EXPLICIT SERRE WEIGHTS FOR GL_2

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Abstract. We give an description of the weight part of Serre's conjecture for GL_2 in terms of Kummer theory, which is completely explicit in that it avoids any mention of p-adic Hodge theory. The key inputs are a description of the reduction modulo p of crystalline extensions in terms of certain "G_K-Artin–Scheier" extensions" and a result of Abrashkin, describes these "G_K-Artin–Scheier" in terms of Kummer theory.

In the unramified case an alternative explicit formulation was previously given by Dembele–Diamond–Roberts in terms of local class field theory. We also show that, in this special case, their description can be recovered directly from ours using the explicit reciprocity laws of Brückner–Shaferevich–Vostokov.

Contents

1. Introduction

Overview. Serre conjectured in [Ser87] that every continuous odd representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ arose as the reduction modulo p of the Galois representation attached to a modular form. Furthermore, Serre predicted the possible weights of the relevant modular forms in terms of the local representation $\overline{\rho}|_{G_{\mathbb{Q}_p}}$. As the following example illustrates, the recipe is extremely explicit. Suppose

$$\overline{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \overline{\chi}^k_{\mathrm{cyc}} & c \\ 0 & 1 \end{pmatrix}, \qquad 0 \leq k \leq p-1$$

with $\overline{\chi}_{\mathrm{cyc}}$ the mod p cyclotomic character and $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ the inertia subgroup. Then Serre expected that $\overline{\rho}$ would be modular of weight $k+1$. The one exception is when $k=1$; in this case $\overline{\rho}$ is modular of weight 2 if and only if the class of c is *peu ramifié*, i.e. contained in the image of the Kummer map

$$\mathbb{Z}_p^\times \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p \to H^1(G_{\mathbb{Q}_p}, \overline{\mathbb{F}}_p(\overline{\chi}_{\mathrm{cyc}})).$$

Otherwise $\overline{\rho}$ will be modular of weight $p+1$. A reason one expects constraints on c only in this special case is because it is the one instance where $H^1(G_{\mathbb{Q}_p}, \overline{\mathbb{F}}_p(\overline{\chi}^k_{\mathrm{cyc}}))$ has dimension >1.

Generalisations of this weight recipe have been made with $\mathbb{Q}$ replaced by a totally real field F (see, for example, [BDJ10; BGG13]). When $\overline{\rho}|_{G_{F_v}}$ is semisimple at each prime v of F dividing p this is an immediate extension of Serre's description. However, the more general setup requires considerably more complicated constraints on the extension classes. The previously mentioned conjectures give a description of these extension classes in terms of reductions of crystalline representations which, while conceptually appealing, is neither explicit or computable.

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The goal of this paper is to describe these constrains on extension classes in the spirit of Serre's original conjecture, using the Kummer map. As a consequence, we obtain an explicit formulation of the weight part of Serre's conjecture which avoids any mention of p-adic Hodge theory.

Crystalline lifts and our main result. The conjectures mentioned above have received significant attention, and have been essentially resolved in a series of papers [GK14; BGG13; New14], culminating in [GLS15]. In particular, for $p > 2$ and a totally real field F, the possible weights of a modular representation $\overline{\rho}\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ can be described in terms of a set of "local" Serre weights $W^{\mathrm{cr}}(\overline{\rho}|_{G_{F_v}})$ defined by way of crystalline lifts of $\overline{\rho}|_{G_{F_v}}$ at places v of F diving p. We remind the reader that elements of $W^{\mathrm{cr}}(\overline{\rho}|_{G_{F_v}})$ are (isomorphism classes of) irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k_v)$ for k_v the residue field of F_v. This is explained in detail in Section 2.

The main goal of this paper is then reduced to the purely local problem of explicitly describing $W^{\mathrm{cr}}(\overline{r})$ for any continuous $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ with $K/\mathbb{Q}_p$ a finite extension. The results of [GLS15] give such an explicit description when $\overline{r}$ is semisimple and in general show that $W^{\mathrm{cr}}(\overline{r}) \subset W^{\mathrm{cr}}(\overline{r}^{\mathrm{ss}})$. See Section 3.

Let f denote the residue field degree of K. To state our main result we fix a uniformiser $\pi \in K$ and a (p^f-1)-th root $\pi^{1/(p^f-1)}$. Set L equal the (p^f-1)-th unramified extension of $K(\pi^{1/(p^f-1)})$. Then L contains a p-th root of unity ϵ_1. If l denotes the residue field of L the Artin–Hasse exponential defines a homomorphism of $\mathbb{Z}_p$-modules

$$vW(l)[[v]] \to 1 + vW(l)[[v]]$$

sending $f \mapsto \sum_{n\geq 0}\left(\frac{\varphi^n(f)}{p^n}\right)$ for φ the $\mathbb{Z}_p$-linear endomorphism of $W(l)[[v]]$ given by $v \mapsto v^p$ and the lift of Frobenius on $W(l)$. Composing with evaluation at $v = \pi^{1/(p^f-1)}$ produces a homomorphism $vW(l)[[v]] \to 1 + \mathfrak{m}_L$, and composing again with the Kummer map gives $vW(l)[[v]] \to H^1(G_L, \mathbb{F}_p)$. Applying $\otimes_{\mathbb{Z}_p}\overline{\mathbb{F}}_p$ we obtain a homomorphism

$$\Psi : vl[[v]] \to H^1(G_L, \overline{\mathbb{F}}_p).$$

See Section 4 for more details on these constructions. The following is our explicit version of the weight part of Serre's conjecture:

Theorem 1.1. *Suppose $p > 2$ and $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous with $\overline{r} \sim \left(\begin{smallmatrix}\chi_1 & c\\ 0 & \chi_2\end{smallmatrix}\right)$. For each $\sigma \in W^{\mathrm{cr}}(\overline{r}^{\mathrm{ss}})$, there exists a constant $C_\sigma \in vl[[v]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$, depending on χ_1, χ_2 and σ only, such that $\sigma \in W^{\mathrm{cr}}(\overline{r})$ if and only if one of the following conditions are satisfied:*

(1) $\chi_1\chi_2^{-1}$ is an unramified twist of the mod p cyclotomic character and $\sigma = \mathrm{Sym}^{p-1}(k \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)^2$ is the Steinberg representation;

(2) $c|_{G_L} \in \Psi(C_\sigma(k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p))$, where $u = v^{p^f-1}$ and k is the residue field of K.

In particular, the subspace $\Psi(C_\sigma(k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p))$ replaces the notion of *peu ramifié* classes in Serre's original conjecture. This result is Theorem 5.3 and the constant C_σ is described explicitly in Definition 5.1.

Relation to Dembélé–Diamond–Roberts. The idea of making the weight part of Serre's conjecture completely explicit in the non-semisimple setting by removing any reference to p-adic Hodge theory was first addressed in [DDR16]. There they assume $K/\mathbb{Q}_p$ is unramified and in this setting formulated a conjectural description of $W^{\mathrm{cr}}(\overline{r})$ using local class field theory to describe subspaces of $H^1(G_K, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$. These predictions were subsequently proven in [CEGM17]. In [Ste22] the second author showed that when $K/\mathbb{Q}_p$ ramifies it is still possible to give an explicit description along the lines of [DDR16] and prove the equivalence of this description to $W^{\mathrm{cr}}(\overline{r})$.

In each of [DDR16; Ste22] the relevant subspaces of $H^1(G_K, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$ are described by first exhibiting a basis of $H^1(G_K, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$ and then defining the subspaces as the span of certain elements of this basis. One issue with this approach is that in certain boundary situations deciding which basis elements should be included in the subspace requires a complicated combinatorial recipe. Even in the unramified case this recipe (see [DDR16, §7.1]) is rather involved. In the presence of ramification finding a simpler and more direct description for which basis elements are to be included becomes a difficult combinatorial problem which is unlikely to have a straightforward general solution (see, for example, [Ste20, Ch. 7] where simpler descriptions are given under several simplifying assumptions). One of the main motivations for this paper was to circumvent these complications.

On the other hand, one can wonder whether Theorem 1.1 could be used to recover the results of [DDR16] in the unramified case. We do this in the last part of the paper, using the explicit reciprocity law of Brückner–Shafervich–Vostokov to pass between the Kummer theoretic description on $H^1(G_L, \overline{\mathbb{F}}_p)$ and that given in terms of local class field theory.

Proposition 1.2. *Assume $K/\mathbb{Q}_p$ is unramified and let $L^{\mathrm{DDR}}_\sigma(\chi_1,\chi_2) \subset H^1(G_K, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$ denote the subspace defined in [DDR16] (see Section 12.2 for more details). Then $L^{\mathrm{DDR}}_\sigma(\chi_1,\chi_2)$ equals the preimage of $\Psi(C_\sigma(k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p))$ in $H^1(G_K, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$.*

Of course this proposition follows immediately given that both $\Psi(C_\sigma(k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p))$ and $L^{\mathrm{DDR}}_\sigma(\chi_1,\chi_2)$ have the same interpretation in terms of crystalline lifts. However, in the spirit of this entire paper, our calculations avoid any p-adic Hodge theoretic description. In particular, our calculations are independent of [CEGM17] and so give a new proof of the results in [DDR16]. We believe it is possible to use a strategy similar to the one we have used here to give a direct comparison of the results of this paper to the results of [Ste22] in the ramified case.

Method of proof. To prove Theorem 1.1 we need to show that, if $\overline{r}$ admits a crystalline lift with Hodge–Tate weights corresponding to σ, then this imposes significant conditions on $\overline{r}$ which can ultimately be formulated as in Theorem 1.1. This is done is three steps:

Step 1. This uses the integral p-adic Hodge theory developed in [GLS15]. If r is a crystalline lift of $\overline{r}$ witnessing $\sigma \in W^{\mathrm{cr}}(\overline{r})$, then [GLS15] gives a description, in terms of σ, of the shape of $\overline{\mathfrak{M}}$, the reduction modulo p of the Breuil–Kisin module associated to r. Concretely, $\overline{\mathfrak{M}}$ is a finite free $k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$-module equipped with a semi-linear Frobenius endomorphism, and [GLS15] describes the matrix of this endomorphism in terms of a nice choice of basis (see Proposition 7.3). Set $K_\infty = K(\pi^{1/p^\infty})$ for π^{1/p^∞} a compatible system of p-th power roots of $\pi \in K$. Then $\overline{\mathfrak{M}}$ and $\overline{r}$ are related using the existence of a φ, G_{K_∞}-equivariant identification

$$\overline{\mathfrak{M}} \otimes_{k[[u]]} C^\flat = \overline{r}^\vee \otimes_{\mathbb{F}_p} C^\flat,$$

where $C^\flat$ denotes a specific algebraic closure of $k((u))$ and G_{K_∞} acts trivially on $\overline{\mathfrak{M}}$. In particular, $\overline{r}^\vee|_{G_{K_\infty}} = (\overline{\mathfrak{M}} \otimes_{\mathbb{F}_p} C^\flat)^{\varphi=1}$. Concretely, if β is a basis of $\overline{\mathfrak{M}}$ and $\alpha = \beta D$ generates $\overline{r}^\vee$ (so that D is a matrix with $\varphi(D^{-1})D$ equals the matrix of the Frobenius relative to β) then the G_{K_∞}-action on α is given by

$$\sigma(\alpha) = \alpha\sigma(D)D^{-1}.$$

From this we deduce a statement of the following shape: if $\overline{r} \sim \left(\begin{smallmatrix} \chi_1 & c \\ 0 & \chi_2 \end{smallmatrix}\right)$, then there exists a subspace $L^{\mathrm{AS}}_{\sigma,\chi_1,\chi_2} \subset H^1(G_{K_\infty}, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$ defined in terms of Artin–Scheier cocycles so that $\sigma \in W^{\mathrm{cr}}(\overline{r})$ implies

$$c|_{G_{K_\infty}} \in L^{\mathrm{AS}}_{\sigma,\chi_1,\chi_2}.$$

We emphasise that everything so far follows, more or less, immediately from [GLS15]. In the unramified case this is the essential tool used to prove the conjecture of [DDR16] in [CEGM17].

Step 2. The second step is to upgrade the description of the G_{K_∞}-action on $\overline{r}$, given in terms of $\overline{\mathfrak{M}}$, to a description of the G_K-action. For this we first recall that the action of G_{K_∞} on $C^\flat$ naturally extends to a G_K-action. Therefore, $C^\flat$-semi-linearly extending the G_K-action on $\overline{r}^\vee \otimes_{\mathbb{F}_p} C^\flat$ we obtain a φ-equivariant G_K-action on $\overline{\mathfrak{M}} \otimes_{k[[u]]} C^\flat$. Since the G_K-action on $\overline{r}$ comes from the reduction modulo p of a crystalline representation this G_K-action must satisfy the following divisibility

$$\sigma(m) - m \in \mathfrak{M} \otimes_{k[[u]]} u^{(e+p-1)/(p-1)}\mathcal{O}_{C^\flat},$$

for all $\sigma \in G_K$ and $m \in \mathfrak{M}$.

On the other hand, ideas from [Bar22] give a procedure which, in good cases, constructs an alternative G_K-action on $\overline{\mathfrak{M}} \otimes_{k[[u]]} C^\flat$. This is done as follows: choose a basis β of $\overline{\mathfrak{M}}$ and define a "naive" G_K-action $\sigma_{\mathrm{naive},\beta}$ on $\overline{\mathfrak{M}} \otimes_{k[[u]]} C^\flat$ by semi-linearly extending the action which fixes $\varphi(\beta)$. In general, this action will not be φ-equivariant. However, one can attempt to produce a φ-equivariant action from it by considering

$$\sigma = \lim_{n\to\infty} \varphi^n \circ \sigma_{\mathrm{naive},\beta} \circ \varphi^{-n}.$$

Typically (i.e. for an arbitrary Breuil–Kisin module) this limit will not converge. However, in our case this limit really exists, due to the special shape of $\overline{\mathfrak{M}}$ (and ultimately the fact that the Hodge–Tate weights of r are sufficiently small). Furthermore, one shows that this is the unique G_K-action satisfying the above divisibility. Therefore, the G_K-action computed by this limit coincides with the G_K-action coming from $\overline{r}$.

Concretely, if $\alpha = \beta D$ is a basis of $\overline{r}^\vee$ then the G_K-action on $\overline{r}^\vee$ is given by

$$\sigma(\alpha) = \alpha\left(\lim_{n\to\infty} \varphi^n(\sigma(D)D^{-1})\right).$$

This allows us to reformulate the implication in the final part of (1); we obtain that $\sigma \in W^{\mathrm{cr}}(\overline{r})$ implies

$$c|_{G_L} \in L^{G_K\text{-AS}}_{\sigma,\chi_1,\chi_2},$$

where now $L^{G_K\text{-AS}}_{\sigma,\chi_1,\chi_2} \subset H^1(G_K, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$ is a subspace of certain "G_K-Artin–Schreier" coycles.

Step 3. The final step is to produce a dictionary between the restriction of these"G_K-Artin–Schreier" cocycles to G_L and Kummer cocycles. This was done in a beautiful computation of Abrashkin [Abr97]. To be precise he considers any $h \in vl[[v]]$ and chooses $h' \in C^\flat$ so that $\varphi(h') - h' = h$. Then he considers the "G_L-Artin–Schreier" cocycle $G_L \to \mathbb{F}_p$ defined by

$$\sigma \mapsto \lim_{n\to\infty} \varphi^n \left(\sigma\left(\frac{h'}{\overline{H}(v)}\right) - \frac{h'}{\overline{H}(v)} \right),$$

where $\overline{H}(v)$ is the reduction modulo p of a polynomial $H(v) \in W(l)[v]$ satisfying $H(\pi^{1/(p^f-1)}) = \epsilon_1$; equivalently this cocycle can be described as sending $\sigma \in G_L$ onto the image of $\sigma\left(\frac{h'}{\overline{H}(v)}\right) - \frac{h'}{\overline{H}(v)}$ under the map $\mathcal{O}_{C^\flat} \to \overline{\mathbb{F}}_p$. The restriction to G_L of those cocycles in $L^{G_K\text{-AS}}_{\sigma,\chi_1,\chi_2}$ all have this form. Abrashkin gives an explicit formula, using the map Ψ from above, which expresses this cocycle as a Kummer cocycle (see Proposition 9.1). From this we deduce, for $\overline{r} \sim \left(\begin{smallmatrix} \chi_1 & c \\ 0 & \chi_2 \end{smallmatrix}\right)$, that $\sigma \in W^{\mathrm{cr}}(\overline{r})$ implies

$$c|_{G_L} \in \Psi(C_\sigma(k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)).$$

It only remains to prove the opposite implication: if $c|_{G_L} \in \Psi(C_\sigma k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$ then we must produce a crystalline lift of $\overline{r}$ witnessing $\sigma \in W^{\mathrm{cr}}(\overline{r})$. We do this in the standard way by producing certain crystalline lifts of the characters χ_1 and χ_2 and then considering the image in $H^1(G_K, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$ of the space of crystalline extensions of these two lifts. By the above this image contains the preimage of $\Psi(C_\sigma(k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p))$ in $H^1(G_K, \overline{\mathbb{F}}_p(\chi_1/\chi_2))$ and we will be done if these two subspaces are equal. This follows by comparing dimensions.

Acknowledgements.

2. Serre weights

Definition 2.1. For a finite extension k of $\mathbb{F}_p$, a Serre weight (for $\mathrm{GL}_2(k)$) is an isomorphism class of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$. Any such class can be represented by

$$\sigma_{a,b} := \bigotimes_{\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)} \left(\det^{b_\tau} \otimes_k \mathrm{Sym}^{a_\tau - b_\tau} k^2\right) \otimes_{k,\tau} \overline{\mathbb{F}}_p$$

for uniquely determined integers a_τ, b_τ with b_τ and $a_\tau - b_\tau \in [0, p-1]$ and not all b_τ equal to $p-1$.

Suppose K is a finite extension of $\mathbb{Q}_p$ with residue field k and V a de Rham representation of G_K on a $\overline{\mathbb{Q}}_p$-vector space. For each $\kappa \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$, the κ-Hodge–Tate weights $\mathrm{HT}_\kappa(V)$ of V is the multiset of integers which contains i with multiplicity

$$\dim_{\overline{\mathbb{Q}}_p}(V \otimes_{\kappa,K} C(-i))^{G_K}.$$

Here $C(i)$ is a completed algebraic closure of K with the twisted G_K-action $\sigma(a) = \chi_{\mathrm{cyc}}(\sigma)^i\sigma(a)$ for χ_{cyc} the p-adic cyclotomic character. In particular, $\mathrm{HT}_\kappa(\chi_{\mathrm{cyc}}) = \{1\}$ for every κ.

Definition 2.2. Let V be a crystalline representation of G_K on a finite free $\overline{\mathbb{Z}}_p$-module. We say that V has Hodge type $\sigma_{a,b}$ if the following condition is satisfied:

- For each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$, we can index the embeddings $\{\kappa \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p) \mid \kappa|_k = \tau\}$ as $\{\kappa_0, \kappa_1, \dots, \kappa_{e-1}\}$ such that

$$\mathrm{HT}_{\kappa_0}(V[\tfrac{1}{p}]) = \{a_\tau + 1, b_\tau\}, \qquad \mathrm{HT}_{\kappa_i}(V[\tfrac{1}{p}]) = \{1, 0\}$$

  for $i = 1, \dots, e-1$.

Definition 2.3. For a continuous $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, we let $W^{\mathrm{cr}}(\overline{r})$ denote the set of Serre weights $\sigma_{a,b}$ for which there exists a crystalline representation of G_K on a finite free $\overline{\mathbb{Z}}_p$-module V with Hodge type $\sigma_{a,b}$ and $V \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong \overline{r}$.

The following motivates the definition of $W^{\mathrm{cr}}(\overline{r})$. Suppose that F is a totally real extension of $\mathbb{Q}$. Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$ be a continuous and absolutely irreducible representation which arises as the reduction modulo p of a p-adic representation associated to a Hilbert modular eigenform of parallel weight 2. For each place v of F dividing p, let k_v denote the residue field of F_v. Let D be a quarternion algebra with centre F and which is split at all places dividing p, and at zero or one infinite place. In [GK14, 5.5.2] it is explained what it means for $\overline{\rho}$ to be modular for D of weight $\sigma = \otimes_{v|p}\sigma_v$ (each σ_v being a Serre weight for $\mathrm{GL}_2(k_v)$).

Theorem 2.4. *Suppose that $p > 2$. Assume also that $\overline{\rho}$ is modular, compatible with D in the sense of [GK14, 5.5.3], and that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. If $p = 5$ assume that the projective image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is not isomorphic to A_5. Then $\overline{\rho}$ is modular for D of weight $\sigma = \otimes_{v|p}\sigma_v$ if and only if $\sigma_v \in W^{\mathrm{cr}}(\overline{\rho}|_{G_{F_v}})$ for each $v \mid p$.*

Proof. See [GLS15, §4.1 and §4.2]. $\square$

3. Explicit Serre weights in the semisimple case

Let K be a finite extension of $\mathbb{Q}_p$ with residue field k. In this section we recall an explicit description of $W^{\mathrm{cr}}(\overline{r})$ from [GLS15] when $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is semisimple. Set $f = [k : \mathbb{F}_p]$ and $e = e(K/\mathbb{Q}_p)$. Fix $\pi \in K$ a uniformiser with $\pi^{1/(p^f-1)}$ a $(p^f - 1)$-th root in a fixed algebraic closure. We can then define a character

$$\omega : G_K \to k^\times$$

by setting $\omega(\sigma)$ equal the image of $\sigma(\pi^{1/(p^f-1)})/\pi^{1/(p^f-1)}$ in $k^\times$. Note that ω depends upon $\pi^{1/(p^f-1)}$, but its restriction to the inertia subgroup I_K does not. For $\tau : k \to \overline{\mathbb{F}}_p$, set $\omega_\tau := \tau \circ \omega$. Then, for every character $\chi : G_K \to \overline{\mathbb{F}}_p^\times$, one can write

$$\chi|_{I_K} = \prod_\tau \omega_\tau^{n_\tau}|_{I_K}$$

for some (not necessarily uniquely determined) integers n_τ. The n_τ are uniquely determined if we further ask that $n_\tau \in [1, p]$ and not every n_τ equals p.

Definition 3.1. Let $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be continuous and semisimple. Following [GLS15, 4.1] we define a set of Serre weights $W^{\exp}(\overline{r})$ by:

- If $\overline{r}$ is irreducible, then $\sigma_{a,b} \in W^{\exp}(\overline{r})$ if there exists a subset $J \subset \mathrm{Hom}_{\mathbb{F}_p}(k_2, \overline{\mathbb{F}})$ (here k_2 denotes the unique degree 2 extension of k) and $0 \le x_\kappa \le e-1$ for each $\kappa \in \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$ so that
$$\overline{r}|_{I_K} \cong \begin{pmatrix} \prod_{\tau \in J} \omega_\tau^{a_\tau+1+x_{\tau|_k}} \prod_{\tau \notin J} \omega_\tau^{b_\tau+e-1-x_{\tau|_k}} & 0 \\ 0 & \prod_{\tau \notin J} \omega_\tau^{a_\tau+1+x_{\tau|_k}} \prod_{\tau \in J} \omega_\tau^{b_\tau+e-1-x_{\tau|_k}} \end{pmatrix}$$
and so that J is such that $\mathrm{Hom}_{\mathbb{F}_p}(k_2, \overline{\mathbb{F}}_p)$ is the disjoint union of J and the set of $\tau \circ \sigma$ for $\tau \in J$, where σ denotes the non-trivial element of $\mathrm{Gal}(k_2/k)$.
- If $\overline{r}$ is a direct sum of two characters, then $\sigma_{a,b} \in W^{\exp}(\overline{r})$ if there exists a subset $J \subset \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$ and integers $0 \le x_\tau \le e-1$ for $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$ so that
$$\overline{r}|_{I_K} \cong \begin{pmatrix} \prod_{\tau \in J} \omega_\tau^{a_\tau+1+x_\tau} \prod_{\tau \notin J} \omega_\tau^{b_\tau+e-1-x_\tau} & 0 \\ 0 & \prod_{\tau \notin J} \omega_\tau^{a_\tau+1+x_\tau} \prod_{\tau \in J} \omega_\tau^{b_\tau+e-1-x_\tau} \end{pmatrix}.$$

Remark 3.2. The following equivalent formulation of $W^{\exp}(\overline{r})$ when $\overline{r}$ is reducible will be more convenient for us: $\sigma_{a,b} \in W^{\exp}(\overline{r})$ if and only if there exist $s_\tau, t_\tau \ge 0$ such that

$$a_\tau - b_\tau + e = s_\tau + t_\tau, \qquad \max\{s_\tau, t_\tau\} \ge a_\tau - b_\tau + 1$$

and such that

$$\overline{r}|_{I_K} \sim \begin{pmatrix} \prod_\tau \omega_\tau^{s_\tau+b_\tau} & 0 \\ 0 & \prod_\tau \omega_\tau^{t_\tau+b_\tau} \end{pmatrix}.$$

To see this, given such (s_τ, t_τ) set $J = \{\tau \mid t_\tau < a_\tau - b_\tau + 1\}$ and

$$x_\tau = \begin{cases} e-1-s_\tau & \text{if } \tau \notin J; \\ e-1-t_\tau & \text{if } \tau \in J. \end{cases}$$

Note $\tau \notin J$ implies $t_\tau \ge a_\tau - b_\tau + 1$. Thus, $s_\tau \le e-1$. Similarly, $\tau \in J$ implies $t_\tau \le e-1$. Therefore, $x_\tau \in [0, e-1]$. For such J and x_τ, $\overline{r}|_{I_K}$ can be written as in Definition 3.1. Similarly, one produces such s_τ, t_τ from $J \subset \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$ and $x_\tau \in [0, e-1]$.

Theorem 3.3 (Gee–Liu–Savitt, Wang)**.** *If $\overline{r}$ is semisimple, then $W^{\exp}(\overline{r}) = W^{\mathrm{cr}}(\overline{r})$.*

Proof. When $p > 2$ this is [GLS15, 5.1.5]. When $p = 2$ the methods of Gee–Liu–Savitt have been adapted by Wang (see [Wan17, Theorem 5.4]). $\square$

When $\overline{r}$ is reducible but not semisimple [GLS15] shows that $W^{\mathrm{cr}}(\overline{r}) \subset W^{\exp}(\overline{r}^{\mathrm{ss}})$. However, this inclusion is rarely an equality. Our goal is to give a an explicit condition on the extension class of $\overline{r}$ which determines whether $\sigma_{a,b} \in W^{\exp}(\overline{r}^{\mathrm{ss}})$ is contained in $W^{\mathrm{cr}}(\overline{r})$.

4. Artin–Hasse exponential

Here we assume $p > 2$. Recall also the uniformiser $\pi \in K$ fixed in the previous section and its (p^f-1)-th root $\pi^{1/(p^f-1)}$. Let L denote the unramified extension of $K(\pi^{1/(p^f-1)})$ of degree p^f-1 and note that L contains a primitive p-th root of unity ϵ_1. Let l denote the residue field of L. Then [Vos79, Proposition 1] constructs an isomorphism of $\mathbb{Z}_p$-modules

$$E^{\mathrm{AH}} : vW(l)[[v]] \xrightarrow{\sim} 1 + vW(l)[[v]],$$

given by

$$x \mapsto \exp\left(\sum_{n\geq 0} (\frac{\varphi}{p})^n(x)\right).$$

Here $\exp(x)$ denotes the formal power series $\sum_{i\geq 0}\frac{x^i}{i!}$ and φ denotes the $\mathbb{Z}_p$-linear operator on $W(l)[[v]]$, which acts as the Witt vector Frobenius on $W(l)$ and which sends $v \mapsto v^p$. Composing this map with evaluation at $v = \pi^{1/(p^f-1)}$, we obtain a homomorphism of $\mathbb{Z}_p$-modules

$$E^{\mathrm{AH}}_{\Theta} : vW(l)[[v]] \to (1+\mathfrak{m}_L)^{\times}$$

and composing with the Kummer map produces a $\mathbb{Z}_p$-module homomorphism

$$vW(l)[[v]] \to H^1(G_K, \mathbb{F}_p)$$

sending f onto the cocycle $c : G_L \to \mathbb{F}_p$ defined by $\sigma(E^{\mathrm{AH}}_{\Theta}(f)^{1/p}) = \epsilon_1^{c(\sigma)} E^{\mathrm{AH}}_{\Theta}(f)^{1/p}$. We obtain

$$\Psi : vl[[v]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \to H^1(G_L, \overline{\mathbb{F}}_p)$$

by applying $\otimes_{\mathbb{Z}_p}\overline{\mathbb{F}}_p$.

5. Explicit Serre weights in the indecomposable case

In this section we assume $p > 2$ (so that Section 4 applies) and give our definition of $W^{\exp}(\overline{r})$ for a possibly non-semisimple

$$\overline{r} \sim \begin{pmatrix} \chi_1 & c \\ 0 & \chi_2 \end{pmatrix}.$$

For this it will be useful to identify $k[[v]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \cong \prod_\tau \overline{\mathbb{F}}_p[[v]]$ via $(\sum a_i v^i) \otimes b \mapsto (\sum \tau(a_i) b v^i)_\tau$, where τ runs over the embeddings in $\mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$. This allows us to express elements of $k[[v]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ as tuples $(y_\tau)_\tau$ with $y_\tau \in \overline{\mathbb{F}}_p[[v]]$.

Definition 5.1. Define $W^{\exp}(\overline{r})$ as follows. If $a_\tau - b_\tau = p-1$ for every τ, then $\sigma_{a,b} \in W^{\exp}(\overline{r})$ if and only if $\overline{r}|_{I_K} \sim \prod_\tau \omega_\tau^{b_\tau} \otimes \left(\begin{smallmatrix} \overline{\chi}_{\mathrm{cyc}} & c \\ 0 & 1 \end{smallmatrix}\right)$ for $\overline{\chi}_{\mathrm{cyc}}$ the mod p cyclotomic character. Otherwise, $\sigma_{a,b} \in W^{\exp}(\overline{r})$ if and only if there exist $s_\tau, t_\tau \geq 0$ with

$$a_\tau - b_\tau + e = s_\tau + t_\tau, \qquad \max\{s_\tau, t_\tau\} \geq a_\tau - b_\tau + 1$$

and such that

(1) $\overline{r}|_{I_K} \sim \left(\begin{smallmatrix} \prod_\tau \omega_\tau^{s_\tau+b_\tau} & c \\ 0 & \prod_\tau \omega_\tau^{t_\tau+b_\tau} \end{smallmatrix}\right)$;

(2) $c|_{G_L} \in \Psi_{s,t} \subset H^1(G_L, \overline{\mathbb{F}}_p)$, where

$$\Psi_{s,t} := \Psi\left((v^{\Omega_{\tau,t-s}+(p^f-1)(\delta_\tau - s_\tau)})\overline{H}(v)k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p\right), \qquad u = v^{p^f-1}$$

for

- $\overline{H}(v) \in l[[v]]$ the reduction modulo p of $z(v)^p - 1$ for $z(v) \in W(l)[v]$ the polynomial of minimal degree with $z(\pi^{1/(p^f-1)}) = \epsilon_1$,
- $\Omega_{\tau,n} := \sum_{i=0}^{f-1} p^i n_{\tau\circ\varphi^i}$ for any tuple $n = (n_\tau)_\tau$,

- $\delta_\tau := a_\tau - b_\tau + 1$ if $t_\tau < a_\tau - b_\tau + 1$ and 0 otherwise.

When $\overline{r}$ is a split extension of two characters this agrees with the definition of $W^{\exp}(\overline{r})$ from Definition 3.1. This follows from Remark 3.2 and, in the special case where $a_\tau - b_\tau = p-1$ for every τ, the fact that $\overline{\chi}_{\mathrm{cyc}}|_{I_K} = \prod_\tau \omega_\tau^{e+p-1}|_{I_K}$.

We also point out that multiple tuples (s,t) can exist satisfying condition (1), and a priori these different (s,t) produce different subspaces $\Psi_{s,t}$. The following proposition makes this precise.

Proposition 5.2. *Assume $\sigma_{a,b} \in W^{\exp}(\overline{r}^{\mathrm{ss}})$. Let $\mathcal{S}(\overline{r},\sigma_{a,b})$ denote the set of (s,t) as in Definition 5.1 satisfying condition (1). Then $\mathcal{S}(\overline{r},\sigma_{a,b})$ contains a unique maximal element for the ordering defined by*

$$(s,t) \leq (s',t') \Leftrightarrow \Omega_{\tau,t-t'}, \Omega_{\tau,s'-s} \in (p^f-1)\mathbb{Z}_{\geq 0} \text{ for all } \tau.$$

Furthermore, $(s,t) \leq (s',t')$ implies $\Psi_{s,t} \subset \Psi_{s',t'}$ so in condition (2) we can assume that (s,t) is the maximal element in $\mathcal{S}(\overline{r},\sigma_{a,b})$.

Proof. The first statement is Lemma 11.1 and the second is Lemma 11.3. □

The following is our main theorem (whose proof is completed in Section 11).

Theorem 5.3. *Recall $p>2$. Then $W^{\exp}(\overline{r}) = W^{\mathrm{cr}}(\overline{r})$.*

6. Some simplifying assumptions

We begin with some easy reductions in the proof of Theorem 5.3.

Lemma 6.1. *For $* \in \{\exp, \mathrm{cr}\}$ one has $\sigma_{a,b} \in W^*(\overline{r}) \Leftrightarrow \sigma_{a-b,0} \in W^*(\overline{r} \otimes \prod_\tau \omega_\tau^{-b_\tau})$. In particular, Theorem 5.3 will follow if we can show*

$$\sigma_{a,0} \in W^{\exp}(\overline{r}) \Leftrightarrow \sigma_{a,0} \in W^{\mathrm{cr}}(\overline{r})$$

for every tuple $a = (a_\tau)_\tau$ with $0 \leq a_\tau \leq p-1$.

Proof. It is clear from the definitions that $\sigma_{a,b} \in W^{\exp}(\overline{r}) \Leftrightarrow \sigma_{a-b,0} \in W^{\exp}(\overline{r} \otimes \prod_\tau \omega_\tau^{-b_\tau})$. To see that $\sigma_{a,b} \in W^{\mathrm{cr}}(\overline{r}) \Leftrightarrow \sigma_{a-b,0} \in W^{\mathrm{cr}}(\overline{r} \otimes \prod_\tau \omega_\tau^{-b_\tau})$ recall that, after fixing embeddings $\tau_0, \ldots, \tau_{e-1} \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ inducing τ on k, $\prod_\tau \omega_\tau^{b_\tau}$ is the reduction modulo p of a crystalline character ψ with τ_0-Hodge–Tate weight b_τ and τ_i-Hodge–Tate weight 0 for $i = 1, \ldots, e-1$. Therefore, if r witnesses $\sigma_{a,b} \in W^{\mathrm{cr}}(\overline{r})$ then $r \otimes \psi^{-1}$ witnesses $\sigma_{a,0} \in W^{\mathrm{cr}}(\overline{r} \otimes \prod_\tau \omega_\tau^{-b_\tau})$, and similarly for the other direction. □

Lemma 6.2. *To prove $\sigma_{a,0} \in W^{\exp}(\overline{r}) \Leftrightarrow \sigma_{a,0} \in W^{\mathrm{cr}}(\overline{r})$, we can assume that, without restricting to I_K, $\overline{r}$ is of the form*

$$\overline{r} \sim \begin{pmatrix} \prod_\tau \omega_\tau^{s_\tau} & c \\ 0 & \prod_\tau \omega_\tau^{t_\tau} \end{pmatrix}$$

for some tuple of non-negative $(s_\tau, t_\tau)_\tau$ satisfying, for all τ,

$$a_\tau + e = s_\tau + t_\tau, \qquad \max\{s_\tau, t_\tau\} \geq a_\tau + 1.$$

Proof. As explained in the previous section, we know $W^{\mathrm{cr}}(\overline{r}) \subset W^{\exp}(\overline{r}^{\mathrm{ss}})$. Therefore, both $\sigma_{a,0} \in W^{\exp}(\overline{r})$ and $\sigma_{a,0} \in W^{\mathrm{cr}}(\overline{r})$ imply $\overline{r}|_{I_K} \sim \left(\begin{smallmatrix} \prod_\tau \omega_\tau^{s_\tau} & c \\ 0 & \prod_\tau \omega_\tau^{t_\tau} \end{smallmatrix}\right)$ for some non-negative integers s_τ, t_τ satisfying $s_\tau + t_\tau = a_\tau + e$ and $\max\{s_\tau, t_\tau\} \geq a_\tau + 1$.

To remove the restriction to inertia we use that $W^{\mathrm{cr}}(\overline{r}) = W^{\mathrm{cr}}(\overline{r}')$ if $\overline{r}'|_{I_K} \cong \overline{r}|_{I_K}$ (see [GLS15, 6.3.1]). Therefore, it suffices to construct

$$\overline{r}' \sim \begin{pmatrix} \prod_\tau \omega_\tau^{s_\tau} & c' \\ 0 & \prod_\tau \omega_\tau^{t_\tau} \end{pmatrix}$$

with $c'|_{G_L} = c|_{G_L}$ and $\overline{r}'|_{I_K} \cong \overline{r}|_{I_K}$, since then $\sigma_{a,0} \in W^{\exp}(\overline{r})$ if and only if $\sigma_{a,0} \in W^{\exp}(\overline{r}')$.

First we note that $c'|_{G_L} = c|_{G_L}$ implies $c|_{I_L} = c'|_{I_L}$. Since I_K/I_L has order prime-to-p, the restriction $H^1(I_K, \chi_1/\chi_2) \to H^1(I_L, \chi_1/\chi_2)$ is injective. Therefore $c|_{I_K}$ and $c'|_{I_K}$ represent the same cohomology class. Thus $\overline{r}'|_{I_K} \cong \overline{r}|_{I_K}$. Therefore, it suffices to find $[c'] \in H^1(G_K, \prod_\tau \omega_\tau^{s_\tau - t_\tau})$ such that $c'|_{G_L} = c|_{G_L}$. Since $\mathrm{Gal}(L/K)$ is prime-to-p, we have $H^2(\mathrm{Gal}(L/K), \chi_1/\chi_2) = 0$. Therefore, inflation-restriction implies that res: $H^1(G_K, \chi_1/\chi_2) \to H^1(G_L, \chi_1/\chi_2)^{\mathrm{Gal}(L/K)} = H^1(G_L, \overline{\mathbb{F}}_p)^{\mathrm{Gal}(L/K)}$ is surjective. The same is true with χ_1/χ_2 replaced by $\prod_\tau \omega_\tau^{s_\tau - t_\tau}$. So we can find a cocycle c' representing a class in $H^1(G_K, \prod_\tau \omega_\tau^{s_\tau - t_\tau})$ with $c'|_{G_L} = c|_{G_L}$, as required. □

Lemma 6.3. *To prove $\sigma_{a,0} \in W^{\exp}(\overline{r}) \Leftrightarrow \sigma_{a,0} \in W^{\mathrm{cr}}(\overline{r})$, we may assume that $(s_\tau, t_\tau) \neq (p+e-1, 0)$ for at least one τ in (1) of Definition 5.1.*

Proof. Suppose one can take $(s_\tau, t_\tau) = (p+e-1, 0)$ in (1) of Definition 5.1. Then $a_\tau = p-1$ for every τ and $\prod_\tau \omega_\tau^{s_\tau - t_\tau}$ is an unramified twist of the cyclotomic character. Thus $\sigma_{a,0} \in W^{\exp}(\overline{r})$, and we have to show $\sigma_{a,0} \in W^{\mathrm{cr}}(\overline{r})$ under these assumptions. Below we sketch the proof of this fact, following [GLS14, 9.4] (which treats the unramified case) and [GLS12, 5.2.9] (which treats the totally ramified case).

Write $\overline{r} \sim \begin{pmatrix} \prod_\tau \omega_\tau^{p+e-1} & c \\ 0 & 1 \end{pmatrix}$ as in Lemma 6.2. As in the proof of Lemma 6.1, there is a crystalline character $\widetilde{\omega} : G_K \to \overline{\mathbb{Z}}_p^\times$ lifting $\prod_\tau \omega_\tau^{p+e-1}$ with $\mathrm{HT}_{\tau'}(\widetilde{\omega}) = \{p\}$ for $\tau' = \tau_0$ and $\mathrm{HT}_{\tau'}(\widetilde{\omega}) = \{1\}$ for $\tau' = \tau_1, \ldots, \tau_{e-1}$ (where $\tau_0, \ldots, \tau_{e-1} : K \to \overline{\mathbb{Q}}_p$ such that $\tau_i|_k = \tau$). For any unramified character ψ with $\psi \equiv 1$ modulo $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$, consider the Block–Kato subgroup $H^1_f(G_K, \psi\widetilde{\omega}) \subset H^1(G_K, \psi\widetilde{\omega})$ classifying crystalline extensions of 1 by $\psi\widetilde{\omega}$. Any such extension has Hodge type $\sigma_{a,0}$, so we will be done if we can show that any class in $H^1(G_K, \prod_\tau \omega_\tau^{p+e-1})$ is contained in the image of the reduction map

$$H^1_f(G_K, \psi\widetilde{\omega}) \to H^1(G_K, \prod_\tau \omega_\tau^{e+p-1})$$

for at least one ψ. In fact, since every Hodge–Tate weight of $\widetilde{\omega}$ is ≥ 0, one has $H^1_f(G_K, \psi\widetilde{\omega}) = H^1(G_K, \psi\widetilde{\omega})$. Therefore, this can be checked using standard techniques from Galois cohomology. $\square$

7. Breuil–Kisin modules

We do not assume that $p > 2$ in this section. Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$, sufficiently large that there is an embedding $k \hookrightarrow \mathbb{F}$. A Breuil–Kisin module $\mathfrak{M}$ over $\mathbb{F}$ is a finite free $\mathfrak{S}_\mathbb{F} := k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$-module equipped with a homomorphism

$$\varphi : \mathfrak{M} \otimes_{\mathfrak{S}_\mathbb{F}, \varphi} \mathfrak{S}_\mathbb{F} \to \mathfrak{M}$$

with cokernel killed by a power of $E(u) \in \mathfrak{S}_\mathbb{F}$.[1] Here φ on $\mathfrak{S}_\mathbb{F}$ denotes $\mathbb{F}$-linear extension of the the p-th power map on $k[[u]]$ and $E(u)$ denotes the (reduction modulo p of the) minimal polynomial over $W(k)$ of π. Thus $E(u) = u^e$.

Definition 7.1. Recall C is a completed algebraic closure C of K, with integers $\mathcal{O}_C$. We assume C contains the extension L/K from Section 4. Set $\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p$ and recall that $\mathcal{O}_{C^\flat}$ multiplicatively identifies with $\varprojlim_{x \mapsto x^p} \mathcal{O}_C$. Fix a choice of compatible system $\pi^{1/(p^f-1)p^\infty} \in C$ of p-th power roots of $\pi^{1/(p^f-1)}$ so that

$$(\pi^{1/(p^f-1)}, \pi^{1/p(p^f-1)}, \pi^{1/p^2(p^f-1)}, \ldots) \in \mathcal{O}_{C^\flat}$$

Then $v \mapsto (\pi^{1/(p^f-1)}, \pi^{1/p(p^f-1)}, \pi^{1/p^2(p^f-1)}, \ldots)$ defines an embedding

$$l[[v]] \hookrightarrow \mathcal{O}_{C^\flat}$$

via which we view $\mathcal{O}_{C^\flat}$ as an $l[[v]]$-algebra. As in Definition 5.1 we set $u = v^{p^f-1}$ so that we also get an embedding $k[[u]] \hookrightarrow \mathcal{O}_{C^\flat}$. Then $\mathcal{O}_C$ is u-adically complete and $C^\flat := \mathcal{O}_{C^\flat}[\frac{1}{u}] = \operatorname{Frac} \mathcal{O}_{C^\flat}$ is algebraically closed. We also fix $\epsilon = (1, \epsilon_1, \epsilon_2, \ldots) \in \mathcal{O}_{C^\flat}$ with each ϵ_i a primitive p^i-th root of unity and ϵ_1 the root of unity fixed in Section 4.

Proposition 7.2. *If r is a crystalline representation over $\mathcal{O}$ with Hodge–Tate weights ≥ 0 and $r \otimes_\mathcal{O} \mathbb{F} \cong \overline{r}$ then there exists a Breuil–Kisin module $\mathfrak{M}$ over $\mathbb{F}$ and a continuous φ-equivariant $C^\flat$-semilinear G_K-action on $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ such that*

- *(D1) $\sigma(x) - x \in \mathfrak{M} \otimes_{k[[u]]} u^{(e+p-1)/(p-1)} \mathcal{O}_{C^\flat}$ for all $\sigma \in G_K$ and $x \in \mathfrak{M}$*
- *(D2) $\sigma(x) = x$ for $\sigma \in G_{K_\infty}$ and $x \in \mathfrak{M}$.*

and such that φ, G_K-equivariantly

$$\mathfrak{M} \otimes_{k[[u]]} C^\flat \cong \overline{r}^\vee \otimes_{\mathbb{F}_p} C^\flat$$

where the Frobenius on the left hand side is that fixing $\overline{r}$ (so we can identify $\overline{r}^\vee = (\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1}$).

Proof. This follows by applying [Bar20, 2.1.12] to the crystalline representation $r^\vee$ (note that Hodge–Tate weights in [Bar20] are normalised to be the negative of those here so $r^\vee$ has Hodge–Tate weights ≥ 0 in the sense of loc. cit.) and base-changing along $\mathcal{O} \to \mathbb{F}$. Here we also use the observation that the image of $\mu := [\epsilon] - 1$ modulo p generates the ideal $u^{ep/(p-1)} \mathcal{O}_{C^\flat}$ (cf. [Fon94, 5.1.3]), and so $[\pi^\flat]\varphi^{-1}(\mu)$ modulo p generates the ideal $u^{1+ep/(p-1)} \mathcal{O}_{C^\flat} = u^{(e+p-1)/(p-1)} \mathcal{O}_{C^\flat}$. $\square$

The following is (a minor alteration of) the key technical result in [GLS15].

[1] One can define Breuil–Kisin modules over more general $\mathbb{Z}_p$-algebras but in this paper we only need to consider p-torsion Breuil–Kisin modules over $\mathbb{F}$.

Proposition 7.3. *Suppose that r has Hodge type $\sigma_{a,0}$ and that $r \otimes_{\mathcal{O}} \mathbb{F}$ satisfies the assumptions from Lemma 6.2 and Lemma 6.3. Let $\mathfrak{M}$ be the mod p Breuil–Kisin module associated to r as in Proposition 7.2.*

Fix any $\tau_0 : k \to \mathbb{F}$. Then there exists integers $0 \le s_\tau, t_\tau \le p$ with

$$s_\tau + t_\tau = a_\tau + e, \qquad \max\{s_\tau, t_\tau\} \ge a_\tau + 1 \tag{1}$$

for $r_\tau = a_\tau + 1$, and an $\mathfrak{S}_{\mathbb{F}}$-basis β of $\mathfrak{M}$ such that

$$\varphi(\beta) = \beta \begin{pmatrix} (u^{t_\tau}) & (y_\tau) \\ 0 & (u^{s_\tau}) \end{pmatrix}$$

for

$$y_\tau \in u^{\delta_\tau}\mathbb{F}[[u]]$$

with δ_τ as in Definition 5.1. Furthermore, we do not have $(s_\tau, t_\tau) = (e+p-1, 0)$ for every τ.

Proof. All the references to [GLS15] here require $p > 2$. However, the same results hold when $p = 2$ by [Wan17] provided that the uniformiser $\pi \in K$ is chosen as in [Wan17, 2.1].

It is shown in [GLS15, 5.1.5] that there are s_τ, t_τ satisfying (1) and a basis β satisfying $\varphi(\beta) = \beta \left(\begin{smallmatrix} x_1(u^{t_\tau}) & (y_\tau) \\ 0 & x_2(u^{t_\tau}) \end{smallmatrix}\right)$ for some $x_1, x_2 \in (k \otimes_{\mathbb{F}_p} \mathbb{F})^\times$ and polynomials y_τ as claimed, except that possibly y_τ contains a non-zero term of degree u^{t_τ}. A straightforward change of basis argument allows us to remove these u^{t_τ} terms from the y_τ at the cost of introducing terms of degree $\ge \delta_\tau$. This is not an issue since we only ask that $y_\tau \in u^{\delta_\tau}\mathbb{F}[[u]]$. Thus, we only have to show that we can take $x_1 = x_2 = 1$ and that $\{s_\tau, t_\tau\} \ne \{0, p\}$ for at least one τ. This is due to the assumptions on $\overline{r}$ made in Lemma 6.2.

To explain this set

$$D = \begin{pmatrix} d_1 & dd_1 \\ 0 & d_2 \end{pmatrix} \in \mathrm{Mat}(C^\flat \otimes_{\mathbb{F}_p} \mathbb{F}) \tag{2}$$

with entries defined by the equations:

$$d_1 = \varphi(d_1)x_1(u^{t_\tau}), \qquad d_2 = \varphi(d_2)x_2(u^{s_\tau}), \qquad \varphi(d) - d = -\frac{d_2}{d_1 x_2}(u^{-s_\tau}y_\tau)$$

Then $D = \left(\begin{smallmatrix} x_1(u^{t_\tau}) & (y_\tau) \\ 0 & x_2(u^{t_\tau}) \end{smallmatrix}\right)\varphi(D)$ and so, if $\alpha = \beta D$ in $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ then $\varphi(\alpha) = \alpha$. Therefore, α is an $\mathbb{F}$-basis of $\overline{r}^\vee = (\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1}$ and (D2) from Proposition 7.2 implies

$$\sigma(\alpha) = \alpha D^{-1}\sigma(D) = \alpha \begin{pmatrix} \frac{\sigma(d_1)}{d_1} & \sigma(d)\frac{\sigma(d_1)}{d_1} - d\frac{\sigma(d_2)}{d_2} \\ 0 & \frac{\sigma(d_2)}{d_2} \end{pmatrix} \tag{3}$$

for $\sigma \in G_{K_\infty}$.

We can explicitly compute the characters appearing on the diagonal of (3). First note that, by an easy calculation, [2] we can write $d_1 = \widetilde{x}_1(v^{-\Omega_{\tau,t}})$ for $\widetilde{x}_1$ satisfying $\widetilde{x}_1 = \varphi(\widetilde{x}_1)x_1$ and $\Omega_{\tau,t} = \sum_{i=0}^{f-1} p^i t_{\tau\circ\varphi^i}$ as in Definition 5.1. Notice also that by definition $\sigma(v)v^{-1} = \omega_\pi(\sigma)$ for $\sigma \in G_{K_\infty}$ and so

$$\sigma((u^{\Omega_{\tau,t}}))(u^{-\Omega_{\tau,t}}) = (\omega_\tau(\sigma)^{\Omega_{\tau,t}})_\tau = \prod_\tau \omega_\tau(\sigma)^{t_\tau}$$

for $\sigma \in G_{K_\infty}$. On the other hand, $\widetilde{x}_1 \in (\overline{k} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$ for G_{K_∞} acts on $\widetilde{x}_1$ by multiplication with an unramified character ψ_1. Therefore

$$\sigma(d_1)d_1^{-1} = \psi_1 \prod_\tau \omega_\tau(\sigma)^{-t_\tau}$$

and similarly with d_1 replaced by d_2 and t_τ with s_τ. By the assumption on $\overline{r}$ from Lemma 6.2 ψ_1 and ψ_2 trivial. Therefore $\widetilde{x}_1, \widetilde{x}_2 \in (k \otimes_{\mathbb{F}_p} \mathbb{F})^\times$. But then we can replace β by $\beta \left(\begin{smallmatrix} \widetilde{x}_1 & 0 \\ 0 & \widetilde{x}_2 \end{smallmatrix}\right)$ and assume $x_1 = x_2 = 1$. ∎

[2] Here is the calculation: we have $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F} \cong \prod \mathbb{F}[[u]]$ via $a \otimes b \mapsto (\tau(a)b)_\tau$. Thus $\varphi((u^{n_\tau})) = (u^{pn_{\tau\circ\varphi}})_\tau$. Therefore

$$\varphi((v^{\Omega_{\tau,t}})_\tau) = (v^{p\Omega_{\tau\circ\varphi,t}})_\tau = (v^{\Omega_{\tau,t}}u^{t_\tau})_\tau$$

with the second equality following from the fact that $p\Omega_{\tau,t} = pt_{\tau\circ\varphi} + \ldots + p^f t_\tau = \Omega_{\tau,t} + (p^f - 1)t_\tau$.

8. Constructing Galois actions

As in the previous section we do not assume here that $p > 2$. The goal is to show that the G_{K_∞}-action on $\bar{r}^\vee$ described in (3) can be extended to a description of the whole G_K-action.

Theorem 8.1. *Assume $\mathfrak{M}$ is a Breuil–Kisin module over $\mathbb{F}$ with shape as in Proposition 7.3. Then there exists a unique continuous φ-equivariant $C^\flat$-semilinear action of G_K on $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ satisfying:*

(D1) $\sigma(x) - x \in \mathfrak{M} \otimes_{k[[u]]} u^{(e+p-1)/(p-1)} \mathcal{O}_{C^\flat}$ for all $\sigma \in G_K$ and $x \in \mathfrak{M}$

(D2) $\sigma(x) = x$ for $\sigma \in G_{K_\infty}$ and $x \in \mathfrak{M}$.

Furthermore, if β is basis of $\mathfrak{M}$ with $\varphi(\beta) = \beta C$ then this G_K-action can be described concretely by $\sigma(\beta) = C_\sigma \beta$ for

$$C_\sigma = \lim_{n\to\infty} \left(C\varphi(C) \dots \varphi^n(C)\varphi^n(\sigma(C^{-1})) \dots \varphi(\sigma(C^{-1}))\sigma(C^{-1}) \right) \in \mathrm{Mat}(C^\flat \otimes_{\mathbb{F}_p} \mathbb{F})$$

We emphasise that the assumption from Proposition 7.3 that we do not have $(s_\tau, t_\tau) = (e + p - 1, 0)$ for every τ is crucial for the uniqueness of this G_K-action. We also mention that this uniqueness has essentially already been proved in the (essentially equivalent) language of $(\varphi, \hat{G})$-modules [GLS15, 6.1.3].

Proof. As we will explain below, in most cases the theorem follows from an application of [Bar22, 11.3]. Unfortunately, these results do not apply in the special case where $(s_\tau, t_\tau) = (0, p + e - 1)$ for every τ. We treat this special case directly at the end of the proof.

For now assume $(s_\tau, t_\tau) \neq (0, p + e - 1)$ for every τ. Since we also do not have $(s_\tau, t_\tau) = (p + e - 1, 0)$ for every τ, there are integers $0 \le q_\tau \le e + p - 1$ not all equal to $e + p - 1$ such that

- $(u^{q_\tau})\mathfrak{M} \subset \mathfrak{M}^\varphi \subset \mathfrak{M}$ for $\mathfrak{M}^\varphi$ the image of the linearised Frobenius on $\mathfrak{M}$.

Indeed, if β is a basis of $\mathfrak{M}$ with $\varphi(\beta) = \beta C$ then $(u^{q_\tau})\mathfrak{M}$ is generated by $\varphi(\beta)(u^{q_\tau})C^{-1}$, and so the assertion is equivalent to asking that $(u^{q_\tau})C^{-1} \in \mathrm{Mat}(k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F})$. Following [Bar22, 11.1], the choice of β also allows us to define a "naive" $C^\flat$-semilinear G_K-action $\sigma_{\mathrm{naive},\beta}$ on $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ by $C^\flat$-semilinearly extending the G_K-action which is trivial on $\varphi(\beta)$. Typically, $\sigma_{\mathrm{naive},\beta}$ will not be φ-equivariant. However, [Bar22, 11.3] (and its proof) asserts that the condition

- $\sigma_{\mathrm{naive},\beta}(x) - x \in \mathfrak{M} \otimes_{k[[u]]} u^{(e+p-1)/(p-1)} \mathcal{O}_{C^\flat}$ for every $x \in \mathfrak{M}$ and $\sigma \in G_K$

together with the previous bullet point imply that the sequence $\varphi^n \circ \sigma_{\mathrm{naive},\beta} \circ \varphi^{-n}$ converges to a φ-equivariant $C^\flat$-semilinear G_K-action on $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ satisfying (D1) and (D2). Furthermore, this limit is the unique such action. It also gives the formula for C_σ in the theorem since, by definition, $\sigma_{\mathrm{naive},\beta}(\beta) = \beta C \sigma(C)^{-1}$, and so

$$\begin{aligned} \varphi^n \circ \sigma_{\mathrm{naive},\beta} \circ \varphi^{-n}(\beta) &= \varphi^n \circ \sigma_{\mathrm{naive},\beta} \left(\beta \varphi^{-1}(C^{-1})\varphi^{-2}(C^{-1}) \dots \varphi^{-n}(C^{-1}) \right) \\ &= \varphi^n \left(\beta C \sigma(C^{-1}) \varphi^{-1}(\sigma(C^{-1})) \varphi^{-2}(\sigma(C^{-1})) \dots \varphi^{-n}(\sigma(C^{-1})) \right) \\ &= \beta \left(C\varphi(C) \dots \varphi^{n-1}(C)\varphi^n(C)\varphi^n(\sigma(C^{-1}))\varphi^{n-1}(\sigma(C^{-1})) \dots \varphi(\sigma(C^{-1}))\sigma(C^{-1}) \right) \end{aligned}$$

We've already seen the first bullet point holds. To apply these results we need to check the second does also.

Concretely, since $\sigma_{\mathrm{naive},\beta}(\beta) = \beta C \sigma(C^{-1})$, the second bullet point is asserting that $C\sigma(C^{-1}) - 1 \in u^{(e+p-1)/(p-1)} \mathrm{Mat}(\mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F})$. To check this we take β a basis as in Proposition 7.3 so that

$$\begin{aligned} C\sigma(C^{-1}) - 1 &= \begin{pmatrix} (u^{t_\tau}) & (y_\tau) \\ 0 & (u^{s_\tau}) \end{pmatrix} \begin{pmatrix} (\sigma(u)^{-t_\tau}) & -(\sigma(y_\tau)\sigma(u)^{e+1-r_\tau}) \\ 0 & (\sigma(u)^{-s_\tau}) \end{pmatrix} - 1 \\ &= \begin{pmatrix} (\frac{u}{\sigma(u)}^{t_\tau} - 1) & (\sigma(u)^{-s_\tau}(y_\tau - \sigma(y_\tau)\frac{u}{\sigma(u)}^{t_\tau}) \\ 0 & (\frac{u}{\sigma(u)}^{s_\tau} - 1) \end{pmatrix} \end{aligned}$$

For the require divisibility we use that $\frac{u}{\sigma(u)} - 1$ is divisible by $\epsilon - 1$ in $\mathcal{O}_{C^\flat}$ and is therefore contained in $u^{ep/(p-1)}\mathcal{O}_{C^\flat}$. This clearly implies the required divisibility for the diagonal entries. For the upper right entry, note that

$$y_\tau - \sigma(y_\tau)\left(\frac{u}{\sigma(u)}\right)^{t_\tau} = y_\tau - \sigma(y_\tau) + \sigma(y_\tau)(1 - \left(\frac{u}{\sigma(u)}\right)^{t_\tau})$$

is divisible by $u^{\delta_\tau + ep/(p-1)}$ and so we just need that $\delta_\tau - s_\tau + ep/(p-1) \ge (e + p - 1)/(p - 1)$. This inequality follows from the observation that $(e+p-1)/(p-1) - ep/(p-1) = -e+1$ and the computation:

$$\delta_\tau - s_\tau = \begin{cases} r_\tau - s_\tau & \text{if } t_\tau < r_\tau, \text{ in which case } r_\tau - s_\tau = t_\tau - e + 1 \ge -e + 1 \\ -s_\tau & \text{if } t_\tau \ge r_\tau, \text{ in which case } -s_\tau = t_\tau - r_\tau - e + 1 \ge -e + 1 \end{cases}$$

This proves the theorem under the assumption that $(s_\tau, t_\tau) \neq (0, e+p-1)$ for every τ.

We conclude by addressing the case where $(s_\tau, t_\tau) \neq (0, e+p-1)$ for every τ. This case is particularly simple because we can choose a basis β of $\mathfrak{M}$ so that[3] $\varphi(\beta) = \beta C$ with $C = \left(\begin{smallmatrix} u^{e+p-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Now suppose that $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ is equipped with a G_K-action as in the theorem. The calculations from the proof of Proposition 7.3 imply $\overline{r}^\vee = (\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1}$ is generated by $\alpha = \beta D$ for $D = \left(\begin{smallmatrix} v^{-\Omega} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\Omega = (p+e-1)(1+\ldots+p^{f-1})$. Furthermore, for $\sigma \in G_{K_\infty}$ we have $\sigma(\alpha) = \alpha \left(\begin{smallmatrix} \prod_\tau \omega_\tau(\sigma)^{-(e+p-1)} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It follows that the G_K-action on α must be of the form $\sigma(\alpha) = \alpha \left(\begin{smallmatrix} \prod_\tau \omega_\tau(\sigma)^{-(e+p-1)} & c(\sigma) \\ 0 & 1 \end{smallmatrix}\right)$ for $c : G_K \to \mathbb{F}$ a cocycle vanishing on G_{K_∞} (in fact non-zero such cocycles occur only in very specific situations by [GLS15, 5.4.2]). But then

$$\sigma(\beta) = \beta \begin{pmatrix} v^{-\Omega} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \prod_\tau \omega_\tau^{-(e+p-1)} & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma(v)^\Omega & 0 \\ 0 & 1 \end{pmatrix} = \beta \begin{pmatrix} \left(\frac{\sigma(v)}{v}\right)^\Omega \prod_\tau \omega_\tau^{-(e+p-1)} & v^{-\Omega} c \\ 0 & 1 \end{pmatrix}$$

Clearly this G_K-action only satisfies condition (D1) if $c = 0$, which proves uniqueness. To finish the proof we just have to show that the limit formula in the theorem converges to a G_K-action as claimed (this G_K-action will then coincide with that given in the previous formula when $c = 0$). But this convergence is clear because

$$C\varphi(C)\ldots\varphi^n(C)\varphi^n(\sigma(C^{-1}))\ldots\varphi(\sigma(C^{-1}))\sigma(C^{-1}) = \begin{pmatrix} \left(\frac{u}{\sigma(u)}\right)^{(e+p-1)(1+\ldots+p^n)} & 0 \\ 0 & 1 \end{pmatrix}$$

converges in $u^{(e+p-1)/(p-1)} \operatorname{Mat}(\mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F})$ as $n \to \infty$ since $\left(\frac{u}{\sigma(u)}\right) - 1 \in u^{(e+p-1)/(p-1)} \mathcal{O}_{C^\flat}$. $\square$

Corollary 8.2. *Suppose $\sigma_{a,0} \in W^{\mathrm{cr}}(\overline{r})$ for $\overline{r}$ as above. Then there exists an $\overline{\mathbb{F}}_p$-basis α of $\overline{r}^\vee$ such that*

$$\sigma(\alpha) = \alpha \begin{pmatrix} 1 & \sigma(d) - d \textit{ modulo } \mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \\ 0 & 1 \end{pmatrix}$$

for $\sigma \in G_L$ and for $d \in C^\flat \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ satisfying

$$\varphi(d) - d = (u^{\Omega_{\tau,t-s}/(p^f-1)+\delta_\tau - s_\tau}) k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p,$$

with s, t and δ as defined in Definition 5.1.

Proof. First choose a finite extension $E/\mathbb{Q}_p$ with integers $\mathcal{O}$ so that the crystalline $\overline{\mathbb{Z}}_p$-representation r witnessing $\sigma_{a,0} \in W(\overline{r})$ is defined over $\mathcal{O}$. Take $\mathbb{F}$ equal the residue field of $\mathcal{O}$. Enlarging E if necessary we can assume that there is an embedding $k \hookrightarrow \mathbb{F}$ so the above results apply. In particular, applying Theorem 7.2 to r produces a Breuil–Kisin module $\mathfrak{M}$ with shape as in Proposition 7.3. By uniqueness the G_K-action on $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ produced in Theorem 8.1 coincides with that induced by the G_K-action on $\overline{r}^\vee$.

In the proof of Proposition 7.3 we described a basis $\alpha = \beta D$ of $\overline{r}^\vee = (\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1}$ with $D = \left(\begin{smallmatrix} d_1 & dd_1 \\ 0 & d_2 \end{smallmatrix}\right)$ for $d_1 = (v^{-\Omega_{\tau,t}})$, $d_2 = (v^{-\Omega_{\tau,s}})$ and

$$\varphi(d) - d = -\frac{d_2}{d_1(u^{s_\tau})} y_\tau \in u^{\Omega_{\tau,t-s}/(p^f-1)+\delta_\tau - s_\tau} k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$$

Using the description of the G_K-action from Theorem 8.1 we will compute $\sigma(\alpha)$ for $\sigma \in G_K$. Since $\alpha = \varphi(\alpha) = \varphi(\beta)\varphi(D)$ we have $\sigma_{\mathrm{naive},\beta}(\alpha) = \alpha\varphi(D^{-1}\sigma(D))$ and so

$$\begin{aligned} \varphi^n \circ \sigma_{\mathrm{naive},\beta} \circ \varphi^{-n}(\beta) &= \varphi^n \circ \sigma_{\mathrm{naive},\beta} \circ \varphi^{-n}(\alpha D^{-1}) \\ &= \alpha\varphi^{n+1}\left(D^{-1}\sigma(D)\right)\sigma(D^{-1}) \\ &= \beta D\varphi^{n+1}\left(D^{-1}\sigma(D)\right)\sigma(D^{-1}) \end{aligned}$$

It follows that $\lim_{n\to\infty} \varphi^n(D^{-1}\sigma(D))$ converges to a matrix D_σ with entries in $\mathbb{F}$ and $\sigma(\alpha) = \alpha D_\sigma$. Observe that, for $x \in C^\flat \otimes_{\mathbb{F}_p} \mathbb{F}$, convergence of $\varphi^n(x)$ is equivalent to asking that $x \in \mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F}$ and,

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[3] A priori we have $C = \left(\begin{smallmatrix} u^{e+p-1} & y \\ 0 & 1 \end{smallmatrix}\right)$ but, via a change of basis, we can replace C by

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{e+p-1} & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \varphi(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u^{e+p-1} & y - x + \varphi(x)u^{e+p-1} \\ 0 & 1 \end{pmatrix}$$

for any $x \in k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$. If $y_0 = y$ and $y_{i+1} = \varphi(y_i)u^{e+p-1}$ then $x = \sum_{i\geq 0} y_i$ converges in $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ satisfies $y - x + \varphi(x)u^{e+p-1}$ which shows we can assume C is diagonal.

if $\overline{x}$ is the image of x under the reduction map $\mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F} \to \overline{k} \otimes_{\mathbb{F}_p} \mathbb{F}$, then $\overline{x} \in \mathbb{F}$. Furthermore, $\lim_{n\to\infty} \varphi^n(x)$ equals the image of $\overline{x}$ under the multiplicative section of this reduction map. Therefore, $D^{-1}\sigma(D) \in \mathrm{Mat}(\mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F})$ and so

$$D_\sigma = \begin{pmatrix} \frac{\sigma(d_1)}{d_1} & \sigma(d)\frac{\sigma(d_1)}{d_1} - d\frac{\sigma(d_2)}{d_2} \\ 0 & \frac{\sigma(d_2)}{d_2} \end{pmatrix} \text{ modulo } \mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F}$$

The lemma then follows from the calculation of $D^{-1}\sigma(D)$ as in (3).

To finish the proof it suffices to show that

$$\begin{pmatrix} 1 & \sigma(d) - d \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \frac{\sigma(d_1)}{d_1} & \sigma(d)\frac{\sigma(d_1)}{d_1} - d\frac{\sigma(d_2)}{d_2} \\ 0 & \frac{\sigma(d_2)}{d_2} \end{pmatrix} \text{ modulo } \mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$$

for $\sigma \in G_L$. We do this by showing that $\sigma(d_i)d_i^{-1} \equiv 1$ modulo $u^{ep/(p-1)}\mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F}$ for $\sigma \in G_L$ and that $d \in u^{-(e+p-1)/(p-1)}\mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F}$. Clearly the first claim implies the required congruences on the diagonal. When combined with the second claim it also implies that $\sigma(d)\frac{\sigma(d_1)}{d_1} - d\frac{\sigma(d_2)}{d_2} \equiv \sigma(d) - d$ modulo $\mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F}$ which finishes the proof.

For the first claim note that $\frac{d\sigma(d_i)}{d_i}$ is divisible by $\epsilon - 1$, which we've already seen generates the ideal $u^{(e+p-1)/(p-1)}\mathcal{O}_{C^\flat}$. For the second claim it suffices to show that $\overline{r}^\vee = (\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1}$ is contained in $\mathfrak{M} \otimes_{k[[u]]} u^{-(e+p-1)/(p-1)}\mathfrak{m}_{C^\flat}$. Take $x \in \overline{r}^\vee$. Then $x = (u^{n_\kappa})m$ for some $m \in \mathfrak{M} \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ and some $n_\kappa \in \mathbb{Q}$. Assume the n_κ are as large as possible. Since $\varphi(x) = x$ it follows that $(u_\kappa^n)m = (u^{pn_{\kappa\circ\varphi}})\varphi(m)$ and so $\varphi(m) = (u^{n_\kappa - pn_{\kappa\circ\varphi}})m$. Recall there exists $q_\tau \leq e+p-1$ with at least one inequality strict and $(u^{q_\kappa})\mathfrak{M} \subset \mathfrak{M}^\varphi$ (where $\mathfrak{M}^\varphi$ denotes the image of the linearised Frobenius). Therefore

$$\varphi(m) = (u^{n_\kappa - pn_{\kappa\circ\varphi} - q_\kappa})\varphi(m') = \varphi((u^{(n_\kappa - pn_{\kappa\circ\varphi} - q_\kappa)/p})m')$$

for some $m' \in \mathfrak{M} \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$. Injectivity of φ ensures that $m \in (u^{(n_\kappa - pn_{\kappa\circ\varphi} - q_\kappa)/p})\mathfrak{M} \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$. We must therefore have

$$n_\kappa - pn_{\kappa\circ\varphi} \leq q_\kappa$$

since otherwise we contradict the maximality of the n_κ. Therefore

$$\begin{aligned}(1-p^{f-1})n_\kappa &= n_\kappa - pn_{\kappa\circ\varphi} + p(n_{\kappa\circ\varphi} - pn_{\kappa\circ\varphi^2}) + \ldots + p^{f-1}(n_{\kappa\circ\varphi^{f-1}} - pn_\kappa) \\ &\leq q_\kappa + pq_{\kappa\circ\varphi} + \ldots + p^{f-1}q_{\kappa\circ\varphi^{f-1}} \\ &< (e+p-1)\left(1 + p + \ldots + p^{f-1}\right) \\ &= (e+p-1)\left(\frac{p^{f-1}-1}{p-1}\right)\end{aligned}$$

We conclude that $n_\kappa > -(e+p-1)/(p-1)$ and so $x \in \mathfrak{M} \otimes_{k[[u]]} u^{-(e+p-1)/(p-1)}\mathfrak{m}_{C^\flat}$ as desired. This finishes the proof. □

9. A result of Abrashkin

In this section it is necessary to assume $p > 2$.

Proposition 9.1 (Abrashkin). *Suppose $d \in C^\flat \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ with*

$$\varphi(d) - d \in \frac{1}{u^{ep/(p-1)}} vl[[v]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$$

Then $\sigma(d) - d \in \mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ for all $\sigma \in G_L$ and

$$\sigma(d) - d \equiv \Psi((\varphi(d) - d)\overline{H}(v)) \text{ modulo } \mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$$

for $\overline{H}(v)$ described in Definition 5.1.

Proof. This is essentially [Abr97, 2.3] specialised to the case $N = 1$. To explain how to recover the formulation here first note that $\mathbb{F}$-linearity it suffices to prove the proposition when $d \in C^\flat$. If $\varphi(d) - d = \frac{h}{\overline{H}(v)}$ for $h \in vl[[v]]$ then [Abr97] shows that $\sigma(d) - d \equiv \Psi(h)$. Therefore, we just need to show that and element of $\frac{1}{u^{ep/(p-1)}} vl[[v]]$ can be written as $\frac{h}{\overline{H}(v)}$ for $h \in vl[[v]]$. This follows from the observation that since $(\epsilon_1 - 1)^p$ has p-adic valuation $p/(p-1)$ it is divisible by $\pi^{ep/(p-1)}$ and so $\overline{H}(v)$ is the reduction modulo p of a polynomial divisible by $u^{ep/(p-1)}$. □

Corollary 9.2. *Suppose $d \in l((v)) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ with $\varphi(d) - d \in \frac{1}{u^{ep/(p-1)}} vl[[v]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. Then*

$$\Psi((\varphi(d) - d)\overline{H}(v)) = 0$$

Proof. By the previous proposition we need to show that $\sigma(d) - d \in \mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F}$ for $\sigma \in G_L$. If $\varphi(d) - d = h$ then $\sigma(d) - d$ is a solution of $\varphi(X) - X = \sigma(h) - h$ and so it suffices to show that $\sigma(h) - h \in \mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \mathbb{F}$. Since $h \in \frac{1}{u^{ep/(p-1)}} vl[[v]]$ this follows from the observation that $\sigma(v^i) - v^i \in v^i(\epsilon - 1)\mathcal{O}_{C^\flat} = v^i u^{ep/(p-1)} \mathcal{O}_{C^\flat}$. $\square$

10. Dimension calculations

We continue to assume $p > 2$ here (in order to apply Corollary 9.2). Let (s, t) be as in Remark 3.2 and fix $\tau_0 : k \hookrightarrow \mathbb{F}$ as in the following

Lemma 10.1. *There exists $\tau_0 : k \to \mathbb{F}$ so that if $\Omega_{\tau_0, s-t} = 0$ then t_τ*

Then define $Y_{s,t} \subset k((u)) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ as the $\overline{\mathbb{F}}_p$-vector space consisting of those $(y_\tau)_\tau$ for which

$$y_\tau = y^{(\tau)}_{-s_\tau} u^{-s_\tau} + y^{(\tau)}_{-s_\tau+1} u^{-s_\tau+1} + \ldots + y^{(\tau)}_0 + y^{(\tau)}_{\frac{\Omega_{\tau,s-t}}{(p^f-1)}} u^{\frac{\Omega_{\tau,s-t}}{(p^f-1)}}$$

with $y^{(\tau)}_i \in \overline{\mathbb{F}}_p$ and

$$y^{(\tau)}_i = 0 \text{ for } \begin{cases} -s_\tau \leq i < a_\tau + 1 - s_\tau & \text{if } t_\tau < a_\tau + 1; \\ i = 0 & \text{if } t_{\tau \circ \varphi^{-1}} \geq a_{\tau\circ\varphi^{-1}} + 1; \\ i = \frac{\Omega_{\tau,s-t}}{(p^f-1)} & \text{if } \tau \neq \tau_0 \text{ or } \Omega_{\tau_0,s-t} \notin (p^f-1)\mathbb{Z}_{\geq 0}. \end{cases}$$

Proposition 10.2.

$$\Psi\left((v^{\Omega_{\tau,t-s} + (p^f-1)(\delta_\tau - s_\tau)})\overline{H}(v)k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p\right) = \Psi\left((v^{\Omega_{\tau,t-s}})\right)\overline{H}(v)Y_{s,t})$$

Proof. By Corollary 9.2 it suffices to find $d \in l((v)) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ so that

$$(v^{\Omega_{\tau,t-s}})x + \varphi(d) - d \in (v^{\Omega_{\tau,t-s}})Y_{s,t}$$

whenever $x \in (u^{\delta_\tau - s_\tau})k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. Setting $d = d_1(v^{\Omega_{\tau,t-s}})$ and dividing the above equation by $(v^{\Omega_{\tau,t-s}})$ shows that this is equivalent to finding $d_1 \in l((v)) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ with

$$x + \varphi(d_1)(u^{t_\tau - s_\tau})_\tau - d_1 \in Y_{s,t}$$

By linearity we can assume that $x = e_\kappa u^{i_0}$ for some $\kappa : k \to \overline{\mathbb{F}}_p$, where e_κ denotes the κ-th idempotent in $k \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p = \prod_\tau \overline{\mathbb{F}}_p$. Define $i_n = pi_{n-1} + t_{\kappa\circ\varphi^{-n}} - s_{\kappa\circ\varphi^{-n}}$ and

$$d^{(n)}_1 = \sum_{j=0}^{n-1} e_{\kappa\circ\varphi^{-j}} u^{i_j}$$

An inductive argument shows that

$$x + \varphi(d^{(n)}_1)(u^{t_\tau - s_\tau})_\tau - d^{(n)}_1 = e_{\kappa\circ\varphi^{-n}} u^{i_n}$$

and so we will be done if we can show $e_{\kappa\circ\varphi^{-n}} u^{i_n} \in Y_{s,t}$ for some $n \geq 0$.

Suppose first that $i_n \geq 0$ for every n. Replacing $x = e_\kappa u^{i_0}$ by $e_{\kappa\circ\varphi^{-n}} u^{i_n}$ for suitable n we can also assume that $\kappa = \tau_0$ for τ_0 the embedding fixed at the beginning of the section. Set $I_n = i_{fn}$ and notice that $I_{n+1} = p^f I_n - \Omega_{\tau_0,s-t}$. Notice also that $I_n > I_{n-1}$ implies $I_{n-1} > \Omega_{\tau_0,s-t}/(p^f-1)$ which implies $I_n > \Omega_{\tau_0,s-t}/(p^f-1)$ and hence that $I_{n+1} > I_n$. Similarly $I_n < I_{n-1}$ implies $I_{n+1} < I_n$. As we've assumed $I_n \geq 0$ for all n it follows that either I_n is constant or I_n is increasing. In the latter case the sequence $d^{(n)}_1$ converges as $n \to \infty$ to d_1 with $x + \varphi(d_1)(u^{t_\tau - s_\tau}) - d_1 = 0$. In the former case $\Omega_{\tau_0,s-t} = (p^f - 1)i_0$ and so, since $i_0 \geq 0$, we have

$$e_{\tau_0} u^{i_0} = e_{\tau_0} u^{\Omega_{\tau_0,s-t}} \in Y_{s,t}$$

as required.

The previous paragraph allows us to assume there is an n with $i_n < 0$. If $n > 0$ and $i_{n-1} > 0$ then $i_n \geq p + t_{\kappa\circ\varphi^{-n}} - s_{\kappa\circ\varphi^{-n}}$. Therefore, $i_n \geq \delta_{\kappa\circ\varphi^{-n}} - s_{\kappa\circ\varphi^{-n}}$ and so

$$e_{\kappa\circ\varphi^{-n}} u^{i_n} \in Y_{s,t}$$

(the condition one needs to check is that $i_n \geq -s_\tau$ if $t_{\kappa\circ\varphi^{-n}} \geq a_{\kappa\circ\varphi^{-n}} + 1$ and $i_n \geq a_{\kappa\circ\varphi^{-n}} + 1 - s_{\kappa\circ\varphi^{-n}}$ if $t_{\kappa\circ\varphi^{-n}} < a_{\kappa\circ\varphi^{-n}} + 1$, but this is the same as asking that $i_n \geq \delta_{\kappa\circ\varphi^{-n}} - s_{\kappa\circ\varphi^{-n}}$). If $n = 0$ then we also have $i_n \geq \delta_{\kappa\circ\varphi^{-n}} - s_{\kappa\circ\varphi^{-n}}$ due to the assumption that $x \in (u^{\delta_\tau - s_\tau})k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$, and so we also have $e_{\kappa\circ\varphi^{-n}} u^{i_n} \in Y_{s,t}$.

The only remaining possibility is that there is an $n > 0$ with $i_n < 0$ and $i_{n-1} = 0$. If $t_{\kappa\circ\varphi^{-n}} < r_{\kappa\circ\varphi^{-n}}$ then $e_{\kappa\circ\varphi^{-n+1}} = e_{\kappa\circ\varphi^{-n+1}} u^{i_{n-1}} \in Y_{s,t}$ so we can also assume $t_{\kappa\circ\varphi^{-n}} \geq r_{\kappa\circ\varphi^{-n}}$. But then $e_{\kappa\circ\varphi^{-n}} u^{i_n} \in Y_{s,t}$ because $-s_{\kappa\circ\varphi^{-n}} \leq i_n = t_{\kappa\circ\varphi^{-n}} - s_{\kappa\circ\varphi^{-n}} < 0$. □

Corollary 10.3.

$$\dim_{\overline{\mathbb{F}}_p} \Psi\left((v^{\Omega_{\tau,t-s} - (p^f-1)(\delta_\tau - s_\tau)})\overline{H}(v)k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p\right) \leq \nu + \sum_\tau \begin{cases} s_\tau - a_\tau & \text{if } t_\tau < a_\tau + 1 \\ s_\tau & \text{if } t_\tau \geq a_\tau + 1 \end{cases}$$

where $\nu = 0$ unless $\Omega_{\tau_0,s-t} \in (p^f - 1)\mathbb{Z}_{\geq 0}$ in which case $\nu = 1$.

11. Proof of the main theorem

We are now ready to put together the proof of Theorem 5.3. Recalling Lemma 6.1 we have to show that $\sigma_{a,0} \in W^{\exp}(\overline{r})$ if and only if $\sigma_{a,0} \in W^{\mathrm{cr}}(\overline{r})$ under the assumptions in Lemma's 6.2 and 6.3. The "if" direction is easy from the results so far: Corollary 8.2 implies that

$$\overline{r}^\vee|_{G_L} \sim \begin{pmatrix} 1 & \sigma(d) - d \text{ modulo } \mathfrak{m}_{C^\flat} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \\ 0 & 1 \end{pmatrix}$$

with $d \in C^\flat \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ satisfying $\varphi(d) - d \in (u^{\Omega_{\tau,t-s} + \delta_\tau - s_\tau})k \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ for $s_\tau, t_\tau, \delta_\tau$ as in Definition 5.1. Proposition 9.1 therefore allows us to write

$$\overline{r}^\vee|_{G_L} \sim \begin{pmatrix} 1 & \Psi(y\overline{H}(v)) \\ 0 & 1 \end{pmatrix}$$

for some $y \in (u^{\Omega_{\tau,t-s} + \delta_\tau - s_\tau})k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. Therefore

$$\overline{r}|_{G_L} \sim \begin{pmatrix} 1 & \Psi(-y\overline{H}(v)) \\ 0 & 1 \end{pmatrix}$$

which shows that $\sigma_{a,0} \in W^{\exp}(\overline{r})$.

It remains to prove the "only if" direction. For this let $\mathcal{S} := \mathcal{S}(\overline{r}, \sigma_{a,0})$ denote the set of tuples $(s,t) = (s_\tau, t_\tau)_\tau$ with s_τ, t_τ as in Remark 3.2. Since $\prod_\tau \omega_\tau^{n_\tau} = \omega_{\tau_0}^{\Omega_{\tau_0, n_\tau}}$, once we fix one $(s', t') \in \mathcal{S}$ then $\mathcal{S}$ consists of (s,t) with

- $s_\tau + t_\tau = a_\tau + e$
- $\max\{s_\tau, t_\tau\} \geq a_\tau + 1$
- $\Omega_{\tau,t-t'}, \Omega_{\tau,s'-s} \in (p^f - 1)\mathbb{Z}$ for all τ

We can also define a partial ordering $\leq$ on $\mathcal{S}$ by asserting that

$$(s,t) \leq (s',t') \Leftrightarrow \Omega_{\tau,t-t'}, \Omega_{\tau,s'-s} \in (p^f - 1)\mathbb{Z}_{\geq 0}$$

The next two lemmas prove Proposition 5.2

Lemma 11.1. *$\mathcal{S}$ contains a unique maximal pair $(s^{\max}, t^{\max})$.*

Proof. This is essentially [GLS15, 5.3.1] with a different formulation. To see this fix $(s', t') \in \mathcal{S}$. Then $\mathcal{S}$ is in bijection with the set $\mathcal{S}_0$ of tuples $t = (t_\tau)$ for which $\Omega_{\tau,t-t'} \in (p^f - 1)\mathbb{Z}$ and

$$t_\tau \in [0, e-1] \cup [a_\tau + 1, a_\tau + e]$$

If $(s,t) \in \mathcal{S}$ then clearly $t \in \mathcal{S}_0$. Conversely, if $t \in \mathcal{S}_0$ then define $s = (s_\tau)$ by setting $s_\tau = a_\tau + e - t_\tau$. Then $\max\{s_\tau, t_\tau\} \geq a_\tau + 1$ and

$$\Omega_{\tau,s'-s} = \sum_{i=0}^{f-1} p^i(s'_{\tau\circ\varphi^i} - s_{\tau\circ\varphi^i}) = \sum_{i=0}^{f-1} p^i(t_{\tau\circ\varphi^i} - t'_{\tau\circ\varphi^i}) = \Omega_{\tau,t-t'}$$

since $s'_{\tau\circ\varphi^i} = a_{\tau\circ\varphi^i} + e - t_{\tau\circ\varphi^i}$. Thus $(s,t) \in \mathcal{S}$. Under this identification the ordering on $\mathcal{S}$ corresponds to the ordering $t \leq t'$ if and only if $\Omega_{\tau,t-t'} \in (p^f - 1)\mathbb{Z}_{\geq 0}$.

On the other hand, in [GLS15, 5.3.1] a set (also denoted $\mathcal{S}$) is considered consisting of rank one Breuil–Kisin modules $\overline{\mathfrak{P}}(t_\tau; 1)$ over $\mathfrak{S}_{\mathbb{F}}$ indexed by $t = (t_\tau) \in \mathcal{S}_0$ (in loc. cit. one sets $r_\tau := a_\tau + 1$). It is shown that this set contains a unique maximal element $\overline{\mathfrak{P}}(t_\tau^{\max}; 1)$ for the ordering defined by $\overline{\mathfrak{P}}(t_\tau; 1) \leq \overline{\mathfrak{P}}(t'_\tau; 1)$ if and only if there exists a non-zero morphism of Breuil–Kisin modules $\overline{\mathfrak{P}}(t_\tau; 1) \to \overline{\mathfrak{P}}(t'_\tau; 1)$. Since

$\overline{\mathfrak{P}}(t_\tau;1) \leq \overline{\mathfrak{P}}(t'_\tau;1)$ if and only if $\Omega_{\tau,t-t'} \in (p^f-1)\mathbb{Z}_{\geq 0}$ by [GLS15, 5.1.2] it follows that $t^{\max} = (t^{\max}_\tau)$ is the unique maximal element in $\mathcal{S}_0$. ☐

Remark 11.2. Note that, using the dictionary described in the proof of Lemma 11.1, the construction of $(s^{\max}, t^{\max})$ in [GLS15, 5.3.1] is constructive, and can be used to compute this pair explicitly.

Lemma 11.3. *If $(s,t) \leq (s',t')$ then $\Psi_{s,t} \subset \Psi_{s',t'}$.*

Proof. Recall $\Psi_{s,t} := \Psi\left((v^{\Omega_{\tau,t-s}+(p^f-1)(\delta_\tau - s_\tau)})\overline{H}(v)k[[u]] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p\right)$. Therefore, it suffices to show that $(s,t) \leq (s',t')$ implies

$$\left((p^f-1)\delta_\tau + \Omega_{\tau,t-s} - (p^f-1)s_\tau\right) - \left((p^f-1)\delta'_\tau + \Omega_{\tau,t'-s'} - (p^f-1)s'_\tau\right) \in (p^f-1)\mathbb{Z}_{\geq 0} \tag{4}$$

since this implies the source of $\Psi_{r,s,t}$ is contained in that of $\Psi_{r,s',t'}$. Notice that

$$\Omega_{\tau,t-s} - \Omega_{\tau,t'-s'} = \Omega_{\tau,s'-s} + \Omega_{\tau,t-t'} \in (p^f-1)\mathbb{Z}_{\geq 0}$$

by the fact that $(s,t) \leq (s',t')$. This shows the divisibility in (4). We are left proving positivity. We can write

$$(p^f-1)(s'_\tau - s_\tau) + \Omega_{\tau,t-t'} + \Omega_{\tau,s'-s} = \Omega_{\tau,t-t'} + p\Omega_{\tau\circ\varphi,s'-s}$$

and we have to show this is $\geq (p^f-1)(\delta'_\tau - \delta_\tau)$. Since $\delta'_\tau - \delta_\tau \leq p$ this is immediate when $\Omega_{\tau\circ\varphi,s'-s} \geq (p^f-1)$ so we can assume $\Omega_{\tau\circ\varphi,s'-s} = 0$. Since $s_\tau + t_\tau = s'_\tau + t'_\tau$ for each τ it follows that $\Omega_{\tau\circ\varphi,t-t'} = 0$ also. Since $(p^f-1)(t_\tau - t'_\tau) = \Omega_{\tau,t-t'} + p\Omega_{\tau\circ\varphi,t-t'} = \Omega_{\tau,t-t'}$ we conclude that $t_\tau \geq t'_\tau$. Therefore $\delta'_\tau - \delta_\tau \leq 0$ and so we are done. ☐

Returning to the proof of the "only if" direction, suppose $\sigma_{a,0} \in W^{\exp}(\overline{r})$. Lemma 11.3 implies that

$$\overline{r} \sim \begin{pmatrix} \prod_\tau \omega_\tau^{s_\tau} & c \\ 0 & \prod_\tau \omega_\tau^{t_\tau} \end{pmatrix}$$

and $c|_{G_L} \in \Psi_{s,t}$ for $(s,t) \in \mathcal{S}$ the maximal pair. We have to produce a crystalline lift r with $\overline{r} \cong r \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$ and Hodge type $\sigma_{a,0}$. To do this choose an indexing of those $\tau_0, \ldots, \tau_{e-1} : K \to \overline{\mathbb{Q}}_p$ lifting τ. We then consider crystalline extensions

$$r \sim \begin{pmatrix} \omega_s & C \\ 0 & \omega_t \end{pmatrix}$$

for ω_s and ω_t crystalline characters lifting $\prod_\tau \omega_\tau^{s_\tau}$ and $\prod_\tau \omega_\tau^{t_\tau}$ with

$$(\mathrm{HT}_{\tau_0}(\omega_s), \mathrm{HT}_{\tau_0}(\omega_t)) = \begin{cases} (a_\tau+1, 0) & \text{if } t_\tau < a_\tau + 1 \\ (0, a_\tau+1) & \text{if } t_\tau \geq a_\tau + 1 \end{cases}$$

and

$$\{(\mathrm{HT}_{\tau_j}(\omega_s), \mathrm{HT}_{\tau_j}(\omega_t))\}_{j=1,\ldots,e-1} = \begin{cases} \underbrace{(1,0),\ldots,(1,0)}_{s_\tau - a_\tau - 1 \text{ times}}, \underbrace{(0,1),\ldots,(0,1)}_{t_\tau \text{ times}} & \text{if } t_\tau < a_\tau + 1 \\ \underbrace{(1,0),\ldots,(1,0)}_{s_\tau \text{ times}}, \underbrace{(0,1),\ldots,(0,1)}_{t_\tau - a_\tau - 1 \text{ times}} & \text{if } t_\tau \geq a_\tau + 1 \end{cases}$$

Any such r has Hodge type $\sigma_{a,0}$. The cocycles C defining such an extension are described by the Bloch–Kato subspace $H^1_f(G_K, \omega_s/\omega_t) \subset H^1(G_K, \omega_s/\omega_t)$. It follows, for example from ??, that the image $L_{\max}$ of $H^1_f(G_K, \omega_s/\omega_t) \subset H^1(G_K, \omega_s/\omega_t)$ under $H^1(G_K, \omega_s/\omega_t) \to H^1(G_K, \prod_\tau \omega_\tau^{s_\tau - t_\tau})$ has $\overline{\mathbb{F}}_p$-dimension

$$\nu' + \sum_\tau \begin{cases} s_\tau - a_\tau & \text{if } t_\tau < a_\tau + 1 \\ s_\tau & \text{if } t_\tau \geq a_\tau + 1 \end{cases}$$

where $\nu' = 0$ unless $\Omega_{\tau, s^{\max} - t^{\max}} \in (p^f-1)\mathbb{Z}$ in which case $\nu' = 1$. It follows from the "if" direction proved above

$$L_{\max} \subset \Psi_{s,t}$$

Recall from Corollary 10.3 that, after fixing τ_0, $\Psi_{s,t}$ has dimension

$$\leq \nu + \sum_\tau \begin{cases} s_\tau - a_\tau & \text{if } t_\tau < a_\tau + 1 \\ s_\tau & \text{if } t_\tau \geq a_\tau + 1 \end{cases}$$

for $\nu = 0$ unless $\Omega_{\tau_0, s-t} \in (p^f-1)\mathbb{Z}_{\geq 0}$, in which case $\nu = 1$. In particular $\nu' \geq \nu$ and so $\dim_{\mathbb{F}} L_{\max} \geq \dim_{\mathbb{F}} \Psi_{s,t}$. Therefore $L_{\max} = \Psi_{s,t}$ and so we can choose C so that $\overline{r} \cong r \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$ as desired. This finishes the proof of Theorem 5.3.

Corollary 11.4. *The inequality in Corollary 10.3 is an equality when (s,t) is maximal.*

12. Explicit comparison with Dembèlè–Diamond–Roberts

In the remainder of this paper we will compare our main result to the explicit conjecture of [DDR16]. As mentioned in the introduction, it follows from the results of this paper and [CEGM17] that the subspace of $H^1(G_K,\overline{\mathbb{F}}_p(\chi_1/\chi_2))$ defined by [DDR16] coincides with the inverse image in $H^1(G_K,\overline{\mathbb{F}}_p(\chi_1/\chi_2))$ of the subspace defined in Prop. 10.2 with (s,t) maximal. This follows since both spaces have the same description through p-adic Hodge theory in terms of crystalline lifts. In the spirit of this paper, we will give a direct proof of the equality of these two spaces using a reciprocity law of Brückner–Shaferevich–Vostokov (see [Vos79, Thm. 4]) without reference to any p-adic Hodge theory. As a consequence we get a new proof of the conjecture of [DDR16].

12.1. **Vostokov's formula.** We keep assuming $p>2$ is a prime (as is assumed in [Vos79]). Let L be a p-adic local field containing a fixed p-th root of unity ϵ_1. We fix a uniformiser $\varpi \in L$. Let l denote the residue field of L and $W(l)$ its ring of Witt vectors.

We have the identification via local class field theory of $\widehat{L^\times} \cong G_L^{\mathrm{ab}}$, where the last term denotes the abelianisation of the absolute Galois group of L. The Hilbert symbol

$$(\ ,\ )\colon L^\times \times L^\times \to \mu_p(L)$$

is then defined as the map $(\alpha,\beta)\mapsto \epsilon_1^{c(\alpha,\beta)}$, where $c(\alpha,\beta)\in \mathbb{Z}/p\mathbb{Z}$ is defined by $\epsilon_1^{c(\alpha,\beta)} = \frac{\sigma_\alpha(\beta^{1/p})}{\beta^{1/p}}$ for $\sigma_\alpha \in G_L$ corresponding to $\alpha \in L^\times$ under local class field theory and $\beta^{1/p}$ any root of $x^p-\beta$ in $\overline{L}$. Note that $c(-,\beta)\in H^1(G_L,\overline{\mathbb{F}}_p)$ is precisely the cocycle corresponding to β under the Kummer map $L^\times \to H^1(G_L,\overline{\mathbb{F}}_p)$.

Vostokov [Vos79] finds an explicit way of expressing the above mentioned Hilbert symbol. Before we state his formula we need to introduce some notation. It is clear that we can find $c_i\in W(l)$ such that $\epsilon_1 = 1+c_1\varpi + c_2\varpi^2+\cdots$. We define the power series $z(v) := 1+\sum_{i\geq 1} c_i v^i \in W(l)[[v]]$.

Recall from §4 that we have an isomorphism of $\mathbb{Z}_p$-modules

$$\begin{aligned} E^{\mathrm{AH}}\colon vW(l)[[v]] &\xrightarrow{\quad\sim\quad} 1+vW(l)[[v]];\\ v &\longmapsto \exp\left(\sum_{n\geq 0}\left(\frac{\varphi}{p}\right)^n (v)\right). \end{aligned}$$

The inverse of this isomorphism will be denoted by L^{AH}. Let $\log(v)$ denote the formal power series $\log(v) := \sum_{n\geq 1}(-1)^{n+1}\frac{(v-1)^n}{n}$. It follows from [Vos79, §1] that we may explicitly define

$$\begin{aligned} L^{\mathrm{AH}}\colon 1+vW(l)[[v]] &\xrightarrow{\quad\sim\quad} vW(l)[[v]];\\ v &\longmapsto \left(1-\frac{\varphi}{p}\right)(\log(v)). \end{aligned}$$

Fix $\alpha,\beta\in L^\times$. Then we can write $\alpha = \varpi^a\theta\varepsilon$ where θ is an element of the system of representatives of the residue field l and ε is a principal unit, and similarly $\beta = \varpi^b\theta'\eta$. We may express

$$\varepsilon = 1+a_1\varpi + a_2\varpi^2 + a_3\varpi^3+\cdots,$$

with the coefficients a_i lying in $W(l)$. This allows us to define a power series $\varepsilon(v)\in W(l)[[v]]$ via $\varepsilon(v) := 1+a_1v+a_2v^2+a_3v^3+\cdots$. We define $A(v)\in W(l)((v))$ as $v^a\theta\varepsilon(v)$. The definitions of $\eta(v)\in W(l)[[v]]$ and $B(v)\in W(l)((v))$ are analogous for β.

Let res_v denote the map $W(l)((v))\to W(l)$ defined by $\sum_r a_r v^r \mapsto a_{-1}$, i.e. the taking the residue of the series. Moreover, on $W(l)((v))$ we let the operator $\frac{d}{dv}$ act as the formal derivative with respect to v. The operator $d_{\log}$ on $W(l)((v))$ will act as the logarithmic derivative, i.e. $f(v)\mapsto \frac{1}{f(v)}\frac{d}{dv}f(v)$. Now we define a value $\gamma\in W(l)$ as

(5)
$$\gamma = \mathrm{res}_v\left(\left(L^{\mathrm{AH}}(\varepsilon(v))\frac{dL^{\mathrm{AH}}(\eta(v))}{dv} - L^{\mathrm{AH}}(\varepsilon(v))d_{\log}(B(v)) + L^{\mathrm{AH}}(\eta(v))d_{\log}(A(v))\right)\left(\frac{1}{z(v)^p-1}\right)\right).$$

Vostokov's formula [Vos79, Thm. 4] now states that

$$c(\alpha,\beta)\equiv \mathrm{Tr}_{W(l)/\mathbb{Z}_p}(\gamma) \bmod p,$$

where $c(\alpha,\beta)$ is the power of ϵ_1 appearing in the Hilbert symbol (α,β).

12.2. **The DDR Conjecture.** Let $p > 2$ and let K be an unramified extension of $\mathbb{Q}_p$. Let k be the residue field of K and $f = [k : \mathbb{F}_p]$ and fix a uniformiser $\pi \in K^\times$. Let $L = M(\pi^{1/(p^f-1)})$ where M/K is a degree-prime-to-p unramified extension and $\pi^{1/(p^f-1)}$ is a (p^f-1)-th root of π. Note that L satisfies the conditions of §12.1 with $\varpi = \pi^{1/(p^f-1)}$. Suppose $\overline{r}\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is reducible and write

$$\overline{r} \sim \begin{pmatrix} \chi_1 & c_{\overline{r}} \\ 0 & \chi_2 \end{pmatrix}$$

for characters $\chi_1, \chi_2\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ and let $\chi := \chi_1\chi_2^{-1}$ as before. Possibly after enlarging M, we may assume that $\chi|_{G_L} = 1$.

Recall from §3 that, for an embedding $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$, we defined a character $\omega_\tau\colon G_K \to \overline{\mathbb{F}}_p^\times$. Then we may write

$$\chi = \psi \prod_{\tau\in\mathrm{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)} \omega_\tau^{a_\tau}$$

where $a_\tau \in \mathbb{Z}$ and ψ an unramified character. This decomposition is unique as long as we require that $a_\tau \in [1, p]$ for all τ with at least one $a_\tau \neq p$, which we will require from now on. This implies that

$$\chi|_{I_K} = (\omega_\tau|_{I_K})^{\sum_{i=0}^{f-1} p^i a_{\tau\circ\varphi^i}},$$

for all τ. Let us define $\Omega_{\tau,a} := \sum_{i=0}^{f-1} p^i a_{\tau\circ\varphi^i}$. For a fixed $\tau\colon k \hookrightarrow \overline{\mathbb{F}}_p$, let $\lambda_{\tau,\psi}$ denote a basis of the one-dimensional $\overline{\mathbb{F}}_p$-vector space $(l \otimes_{k,\tau} \overline{\mathbb{F}}_p)^{\mathrm{Gal}(L/K)=\psi}$.

Write $\varpi := \pi^{1/(p^f-1)}$. Let $\Theta\colon 1 + vW(l)[[v]] \to \mathcal{O}_L^\times$ denote the evaluation map given by $v \mapsto \varpi$. Together with the $\mathbb{Z}_p$-linear map $\psi_r\colon W(l) \to vW(l)[[v]]$ given by $x \mapsto xv^r$, for $r \geq 1$, we can take the composition $\Theta \circ E^{\mathrm{AH}} \circ \psi_r$. After applying $- \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$ this gives the homomorphism

$$\begin{aligned} \varepsilon_{\varpi^r}\colon l \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p &\to \mathcal{O}_L^\times \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p; \\ a \otimes b &\mapsto E^{\mathrm{AH}}([a]\varpi^r) \otimes b. \end{aligned}$$

In the paper [DDR16] an explicit basis of $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$ is defined, which we will define now. For each $\tau\colon k \hookrightarrow \overline{\mathbb{F}}_p$, we will define an embedding τ' and an integer n'_τ. If $a_{\tau\circ\varphi} \neq p$, then $\tau' := \tau \circ \varphi$ and $n'_\tau = \Omega_{\tau\circ\varphi,a}$. If $a_{\tau\circ\varphi} = p$, then we let j be the smallest integer such that $j > 1$ and $a_{\tau\circ\varphi^j} \neq p - 1$. We set $\tau' := \tau \circ \varphi^j$ and $n'_\tau = \Omega_{\tau\circ\varphi^j,a} - (p^f - 1)$. Then we define

$$u_\tau := \varepsilon_{\varpi^{n'_\tau}}(\lambda_{\tau',\psi}) \in \mathcal{O}_L^\times \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p,$$

for all $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$. If $\chi = 1$, we additionally define $u_{\mathrm{triv}} := \varpi \otimes 1 \in \mathcal{O}_L^\times \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$. If χ is cyclotomic, we additionally define $u_{\mathrm{cyc}} := \varepsilon_{\varpi^{p(p^f-1)/(p-1)}}(b \otimes 1)$, where $b \in l$ is any element with $\mathrm{Tr}_{l/\mathbb{F}_p}(b) \neq 0$. In [DDR16, §5] it is shown that $\{u_\tau \mid \tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)\}$, together with u_{triv} if χ is trivial and u_{cyc} if χ is cyclotomic, forms a basis of the $\overline{\mathbb{F}}_p$-vector space

$$U_\chi := \left(L^\times \otimes \overline{\mathbb{F}}_p(\chi^{-1})\right)^{\mathrm{Gal}(L/K)},$$

which is shown to be the $\overline{\mathbb{F}}_p$-dual of $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$ via the Artin map Art_L of local class field theory. Now it is clear that if we define c_τ, c_{triv} and c_{cyc} in $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$ to be the $\overline{\mathbb{F}}_p$-duals to u_τ, u_{triv} and u_{cyc}, respectively, then we get a basis for $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$.

We fix a Serre weight $\sigma := \sigma_{a,0}$ and let us write $r_\tau = a_\tau + 1$, for all $\tau\colon k \to \overline{\mathbb{F}}_p$. Let (s, t) denote the unique maximal element of $S(\overline{r})$ as in Prop. 5.2. We note that, since $K/\mathbb{Q}_p$ is unramified, we have that $t_\tau \in \{0, r_\tau\}$ and $s_\tau = r_\tau - t_\tau$, for all τ. We define $J := \{\tau \mid t_\tau < r_\tau\}$. We refer to [DDR16, §7.1] for the definition of the shift function $\mu\colon \wp\left(\mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)\right) \to \wp\left(\mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)\right)$ that operates on subsets of embeddings, such as J.

In [DDR16] the subspace $L_\sigma^{\mathrm{DDR}}(\chi_1, \chi_2) \subseteq H^1(G_K, \overline{\mathbb{F}}_p(\chi))$ is defined as the span of c_τ for $\tau \in \mu(J)$ together with c_{triv} when χ is trivial and c_{cyc} when χ is cyclotomic, $J = \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$ and $r_\tau = p$ for all τ. Define a set Serre weights $W^{\mathrm{DDR}}(\overline{r})$ via

$$\sigma \in W^{\mathrm{DDR}}(\overline{r}) \iff c_{\overline{r}} \in L_\sigma^{\mathrm{DDR}}(\chi_1, \chi_2).$$

The main conjecture of [DDR16] (which is the main theorem of [CEGM17]) says that $W^{\mathrm{DDR}}(\overline{r}) = W^{\mathrm{cr}}(\overline{r})$. Recall from [DDR16] that the definition of $W^{\mathrm{DDR}}(\overline{r})$ is already equal to $W^{\exp}(\overline{r})$ when $\overline{r}$ is semisimple.

12.3. **Explicit Comparison.** For a direct comparison of the set of weights $W^{\exp}(\overline{r})$ (see Definition. 5.1) and $W^{\mathrm{DDR}}(\overline{r})$ we use the explicit reciprocity law of Brückner–Shaferevic–Vostokov as described in §12.1. As mentioned earlier the equality of these two sets of weights follows formally since they both equal $W^{\mathrm{cr}}(\overline{r})$, but here our aim here is to prove the equality explicitly without reference to the full machinery of p-adic Hodge theory. Under the restriction map, we may consider $L_\sigma^{\mathrm{AH}}(\chi_1,\chi_2)$ to be a subspace of $H^1(G_L,\overline{\mathbb{F}}_p)$. Then we will prove that $L_\sigma^{\mathrm{AH}}(\chi_1,\chi_2)=\Psi_{s,t}$ when (s,t) is maximal (cf. Definition. 5.1).

12.3.1. *Calculations.* To prove this statement we will use Vostokov's formula (5) for the Hilbert symbol. Since the trace commutes with reduction modulo p, we may tensor this formula with $\otimes_{\mathbb{Z}_p}\overline{\mathbb{F}}_p$. By applying $\otimes_{\mathbb{Z}_p}\overline{\mathbb{F}}_p$ to E^{AH} and L^{AH} we obtain $\overline{E}^{\mathrm{AH}}$ and $\overline{L}^{\mathrm{AH}}$, resp. Moreover, let

$$\begin{aligned}\varepsilon_{v^r}: l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p&\to(1+vW(l)[[v]])\otimes_{\mathbb{Z}_p}\overline{\mathbb{F}}_p\\ a\otimes b&\mapsto\overline{E}^{\mathrm{AH}}([a]v^r)\otimes b.\end{aligned}$$

We will write $\hat{u}_\tau:=\varepsilon_{v^{n'_\tau}}(\lambda_{\tau',\psi})$. Recall that $e_\tau\in k\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p$ denotes the τ-th idempotent. Furthermore, note that it follows from the definition that $\frac{p^f-1}{p-1}\le n'_\kappa<\frac{p(p^f-1)}{p-1}$.

Proposition 12.1. *Suppose $p>2$ and $K/\mathbb{Q}_p$ is unramified. Then $W^{\exp}(\overline{r})=W^{\mathrm{DDR}}(\overline{r})$.*

Proof. These sets of weights are identically defined when $\overline{r}$ is semisimple. Assume without loss of generality that $\overline{r}\colon G_K\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is reducible and write

$$\overline{r}\sim\begin{pmatrix}\chi_1 & c_{\overline{r}}\\ 0 & \chi_2\end{pmatrix}.$$

As in Lem. 6.1 it suffices to prove this statement only for Serre weights with vanishing determinant.

Suppose χ is cyclotomic, $J=\mathrm{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)$ and $a_\tau=p-1$ for all τ. Then it follows from [DDR16] that $\sigma_{a,0}\in W^{\mathrm{DDR}}(\overline{r})$. Similarly, it follows from Definition. 5.1 that $\sigma_{a,0}\in W^{\exp}(\overline{r})$. Therefore, if χ is cyclotomic and $J=\mathrm{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)$, we may assume $r_\tau<p$ for at least one τ. Note that $\Omega_{\tau,t-s}\ge-\frac{p(p^f-1)}{p-1}$. If we have an equality, then $(s_\tau,t_\tau)=(p,0)$ for all τ. This implies that χ is cyclotomic, $J=\mathrm{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)$ and $a_\tau=p-1$ for all τ. Therefore, we may assume $\Omega_{\tau,t-s}>-\frac{p(p^f-1)}{p-1}$.

From now on, we will fix a Serre weight $\sigma=\sigma_{a,0}$ and a maximal tuple (s,t) as in Prop. 5.2. We want to show that $\Psi_{s,t}=L_\sigma^{\mathrm{AH}}(\chi_1,\chi_2)$, where the second space is considered as a subspace of $H^1(G_L,\overline{\mathbb{F}}_p)$ under the injective restriction map. Since both spaces have the same dimension, it suffices to prove one inclusion. To prove $\Psi_{s,t}\subset L_\sigma^{\mathrm{AH}}(\chi_1,\chi_2)$, it thus suffices to show

$$c(u_\kappa,E_\Theta^{\mathrm{AH}}(x))\equiv 0 \bmod p$$

for all $\kappa\notin\mu(J)$ and for all $x\in(v^{\Omega_{\tau,t-s}})\overline{H}(v)\left(k[[u]]\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p\right)$ and, additionally, if χ is cyclotomic and either $J\neq\mathrm{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)$ or $r_\tau<p$ for some τ, then we need to show that $c(u_{\mathrm{cyc}},E_\Theta^{\mathrm{AH}}(x))\equiv 0\bmod p$ for all $x\in(v^{\Omega_{\tau,t-s}})\overline{H}(v)\left(k[[u]]\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p\right)$. (Note that the exponent term $\delta_\tau-s_\tau$ in Definition 5.1 vanishes for all τ when K is unramified.)

We note that by Prop. 10.2 it is sufficient to prove

$$c\left(u_\kappa,\overline{E}^{\mathrm{AH}}\left(e_\tau\overline{H}(X)(X^{\Omega_{\tau,t-s}})_\tau\right)\right)=0\text{ for }\kappa\notin\mu(J)\text{ and }\tau\circ\varphi^{-1}\in J.$$

and, additionally, to prove that if $\tau=\tau_0$ and $\Omega_{\tau,s-t}\in(p^f-1)\mathbb{Z}_{\ge0}$, then

$$c\left(u_\kappa,\overline{E}^{\mathrm{AH}}\left(e_\tau u^{\frac{\Omega_{\tau,s-t}}{p^f-1}}\overline{H}(X)(X^{\Omega_{\tau,t-s}})_\tau\right)\right)=0\text{ for }\kappa\notin\mu(J)$$

and if χ is cyclotomic and either $J\neq\mathrm{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)$ or $r_\tau<p$ for at least one τ, then

$$c\left(u_{\mathrm{cyc}},\overline{E}^{\mathrm{AH}}\left(e_\tau\overline{H}(X)(X^{\Omega_{\tau,t-s}})_\tau\right)\right)=0\text{ for }\tau\circ\varphi^{-1}\in J.$$

(We note that since $p>2$ it is not possible (in the unramified case) for χ to be trivial and cyclotomic at the same time so that the two exceptional cases cannot occur at the same time.) Since the trace is additive, we may treat the three terms inside inside the residue in Equation (5) separately.

Let us start by proving that, for $\kappa\notin\mu(J)$ and $\tau\circ\varphi^{-1}\in J$, we have that

$$\mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\mathrm{Res}_v\left(\overline{L}^{\mathrm{AH}}(\hat{u}_\kappa)\frac{d\left(e_\tau\overline{H}(v)\left(v^{\Omega_{\tau,t-s}}\right)_\tau\right)}{dv}\frac{1}{\overline{H}(v)}\right)\right)=0.$$

It follows from the definition $H(v) = z(v)^p - 1$ that $\frac{d\overline{H}(v)}{dv} = 0$. Also note that $\overline{L}^{\mathrm{AH}}(\hat{u}_\kappa) = \lambda_{\kappa',\psi} v^{n'_\kappa}$. Therefore, this reduces to showing that

$$\mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\mathrm{Res}_v\left(\lambda_{\kappa',\psi} v^{n'_\kappa}\Omega_{\tau,t-s} e_\tau v^{\Omega_{\tau,t-s}-1}\right)\right) = 0.$$

For this to be non-zero, it is clear that we require $n'_\kappa + \Omega_{\tau,t-s} = 0$, since the residue vanishes otherwise, and $\kappa' = \tau$, since $\lambda_{\kappa',\psi} e_\tau = 0$ otherwise. It follows from Proposition 12.2 (with $m = 0$) that these two conditions imply that $\kappa \in \mu(J)$.

Next let us prove that

$$\mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\mathrm{Res}_v\left(\overline{L}^{\mathrm{AH}}(\hat{u}_\kappa) d_{\log}\left(\overline{E}^{\mathrm{AH}}\left(e_\tau \overline{H}(v)\left(v^{\Omega_{\tau,t-s}}\right)_\tau\right)\right)\frac{1}{\overline{H}(v)}\right)\right) = 0$$

for all $\kappa \notin \mu(J)$ and $\tau\circ\varphi^{-1} \in J$. If $x \in vl[[v]]\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p$, then $d_{\log}\left(\overline{E}^{\mathrm{AH}}(x)\right) = (1 + x + x^p + \cdots)\frac{d}{dv}(x)$. It follows that we need to evaluate

$$\mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\mathrm{Res}_v\left(\lambda_{\kappa',\psi} v^{n'_\kappa}\Omega_{\tau,t-s} e_\tau v^{\Omega_{\tau,t-s}-1}\left(\sum_{m\geq 0} e_{\tau\circ\varphi^{-m}}\left(\overline{H}(v) v^{\Omega_{\tau,t-s}}\right)^{p^m}\right)\right)\right).$$

By the proof of Prop. 9.1, we know $\overline{H}(v) \in v^{p(p^f-1)/(p-1)} l[[v]]$. Since $\Omega_{\tau,t-s} > -\frac{p(p^f-1)}{p-1}$, we see $\overline{H}(v)v^{\Omega_{\tau,t-s}} \in vl[[v]]$. Since $e_\tau e_{\tau\circ\varphi^{-m}} \neq 0$ if and only if $m \equiv 0 \bmod f$, we may assume $m \in f\mathbb{Z}_{\geq 0}$. Suppose that $m \geq f$. Then $v_v(v^{n'_\kappa+\Omega_{\tau,t-s}}(\overline{H}(v)^{\Omega_{\tau,t-s}})^{p^m}) > \frac{p^f-1}{p-1} - \frac{p(p^f-1)}{p-1} + p^f = 1$. Therefore, we will not get a non-zero residue from the terms with $m \geq f$. Thus, we may assume $m = 0$. We need to evaluate

$$\mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\mathrm{Res}_v\left(\lambda_{\kappa',\psi} e_\tau \Omega_{\tau,t-s}\overline{H}(v) v^{n'_\kappa+2\Omega_{\tau,t-s}-1}\right)\right)$$

Since $\lambda_{\kappa',\psi}e_\tau = 0$ if $\kappa' \neq \tau$, we require $\kappa' = \tau$ for the trace to be non-zero. It follows from the definitions that $\chi|_{I_K} = (\omega_{\kappa'}|_{I_K})^{n'_\kappa}$ and $\chi|_{I_K} = (\omega_\tau|_{I_K})^{\Omega_{\tau,s-t}}$, so $n'_\kappa \equiv \Omega_{\tau,s-t} \bmod (p^f-1)$. If $\Omega_{\tau,t-s} > -\frac{p^f-1}{p-1}$, then $v_v(\overline{H}(v)v^{n'_\kappa+2\Omega_{\tau,t-s}}) > 0$. So we may assume $-\frac{p(p^f-1)}{p-1} < \Omega_{\tau,t-s} \leq -\frac{p^f-1}{p-1}$. From the congruence it follows that $n'_\kappa = -\Omega_{\tau,t-s}$. Then $v_v(\overline{H}(v)v^{n'_\kappa+2\Omega_{\tau,t-s}}) = v_v(\overline{H}(v)v^{\Omega_{\tau,t-s}}) \geq 1$, and the residue vanishes.

Now we need to show that, for all $\kappa \notin \mu(J)$ and $\tau\circ\varphi^{-1} \in J$, we have

$$\mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\mathrm{Res}_v\left(e_\tau\overline{H}(v)(v^{\Omega_{\tau,t-s}})_\tau d_{\log}(\hat{u}_\kappa)\frac{1}{\overline{H}(v)}\right)\right) = 0.$$

If $x \in vl[[v]]\otimes\overline{\mathbb{F}}_p$, then $d_{\log}\left(\overline{E}^{\mathrm{AH}}(x)\right) = (x + x^p + x^{p^2} + \cdots)d_{\log}(x)$. Moreover, $d_{\log}(\lambda v^n) = nv^{-1}$. Therefore,

$$e_\tau\overline{H}(v)(v^{\Omega_{\tau,t-s}})_\tau d_{\log}(\hat{u}_\kappa)\frac{1}{\overline{H}(v)} = \sum_{m\geq 0} n'_\kappa \varphi^m(\lambda_{\kappa',\psi}) v^{p^m n'_\kappa - 1} v^{\Omega_{\tau,t-s}} e_\tau.$$

For the residue of this sum to be non-zero, we require that $p^m n'_\kappa + \Omega_{\tau,t-s} = 0$ for some $m \geq 0$. Suppose this condition is satisfied for m. Then the trace evaluates to

$$n'_\kappa \mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\varphi^m(\lambda_{\kappa',\psi})e_\tau\right).$$

Since $\varphi^m(\lambda_{\kappa',\psi})e_\tau = \varphi^m(\lambda_{\kappa',\psi}e_{\tau\circ\varphi^m})$ and $\lambda_{\kappa',\psi}e_{\tau\circ\varphi^m} = 0$ if $\kappa' \neq \tau\circ\varphi^m$, we see that for the trace to be non-zero we require $p^m n'_\kappa + \Omega_{\tau,t-s} = 0$ and $\kappa' = \tau\circ\varphi^m$. These two conditions imply that $\kappa \in \mu(J)$ by Proposition 12.2, therefore the trace is zero.

Lastly, we turn our attention to the trivial and cyclotomic exceptional cases. Firstly, for the trivial case, suppose that $\tau = \tau_0$ and $\Omega_{\tau,s-t} \in (p^f-1)\mathbb{Z}_{\geq 0}$. Since $u = v^{p^f-1}$, we need to show that, for $\kappa \notin \mu(J)$, we have

$$c\left(u_\kappa, \overline{E}^{\mathrm{AH}}\left(e_\tau\overline{H}(v)\right)\right) = 0.$$

Again, using Equation (5) and the additivity of the trace, this follows from the three statements

$$\mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\mathrm{Res}_v\left(\overline{L}^{\mathrm{AH}}(\hat{u}_\kappa)\frac{d\left(e_\tau\overline{H}(v)\right)}{dv}\frac{1}{\overline{H}(v)}\right)\right) = 0,$$

$$\mathrm{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\mathrm{Res}_v\left(\overline{L}^{\mathrm{AH}}(\hat{u}_\kappa)d_{\log}\left(\overline{E}^{\mathrm{AH}}\left(e_\tau\overline{H}(v)\right)\right)\frac{1}{\overline{H}(v)}\right)\right) = 0$$

and

$$\operatorname{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\operatorname{Res}_v\left(e_\tau\overline{H}(v)d_{\log}(\hat{u}_\kappa)\frac{1}{\overline{H}(v)}\right)\right)=0.$$

The first two statements follow immediately from $\frac{d\overline{H}(v)}{dv}=0$. As before, we may rewrite the third statement as

$$\operatorname{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\operatorname{Res}_v\left(\sum_{m\geq 0}n'_\kappa\varphi^m(\lambda_{\kappa',\psi})v^{p^m n'_\kappa-1}e_\tau\right)\right)=0.$$

Since $n'_\kappa\geq\frac{p^f-1}{p-1}$, we find that $p^m n'_\kappa>0$. Therefore, also the residue in the third statement vanishes.

For the cyclotomic exceptional case, suppose χ is cyclotomic and either $J\neq\operatorname{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)$ or $r_\tau<p$ for some τ. Then we need to show that $c\left(u_{\mathrm{cyc}},\overline{E}^{\mathrm{AH}}\left(e_\tau\overline{H}(X)(X^{\Omega_{\tau,t-s}})_\tau\right)\right)=0$ for $\tau\circ\varphi^{-1}\in J$. As before, using Equation (5), this follows from the three statements

$$\operatorname{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\operatorname{Res}_v\left((b\otimes 1)v^{\frac{p(p^f-1)}{p-1}}\Omega_{\tau,t-s}e_\tau v^{\Omega_{\tau,t-s}-1}\right)\right)=0,$$

$$\operatorname{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\operatorname{Res}_v\left((b\otimes 1)v^{\frac{p(p^f-1)}{p-1}}\Omega_{\tau,t-s}e_\tau v^{\Omega_{\tau,t-s}-1}\left(\sum_{m\geq 0}e_{\tau\circ\varphi^{-m}}\left(\overline{H}(v)v^{\Omega_{\tau,t-s}}\right)^{p^m}\right)\right)\right)=0$$

and

$$\operatorname{Tr}_{l\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p/\overline{\mathbb{F}}_p}\left(\operatorname{Res}_v\left(\sum_{m\geq 0}\left(\frac{p(p^f-1)}{p-1}\right)\varphi^m(b\otimes 1)v^{p^m\left(\frac{p(p^f-1)}{p-1}\right)-1}v^{\Omega_{\tau,t-s}}e_\tau\right)\right)=0.$$

The residue in the first statement vanishes since $\frac{p(p^f-1)}{p-1}+\Omega_{\tau,t-s}>0$. The residue in the second statement vanishes since $v^{\frac{p(p^f-1)}{p-1}+\Omega_{\tau,t-s}}\left(\overline{H}(v)v^{\Omega_{\tau,t-s}}\right)^{p^m}\in vl[[v]]$ for all $m\geq 0$. The residue in the third statement vanishes since $p^m\left(\frac{p(p^f-1)}{p-1}\right)+\Omega_{\tau,t-s}>0$ for all $m\geq 0$. This completes the proof. □

12.3.2. *Combinatorics.* Here we will prove the following combinatorial statement that we have used in the proof above.

Proposition 12.2. *Suppose $\kappa\in\operatorname{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)$ and $\tau\circ\varphi^{-1}\in J$ satisfy*

(1) $p^m n'_\kappa=\Omega_{\tau,s-t}$;
(2) $\kappa'=\tau\circ\varphi^m$.

for some $m\in\mathbb{Z}_{\geq 0}$. Then $\kappa\in\mu(J)$.

Proof. Suppose $\kappa\in\operatorname{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)$ and $\tau\circ\varphi^{-1}\in J$ satisfy the two conditions from the statement for a fixed $m\geq 0$. Write $\tau'=\tau\circ\varphi^{-1}$. Then it follows that $\kappa'=\tau'\circ\varphi^{m+1}$. Moreover, since $\Omega_{\tau',s-t}=p\Omega_{\tau,s-t}-(p^f-1)(s_{\tau'}-t_{\tau'})$, we see that the first condition of the statement becomes

$$p^{m+1}n'_\kappa=\Omega_{\tau',s-t}+(p^f-1)(s_{\tau'}-t_{\tau'}).$$

Since $\tau'\in J$, we have $s_{\tau'}-t_{\tau'}=r_{\tau'}$. We conclude that κ satisfies the two conditions

(1) $p^{m+1}n'_\kappa=\Omega_{\tau',s-t}+r_{\tau'}(p^f-1)$;
(2) $\kappa'=\tau'\circ\varphi^{m+1}$.

Then it follows immediately from [CEGM17, Prop. 3.6.7] that $\kappa\in\mu(J)$, as required. □

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