Comparing Probabilistic Models: IC, IAC, IANC

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Abstract

One possible way of analyzing voting rules is to compute the probability of different events like choosing a Condorcet winner or manipulation. One needs to carry out Monte-Carlo experiments using some probabilistic model which determines how to generate voters' preferences. This paper aims to compare two well-known probabilistic models, impartial culture (IC) and impartial anonymous culture (IAC), with recently introduced impartial anonymous and neutral culture model (IANC). We answer the question how results of the same experiment differ in distinct probabilistic models. We show that IAC and IANC models yield almost the same results when the number of alternatives or the number of voters exceeds 10; while maximal difference between IC and IAC/IANC tends to 1 as the number of voters grows, and to 0 as the number of alternatives tends to infinity.

Keywords IC, IAC, IANC, probabilistic models, manipulability

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1 Introduction

Social choice theory studies various properties of voting rules, but it often occurs that a particular rule does not satisfy this or that important property. For example, we say that a rule is not Condorcet-consistent if it fails to choose a Condorcet winner at least once. Then it is of interest to know what the share of such failures in the total number of situations is. It could be also considered as a probability, and further, for convenience, we will call it index. To define what should be considered as a "`situation"' one needs to use some probabilistic model.

The basic model called impartial culture model (IC) takes the set of all preference profiles as a sample space. It was introduced in (Guilbaud 1952) and used in a wide number of studies. The main IC assumptions are: voters choose their preference independently, both names of voters and names of alternatives matter. Another important probabilistic model is the impartial anonymous culture model (IAC), first described in (Kuga and Nagatani 1974) and (Gehrlein and Fishburn 1976). It assumes that voters are anonymous and we could not distinguish between them. In other interpretation, voters tend to choose similar preferences. Since there is no difference between names of voters, then there is no difference between preference profiles that have the same collection of individual preferences. Such families of preference profiles are called anonymous equivalence classes (AECs) and assumed to be equally likely in IAC.

In this paper, we consider the impartial anonymous and neutral culture model (IANC), in which both names of voters and names of alternatives do not matter. In this model, such preference profiles that differ in permutation of voters and/or alternatives are regarded as equivalent. Therefore, the set of all preference profiles splits up into anonymous and neutral equivalence classes (ANECs) which are equally likely. The investigation of this model was started in (Eğecioğlu 2005) and extended in (Eğecioğlu and Giritligil 2009). They introduced a way of calculating the number of ANECs and an algorithm for their uniform random generation.

One of the arguments for considering such model is that every unbiased social choice rule satisfies both anonymity and neutrality. It means that any two preference profiles from the same ANEC will be both either manipulable or not with respect to those rules. We can regard an ANEC as a type of group preference. Thus, considering only representatives of ANEC, we do not count preference profiles of the same type twice. For example, if one looks for rules minimizing the number of public preference types that admit manipulation, then manipulability index should be considered in IANC.

Another argument concerns the computational aspect. The number of preference profiles grows exponentially with the number of voters and as factorial with the number of alternatives, while the growth of the number of anonymous equivalence classes (AECs) and anonymous and neutral equivalence classes (ANECs) is polynomial (see figure 1). It means that in some cases total enumeration of ANECs is possible, while the enumeration of preference profiles is not. For example, when we have 4 alternatives and 7 voters, the number of ANECs is 84825, and the number of preference profiles is $4.586 \cdot 10^9$. Monte-Carlo scheme in IANC will give more accurate results than in IC.

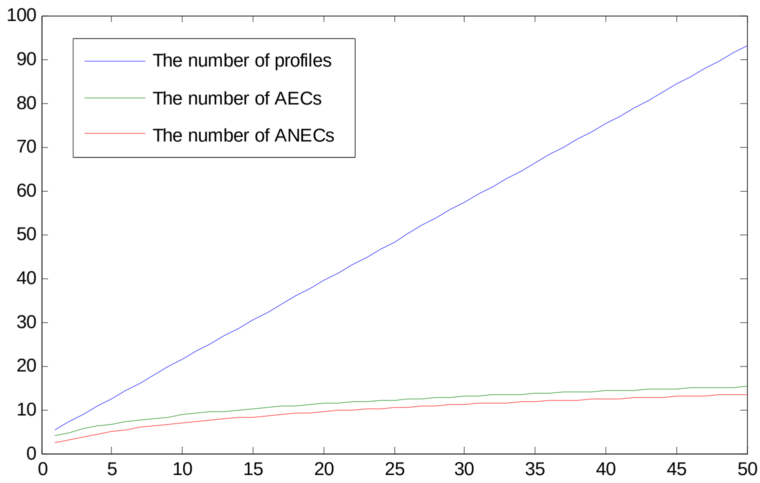

Fig.1 The number of preference profiles, AECs, and ANECs, log-scale.

To conduct computational experiments in these models one needs to carry out random uniform generation of preference profiles (in IC) or equivalence classes (in IAC and IANC). However, we should know whether the results of computational experiments in IANC differ from those in the basic IC model. Assume that in some cases the upper bound of difference between the values of the index in IC and IANC is almost zero. Then, on the one hand, there is a plenty of results already calculated in IC, and we do not need additional computations in IANC. On the other hand, we could do computations in IANC first, because they will give more accurate results for large parameters, and put corresponding indexes in IC equal to those in IANC.

The difference between IC and IANC models was investigated for small number of alternatives and voters in (Veselova 2012). By “difference between models A and B” we mean "maximal difference between index in A and the same index in B over all possible indexes". This paper unites all three probabilistic models in the whole picture and completely defines their position relatively to each other. First, we show that there is almost no difference between the IANC and IAC model when the number of voters or alternatives exceeds 10. Knowing maximal difference between IC and IAC we obtain the asymptotic difference between IC and IANC.

Finally, we illustrate the theoretical study with examples of the computation of Nitzan-Kelly's manipulability index in the IC and IANC model for four social choice rules with 3 alternatives and up to 30 voters.

2 Definitions, notions, and theoretical basis

We use the notions for the impartial anonymous and neutral culture model introduced in (Eğecioğlu 2005). First, there is a set of alternatives A, consisting of m elements, and a set of voters $N = \{1, 2, ..., n\}$ with n elements. A preference profile is an ordered set of individual preferences, $\vec{P} = (P_1, P_2, ..., P_n)$, where P_i is a preference of the i-th individual, which is a linear order.

The number of different preference profiles is $(m!)^n$. The IC model assumes that each voter selects his or her preferences out of $m!$ possible linear orders independently and each of $(m!)^n$ preference profiles is equally likely. The set of all preference profiles with n voters and m alternatives is denoted by Ω.

As mentioned above, in the IANC model there is no difference between names of voters and between names of alternatives. For example, in this model the following three profiles are considered as the same representation of preferences:

$$\vec{P} = \begin{array}{cc} \underline{P_1} & \underline{P_2} \\ y & y \\ x & z \\ z & x \end{array} \quad \vec{P}' = \begin{array}{cc} \underline{P_1'} & \underline{P_2'} \\ x & x \\ y & z \\ z & y \end{array} \quad \vec{P}'' = \begin{array}{cc} \underline{P_1''} & \underline{P_2''} \\ y & y \\ z & x \\ x & z \end{array}.$$

Therefore, we have a partition of Ω into ANECs. Any preference profile from a given ANEC can be taken as the representative profile (or root). In other words, ANEC is a set of preference profiles that can be generated from each other by permuting voters and renaming alternatives.

The permutation of voters (or columns), an element of the symmetric group S_n, is denoted by σ and a permutation on the set of alternatives is $\tau \in S_m$, τ_0 and σ_0 are the identity permutations. A pair of permutations τ and σ is denoted by $g = (\tau, \sigma)$. $G = S_m \times S_n$ is the group of all ordered pairs of permutations $g = (\tau, \sigma)$, and it acts on the set of all preference profiles. There are $m!$ permutations of alternatives and $n!$ permutations of voters and, therefore, the number of pairs of permutations $g = (\tau, \sigma)$ is

$$|G| = m!n!.$$

A partition μ of m is defined as a weakly decreasing sequence of positive integers $\mu = (\mu_1, \mu_2, ..., \mu_\beta)$, such that $\mu_1 \geq \mu_2 \geq ... \geq \mu_\beta$ and $\mu_1 + \mu_2 + ... + \mu_\beta = m$, where μ_i is a part of μ. For example, (3,2,1,1) is a partition of 7 into 4 parts. Π_m is the set of all partitions of the integer m.

Each permutation can be represented via cycle decomposition. Therefore, τ defines a partition $\mu \in \Pi_m$ in such a way that parts of a partition μ are the lengths of cycles in τ. For any partition μ we define a number

$$z_\mu = 1^{\beta_1} 2^{\beta_2} ... m^{\beta_m} \beta_1! \beta_2! ... \beta_m!,$$

where β_i is the number of parts of length i, $\beta_1 + \beta_2 + ... + \beta_m = \beta$.

The image of a profile $\vec{P}$ under a permutation $g = (\tau, \sigma)$ is denoted by $\vec{P}^g$. Anonymous and neutral equivalence class $\theta_{\vec{P}}$ can be defined as a subset of Ω: $\{\vec{P}^g \mid g \in G\}$. Profiles $\vec{P}_1, \vec{P}_2$ are regarded as equivalent if there exists $g \in G$, such that $\vec{P}_1^g = \vec{P}_2$.

An indicator function $\chi(S)$ of statement S is defined as follows

$$\chi(S) = \begin{cases} 1, & if \quad S \quad is \quad True, \\ 0, & if \quad S \quad is \quad False. \end{cases}$$

$LCM(\lambda)$ is the least common multiple of the parts of λ. The number of anonymous and neutral equivalence classes for n voters and m alternatives was found in (Eğecioğlu 2005). It is given by

$$R(m,n) = \sum_{\mu \in \Pi_m} z_\mu^{-1} \begin{pmatrix} \frac{n}{t} + \frac{m!}{t} - 1 \\ \frac{m!}{t} - 1 \end{pmatrix},$$

where $t = LCM(\mu)$ and the binomial coefficient for integer k, $0 \le k \le x$ is defined as

$$\binom{x}{k} = \begin{cases} \dfrac{x!}{k!(x-k)!}, x \in \mathbb{N}, \\ 0, x \notin \mathbb{N}. \end{cases}$$

For n and $m!$ being relatively prime, the number of equivalence classes is

$$R(m,n) = \frac{1}{m!}\binom{n+m!-1}{m!-1}.$$

In the IAC model there is no difference between names of voters. Thus, all preference profiles generated from $\vec{P}$ via permuting columns form an AEC, $\rho_{\vec{P}} = \{\vec{P}^{\sigma} \mid \sigma \in S_n\}$. Similarly to IANC, the set of all preference profiles with *m* rows and *n* columns Ω is split up by the group action of S_n into a disjoint union of orbits (AECs)

$$\Omega = \rho_1 + \rho_2 + ... + \rho_K .$$

The number of AECs is equal to the number of ways to choose n columns from $m!$ column types with repetitions (Feller, 1957)

$$K(m,n) = \binom{m!+n-1}{m!-1}.$$

For each AEC ρ_j define an anonymous preference profile $\vec{p}_{\rho_j} = (n_1, n_2, ..., n_{m!})$, where n_i is the number of voters with a preference order of the i-th type, $n_1 + n_2 + ... + n_{m!} = n$. Now consider the set of anonymous preference profiles as the initial set and the group action of S_m on it. $\vec{p}^{\tau}_{\rho_j}$ denotes the image of a profile $\vec{p}_{\rho_j}$ under a permutation τ. Now each ANEC includes such anonymous preference profiles as elements and could be generated from one of them the following way. We take a representative $\vec{p}_{\rho_j}$ of AEC ρ_j and permute alternatives all possible ways

$$\theta_{\rho_j} = \{\vec{p}^{\tau}_{\rho_j} \mid \tau \in S_m\}.$$

The stabilizer for ρ_j is

$$G_{\rho_j} = \{\tau \in S_m \mid \vec{p}^{\tau}_{\rho_j} = \vec{p}_{\rho_j}\}.$$

Thus, the number of anonymous preference profiles in ANEC is defined by the formula

$$\left|\theta_{\rho_j}\right| = \frac{|S_m|}{|G_{\rho_j}|}.$$

Obviously, the minimal number of permutations in G_{ρ_j} is 1, then

$$\left|\theta_{\rho_j}\right| = \frac{m!}{|G_{\rho_j}|} \le m!.$$

The following figure is an illustration of the example with 3 alternatives and 4 voters. Columns are the ANECs and the y-axis show the number of anonymous preference profiles in ANEC. Since ANECs do not intersect and the same holds for AECs (two distinct AECs could not be represented by the same anonymous preference profile), the number of AECs representatives in all ANECs is equal to $K(m,n)$ (the area of all columns on the Figure 2, filled with grey).

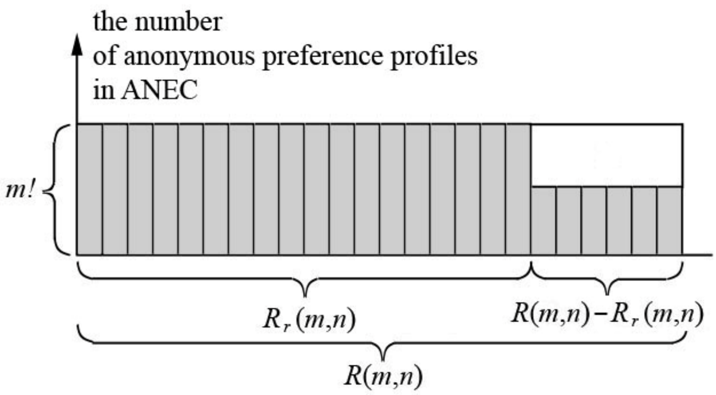

Fig. 2 The set of ANECs and the number of anonymous preference profiles in ANECs for 3 alternatives and 4 voters

3 Estimating differences

Let us denote the difference between model A and model B by Δ_{A-B}. Each of Δ_{IC-IAC}, $\Delta_{IAC-IANC}$, and $\Delta_{IC-IANC}$ is characterized by a certain value from [0, 1] for some predefined *m* and *n*. Thus, we consider IC, IAC, and IANC models as points and Δ_{IC-IAC}, $\Delta_{IAC-IANC}$, $\Delta_{IC-IANC}$ as distances between them. Applying triangular inequality we get

$$|\Delta_{IC-IAC} - \Delta_{IAC-IANC}| < \Delta_{IC-IANC} < \Delta_{IC-IAC} + \Delta_{IAC-IANC}.$$

First, we estimate $\Delta_{IAC-IANC}$. We use the same method as in (Veselova 2012) counting the number and the cardinality of those ANECs which are greater than average in cardinality. The average cardinality of ANEC is close to $m!$. Then the only classes whose cardinality is greater than average are the classes of cardinality $m!$. The number of such classes is denoted by $R_r(m,n)$. Then the difference between IAC and IANC models is

$$\Delta_{IAC-IANC} = \frac{m!R_r(m,n)}{K(m,n)} - \frac{R_r(m,n)}{R(m,n)}.$$

Theorem 1 finds the limit of this difference.

Theorem 1 Difference between IAC and IANC models, $\Delta_{IAC-IANC}$, tends to zero as $n \to \infty$ or $m \to \infty$.

Thus, IAC and IANC become closer and closer while the number of voters or the number of alternatives grows. If $n > 10$ or $m > 10$, $\Delta_{IAC-IANC}$ is less than 0.02. Using inequality

$$|\Delta_{IC-IAC} - \Delta_{IAC-IANC}| < \Delta_{IC-IANC} < \Delta_{IC-IAC} + \Delta_{IAC-IANC}$$

we can conclude that in this case the difference between IC and IANC models is almost the same as between IC and IAC.

$$\Delta_{IC-IAC} - 0.02 < \Delta_{IC-IANC} < \Delta_{IC-IAC} + 0.02$$

And the latter is evaluated easily even for large number of voters using Monte-Carlo generation of AEC representatives.

Figure 3 illustrates the difference between IC and IANC which is the same as IC-IAC difference.

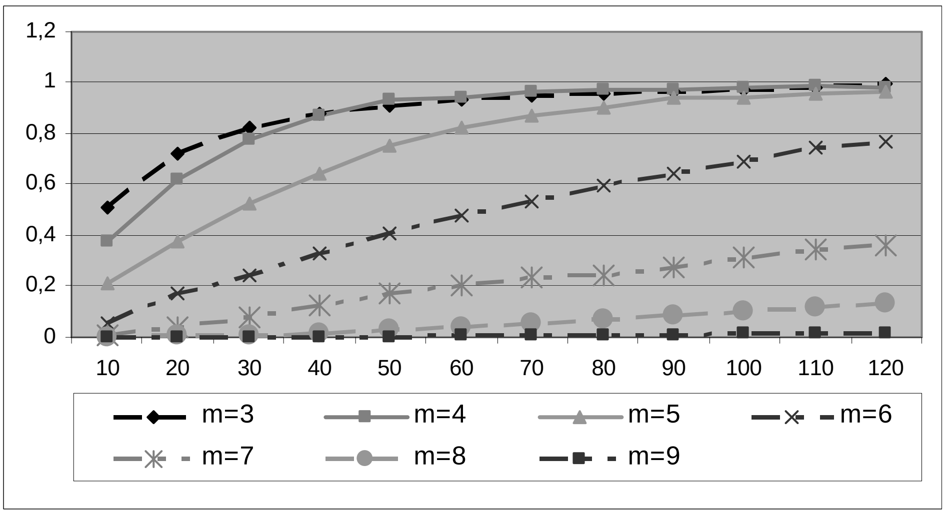

Fig. 3 Maximal difference of indexes in the IC and IANC models

There is almost no difference between IC and IANC models when the number of alternatives exceeds 9 and the number of voters is less than 120. At the same time, the maximal difference tends to one, which may cause significant changes in the relative manipulability of social choice rules. In practice, however, the difference of indexes will not be so high, but it is still possible for some probabilistic measures.

4 Manipulability of social choice rules in IC and IANC models

In this Section we calculate the actual difference of the Nitzan-Kelly's indexes in the IC and IANC models for four social choice rules and compare it with maximal difference found in the previous Section. First, we give a formal definition of these rules.

In addition, we give some definitions on manipulability. A preference profile where all individuals express their true preferences except the *i*-th individual is denoted by $\vec{P}_{-i}(P_i') = (P_1, ..., P_{i-1}, P_i', P_{i+1}, ..., P_n)$. P_i' is the deviation of the *i*-th individual from his true preferences P_i.

A social choice (or the outcome of aggregating procedure) with respect to the profile $\vec{P}$ is denoted by $C(\vec{P})$. As in (Aleskerov et al 2010), the case of multiple choice is considered. That means that the result of an aggregating procedure might consist of several elements. Consider a preference profile $\vec{P}''$ from the example above. How can one decide, what is better for the first individual: the set $\{x, y, z\}$ or $\{z\}$? To answer this question, several methods of expanding preferences are proposed. In this study, Leximin and Leximax, two lexicographic methods introduced in (Pattanaik 1978), are considered.

On the basis of a linear order representing voter's preferences on the set of alternatives, expanded preferences order all the subsets of A. In the *Leximin* method, first, the worst alternatives of two sets are compared, and the set where the better alternative is contained is considered as a better set. If they are the same, then the second-worst alternatives are compared and so on. In the *Leximax* method of expanding preferences, the best alternatives of two sets are compared, then the second-best alternatives and so on. EP_i denotes the expanded preferences of individual *i*.

For example, if voter i prefers alternative x to alternative y and alternative y to alternative z, then, according to the *Leximin* method,

$$\{x\}EP_i\{x,y\}EP_i\{y\}EP_i\{x,z\}EP_i\{x,y,z\}EP_i\{y,z\}EP_i\{z\}.$$

According to *Leximax*

$$\{x\}EP_i\{x,y\}EP_i\{x,y,z\}EP_i\{x,z\}EP_i\{y\}EP_i\{y,z\}EP_i\{z\}.$$

In the case of multiple choice manipulation is defined as follows: if there exists such insincere preference P_i' that for individual i $C(\vec{P}_{-i}(P_i'))EP_iC(\vec{P})$, then manipulation takes place.

The Nitzan-Kelly's index of manipulability in IC model is the ratio

$$NK_{IC}=\frac{d_0}{(m!)^n},$$

where d_0 is the number of profiles in which manipulation is possible.

In the IANC model this index is calculated over the set of roots of equivalence classes

$$NK_{IANC}=\frac{r_0}{R(m,n)},$$

where r_0 is the number of roots in which manipulation is possible

1. Plurality Rule. This rule chooses the best alternatives for the maximal number of voters.

$$a\in C(\vec{P})\Leftrightarrow[\forall x\in A\ \ n^+(a,\vec{P})\geq n^+(x,\vec{P})]$$

where $n^+(a,\vec{P})=card\{i\in N\mid\forall y\in A\ \ aP_iy\}$.

2. Approval Voting. Social choice is a set of alternatives which are among the q best alternatives in the preferences of the maximal number of voters.

$$a\in C(\vec{P})\Leftrightarrow[\forall x\in A\ \ n^+(a,\vec{P},q)\geq n^+(x,\vec{P},q)]$$

where $n^+(a,\vec{P},q)=card\{i\in N\mid card\{y\in A\mid yP_ia\}\leq q-1\}$.

3. Borda's Rule. For each alternative in the i-th voter's preferences the number $r_i(x,\vec{P})$ is calculated as follows

$$r_i(x,\vec{P}) = card\{b \in A : xP_ib\}.$$

The sum of $r_i(x,\vec{P})$ over all $i \in N$ is called a Borda's count.

$$r(a,\vec{P}) = \sum_{i=1}^{n} r_i(a,\vec{P}).$$

Borda's rule chooses alternatives with the maximal Borda's count.

$$a \in C(\vec{P}) \Leftrightarrow [\forall b \in A, \quad r(a,\vec{P}) \geq r(b,\vec{P})].$$

4. Black's procedure. Chooses a Condorset winner, if it exists, and, if it does not exist, the winner of Borda's rule.

We compute the Nitzan-Kelly's indexes both in the IC and IANC model using the Leximin and Leximax method of expanding preferences in 3-alternatives voting. To conduct experiments in IANC, we use uniform ANEC generation algorithms introduced in (Eğecioğlu, 2005). After that, we calculate the difference of these indexes

$$\Delta K_{IANC} = \frac{d_0}{(m!)^n} - \frac{r_0}{R(m,n)},$$

Figures 4 and 5 represent Nitzan-Kelly's index for $m = 3$ and $3 \leq n \leq 30$ in IC and IANC model with Leximin method (for graphs with Leximax method, see Appendix B). Black's procedure is the least manipulable in both cultures. Plurality rule and approval voting rule interchanged their positions relatively to other voting rules. The approval rule turns out to be the most manipulable in the IANC model, while under the IC assumption plurality rule has the highest value of manipulability index in most cases.

For further exploration of this phenomenon let us consider the differences of indexes plotted in Figure 6. As it can be seen from this graph, approval voting rule has the lowest and sometimes negative values of difference. This fact can be explained as follows: preference profiles in which manipulation is possible often belong to equivalence classes with a small number of elements. The plurality rule has the highest level of difference in all cases considered. These two facts cause changes in the relative manipulability of social choice rules.

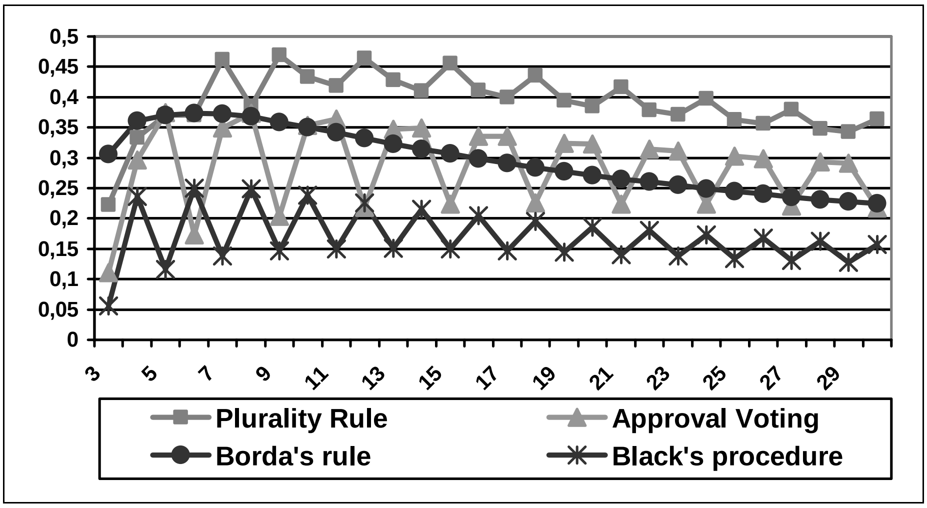

Fig. 4 The Nitzan-Kelly's index for the Leximin method in the IC model

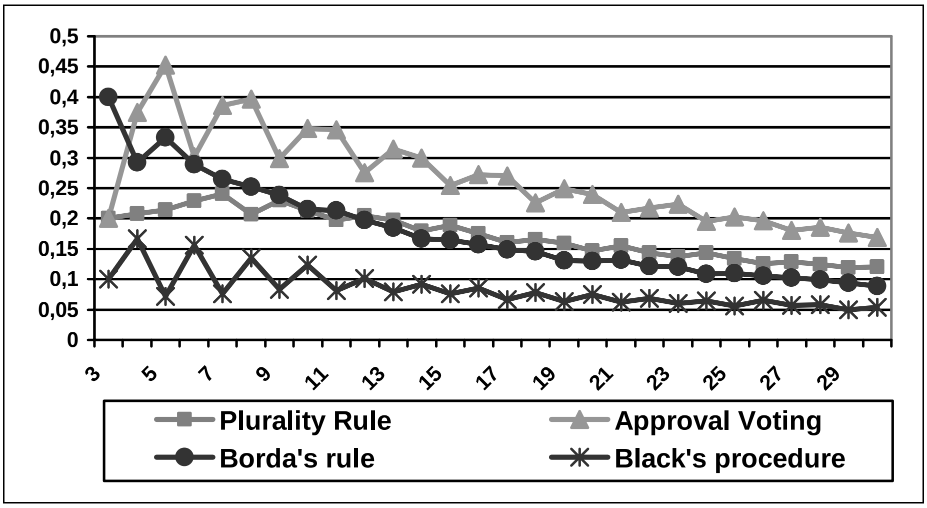

Fig. 5 The Nitzan-Kelly's index for the Leximin method in the IANC model

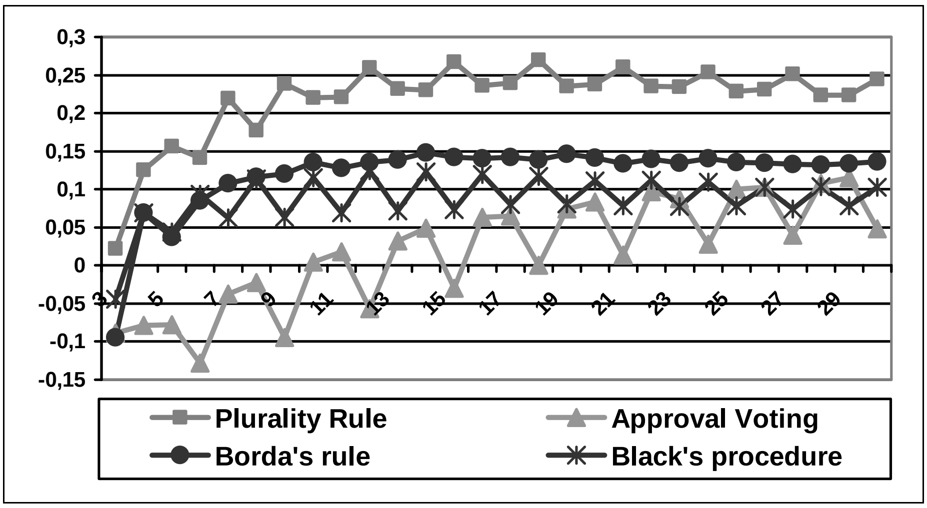

Fig. 6 The difference of the Nitzan-Kelly's index in IC and IANC, Leximin

6 Concluding remarks

Anonymity and neutrality are the basic axioms in social choice theory. The IANC model, based on these axioms, assumes that both names of voters and names of alternatives do not matter. In this model preference profiles that differ in permutation of voters or permutation of alternatives are united in the same anonymous and neutral equivalence class. The set of these classes is a convenient domain for social choice rules that satisfy both anonymity and neutrality. Some properties of social choice rules such as manipulability are measured as probabilities. In IC model a sample space is the set of all preference profiles, in IANC model it is the set of ANEC representatives.

The aim of this study is to answer the question: how do these probabilities in IC, IAC, and IANC model differ from each other? Does it make sense to do computational experiments in the new probabilistic model, IANC, if they are often more complex than experiments in the models IC and IAC? We compare IANC model with IAC and reveal that the maximal difference between them does not exceed 0.02 when the number of voters or the number of alternatives is greater than 10. It means that actual difference of probabilistic measures would be even less than 0.02, thus, we could consider them as equal. As a consequence, difference between IANC and IC will be almost the same as between IC and IAC. The second one is easy to calculate: it tends to 1 as the number of voters grows, and to 0 as the number of alternatives grows. Surprisingly, all three models yield the same result in a large number of cases.

Finally, we provide results of computational experiments in IC and IANC model calculating Nitzan-Kelly’s manipulability index for four voting rules in the case of 3 alternatives. It is shown that although the actual difference of manipulability indexes does not reach its upper bound, for some rules it is high enough to change the relative manipulability of these rules in the IANC model. Without conducting an experiment, we could say that analogous results for IAC model would be almost the same for $n > 10$.

Acknowledgements

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Appendix A

Theorem 1 Maximal difference of indexes in the IAC and IANC model $\Delta_{IAC-IANC}$ tends to zero as $n \to \infty$ or $m \to \infty$.

Proof.

1) First prove that

$$\lim_{n\to\infty} \frac{K(m,n)}{m!R(m,n)} = 1 \text{ and } \lim_{m\to\infty} \frac{K(m,n)}{m!R(m,n)} = 1 .$$

$$\frac{K(m,n)}{m!R(m,n)} = K(m,n) \Bigg/ \left(m! \cdot \frac{1}{m!} \binom{m!+n-1}{m!-1} + m! \sum_{\mu\in\Pi_m \setminus (1,1,...,1)} z_\mu^{-1} \binom{n/t+m!/t-1}{m!/t-1} \right) =$$

$$= K(m,n) \Bigg/ \left(K(m,n) + m! \sum_{\mu\in\Pi_m \setminus (1,1,...,1)} z_\mu^{-1} \binom{n/t+m!/t-1}{m!/t-1} \right) =$$

Let us denote the sum in the denominator by $\sum_{\mu\in\Pi_m \setminus (1,1,...,1)} A_\mu$, where

$$A_\mu = z_\mu^{-1} \binom{n/t+m!/t-1}{m!/t-1} \text{ for some } \mu \in \Pi_m \setminus (1,1,...,1) .$$

Then

$$\frac{K(m,n)}{K(m,n) + m! \sum_{\substack{\mu\in\Pi_m \setminus \\ (1,1,...,1)}} A_\mu} = \frac{1}{1 + \sum_{\substack{\mu\in\Pi_m \setminus \\ (1,1,...,1)}} \frac{m! A_\mu}{K(m,n)}} .$$

$$\frac{m! A_i}{K(m,n)} = \frac{m!}{z_\mu} \cdot \frac{\binom{n/t+m!/t-1}{m!/t-1}}{\binom{m!+n-1}{m!-1}} = \frac{m!}{z_\mu} \cdot \frac{(n/t+m!/t-1)!(m!-1)!n!}{(m!/t-1)!(n/t)!(m!+n-1)!}$$

2) Fix the value of m and let $n \to \infty$

$$\lim_{n\to\infty} \frac{m!}{z_\mu} \cdot \frac{(n/t+m!/t-1)!(m!-1)!n!}{(m!/t-1)!(n/t)!(m!+n-1)!} = \frac{m!(m!-1)!}{z_\mu (m!/t-1)!} \lim_{n\to\infty} \frac{(n/t+m!/t-1)!n!}{(n/t)!(m!+n-1)!} =$$

$$= \frac{m!(m!-1)!}{z_\mu (m!/t-1)!} \lim_{n\to\infty} \frac{(n/t+m!/t-1)!}{(n/t)!} \cdot \frac{n!}{(m!+n-1)!} =$$

$$= \frac{m!(m!-1)!}{z_\mu (m!/t-1)!} \lim_{n\to\infty} \frac{(n/t+1)\cdot ... \cdot (n/t+m!/t-1)}{(n+1)\cdot ... \cdot (m!+n-1)} = 0 .$$

Now fix n and let $m \to \infty$

$$\lim_{m\to\infty}\frac{m!}{z_\mu}\cdot\frac{(n/t+m!/t-1)!(m!-1)!n!}{(m!/t-1)!(n/t)!(m!+n-1)!}=\frac{n!}{(n/t)!}\lim_{n\to\infty}\frac{m!}{z_\mu}\cdot\frac{(n/t+m!/t-1)!(m!-1)!}{(m!/t-1)!(m!+n-1)!}$$

$$\frac{m!}{z_\mu}\cdot\frac{(n/t+m!/t-1)!(m!-1)!}{(m!/t-1)!(m!+n-1)!}<\frac{m!(n/t+m!/t-1)!(m!-1)!}{(m!/t-1)!(m!+n-1)!},$$

$$\lim_{m\to\infty}\frac{m!(n/t+m!/t-1)!(m!-1)!}{(m!/t-1)!(m!+n-1)!}=\lim_{m\to\infty}\frac{m!(n/t+m!/t-1)\cdot\ldots\cdot(m!/t)}{(m!+n-1)\cdot\ldots\cdot(m!+1)m!}=$$

$$=\lim_{m\to\infty}\frac{(n/t+m!/t-1)}{(m!+n-1)}\cdot\ldots\cdot\frac{(m!/t)}{(m!+n-n/t)}\cdot\frac{1}{(m!+n-n/t-1)\cdot\ldots\cdot(m!+1)}=0.$$

Consequently,

$$\lim_{m\to\infty}\frac{m!}{z_\mu}\cdot\frac{(n/t+m!/t-1)!(m!-1)!n!}{(m!/t-1)!(n/t)!(m!+n-1)!}=0.$$

Finally,

$$\lim_{m\to\infty}\sum_{\substack{\mu\in\Pi_m\backslash\\(1,1,\ldots,1)}}\frac{m!A_\mu}{K(m,n)}=0 \text{ and } \lim_{n\to\infty}\sum_{\substack{\mu\in\Pi_m\backslash\\(1,1,\ldots,1)}}\frac{m!A_\mu}{K(m,n)}=0,$$

$$\lim_{n\to\infty}\frac{K(m,n)}{K(m,n)+m!\sum\limits_{\substack{\mu\in\Pi_m\backslash\\(1,1,\ldots,1)}}A_\mu}=1 \text{ and } \lim_{m\to\infty}\frac{K(m,n)}{K(m,n)+m!\sum\limits_{\substack{\mu\in\Pi_m\backslash\\(1,1,\ldots,1)}}A_\mu}=1,$$

$$\lim_{n\to\infty}\frac{K(m,n)}{m!R(m,n)}=1 \text{ and } \lim_{m\to\infty}\frac{K(m,n)}{m!R(m,n)}=1.$$

3) The maximal difference between IAC and IANC models is

$$\frac{m!R_r(m,n)}{K(m,n)}-\frac{R_r(m,n)}{R(m,n)}=\frac{R_r(m,n)}{R(m,n)}\left(\frac{m!R(m,n)}{K(m,n)}-1\right).$$

Since

$$\frac{R_r(m,n)}{R(m,n)}<1,$$

then the limit of this difference using the previous result

$$\lim_{n\to\infty}\frac{R_r(m,n)}{R(m,n)}\left(\frac{m!R(m,n)}{K(m,n)}-1\right)=0.$$

And the same holds for $m\to\infty$

$$\lim_{m\to\infty}\frac{R_r(m,n)}{R(m,n)}\left(\frac{m!R(m,n)}{K(m,n)}-1\right)=0. \ \square$$

Appendix B

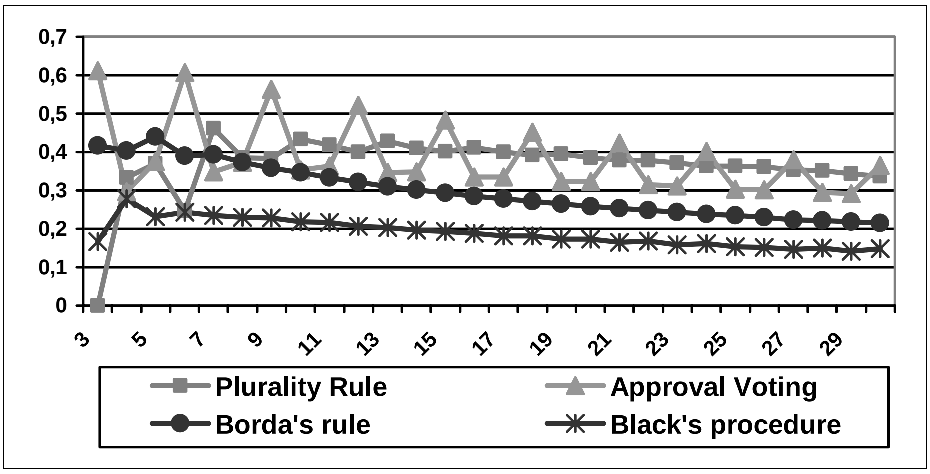

Fig. 8 The Nitzan-Kelly's index for the Leximax method in the IC model

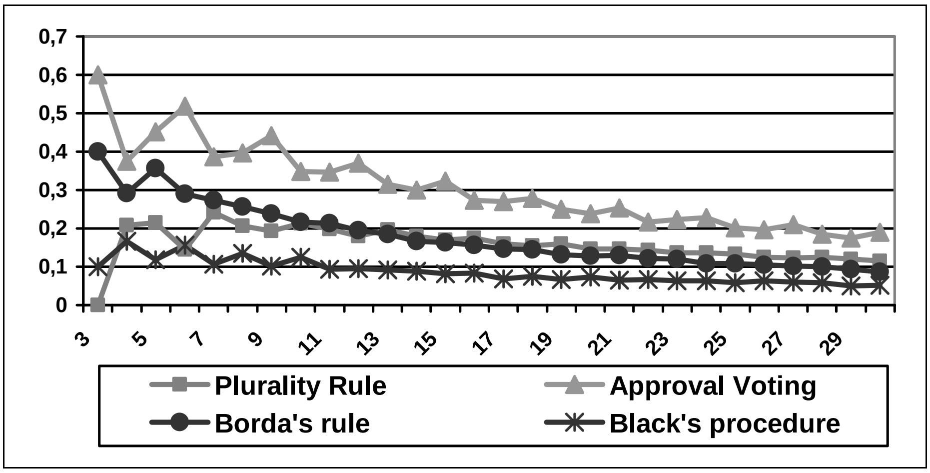

Fig. 9 The Nitzan-Kelly's index for the Leximax method in the IANC model

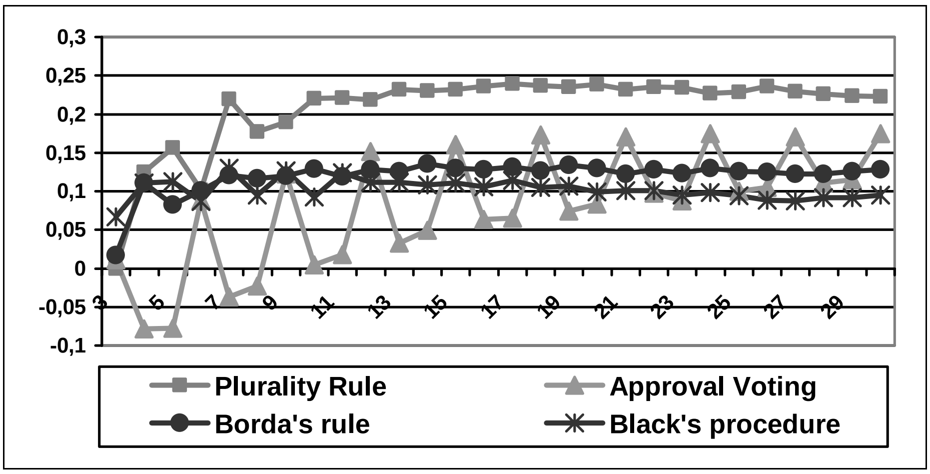

Fig. 7 The difference of the Nitzan-Kelly's index in IC and IANC, Leximax

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