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RESEARCH ARTICLE

On the spatial representation of preference profiles

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Abstract Given a set of alternatives with multiple attributes, I characterize the set of preference profiles that are representable by weighted versions of a class of utility functions indexed by a parameter $\delta > 0$, where $\delta \geq 1$ corresponds to the set of Minkowski's (1886) metric functions. In light of the starkly different consequences between representability with $\delta \leq 1$ or with $\delta > 1$, I propose a test to empirically estimate δ and I discuss the theoretical and empirical implications for spatial models of political competition.

Keywords Utility representation · Spatial models · Multidimensional preferences · Spatial representation

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1 Introduction

Multidimensional spatial models are useful to represent preferences over alternatives with multiple attributes. Political economy is a prominent application, where each

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dimension corresponds to a different policy issue. Starting with Davis et al. (1972), who extended Hotelling's (1929) model in the real line to multiple dimensions, the standard assumption is that agents have a most preferred alternative in the policy space, and utilities that are decreasing in the Euclidean distance to this point, typically with a quadratic loss function.[1]

However, many policy issues take values that are not objectively quantifiable, so the set of alternatives is not a subset of a vector space. For instance, given a set of policies on moral issues such as abortion or gay rights, it may be possible to order these policies from those that grant most expansive to those that grant most restrictive rights, but it is not possible to quantify each policy in units of rights, because rights are not endowed with a cardinality measure. When we assign numeric values to these policies and represent them in a vector space, the spatial representation of the set of alternatives is an object of choice for the theorist and the chosen spatial representation of the set of alternatives determines the shape of the utility representation of preferences.

I characterize the set of preference profiles that are representable (in some carefully chosen space) using utility functions that are weighted generalizations of Minkowski's (1886) metric functions. I consider a family of functions parameterized by a single parameter $\delta > 0$, and for each value of the parameter I consider a class of functions that allows for different weights for each attribute. The ubiquitous quadratic Euclidean representation is a special case with $\delta = 2$ and equal weights.

The conditions on the preference profile that are necessary and sufficient to represent the preferences by utility functions of any given shape depend on the desired shape, captured by δ. If the preference profile is representable by utilities in the class of functions with $\delta > 1$, the standard assumptions of differentiability and concavity of the utility functions are appropriate, and even if Euclidean utilities are an imperfect simplification, the may capture well enough the qualitatively relevant insights. However, if the preference profile satisfies the conditions that make it representable by utility functions with $\delta < 1$, then utilities are neither differentiable nor quasiconcave (let alone concave), with great implications for political economy; for instance, the classic negative results on the generic emptiness of the core and generic intransitivity of social preferences under simple majority rule (Plott 1967; McKelvey and Schofield 1987; McKelvey 1976, 1979) break down.

In applications where attributes do not have objective units of measure, it is problematic to make assumptions on the shape of indifference curves, because these shapes are not primitives, but rather, they are endogenously derived from the chosen spatial representation of the set of alternatives. It is more appropriate to make assumptions on primitives, such as the preference profile with respect to the abstract set of alternatives, and then we can ask if the preference profile is such that we can find some spatial representation under which the utility functions take any given desired form. In this paper, I identify the assumptions on preference profiles that are necessary and sufficient in order for the profiles to be representable by utilities that belong to a large family of functions that generalizes the set of Minkowski (1886) metric functions, and contain the standard quadratic Euclidean utilities as a special case.

[1] See for instance Kramer (1977), Enelow and Hinich (1981), Feddersen (1992), Gomberg et al. (2004), Schofield and Sened (2006), Schofield Schofield (2007), Baron et al. (2011) or Eguia (2011b).

In related literature, Kannai (1977) and Richter and Wong (2004) find conditions such that preferences in a given space can be represented by a concave utility function. Kalandrakis (2006) finds sufficient conditions on the utility functions of the agents such that each equilibrium in a class of bargaining games can be expressed as a continuous function of the parameters of the model; these conditions require utility functions to be concave and differentiable. D'Agostino and Dardanoni (2009) characterize the Euclidean distance function in terms of five invariance and monotonicity axioms; Degan and Merlo (2009) question whether the hypothesis that voters vote according to a utility function that is decreasing in the Euclidean distance is empirically falsifiable when the ideal point of the voter is unknown; and Azrieli (2009) finds conditions such that the rankings of candidates by an infinite number of voters are consistent with voters having utility functions that are additive in a valence term for each candidate and the square of the Euclidean distance from the candidate to the voter.[2]

The literature that explicitly endogenizes the spatial representation of the set of alternatives for the purpose of representing preferences by means of specific utility functions is very recent. Knoblauch (2010) develops an algorithm that constructs a Euclidean representation in one dimension of a preference profile, if such representation exists.

The closest references are Bogomolnaia and Laslier (2007) and Eguia (2011a). Bogomolnaia and Laslier (2007) find how many dimensions must be used in the spatial representation of the preference profile to represent any ordinal preference profile over a finite number of alternatives using Euclidean preferences in the chosen space. They find that any preference profile can be represented by Euclidean preferences using as many dimensions as there are agents. More strikingly, they find that any preference profile can be represented by some convex preferences using only two dimensions.

In the construction by Bogomolnaia and Laslier (2007), dimensions are object of choice for the theorist; they do not have any exogenous meaning. Alternatives are assigned values in each dimension solely for convenience to make it possible to represent preferences using Euclidean utilities, without any further constraint. I take a different approach with more constraints: I assume that there exist exogenously given issues, and that alternatives are ordered in each of these issues. To represent the set of alternatives, I assign one dimension to each exogenous issue, and in each dimension I assign values to each alternative that respect the order of alternatives in the corresponding issue. This spatial representation then has an intuitive interpretation as a geometric representation of the underlying issues. In my framework, only the cardinality along each dimension is endogenous in the spatial representation, while the order of alternatives and the number of dimensions is exogenously given by the underlying issues. In Bogomolnaia and Laslier (2007), cardinality, ordinality, and number of dimensions are all endogenous, and they are able to represent all preference profiles by adding as many dimensions as necessary and rearranging the order of alternatives at will within each dimension. Because of the additional constraints in my problem,

[2] Other contributions to the field of utility theory (Candeal-Haro and Induráin-Eraso 1995, or Evren 2008 among others) are technically related to all these papers, but are less directly applicable to the substantive question of how to represent spatial preferences over policy issues.

there are many preference profiles that we cannot represent with the desired utility function: I characterize the set of preference profiles that we can represent as desired.

In Eguia (2011a) I solve the special cases with only one agent, and with $\delta = 1$ and city block utility functions for n agents. Conditions for these special cases are very simple. In the current paper, I generalize the results to any $\delta > 0$, allowing us to consider utilities of any shape within the weighted generalization of the family of Minkowski (1886) metric functions.

2 Theory

Let A be a set of attributes, of size K. For each attribute $k \in A = \{1, \ldots, K\}$, let X_k be the set of possible values on attribute k. This set can be finite, countable or uncountable with the same cardinality as $\mathbb{R}$. Let the elements of X_k be ordered by a linear order $\geq_k$ with a maximal element $x_k^{\max}$ and a minimal element $x_k^{\min}$. Let $>_k$ be the strict order derived from $\geq_k$. Given the set of possible values on each attribute, let the set of alternatives be the Cartesian product $X = X_1 \times X_2 \times \cdots \times X_K$ and let ΔX be the set of simple lotteries on X.[3] In a political economy application, each attribute $k \in A$ is a policy issue and X is the set of alternative policy bundles.

For any given lottery $p \in \Delta X$, let $p(x)$ denote the probability that p assigns to $x \in X$. For any $p \in \Delta X$, the support of p is the set of values to which p assigns positive probability, $\{x \in X : p(x) > 0\}$. Slightly abusing notation, let $x, y, z, w \in X$ denote as well degenerate lotteries, so they belong to ΔX. Let x_k denote the $k - th$ element of the ordered list x, let $X_{-k} = X_1 \times \cdots \times X_{k-1} \times X_{k+1} \times \cdots \times X_K$ and let $x_{-k} \in X_{-k}$ denote the ordered list of length $K - 1$ that contains all attribute values of alternative x except x_k. Then, we can write x as $x = (x_k, x_{-k})$. Let $p_k \in \Delta X_k$ be a simple lottery on X_k, let $p_k(x_k)$ be the probability that p_k assigns to x_k and let $(p_k; x_{-k}) \in \Delta X$ be the lottery over alternatives that runs lottery p_k on attribute k and yields x_{-k} with certainty on all other attributes.

Let N be a society with n agents. For any $i \in N$, let $\succsim^i$ be a complete and transitive binary relation on ΔX representing the weak preferences of agent i over lotteries on X. Let the strict preference relation ($x \succsim^i y$, not $y \succsim^i x$) be denoted by $x \succ^i y$ and let the indifference relation ($x \succsim^i y$, $y \succsim^i x$) be denoted by $x \sim^i y$. Let $\succsim \equiv (\succsim^i, \ldots, \succsim^n)$ a preference profile and assume that each agent i has a unique preferred alternative denoted $x^i \in X$ so that $x^i \in \Delta X$ is the maximal element of the order $\succsim^i$. Let $\succsim^i$ satisfy the independence and Archimedean axioms due to von Neumann and Morgenstern (1944).

Axiom 1 (Archimedean) *For any* $p, q, r \in \Delta X$ *such that* $p \succ^i q \succ^i r$*, there exists* $\alpha \in (0, 1)$ *such that* $\alpha p + (1 - \alpha) r \sim^i q$.

Axiom 2 (Independence) *For any* $p, q, r \in \Delta X$ *and any* $\alpha \in (0, 1)$, $p \succsim^i q$ *if and only if* $\alpha p + (1 - \alpha) r \succsim^i \alpha q + (1 - \alpha) r$.

[3] A simple lottery is a lottery with finite support, that is, a lottery that assigns positive probability only to a finite number of alternatives.

Then, $\succsim^i$ can be represented by a utility function $u^i : X \longrightarrow \mathbb{R}$ such that for any lotteries $p, q \in \Delta X$, agent i weakly prefers p to q if and only if $\sum_X p(x)u(x) \geq \sum_X q(x)u(x)$. This is part of the celebrated expected utility theorem by Von Neumann and Morgenstern.

A spatial representation of X is a vector valued function $f = (f_1, f_2, \ldots, f_K)$ such that $f_k : X_k \longrightarrow \mathbb{R}$ is strictly increasing in the exogenous order $\geq_k$ for each $k \in A$ and $f(x) \in \mathbb{R}^K$ represents alternative $x \in X$. Let $\mathcal{F}$ be the set of all possible spatial representations satisfying this monotonicity requirement. The motivating question is what conditions must the preference profile $\succsim$ satisfy so that there exists a spatial representation f such that the preference profile can be represented by a given utility function that has a desirable shape in $f(X) \in \mathbb{R}^K$.

In addition to the standard expected utility axioms, the first axiom that I introduce is a separability condition that guarantees that agents evaluate attributes independently, so that their preferences can be represented by an additively separable utility function, as shown by Fishburn (1970). Let $L(x, y)$ be a lottery that assigns equal probability to x and y. Let $x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_K, y_K\})$ and $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_K, y_K\})$ be the join and the meet of x and y.

Axiom 3 (Modularity) *For all $x, y \in X$, $L(x, y) \sim^i L(x \vee y, x \wedge y)$.*

With only two attributes, modularity is equivalent to the standard separability condition by Fishburn (1970), Theorem 11.1, by which an agent is indifferent over two 50–50 lotteries if each of these lotteries induces the same probability distribution over outcomes on each attribute. For example, $L((a, c), (b, d))$ and $L((a, d), (b, c))$ both assign probability 0.5 to outcomes a and b in the first attribute and to outcomes c and d in the second, so the agent must be indifferent. With three or more attributes, modularity is simpler and a (negligibly) weaker assumption: Fishburn's separability implies modularity, and modularity together with transitivity implies Fishburn's separability.

Axiom 4 (Multi-attribute single-peakedness) *There exists an ideal alternative $x^* \in X$ such that for each attribute k and any values $x_k^a, x_k^b, x_k^c, x_k^d \in X_k$:*

$$x_k^a <_k x_k^b \leq_k x_k^* \leq_k x_k^c <_k x_k^d \Longrightarrow (x_k^b, x_{-k}^*) \succ^i (x_k^a, x_{-k}^*) \text{ and } (x_k^c, x_{-k}^*) \succ^i (x_k^d, x_{-k}^*).$$

A multi-attribute single-peaked preference relation has a best alternative such that, moving away from the peak on any given attribute, preferences decrease, as in a unidimensional single-peaked relation.

Consider the distance function $d^\delta(y, y') = \left(\sum_{k=1}^K |y_k - y_k'|^\delta\right)^{1/\delta}$. For $\delta = 2$, it is the Euclidean distance. For any $\delta \geq 1$, this function is a Minkowski (1886) metric; whereas, if $\delta < 1$, it is not a metric because it violates the triangle inequality, but for any $\delta > 0$ it is a semi-metric. Axioms $A1$–$A4$ characterize the set of preferences such that for any $\delta > 0$, there exists a spatial representation f (which depends on δ) such that the preference of a single individual can be represented by a utility function $u^i(x) = -\sum_{k=1}^K |f_k(x_k) - f_k(x_k)|^\delta$.

Proposition 1 (Eguia 2011a) *For any positive real number δ, a spatial representation $f^\delta = (f_1^\delta, \ldots, f_K^\delta) \in \mathcal{F}$ such that the utility function $u^i(x) = -\sum_{k=1}^K |f_k^\delta(x_k) -$*

$f_k^\delta(x_k^i)|^\delta$ *represents the preference order* $\succsim^i$ *exists if and only if* $\succsim^i$ *satisfies axioms* $A1$–$A4$.

Decomposing the utility function $u^i(x)$ into a loss function l, a distance function d and a spatial representation f so that $u^i(x) = l \circ d \circ f(x)$, it follows from proposition 1 that $\succsim^i$ satisfies axioms $A1$–$A4$ if and only if for any positive real number δ we can represent $\succsim^i$ by a utility function that decreases in the Minkowski δ semi-metric, with a loss function that is a power function with power δ.

Agents may care more about some attributes than others, and different agents may care more about different issues. In this case, it is necessary to introduce weights. Agents with single-peaked preferences may also have asymmetric preferences, caring more about deviations in one direction than the other away from their peak. Documented examples of asymmetric single-peaked preferences are the monetary policy preferences of Central banks such as the US Fed or the European ECB (Blinder 1997; Ruge-Murcia 2003; Surico 2007) and the preferences over taxation and spending by countries that receive foreign aid (Heller 1975; Feeny 2006). In recent work in Political Science, Krasa and Polborn (2010) propose a theory of elections with voters who have asymmetric single-peaked preferences. I accommodate asymmetric preferences by allowing the weights on each dimension not to be a constant, but to be a function of the side of the half space given by the ideal value of the agent in this dimension. Asymmetric distance functions are no longer semi-metric because they violate symmetry: they are then a quasi-metric if $\delta \geq 1$, or a quasi-semi-metric regardless of whether triangle inequality and symmetry are violated. I refer to quasi-semi-metric functions as distances, consistent with the intuition that they measure the separation or difficulty to travel from a point to another. For a purely geographical interpretation of symmetry violations, if A is a point uphill and B is a point downhill, the walking distance from A to B is less than the walking distance from B to A.

I seek to characterize the set of preference profiles that are representable by n utility functions such that for some $\delta > 0$ and some spatial representation f, the utility function of each agent i satisfies the following definition:

Definition 1 A utility function $u^i : X \longrightarrow \mathbb{R}$ is a Weighted Minkowski δ-distance function on $f(X)$ if

$$u^i(x) = -\sum_{k=1}^{K} w_k^i(x_k, x_k^i)|f_k(x_k) - f_k(x_k^i)|^\delta, \text{ where } \delta \in \mathbb{R}_{++} \text{ and} \tag{1}$$

$$w_k^i(x_k, x_k^i) = \{w_{k-}^i \text{ if } x_k \leq_k x_k^i,\ w_{k+}^i \text{ if } x_k \geq_k x_k^i\},$$
$$\text{for some } (w_{1-}^i, w_{1+}^i, \ldots, w_{K-}^i, w_{K+}^i) \in \mathbb{R}_+^{2K}.$$

Agent i assigns weight w_{k-}^i to deviations toward a value of attribute k below her ideal, and weight w_k^i to deviations toward a value of attribute k above her ideal. Linear city block and quadratic Euclidean utilities are, respectively, the Weighted Minkowski 1-distance and Weighted Minkowski 2-distance functions that assign equal weights to all attributes and directions for each agent. I also refer to the class of Weighted Minkowski δ-distance functions without specifying the particular map in which they

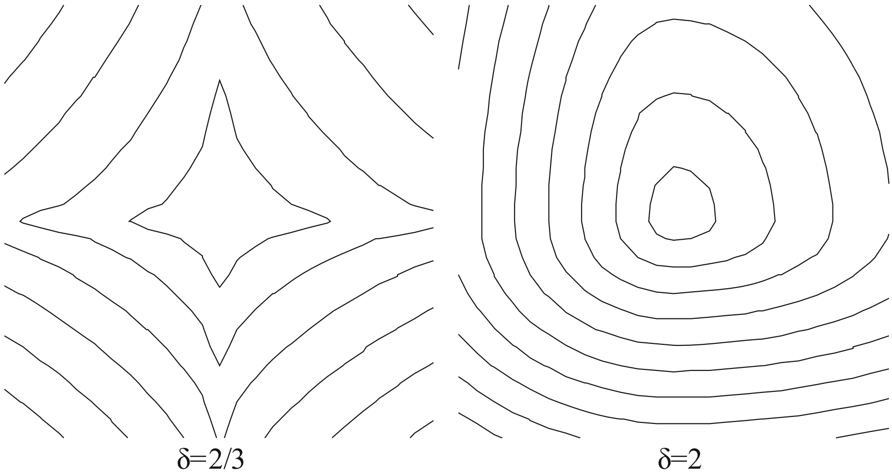

Fig. 1 Weighted Minkowski δ-distances

adopt the desired shape; this class is just the union of all utility functions that are Weighted Minkowski δ-distance functions on $f(X)$ for some spatial representation $f \in \mathcal{F}$. Similarly, I refer to the class of utility functions that are Weighted Minkowski distances without specifying δ or f as the union of all utility functions that are Weighted Minkowski δ-distance functions on $f(X)$ for some parameter $\delta \in \mathbb{R}_{++}$ and some spatial representation $f \in \mathcal{F}$.

Figure 1 depicts the indifference curves over $f(X)$ for an arbitrary Weighted Minkowski 2/3-distance function and for a Weighted Minkowski 2-distance. If $\delta > 1$, indifference curves are smooth and upper contour sets are strictly convex, as in the standard case with Euclidean circles. If $\delta \leq 1$, indifference curves are not smooth. If $\delta < 1$, upper contour sets are not convex.

Axioms $A1$–$A4$ are necessary and sufficient to be able to represent a single preference relation by some utility function that is a Weighted Minkowski δ-distance function on $f(X)$. More generally, if all the preference relations in a preference profile $\succsim = (\succsim^1, \ldots, \succsim^n)$ satisfy axioms $A1$–$A4$, then it is possible to represent each preference relation $\succsim^i$ by a utility function u^i that is a Weighted Minkowski δ-distance on $f^i(X) \ldots$ but notice that each agent $i \in N$ needs her own idiosyncratic spatial representation mapping of the set of alternatives into a vector space. This is not a useful approach: We want to represent the preferences of all agents in the same vector space. The goal in this paper is to find the additional conditions that are necessary and sufficient to represent the preferences of every agent using utility functions that are all Weighted Minkowski δ-distance functions on $f(X)$, for *some common spacial representation* $f(X)$. Much less formally, and applied only to problems with two attributes, I seek to find under what conditions we can draw all preferences in the same figure using indifference curves like those in Fig. 1.

For each $k \in A$, let $l_k, h_k \in N$ be such that $x_k^{l_k} \leq_k x_k^i \leq_k x_k^{h_k} \forall i \in N$. These are the agents with the lowest and highest ideal value on attribute k. Given a preference profile $\succsim$ that satisfies axioms $A1$–$A4$, for any attribute $k \in A$, for any values on all other attributes $x_{-k} \in X_{-k}$, and for any agent $i \in N$, let $\gamma_k^i \in [0, 1]$ and lottery $p_k \in \Delta X_k$ with $p_k(x_k^{\max}) = (\gamma_k^i)^\delta$ and $p_k(x_k^{l_k}) = 1 - \left(\gamma_k^i\right)^\delta$ be such that agent l_k is indifferent between lottery p_k and value x_k^i, that is, $(p_k; x_{-k}) \sim^{l_k} (x_k^i, x_{-k})$ and let $\gamma_k^0 \in \mathbb{R}$ and lottery $q_k \in \Delta X_k$ with $q_k(x_k^{\min}) = \left(\frac{\gamma_k^{h_k}}{\gamma_k^{h_k}+\gamma_k^0}\right)^\delta$ and $q(x_k^{h_k}) = 1 - \left(\frac{\gamma_k^{h_k}}{\gamma_k^{h_k}+\gamma_k^0}\right)^\delta$ be such that agent h_k is indifferent between $(q_k; x_{-k})$ and $(x_k^{l_k}, x_{-k})$.

Lottery p_k yields the highest feasible value on attribute k with probability $(\gamma_k^i)^\delta$ and the lowest ideal value otherwise. The graphical intuition is that if we normalize the spatial representation f to map $x_k^{l_k}$ to $f_k(x_k^{l_k}) = 0$ and to map $x_k^{\max}$ to $f_k(x_k^{\max}) = 1$, then γ_k^i is the point in the real line into which we must map the ideal value of agent i on attribute k in order for u^{l_k} to be a Weighted Minkowski δ-distance on $f(X)$. Once we have mapped the ideal value of every agent, $\gamma_k^{h_k}$ is the distance from the point into which we have mapped the highest ideal value, and the point (zero) into which we have mapped the lowest ideal value. Lottery q_k proposes a lottery between the lowest feasible value and the ideal value of h_k designed so that $-\gamma_k^0$ is the point in the real line into which we must map $x_k^{\min}$ in order for u^{h_k} to be a Weighted Minkowski δ-distance on $f(X)$, given that we have already fixed $f_k(x_k^{l_k}) = 0$ and $f_k(x_k^{\max}) = 1$.

In the following theorem, I detail the additional conditions that guarantee that the preferences of every agent are such that their utility representations acquire the desired shapes in $f(X)$. After the formal statement, I provide an intuition for the cumbersome conditions, and an example.

Theorem 1 *For any $\delta \in \mathbb{R}_{++}$, a spatial representation $f \in \mathcal{F}$ and a utility function $u^i(x)$ for each $i \in N$ such that $u^i(x)$ represents $\succsim^i$ and is a Weighted Minkowski δ-distance on $f(X)$ exist if and only if axioms A1–A4 and the following conditions hold for any attribute k, any $x_{-k} \in X_{-k}$, any $\alpha_i, \alpha_{l_k} \in [0, 1]$ and any agent $i \in N$:*

(i) For any $x_k^a \geq_k x_k^i$ and $p_k, q_k \in \Delta X_k$ such that $p_k(x_k^{\max}) = \alpha_i$, $p_k(x_k^i) = 1-\alpha_i$, $q_k(x_k^{\max}) = \alpha_{l_k}$, $q_k(x_k^i) = 1 - \alpha_{l_k}$, and $(p_k; x_{-k}) \sim^i (x_k^a, x_{-k})$,

$$(q_k; x_{-k}) \sim^{l_k} (x_k^a, x_{-k}) \iff \alpha_{l_k} = \frac{\left(\gamma_k^i + (1-\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^i)^\delta}{1-(\gamma_k^i)^\delta}.$$

(ii) For any $x_k^b \in X_k$ such that $x_k^{l_k} \leq_k x_k^b \leq_k x_k^i$, and $p_k, q_k \in \Delta X_k$ such that $p_k(x_k^i) = \alpha_i$, $p_k(x_k^{l_k}) = 1 - \alpha_i$, $q_k(x_k^{l_k}) = \alpha_{l_k}$, $q_k(x_k^i) = 1 - \alpha_{l_k}$, and $(p_k; x_{-k}) \sim^i (x_k^b, x_{-k})$,

$$(q_k; x_{-k}) \sim^{l_k} (x_k^b, x_{-k}) \iff \alpha_{l_k} = 1 - \left(1 - (1-\alpha_i)^{1/\delta}\right)^\delta.$$

(iii) *For any* $x_k^c \leq_k x_k^i$ *and* $p_k, q_k \in \Delta X$ *such that* $p_k(x_k^{\min}) = \alpha_i$, $p_k(x_k^i) = 1 - \alpha_i$, $q_k(x_k^{\min}) = \alpha_{h_k}$, $q_k(x_k^i) = 1 - \alpha_{h_k}$, *and* $(p_k; x_{-k}) \sim^i (x_k^c, x_{-k})$,

$$(q_k; x_{-k}) \sim^{h_k} (x_k^c, x_{-k}) \Longleftrightarrow \alpha_{h_k} = \frac{\left(\gamma_k^{h_k} - \gamma_k^i + (\gamma_k^0 + \gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^{h_k} - \gamma_k^i)^\delta}{(\gamma_k^0 + \gamma_k^{h_k})^\delta - (\gamma_k^{h_k} - \gamma_k^i)^\delta}.$$

To provide an intuition for these conditions, note that under the modularity axiom, preferences are separable across attributes, or, equivalently, the utility function is additively separable across dimensions. Take an arbitrary attribute k and fix arbitrary values x_{-k} on all other attributes. Further, consider a case with only two agents l and h such that $x^{\min} <_k x_k^l <_k x_k^h <_k x^{\max}$. That is, agent l ideally prefers less of attribute k than agent h, but neither has an extreme top preference on this attribute. Start by fixing $f_k(x_k^l) = 0$ and $f_k(x_k^{\max}) = 1$. This is just a normalization.

The ideal values x_k^l and x_k^h divide the values on attribute k into three subsets: higher values $\{x_k : x_k^h <_k x_k\}$, intermediate values $\{x_k\colon x_k^l \leq_k x_k \leq_k x_k^h\}$, and lower values $\{x_k : x_k <_k x_k^l\}$.

Consider lotteries with support entirely in the third subset, corresponding to condition (iii) with values below what both l and h want. A symmetric intuition holds for the first subset with values above with what both l and h want, and I discuss the intuition for the second subset below, after an example. Both agents agree ordinally within this set, that is, they agree on their preferences over any degenerate lottery with support in this set. A necessary condition to represent their preferences with utility functions with $\delta = 1$ is that they agree as well over any nondegenerate lottery over this set. That is, their preferences must agree cardinally, not just ordinally.

Suppose agents do not agree cardinally. If they disagree in a systematic way, their preferences may be representable by utilities with $\delta \neq 1$. To represent their preferences with utility functions with $\delta > 1$, agent l must be more averse to risk than agent h over lotteries whose support lies below x_k^l (lotteries over the subset of low values). Given that we have mapped x_k^l to 0 and $x_k^{\max}$ to 1, choosing any particular $\delta \in \mathbb{R}_{++}$ pins down a unique point $f_k(x_k^h)$ in the real line that is consistent with l having a utility function that decreases linearly in the power δ of the distance between $f_k(x_k^l)$ and $f_k(x_k)$; this point receives the notation γ_k^h in the theorem, while by our normalization, $\gamma_k^l = 0$. The difference in $l's$ and $h's$ attitude toward risk that is necessary to represent their preferences by utilities with parameter δ is increasing in $f_k(x_k^h)$. If $f_k(x_k^h) = \gamma_k^h$ is very close to $f_k(x_k^l)$, then their risk attitudes must be very similar; and indeed, check that given $\gamma_k^l = 0$ and $\gamma_k^h \longrightarrow 0$, the expression in condition iii) converges in the limit to $\alpha_{h_k} = \alpha_i$ for $i = l$. On the other hand, if $f_k(x_k^h) - f_k(x_k^l)$ is large, their risk attitude must become more dissimilar, with l being less willing to endure risk than h.

Further, this difference is also increasing in the desired δ, that is, if we wish to represent preferences using utility functions with high δ, then l must become more averse to risk than h at a faster rate with increases in distance $f_k(x_k^h) - f_k(x_k^l)$.

Conversely, to represent preferences using utility functions with $\delta < 1$, it must be that l exhibits a greater preference for risk than h over lotteries whose support lies below the ideal value of l.

Learning the preferences over nondegenerate lotteries with support in a set of values on a single attribute over which both agents agree on degenerate lotteries pins down a unique value of δ under which it may be possible (depending on their preferences over lotteries in the other sets) to represent the preferences of both agents by a utility function of the desired class.

The loose intuition is that $\delta > 1$ requires that the agent with an ideal point nearer the support of the lottery must be more averse to risk than those with an ideal point far from the lottery, given that both agents' ideal points fall to the same side away from the support of the lottery. Whereas, $\delta < 1$ requires the closer agent to be more willing to accept risk.

For instance, representation by utilities with $\delta > 1$, including the ubiquitous Euclidean utility functions implicitly assumes that moderate conservatives must be more averse to risk than their extreme colleagues over an election contested by an extreme progressive, a moderate progressive, and a moderate conservative. Here is an example with one attribute, five values, and two agents.

Example 2 Let $X = \{A, B, C, D, E\}$, where alternatives are ordered alphabetically. Let $N = \{i, j\}$, $C \succ^i B \succ^i A$ and $C \succ^i D \succ^i E$, and $D \succ^j C \succ^j B \succ^j A$ and $D \succ^j E$. We start by normalizing $f(C) = 0$ and $f(E) = 1$. Suppose i is indifferent between a degenerate lottery that yields D with certainty and a fair lottery between C and E. Given some $\delta > 0$, we want $u^i(x)$ to take the form $u^i(x) = -|f(x) - f(C)|^\delta$ for any $x \in X$, and $u^j(x) = -|f(x) - f(D)|^\delta$ for $x \in \{A, B, C, D\}$ and $u^j(E) = -w^j|f(E) - f(D)|^\delta$ where w^j corresponds to the weight for deviations to the right for j (all other weights are normalized to one without loss of generality). If we want to use $\delta = 1$, then we must map D to $f(D) = 0.5$; whereas, if we want $\delta = 2$ we must map D to $f(D) = \frac{1}{\sqrt{2}} \approx 0.7$ (note that $f(D)$ corresponds to γ_k^j). I next find where $f(A)$ must be. Suppose that j is indifferent between C for sure and a lottery that yields A with probability $2/3$ and D with probability $1/3$. Then, if we are using $\delta = 1$, given $f(C) = 0$ and $f(D) = 0.5$, it must be $f(A) = -1$ in order for u^j to have the desired shape (note that $f(A)$ corresponds to $-\gamma_k^0$), while if we use $\delta = 2$, given $f(D) = \frac{1}{\sqrt{2}}$ it must be $f(A) = \frac{\sqrt{2}-\sqrt{3}}{2} \approx -0.16$ so that $u^j(A) = -\frac{3}{4}$. Finally, I must locate $f(B)$.

Suppose that i is indifferent between a degenerate lottery that yields B or a fair lottery between A and C (which is equivalent to supposing $\alpha_i = 0.5$ in condition iii) and suppose that j is indifferent between a degenerate lottery that yields B for sure, and a lottery that yields A with probability α_j and C with probability $1 - \alpha_j$. In order to be able to represent preferences by our desired utility functions using $\delta = 1$, we must map B to $f(B) = -\frac{1}{2}$ and it must be that $\alpha_j = \alpha_i = \frac{1}{2}$; whereas, in order to represent them with $\delta = 2$, we must map B to $f(B) = \frac{\sqrt{2}-\sqrt{3}}{2\sqrt{2}} \approx -0.06$ so $u^j(B) = -\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{2}-\sqrt{3}}{2\sqrt{2}}\right)^2 \approx -0.67$ and it must be that α_j solves $\alpha_j u^j(A) + (1-\alpha_j)u^j(C) = u^j(B)$ which implies $\alpha_j = \frac{(2+\sqrt{3}-\sqrt{2})^2-4}{2} = 0.686$. Notice that given $\delta = 2$, given

$\gamma_k^0 = -f(A) = \frac{\sqrt{3}-\sqrt{2}}{2}$, given $\gamma_k^i = f(C) = 0$, given $\gamma_k^j = \gamma_k^{h_k} = f(D) = \frac{1}{\sqrt{2}}$ and given $\alpha_i = 0.5$, indeed condition (iii) reduces to

$$\alpha_{h_k} = \frac{\left(\frac{1}{\sqrt{2}} + \left(\frac{\sqrt{3}-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)^{1/2}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2}{\left(\frac{\sqrt{3}-\sqrt{2}}{2} + \frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2} = 0.686$$

as desired, where $h_k = j$ in our example. The relevant insight is that representation by $\delta = 1$ requires i and j to have the same evaluation of lotteries involving A, B and C, whereas representation by $\delta > 2$ requires j to be willing to bear more risk than i, and in particular for $\delta = 2$, agent j must accept up to a 0.68 probability of getting the bad outcome (A) when i only accepts up to 0.5 probability.

Condition (ii) in Theorem 1 addresses preferences over lotteries with support in between the ideal values of two agents. In this subset of intermediate values $\{x_k\colon x_k^l \leq_k x_k \leq_k x_k^h\}$, agents i and h have opposite ordinal preferences over degenerate lotteries. A necessary condition to represent their preferences with utility functions with $\delta = 1$ is that their cardinal preferences be mirror images of each other: if l is indifferent between x_k or a lottery that delivers her ideal value x_k^l with probability α and the value in this subset that is least preferred by l with probability $1 - \alpha$, then h is indifferent between x_k and a lottery that delivers her ideal value x_k^h with probability $1 - \alpha$ and the value in this subset that is least preferred by h with probability α.

For any x_k^b in the subset of intermediate values, suppose agent l is indifferent between x_k^b and a lottery that delivers her ideal value with probability α_l and her least preferred value in the subset otherwise and h is indifferent between x_k^b is indifferent between x_k^b and a lottery that delivers his ideal value with probability α_h and his least preferred value in the subset otherwise. Representation by utilities with parameter $\delta > 1$ (respectively, $\delta < 1$) requires that $\alpha_l + \alpha_h > 1$ (respectively, $\alpha_l + \alpha_h < 1$); condition (ii) pins down the exact relation. The intuition is that representation by concave utility functions implies that both agents prefer the intermediate value x_k^b to any lottery that delivers x_k^l with probability $\alpha \in (1 - \alpha_h, \alpha_l)$ and x_k^h otherwise, while representation by convex utility functions implies that both agents prefer any lottery that delivers x_k^l with probability $\alpha \in (\alpha_l, 1 - \alpha_h)$ and the intermediate value x_k^b otherwise. I develop this intuition toward the end of the paper to obtain simpler necessary conditions for representability by concave functions.

While conditions $(i$–$iii)$ may seem unduly restrictive, note that together with axioms $A1$–$A4$ they characterize the set of preference profiles representable by a family of utility functions that is much more general than the quadratic Euclidean utility functions routinely used in multidimensional political economy models. This family of functions introduces $1 + (2K - 1)(n - 1)$ degrees of freedom. One degree of freedom, given by parameter δ, refers to the shape of the indifference curves and the concavity of the loss function. The weights on each attribute and on each direction within each attribute introduce $2K$ parameters for each agent, but all the weights for one arbitrary agent can be normalized to one by adjusting the spatial representation

f, and for each of the other $(n-1)$ agents, one weight can be normalized to one, so that $2K-1$ degrees of freedom remain for each of these $(n-1)$ agents.

Representability by utility functions that are Weighted Minkowski δ-distances with only one weight per agent and attribute so that $w^i_{k+} = w^i_{k-} = w^i_k \in \mathbb{R}_+$ for each $k \in A$ and $i \in N$ reduces the degrees of freedom to $1+(K-1)(n-1)$, and it requires additional restrictions on preferences.

Proposition 2 *For any $\delta \in \mathbb{R}_{++}$, a spatial representation $f \in \mathcal{F}$ and a utility function $u^i(x)$ that represents $\succsim^i$ and is a Weighted Minkowski δ-distance on $f(X)$ with equal weights in each direction within any given attribute for each agent $i \in N$ exist if axioms A1–A4, conditions i–iii in Theorem 1 and the following conditions hold $\forall k \in A$, $\forall x_{-k} \in X_{-k}$, $\forall \alpha_i, \alpha_{l_k} \in [0,1]$ and $\forall i \in N$:*

(iv) For any $i \in N$ s.t. $x^{l_k}_k <_k x^i_k <_k x^{\max}_k$, $\exists \beta_i, \rho \in \mathbb{R}_{++}$ such that given $p_k, q_k, r_k \in \Delta X_k$ s.t $p_k(x^{\max}_k) = \rho\beta_i$, $p_k(x^i_k) = 1-\rho\beta_i$, $q_k(x^{l_k}_k) = \rho$, $q_k(x^i_k) = 1-\rho$, $r_k(x^{\max}_k) = \frac{\beta_i}{(1+\beta_i^{1/\delta})^\delta}$, $r_k(x^{l_k}_k) = 1-\frac{\beta_i}{(1+\beta_i^{1/\delta})^\delta}$,

$$(p_k; x_{-k}) \sim^i (q_k; x_{-k}) \text{ and } (r_k; x_{-k}) \sim^{l_k} (x^i; x_{-k}).$$

(v) For any $i \in N$ s.t. $x^i_k =_k x^{l_k}_k$ and any lotteries $p_k, q_k \in X_k$, $(p_k; x_{-k}) \succsim^i (q_k; x_{-k}) \iff (p_k; x_{-k}) \succsim^{l_k} (q_k; x_{-k})$.

A partial intuition of condition iv) for the simplest case with $\delta = 1$ is that if agent i is indifferent between moving from x^i_k to $x^{\max}_k$ with probability $\beta^i\delta$ or moving from x^i_k to $x^{l_k}_k$ with probability δ, it means that i regards moving from x^i_k to $x^{\max}_k$ to be $1/\beta^i$ times as bad as moving from x^i_k to $x^{l_k}_k$ ... and agent l_k must agree that moving from $x^{l_k}_k$ to $x^{\max}_k$ is $(1+\frac{1}{\beta^i}) = \frac{1+\beta^i}{\beta^i}$ times worse than moving only to x^i_k, which means that l_k must be indifferent between moving from her ideal point to x^i_k for sure, or all the way to $x^{\max}_k$ with probability $\frac{\beta}{1+\beta}$. Construct a spatial representation f such that agent l_k has a constant weight along dimension k; then under condition iv) any agent with an ideal value different from $x^{l_k}_k$ also has a constant weight along dimension k given representation f. Condition v) deals with agents with the same ideal value as l_k, requiring them to have the same preferences as l_k over any lottery on attribute k.

Preferences representable by a quadratic Euclidean utility function without any weights must satisfy not only conditions $i-v$ for $\delta = 2$, but also the altogether implausible additional condition that all agents assign the same relative importance to each attribute. Notice however that if all agents assign the same relative importance to each attribute, there is no need to introduce different weights to different dimensions, since all the common weights can be normalized to one to represent all preferences by circles. Nevertheless, it is unrealistic to believe that preferences can be accurately represented by Euclidean utilities, which are at best a stylized approximation, and while the family of Weighted Minkowski δ-distance utility functions is a large generalization, the joint assumption of modularity, single-peakedness and conditions (i–iii) remains a rather stringent restriction on preferences.

A further generalization is possible, endogenizing the attributes or orders $\geq_k$. Perhaps preferences are not modular or not single-peaked given these exogenous attributes, but they are modular and single-peaked with respect to some different orders obtained endogenously, as in factor analysis.

Conceive of alternatives as being elements of an abstract set Y. We can then generalize the definition of the modularity and single-peakedness axioms to make them order specific, so that for any T arbitrary orders of the set of alternatives in Y, we define modularity of a preference relation with respect to such T orders, and single-peakedness with respect to the T orders. Then, a generalization of Theorem 1 can state that a spatial representation such the utility functions have the desired shape in $\mathbb{R}^K$ exists if and only axioms $A1$–$A2$ hold and there exist K orders on Y such that modularity, multi-attribute single-peakedness and conditions (*i*) (*ii*) and (*iii*) hold for these K orders.

For a graphic intuition of this further generalization, return to Fig. 1 and imagine that the exogenous attributes correspond not to the axes, but to the diagonals (i.e., the upright corner is the highest value on attribute one, and the up-left corner is the highest value on attribute two). The depicted preferences are not modular with respect to those attributes. Choosing endogenous attributes that correspond to the axes in the figure and not to its diagonals, it is possible to represent preferences with the desired shape. Note that this is a step toward Bogomolnaia and Laslier's (2007) approach of endogenizing the dimensions, for the sake of accommodating preference profiles that cannot otherwise be represented by utility functions of attractive shape.[4]

Leaving the cumbersome expressions and possible generalizations aside, the relevant insight of Theorem 1 is that there is a family of classes of utility functions parameterized by δ that correspond to a continuum of classes of preferences distinguished by the difference in risk attitudes over values "near" the ideal of an agent, and risk attitudes over values "far" from the ideal point. The standard Euclidean representations with the specific value $\delta = 2$, do not seem a priori to be any more plausible than representations with any other value of parameter δ.

2.1 A test to choose the appropriate representation

Whether preference profiles are more accurately represented by smooth, strictly quasiconcave utility functions with $\delta > 1$, or by utility functions with $\delta \leq 1$ that are neither smooth, nor strictly quasiconcave, is an empirical question. From a rigorous point of view, the conditions for Theorem 1 to hold are very restrictive for any δ. It is implausible that they hold exactly. The relevant question is which δ and which functional forms are the best approximation to an accurate utility representation of the true preference profile. I propose a simple test to answer this question.

[4] It is straightforward to generalize Theorem 1 in a different direction, to characterize the set of preference profiles that are representable by utility functions that are Weighted Minkowski δ^i-distances on $f(X)$, so that the spatial representation of the set of alternatives is common to all agents, but the shape of the utility functions are agent-specific. For any vector $\delta = (\delta^1, \ldots, \delta^n) \in \mathbb{R}^n_{++}$, the statement and proof of the more general theorem convey limited new intuition, as they merely substitute δ for the appropriate δ^i at each step. This is omitted, but available from the author.

For an intuition for this test, choose an arbitrary attribute k, fix the values on all other attributes at x_{-k}, assume that there is a continuum of values in X_k that can be indexed by real numbers so that $X_k = [0, 1]$, assume that preferences are continuous in X_k, and let l_k be denoted simply by l. Then for any $i \in N$, we can find a value $x_k^{m_i}$ between the ideal values of agents l and i such that each l and i are indifferent between $x_k^{m_i}$ and a lottery on attribute k that grants $j \in \{l, i\}$ her ideal value x_k^j with probability α and the ideal point of the other player with probability $1 - \alpha$. Formally, let $p_k, q_k \in \Delta X_k$ be such that $p_k(x_k^l) = \alpha$, $p_k(x_k^i) = 1 - \alpha$, $q_k(x_k^i) = \alpha$ and $q_k(x_k^l) = 1 - \alpha$; then $(x_k^{m_i}, x_{-k}) \sim^l (p_k; x_{-k})$ and $(x_k^{m_i}, x_{-k}) \sim^i (q_k; x_{-k})$.

A representation $f_k(x_k)$ that maps $x_k^{m_i}$ to the midpoint between $f_k(x_k^l)$ and $f_k(x_k^i)$ generates utility functions with the same curvature for agents i and j over the set of points $\{x_k^l, x_k^{m_i}, x_k^i\}$. The concavity or convexity of an individual utility function risk given an endogenous spatial representation is not meaningful in terms of attitudes toward risk: It is always possible to make the utility function of an individual agent convex (risk loving) or concave (risk averse) on any given dimension by changing the spatial representation of alternatives. However, if we construct a spatial representation such that the utility functions of the agents all have the same curvature, we can interpret this curvature as an aggregate measure of risk attitudes in society. The value of α that makes agents indifferent between $x_k^{m_i}$ for sure or a lottery between their ideal or the other agent's ideal value identifies the only risk attitude that is consistent with both agents having the same risk attitude. For instance, if $\alpha > 0.5$, then it is not possible to represent both i and j with linear loss utility functions, whereas if $\alpha \leq 0.5$, it is not possible to represent both i and j with concave utility functions.

Proposition 3 *Given an arbitrary attribute k, arbitrary values x_{-k} on all other attributes, a pair of agents i and j and any $\alpha \in (0, 1)$, let the lotteries $p_k^\alpha, q_k^\alpha \in \Delta X_k$ be such that $p_k^\alpha(x_k^i) = \alpha$, $p_k^\alpha(x_k^j) = 1 - \alpha$, $q_k^\alpha(x_k^j) = \alpha$ and $q_k^\alpha(x_k^i) = 1 - \alpha$. Let $u^i(x)$ and $u^j(x)$ be utility functions that represent $\succsim^i$ and $\succsim^j$. Given a spatial representation $f \in \mathcal{F}$, let $v^h : \mathbb{R}^K \longrightarrow \mathbb{R}$ be such that $v^h(f(x)) = u^h(x)$ for $h \in \{i, j\}$.*

If there exist a value $x_k^m \in X_k$ and $\alpha \in (0, \frac{1}{2})$ such that given x_{-k} agent i is indifferent between x_k^m and lottery p_k^α on attribute k, and agent j is indifferent between x_k^m and lottery q_k^α on attribute k, then for any spatial representation $f \in \mathcal{F}$, either $v^i(f(x))$ or $v^j(f(x))$ is not concave.

Notice that if preferences are continuous in X_k, then the pair (x_k^m, α) exists. We only need to empirically estimate their values. If we find that the estimate for α is $\hat{\alpha} < 0.5$, utility functions with a linear or concave loss function cannot represent the preferences, calling into question the received wisdom in the literature that assumes these utility functions.

Proposition 3 applies to preferences that are not necessarily modular or multi-attribute single-peaked. If we restrict preferences to be modular and multi-attribute single-peaked, and we assume that the conditions in Theorem 1 hold so that the preferences are representable by utility functions that are Weighted Minkowski δ-distances, we can use the same test to estimate the parameter δ and the appropriate shape of the indifference curves.

Proposition 4 *Let there exist $\delta > 0$ and a spatial representation $f \in \mathcal{F}$ such that each utility function $u^i(x)$ is a Weighted Minkowski δ-distance on $f(X)$. Given an arbitrary attribute k, agents i, j such that $x^i_k <_k x^j_k$ and any $\alpha \in (0, 1)$, let the lotteries $p^\alpha_k, q^\alpha_k \in \Delta X_k$ be such that $p^\alpha_k(x^i_k) = \alpha$, $p^\alpha_k(x^j_k) = 1 - \alpha$, $q^\alpha_k(x^j_k) = \alpha$ and $q^\alpha_k(x^i_k) = 1 - \alpha$.*

If there exist a value $x^m_k \in X_k$ and $\alpha \in (0, 1)$ such that given arbitrary values x_{-k} on all other attributes, agent i is indifferent between x^m_k and lottery p^α_k and agent j is indifferent between x^m_k and lottery q^α_k on attribute k, then

$$\delta = \frac{-\ln(1-\alpha)}{\ln 2}.$$

Once again, if preferences are continuous in X_k, the pair (x^m_k, α) exists. Choosing an arbitrary pair of agents and an arbitrary attribute, we obtain a simple necessary condition for representability of the preference profile by indifference curves of a particular curvature within the family of Weighted Minkowski shapes. City block linear shapes require $\alpha = \frac{1}{2}$, while weighted Euclidean shapes require $\alpha = \frac{3}{4}$. More generally, quasiconcave and smooth shapes require $\alpha > \frac{1}{2}$.

3 Discussion

Given alternatives that are objects with multiple attributes, if the values within each attribute are not objectively quantifiable, any spatial representation is subjective, arbitrary, or made for convenience. The primitive space of objects is a subset of the Cartesian product of the set of possible values in each attribute. Any assumption on preferences over alternatives in the spatial representation of the space of objects is a joint assumption on preferences over alternatives, and on the chosen spatial representation of the preferences.

I characterize the set of preference profiles such that there exists a spatial representation $f(X)$ common to all agents such that the utility of every agent is linearly decreasing in the δ power of a weighted version of a generalization of a Minkowski (1886) distance function.

While the necessary and sufficient conditions for representability by some utility functions in the class given by parameter δ are cumbersome, I have identified a much simpler necessary condition, which relies on a single value $\alpha \in (0, 1)$ on the relation between the preferences over lotteries by a given pair of agents. Representability by generalized weighted linear city block utilities requires $\alpha = 0.5$, while generalized quadratic Euclidean requires $\alpha = 0.75$. Finding the value of α for any pair of agents and any attribute is a testable empirical question.

Evidence that $\alpha \approx 0.75$ would support the assumption of quadratic Euclidean preferences, standard in the literature. If $\alpha > 0.5$, Euclidean representations may be a good enough approximation that yields the correct qualitative insights: Utility functions are smooth and strictly quasiconcave, and circles (or weighted ovoids) are just a simplification of more nuanced shapes of indifference curves.

On the other hand, evidence that $\alpha \leq 0.5$ would imply that the assumption of quadratic Euclidean preferences is unwarranted and qualitatively misleading, calling into question much of the received wisdom that relies on concavity, quasiconcavity or differentiability assumptions, such as classic results on the generic emptiness of simple majority core (Plott 1967), the intransitivity of social preferences (McKelvey and Schofield 1987; McKelvey 1976, 1979), or results on equilibria of games of repeated elections (Banks and Duggan 2008), on the falsifiability of the assumption that voters vote ideologically (Degan and Merlo 2009) or on common agency lobbying (Baron and Hirsch 2011). In another example of how classic results are overturned if utility functions are convex, Kamada and Kojima (2010) shows that the converge to the median in electoral competition with two office-motivated candidates (Downs 1957) breaks down in a probabilistic voting model.

While the theoretical implications are contingent on the results of the suggested empirical test, the implication for applied research has more immediate consequences: Empirical methods that seek to locate legislators in the space will obtain better results if instead of assuming that $\alpha = 0.75$ and $\delta = 2$ and using only Euclidean indifference curves, they take a more agnostic approach with respect to the shape of indifference curves, and they try to estimate the value of δ, perhaps by searching on a grid of values at discrete intervals from 0 to some finite number, and then choosing the indifference curves with the best fit.

Such applications of spatial models include ideal point estimation (Clinton et al. 2004; Poole and Rosenthal 1985), and models of elections (Schofield and Sened 2006; Schofield 2007), the formation of government coalitions (Sened 1996), or the legislative process (Bianco and Sened 2005). In all these applications, the models' fit and predictive power can increase by the simple expedient of searching for the best parameter values. Indeed, Grynaviski and Corrigan (2006) and Westholm (1997) obtain better results fitting electoral outcomes when they discard the Euclidean utilities and use instead a city block ($\delta = 1$) utility function. Zakharov and Fantazzinni (2008) report an better fit using different weights for different agents. However, neither of these authors considers utility functions with a value of δ other than $\delta = 1$ or $\delta = 2$.

I have provided an axiomatic foundation for the use of a more general class of utility representations. Using the best representation of preferences within this larger class ought to help applied researchers to obtain better results in the future.

Appendix: Proof of Theorem 1

Proof ($\Longrightarrow$). Suppose that $u^i(x) = -\sum_{k=1}^{K} w_k^i(f_k(x_k) - f(x_k^i))[f_k(x_k) - f(x_k^i)]^\delta$ for every $i \in N$. Since the utility function depends on the position of points relative to other points, we can translate the map. Since utilities do not depend on the size of units, we can also rescale the map. Hence, without loss of generality, let $f_k(x_k^{l_k}) = 0$ and $f_k(x_k^{\max}) = 1$. By assumption, $p(x_k^{\max}, x_{-k}) = \left(\gamma_k^i\right)^\delta$ and $p(x_k^{l_k}, x_{-k}) = 1 - \left(\gamma_k^i\right)^\delta$ imply $p \sim^{l_k} (x_k^i, x_{-k})$. Let l be a shorthand notation for l_k. In utility terms,

$$\left(\gamma_k^i\right)^\delta u^l((x_k^{\max}, x_{-k})) + (1 - \left(\gamma_k^i\right)^\delta)u^l((x_k^l, x_{-k})) = u^l((x_k^i, x_{-k})).$$

$$\left(\gamma_k^i\right)^\delta w_{k+}^l (f_k(x_k^{\max}) - f_k(x_k^l))^\delta = w_{k+}^l (f_k(x_k^i) - f_k(x_k^l))^\delta$$
$$\gamma_k^i = f_k(x_k^i).$$

The second equality follows the first because the only difference between the three utility terms is on attribute k, and the second term on the left-hand side cancels out because x_k^l is the ideal value of l on attribute k. The third equality follows the second because $f_k(x_k^{l_k}) = 0$ and $f_k(x_k^{\max}) = 1$.

In condition i), in utility terms, $(p_k; x_{-k}) \sim^i (x_k^a, x_{-k}^i)$ if and only if

$$\begin{aligned}\alpha_i u^i((x_k^{\max}, x_{-k})) + (1-\alpha_i)u^i((x_k^i, x_{-k})) &= u^i((x_k^a, x_{-k}))\\ -\alpha_i w_{k+}^i [f_k(x_k^{\max}) - f_k(x_k^i)]^\delta &= -w_{k+}^i [f_k(x_k^a) - f_k(x_k^i)]^\delta\\ (\alpha_i)^{1/\delta}(1-\gamma_k^i) &= f_k(x_k^a) - \gamma_k^i\\ f_k(x_k^a) &= (\alpha_i)^{1/\delta}(1-\gamma_k^i) + \gamma_k^i.\end{aligned}$$

Similarly, $(x_k^a, x_{-k}) \sim^l (q_k; x_{-k})$ implies

$$\begin{aligned}u^l((x_k^a, x_{-k})) &= \alpha_l u^l((x_k^{\max}, x_{-k})) + (1-\alpha_l)u^l((x_k^i, x_{-k}))\\ -w_{k+}^l [f_k(x_k^a) - f(x_k^l)]^\delta &= -\alpha_l w_{k+}^l [f_k(x_k^{\max}) - f_k(x_k^l)]^\delta - (1-\alpha_l) w_{k+}^l\\ &\quad \times [f_k(x_k^i) - f_k(x_k^l)]^\delta\\ (f_k(x_k^a))^\delta &= \alpha_l + (1-\alpha_l)\left(\gamma_k^i\right)^\delta\\ [(\alpha_i)^{1/\delta}(1-\gamma_k^i) + \gamma_k^i]^\delta &= \alpha_l\left(1 - \left(\gamma_k^i\right)^\delta\right) + \left(\gamma_k^i\right)^\delta\\ \alpha_l &= \frac{[(\alpha_i)^{1/\delta}(1-\gamma_k^i) + \gamma_k^i]^\delta - \left(\gamma_k^i\right)^\delta}{1 - \left(\gamma_k^i\right)^\delta}.\end{aligned}$$

In condition *(ii)*, $(p_k; x_{-k}) \sim^i (x_k^b, x_{-k})$ if and only if

$$\begin{aligned}(1-\alpha_i)\left(\gamma_k^i\right)^\delta &= (\gamma_k^i - f_k(x_k^b))^\delta\\ f_k(x_k^b) &= \gamma_k^i - (1-\alpha_i)^{1/\delta}\gamma_k^i\\ f_k(x_k^b) &= \gamma_k^i[1 - (1-\alpha_i)^{1/\delta}]\end{aligned}$$

and $(q_k; x_{-k}) \sim^l (x_k^b, x_{-k})$ implies

$$\begin{aligned}(1-\alpha_l)(\gamma_k^i)^\delta &= (f_k(x_k^b))^\delta\\ (1-\alpha_l)^{1/\delta}\gamma_k^i &= \gamma_k^i[1 - (1-\alpha_i)^{1/\delta}]\\ (1-\alpha_l)^{1/\delta} &= 1 - (1-\alpha_i)^{1/\delta}\end{aligned}$$

$$\begin{aligned}1-\alpha_l &= [1-(1-\alpha_i)^{1/\delta}]^\delta\\ \alpha_l &= 1-[1-(1-\alpha_i)^{1/\delta}]^\delta.\end{aligned}$$

In condition *(iii)*, $(p_k; x_{-k}) \sim^i (x_k^c, x_{-k})$ if and only if

$$\begin{aligned}\alpha_i u^i((x_k^{\min}, x_{-k})) + (1-\alpha_i)u^i((x_k^i, x_{-k})) &= u^i((x_k^c, x_{-k}))\\ \alpha_i(\gamma_k^i - f_k(x_k^{\min}))^\delta &= (\gamma_k^i - f_k(x_k^c))^\delta.\end{aligned} \tag{2}$$

By assumption, $q(x_k^{\min}, x_{-k}) = \left(\frac{\gamma_k^{h_k}}{\gamma_k^{h_k}+\gamma_k^0}\right)^\delta$ and $q(x_k^{h_k}, x_{-k}) = 1-\left(\frac{\gamma_k^{h_k}}{\gamma_k^{h_k}+\gamma_k^0}\right)^\delta$ imply $q \sim^{h_k} (x_k^l, x_{-k})$. Let h denote h_k. In utility terms,

$$\begin{aligned}\left(\frac{\gamma_k^h}{\gamma_k^h+\gamma_k^0}\right)^\delta |f_k(x_k^{\min}) - \gamma_k^h|^\delta &= (\gamma_k^h)^\delta\\ \frac{1}{\gamma_k^h+\gamma_k^0}(\gamma_k^h - f_k(x_k^{\min})) &= 1\\ f_k(x_k^{\min}) &= -\gamma_k^0.\end{aligned}$$

Therefore, Eq. 2 becomes

$$\begin{aligned}\alpha_i(\gamma_k^i+\gamma_k^0)^\delta &= (\gamma_k^i - f_k(x_k^c))^\delta\\ -(\alpha_i)^{1/\delta}(\gamma_k^0+\gamma_k^i)+\gamma_k^i &= f_k(x_k^c).\end{aligned}$$

Furthermore, $(q_k; x_{-k}) \sim^h (x_k^c, x_{-k})$ implies

$$\begin{aligned}\alpha_h u^h((x_k^{\min}, x_{-k})) + (1-\alpha_h)u^h((x_k^i, x_{-k})) &= u^h((x_k^c, x_{-k}))\\ \alpha_h(\gamma_k^h+\gamma_k^0)^\delta + (1-\alpha_h)((\gamma_k^h-\gamma_k^i)^\delta &= (\gamma_k^h + (\alpha_i)^{1/\delta}(\gamma_k^0+\gamma_k^i) - \gamma_k^i)^\delta\\ \alpha_h[(\gamma_k^0+\gamma_k^h)^\delta - (\gamma_k^h-\gamma_k^i)^\delta] + (\gamma_k^h-\gamma_k^i)^\delta &= (\gamma_k^h + (\alpha_i)^{1/\delta}(\gamma_k^0+\gamma_k^i) - \gamma_k^i)^\delta\\ \alpha_h &= \frac{(\gamma_k^h - \gamma_k^i + (\gamma_k^0+\gamma_k^i)(\alpha_i)^{1/\delta})^\delta - (\gamma_k^h-\gamma_k^i)^\delta}{(\gamma_k^0+\gamma_k^h)^\delta - (\gamma_k^h-\gamma_k^i)^\delta}.\end{aligned}$$

($\Longleftarrow$). Following Fishburn (1970) chapter 11, given any $p \in \Delta X$, let $\hat{p}_k(S_k) = p(\{x : x \in X, x_k \in S_k\})$ for any $S_k \subseteq X_k$. Then, $\hat{p}_k$ is the probability measure on X_k induced by lottery p on X. Fishburn (1970), Theorem 11.1 shows that $u^i(x)$ is additively separable if and only if:

Condition 1 (Separability, Fishburn 1970) *For any* $p, q \in \Delta X$ *such that* $\hat{p}_k = \hat{q}_k \forall k \in A$, *and such that* $p(x), q(x) \in \{0, 1/2, 1\}$ $\forall x \in X$,

$$p \backsim^i q.$$

Lemma 3 *If $\succsim^i$ is transitive and modular, it satisfies Fishburn's separability condition.*

Proof Let p, q be such that $p(x) = p(y) = \frac{1}{2}$ and $q(w) = q(z) = \frac{1}{2}$, where $\{w_k, z_k\} = \{x_k, y_k\}$ for each $k \in A$. By modularity, $p = L(x, y) \backsim^i L(x \vee y, x \wedge y)$ and $q = L(w, z) \backsim^i L(w \vee z, w \wedge z)$. Since $w \vee z = x \vee y$ and $w \wedge z = x \wedge y$, by transitivity of $\succsim^i$, it follows $p \backsim^i q$, Fishburn's separability condition is satisfied, and $u(x)$ is additively separable. □

Since $\succsim^i$ is transitive and modular, the utility function $u^i(x)$ that represents $\succsim^i$ is therefore additively separable, $u^i(x) = \sum_{k=1}^{K} u_k^i(x_k)$ and we can normalize to let $u_k^i(x_k^i) = 0$ for each $i \in N$ and $k \in A$. For an arbitrary $k \in A$, let $l_k, h_k \in N$ be such that $x^{l_k} \leq_k x^i \leq_k x^{h_k}$. If this does not uniquely define l_k, arbitrarily choose one of the agents with the lowest ideal value on attribute k and label her l_k. Similarly for h_k. Fix $f_k(x_k^{l_k}) = 0$ and $f_k(x_k^{\max}) = 1$, and for any $x_k \geq_k x_k^{l_k}$, let $f_k(x_k)$ be such that $u_k^{l_k}(x_k)$ is linearly decreasing in $|f_k(x_k) - f_k(x_k^{l_k})|^\delta$. Then, $f_k(x_k^i) = \gamma_k^i$, where γ_k^i is defined by the lottery $p \sim^{l_k} (x^i, x_{-k})$ and $p(x_k^{\max}, x_{-k}) = \left(\gamma_k^i\right)^\delta$ and $p(x_k^{l_k}, x_{-k}) = 1 - \left(\gamma_k^i\right)^\delta$.

To check that $u_k^i(x_k)$ is linearly decreasing in $|f_k(x_k) - f_k(x_k^i)|^\delta$ for any $i \in N$ and for any $x_k \geq_k x_k^i$, let $x_k^a \geq x_k^i$ and $p \in \Delta X$ be such that $p(x_k^{\max}, x_{-k}) = \alpha_i$, $p(x_k^i, x_{-k}) = 1 - \alpha_i$ and $p \sim^i (x_k^a, x_{-k})$, so

$$u_k^i(x_k^a) = \alpha_i u_k^i(x_k^{\max}) + (1 - \alpha_i) u_k^i(x_k^i) = \alpha_i u_k^i(x_k^{\max})$$

Since $f_k(x_k^i) = \gamma_k^i$ and $f_k(x_k^{\max}) = 1$, in order for $u_k^i(x_k^a)$ to be linearly decreasing in $|f_k(x_k^a) - f_k(x_k^i)|^\delta$ for any $x_k^a \geq_k x_k^i$, we want to show that

$$\begin{aligned}(f_k(x_k^a) - \gamma_k^i)^\delta &= \alpha_i (1 - \gamma_k^i)^\delta \\ f_k(x_k^a) &= (1 - \gamma_k^i)\,(\alpha_i)^{1/\delta} + \gamma_k^i\end{aligned}$$

Since

$$q(x_k^{\max}, x_{-k}) = \frac{\left(\gamma_k^i + (1 - \gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^i)^\delta}{1 - (\gamma_k^i)^\delta} \text{ and}$$

$$q(x_k^i, x_{-k}) = 1 - \frac{\left(\gamma_k^i + (1 - \gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^i)^\delta}{1 - (\gamma_k^i)^\delta}$$

together imply $q \sim^{l_k} (x_k^a, x_{-k})$,

$$\begin{aligned}&\frac{\left(\gamma_k^i + (1 - \gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^i)^\delta}{1 - (\gamma_k^i)^\delta} + \left(1 - \frac{\left(\gamma_k^i + (1 - \gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^i)^\delta}{1 - (\gamma_k^i)^\delta}\right)(\gamma_k^i)^\delta \\ &\quad = \left(f_k(x_k^a)\right)^\delta\end{aligned}$$

$$\left(\frac{\left(\gamma_k^i + (1-\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^i)^\delta}{1-(\gamma_k^i)^\delta}\right)(1-(\gamma_k^i)^\delta) + (\gamma_k^i)^\delta = \left(f_k(x_k^a)\right)^\delta$$

$$\left(\gamma_k^i + (1-\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta = \left(f_k(x_k^a)\right)^\delta$$

$$\gamma_k^i + (1-\gamma_k^i)\alpha_i^{1/\delta} = f_k(x_k^a)$$

as desired. Hence for every $i \in N$, $u_k^i(x_k)$ is linearly decreasing in $|f_k(x_k^a) - f_k(x_k^i)|^\delta$ from x_k^i to $x_k^{\max}$.

Similarly, for any $x_k^b \in X_k$ such that $x_k^{l_k} \leq_k x_k^b \leq_k x_k^i$ and $p \in \Delta X$ such that $p(x_k^i, x_{-k}) = \alpha_i$, $p(x_k^{l_k}, x_{-k}) = 1 - \alpha_i$ and $p \sim^i (x_k^b, x_{-k})$,

$$\begin{aligned} \alpha_i u_k^i(x_k^i) + (1-\alpha_i)u_k^i(x_k^{l_k}) &= u_k^i(x_k^b) \\ (1-\alpha_i)u_k^i(x_k^{l_k}) &= u_k^i(x_k^b). \end{aligned}$$

Since $f_k(x_k^l) = 0$ and $f_k(x_k^i) = \gamma_k^i$, we want to show that

$$\begin{aligned} (1-\alpha_i)(\gamma_k^i)^\delta &= (\gamma_k^i - f_k(x_k^b))^\delta \\ (1-\alpha_i)^{1/\delta}\gamma_k^i &= \gamma_k^i - f_k(x_k^b) \\ f_k(x_k^b) &= \gamma_k^i(1-(1-\alpha_i)^{1/\delta}). \end{aligned}$$

Since $q(x_k^l, x_{-k}) = 1 - \left(1-(1-\alpha_i)^{1/\delta}\right)^\delta$, $q(x_k^i, x_{-k}) = \left(1-(1-\alpha_i)^{1/\delta}\right)^\delta$ imply $q \sim^l (x_k^b, x_{-k})$,

$$\begin{aligned} \left(1-(1-\alpha_i)^{1/\delta}\right)^\delta (\gamma_k^i)^\delta &= (f_k(x_k^b))^\delta \\ \gamma_k^i[1-(1-\alpha_i)^{1/\delta}] &= f_k(x_k^b). \end{aligned}$$

Hence, $u_k^i(x_k)$ is linearly decreasing in $|f_k(x_k^b) - f_k(x_k^i)|^\delta$ for any x_k^b between x_k^l and x_k^i, and, as shown earlier, for any x_k between x_k^i and $x_k^{\max}$. It remains to be shown that $u_k^i(x_k)$ is linearly decreasing in $|f_k(x_k^b) - f_k(x_k^i)|^\delta$ for any $x_k \leq_k x_k^l$, and with the same slope as between x_k^l and x_k^i. Let h denote h_k. For any $x_k \leq_k x_k^l$, construct $f_k(x_k)$ such that $u_k^h(x_k)$ is linearly decreasing in $|f_k(x_k^h) - f_k(x_k)|^\delta$ for any $x_k \leq_k x_k^h$. Then $f_k(x_k^{\min}) = -\gamma_k^0$. For any $i \in N$ and for any $x_k^c \leq_k x_k^i$, given $p(x_k^{\min}, x_{-k}) = \alpha_i$ and $p(x_k^i, x_{-k}) = 1 - \alpha_i$, if $p \sim^i (x_k^c, x_{-k})$, then

$$\alpha_i u_k^i(x_k^{\min}) = u_k^i(x_k^c).$$

Since $f_k(x_k^i) - f_k(x_k^{\min}) = \gamma_k^i + \gamma_k^0$, we want to show

$$\alpha_i(\gamma_k^i + \gamma_k^0)^\delta = (\gamma_k^i - f_k(x_k^c))^\delta$$
$$f_k(x_k^c) = \gamma_k^i - (\gamma_k^i + \gamma_k^0)(a_i)^{1/\delta}$$

Given $q(x_k^{\min}, x_{-k}) = \frac{\left(\gamma_k^h - \gamma_k^i + (\gamma_k^0+\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^h - \gamma_k^i)^\delta}{(\gamma_k^0+\gamma_k^h)^\delta - (\gamma_k^h-\gamma_k^i)^\delta}$ and $q(x_k^i, x_{-k}) = 1 - \frac{\left(\gamma_k^h - \gamma_k^i + (\gamma_k^0+\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^h - \gamma_k^i)^\delta}{(\gamma_k^0+\gamma_k^h)^\delta - (\gamma_k^h-\gamma_k^i)^\delta}$, $(x_k^c, x_{-k}) \sim^h q$ implies

$$\begin{aligned}
(\gamma_k^h - f_k(x_k^c))^\delta &= \frac{\left(\gamma_k^h - \gamma_k^i + (\gamma_k^0+\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^h - \gamma_k^i)^\delta}{(\gamma_k^0+\gamma_k^h)^\delta - (\gamma_k^h-\gamma_k^i)^\delta}(\gamma_k^h + \gamma_k^0)^\delta \\
&\quad + \left(1 - \frac{\left(\gamma_k^h - \gamma_k^i + (\gamma_k^0+\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^h - \gamma_k^i)^\delta}{(\gamma_k^0+\gamma_k^h)^\delta - (\gamma_k^h-\gamma_k^i)^\delta}\right) \\
&\quad \times (\gamma_k^h - \gamma_k^i)^\delta; \\
(\gamma_k^h - f_k(x_k^c))^\delta &= (\gamma_k^h - \gamma_k^i)^\delta + \frac{\left(\gamma_k^h - \gamma_k^i + (\gamma_k^0+\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_k^h - \gamma_k^i)^\delta}{(\gamma_k^0+\gamma_k^h)^\delta - (\gamma_k^h-\gamma_k^i)^\delta} \\
&\quad \times \left[(\gamma_k^h + \gamma_k^0)^\delta - (\gamma_k^h - \gamma_k^i)^\delta\right] \\
(\gamma_k^h - f_k(x_k^c))^\delta &= \left(\gamma_k^h - \gamma_k^i + (\gamma_k^0+\gamma_k^i)\alpha_i^{1/\delta}\right)^\delta \\
\gamma_k^h - f_k(x_k^c) &= \gamma_k^h - \gamma_k^i + (\gamma_k^0+\gamma_k^i)\alpha_i^{1/\delta} \\
\gamma_k^i - (\gamma_k^0+\gamma_k^i)\alpha_i^{1/\delta} &= f_k(x_k^c)
\end{aligned}$$

as desired.

Thus, for every $i \in N$, there exists weights w_{k+} and w_{k-} such that $u_k^i(x_k) = -w_{k+}|f_k(x_k) - f_k(x_k^i)|^\delta$ for every $x_k \geq_k x_k^i$ and $u_k^i(x_k) = -w_{k-}|f_k(x_k) - f_k(x_k^i)|^\delta$ for every $x_k \leq_k x_k^i$. Choosing the appropriate relative weights for each attribute k, we obtain $u^i(x) = -\sum_{k=1}^{K} w_k(x_k, x_k^i)|f_k(x_k) - f_k(x_k^i)|^\delta$, where $w_k(x_k, x_k^i) = \{w_{k-}$ if $x_k \leq_k x_k^i$ and w_{k+} if $x_k >_k x_k^i\}$. □

Proof of Proposition 2

Proof ($\Longrightarrow$). Suppose that $u^i(x) = -\sum_{k=1}^{K} w_k^i[f_k(x_k) - f(x_k^i)]^\delta$ for every $i \in N$. By Theorem 1, axioms $A1$–$A4$ and conditions $(i$–$iii)$ are satisfied.

Suppose first that $x_k^i =_k x_k^{l_k}$; then $(p_k; x_{-k}) \succsim^i (q_k; x_{-k})$ if and only if

$$-w_k^i \sum_{X_k} p_k(x_k)[f_k(x_k) - f(x_k^{l_k})]^\delta \geq -w_k^i \sum_{X_k} p_k(x_k)[f_k(x_k) - f(x_k^{l_k})]^\delta$$
$$-w_k^{l_k} \sum_{X_k} p_k(x_k)[f_k(x_k) - f(x_k^{l_k})]^\delta \geq -w_k^{l_k} \sum_{X_k} p_k(x_k)[f_k(x_k) - f(x_k^{l_k})]^\delta$$

which holds if and only if $(p_k; x_{-k}) \succsim^{l_k} (q_k; x_{-k})$, hence condition v is satisfied.

Suppose instead that $x_k^{l_k} <_k x_k^i <_k x_k^{\max}$ and choose β_i such that $(r_k; x_{-k}) \sim^{l_k} (x^i; x_{-k})$. Then,

$$[f_k(x_k^i) - f(x_k^{l_k})]^\delta = \frac{\beta_i}{(1+\beta_i^{1/\delta})^\delta}[f_k(x_k^{\max}) - f(x_k^i) + f_k(x_k^i) - f(x_k^{l_k})]^\delta$$
$$[f_k(x_k^i) - f(x_k^{l_k})](1+\beta^{1/\delta}) = \beta_i^{1/\delta}[f_k(x_k^{\max}) - f(x_k^i) + f_k(x_k^i) - f(x_k^{l_k})]$$
$$f_k(x_k^i) - f(x_k^{l_k}) = \beta_i^{1/\delta}[f_k(x_k^{\max}) - f(x_k^i)]$$

and therefore

$$u_k^i(x_k^i) - u(x_k^{l_k}) = \beta_i[u(x_k^{\max}) - u(x_k^i)]$$

and hence $(p_k; x_{-k}) \sim^i (q_k; x_{-k})$.

($\Longleftarrow$) By Theorem 1, we know that there exist $f \in \mathcal{F}$ and weights $w^i \in \mathbb{R}_+^{2K}$ for each $i \in N$ such that $u^i(x) = -\sum_{k=1}^{K} w_k^i(x_k, x_k^i)|f_k(x_k) - f_k(x_k^i)|^\delta$. We want to show that $w_k^i(x_k, x_k^i) = w_k^i$ for any $x_k \in X_k$, any $k \in A$ and any $i \in N$. By separability, it suffices to show that it holds for an arbitrary k. Construct f such that agent l_k has a constant weight over attribute k. By condition v, any $i \in N$ such that $x_k^i = x_k^{l_k}$ also has a constant weight. Consider any $i \in N$ such that $x_k^{l_k} <_k x_k^i <_k x_k^{\max}$. Let β_i and δ be such that $(p_k; x_{-k}) \sim^i (q_k; x_{-k})$ and $(r_k; x_{-k}) \sim^{l_k} (x^i; x_{-k})$. Then, $(r_k; x_{-k}) \sim^{l_k} (x^i; x_{-k})$ implies $f_k(x_k^i) - f(x_k^{l_k}) = \beta_i^{1/\delta}[f_k(x_k^{\max}) - f(x_k^i)]$, while $(p_k; x_{-k}) \sim^i (q_k; x_{-k})$ implies

$$-\rho w_{k-}^i[f_k(x_k^i) - f(x_k^{l_k})]^\delta = -\rho\beta_i w_{k+}^i[f_k(x_k^{\max}) - f(x_k^i)]^\delta$$
$$w_{k-}^i[\beta_i^{1/\delta}[f_k(x_k^{\max}) - f(x_k^i)]]^\delta = \beta_i w_{k+}^i[f_k(x_k^{\max}) - f(x_k^i)]^\delta$$
$$w_{k-}^i = w_{k+}^i.$$

Finally, the weight w_{k+}^i of any agent such that $x_k^i =_k x_k^{\max}$ is arbitrary, so we can let it be $w_{k+}^i = w_{k-}^i$. □

Proof of Proposition 3

Proof Normalize to let $f_k(x_k^i) = 0$, $f_k(x_k^j) = 1$, $u^i(x_k^i, x_{-k}) = 1$, $u^i(x_k^j, x_{-k}) = 0$, $u^j(x_k^j, x_{-k}) = 1$ and $u^j(x_k^j, x_{-k}) = 0$. The indifference of agent i between x_k^m and lottery p_k^α implies that $u^i(x_k^m, x_{-_k}) = \alpha$ and similarly the indifference of j between x_k^m and the lottery implies $u^j(x_k^m, x_{-_k}) = \alpha$. Concavity of v^i requires

$$\begin{aligned} v^i(f_k(x_k^m), f_{-k}(x_{-k})) &= u^i(x_k^m, x_{-_k}) \geq f_k(x_k^m)u^i(x_k^j, x_{-k}) + (1 - f_k(x_k^m))u^i(x_k^i, x_{-k}) \\ &= 1 - f_k(x_k^m) \\ f_k(x_k^m) &\geq 1 - \alpha \end{aligned}$$

Concavity if v^j requires

$$\begin{aligned} v^j(f_k(x_k^m), f_{-k}(x_{-k})) &= u^j(x_k^m, x_{-_k}) \geq f_k(x_k^m)u^j(x_k^j, x_{-k}) \\ &\quad + (1 - f_k(x_k^m))u^j(x_k^i, x_{-k}) = f_k(x_k^m) \\ f_k(x_k^m) &\leq \alpha \end{aligned}$$

If $\alpha < \frac{1}{2}$, both requirements are incompatible. □

Proof of Proposition 4

Proof Since $u^i(x)$ is additively separable, let $(u_1^i, \ldots, u_K^i)$ be such that $u^i(x) = \sum_{k=1}^{K} u_k^i(x_k)$ and $u_k^i(x_k^i) = 0$.

Let $v^i(x)$ be a different utility representation of $\succsim^i$, such that

$$v_k^i(x_k) = \frac{u_k^i(x_k) - u_k^i(x_k^j)}{-u_k^i(x_k^j)} = 1 + \frac{w_k^i(x_k, x_k^i)}{u_k^i(x_k^j)} |f_k(x_k) - f_k(x_k^i)|^\delta. \tag{3}$$

Note $v_k^i(x_k^j) = 0$ and $v_k^i(x_k^i) = 1$. Then, $(x_k^m, x_{-k}) \sim^i (p_k^\alpha; x_k)$ implies $v_k^i(x_k^m) = \alpha$. Similarly, let $v^j(x)$ be a utility representation of $\succsim^j$ such that

$$v_k^j(x_k) = \frac{u_k^j(x_k) - u_k^j(x_k^i)}{-u_k^j(x_k^i)} = 1 + \frac{w_k(x_k, x_k^j)}{u_k^j(x_k^i)} |f_k(x_k) - f_k(x_k^j)|^\delta. \tag{4}$$

Then, $v_k^j(x_k^i) = 0$ and $v_k^j(x_k^j) = 1$, and $(x_k^m, x_{-k}) \sim^j (q_k^\alpha; x_k)$ implies $v_k^j(x_k^m) = \alpha$. Without loss of generality, let $f_k(x_k^i) = 0$ and $f_k(x_k^j) = 1$. Then by Eq. 3, $v_k^i(x_k^j) = 0$

implies

$$\begin{aligned}0 &= 1 + \frac{w_k(x_k^j, x_k^i)}{u_k^j(x_k^i)}\\ \frac{w_{k+}^i}{u_k^i(x_k^j)} &= -1\\ w_{k+}^i &= -u_k^i(x_k^j)\end{aligned}$$

and therefore, substituting back in Eq. 3, $v_k^i(x_k) = 1-|f_k(x_k)|^\delta$ for any $x_k \geq_k x_k^i$. Similarly, from Eq. 4 and $v_k^j(x_k^i) = 0$, we obtain $w_{k-}^j = -u_k^j(x_k^i)$, and thus $v_k^j(x_k) = 1-|1 - f_k(x_k)|^\delta$ for any $x_k \leq_k x_k^j$.

In particular, x_k^m is such that $x_k^i \leq_k x_k^m \leq x_k^j$, and $v_k^i(x_k^m) = v_k^j(x_k^m) = \alpha$. Therefore,

$$1 - |f_k(x_k^m)|^\delta = \alpha = 1 - |1 - f_k(x_k^m)|^\delta. \tag{5}$$

Since $f_k(x_k)$ is strictly increasing in $\geq_k$, and since $f_k(x_k^i) = 0$, $f_k(x_k^j) = 1$ and $x_k^i \leq_k x_k^m \leq_k x_k^j$, it follows $f_k(x_k^m) \in [0, 1]$ Equation 5 then implies

$$\begin{aligned}(f_k(x_k^m))^\delta &= (1 - f_k(x_k^m))^\delta\\ f_k(x_k^m) &= 1 - f_k(x_k^m)\\ f_k(x_k^m) &= 1/2.\end{aligned}$$

Then, $1 - f_k(x_k^m)^\delta = \alpha$ implies

$$\begin{aligned}\frac{1}{2^\delta} &= 1 - \alpha\\ \ln\frac{1}{2^\delta} &= \ln(1-\alpha)\\ -\delta \ln 2 &= \ln(1-\alpha)\\ \delta &= \frac{-\ln(1-\alpha)}{\ln 2}.\end{aligned}$$

□

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