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In quest of the banks set in spatial voting games

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Abstract The *Banks set* (1(4):295–306, 1985) is one of the more important concepts in voting theory since it tells us about the sophisticated outcomes of standard amendment voting procedures commonly in use throughout the English speaking world (and elsewhere as well). While the properties of the Banks set for finite voting games have been extensively studied, little is known about how to find members of this set for majority rule spatial voting games involving possibly infinite agendas. We look at this question for two-dimensional games where voters have Euclidean preferences, and offer a variety of new results that delimit areas of the space that can be shown to lie within the Banks set, such as the *Schattschneider set*, the *tri-median set*, and the *Banks line set*—geometric constructs which we show to be nested within one another.

1 Introduction

In this essay we focus on the geometry of the *Banks set*. The *Banks set* is one of the most important ideas in the intersection of game theory and social choice theory.[1]

[1] There has been a proliferation of solution concepts and geometric constructs that help define the internal structure of majority rule (and quota rule) spatial voting games. Among the most important of these are the *minmax set* (Kramer 1972); the *multidimensional median* (Shepsle and Weingast 1981); the *yolk* (McKelvey 1986), in two dimension the smallest circle that touches all median line; and the

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It may be thought of as a solution concept for any voting process which makes use of a *king-of-the-hill rule* for sequential committee decisions. In such a process the current king of the hill is paired against another alternative and the winner then becomes (remains) king of the hill. Alternatives are usually treated as coming up for consideration in some prespecified order, defined by some (finite) *agenda*. Votes continue in this manner until all (feasible) alternatives have been exhausted, or until some particular *stopping rule* is invoked. The best known variant of king-of the-hill voting is one where the last vote pits the reigning king of the hill against a predetermined alternative, a.k.a. the *status quo*. This king-of-the-hill variant was labeled *standard amendment procedure* (SAP) by Black (1958) because it is the form of voting most common in legislatures and private organizations on bills, and amendments thereto, in countries with a British heritage.

Shepsle and Weingast (1984) showed that, for fixed finite agendas under SAP, when voting is *sophisticated* (in the sense of Farquharson 1970), outcomes are restricted to the *uncovered set*. The uncovered set may be defined in several equivalent ways. Without considering the possibility of ties, two of the most useful definitions are below:

(a) The set of alternatives for which there is no other alternative that beats it and everything it beats (i.e. "covers it").
(b) The set of alternatives which can defeat each and every other alternative either directly, or indirectly at one remove, by beating something which beats the other (Miller 1980).

However, not all elements in the uncovered set need be feasible outcomes when we have sophisticated voting under SAP with a given agenda. Banks (1985) was able to precisely delimit the set of feasible outcomes of sophisticated voting under SAP (or any other king-of-of-the hill variant) by specifying a subset of the uncovered set, initially referred to as M*, which has come to be called the *Banks set* in his honor (Miller et al. 1990). For finite voting games of a large enough size the Banks set may be a proper subset of the uncovered set (Moulin 1986).

Footnote 1 continued
heart (Schofield 1995). In addition, spatial versions of well-known solution concepts such as the *Borda winner* (Black 1958; Saari 1994), the *Copeland winner* (Straffin and Philip 1980), the *uncovered set* (Fishburn 1977; Miller 1980; Moulin 1986), and the *Banks set* (Banks 1985; Miller et al. 1990; Banks et al. 2002), have been identified (Feld and Grofman 1988a,b; Owen and Shapley 1989; Shepsle and Weingast 1984; McKelvey 1986; Cox 1987; Hartley and Kilgour 1987; Feld et al. 1987; Penn 2006a,b; Godfrey et al., forthcoming; Feld et al., forthcoming)· And other less well-known concepts that are part of the geometric structure of majority rule, such as the *Schattschneider set*, the locus of all possible multidimensional medians (Feld and Grofman 1988b), and the *finagle circle*, the smallest circle that is a *Von Neumann-Morgenstern externally stable solution set* (Wuffle et al. 1989), have also been proposed. However, until quite recently, most of these concepts lacked direct applicability in games with more than a limited number of (weighted) voters because of the lack of computer algorithms to identify the relevant geometry. That situation has begun to change dramatically due to the work of Bianco, Sened, and colleagues (Bianco et al. 2004, 2006). These authors have developed algorithms for finding the *yolk* and the *uncovered set* for large n data sets, and they have illustrated the remarkable power of the uncovered set as a solution concept for experimental majority rule voting games, and as a means to elucidate the nature of historical changes in the structure of legislative voting in the US. Congress. Building on the work of Bianco and Sened, and the seminal theoretical essays that preceded them, Godfrey (2007) has developed a computer package that can locate virtually all the constructs previously identified for the case of spatial voting games using Euclidean or city-block preferences.

While there have been a number of papers that look at properties of the Banks set in finite voting games and compare its properties to those of other proposed tournament solution concepts (see Laslier 1997 for a thorough overview, and Moser 2008 for more recent work), almost nothing is known about the Banks set in the context of spatial voting. Indeed, great care is needed to define the Banks set in spatial voting games with a continuum of policies (Penn 2006b). This paper is an attempt to remedy that lack of knowledge.[2]

While Penn (2006b) found the Banks set analogue for a divide the dollar game involving three voters in terms of the two dimensional simplex representation of that game, all that has been known about the location of Banks set in the general case of majority rule spatial voting games is that, in such games as in finite voting games, the Banks set must be a (perhaps proper) subset of the uncovered set. But, since, until quite recently (Bianco et al. 2004, 2006; Godfrey 2005) no one knew how to find the uncovered set in spatial voting games except for the special case of three voters (Feld et al. 1987; Hartley and Kilgour 1987); and even in the three voter case it was not known for certain what portions of the uncovered set were also in the Banks set,[3] it is reasonable to say we knew essentially nothing about the Banks set for spatial voting games.

Here we go on a quest to locate the Banks set in majority rule spatial voting games. To avoid knife-edge complexities, we assume an odd number of voters and that there are no two voters with identical locations. While we still have not quite pinned down the exact location of the Banks set we have made considerable progress in majority rule spatial voting games involving *Euclidean preferences*, i.e., circular indifference curves.

Our analyses build on the *Schattschneider set*, which is the geometric locus of all multidimensional medians—and thus represents the potential for agenda manipulation through the creation of orthogonal single dimensional committee jurisdictions (Feld and Grofman 1990)—by beginning with the link between it and the Banks set, and then extending from that insight.

- We prove that all point*s* in the *Schattschneider* s*et* must be in the Banks set.
- We then define a new geometric construction, the *tri-median set*, which contains the Schattschneider set, and is, in turn, contained in the Banks set; and we show that, in a three voter situation, the tri-median set is the interior of the space bounded by the Schattschneider set plus the perimeter that consists of Schattschneider points.
- We define another new geometric construction, the *Banks line set*, which contains the tri-median set, and we show that it, in turn, is contained in the Banks set. We offer a simple sequential geometric construction to find the *Banks line set* in the three voter case, and offer methods which can be extended to the general case using computer-based search.
- In a three voter situation, we show that the Banks line set is a proper subset of the entire Banks set

[2] Companion essays by the present authors, such as one on the Shapley–Owen value (Owen and Shapley 1989), forthcoming in a volume co-edited by Maurice Salles honoring the contributions of Moshe Machover and Daniel Felsenthal, and one on the spatial Copeland winner, forthcoming in *Political Analysis*, look at the geometric structure of (and computer programs to calculate) other important spatial solution concepts.

[3] Feld et al. (1987) conjectured that, in the three voter case, the Banks set and the uncovered set were one and the same, but their proof of this was informal.

In the next section we provide the necessary formal apparatus for these results. In the first subsection of Section II we provide a glossary for our most important notation. In the succeeding subsection we provide an overview of some basic geometric and game theoretic concepts, drawing heavily on the framework provided in Owen and Shapley (1989). The reader who is already familiar with spatial modeling ideas can skim through this subsection. In the third subsection we provide a formal definition of the Banks set for the case of spatial voting games, dealing with some important technicalities about how to extend the definition from the finite to the infinite case. In the fourth subsection we offer the material that is original with this article, in which we specify the geometric constructions that are (nested) subsets of the Banks set. We suggest that a useful way to quickly understand the contributions of this fourth subsection is to look at the five statements denoted as theorems, and then to examine the figures, especially Figs. 1, 2, 3, for the location of the Schattschneider set; Figs. 6 and 7, for the location of the tri-median set (and the comparisons with the Schattschneider set); Fig. 8, for the construction to identify the Banks line set in the three voter case; and Fig. 9, to see how much of the Banks set is occupied by the Banks line set for the case of three voters.

2 Results and geometric constructions

2.1 Notation and glossary

A	set
$\|A\|$	cardinality of set A
2^A	power set, set of all subsets of A
∂A	boundary of set A
Hull(A)	the smallest convex set containing A, i.e., the convex hull
{a,b,c}	set with explicit elements a,b,c
{x|Cx}	set of elements x for which the condition Cx is true, C ∼ predicate
∅	empty set
R	real numbers
R^2	R xR ∼ real plane
[0, 1]	closed unit interval on R
(0,1)	open unit interval on R
A	point, vector, or matrix variable, or a set with one element ≠ ∅
A + B	vector addition (note: points are vectors)
(x,y)	tuple (not be confused with an interval)
a,b,c	scalar variables, or elements of a set
iff	"if and only if"
d(X,Y)	metric, distance between X and Y
AB	line segment with endpoint A and B
F()	functions: F: $A \to B$, F maps elements of A to elements of B
E	Euclidean plane, i.e., R^2 as a vector space with Euclidean metric
N	set of voters

P	set of ideal points (associated with voters)
U	unit directional vectors, U(θ)
L	set of lines
M	set of median lines
(A,t)	an *agenda* (A,t) is a subset $A \rightarrow E$ with order parameter t: $A \rightarrow [0,1]$
S	Schattschneider set
T	tri-median set
B	Banks set

2.2 Basic geometric and game theory concepts

In the following discussion we have highlighted the most important assertions and definitions as consecutively numbered items. Previously known results are generally identified as Propositions, abbreviated Prop. Major results new to this paper are designated as Theorems (Thm.) or Corollaries (Cor), with ancillary results designated as Lemmas. All other numbered statements are either Definitions, abbreviated Def., or key Assumptions.

In this subsection we identify the basic elements of spatial voting games and define terms such as win set and median line.

Following Owen and Shapley (1989), we take as primitive a finite collection of individuals or voters, N, $|N| = \mathrm{n}$, and a collection of subsets W of N, $W \subseteq 2^N$, the *winning* coalitions, such that $\emptyset \notin W$ and $N \in W$. A *losing* coalition is a subset S of N such that $S \notin W$. The structure $V = (N, W)$ is regarded as an *abstract voting game*.
Def. A *proper* voting game satisfies

(1) If $S \in W$ and $T \in W$, then $S \cap T \neq \emptyset$
(2) If $S \in W$ and $S \subset T$, then $T \in W$.

Def. A proper voting game is *decisive* if

(3) For all $S \in N$ either $S \in W$ or $N - S \in W$

Def. Let $<<$ be a strict linear ordering of N. For i,j$\in N$, let $Q(\mathrm{i},<<) = \{j | j << i\}$.

(4) i$\in N$ is a *pivot* for $<<$ iff $Q(\mathrm{i},<<) \notin W$ and $Q(\mathrm{i},<<) \cup \{\mathrm{i}\} \in W$

Define a *spatial* voting game (of dimension 2) as the triple (N, W, P), where (N, W) is an abstract voting game and P is a set of n *points* in E, $P = \{\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_\mathrm{n}\}$, called *ideal points* (of the voters).

In this paper all definitions and theorems apply to a 2-dimensional Lebesgue measureable, Euclidean metric space, E, with metric d, and endowed with an inner product, i.e., the dot or scalar product of vectors. Note that E is equivalent topologically to R^2 (the real plane) and will be used interchangeably with R^2. This space represents policies, X, that can be represented by two independent, unbounded, real-valued continuous attributes, x and y, i.e., X = (x, y).

It will be useful to consider *unit directional vectors*, U(θ), i.e., U(θ) = (cosθ, sinθ), with inner product satisfying $<\mathrm{U},\mathrm{U}> = 1$. Each unit vector induces an ordering on N according to i $<<$ j iff $<\mathrm{U},\mathrm{P}_\mathrm{i}> \leq <\mathrm{U},\mathrm{P}_\mathrm{j}>$.

Owen and Shapley (1989) prove several propositions regarding U that will be relevant to our consideration of the Banks set:

Propositions:

(5) The set of U such that the order induced by U is *not* strict has measure zero
(6) The vectors U and—U induce reverse orders
(7) The set of U inducing a given strict order $<<$ is open

Following Owen and Shapley we posit
Assumptions:

(8) The game (N, W) is decisive
(9) The points P are distinct: $P_i = P_j \Rightarrow i = j$ (or equivalently $|N| = |P|$)
(10) $|P| > 2$ and odd

Assumption (9) simplifies our analysis considerably. In practice, due to limitations in scale resolution, coincidence is a common feature of ideal point data sets. There are two choices to handle this. One is to allow such coincidence in our definitions and theorems, resulting in a more awkward and harder to follow analysis. Or we can posit that ideal points in any data set are separated by some unobservable distance *epsilon*. For our analysis the identification of ideal points with specific "named" agents is not needed, and so in this paper we adopt the later approach.[4]

Given two policies, X and Y, Owen and Shapley distinguish three types of points in E

Definitions:

(11) $T_w(X,Y) = \{j|d(P_j, X) < d(P_j, Y)\}$
(12) $T_w(Y,X) = \{j|d(P_j, Y) < d(P_j, X)\}$
(13) $T_e(X,Y) = \{j|d(P_j, X) = d(P_j, Y)\}$

These sets are disjoint and jointly encompass E. Owen and Shapley offer the following construction to understand these three sets.

Let MM' be the perpendicular bisector of XY. Then $T_w(X,Y)$ consists of those j such that P_j is on the *same side* of MM' as X, i.e., those j who *prefer* X to Y, $T_w(Y,X)$ of those j with P_j on the same side as Y (the *other side* of MM' from X), i.e., those j who prefer Y to X, and $T_e(X,Y)$ of those with P_j *on* the line MM', i.e., those j who are *indifferent* between X and Y.

If $T_w(X, Y) \in W$, then X *dominates* Y, $X > Y$. If both $T_w(X,Y)$ and $T_w(Y,X)$ are losing coalitions, then there is a *standoff* between X and Y, $X \sim Y$. Again, Owen and Shapley prove several propositions relevant to our work here

Propositions:

(14) For given X, the set of Y for which $T_e(X,Y) \neq \emptyset$ has Lebesgue measure 0.
(15) If $X \sim Y$, then $T_e(X,Y) \neq \emptyset$
(16) For a given X, E can be partitioned into three (disjoint) sets

$\mathrm{Dom}(Y) = \{Y|X > Y\}$

[4] In general, however, computational algorithms must be crafted so as to be able to deal with the possibility of several coincident agents (or of single actors with non-identical weights).

Def(X) = {Y|Y $>$ X}
Stand(X) = {Y|X $\sim$ Y}
with Stand(X) having measure 0 and Def(X) having a finite measure.

Another name for Def(X) is the *win set* of X, W(X), with Stand(X) being the *boundary* of W(X), written as ∂W(X). We introduce R(X) = W(X) $\cup\partial$W(X), i.e., the *closure* of W(X).

The case of simple majority rule will be of particular interest, and will be assumed hereafter unless otherwise indicated.

Definition:

(17) X $>$ Y by *simple majority rule* iff $\mathrm{T_w}$(X,Y)$\in$ W and $|\mathrm{T_w}\,(\mathrm{X,Y})\,| \geq (|\,N\,|+1)/2$

Observe that if X $\sim$ Y, then $\mathrm{T_e}(\mathrm{X,Y}) \neq \emptyset$. This follows by simple counting. Since both $\mathrm{T_w}$(X,Y) and $\mathrm{T_w}$(Y,X) are losing coalitions by definition we have:

$$|\mathrm{T_w}(\mathrm{X,Y})| < (|N| + 1)/2 \text{ and } |\mathrm{T_w}(\mathrm{Y,X})| < (|N| + 1)/2.$$

In fact, since $|\mathrm{T_w}(\mathrm{X,Y})|$ and $|\mathrm{T_w}(\mathrm{Y,X})|$ must be integers (voters are indivisible), we can assert a stronger bound

$$|\mathrm{T_w}(\mathrm{X,Y})| < |N|/2 \text{ and } |\mathrm{T_w}(\mathrm{Y,X})| < |N|/2$$

And so

$$|\mathrm{T_w}(\mathrm{X,Y})| + |\mathrm{T_w}(\mathrm{Y,X})| < |N|$$

But

$$|\mathrm{T_w}(\mathrm{X,Y})| + |\mathrm{T_w}(\mathrm{Y,X})| + |\mathrm{T_e}(\mathrm{X,Y})| = |N|$$

And so

$$|\mathrm{T_e}(\mathrm{X,Y})| > 0.$$

Observe that since P is finite and two points define a line, there is a finite set of lines between pairs of points in P. This set, being finite has measure 0.

An especially important object in spatial voting games is a *line* (formally defined by (20) below) that separates voters into two equal groups, modulo voters on the line, called a *cut* or *median* line:

Definition.

(18) A *median line*, L_{M} is a line in E such that for X $\sim$ Y: $L_{\mathrm{M}} \cap \mathrm{T_e}(\mathrm{X,Y}) \neq \emptyset$.

Except for a set of measure 0 a median line contains only one voter. If more than one voter is present on the median line this number must be odd, as $|P| \sim$ odd. Also, observe that at least one median line always exists.

Proposition.

(19) In a spatial voting game the set of median lines, M, is not empty, $M \neq \emptyset$.

The proof that $M \neq \emptyset$ introduces tools that will be of later use and so is elaborated here.
Def. Given a point P in E and a unit vector U, a *line* is defined as a set of points L:

(20) $L = \{X| < \mathrm{U}, \mathrm{X} - \mathrm{P} >= 0\}$,

i.e., U is *orthogonal* to the vector $X - P$, written, $U \perp X - P$, and is called the *normal* (to the line). Two lines are orthogonal if their normals are orthogonal.

The parametric definition of a line is often useful. Given two points on a line, P and Q, with P $\neq$ Q, let $D = P - Q$, then any other point on the line can be found using

X(t) = P + tD, where t is in R. Note that U $\perp$ D.

We will refer to D as the *line directional vector* and t as the *order parameter*. Note: an *open line segment* is: $\{\mathrm{X}(\mathrm{t}_1), \mathrm{X}(\mathrm{t}_2)\} = \{\mathrm{X}|\mathrm{X}(\mathrm{t}) = \mathrm{P}+\mathrm{tD};\ \mathrm{t}_1 < \mathrm{t} < \mathrm{t}_2\}$. The segment is closed if "<" is replaced with "$\leq$".

Let U be a unit directional vector, and P an arbitrary point in E. Associated with U and P is a line, L_U, with normal $\mathrm{U}_\perp$, such that $\mathrm{U}_\perp \perp$ U, i.e., $L_U = \{\mathrm{X}| < \mathrm{U}_\perp, \mathrm{X} - \mathrm{P} >= 0\}$. Recall that
Prop. U induces an ordering $<<$ on N according to

(21) i $<<$ j iff $<\mathrm{U},\mathrm{P}_\mathrm{i} > \leq < \mathrm{U},\mathrm{P}_\mathrm{j} >$.

Since we are considering simple majority rule with $|N|$ odd, there will be a pivot according to (4) and (5), except for a set of measure zero. Since the set of non-strict U has measure zero we can avoid such U by a small perturbation (or simply starting all over). Therefore, assume the order induced by U is strict. Let the pivot be labeled as m, for the *median* (voter), P_M. Next, consider the line defined by $L_{U\perp} = \{\mathrm{X}| < \mathrm{U}, \mathrm{X} - \mathrm{P}_\mathrm{m} >= 0\}$. $L_{U\perp}$ is orthogonal to L_U, in the sense that $\mathrm{U}_\perp \perp$ U and passes through P_M. If it were to pass through another ideal point, this would contradict the strictness of U (and so we would perturb U to make it strict). This line is the median line we seek. To see that it is a median line, let X = P_i for m $<<$ i and Y = P_j for j $<<$ m. Then T_e(X,Y) = j and both T_w(X,Y) and T_w(Y,X) are losing coalitions, i.e., X $\sim$ Y and $L_\mathrm{M} \cap \mathrm{T}_\mathrm{e}(\mathrm{X},\mathrm{Y}) = \mathrm{j} \neq \emptyset$. □

2.3 Defining the Banks Set for Spatial Voting Games

We turn now to defining the Bank set and structures specifically relevant to the Banks set.

Definition

(22) An *agenda* (A,t) is a subset $A \subset E$ with order parameter t: $A \rightarrow [0,1]$.

Note that the agenda is assumed to include both endpoints, i.e., the agenda is compact. Elements of A are said to be *on* the agenda; elements of A^c are *off* the agenda. Penn requires $|A|$ be finite. We do not. In those cases where $|A|$ is countable or finite we shall refer to the agenda as a *countable* or *finite agenda* and the order parameter can be defined in these cases as t: $A \rightarrow \{1, 2, \ldots, |A|\}$ where $|A|$ is either finite or countable. Agendas that are not countable will be referred to as *continuous*. Consideration of the continuous case is needed in this paper when defining certain agendas, i.e., Banks lines, to ensure that no policies outside the agenda defeat all policies within the agenda.

Definition.

Individual elements of A on the agenda are represented as A(t), with a strict order $<<$ (not to be confused with dominance, $<$) defined by

(23) A(t) $<<$ A(s) when t $<$ s.

Elements of a countable agenda are evaluated by *strict successive elimination*: given $t_j > t_i$, if $T_w(A(t_j), A(t_i))$ is a winning coalition, then $A(t_j)$ dominates $A(t_i)$, and $A(t_i)$ is eliminated. Otherwise, $A(t_j)$ is eliminated, i.e., ties favor the status quo. The process begins by comparing A(0) with A(1) and then considers in order $t = 2, 3, \ldots, |A|$. Note that this definition differs from Penn's *successive elimination* in which ties favor the challenger. In Penn (2006b), polices are ordered by the weak order R(X), i.e., the closure of the win set W(X) not by the strict order W(X). We have added the qualifier *strict* to underscore this difference from Penn.

Strict successive elimination cannot be realized (i.e., is not computable) for continuous agendas. Starting with A(0), successive elimination requires the *next* A(t) be selected and T_w(A(0),A(t)) evaluated; but there is no *next* t. The real interval [0,1] is dense. Between any two numbers in [0,1] another can always be found. Nevertheless for any s and t, s $<$ t we can determine whether T_w(A(t),A(s)) is a winning coalition. We will be interested in agendas for which T_w(A(t),A(s)) is a winning coalition whenever s $<$ t, i.e., these will be what we will later define as *Banks chains*, with A(1) in the *Banks set*.

Definition.

(24) A *sophisticated agenda* (A,t) is an agenda such that A(t) $>$ A(s) iff t $>$ s.

Penn refers to sophisticated agendas as sophisticated*equivalent* agendas, in so far as amendments to policies on the agenda defeated by policies of the agenda are ignored. Penn, of course, also restricts consideration to finite agendas, where we do not. And, finally, in Penn *weak dominance*, A(t) $\geq$ A(s), is used where we require (strong) *dominance* A(t) $>$ A(s), i.e., the status quo cannot be defeated by a tie.

A sophisticated agenda is considered *stable* if there is no policy off the agenda that dominates all policies on the agenda, i.e., if A $>$ A(t) for all t: $0 \leq t < 1$, then A = A(1), i.e., A$\in$ A, The element A(1) of a sophisticated agenda is called the *maximal element*.

These preliminary definitions and assumptions now allow us to state a key definition of our paper.

Definition.

(25) A *Banks chain* is a stable sophisticated agenda.

With this definition in hand, we can now define the *Banks set, B,* as the set of all policies that are the maximal element of a Banks chain in a spatial voting game,[5] i.e.,

Definition.

(26) Given (N, W, P), $B = \{\mathrm{X}|\mathrm{X} = \mathrm{X}(1), (\mathrm{X}, \mathrm{t}) \sim \text{a Banks chain}, \mathrm{X} \in E\}$

While the Banks set has been identified for various finite cases an explicit description of the Banks set in the spatial setting is unknown. Also unknown is a general method for constructing the Banks set in the spatial context. We do know, however, that the (finite) Banks set is a subset of the uncovered set, as defined by Miller (1980, see also Penn 2006a).

We now observe that the Banks set, B, as defined here for the infinite case, must also be a subset of the standard definition of the uncovered set (Miller 1980), under strict successive elimination. The reason is that a covered policy, X, is one whose win set, W(X), is dominated by a policy Y, with W(Y) $\subset$ W(X). The containment is strict since win sets are open under strict order. Thus if there were a policy X $\in B$ that was covered, it would be dominated by a policy Z $\in$ W(Y), i.e., X would not be the limit point of a Banks chain.[6]

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[5] Penn (2006b) defines the Bank set by weakly specifying the functional form of voter preferences as a quasi-concave, continuous utility function. As noted earlier, we have chosen in this paper to restrict this utility function further to the Euclidean metric, d.

[6] Care must be taken when considering the relationship of the Banks set with the uncovered set, as there are alternate definitions of each in the literature (Penn 2006a). In fact, our definition of the Bank's set differs slightly from Penn's in regard to how standoffs are resolved. Penn's definition of the Banks set assumes closed win sets whereas our definition assumes open win sets. We are compelled to use open win sets for a technical reason. With a continuous agenda and closed win sets it would be possible to get stuck in a cycle between two policies residing on the boundary of the win set. With open win sets, a standoff favors the status quo, so that in the back-and-forth procedure we use to demonstrate that any Schattschneider point is Banks point (see subsection immediately below), we are always advancing in toward the Schattschneider point. If the win sets were closed, i.e., standoffs favors the challenger, the agenda could get stuck on the win set boundary - bouncing back-and-forth between the same two policies. Requiring the win sets to be open forces each successive agenda item to be infinitesimally closer to the relevant Schattschneider point than the previous agenda item. Although we are compelled to assume open sets, we also believe this to be the more natural assumption. The assumption of open or closed win sets devolves to assumptions about how voters behave under indifference. Recall that a win set is constructed by considering the indifference curves of voters. There are two alternatives for how a voter will vote in regard to policies on his/her indifference curve: "yes" and "no." There are then three cases (models) for how groups of voters might behave: (1) all vote "yes", (2) all vote "no", (3) votes that are a mix of "yes" and "no". Penn assumes case (1). We assume case (2). The set of policies preferred by a voter is closed under case (1) and open under case (2). The resulting win sets are then correspondingly closed or open. Voters, as human beings, can chose to comply with any of these models and still be "rational." However, case (2) seems to us more naturally aligned with the meaning of an "indifference" curve.

For finite agendas, or even countable agendas, the distinction between open and closed win sets does not appear to be significant, i.e., the boundary has zero measure. In the case of continuous agendas, however, the open vs closed model of win sets could substantially impact the specification of the Bank's set. We require a continuous agenda in our proof to ensure that no point off the agenda dominates all points on the agenda. The specific move is to assume such a point, X, and then consider its projection on the agenda, X'. If the agenda is continuous X' is guaranteed to exist and to dominate X. We can weaken our argument by considering geometries based on algebraic fields. One classic example, well-adapted to concerns of spatial voting, are the (quadratic) "surds," a field-extension of the integers obtained by allowing any order of square root. Since the integers are countable and the process of extracting square roots to any order is a countable process, the resulting field is itself countable and dense, i.e., between any two surds one can always

2.4 Geometric constructs that are subsets of the banks set in spatial voting games

We now set about identifying significant subsets of the Banks set, in our effort to more explicitly locate this elusive object in the spatial context.

We begin by introducing the Schattschneider set (Feld and Grofman 1988b).[7] A *Schattschneider point* is the intersection of two orthogonal median lines

(27) *Def.* Given $L_1. L_2 \in M$, such that $L_1 \perp L_2$, $\mathrm{S} = L_1 \cap L_2$, is a *Schattschneider point.*

Def. The *Schattschneider set, S,* is the set of all Schattschneider points, i.e.,

(28) $S = \{\mathrm{S} = L_1 \cap L_2 | L_1. L_2 \in M,\ L_1 \perp L_2\ \}$

A key claim is that this set is not empty.

(29) *Proposition*. In a spatial voting game the Schattschneider set, S, is not empty, i.e., $S \neq \emptyset$.

The strategy of the proof is to pick any line, L_1. Project the ideal points on to L_1. Next, find the median among these points. Consider the line normal to the first line passing through this median, L_2. Then project ideal points on to L_2, find the median and consider a third line normal to this second line, L_3. L_2 and L_3 is each a median line, and so their intersection is a Schattschneider point. Since the point can be explicitly found starting from any initial line, L_1, the Schattschneider set is never empty, given an odd number of ideal points > 2.
Proof:

Pick an arbitrary point, P. Let $L_1 \sim < \mathrm{N}_1, X - P >$ be some line in E containing P. Project on to L_1 the ideal points of E. Specifically, let Y be an ideal point with orthogonal projection $\mathrm{Y}_\perp$ on L_1, i.e., $< \mathrm{P} - \mathrm{Y}_\perp, \mathrm{Y} - \mathrm{Y}_\perp > = 0$. Note that $L_1 \sim < \mathrm{Y}_\perp - \mathrm{P}, \mathrm{N}_1 >$ which can be expressed parametrically as $\mathrm{P(s)} = \mathrm{Y}_\perp + \mathrm{s}(\mathrm{P} - \mathrm{Y}_\perp)$. Let N_2 be a vector normal to N_1. Define a line, $L = < Y - \mathrm{Y}_\perp, \mathrm{N}_2 >$ or, parametrically $\mathrm{P(t)} = \mathrm{Y}_\perp + \mathrm{t}(\mathrm{Y} - \mathrm{Y}_\perp)$. At the point of intersection $P(s) = P(t)$:

$$\mathrm{s}(\mathrm{P} - \mathrm{Y}_\perp) = \mathrm{t}(\mathrm{Y} - \mathrm{Y}_\perp) => (\mathrm{t} - \mathrm{s})\mathrm{Y}_\perp = \mathrm{tY} - \mathrm{sP}$$

For $\mathrm{t} \neq \mathrm{s}$: $\mathrm{Y}_\perp = (\mathrm{tY} - \mathrm{sP})/(\mathrm{t} - \mathrm{s})$. For t = s, we would have P = Y which is impossible unless Y is on the line L. Repeat this computation for all ideal points. Next, using s as an order parameter, rank the projected ideal points on L_1. Let M_1 be the median

Footnote 6 continued
find another surd. In the Euclidean plane, if one starts with two given points, these points defining a unit interval, and then admits any other point attainable through an (unmarked) ruler and compass construction, the resulting set of points defines the "surd plane." (Algebraically, the intersection of lines and circles is quadratic expression.) The surd plane is dense and countable. We may then consider a Bank's line formed from surds. Given a surd point off the line one can, by ruler and compass, construct its projection on the line. Since the line is a surd agenda, the projection will exist and dominate. Furthermore, having open win sets, the fact that the surds are dense allows one to implement the back-and-forth procedure with each successive point being (infinitesimally) closer to the Schattschneider point.

[7] The Schattschneider set was named for the political scientists E. E. Schattschneider by Feld and Grofman (1988b) because his work (Schattschneider 1968), though completely non-technical, demonstrated the importance of cleavage lines in defining the structure of winning and losing coalitions.

among these points. Consider the line, L_2, $P(r) = M + r(M - \mathrm{Y_M})$, where Y_M is an ideal point with projection $\mathrm{Y_M}$ on L_1. This line is normal to $P(s) = \mathrm{Y}_\perp + s(P - \mathrm{Y}_\perp)$. And is a median line, by construction. Now, project on to L_2 the ideal points of E. Rank these points by the order parameter r and select the median among these projected points on L_2, S. Let Y_S be an ideal point with projection S on $L_{2.}$ The line L_3, P(q) $=$ S $+$ q($\mathrm{Y}_S -$ S) is itself a median line, by construction. Thus starting from an arbitrary line L_1, we have found two median lines, L_2 and L_3, normal to each other, intersecting at S. Thus S is in S and so $S \neq \{\emptyset\}$. □

(30) *Cor*. Every median line contains a Schattschneider point.

Proof:

Let L be a median line. Project on to L the ideal points P. Using the order parameter of L, rank these points. Select the median, X, and consider the line $L_\perp$ passing through X and orthogonal to L. The line $L_\perp$passes through the image of X, an odd number of points in P, per remarks following (18). Whence $L_\perp$ is also a median line and X is therefore a Schattschneider point. □

Def. A useful construct is the *Pareto set* of a spatial voting game. It is defined as the minimal convex set that contains the set of points P, often called the *convex hull* of P.

(31) The Pareto set of the spatial voting game, (N, W, P), is the minimal convex set of P, Hull(P).

We may use some simple geometry to find the locus of the Pareto set. Given P and a median line, M, project P onto this line. Rank these projected points according to the order parameter of M. The projected points with least and highest rank define a line segment that contains all points of P projected on to M. The points of P corresponding to the projected points on M with highest and lowest rank are on Hull(P). By considering all possible median lines we determine the points of P on Hull(P). An algorithm that implements this idea efficiently is the Graham Scan (Graham 1972).

From this construction we can readily ascertain that a Schattschneider point is within Hull(P). If it were not, the Schattschneider point could not be the intersection of two orthogonal median lines. A median line, M, cuts P in half, i.e., $P = P_1 \cup P_2 \cup (M \cap P)$ with $P_1 \cap P_2 = \emptyset$, $P_1 \cap M = \emptyset$, and $P_2 \cap M = \emptyset$. Let S be the median on such a line. If S were outside Hull(P) it would not be possible to construct a second median line M', orthogonal to M and passing through S.

We use the Pareto set as a "reference frame" when discussing a spatial voting game. Its principal axes determine the "scale." In the two-dimensional setting, we can then speak of points *far* from the the Pareto set as points whose distance from the Pareto set frontier is much larger than the length of either principal axis.

We next show that all Schattschneider points must be in the Banks set. This is the first key theorem of the paper.

(32) *Thm 1.* $S \subset B$.

Proof:

We begin by defining a continuous sophisticated agenda, with the Schattschneider point, S, defined by two orthogonal median lines M and N, dominating all elements of the agenda. Select a point X_0 on M far from the Pareto set. Recall that

Def(X) = {Y|Y > X}. Consider Def(X)0). Observe that $M \cap$ Def(X_0) is an open set of M containing S, i.e., Def(X)0) is open and S is the median on M. Consider the limit points of $M \cap$ Def(X)0), X_0 and X'_0, i.e., X'_0 is the reflection of X_0 on M across N. Observe that $d(X_0, X'_0) = 2d(X_0,S)$. Select a point $X_1 \in M \cap$ Def(X)0) a small distance ε from X'_0, i.e., $\varepsilon << d(X_0,S)$. Observe that $X_1 \in$ Def(X)0). Next consider the reflection of X_1 on M across N, X'_1. Note that X'_1 is a limit point of Def(X)1). Using the same ε, select a point in Def(X)1) a distance ε from X'_1. Observe that $X_1 \in$ Def(X)0). After some finite $n \sim d(X_0,S)/\varepsilon$ we select a point X_n such that $d(X_n,S) \leq \varepsilon$. Observe that we have defined a sophisticated agenda with $X_i < X_j$ for $i < j$. If we include S in this agenda, observe that $X_i < S$ for all i in the agenda. Now, let $\varepsilon \rightarrow 0$. In the limit when $\varepsilon = 0$, we have a continuous agenda with X < S for all X in the agenda, X ≠ S.

Next we show that S is a Banks point. Consider an arbitrary point Y not on M. Let Y' be the projection of Y on M. Since M is a median line, $Y' \in$ Def(Y), showing there exists an X on M, such that Y< X. If $Y' \in \{X_0, X'_0\}$ we are done. If $Y' \notin \{X_0, X'_0\}$ we need to find X in $\{X_0, X'_0\}$ such that $X \in$ Def(Y). Observe that since N is a median line, the reflection of Y' on M across N, Y'' also dominates Y. But $\{X_0, X'_0\} \subset \{Y', Y''\}$, and so X > Y for any $X \in \{X_0, X'_0\}$. Thus S is the only point such that for $X \in \{X_0, X'_0\}$, X ≠ S, S >X. □

The intuition underlying this proof is important to grasp, since it is easy to lose sight of it amidst the formalism. The proof strategy consists in constructing an agenda along the first of two orthogonal median lines in which successive amendments are on opposite sides of the second median line with each amendment closer to the Schattschneider point S than previous amendment. In this way each amendment is seen to defeat all prior amendments. The difficulty is that the continuum limit must be taken and in so doing the notion of a "next" amendment "closer" to S than the previous amendment becomes meaningless. The proof must overcome this challenge, and this fact accounts for the technical complexity of the proof.[8]

We can summarize the logic of the proof as follows. By definition, if S is a Schattschneider point, then it lies at the orthogonal intersection of two median lines, M and N. Consider a trajectory over the line M that begins far off in one direction and alternates back and forth across S, getting incrementally closer to S, and ending at S. Each such point must dominate all previous points because it is closer to the median line, N, which determines the majority in that direction. Then, we only need to prove that there is no other point that dominates all the points in this trajectory on M. Consider any arbitrary point, Q, off the line M. Because M itself is a median line, the projection of Q on M, Q', must dominate Q. Therefore, Q cannot dominate all points on the trajectory.

In the three voter case, it is known that a very simple construction based on overlapping circles defines the Schattschneider set (Feld and Grofman 1988b). All median lines pass through a voter ideal point. The locus of points of orthogonal intersection of lines passing through any two points is the circle through those two points around the midpoint between those two points. Therefore, the only points that can fall at orthog-

[8] An informal version of this proof is given in Feld and Grofman (1988b).

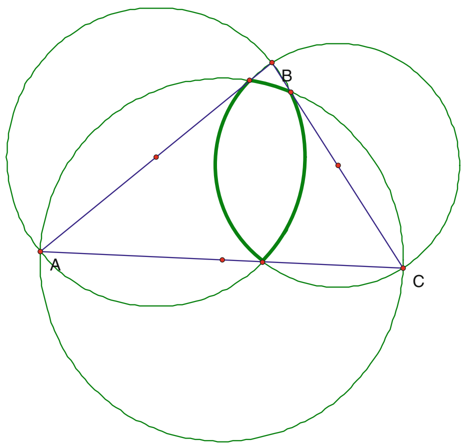


Fig. 1 Typical Schattschneider set construction in the three voter case

onal intersections of two median lines must fall on the three circles passing through pairs of voter ideal points centered between them, as shown in Fig. 1. Figure 1 shows three voter ideal points, ABC, the triangle created by lines connecting them, and the circles through pairs of points centered between them. Since only the lines through voter ideal points that pass through the triangle are median lines, then the only parts of the circles that are Schattschneider points are arcs the define what we may think of as an arc triangle. Those arcs connect with one another at the edges of the triangle as shown in Fig. 1.

Figure 1 shows a typical case. However, when one of the angles in the triangle is greater than $\pi/2$ (90 degrees), then the circle surrounding the line opposite it in the triangle is completely outside the triangle, and therefore contains no Schattschneider points. In that situation, the Schattschneider set is composed entirely of arcs from only two circles, as shown in Fig. 2.

We next illustrate the Schattschneider set geometric construction for more complex situations with more than three voters. Figure 3 shows the Schattschneider set of arcs for an arbitrary five voter example. The five voters are arrayed around the outside of the Pareto set. The Schattschneider arcs go from a to b to c to d to e to f and back to a. In general, the Schattschneider arcs can crisscross one another in all kinds of odd ways; However, they tend to be somewhat "central" to the Pareto set.

Fortunately, some of the seeming "strangeness" shown in the Schattschneider set construction in Fig. 3 becomes less problematic when we introduce a larger geometric construction that always includes the Schattschneider points, but replicates the property of the Schattschneider set of being contained within the Banks set.

An important fact we shall need is that, for a spatial voting game (N, W, P), with $|P| \sim$ odd, except possibly for voter ideal points, there is an odd number of median lines through any point in $E - P$. In fact, for points outside the Pareto set there is one and only one median line through each point in $E - P$. Inside the Pareto set this number can be > 1, but must still be odd when the number of lines is finite. The number can be infinite, but only for ideal points, i.e., points in P. In these cases the notion of odd or even is meaningless.

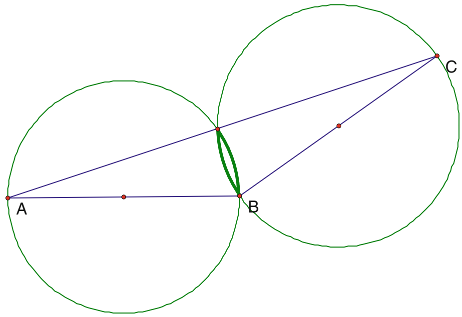
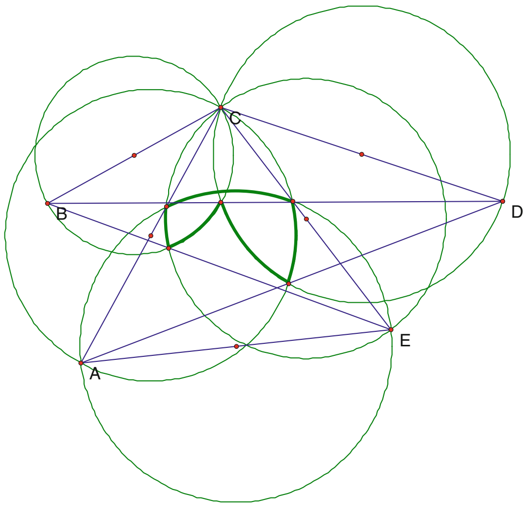


Fig. 2 Schattschneider set construction in the three voter case when one angle is more than ninety degrees

(a) construction showing circles



(b) same construction without the circles being shown

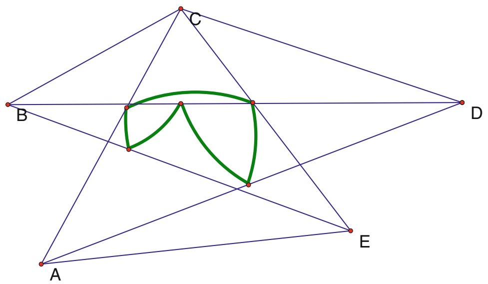


Fig. 3 Schattschneider set construction in the five voter case when all voters are on the boundary of the Pareto set **a** construction showing *circles* and **b** same construction without the *circles* being shown

The following proposition asserts that any point in E is contained in some median line. Under certain conditions a stronger claim is possible; these are presented as corollaries. The basic intuition is to take any point, X, treat it as the origin in a transformation to polar coordinates, and then realize that the ideal points P will be subtended by some arc. There will be one angle which divides P into two halves. This angle defines a median line that contains X. There may be more than one such angle when the angular measure of the arc is greater than π. Care must be taken to distinguish three cases, points outside the Pareto set (one median line), points inside the Pareto set but not ideal points (finite and odd number of median lines), and ideal points (possibly infinite number of median lines).

(33) *Proposition.* For all X$\in E$, there exists $L \in M$, $L = \{\text{Y}| < \text{N}, \text{Y} - \text{X} >= 0\}$

Proof:

Let X$\in E$. Select a polar coordinate system. Associate with each point Z$\in P$ an angle θ_Z. Rank the points Z by θ_Z. Let $P_M = \{Z|Z \in P \text{ and } \theta_Z = \theta_M \sim \text{median of } \theta_Z\}$, θ_M and the corresponding Z are unique with respect to X, except for a set of measure 0, i.e., the finite number of lines that contain X defined by pairs of points in P. Let U = ($\sin\theta_Z$, $\cos\theta_Z$), with N normal to U. Let $L = \{\text{Y}| < \text{N}, \text{Y} - \text{X} >= 0\}$. Let $P_< = \{Z|Z \in P \text{ and } \theta_Z < \theta_M\}$ and $P_> = \{Z|Z \in P \text{ and } \theta_Z > \theta_M\}$. Observe that $|P_<| = |P_>|$, with $|P_M| \sim$ odd. Thus: $L \in$ M. □

(34) *Cor.* For all X$\notin$Hull(P), X$\notin P$, $L \in M, L = \{\text{Y}| < \text{N}, \text{Y} - \text{X} > = 0\}$: $|\{L\}| = 1$

Proof:

Let X$\in E$ with X$\notin$Hull(P). Select a polar coordinate system. Associate with each point Z$\in P$ an angle θ_Z. Rank the points Z by θ_Z. Let θ_{max} = max $\{\theta_Z\}$; θ_{min} = min $\{\theta_Z\}$. Since X$\notin$ Hull(P), $\theta_{max} - \theta_{min} < \pi$. Only θ in $\theta_{max} > \theta > \theta_{min}$ is median. Hence $|\{L\}| = 1$. □

(35) *Cor.* For all X$\in$Hull(P), X$\notin$P, L$\in$ M, L $= \{\text{Y}| < \text{N}, \text{Y} - \text{X} >= 0\}$: $|\{L\}| \sim$ odd

Proof:

Let X$\in E$ with X$\in$ Hull(P) and X$\notin P$. Select a polar coordinate system. Associate with each point Z$\in P$ an angle θ_Z. Rank the points Z by θ_Z. Let θ_{max} = max $\{\theta_Z\}$; θ_{min} = min $\{\theta_Z\}$. Since X$\in$Hull(P), $\pi < \theta_{max} - \theta_{min} < 2\pi$. When $\theta_{max} \neq \theta_{min}$ more than one median line exists. However, each median line must separate voters into equal groups, modulo voters on the median line. Two voters cannot be on a median line as $|P| \sim$ odd. Therefore, there are n $\leq |P|$ distinct possible median lines, where n $\sim$ odd. Thus: $|\{L\}| \sim$ odd (and finite). □

(36) *Cor.* For X$\in P$ and $L \in M$ with $L = \{Y| < \text{N}, \text{Y} - X >= 0\}$: $|\{L\}| \leq \infty$

Proof:

Let X$\in E$ with X$\in$ Hull(P) and X$\in P$. Select a polar coordinate system. Associate with each point Z$\in P$ an angle θ_Z. Rank the points Z by θ_Z. Let $\theta_{max} =$ max $\{\theta_Z\}$; θ_{min} = min $\{\theta_Z\}$. As there is at least one median line through any point of E, let L be such a line with direction angle θ_L. Let $\theta_1$be the angle for the first point in P with $\theta_Z \leq \theta_L$, i.e., $\theta_1 \leq \theta_{L;}$ and let Let θ_2 be the angle for the first point in P with $\theta_Z \geq \theta_L$, i.e., $\theta_2 \geq \theta_{L.}$ Except for possibly a set of measure 0 when points in P are collinear, $\theta_1 < \theta_2$. Any line with direction angle $\theta_1 < \theta < \theta_2$ is then a median line. This set has positive, finite measure. Hence, in this case $|\{L\}| = \infty$. For those points in P, where $\theta_1 = \theta_2$, $|\{L\}| < \infty$. Thus: $|\{L\}| \leq \infty$.. □

In (36), the case where $|\{L\}| < \infty$ we shall see corresponds to ideal points with a Shapley-Owen value (Owen and Shapley, 1992) of 0. All other ideal points have non-zero Shapley-Owen value

The Schattschneider set is the building block of our subsequent constructions. The next geometric construct we introduce, the *tri-median set*, will turn out to have the property that it includes the Schatttschneider set but it is still included within the Banks set. Just as the Banks set consists of Banks points, the *tri-median set*, consists of *tri-median points*. A *tri-median point* is a point at the intersection of three median lines such that none of the three angles between the lines is $> \pi/2$, i.e., the angle between each normal is $\geq \pi/2$.

Definition.

(37) A *tri-median point* is a point T, such that T = $L_1 \cap L_2 \cap L_3$, where for i,j = 1,2,3 $L_i \in M$; $L_i = \{\mathrm{X}| < \mathrm{U_i}, < \mathrm{X} - \mathrm{T} >= 0\}$; $< \mathrm{U_i}, \mathrm{U_j} > \leq 0$, i $\neq$ j; $< \mathrm{U_i}, \mathrm{U_j} > = 1$, i=j.

Equivalently, no two of the angles defined by the intersection of these three median lines are less than $\pi/4$.

Figure 4 illustrates a hypothetical intersection of three median lines.

Definition.

The *tri-median set* is the set of all tri-median points.

(38) $T = \{\mathrm{X}\,|\mathrm{X} = L_1 \cap L_2 \cap L_{3;} L_i \in M; L_i = \{\mathrm{Y}| < \mathrm{U_i}, \mathrm{Y} - \mathrm{X} >= 0\}; < \mathrm{U_i}, \mathrm{U_j} > \leq 0$, i $\neq$ j; $< \mathrm{U_i}, \mathrm{U_j} > = 1$, i=j; i,j = 1,2,3}

A key claim is that this set is not empty.

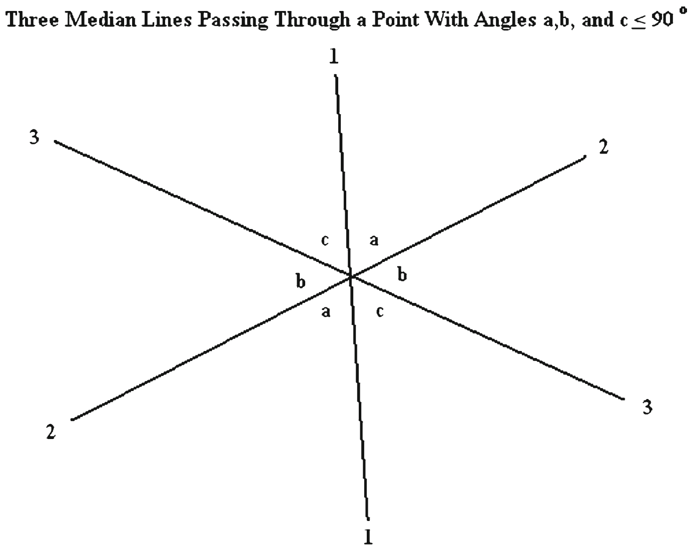


Fig. 4 Trimedian set elements

(39) *Lemma*. In a spatial voting game the tri-median set, T, is not empty, $T \neq \emptyset$.

The proof turns on the fact previously established that an odd or infinite number of median lines pass through any point.

Proof:

Let S be a Schattschneider point. Then two median lines pass through S. By corollaries (35) and (36) there must be at least one more median line, possibly an infinite number, through S, as this number must be odd or infinite. Thus S is a tri-median point point and so T is not empty, $T \neq \emptyset$. □

In fact, we have proven that the tri-median set contains the Schattschneider set.

(40) *Lemma*. $S \subset T$

Now we come to our next key result by showing that the tri-median set is contained in the Banks set.

(41) *Thm* 2. $T \subset B$

The proof follows the same strategy as proving containment of the Schattschneider set in the Banks set, but requires a bit more work.
Proof:

Let T be a tri-median point. Then there are at least three median lines, L_1, L_2, and L_3, interesting at T, T = $L_1 \cap L_2 \cap L_3$.

Let the angle between these lines be denoted by α_{12}, α_{23}, α_{31}. Since T is tri-median each angle $\alpha \leq \pi/2$, where $\alpha = \alpha_{12}$, α_{23}, and α_{31}. Observe also that $\alpha_{12} + \alpha_{23} + \alpha_{31} = \pi$. If any of $\alpha = \pi/2$ for any of α_{12}, α_{23}, and α_{31}, then T is a Schattschneider point and we have that it is a Banks point. Therefore, we assume $\alpha < \pi/2$ for each of of α_{12}, α_{23}, and α_{31}.

Next, consider a line orthogonal to L_1 passing through T, $L_\perp$. Let θ be the angle between $L_\perp$ and L_2, i.e., $\theta = \pi/2 - \alpha_{12}$. Consider a sophisticated agenda define on $L_\perp$ starting at some point far from the Pareto set and approaching T by alternating across L_1, each successive point being some small distance ε closer to T than the previous point in the agenda. As each point is closer to the median than the all previous points, it dominates all previous points. Taking the limit $\varepsilon \rightarrow 0$ we arrive at a continuous sophisticated agenda dominated by T. All that remains to show that T is a Banks point is to demonstrate that no other point than T dominates all points in the agenda.

Consider some arbitrary point V between$L_\perp$ and L_2. An exactly similar analysis applies to another point picked between L_1 and L_2 (and so will be omitted). Assuming a coordinate system in which $l_\perp$ follows the x-axis, Fig. 5 can be used to visualize these relationships, but the analysis does not depend on this specific representation.

Next we introduce various objects that will be used in the remainder of the proof. First, let U = $(\cos\theta, \sin\theta)$ be the direction vector for L_2. The equation for L_2 is then $X = T + \mathrm{T}_2 U$. Let L_2' be a line parallel to l_2 passing through V, given by equation $\mathrm{X} = \mathrm{V}'' + \mathrm{t}_{2'} U$. And finally let V′ be the projection of V onto L_2, the line passing through V and V′ being given by $X = V + \mathrm{T}_\perp \mathrm{U}^\perp$, where $U^\perp = (-\sin\theta, \cos\theta)$. Observe that $\mathrm{V}'' \sim \mathrm{V}$, as V″ and V are on the same line L_2', parallel to the median line L_2.

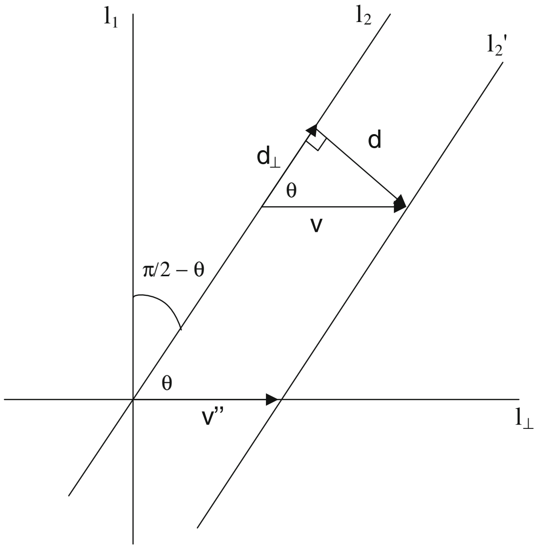


Fig. 5 Trimedian set construction

We now compute V'' and select a point on $L_\perp$ a small distance ε closer to L_2, thereby dominating V'' and so V, thereby showing that V does not dominate all points on $l_\perp$. Let $d(V', V) = d$. Then $V' = V + d\ U^\perp$. Observe that $V'' = V - V' + d\cot\theta$. Combining these equations $V'' = d\cot\theta\, U - dU^\perp = d(cot\,\theta\, U - U^{\perp)} = d(1/\sin\theta, 0)$, i.e., a vector of magnitude d/sinθ, directed along $L_\perp$. Let $\varepsilon < d/\sin\theta$. Consider the point $V^* = (d/\sin\theta - \varepsilon, 0)$. Observe that $V^* > V''$, by virtue of being closer to L_2. Hence $V^* > V$. This argument applies to any point V we might select, showing that no point other than T dominates all points on the continuous sophisticated agenda defined on $L_\perp$. Thus T is a Banks point. Finally, this argument can be applied to any tri-median point, showing $T \subset B$. □

We next examine the special case of three voter spatial voting games, (N, W, P), with $|P| = 3$. In such three voter games we can provide a precise a precise delineation of the tri-median set, by showing that the Schattschneider set, S, *and* all points interior to it make up the tri-median set, T. Discussion of the points interior to S is subtle, however, in so far as S can in general be a closed, self-intersecting curve. If it were not self-intersecting the Jordan Curve Theorem could be invoked to identify the set of points interior to S. Fortunately, for the case of three voters, S is not self-intersecting.

(42) *Lemma*. For (N, W, P), with $|P| = 3$, S is a closed, non-intersecting curve.

Proof:

The proof consists in observing that S consists of two or three arcs, each pair of arcs intersecting the triangle defined by P, i.e., $\partial\mathrm{Hull}(P)$. There are three cases, one where P defines an acute triangle, second when P defines an obtuse triangle, and third when P defines a right triangle, a limit case for either an acute or obtuse triangle. In the case that P is defines an acute triangle, there are three arcs. The intersection of each pair of arcs is determined as follows. Let L(X,Y) be the line containing points X and Y, i.e.,

$$< \mathrm{N}, \mathrm{Y} - \mathrm{X} >= 0, \text{ where } L = \{\mathrm{Z} | < \mathrm{N}, \mathrm{Z} - \mathrm{X} >= 0\}.$$

Begin by considering the acute case with points $\mathrm{P_i} \in P$, i=1,2,3. Consider the (limiting) median line $L(\mathrm{P_i}, \mathrm{P_j})$, i≠j, and let $L^{\perp}$ be the line orthogonal to $L(\mathrm{P_i}, \mathrm{P_j})$ passing through $\mathrm{P_k}$, k≠i, j. Since P defines an acute triangle, $L^{\perp}$ is also a median line, and so the intersection of L and $L^{\perp}$, P', is in S. Now consider a line L_{Δ} through $\mathrm{P_k}$ which differs from $L^{\perp}$ by some small angle $\Delta\,\theta$, where θ is the direction of $L^{\perp}$. Next find the orthogonal median line intersecting L_{Δ}. The point of intersection is in S and is interior to ∂Hull(P). Depending on the sign of$\Delta\,\theta$, the point will be on different arcs that comprise S. Each arc is traced out by repeating this construction for larger values of $|\Delta\,\theta|$ until the arc intersects $\partial\mathrm{Hull}(P)$. By symmetry and the fact $|P| = 3$, there are 3 such intersections with $\partial\mathrm{Hull}(P)$ and 3 arcs. These arcs, which comprise S, do not intersect except where they intersect $\partial\mathrm{Hull}(P)$.

Next consider the obtuse case. Let $\mathrm{P_k}$ be the the point whose angle in the triangle, i.e., $\partial\mathrm{Hull}(P)$ is obtuse. Then the line $L(\mathrm{P_i}, \mathrm{P_j})$, i≠ j ≠k is a (limiting) median line. Let $L^{\perp}$ be the line orthogonal to $L(\mathrm{P_i}, \mathrm{P_j})$ passing through $\mathrm{P_k}$, which is also a median line. The intersection of these lines, P', is in S and also $\partial\mathrm{Hull}(P)$. Now, from P' trace out two arcs of S following the same procedure as in the acute case. Observer, however, that these arcs cannot intersect $\partial\mathrm{Hull}(P)$ except at P' and possibly $\mathrm{P_k}$, i.e., lines orthogonal to the (limiting) median lines $L(\mathrm{P_i}, \mathrm{P_k})$ and $L(\mathrm{P_j}, \mathrm{P_k})$ intersect outside $\partial\mathrm{Hull}(P)$. Indeed these arcs must intersect at $\mathrm{P_k}$ as $\mathrm{P_k}$ is in S. Consider the (limiting) median line $L(\mathrm{P_i}, \mathrm{P_k})$. Because $\mathrm{P_k}$ is the oblique point of the triangle P, the line orthogonal to $L(\mathrm{P_i}, \mathrm{P_k})$ passing through $\mathrm{P_k}$ is also median line, showing that $\mathrm{P_k}$ is in S. Thus the two arcs comprise S.

The final case of a right triangle is simply a limiting case of either of the previous two cases, i.e., S is comprised of two arcs. Thus, S is a closed, non-self-intersecting curve. □

Now we are ready to prove, for the three voter case, that not just the Schattschneider set, S, but also all points interior to it, are contained in the tri-median set, T. And, since we know how to geometrically specify the locus of the Schattschneider set we are now further on our way to specifying (some of) the geometry of the tri-median set which, because it includes the Schattschneider set, will constitute a larger part of the Banks set, the geometry of which is our ultimate quest.

Let $C \subset E$ be a simple non-intersecting curve. Denote by int(C) the open set of points interior to C, and [int(C)] = int(C) ∪ C.

(43) *Thm 3*. For a spatial voting game (N, W, P), with $|P| = 3$, [int(S)] $\subset T$.

Proof:

From (40) we have $S \subset T$. So all we need to show now is int(S) $\subset T$. By (35) any point P in Hull(P) must be the intersection of an odd number of median lines. In fact, since $\partial\mathrm{Hull}(P)$ is a triangle each line $L(\mathrm{P_i}, P)$, i=1,2,3, is a median line. Select some vertex of $\partial\mathrm{Hull}(P)$ such that $\partial\mathrm{Hull}(P) \cap \mathrm{S} = \emptyset$, i.e., $\mathrm{P_k}$ is an acute vertex. Let S' be the closest point on S to $\mathrm{P_k}$, and consider the line $L(\mathrm{P_k}, \mathrm{S}')$. Next select some point P' in a small distance from $\mathrm{P_k}$ on $L(\mathrm{P_k}, \mathrm{S}')$, i.e., $\mathrm{d}(\mathrm{P_k}, \mathrm{P}') << \mathrm{d}(\mathrm{P}', \mathrm{S}')$. Observe that the unit normals for the three median lines intersecting at P′ satisfy $< U_i, U_j >> 0$ for i≠ j ≠k, whereas $< U_i, U_k >< 0$, for i≠k. Indeed, for all points between P_k and S′ these conditions must hold. Only when S′ is attained does $< U_i, U_j > = 0$. And then, for points in int(S), by continuity, $< U_i, U_j >< 0$ while $< U_i, U_k >< 0$, for i≠k continues to hold. Thus, $L(P_k, S') \cap [\text{int}(S)] \subset T$. Now by varying S’ along the arc containing the original S’ we show $[\text{int}(S)] \subset T$. □

We show in Fig. 6 that, for three voters, the tri-median set is simply the interior of the Schattschneider set and its boundary.

However, for more complex situations with more voter ideal points, the tri-median set can extend far beyond the Schattschneider set in various directions, as shown in Fig. 7.

In our next results we provide a still more general construction, the Banks line set, that is included inside the Banks set, and within which the tri-median set is nested.

For a given spatial voting game (N, W, P) we introduce three structures, the *Banks line*, *Banks line trajectory*, and *Banks line set*. The Banks line set identifies a set of Banks points, *Banks line points*, that we can show to be a very substantial component of the Banks set for the case of three voters, but which is still not the entire Banks set.

Definition.

(44) Let C be a *Banks chain*, if for all A∈ C A∈ L(A(0), A(1)), then L is a *Banks line*.

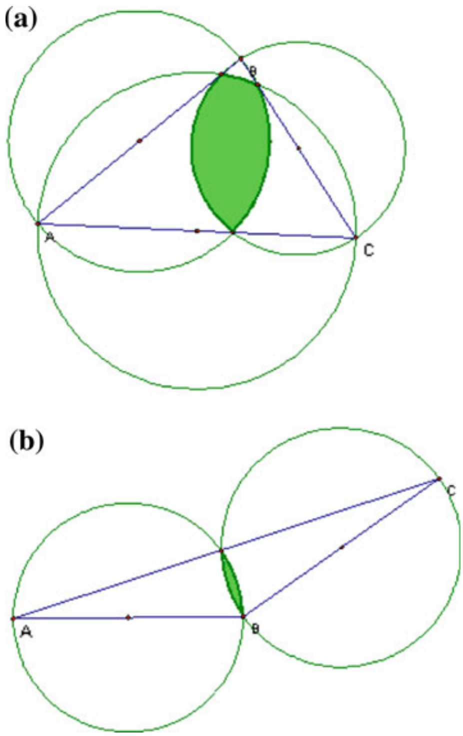


Fig. 6 **a** Redrawn to show both the Schattschneider set and the Tri-median set. **b** Redrawn to show both the Schattschneider set and the Tri-median set

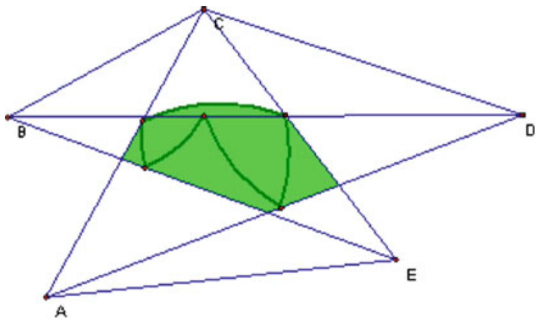


Fig. 7 Redrawn to show both the Schattschneider set and the Tri-median set

A *Banks line* is just a line such that there is no point off the line that dominates everything on the line. Intuitively, a *Banks line trajectory* is defined as a series of points along a Banks line, alternating back and forth across the median on that line, each such point being successively closer to that median. Representing this intuition formally is complicated by the fact that the trajectory is continuous, i.e., cannot be *well-ordered* using the usual order relation on the reals numbers (cf. Zorn's Lemma). In place of this intuition we define the Banks chain, (A,t), as consisting of elements A(t) such that in a sufficiently small neighborhood $\varepsilon > 0$ of t, $0 < t \pm \varepsilon < 1$, there exists $0 < t' < 1$ such that A(t) and A(t′) are separated by the median on the Banks line.

Definitions.

(45) Let L be a line that intersects a median line at point P_M. Let (A,t) be a Banks chain with $A(t) \in (A,t) \Rightarrow A(t) \in L$, $A(1) = P_M$, $d(A(x),P_M) < d(A(y), P_M)$ iff $x > y$, and for any $0 < t < 1$ and a sufficiently small neighborhood of t, $0 < t \pm \varepsilon < 1$, there exists $0 < t' < 1$ such that A(t) and A(t′) are separated by the median. Then (A,t) is a *Banks line trajectory.*

(46) The *Banks line set* consists of all lines containing a Banks line trajectory.

(47) A *Banks line point* is the Banks point of a Banks line trajectory.

The proofs that Schattschneider points and tri-median points are in the Banks set rely on showing that there exists a Banks line trajectory consisting of points selected by alternating across the median on the line, each successively closer to the median on that line, such that any point off the Banks line trajectory is dominated by some point on the Banks line trajectory. The argument relies on taking a limit such that the distance from the median of successively chosen points tends to zero, i.e., the Banks line trajectory is a continuous agenda.

We can now state and prove a series of theorems concerning Banks lines, the Banks line set and Banks line points. These results are useful in allowing us to show that not all Banks line points are also tri-median points, i.e., the tri-median set is a proper subset of the Banks line set, just as the Schattschneider set is a proper subset of the tri-median set.

First we show that every Banks line must intersect the Pareto set, Hull(P).

(48) *Lemma*. Let L be a Banks line, then $L \cap \text{Hull}(P) \neq \emptyset$.

Proof:

Let a L be a line such that $L \cap \text{Hull}(P) = \emptyset$. Project on to L the ideal points P. Then, using the order parameter of L, rank these points and select the median, M. Next, select P$\in \partial\text{Hull}(P)$ such that d(M,P) = c is minimal. Next, let X$\in L$ and Y$\in \partial\text{Hull}(P)$, then d(X,Y) $\geq$ d(X,P_i) $\geq$ d(M,P), where $\text{P}_\text{i} \in P$ or $\text{i} = 1, 2, \ldots, |P|$. Thus P dominates any X in L. Hence, if L is to be Banks line, then $L\cap$ Hull$(P) \neq \emptyset$. □

Next we show that any line between two parallel Banks lines must also be a Banks line.

(49) *Lemma*. If two parallel lines are Banks lines, then every parallel line between them is a Banks line.

Proof:

Let L_1 and L_2 be parallel Banks lines. Suppose L is a line *between* L_1 and L_2 and parallel to both, given by $L = \{X| < \text{U}, \text{X} - \text{P} >= 0\}$. That L is between L_1 and L_2 is determined by considering the line $L^{\perp} = \{\text{X}| < \text{U}^{\perp}, \text{X} - \text{P} >= 0\}$, where $< \text{U}, \text{U}^{\perp} >= 0$. Recall that $L^{\perp}$ can be represented as $\text{X(t)} = \text{P} + \text{t U}^{\perp}$. We say L is *between* L_1 and L_2 when for $\text{X}(\text{T}_1) \in L_1$, $\text{X}(\text{T}_2) \in L_2$, and $\text{X(t)} \in L$ then $\text{T}_1 < \text{t} < \text{T}_2$ or $\text{T}_1 > \text{t} > \text{T}_2$. Suppose further that L is *not* a Banks line. Then there must be a point, P', that dominates all points on L. Consider the line $L^{\perp}$ passing through P', $\text{X(t)} = \text{P}' + \text{t U}^{\perp}$. Choose t such that $\text{T}_1 < \text{t} < \text{T}_2$, where $\text{X}(\text{T}_1) \in L_1$, $\text{X}(\text{T}_2) \in L_2$, and and $\text{X}(\text{T}_0) \in L$. Observe that X(t) =$\text{P}'$ when t = 0. Suppose $\text{t} > 0$ (if not reverse the sign of t). Since P' dominates L there must be a median line, L_M, parallel to L with t < 0. But then d(P', L_M) $<$ d(P', L) and d(L, L_2) $<$ d(P', L_2), i.e., $L_\text{M} < \text{P}' < L < L_2$, where "<" is an order relation defined by the metric as given. So if P' dominates L is must also dominate L_2, contradicting the condition that L_2 is a Banks line. Thus L must also be a Banks line. □

Consider a median line L_M and the set of parallel lines, K, orthogonal to L_M, i.e., $\text{U}_L \perp \text{U}_\text{M}$, where U_M is the unit normal of L_M and U_L is the unit normal of the parallel lines. The points of intersection of L_M and lines in K constitute a continuous set of points, i.e.,L_M itself. A subset of K constitutes a set of Banks lines, e.g., one such line intersects l_M at a Schattschneider point and is a Banks line (cf. (32) above). All points on lines in K outside the Pareto set, Hull(P), are dominated by some point inside the Pareto set and so cannot be Banks lines. Thus the set of Banks lines orthogonal to $L_M$is bounded. We now prove that the Banks line points, arising from a median line, compose a continuous set along that median line:

(50) *Lemma*. Given a median lineL_M, let $K = \{L|L = \{\text{X}| < \text{U}, \text{X} - \text{P}_L >; L_\text{M} = \{\text{X}| < \text{U}_\text{M}, \text{X} - \text{P}_\text{M} >; \text{U}_L \perp \text{U}_\text{M}\}$ and $M = \{\text{X}|\text{X} = L \cap L_\text{M}, L \in K\}$, then $B_L \cap M \neq \emptyset$ and continuous.

Proof:

First, since L_M is a median line there is some X$\in L_\text{M}$ that is also a Schattschneider point. Let L' be the line orthogonal to L_M intersecting L' at X. Observe that $L' \in K$ and so we see that $B_L \cap M \neq \emptyset$. Next, by (47), observe that any line $L \in K$ such that $L\cap$ Hull$(P) = \emptyset$ is dominated by some point in Hull(P). Thus $B_L \cap M$ is bounded at least by $L_{sup}, L_{inf} \in K$ where $L_{sup} \neq L_{inf}$ and $|L_{sup} \cap \text{Hull}(P)| = |L_{inf} \cap \text{Hull}(P)| = 1$. Next, suppose $L_1, L_2 \in K$, $B_L \cap L_1 \neq \emptyset$, and $B_L \cap L_2 \neq \emptyset$, then by (48) $B_L \cap M$ is continuous (simply connected). □

(51) *Lemma.* Suppose L is not a median line. The *majority side* of L is the half-plane of E bounded by L that contains at least $(|N| + 1)/2$ points in P, whereas the *minority side* of L is the half-plane of E bounded by L that contains no more than $(|N| - 1)/2$ points in P.

Proof:

Observe that a line L separates the E into to half-planes. Unless L is a median line, one of these half-planes separated by L will contain at least $(|N| + 1)/2$ of points in P. □

On the other hand, if L is a median line it contains at least one point in P. Accordingly neither half-plane separated by L can contain more than $(|N| - 1)/2$ points in P, i.e., both half-planes are minority sides.

(52) *Lemma.* Suppose L is not a median line, then L is a Banks line iff there is no point Y on the majority side of L that dominates every point on, i.e., such that for all X∈L, Y > X.

Proof:

Every point X on the minority side of L is dominated by the projection of X on L. Therefore, unless there is a point Y on the majority side of L that dominates every point on L, L is a Banks line. □

Certain lines, namely the median lines, constitute Banks lines, with their corresponding Schattschneider point being the Banks Line point.

(53) Every median line itself is a Banks line, and the Schattschneider point on that line is in the Banks line set.

Proof:

There is no majority side of a median line. Consequently, there is no point that dominates every point on that line. The median point on a median line is a Schattschneider point. Since a median line is a Banks line, the median on that line, the Schattschneider point, is a Banks line point. □

Now we have developed the apparatus we need for our most general theorem.

(54) *Theorem 4*. The Banks line set can be found by determining the extent of the Banks line set in each direction of each median line from each Schattschneider point.

Proof:

From (50) the Banks line points arising from a median line are a continuous set along that median line. From (30) each median line passes through some Schattschneider point. From (32) each Schattschneider point is a Banks line point for the two median lines that intersect at that Schattschneider point. Therefore, the sections of Banks line points on each median line containing the Schattschneider point contain all the Banks line points. □

We can make this rather abstract statement much more specific for the special case of three voters. In so doing we get a rather elegant geometric result.

(55) *Theorem 5.* Given a spatial voting game, (N, W, P), with $|N| = 3$, a line L with$L \cap P = \emptyset$ and$L \cap \text{Hull}(P) \neq \emptyset$, is a Banks line iff points in P on the majority side of L have a distance to one another that is greater than the sum of their distances to the line.

For three voters the three ideal points form a triangle, i.e., $\partial\text{Hull}(P)$ is a triangle. A line that intersects the triangle but no ideal point, and such that the two ideal points on the majority side of the line are closer to each other than to the line, is a Banks line, as shown by the proof. In Fig. 8, line j meets these conditions and thus is a Banks line, but parallel line k does not meet these conditions and is not a Banks line.

Proof:

Consider Fig. 8. If the two points are further from one another than the sum of their distances to the line, as is true for line j, then any point D is further from one of them than is the line. In Fig. 8, the point D is further from both A and C than they are from line j. Consequently, the win set "petal" (a subset of the win set that is preferred by a winning coalition) formed by that voter (actually both for A and C in this example) and the voter on the other side of the line (B) cuts the line as shown shaded in Fig. 8. Therefore, there is some point on the line that is in that win set petal of D and dominates D.

Note that there is no point on line k that dominates D.

The property identified in (55) can be used to directly construct the Banks Line set for three voter situations. Figure 9 shows an example of the Banks Line set for three voters, for an equilateral triangle. As this example illustrates, the Banks Line set may be a proper subset of the uncovered set.

While we have only found the Banks line set analytically for the case of three voters, it is possible to find it in the general case by the method described in (54), by

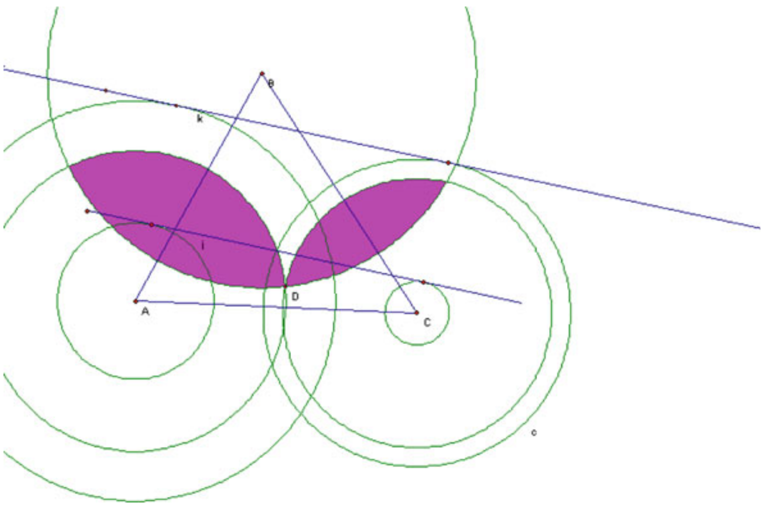


Fig. 8 Construction used to identify Banks line points

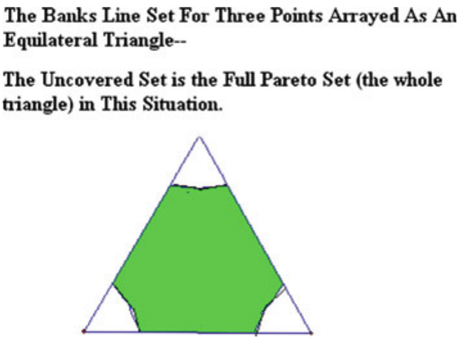


Fig. 9 Construction to show that the Banks line set (shown *colored*) can be a proper subset of the uncovered set (Color figure online)

first finding the Schattschneider set, and then the Banks line points extending from the Schattschneider set in each direction along pairs of orthogonal median lines.

(56) *Prop.* Given a spatial voting game, (N, W, P), with $|N| = 3$, the Banks set is the full uncovered set.

Proof sketch:

For any uncovered point, draw its win set as shown in Fig. 10 for voters A, B, C and point p. Create a Banks trajectory by starting at 1, the tip of one of the petals of the win set, and continuing to 2, the tip of a second petal. 2 dominates 1 if 2 is incrementally inside of the petal, because B would then prefer to 2 to 1; meanwhile C prefers 2 to 1 and A prefers 1 to 2; so with C as the decider, 2 dominates 1. Similarly, 3 dominates 2 and 3 dominates 1 as long as it is incrementally moved inward; in like manner 3 dominates 2 because C marginally prefers 3 to 2, and 3 dominates 1 because A marginally prefers it. Then, 4 dominates 3, 2, and 1, because A, B, and C margin-

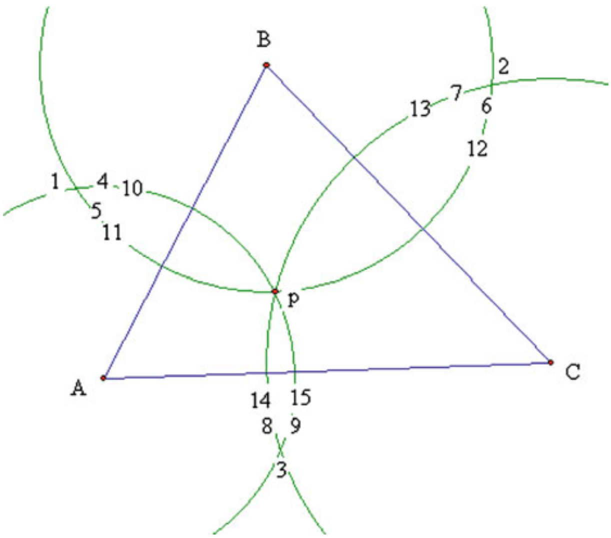


Fig. 10 Construction to show that the Banks set and the uncovered set coincide in the case of three voters

ally prefer 4 respectively. Then, 5 dominates 4,3,2,1, because C, A, B, and C prefer 5 respectively. The process continues to 6, 7, 8… in a similar way, converging on p. As the process proceeds, the trajectory comes to include all points on the boundary of the three petals. Then, since p is an uncovered point, there is no point that has a win set inside the win set of p. For any other point, if its win set is not inside the win set of p, then it must overlap with the win set of p and therefore must intersect the boundary of the win set of p and therefore lose to some point in the Banks trajectory we have just created.

3 Discussion

While we still have not completed our quest to find a full geometric characterization of the Banks set, we have made substantial progress toward this goal. We have been able to show geometrically how to locate ever more general nested subsets of the Banks set, namely the Schattschneider set, the tri-median set, and the Banks line set, with the latter two geometric constructions original with this essay.

In the three voter case, the only case where we know for sure the location of the entire Banks set, we found that the Banks Line set was a proper subset of the Banks set. We conjecture that this result will also hold for cases with more than three voters. Note that the Banks set is bounded from above by the uncovered set and by the Banks line points from below. In the case of an equilateral triangle, such as Fig. 9, Banks line points fill the interior of the triangle up to a relatively small neighborhood of each vertex, the uncovered set being the entire Pareto set for an equilateral triangle (Feld et al. 1987). Thus, in the equilateral case, the Banks set can differ by only a "small" amount from the Banks line points, i.e., the difference must fall in the neighborhood between the uncovered set (here the whole triangle) and the set of Banks line points (as shown in Fig. 9). The uncovered set, however, is the full triangle (excluding ideal points) only for the equilateral case. In general, the more obtuse the vertex angle, the closer to the vertex is the uncovered set, whereas the more acute the angle the larger further away. For acute angles the uncovered set is separated from the vertex by a non-trivial distance (Feld et al. 1987).

While we do not have a complete geometric characterization of the Banks set for majority rule spatial voting games, the particular subsets of the Banks set we identified and showed how to locate geometrically are each important, we believe, in their own right, because the Schattschneider set, the tri-median set, and the Banks line set each may result from plausible real world decision processes. In particular, a group that decides to vote on orthogonal dimensions one dimension at a time will always arrive at a Schattschneider point. And, a group that focuses on a single dimension of variation, incrementally closing in on the median point on that dimension, will find a Banks line point. Moreover, within the set of Banks line points, the collectivity is more likely to find a tri-median point than other Banks line points, because one can get to the same tri-median point in multiple ways, i.e., from more than one Banks line. Relatedly, we conjecture that the points in the Banks line set will, on average, have smaller win sets than points in the overall uncovered set, and thus be more likely, *ceteris paribus*, to be chosen than other points in the uncovered set in many majority rule processes (cf. Feld et al., forthcoming). Thus, we suggest that, even though all Banks points are theoretically possible outcomes of sequential agendas, the particular Banks points that we have identified may be more likely outcomes than other Banks points.

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