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# Foundations of spatial preferences☆

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ABSTRACT

I provide an axiomatic foundation for the assumption of specific utility functions in a multidimensional spatial model, endogenizing the spatial representation of the set of alternatives. Given a set of objects with multiple attributes, I find simple necessary and sufficient conditions on preferences such that there exists a mapping of the set of objects into a Euclidean space where the utility function of the agent is linear city block, quadratic Euclidean, or more generally, it is the $\delta$ power of one of Minkowski (1886) metric functions. In a society with multiple agents I characterize the set of preferences that are representable by weighted linear city block utility functions, and I discuss how the result extends to other Minkowski utility functions.



Consider objects that have multiple attributes, and values within each attribute have a natural order. A multidimensional spatial model is useful to analyze preferences and choices over these objects. The original spatial model was presented by Hotelling (1929) to study product differentiation in the real line. Morris et al. (1972) adapted the multidimensional spatial model to study political competition over multiple policy issues, letting each policy issue correspond to a dimension in a vector space. The standard approach in political economy is to assume that each agent has an ideal policy bundle represented by a most preferred point in the vector space, and that preferences over policy bundles are representable by a utility function that is quadratic or more generally concave in the Euclidean distance to the ideal point of the agent.[1]

In this paper I characterize the set of preferences over alternatives that are representable by city block utility functions, by Euclidean utility functions, or more generally, by utility functions based on one of Minkowski (1886) metric functions. I highlight that the map of the set of alternatives into a vector space is just a representation, much like a utility function is a representation of preferences, and both the spatial mapping and the utility function on this space are endogenously chosen for their convenient role as objects that are more tractable than a binary relation over the primitive set of alternatives. The characterization provides axiomatic micro-foundations for the existing multidimensional spatial models in the literature, improving our understanding about their assumptions.

In many applications, attributes are not objectively quantifiable and the choice set is not a subset of a vector space. For instance, policies on social values or foreign policy do not have natural units of measure. Policies can be represented as vectors, but this representation is endogenous, and any assumption on the utility of an agent that depends on an endogenous spatial representation and not on the primitives is suspect, because the same preferences can also be represented by a utility function of a very different shape using another spatial representation of the set of alternatives.

The substantive implications of assuming a specific shape of the indifference curves are great. If indifference curves are Euclidean or more generally smooth, for a generic distribution of ideal points the core of simple majority is generically empty and majority rule is globally intransitive.[2] On the other hand, if indifference curves are given by the city block distance, then under more general con-

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[1] See for instance Kramer (1977), Enelow and Hinich (1981), Feddersen (1992), Schofield and Sened (2006) or Schofield (2007) for a small sample of highly cited work that uses quadratic Euclidean preferences.

[2] See Plott (1967), McKelvey and Schofield (1987) and McKelvey (1976, 1979).

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ditions the majority rule core is not empty and there exists a stable policy outcome.[3] Therefore, for any given utility function, it is crucial to identify the class of preferences that can be represented by this function. That is the purpose of this paper.

In related work, Kannai (1977) and Richter and Wong (2004) find conditions such that preferences in a given space can be represented by a concave utility function. D'Agostino and Dardanoni (2009) characterize the Euclidean distance function in terms of five invariance and monotonicity axioms; Knoblauch (2000) develops an algorithm that constructs a Euclidean representation in one dimension of a preference profile, if such representation exists; Kalandrakis (2000) investigates whether the incomplete preferences revealed by a finite number of binary voting choices is consistent with a concave utility representation of these preferences; Degan and Merlo (2009) question whether the hypothesis that voters vote according to a utility function that is decreasing in the Euclidean distance is empirically falsifiable when the ideal point of the voter is unknown; and Azrieli (2009) finds conditions such that the rankings of candidates by an infinite number of voters are consistent with voters having utility functions that are additive in a valence term for each candidate and the square of the Euclidean distance from the candidate to the voter.

The closest reference is by Bogomolnaia and Laslier (2007). They find how many dimensions must be used to represent any ordinal preference profile over a finite number of alternatives using Euclidean preferences. Because their set of alternatives is finite, for a single individual, their problem is trivial. Any preference can be represented in just one dimension by assigning alternatives to natural numbers according to the preference order of the agent. By contrast, I consider an infinite number of alternatives by studying lotteries over alternatives. A more substantive difference is that in their theory, alternatives are not defined as a collection of attributes, and hence the dimensions of the space are fully endogenous.

My motivation to consider alternatives with multiple attributes is political competition over multiple political issues, issues such as income taxation, public health care provision, or immigration. I treat each of this issues as an attribute, so that an alternative is a policy bundle with a policy prescription on each issue. I consider all spatial representation in $K$ dimensions that are consistent in each dimension with the natural order of values within each of the $K$ exogenously given attributes. Given a class of utility functions that includes linear city block and quadratic Euclidean utilities as special cases, I find conditions on the preference relation under which, using one of these spatial representations to represent the set of alternatives, it is possible to represent the preference relation by the desired utility function.

## 1. Theory

Let $A$ be a set of attributes, of size $K$. For each attribute $k \in A = \{1, \ldots, K\}$, let $X_k$ be the set of possible values on attribute $k$. This set can be finite, countable or uncountable with the same cardinality as $\mathbb{R}$. Let the elements of $X_k$ be ordered by a linear order $\geq_k$ with a maximal element $x_k^{\max}$ and a minimal element $x_k^{\min}$. Let $>_k$ be the strict order derived from $\geq_k$. Given the set of possible values on each attribute, let the set of alternatives be the Cartesian product $X = X_1 \times X_2 \times \ldots \times X_K$ and let $\Delta X$ be the set of simple lotteries on $X$.[4] In a political economy application, each attribute $k \in A$ is a policy issue and $X$ is the set of alternative policy bundles.

For any given lottery $p \in \Delta X$, let $p(x)$ denote the probability that $p$ assigns to $x \in X$. For any $p \in \Delta X$, let $supp(p) = \{x \in X : p(x) > 0\}$ be the support of $p$. Slightly abusing notation, let $x, y, z, w \in X$ denote as well degenerate lotteries, so they belong to $\Delta X$. Let $x_k$ denote the $k$-th element of the ordered list $x$, let $X_{-k} = X_1 \times \ldots \times X_{k-1} \times X_{k+1} \times \ldots \times X_K$ and let $x_{-k} \in X_{-k}$ denote the ordered list of length $K-1$ that contains all attribute values of alternative $x$ except $x_k$. Then we can write $x$ as $x = (x_k, x_{-k})$. Let $p_k \in \Delta X_k$ be a simple lottery on $X_k$, let $p_k(x_k)$ be the probability that $p_k$ assigns to $x_k$ and let $(p_k; x_{-k}) \in \Delta X$ be the lottery over alternatives that runs lottery $p_k$ on attribute $k$ and yields $x_{-k}$ with certainty in all other attributes.

Let $\succsim$ be a complete and transitive binary relation on $\Delta X$ representing the weak preferences of agent $i$ over lotteries on $X$. Let $x \succ y$ denote ($x \succsim y$, not $y \succsim x$) and let $x \sim y$ denote ($x \succsim y$, $y \succsim x$). Let $\succsim$ satisfy the independence and Archimedean axioms due to Von Neumann and Morgenstern (1944).

**Axiom 1.** (Archimedean): If $p, q, r \in \Delta X$ such that $p \succ q \succ r$, then there is an $\alpha \in (0,1)$ such that $\alpha p + (1-\alpha) r \sim q$.

**Axiom 2.** (Independence): For all $p, q, r \in \Delta X$ and any $\alpha \in (0,1)$, then $p \succsim q$ if and only if $\alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r$.

Then the preferences over lotteries can be represented by a utility function $u : X \to \mathbb{R}$ such that for any $p, q \in \Delta X$, $p \succsim q$ if and only if $\sum_X p(x) u(x) \geq \sum_X q(x) u(x)$. This is part of the celebrated expected utility theorem by Von Neumann and Morgenstern (1944).

A spatial representation of $X$ is a vector valued function $f = (f_1, f_2, \ldots, f_K)$ such that $f_k : X_k \to \mathbb{R}$ is strictly increasing in $\geq_k$ for each $k \in A$ and $f(x) \in \mathbb{R}^K$ represents alternative $x \in X$. Let $\mathcal{F}$ be the set of all possible spatial representations satisfying this monotonicity requirement. The motivating question is under what conditions on $\succsim$ there exists a spatial representation $f$ such that the preferences over $f(X) \subseteq \mathbb{R}^K$ can be represented according to a given utility function.

In addition to the standard expected utility axioms, the first axiom that I introduce is a separability condition that guarantees that agents evaluate attributes independently, so that their preferences can be represented by an additively separable utility function, as shown by Fishburn (1970). Let $L(x, y)$ be a lottery that assigns equal probability to $x$ and $y$. Let $x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_K, y_K\})$ and $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_K, y_K\})$ be the join and the meet of $x$ and $y$.

**Axiom 3.** (Modularity) For all $x, y \in X$, $L(x, y) \sim L(x \vee y, x \wedge y)$.

With only two attributes, modularity is equivalent to the standard separability condition by Fishburn (1970), Theorem 11.1, by which an agent is indifferent over two 50–50 lotteries if each of these lotteries induces the same probability distribution over outcomes on each attribute. For example $L((a, c), (b, d))$ and $L((a, d), (b, c))$ both assign probability 0.5 to outcomes $a$ and $b$ in the first attribute and to outcomes $c$ and $d$ in the second, so the agent must be indifferent. With three or more attributes, modularity is simpler and a (negligibly) weaker assumption: Fishburn's separability implies modularity, and modularity together with transitivity implies Fishburn's separability.[5]

[3] See Rae and Taylor (1971), Wendell and Thorson (1974), McKelvey and Wendell (1976) and Humphreys and Laver (2009).

[4] A simple lottery is a lottery with finite support, that is, a lottery that assigns positive probability only to a finite number of alternatives.

[5] Birkhoff (1948) calls a function $f$ satisfying $f(x) + f(y) = f(x \vee y) + f(x \wedge y)$ a *valuation*. See Kreps (1982), Milgrom and Shannon (1994) and Topkis (1998) for related ordinal and cardinal definitions of modularity.

**Axiom 4.** (Multi-attribute single peakedness) $\exists x^* \in X$ such that for each $k \in \{1, 2, \ldots, K\}$, and any $x_k^a, x_k^b, x_k^c, x_k^d \in X_k$:

$$x_k^a <_k x_k^b \leq_k x_k^* \leq_k x_k^c <_k x_k^d \Rightarrow (x_k^b, x_{-k}^*) \succ (x_k^a, x_{-k}^*) \text{ and } (x_k^c, x_{-k}^*) \succ (x_k^d, x_{-k}^*).$$

A multi-attribute single peaked preference relation has a best policy such that, moving away from the peak on any given attribute, preferences decrease, as in a unidimensional single peaked relation. This condition of single-peakedness is weaker than the multi-dimensional single peakedness used by Barberà, Gul and Stacchetti (1993), but together with modularity, it suffices to guarantee that their stricter restriction is also satisfied, and that alternatives and preferences can be represented in a vector space such that the utility of the agent is a decreasing function of any of Minkowski's Minkowski (1886) norms.

**Theorem 1.** *Assume $\succsim$ has a unique maximal element and is representable by the expected utility of $u : X \to \mathbb{R}$. For any $\delta \in (0, \infty)$, there exists a spatial representation $f^\delta = (f_1^\delta, \ldots, f_K^\delta) \in \mathcal{F}$ such that*

$$u(x) = -\sum_{k=1}^{K} |\overset{\delta}{\underset{k}{f}}(x_k)|^\delta.$$

*if and only if $\succsim$ is multi-attribute single peaked and modular.*

This and all other proofs are in the appendix. Theorem 1 is a straightforward implication of Fishburn's (1970) results on expected utility theory for individual decision making. Most relevant in applications, it says that if preferences are modular and multi-attribute single peaked, we can represent alternatives and preferences in a specific vector space fixing the ideal alternative of the agent at the origin of coordinates and using a utility function that is linear in the city block norm, or we can represent them in a different space using a utility function that is quadratic in Euclidean norm. More generally, for any $\delta > 0$, let $d^\delta(y, y') = \left(\sum_{k=1}^{K} |y_k - y_{k'}|^\delta\right)^{1/\delta}$. Note that if $\delta \geq 1$, the function $d^\delta(y, y')$ is a Minkowski (1886) metric, whereas if $\delta < 1$, it is not a metric because it violates the triangle inequality. I nevertheless refer to it as a distance function in an informal use of the term *distance* because for a fixed ideal point $y'$, the function gives an intuitive notion of proximity to this ideal point. I reserve the term *metric* for the mathematical notion of distance satisfying symmetry and triangle inequality. For any $\delta > 0$, mapping the most preferred alternative of the agent to the origin of coordinates and choosing the appropriate spatial representation $f^\delta(X)$, we can decompose the utility function $u(x) = l \circ d \circ f^\delta(x)$, where the distance function $d(\cdot)$ is equal to $d^\delta(f^\delta(x), 0)$ and the loss function $l(\cdot)$ is a power function of degree $\delta$. What we cannot do is represent the preferences in any space using a utility function that is linear or exponential in the Euclidean distance, such as the one used in the celebrated D-NOMINATE method to estimate the location of the ideal policy in two dimensions of US legislators devised by Poole and Rosenthal (1997). Euclidean utility functions that are not quadratic in the Euclidean distance are inconsistent with preferences satisfying the modularity assumption. I state this as a more general claim that follows as a corollary from the proof of Theorem 1.

**Claim 2.** *Assume $K \geq 2$ and $\succsim$ is modular and representable by the expected utility of a function $u : X \to \mathbb{R}$. Assume that there exist a spatial representation $f = (f_1, \ldots, f_K) \in \mathcal{F}$ and a strictly increasing loss function $l : \mathbb{R} \to \mathbb{R}$, such that*

$$u(x) = -l\left(\left(\sum_{k=1}^{K} |f_k(x_k)|^\delta\right)^{1/\delta}\right) \text{ for some } \delta \in \mathbb{R}_{++}.$$

*Then $l(d) = \alpha + \beta(d)^\delta$ for some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_{++}$.*

The parameters $\alpha$, $\beta$ merely note that utility functions are unique only up to affine transformations; normalize $\alpha = 0$ and $\beta = 1$ for the simplest expression.

Modularity is a separability assumption that requires agents to treat issues independently, assessing their preferences over lotteries over policies on one issue in the same manner regardless of the policies in any other issue. See Lacy (2001) for an empirical test on whether preferences across policy issues are separable.

It may seem surprising that the same preference relation can be represented using a city block utility function, or using a quadratic Euclidean utility function, particularly in light of the results on generic inexistence of majority voting core outcomes if preferences are Euclidean and the more positive results on the existence of core outcomes under majority voting with city block preferences. The results on existence of core outcomes rely on a common space for all agents in a society with at least three agents. The need for a common spatial representation imposes further restrictions.

### 1.1. A common space for multiple agents

I extend Theorem 1 to a society with multiple agents for the city block case $\delta = 1$.

In a society $N$ with $n$ agents, the new primitive are $n$ complete and transitive binary relations defined on $\Delta X$ that satisfy the Archimedean and independence axiom, so that for each $i \in N = \{1, \ldots, n\}$, the preference relation $\succsim_i$ is representable by the expected utility of a utility function $u^i$ defined over $X$. I assume that each $\succsim_i$ also satisfies modularity and multi-attribute single peakedness. The challenge in this section is to find the additional conditions so that the preferences of every agent are representable by the desired utility functions with a spatial representation common to all agents. Let $\succsim_N \equiv (\succsim_1, \ldots, \succsim_n)$ and assume that each agent $i$ has a unique preferred alternative denoted $x^i \in X$ so that $x^i \in \Delta X$ is the maximal element of the order $\succsim_i$.

Agents may care more about some attributes than others. With a single agent, this is easily solved by appropriately rescaling the units of the spatial representation of the alternatives. With multiple agents, it is necessary to introduce weights for each dimension. Given any $y, y' \in \mathbb{R}^K$, the standard $l_1$ norm is $\|y\|_1 = \sum_{k=1}^{K} |y_k|$ with its associated $l_1$ metric $\|y - y'\|_1$. I consider a generalization that assigns different weights to each dimension, and to each direction away from the ideal point on each dimension. Allowing the weights on each dimension not to be a constant, but to be a function of the side of the half space given by the ideal value of the agent in this dimension, the distance functions are no longer a mathematical metric, since they violate symmetry. I nevertheless refer to them as distances, consistent with the intuition that they measure the separation or difficulty to travel from a point to another. For a purely geographical interpretation, if $A$ is a point uphill and $B$ is a point downhill, the walking distance from $A$ to $B$ is less than the walking distance from $B$ to $A$.

**Definition 1.** Given any vector of weights $\lambda \equiv (\lambda_{1+}, \lambda_{1-}, \lambda_{2+}, \lambda_{2-}, \ldots, \lambda_{K+}, \lambda_{K-}) \in \mathbb{R}_+^{2K}$, a generalized weighted city block distance with weights $\lambda$ is a function $g : \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}_+$

such that for any $y, y\prime \in \mathbb{R}^K$,

$$g(y, y\prime) = \sum_{k=1}^{K} \lambda_k(y_k, y_{k\prime})|y_k - y_{k\prime}|, \text{ where } \lambda_k(y_k, y_{k\prime}) = \{\lambda_{k+} \text{ if } y_k \geq y_{k\prime},\ \lambda_{k-} \text{ if } y_k < y_{k\prime}\}\ \forall k \in A.$$

Given agent $i\prime s$ ideal point $y^i \in \mathbb{R}^K$, generalized weighted city block preferences are generated in the standard way, letting $y$ be preferred to $y'$ if and only if $y$ is closer than $y'$ to $y^i$. Weighted city block preferences in $\mathbb{R}^2$ have indifference curves that are rhombi with diagonals along the axes. Allowing weights to differ on either side of the ideal point creates indifference curves that in $\mathbb{R}^2$ are tangential quadrilaterals with diagonals along the axes, but sides of unequal length.

Since the space of alternatives $X$ is not originally contained in $\mathbb{R}^K$, the distance is defined over the mapping $f(X)$ of alternatives into an Euclidean space, so given her ideal object $x^i$, agent $i$ has generalized weighted city block preferences over $f(X)$ if for any $x$, $x' \in X$, agent $i$ prefers $x$ to $x'$ if and only if the generalized weighted city block distance from $f(x)$ to $f(x^i)$ is smaller than the generalized weighted city block distance from $f(x')$ to $f(x^i)$.

Given any attribute $k \in A$, let $l_k$, $h_k \in N$ be such that $x_k^{l_k} \preceq_k x_k^i \preceq_k x_k^{h_k} \forall i \in N$. These are the agents with a lowest and highest ideal value on attribute $k$. Given any two agents $i$, $j$ with preferred alternatives $x^i$ and $x^j$, let $x_k^{h(i,j)} \equiv \max\{x_k^i, x_k^j\}$ and let $x_k^{l(i,j)} \equiv \min\{x_k^i, x_k^j\}$.

I find the necessary and sufficient conditions to represent the preferences of every agent by utility functions that are linear in generalized weighted city block distances in some space that is common to all agents.

**Condition 1.** (Linear representability) For any $i, j \in N$, $\forall k \in A$, $\forall x_k^a, x_k^b, x_k^c \in X_k$ such that $x_k^a \preceq_k x_k^{l(i,j)} \preceq_k x_k^b \preceq_k x_k^{h(i,j)} \preceq_k x_k^c$, $\forall x_{-k} \in X_{-k}$ and $\forall \alpha \in [0,\ 1]$, given $p_k^a, p_k^b, p_k^c \in \Delta X_k$ such that $p_k^a(x_k^{\min}) = \alpha$, $p_k^a(x_k^{l(i,j)}) = 1 - \alpha$, $p_k^b(x_k^{h(i,j)}) = \alpha$, $p_k^b(x_k^{l(i,j)}) = 1 - \alpha$, $p_k^c(x_k^{\max}) = \alpha$ and $p_k^c(x_k^{h(i,j)}) = 1 - \alpha$,

$$(p_k^z; x_{-k}) \backsim_i (x_k^z, x_{-k}) \Leftrightarrow (p_k^z; x_{-k}) \backsim_j (x_k^z, x_{-k}) \text{ for any } z \in \{a, b, c\}.$$

*Linear representability* has a simple interpretation. Fixing the value of all attributes except $k$, and evaluating lotteries that assign different values to attribute $k$, if agent $l_k$ and agent $i$ agree in their ordinal preference among all the possible outcomes of the lotteries, then they agree on their ranking of the lotteries as well. Agents $l_k$ and $i$ share the same ranking among all lotteries on attribute $k$ that assign positive probability only to values that are no greater than the ideal value of $l_k$. Similarly, for lotteries that are in any event above $x_k^i$, agents agree that they want less of attribute $k$, and they agree on their ranking of these lotteries. In the intermediate interval between their two ideal policies, the agents have opposite rankings over sure outcomes: Agent $l_k$ wants less, agent $i$ wants more. Linear representability requires that if agent $l_k$ is indifferent between a lottery in this interval and a sure outcome, agent $i$ must be indifferent as well. An intuition is that in this region the agents are in a zero-sum game: Whatever $l_k$ gains, $i$ loses, so if $l_k$ is indifferent between two lotteries, $i$ must be indifferent as well.

Linear representability, together with modularity, multidimensional single peakedness and the standard expected utility conditions assumed throughout this section, guarantees that it is possible to construct a spatial representation $f(X)$ such that we can represent the preferences of every $i \in N$ by a function that is linearly decreasing in the generalized weighted city block distance to the ideal point of the agent.

**Proposition 1.** *A common spatial representation $f \in \mathcal{F}$ such that for each $i \in N$ the utility function $u^i(x)$ is linearly decreasing in the generalized weighted city block distance to $f(x^i)$ for some vector of weights $\lambda^i \in \mathbb{R}_+^{2K}$ exists if and only if preferences $\succsim_N$ satisfy modularity, multi-attribute single peakedness and linear representability.*

Succinctly, and a bit informally, if agents agree on lotteries when they agree on sure outcomes and they have exactly opposite preferences over lotteries when they have exactly opposite preferences over sure outcomes, then their ordinal preferences over multi-attribute objects can be represented in a common space such that these preferences can all be represented by utility functions that are linearly decreasing in a generalized weighted city block distance to the respective ideal points.

Proposition 1 has very important consequences in political competition over policy bundles with multiple policy dimensions: If agents have weighted city block preferences, for an open set of distributions of weights, there exist policy bundles that are in the majority voting rule core, so they cannot be defeated by any other policy.[6]

I analyze the – more complicated – conditions for representability by quadratic Euclidean utility functions or by utility functions that are weighted Minkowski distances in a separate working paper, available from the author.

## 2. Conclusion

Given alternatives that are objects with multiple attributes, it is convenient to represent these objects as points in a vector space. However, unless the values within each attribute are objectively quantifiable, any spatial representation is subjective. The primitive space of objects is a subset of the Cartesian product of the set of possible values in each attribute. Any assumption on preferences over alternatives in the spatial representation of the space of objects is a joint assumption on preferences over alternatives, and on the chosen spatial representation of the preferences.

I show that choosing the appropriate spatial representation $f(X)$, the preferences of a single agent can be represented by a utility function $u(x) = -[d^\delta(x, 0)]^\delta = -\sum_{k=1}^{K} f_k(x_k)^\delta$ for any $\delta > 0$ if and only if the preferences are separable across attributes, and they are single peaked within each attribute.

In a society with multiple agents, an additional condition on preferences guarantees that there exists a spatial representation $f(X)$ common to all agents such that the utility of every agent is weighted linear city block.

## Appendix A.

### Proof of Theorem 1

[6] See Rae and Taylor (1971), Wendell and Thorson (1974), McKelvey and Wendell (1976) and Humphreys and Laver (2009).

**Proof.** $(\Rightarrow)$ Assume $u(x)=-\sum_{k=1}^{K}|f_k^\delta(x_k)|^\delta$. For any $x^a$, $x^b\in X$, let $x^c\equiv x^a\vee x^b$ and $x^d\equiv x^a\wedge x^b$. Then,

$$\begin{aligned}u(x^a)+u(x^b) &= -\sum_{k=1}^{K}|f_k^\delta(x_k^a)|^\delta-\sum_{k=1}^{K}|f_k^\delta(x_k^b)|^\delta=-\sum_{k=1}^{K}\left(|f_k^\delta(x_k^a)|^\delta+|f_k^\delta(x_k^b)|^\delta\right)\\ &=-\sum_{k=1}^{K}\left(|f_k^\delta(x_k^c)|^\delta+|f_k^\delta(x_k^d)|^\delta\right)=-\sum_{k=1}^{K}|f_k^\delta(x_k^c)|^\delta\\ &\quad-\sum_{k=1}^{K}|f_k^\delta(x_k^d)|^\delta=u(x^c)+u(x^d),\end{aligned}$$

so $L(x^a,x^b)\backsim L(x^a\vee x^b, x^a\wedge x^b)$ and modularity is satisfied.

Since $\succsim$ is representable by a Von Neumann and Morgenstern (1944) expected utility function, the maximal element $p\in\Delta X$ such that $p\succsim q$ for any $q\in\Delta X$ must be a degenerate lottery. Denote the maximal element by $x^*\in X$. Since $(u(x^*)\geq u(x)$ for any $x\in X)$ implies ( $f_k^\delta(x_k^*)=\min_{\{x_k\in X_k\}}|f_k^\delta(x_k)|$ for each $k\in A$) and $f_k^\delta$ is strictly increasing in $>_k$ for any $k\in A$, it follows that for any $k\in A$ and any $x^e=(x_k^e,x_{-k}^*)$ and $x^g=(x_k^g,x_{-k}^*)$ such that either $x_k^e<_k x_k^g\leq_k x_k^*$ or $x_k^*\leq_k x_k^g<_k x_k^e$, $(|f_k^\delta(x_k^g)|)/(|f_k^\delta(x_k^e)|)<1$. Then

$$\begin{aligned}u(x^e)-u(x^g) &= -\sum_{h=1}^{K}|f_h^\delta(x_h^e)|^\delta+\sum_{h=1}^{K}|f_h^\delta(x_h^g)|^\delta\\ &= -|f_k^\delta(x_k^e)|^\delta+|f_k^\delta(x_k^g)|^\delta<0,\end{aligned}$$

so $x^g\succ x^e$. Therefore, multi-attribute single peakedness is satisfied.

$(\Leftarrow)$ Assume that $\succsim$ satisfies modularity and multi-attribute single peakedness. Following Fishburn (1970) Chapter 11, given any $p\in\Delta X$, let $p_k(S_k)=p(\{x:x\in X,\ x_k\in S_k\})$ for any $S_k\subseteq X_k$. Then $p_k$ is the probability measure on $X_k$ induced by lottery $p$ on $X$. Fishburn (1970), Theorem 11.1 shows that $u(x)$ is additively separable if and only if:

**Condition 2.** (Separability, Fishburn (1970)) For any $p$, $q\in\Delta X$ such that $p_k=q_k\ \forall k\in A$, and such that $p(x), q(x)\in\{0,1/2,1\}\ \forall x\in X$,

$p\backsim q$.

I want to show that modularity implies Fishburn's separability condition, and hence additive separability of $u(x)$.

Let $p$, $q$ be such that $p(x)=p(y)=(1/2)$ and $q(w)=q(z)=(1/2)$, where $\{w_k,z_k\}=\{x_k,y_k\}$ for each $k\in A$. By modularity, $p=L(x,y)\backsim L(x\vee y, x\wedge y)$ and $q=L(w,z)\backsim L(w\vee z, w\wedge z)$. Since $w\vee z=x\vee y$ and $w\wedge z=x\wedge y$, by transitivity of $\succsim$, it follows $p\backsim q$, Fishburn's separability condition is satisfied, and $u(x)$ is additively separable.

It remains to be shown that given $u_k:X_k\to\mathbb{R}$ for each $k\in A$ such that $u(x)=\sum_{k=1}^{K}u_k(x_k)$, there exists $f^\delta\in\mathcal{F}$ such that $u_k(x_k)=-|f_k^\delta(x_k)|^\delta$ for each $k\in A$. Without loss of generality, fix $u_k(x_k^*)=0$. Then construct $f_k^\delta$ as follows: $f_k^\delta(x_k^*)=0$; for any $x_k^a\in X_k$ such that $x_k^a<_k x_k^*$, $f_k^\delta(x_k^a)=-|u_k(x_k^a)|^{1/\delta}$; and for any $x_k^b\in X_k$ such that $x_k^b>_k x_k^*$, $f_k^\delta(x_k^b)=|u_k(x_k^b)|^{1/\delta}$. Since $\succsim$ is multi-attribute single peaked, $u_k(x_k)<0$ for any $x_k\neq x_k^*$ and thus, for any $x_k^a<_k x_k^*<x_k^b$, $f_k^\delta(x_k^a)<0<f_k^\delta(x_k^b)$. Then

$$\begin{aligned}&f_k^\delta(x_k^a) &&= -|u_k(x_k^a)|^{1/\delta}\Rightarrow|f_k^\delta(x_k^a)|=|u_k(x_k^a)|^{1/\delta}\Rightarrow|f_k^\delta(x_k^a)|^\delta=|u_k(x_k^a)|=-u_k(x_k^a)\\ &\text{and } f_k^\delta(x_k^b) &&= |u_k(x_k^b)|^{1/\delta}\Rightarrow|f_k^\delta(x_k^b)|^\delta=|u_k(x_k^b)|\Rightarrow|f_k^\delta(x_k^b)|^\delta=-u_k(x_k^b).\end{aligned}$$

□

**Proof of Claim 2**

**Proof.** Since $l$ is strictly increasing, the shape of the indifference curves is given by $\left(\sum_{k=1}^{K}|f_k(x_k)|^\delta\right)^{1/\delta}$, or, equivalently, by $\sum_{k=1}^{K}|f_k(x_k)|^\delta$, so preferences are multi-attribute single peaked and modular. Theorem 1 then applies and preferences are representable by the expected utility of $v(x)=-\sum_{k=1}^{K}|f_k^\delta(x_k)|^\delta$. Since the indifference curves given by $\sum_{k=1}^{K}|f_k(x_k)|^\delta$ and by $\sum_{k=1}^{K}|f_k^\delta(x_k)|^\delta$ have the same shapes, the spatial representations $f_k(x_k)$ and $f_k^\delta(x_k)$ must coincide up to a rescaling $f_k^\delta(x_k)=\gamma f_k(x_k)$ for an arbitrary $\gamma\in\mathbb{R}_{++}$. Since preferences are representable by the expected utility of $v(x)$, by Von Neumann and Morgenstern (1944) expected utility theorem, preferences are representable by the expected utility of another function $u(x)$ if and only if $u(x)$ is a positive affine transformation of $v(x)$, which requires $l(d)=\alpha+\beta(d)^\delta$ with $\beta\in\mathbb{R}_{++}$. □

**Proof of Proposition 1**

**Proof.** $(\Rightarrow)$ By contradiction. Let $d_{\lambda^i}(y,y^i)$ denote the generalized weighted city block distance with weights $\lambda^i\in\mathbb{R}^{2K}$. Suppose that $u^i(x)=-d_{\lambda^i}(f(x),f(x^i))$ for every $i\in N$, but linear representability fails for the case $z=a$ in the statement of linear representability. Then, $\exists\{k\in A, x_{-k}\in X_{-k}, i,j\in N, \alpha\in[0,1]$ and $x_k^a\leq_k x_k^{l(i,j)}\}$ such that given $p_k^a\in\Delta X_k$ with $p_k^a(x_k^{\min})=\alpha$ and $p_k^a(x_k^{l(i,j)})=1-\alpha$, $(p_k^a;x_{-k})\backsim_i(x_k^a,x_{-k})$ and $(p_k^a;x_{-k})\not\backsim_j(x_k^a,x_{-k})$. Simplify notation to let $l=l(i,j)$. In utility terms, $(p_k^a;x_{-k})\backsim_i(x_k^a,x_{-k})$ implies

$$\alpha u^i((x_k^{\min},x_{-k}))+(1-\alpha)u^i((x_k^l,x_{-k}))=u^i((x_k^a,x_{-k})),$$

which, since $u^i(x)$ is linearly decreasing in

$$d_{\lambda^i}(f(x),f(x^i))=\lambda_k^i|f_k(x_k)-f_k(x_k^i)|+\sum_{m\neq k}\lambda_m^i|f_m(x_m)-f_m(x_m^i)|,$$

implies

$$\begin{aligned}\alpha|f_k(x_k^{\min})-f_k(x_k^i)|+(1-\alpha)|f_k(x_k^l)-f_k(x_k^i)| &= |f_k(x_k^a)-f_k(x_k^i)|\\ \alpha f_k(x_k^i)-\alpha f_k(x_k^{\min})+(1-\alpha)f_k(x_k^i)-(1-\alpha)f_k(x_k^l) &= f_k(x_k^i)-f_k(x_k^a)\\ \alpha(f_k(x_k^l)-f_k(x_k^{\min})) &= f_k(x_k^l)-f_k(x_k^a).\end{aligned}$$

In utility terms, $(p_k^a;x_{-k})\not\backsim_j(x_k^a,x_{-k})$ implies

$$\begin{aligned}\alpha|f_k(x_k^{\min})-f_k(x_k^j)|+(1-\alpha)|f_k(x_k^l)-f_k(x_k^j)| &\neq |f_k(x_k^a)-f_k(x_k^j)|\\ \alpha f_k(x_k^j)-\alpha f_k(x_k^{\min})+(1-\alpha)f_k(x_k^j)-(1-\alpha)f_k(x_k^l) &\neq f_k(x_k^j)-f_k(x_k^a)\\ \alpha(f_k(x_k^l)-f_k(x_k^{\min})) &\neq f_k(x_k^l)-f_k(x_k^a),\end{aligned}$$

a contradiction. The cases for $z=b$ and $z=c$ follow an analogous argument.

$(\Leftarrow)$ As shown in Theorem 1, for any $i\in N$, $u^i(x)$ that represents $\succsim_i$ is additively separable, so that there exist $(u_1^i,\ldots,u_K^i)$ such that $u^i(x)=\sum_{k=1}^{K}u_k^i(x_k)$ and $u_k^i(x_k^i)=0$. For each attribute $k\in A$, recall that $l_k$ is the agent with the lowest ideal value on attribute $k$. By Theorem 1, there exists $f^{l_k}\in\mathcal{F}$ such that $u_k^{l_k}(x_k)$ is linearly decreasing in $|f_k^{l_k}(x_k)-f_k^{l_k}(x_k^{l_k})|$. Then, by linear representability case $z=c$, $\forall i\in N$ and for any $x_k^c\in[x_k^i,x_k^{\max}]$, $u_k^i(x_k^c)$ is also linearly decreasing in $f_k^{l_k}(x_k^c)-f_k^{l_k}(x_k^i)$. Also note that by linear representability case $z=b$, $u_k^{h_k}(x_k)$ is linearly increasing in $f_k^{l_k}(x_k^b)$ for any $x_k^b\in[x_k^{l_k},x_k^{h_k}]$.

Construct $f_k$ by letting $f_k(x_k) = f_k^{l_k}(x_k)$ if $x_k \geq_k x_k^{l_k}$ and choosing $f_k(x_k)$ for $x_k \leq_k x_k^{l_k}$ in such manner that the utility of $h_k$ is linear from $f_k(x_k^{\min})$ to $f_k(x_k^{h_k})$. Then, by linear representability case $z = a$, for any $i \in N$ and any $x_k^a \in [x_k^{\min}, x_k^i]$, since agents $i$ and $h_k$ agree in their preference, $u_k^i(x_k^a)$ is also linearly decreasing in $f_k(x_k^i) - f_k(x_k^a)$. Therefore, every $i \in N$ has preferences $\succsim_i$ such that given $f_k(x_k)$, the utility function $u_k^i(x)$ on attribute $k$ is linearly decreasing in $f_k(x_k^i) - f_k(x_k)$ for any $x_k \leq_k x_k^i$ and is linearly decreasing in $f_k(x_k) - f_k(x_k^i)$ for any $x_k \geq_k x_k^i$. Since $k$ is arbitrary, assigning the appropriate weights to each direction in each dimension, and to each dimension, the utility function $u^i(x)$ is linearly decreasing in a generalized weighted city block distance from $f(x)$ to $f(x^i)$. $\square$

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