Symmetrical Positional Scoring Rules

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Abstract

Young (1974) developed a classic axiomatization of the Borda rule. He proved that the normative properties of Neutrality, Consistency, and Faithfulness requires that any proper Social Choice Function must be a positional scoring rule. Then adding a final condition of Cancellation, uniquely defines the Borda rule. We revisit the Cancellation property and explain that the conditions under which this property can be invoked are so strict that no voting rule is likely to violate the property in practice. We present simulations which corroborate our hypothesis. We then replace Cancellation with a new property which limits the set of feasible Social Choice functions to the class of symmetrical positional systems, of which the Borda rule is a special case.

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1 Introduction

May (1952) has shown that majority rule on a binary choice set is the only rule which satisfies the four specific properties of decisiveness, anonymity, neutrality, and positive responsiveness. May's properties hold for any rule which simplifies to majority rule in the case of two alternatives. One class of such rules are scoring rules which requires voters to rank all (or a subset of) the alternatives which are then aggregated by weights dependent on their relative ranking by each voter and the alternative with the most collected points is selected as the winner.

Positional scoring rules assign more (or at least as many) points to an alternative the higher it is ranked by a voter. It would be very unusual for a socring rule not to be positional and in practice, the terms "scoring rule" and "positional system" are often used interchangeably. We keep them distinct for a later reference below.

The two positional scoring rules which receive the most attention are plurality and Borda which represent opposite ends of spectrum in terms of voter information supplied. Under plurality, a voter's most preferred alternative is given one point and all other alternatives are treated equally by receiving zero points. For Borda, every alternative is assigned a different number of points. Specifically, an alternative is awarded one point for each competitor ranked below it in the voter's ranking. It should be clear that for the limiting case of two alternatives, both rules assign one point to the alternative preferred by the voter and zero to the other. The alternative with the most points must then be preferred by a majority of voters unless there is a tie, making these procedures identical to majority rule.

The rules differ in their methodologies when there are more than two alternatives. Young (1974) proved that Borda was the only rule which simultaneously satisfies the properties of neutrality, cancellation, faithfulness, and consistency, and properly defines a choice set of "best" elements for any number of alternatives under consideration. By reverse logic, because Borda is equivalent to majority rule in the limiting case of two alternatives, then any rule which reduces to majority rule in that case will also satisfy Young's properties when there are only two alternatives. But when there are more than two alternatives, every rule other than Borda must violate at least one of these properties. The implication thus being that if violations of these properties are considered unacceptable then Borda must be used.

In this paper we revisit these properties. In particular, we note that probabilistically, the likelihood of violating cancellation is very low if not nearly impossible and simulations show the probability of *any* rule violating cancellation is effectively zero. The rest of the properties are guaranteed to be satisifed by any positional scoring rule. We introduce a new property called *inverse clone contraction* which is weaker than cancellation and which, when combined with the other properties, requires the positional system to be symmetric. Simulations show the propensity of various asymmetric positional systems to violate ICC.

2 Borda Rule properties

Let $N = \{1, ..., n\}$ represent the set of voters deciding on the set of alternatives $Z = \{1, ..., z\}$. Each voter i has a preference order D_i consisting of a strong ordering of all possible alternatives. $D = \{D_1, ..., D_n\}$ then represents the vector of preference orderings of D_i on N. Let aP_ib stand for voter i "prefers" alternative a over alternative b. Let n^{ab} represent the number of voters who prefer a over b.

Scoring rules represent a general class of rules such that each voter ranks the alternatives, each of which receives σ_r points from a voter if the candidate is ranked r^{th} by the voter. Denote $\sigma = \{\sigma_1, ..., \sigma_z\}$ as the vector of points assigned to each rank. Define the total number of points acumulated by any alternative a as Ω^a. A *choice set* $\zeta(D)$ is comprised of alternatives that are at least as good as everything else in the sense that that they are ranked at least as high by the voting mechanism as all the other alternatives. For scoring rules, alternative $a \epsilon \zeta(D)$ if and only if $\Omega^a \geq \Omega^b \ \forall b \epsilon Z$. A *Social Choice Function* (SCF) is a function Γ which always returns a non-empty choice set for any profile D. Scoring rules are valid SCFs.

Positional systems are scoring rules in which $\sigma_r \geq \sigma_h \ \forall r < h$, and $\sigma_1 > \sigma_z$. Thus, positional scoring rules ensure that alternatives ranked higher by a voter receive at least as many points as those ranked lower. Scoring rules which are not positional are clearly perverse and are not used in practice or discussed in the general literature. As such, scholars often do not distinguish between positional systems and the more general scoring rules, typically using the term scoring rule with the implicit (or explicit) understanding that the scoring rule is positional. A *strong positional system* is one in which $\sigma_r > \sigma_h \ \forall r < h$. A *weak positional system* is a scoring rule where $\sigma_r \geq \sigma_h \ \forall r < h$, and$\exists s, h$ such that $\sigma_s = \sigma_h$, and $\sigma_1 > \sigma_z$.

The two positional systems which generate the most attention are the simple plurality rule and the Borda count. Plurality rule assigns points such that $\sigma_1 = 1$, $\sigma_r = 0 \ \forall r \neq 1$ whereas Borda rule assigns points such that $\sigma_r = z - r \ \forall r = \{1, ..., z\}$. We note for later purpose that any scoring rule in which $\sigma_r = k - wr$ $w > 0, \forall r = \{1, ..., z\}$ yields the same weak ordering of alternatives and is functionally equivalent to the Borda count as described here, which can be derived by setting $k = z$ and $w = 1$ whereas Young (1974) assumed $k = z + 1, \ w = 2$ in order to center the weights around zero. The critical point for Bordian rules is that the difference in points assigned to each subsequent rank remains constant.

Because plurality assigns the same number of points to all alternatives ranked below the top preference, a full ordering is not actually needed as long as the voter's most preferred alternative is expressed. For Borda, however, a full ranking of all alternatives is required. A "truncated" Borda is sometimes used in practice when there is a large number of alternatives where each voter only ranks a pre-determined number of the alternatives. Points are then assigned such that $\sigma_r = t - r + 1$ $\forall r = \{1, ..., t\}$, $\sigma_r = 0$ $\forall r > t$, $t < z$. The truncated Borda rule is thus a mixture of Borda and plurality. Indeed, plurality can be viewed as the ultimate truncation of Borda by setting $r = 1$. Borda is a strong positional system whereas truncated versions are not when $t < a - 1$.

Young (1974) proved that Borda is the only voting rule which satisfies the properties of neutrality, consistency, faithfulness, and, cancellation, and properly defines a choice set for any set of profiles D. He also proved that faithfulness and consistency imply Pareto, and any SCF which satisfies consistency and cancellation is also anonymous.

To be *anonymous*, a SCF must treat all voters equally, or what May (1952) referred to as egalitarianism. The social choice set depends only on the number of voters having each particular preference ordering. Thus, any permutation on voter names results in the same choice set. Let $D_i^\sim = D_j$, $D_j^\sim = D_i$, and $D_k^\sim = D_k$, $\forall k \neq i, j$. Then $\Gamma$is anonymous if $\zeta(D^\sim) = \zeta(D)$. Switching the preferences of any two voters cannot affect the choice set if they have equal impact on the outcome.

The *neutrality* criteria, similar to May's definition for two alternatives, prescribes that no single alternative is favored by the voting procedure over any of the other alternatives. The switching of any alternatives by every voter leads to a duplicate switching of these alternatives in the group ranking. Formally, neutrality requires $\zeta(\psi[D]) = \psi[\zeta(D)]$, where the permutation on alternative names, ψ, results in the same permutation on the resulting choice set.

Consistency requires that if there is any overlap in the winning alternatives from two separate groups of voters, then the merging of these groups yields the winning alternatives to be the ones that the separate groups held in common.[1] Denote D' and D'' as the vector of preference orderings on N' and N'', where $N' \cap N'' = \emptyset$ and $N' \cup N'' = N$. Then a voting rule is consistent if $\zeta(D') \cap \zeta(D'') \neq \emptyset \rightarrow \zeta(D') \cap \zeta(D'') = \zeta(D)$.

Theorem Positional scoring rules are the only anonymous, neutral and consistent voting mechanisms which guarantee a non-empty choice set (Young 1974).

Faithfullness requires that when there is only one individual in the group the choice set consists only of that person's most preferred alternative. Thus, if $n = 1$, Γ is faithful if $\zeta(D) = \{a\}$ if and only if aP_1b $\forall b \neq a$. In essence, an SCF which is faithful allows anyone acting on their own to be able to decide for themself. Furthermore, any SCF which is both consistent and faithful is also *Pareto*, meaning that if the same alternative is most preferred by everyone in the group, than it will be the only alternative in the choice set. Any scoring rule which sets $\sigma_1 > \sigma_r$ $\forall r \neq 1$ is faithful. Thus all strong positional systems are faithful.

Finally, consider a weaker form of anonymity referred to as *cancellation*. Define the profile D^* to be one

[1] Young (1995) would later refer to this property as *reinforcement*. See also Heckelman and Chen (2013) and Pivato (2013).

in which $n^{ab} = n^{ba}\ \forall a, b \epsilon Z$. Then under cancellation $\zeta(D^*) = Z$. If for all pairs of alternatives the number of people preferring one to the other is the same, then Γ must declare a tie among all the alternatives. Voters are being treated equally because the social choice set only depends on the number of people favoring each alternative, and not which particular persons are on either side.[2] The condition differs from anonymity, however, in that it only applies in certain profiles.[3] It is precisely this limitation which will be the focus of our paper.

Plurality, for example, is anonymous but does not respect cancellation. Suppose $n = 2$ and $D^* = \{xP_1yP_1z,\ zP_2yP_2x\}$. All pairwise majority votes would result in a tie. Yet under plurality, $\zeta(D^*) = \{x, z\}$ violating cancellation. More generally, any weak positional system, and every strong positional system other than Borda, violates cancellation.

Theorem The only SCF which is neutral, consistent, faithful, and obeys cancellation is the Borda count. (Young 1974)

The question remains how often a profile which would lead a non-Bordian SCF to violate cancellation would materalize. Because Young's proof is conditional on an unrestricted domain, it only identifies that any rule other than Borda *could* potentially violate one of the identified axiomatic properties, not that it will for any given profile. Using simulations, we show in the next section that the likelihood of cancellation being invoked is close to nil and thus there remain a large variety of positional systems would could be utilized that will respect all the discussed properties with near certainty in practice. We then describe the new property of inverse clone contraction (ICC) which we show is much more likely to be applicable to any randomly generated profile and that when replacing cancellation with ICC, in combination with the other properties, requires the SCF to be a symmetrical positional system. That is also shown to be true when removing cancellation and replacing neutrality with a stronger version.

3 Likelihood of violating cancellation

4 Inverse clone contraction

Persons i and j are inverse clones if $aP_ib \leftrightarrow bP_ja\ \forall a, b \epsilon Z$. Γ is *inverse clone contractionary* (ICC) if for any pair of inverse clones $i, j \epsilon N$, $\zeta(D \backslash \{D_i, D_j\}) = \zeta(D)$. An ICC SCF treats all inverse clones equally in that

[2] Another fairness interpretation can be deduced by noting that for a profile such as D^* described by cancellation, that every alternative must have the same average rank. Thus requiring a tie among all alternatives shows no favoritism toward any particular alternative and can be thought of as weak form of neutrality.

[3] This is true of the other properties as well. Faithfulness only applies when there is a profile consisting of one person and consistency only applies when there are two separate profiles which have some overlap in their choice sets. Similarly, Pareto only applies when everyone in the group has the same top preference. These properties are trivially satisfied for a given profile by all voting mechanisms when these specific conditions do not exist for that profile.

they directly cancel each other out. An SCF which treats voters differently prior to preference revelation cannot be ICC but one in which voter weights are determined after (voter identified) ballots are examined can be ICC. For example, suppose Γ_0 makes voter 1 the dictator if there are no revealed inverse clones, but otherwise the Borda count will be implemented. Γ_0 is ICC but not anonymous (it will be proved below that Borda is ICC).

The property is weaker than cancellation and anonymity in that it does not refer to all voters. Yet there are anonymous rules such as plurality which do not respect ICC (to be shown below). It is also more likely to be relevant to a random profile than is cancellation and is therefore stronger than cancellation in a probabilistic sense.

Cancellation relates to ICC in the following way. Denote D^{**} as any profile where for all unique pairs $i, j \epsilon N$, $aP_ib \leftrightarrow bP_ja \ \forall a, b\epsilon Z$, that is as every voter can be paired with an inverse clone. Under such a profile, $n^{ab} = n^{ba}$, $\forall a, b\epsilon\, Z$. Cancellation requires that the choice set be compised of the entire set of alternatives, thereby not selecting any one alternative over another. Under ICC the outcome must be the same if the entire group votes or any pair of inverse clones are removed from the group. Eventually, there will be no one left. Technically, of course, no choice set can be derived without an existing domain of preferences (unless the rule is null) but prior to the creation of a choice set no alternative has yet been ruled out. (Lack of a choice set being different from an empty choice set.) For Γ to be ICC under the original profile, it must be that no alternative is ruled out by the resulting choice set. In other words, both Cancellation and ICC require $\zeta(D^{**}) = Z$. Note however, there can be other profiles in which the conditions to invoke cancellation exist but where the conditions to invoke ICC are not met (so that ICC cannot be violated and is trivially satisfied), and vice versa. But we show below that in comparison to the simulations of the previous section, the necessary profile conditions for ICC to be applicable are more likely to occur in most group settings than for cancellation, making ICC a more relevent normative property in practice.

A positional scoring rule is *(strongly) symmetric* if $\sigma_r > \sigma_h \ \forall r < h \leq z$, and

$$\sigma_r + \sigma_{z-r+1} = \begin{cases} 2\sigma_{\frac{z+1}{2}} & \text{for } z \text{ even,} \\ \sigma_{\frac{z}{2}} + \sigma_{\frac{z}{2}+1} & \text{for } z \text{ odd.} \end{cases}$$

A weak symmetry defintion can be constructed by substituting $\sigma_r \geq \sigma_h$ for $\sigma_r > \sigma_h$, and adding the additional stipulation $\sigma_1 > \sigma_z$ (which is complicit within the strong version). Examples of strongly symmetric positional systems for $z = 5$ include $\sigma = \{10, 6, 5, 4, 0\}$ or $\sigma = \{7, 6, 0, -6, -7\}$, and for $z = 6$ include $\sigma = \{8, 6, 5, 4, 3, 1\}$ or $\sigma = \{16, 14, 9, 7, 2, 0\}$. The positive/negative vote rule where voters vote for one alternative and against one alternative is an example of a weakly symmetric rule. Its weights as a scoring rule can be denoted by (for example) $\sigma = \{1, 0, ..., 0, -1\}$. A positional system is symmetric when the median (or average of the two medians) weight is the same as the average of the largest and smallest weights, and this also holds for the average weight of each pairing of weights when moving one position at a time away from both ends toward the median(s). It should be clear that the Borda count is symmetrical but plurality (or any truncated Borda) is not. Because strongly symmetrical positional systems are neutral and consistent SCFs, by virtue of being a positional system, and faithful because σ_1 is greater than any other weight, they must violate cancellation unless $\sigma_r - \sigma_{r+1} = c \; \forall r = \{1, ..., z-1\}$, where c is any arbitrary positive constant thereby making it functionally equivalent to the Borda count.

The table below presents an example profile D^* for which a positional system does not satisfy cancellation despite the weights being symmetrical.

Table 1: Cancellation profile

D_1	D_2	D_3	D_4
x	y	v	x
y	u	u	u
v	v	y	v
u	x	x	y

For every pairwise comparison in Table 1 the outcomes are tied 2-2. Cancellation thus requires every alternative to be an element in the choice set. For Borda, $\Omega^x = \Omega^y = \Omega^u = \Omega^v = 6$ and thus $\zeta(D^*) = \{x, y, u, v\}$ satisfying cancellation. Now consider the alternative symmetrical positional system where $\sigma = \{6,\ 5,\ 2,\ 1\}$ which yields $\Omega^u = 16 > \Omega^x = 14 = \Omega^y = 14 > \Omega^v = 12$ and thus $\zeta(D^*) = \{u\}$ thereby violating cancellation. Yet as noted in the previous section, the likelihood of obtaining a profile such as D^* is remote and thus the conditon for cancellation to apply will rarely be triggered. Note that there are no inverse clones in D^*. Thus any outcome is consistent with ICC for this profile and is satisfied trivially by any voting rule.

Yet in general the profile conditions necessary to trigger the ICC property are more likely to appear than the profile conditions needed to invoke cancellation. [simulations here]

Theorem The only SCFs which are neutral, consistent, faithful and ICC are strongly symmetrical positional systems. [Proof to follow]

5 More nonsense

Now consider a stronger version of neutrality which not all positional systems will satisfy. May's (1952) introduction of this property was defined only for the limiting case of $z = 2$. It has been generalized to the multiple-alternative case in a variety of ways (Young 1974, Goodin and List 2006, Dasgupta and Maskin 2008, Pivato 2013).

May's formal definition of neutrality stated that when everyone reversed their preference then the opposite outcome should arise. Suppose $Z = \{x, y\}$. If every individual voter in profile D were replaced by an inverted clone in profile D^{-1}, then Γ is neutral if and only if $\zeta(D) = \{x\} \longleftrightarrow \zeta(D^{-1}) = \{y\}$, $\zeta(D) = \{y\} \longleftrightarrow \zeta(D^{-1}) = \{x\}, \zeta(D) = \{x, y\} \longleftrightarrow \zeta(D^{-1}) = \{y, x\}$. Of course the ordering of the choice set list is irrelevant so a tie in the last scenario remains a tie for both profiles. In the multiple-alternative case, it is not always clear what *the* 'opposite' outcome is, and a tie can occur among a subset of the alternatives so there are more than just three potential social choice sets derived from any profile when $z > 2$. One way to express May's neutrality property is that any alternative selected by Γ from one profile cannot also be selected under its inverse clone profile, unless all alternatives are selected by both. Let $Z' \cap Z'' = \emptyset$ and $Z' \cup Z'' = Z$. We say Γ satisfies *strong neutrality* when $\zeta(D) = Z' \longleftrightarrow \zeta(D^{-1}) = Z''' \subseteq Z''$, and $\zeta(D) = Z \longleftrightarrow \zeta(D^{-1}) = Z$.

Theorem The only SCFs which are consistent, faithful and strongly neutral are symmetrical positional systems

Recall that plurality rule is neutral under Young's definition but is not symmetric. Denote the profile in Table 1 as D. Under plurality, $\zeta(D) = \zeta(D^{-1}) = \{x\}$thus violating strong neutrality.

[etc etc]

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