One voter, *a*, is picked at random from the set of all voters to have an interaction with another voter, *b*, also picked at random. Following this interaction, *a*'s ideal point follows a random walk, though this is very slightly biased in favor of moving *a*'s ideal point closer to *b*'s ideal

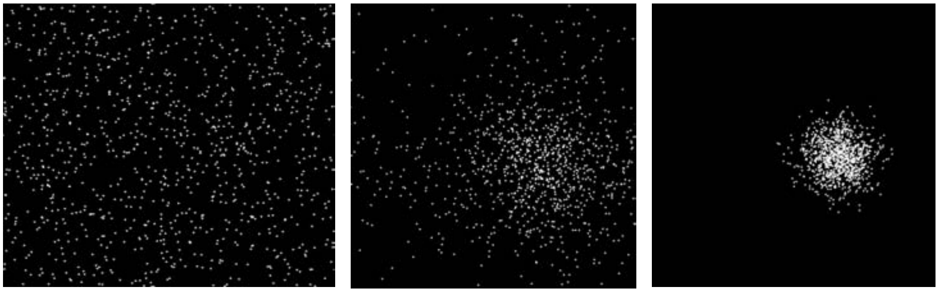

Figure 3.4. An emerging bivariate normal distribution of one thousand voter ideal points arising from random social interactions.

point, rather than further away from this. (We return shortly to discuss people whose views are unmoved by any social interaction.) In line with much of the social psychological evidence about social conformity and personal influence (Baylis 1996; Amorin Neto and Ström 2006), the rationale for this is that social interactions, all other things equal, are more likely to result in the views of those who interact moving closer together rather than further apart. If we iterate such a model continuously, then the pattern that evolves is that the set of ideal points tends to cluster and that, in line with the central limit theorem, their distribution tends to be binomial—approximating a normal distribution in large populations.

Watching such simulations in motion, we clearly observe the evolutionary process. An initial arbitrary scatter of ideal points, for example that in the left panel of Figure 3.4, at first just appears to follow a random walk as a result of the interactions between random pairs of voters. These random interactions also generate small local clusters of ideal points, however. After a period of time, some local cluster of voters emerges at random with sufficient critical mass to act as an attractor for other voters. We can see this beginning to happen in the middle panel of Figure 3.4, which shows a screen shot taken during a simulation run using a NetLogo implementation of our simple model of the evolution of voter ideal points. Random interactions between voters are now more likely to arise with voters in the denser local cluster, and this biases ideal point movement toward the cluster. The clustering process thereby becomes self-reinforcing. The centroid of the evolving distribution of ideal points then "locks in" on the attracting cluster, which is like a grain of sand in an oyster. The precise location of this centroid is entirely arbitrary. The local random cluster of ideal points that eventually emerges as being dense enough to attract other ideal points in the interaction process can arise anywhere at all in the policy space. Indeed, the same initial scatter of ideal points can evolve toward different distributions, given different random social interactions along the way. As the process of social interaction continues, the ideal points evolve to have a bivariate normal distribution, seen in the right panel of Figure 3.4. Strikingly, the emergent normal distribution of voter ideal points is a stochastic steady state—a dynamic equilibrium in which distributions vary periodically around a long-run stationary mean. Although alas this cannot be seen on a static page, *every ideal point in the right panel of Figure 3.4 is in continual motion, but the distribution of the set of ideal points remains the same*.

If we find it substantively unattractive to model social interactions that can result in distributions of voter ideal points with entirely arbitrary centroids, we in effect admit the possibility that these centroids should be substantively "anchored." One such anchor arises if we characterize some subset of voters as "fundamentalists" *whose ideal points do not change as a result of social interaction*. We may further assume that fundamentalists' ideal points are concentrated around some particular spatial location that is substantively important to them. Given the non-evolution of fundamentalists' ideal points, the existence of a very small fundamentalist hardcore can have a big impact on the substantive location of the centroid of any evolved ideal point distribution. Figure 3.5 shows evolved normal distributions of voter ideal points, of which a mere 3 percent were designated fundamentalists, with fixed ideal points drawn from a bivariate normal distribution centered on the origin. The remaining 97 percent were given arbitrary ideal points in a random start and subsequently adapted these on the basis of iterated random interaction. Having "seeded" the voter population with a very small set of inflexible fundamentalists, the centroid of the ideal points of the remaining voters evolves to a position very close to the mean ideal point location of those fundamentalists.

We can generalize this simple model by assuming voters are partitioned into subpopulations. The left panel of Figure 3.6 shows a random start with a uniform random distribution of voter ideal points in two subpopulations, colored white and gray. If we assume random social interactions take place *within but not between subpopulations*, then ideal points *in each subpopulation* evolve under our model of segregated social interaction to have distinctive normally distributed sets of ideal points.[25] The right panel of Figure 3.6 shows a screen shot of quite distinct normal distributions in the white and gray subpopulations, each distribution

[25] We by no means need to make such a strong assumption to derive the type of result we report here. Since our main focus in this book is not on endogenous ideal point distributions, however, we use the strong assumption, for didactic reasons, to make our point in the clearest possible terms.

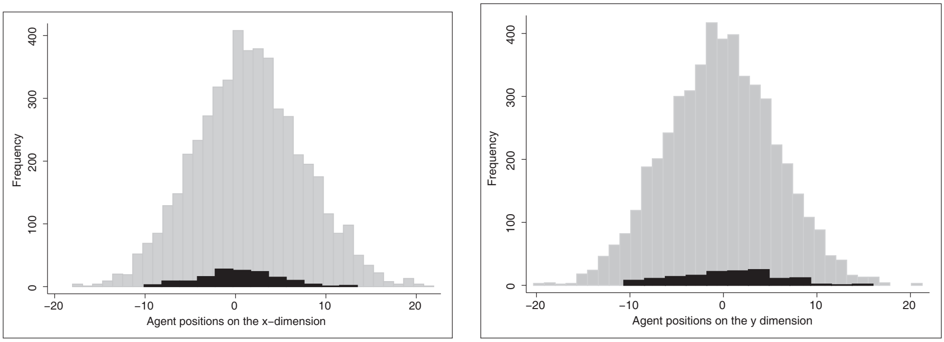

Figure 3.5. Distributions of voter ideal points on x- and y-dimensions, plotted in gray, following repeated random interaction from a random start in which 3 percent of voters were designated fundamentalists, plotted in black, with unmoving ideal points drawn from a bivariate normal distribution with mean (0, 0) and standard deviation 5.0.

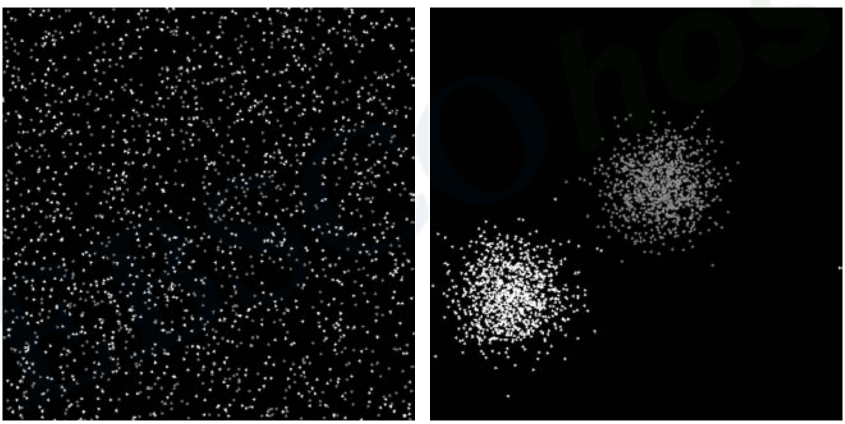

Figure 3.6. Emerging normal distributions in two subpopulations of ideal points.

evolving from the random start in the left panel of the same figure as a result of interaction within but not between subpopulations. Combining subpopulations into an overall voter population, the aggregate distribution of voter ideal points will be asymmetric and may be multimodal. Once again, the emergent ideal point distribution is a stochastic steady state with individual ideal points in continual motion, and subpopulation ideal point centroids are entirely arbitrary.

If we prefer to model subpopulation ideal point centroids as having substantive meaning, rather than being an entirely arbitrary result of

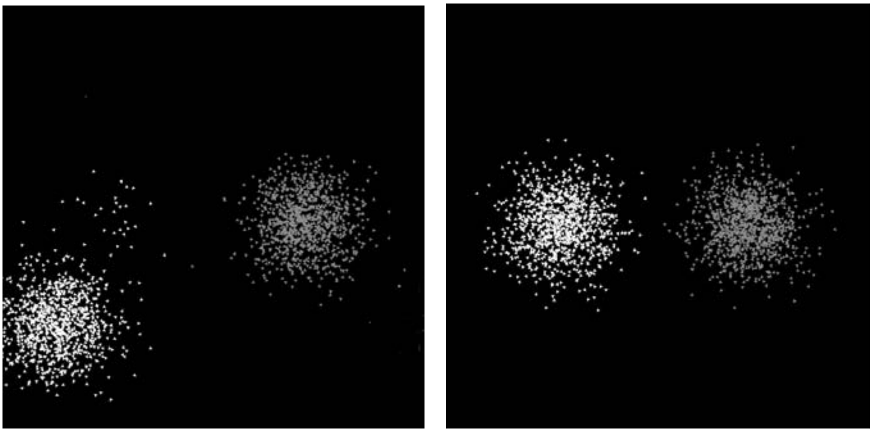

Figure 3.7. Emerging normal distributions in two subpopulations of ideal points, 3 percent of each subpopulation being fundamentalists whose ideal point centroids are at –20 (white) and +20 (gray) on the x-dimension.

random social interactions, then we can again add fundamentalists to our model. Assume that just 3 percent of each subpopulation are fundamentalists who never change their views as a result of social interaction and that the two sets of fundamentalists have distinctive positions on the *x*-dimension, with no difference between them on the *y*-dimension. White fundamentalists are on the left and gray fundamentalists are on the right. The left panel of Figure 3.7 shows an interim screen shot at a point in the simulation where the two subpopulations have evolved to have distinctive distributions of ideal points. This is not a stochastic steady state, however, because the unmoving fundamentalists continue to exert a "pull," as a result of the process of random social interaction, on the ideal point distributions of their respective subpopulations.

Thus, in the left panel of Figure 3.7, we can see the white fundamentalists away from the rest of the white subpopulation, with ideal points scattered in the center-right of the space. These never change position and are thus never drawn by social interaction toward the larger distribution of ideal points in "their" subpopulation. Indeed, the reverse happens; *the fundamentalists slowly but surely draw their subpopulation toward them* as they interact with other subpopulation members. The right panel of Figure 3.7 shows the same simulation many thousands of iterations after the interim state shown in the left panel. The evolved normal distributions of subpopulation ideal points are now centered on the centroids of their respective fundamentalists' ideal points. These distributions are now in stochastic steady states. All nonfundamentalists' ideal points are in continual motion, but the aggregate normal distributions of these remain essentially the same, with each subpopulation ideal point centroid centered on the ideal point centroid of its fundamentalists. *While we do not explore this here, we consider this general type of result to have huge implications for the evolution of public opinion in societies in which social interaction is strongly structured by ethnicity*.

This result does not depend on having 3 percent, or indeed any particular positive proportion, of fundamentalists. Any proportion will do, although the process can take a very long time if there are very few fundamentalists. In the end, however, just as a lone donkey can pull an ocean liner, albeit very slowly, by applying a continuous force in the same direction, even a small number of fundamentalists can eventually move a large subpopulation with which they interact. What we have reported here are essentially model sketches and doodles. We do not have the space to specify and investigate a full-scale model of the endogenous evolution of voter deal points, and this would indeed distract us from our core focus on decision making by party leaders. We leave this as a matter for future work, contenting ourselves here with demonstrating that normal distributions of voter ideal points are not the brute force assumptions they might superficially appear to be.