POTENTIALLY DIAGONALISABLE LIFTS WITH CONTROLLED HODGE–TATE WEIGHTS

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Abstract. Motivated by the weight part of Serre's conjecture we consider the following question. Let $K/\mathbb{Q}_p$ be a finite extension and suppose $\overline{\rho}: G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ admits a crystalline lift with Hodge–Tate weights contained in the range $[0,p]$. Does $\overline{\rho}$ admits a potentially diagonalisable crystalline lift of the same Hodge–Tate weights? We answer this question in the affirmative when $K = \mathbb{Q}_p$ and $n \leq 5$, and $\overline{\rho}$ satisfies a mild 'cyclotomic-free' condition. We also prove partial results when $K/\mathbb{Q}_p$ is unramified and n is arbitrary.

Contents

1. Introduction

Let p be a prime and $K/\mathbb{Q}_p$ a finite extension. A potentially diagonalisable p-adic representation of G_K is one which is potentially crystalline and which, possibly after restriction to $G_{K'}$ for an extension K'/K, lies in the same irreducible component of a potentially crystalline deformation ring as a representation which is a sum of crystalline characters. The notion was introduced in [BLGGT14] where very general change of weight theorems for automorphic Galois representations were proven under the assumption that the representations in question are potentially diagonalisable above p, cf. [BLGGT14, Theorem E].

One motivation for these theorems comes from the following question, which is a generalisation of Serre's classical modularity conjecture. If F is a CM field and $\overline{r}: G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is continuous and irreducible, then what are the possible weights for which there exists an automorphic representation giving rise to $\overline{r}$? Very little is currently known about this question. However if one assumes that $\overline{r}$ is automorphic of *some* weight then the outlook is better—in this case if $\overline{r}_v$ admits a potentially diagonalisable crystalline lift at each place $v \mid p$ of F then, under a Taylor–Wiles hypothesis, it can be shown that $\overline{r}$ is automorphic with weight equal to the Hodge–Tate weights of the lifts of $\overline{r}_v$, cf. [BLGG18, Theorem 3.1.3] or [BG19, Theorem

5.2.1]. Since representations associated to automorphic forms which are unramified above p are crystalline above p, one is led to the following, purely local, question:

Question. Let $K/\mathbb{Q}_p$ be a finite extension and let $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a continuous representation. If $\overline{\rho}$ has a crystalline lift[1] then does $\overline{\rho}$ have a potentially diagonalisable crystalline lift with the same Hodge–Tate weights?

Hui Gao and Tong Liu [GL14] have shown that, when $K/\mathbb{Q}_p$ is unramified, any crystalline representation with Hodge–Tate weights contained in $[0, p-1]$ is potentially diagonalisable, answering our question in the affirmative. A positive answer is also known when $n = 2$ and the weights are contained in $[0, p]$, by work of Toby Gee, Tong Liu and David Savitt [GLS14, GLS15] (see also [Wan17] which extends their proof to the case $p = 2$). These methods have since been generalised in [Bar20a] to answer our question in the affirmative for representations in any dimension which are semi-simple and have weights contained in $[0, p]$.

In this paper we discuss extensions of these results to representations which are not necessarily semi-simple. Progress in this direction has previously been made by Hui Gao [Gao17, Gao18] assuming that every Jordan–Holder factor of $\overline{\rho}$ is one-dimensional (and under some additional technical assumptions). The main innovation of this paper is to put these calculations in a more conceptual framework. This allows us to consider representations whose Jordan–Holder factors are not one-dimensional, and to remove some of the conditions appearing in Gao's work.

A crucial assumption that is necessary for our methods is that $\overline{\rho}$ be cyclotomic-free (cf. Definition 2.1.1). This is an n-dimensional generalisations of the avoidance of representations of the shape $\left(\begin{smallmatrix} \chi_{\mathrm{cyc}} & * \\ 0 & 1 \end{smallmatrix}\right)$. Our first theorem is then the following.

Theorem 1.0.1. *Let $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}})$ be continuous and cyclotomic-free. Suppose there exists a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ with $\overline{\rho} \cong \rho \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$ and with Hodge–Tate weights $\in [0, p]$.*

If $K = \mathbb{Q}_p$ and $n \leq 5$ then there exists a potentially diagonalisable crystalline representation ρ' with $\overline{\rho} \cong \rho' \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$ and with Hodge–Tate weights equal to those of ρ.

In particular the question has the answer yes for $\overline{\rho}$ as in Theorem 1.0.1. As we shall explain below, the assumptions $K = \mathbb{Q}_p$ and $n \leq 5$ ensure that certain irreducible semilinear objects have a particularly simple form and admit crystalline lifts. Beyond these low dimensional cases the situation is more complicated; this is what prevents us from answering the question in greater generality. However, when the Jordan–Holder factors of $\overline{r}$ are one-dimensional this issue does not arise and we are also able to prove:

Theorem 1.0.2. *Suppose $K/\mathbb{Q}_p$ is unramified. Let $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}})$ be continuous and cyclotomic-free. Suppose there exists a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ with $\overline{\rho} \cong \rho \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$ and with τ-Hodge–Tate weights in $[0, p]$ for each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$.*

If every Jordan–Holder factor of $\overline{\rho}$ is one-dimensional then there exists a potentially diagonalisable crystalline representation ρ' with $\overline{\rho} \cong \rho' \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$ and with τ-Hodge–Tate weights equal to those of ρ for every $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)$.

[1]By a crystalline lift we mean a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ such that $\rho \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong \overline{\rho}$

This gives a more general version of the main results of [Gao17, Gao18].

We now explain how potentially diagonalisable lifts may be produced. Every mod p representation is a successive extension of irreducible representations, each of which is induced over an unramified extension of K from a character. Thus the standard method for producing potentially diagonalisable lifts is to consider lifts obtained in the same way, by taking successive crystalline extensions of irreducible representations obtained by inducing crystalline characters. Such lifts are called obvious lifts (following terminology introduced in [GHS18, Subsection 7.1]) and it is straightforward to see that obvious lifts are potentially diagonalisable (see the beginning of Section 7). In both Theorem 1.0.1 and 1.0.2 the ρ' constructed will be obvious lifts.

Remark 1.0.3. Since writing this paper the author has used the technique developed here to substantially generalise Theorems 1.0.1 and 1.0.2. In [Bar20b] it is shown that, under a cyclotomic-freeness assumption on the reduction modulo p (a slight variant of that considered in this paper), *every* crystalline representation with Hodge–Tate weights contained in $[0, p]$ is potentially diagonalisable. This is done by showing that every such crystalline representation is contained in the same component of a crystalline deformation ring as one which is an obvious lift. A key step is to prove that certain moduli spaces of Breuil–Kisin modules over these deformation rings are smooth, and this relies on the extension group computations made in Section 4.

There are two issues with this method of producing obvious lifts. Firstly, it is probably not true that every mod p representation has an obvious lift. Recent work of Matthew Emerton and Toby Gee [EG19] shows, using geometric methods, that potentially diagonalisable lifts can always be obtained, but their methods involve an inductive process where obvious lifts are allowed to vary inside irreducible components of deformation rings. We avoid this problem by restricting attention to cyclotomic-free representations. Secondly, even if one can produce obvious lifts, the Hodge–Tate weights of such lifts seem to be very restrictive. They depend upon $\overline{\rho}$ in a non-obvious way, and it is unclear that if $\overline{\rho}$ has a crystalline lift of some weight then it will be possible to construct an obvious lift of the same weight.

In this paper we resolve the second issue, not by producing obvious crystalline lifts of $\overline{\rho}$, but instead by producing lifts of a semilinear object related to $\overline{\rho}$. More precisely, if $\overline{\rho}$ admits a crystalline lift then Kisin's work in integral p-adic Hodge theory [Kis06] associates to this lift a Breuil–Kisin module. We produce obvious crystalline lifts of the mod p reduction of this Breuil–Kisin module (i.e. we produce obvious lifts whose associated Breuil–Kisin module is congruent modulo p to the one arising from the previous crystalline lift). Our assumption that $\overline{\rho}$ is cyclotomic-free means that this obvious lift will also be an obvious lift of $\overline{\rho}$. The key ingredient which makes this possible is a theorem of Gee–Liu–Savitt which says that the Breuil–Kisin module of a crystalline representation with Hodge–Tate weights contained in $[0, p]$ is of a particularly nice form; in particular its reduction modulo p sees the Hodge–Tate weights of the crystalline representation it was obtained from.

We conclude our introduction by explaining the content of this paper. Section 2 discusses the cyclotomic-freeness condition and its consequences. The main result is that, if K_∞ is the extension of K obtained by adjoining a compatible system of p-th power roots of a uniformiser of K, then any G_{K_∞}-equivariant morphism between cyclotomic-free G_K-representations is G_K-equivariant.

In Section 3 we recall the notion of a Breuil–Kisin module. We state Kisin's construction which associates a Breuil–Kisin module to a crystalline representation and the result of Gee–Liu–Savitt which controls the shape of such Breuil–Kisin modules. We also recall from [Bar20a] the notion of strong divisibility for p-torsion Breuil–Kisin modules, and some of the properties such modules satisfy.

In Section 4 we compute the space of extensions of strongly divisible Breuil–Kisin modules. The dimension is closely related to the dimension of crystalline extensions in characteristic zero, and in Section 5 we use this to show that extensions between strongly divisible Breuil–Kisin modules admit lifts by crystalline extensions.

Section 5 reduces the problem of lifting strongly divisible Breuil–Kisin modules to that of lifting irreducible such modules. In general the structure of such modules is complicated, and we do not know how to produce such lifts. However we show in Section 6, by explicit computation, that in low dimensional situations (when $K = \mathbb{Q}_p$ and $n \leq 4$) their structure can be controlled so that crystalline lifts can be produced. In the final section we put these results together to prove the theorems stated above.

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2. Cyclotomic-free representations

For this section let $K/\mathbb{Q}_p$ be any finite extension. Let $K_\infty = K(\pi^{1/p^\infty})$ where π^{1/p^∞} is a fixed choice of compatible system of p-th power roots of a uniformiser $\pi \in K$. Our aim is to understand restriction from G_K to G_{K_∞} for a class of representations we call cyclotomic-free.

If G is a topological group denote by $\mathrm{Rep}(G)$ the category of continuous representations of G on finite dimensional $\overline{\mathbb{F}}_p$-vector spaces. In this section all unadorned tensor products are over $\overline{\mathbb{F}}_p$.

2.1. Cyclotomic-free representations.

For any field $\mathbb{F}$ of characteristic p let $\mathbb{F}(1)$ denote the G_K-representation whose underlying vector space is $\mathbb{F}$, with G_K-action given by the mod p cyclotomic character χ_{cyc}.

Definition 2.1.1. $V \in \mathrm{Rep}(G_K)$ is cyclotomic-free if V admits a composition series $0 = V_n \subset \ldots \subset V_0 = V$ such that $V_i/V_{i+1} \otimes \overline{\mathbb{F}}_p(1)$ is not a Jordan–Holder factor of V_{i+1} for any i. Pictorially

$$V \sim \begin{pmatrix} \ddots & * & * \\ 0 & V_1/V_2 & * \\ 0 & 0 & V_0/V_1 \end{pmatrix}$$

and we ask that for each i no block above V_i/V_{i+1} is isomorphic to $V_i/V_{i+1} \otimes \overline{\mathbb{F}}_p(1)$. In particular cyclotomic-freeness is ruling out representations of the form $\left(\begin{smallmatrix} \chi_{\mathrm{cyc}} & * \\ 0 & 1 \end{smallmatrix}\right)$.

Note in the definition of cyclotomic-freeness we require that one composition series of V satisfies the conditions describes in (2.1.1), not that every one does.

Lemma 2.1.2. *If $V \in \mathrm{Rep}(G_K)$ is cyclotomic-free then any subquotient of V is cyclotomic-free also.*

Proof. Suppose $f : V \to W$ is surjective and let $(V_i)_i$ be a composition series as in Definition 2.1.1. Set $W_i = f(W_i)$. Then f induces surjective maps $V_i/V_{i+1} \to W_i/W_{i+1}$ and so $W_i/W_{i+1} \cong V_i/V_{i+1}$ or $W_i/W_{i+1} = 0$. Thus, after re-indexing, the $(W_i)_i$ form a composition series as in Definition 2.1.1, and W is cyclotomic-free. If instead $W \subset V$ is a G_K-stable subspace set $W_i = W \cap V_i$. Then $W_i/W_{i+1} \hookrightarrow V_i/V_{i+1}$ and so either W_i/W_{i+1} is isomorphic to V_i/V_{i+1}, or is zero. Re-indexing we obtain a composition series $(W_i)_i$ as in Definition 2.1.1. □

The class of cyclotomic-free representations is not closed under extensions. For example, if the mod p cyclotomic character is trivial it is not even closed under direct sums.

The main result of this section is:

Theorem 2.1.3. *Let V and W be cyclotomic-free G_K-representations and $f : V \to W$ a morphism of G_{K_∞}-representations. Then f is G_K-equivariant.*

The proof is given over the next three subsections. Results in a similar direction appear in [LLHLM18, Lemmas 3.10 and 3.11], and can be compared with Lemma 2.3.5.

2.2. **Restriction for irreducibles.** Let K^{t} be the maximal tamely ramified extension of K. Since K_∞ is totally wildly ramified $K_\infty \cap K^{\mathrm{t}} = K$. Galois theory then tells us that, if $K^{\mathrm{t}}_\infty = K_\infty K^{\mathrm{t}}$, then the restriction map

$$\mathrm{Gal}(K^{\mathrm{t}}_\infty/K_\infty) \to \mathrm{Gal}(K^{\mathrm{t}}/K)$$

is an isomorphism. By [Ser72, Proposition 4] the action of G_K on any semi-simple object of $\mathrm{Rep}(G_K)$ factors through $\mathrm{Gal}(K^{\mathrm{t}}/K)$. Likewise the G_{K_∞}-action on any semi-simple object of $\mathrm{Rep}(G_{K_\infty})$ factors through $\mathrm{Gal}(K^{\mathrm{t}}_\infty/K_\infty)$. Thus we deduce:

Lemma 2.2.1. *Restriction induces an equivalence between the category of semi-simple $V \in \mathrm{Rep}(G_K)$ and the category of semi-simple $V \in \mathrm{Rep}(G_{K_\infty})$.*

This implies Theorem 2.1.3 holds when V and W are irreducible. It also implies that any G_K-composition series of $V \in \mathrm{Rep}(G_K)$ is also a G_{K_∞}-composition series. In particular, if V is cyclotomic-free in the sense of Definition 2.1.1 then $V|_{G_{K_\infty}}$ is cyclotomic-free in the following sense.

Definition 2.2.2 (Cyclotomic-freeness for G_{K_∞}-representations)**.** $V \in \mathrm{Rep}(G_{K_\infty})$ is cyclotomic-free if V admits a composition series $0 = V_n \subset \ldots \subset V_0 = V$ such that $V_i/V_{i+1} \otimes \overline{\mathbb{F}}_p(1)$ is not a Jordan–Holder factor of V_{i+1}.

2.3. **Restriction on Galois cohomology.** The following lemma is well-known (cf. for example. [Bar20a, Lemma 2.1.2]).

Lemma 2.3.1. *Every irreducible $V \in \mathrm{Rep}(G_K)$ is isomorphic to $\mathrm{Ind}_L^K \chi$ where L/K is an unramified extension and $\chi : G_L \to \overline{\mathbb{F}}_p^\times$ is a continuous character.*

We have written Ind_L^K in place of $\mathrm{Ind}_{G_L}^{G_K}$ and we continue with this notation throughout.

If G is a topological group and $V, W \in \mathrm{Rep}(G)$ write $\mathrm{Hom}(V, W) \in \mathrm{Rep}(G)$ for the representation with underlying vector space $\mathrm{Hom}_{\overline{\mathbb{F}}_p}(V, W)$ and G-action given by $\sigma \cdot f = \sigma \circ f \circ \sigma^{-1}$. Note that $\mathrm{Hom}(V,W) = V^\vee \otimes W$ where $V^\vee = \mathrm{Hom}(V, \overline{\mathbb{F}}_p)$. We shall use that if $W \in \mathrm{Rep}(G_L)$ and $V \in \mathrm{Rep}(G_K)$ then there are isomorphisms $\mathrm{Ind}_L^K(V|_L \otimes W) \cong V \otimes \mathrm{Ind}_L^K W$.

If G is profinite we let $H^*(G,V)$ denote the continuous cohomology groups valued in V.

Lemma 2.3.2. *If $V, W \in \mathrm{Rep}(G_K)$ are irreducible then the restriction map*

$$H^1(G_K, \mathrm{Hom}(V,W)) \to H^1(G_{K_\infty}, \mathrm{Hom}(V,W)) \tag{2.3.3}$$

is injective unless $W \cong V \otimes \overline{\mathbb{F}}_p(1)$.

Proof. First, suppose that $\mathrm{Hom}(V,W)$ in (2.3.3) is replaced by $\mathrm{Ind}_L^K Z$ with Z one dimensional. Recall that $\mathrm{Ind}_L^K Z$ is the vector space of continuous functions $f : G_K \to Z$ such that $f(hg) = h \cdot f(g)$ for all $h \in G_L, g \in G_K$. Since $G_{L_\infty} = G_{K_\infty} \cap G_L$, restriction of functions describes a map $\mathrm{Ind}_L^K Z \to \mathrm{Ind}_{L_\infty}^{K_\infty} Z$ fitting into the G_{L_∞}-equivariant diagram

$$\begin{array}{ccc} \mathrm{Ind}_L^K Z & \to & \mathrm{Ind}_{L_\infty}^{K_\infty} Z \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z|_{L_\infty} \end{array}$$

in which the vertical arrows are evaluation at 1. After taking cohomology, the vertical arrows induce isomorphisms and the horizontal arrows induce restriction, cf. [Ser02, Section 2.5]. By [GLS15, Lemma 5.4.2] $H^1(G_L, Z) \to H^1(G_{L_\infty}, Z)$ is injective unless $Z \cong \overline{\mathbb{F}}_p(1)$, and so the same is also true for the restriction map $H^1(G_K, \mathrm{Ind}_L^K Z) \to H^1(G_{K_\infty}, \mathrm{Ind}_L^K Z)$.

Return to the statement of the lemma. As W is irreducible $W \cong \mathrm{Ind}_L^K Z$ for a one-dimensional Z. The projection formula gives $\mathrm{Hom}(V,W) \cong \mathrm{Ind}_L^K(Z \otimes V^\vee|_L)$. Since V is irreducible $V^\vee$ is irreducible also, and so $V^\vee \cong \mathrm{Ind}_F^K Y$ for a one-dimensional Y. Mackey's theorem implies $V^\vee|_L \cong \bigoplus_\gamma \mathrm{Ind}_{FL}^L Y^{(\gamma)}$ with γ running over a subset of G_L and $Y^{(\gamma)} = Y$ as vector spaces with G_{FL}-action given by $\sigma(y) = (\gamma\sigma\gamma^{-1})(y)$. Therefore

$$\mathrm{Hom}(V,W) \cong \bigoplus_\gamma \mathrm{Ind}_L^K(Z \otimes \mathrm{Ind}_{FL}^L Y^{(\gamma)}) = \bigoplus_\gamma \mathrm{Ind}_{FL}^K(Z \otimes Y^{(\gamma)}) \tag{2.3.4}$$

Thus (2.3.3) is a direct sum of restriction maps $H^1(G_K, \mathrm{Ind}_{FL}^K(Z \otimes Y^{(\gamma)})) \to H^1(G_{K_\infty}, \mathrm{Ind}_{FL}^K(Z \otimes Y^{(\gamma)}))$. By the previous paragraph, these maps are injective unless $Z \otimes Y^{(\gamma)} \cong \overline{\mathbb{F}}_p(1)$. However if this is the case then (2.3.4) implies $\mathrm{Hom}(V \otimes \overline{\mathbb{F}}_p(1), W)$ has G_K-fixed points, i.e. $W \cong V \otimes \overline{\mathbb{F}}_p(1)$. □

Lemma 2.3.5. *Suppose W is cyclotomic-free, V is irreducible, and $V \otimes \overline{\mathbb{F}}_p(1)$ is not a Jordan–Holder factor of W. Then*

$$H^i(G_K, \mathrm{Hom}(V,W)) \to H^i(G_{K_\infty}, \mathrm{Hom}(V,W))$$

is an isomorphism if $i = 0$ and is injective if $i = 1$.

Proof. Induct on the length of W. If W is irreducible then the claim follows from Lemma 2.2.1 and Lemma 2.3.2. If W is not irreducible we can fit W into an exact sequence $0 \to W_1 \to W \to W_2 \to 0$ where W_1 is cyclotomic-free, W_2 is irreducible and non-zero, and where $W_2 \otimes \overline{\mathbb{F}}_p(1)$ not a Jordan–Holder factor of W_1. Passing to cohomology gives the diagram

$$\begin{array}{ccccccc} H^0(G_K,\mathrm{Hom}(V,W_2)) & \longrightarrow & H^1(G_K,\mathrm{Hom}(V,W_1)) & \longrightarrow & H^1(G_K,\mathrm{Hom}(V,W)) & \longrightarrow & H^1(G_K,\mathrm{Hom}(V,W_2)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(G_{K_\infty},\mathrm{Hom}(V,W_2)) & \to & H^1(G_{K_\infty},\mathrm{Hom}(V,W_1)) & \to & H^1(G_{K_\infty},\mathrm{Hom}(V,W)) & \to & H^1(G_{K_\infty},\mathrm{Hom}(V,W_2)) \end{array}$$

which commutes and has exact rows. By the inductive hypothesis the first vertical arrow is an isomorphism, and the second and fourth are injective. A diagram chase shows that the third is injective also. Similarly we obtain the diagram

$$\begin{array}{ccccccc} H^0(G_K,\mathrm{Hom}(V,W_1)) & \to & H^0(G_K,\mathrm{Hom}(V,W)) & \longrightarrow & H^0(G_K,\mathrm{Hom}(V,W_2)) & \longrightarrow & H^1(G_K,\mathrm{Hom}(V,W_1)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(G_{K_\infty},\mathrm{Hom}(V,W_1) & \to & H^0(G_{K_\infty},\mathrm{Hom}(V,W)) & \to & H^0(G_{K_\infty},\mathrm{Hom}(V,W_2)) & \to & H^1(G_{K_\infty},\mathrm{Hom}(V,W_1)) \end{array}$$

which commutes and has exact rows. Again the inductive hypothesis implies the first and third vertical arrows are isomorphisms, and the fourth is injective. A diagram chase shows the second is an isomorphism, which finishes the proof. □

Corollary 2.3.6. *Let V and W be as in Lemma 2.3.5 and let $0 \to W \to Z \to V \to 0$ be an exact sequence in $\mathrm{Rep}(G_K)$. Then any G_{K_∞}-equivariant splitting $s : V \to Z$ of this sequence is G_K-equivariant.*

Proof. If such a splitting s exists then the $i = 1$ part of Lemma 2.3.5 implies $0 \to W \to Z \to V \to 0$ is split in $\mathrm{Rep}(G_K)$, say by a G_K-equivariant map $s' : V \to Z$. Then $s' - s \in H^0(G_{K_\infty}, \mathrm{Hom}(V, W))$ and so, by the $i = 0$ part of Lemma 2.3.5,

$$s' - s \in H^0(G_K, \mathrm{Hom}(V, W))$$

Thus s is G_K-equivariant. □

2.4. **Proving the theorem.**

Lemma 2.4.1. *Let W be a cyclotomic-free G_K-representation and V an irreducible G_K-representation. Then any G_{K_∞}-equivariant map $f : V \to W$ is G_K-equivariant.*

Proof. Let $(W_j)_j$ be a composition series as in Definition 2.1.1. There is a largest j such that f factors through $W_j \hookrightarrow W$; so long as $f \neq 0$ (in which case the lemma is trivial) the composite $g : V \xrightarrow{f} W_j \to W_j/W_{j+1}$ is then non-zero. Lemma 2.2.1 implies g is a G_K-equivariant isomorphism. Now $f \circ g^{-1}$ is a G_{K_∞}-splitting of $0 \to W_{j+1} \to W_j \to W_j/W_{j+1} \to 0$ and so $f \circ g^{-1}$ is G_K-equivariant by Corollary 2.3.6. Thus f is G_K-equivariant. □

Proof of Theorem 2.1.3. Induct on the length of V. When V is irreducible the theorem is given by Lemma 2.2.1. If V is not irreducible choose a composition series $(V_i)_i$ as in Definition 2.1.1. As V_1 is cyclotomic-free our inductive hypothesis implies $f : V_1 \to W$ is G_K-equivariant. In particular $\ker(f|_{V_1}) \subset V$ is G_K-stable. If $\ker(f|_{V_1}) \neq 0$ then, as f factors through $V \to V/\ker(f|_{V_1})$ and $V/\ker(f|_{V_1})$ is cyclotomic-free by Lemma 2.1.2, the result follows from our inductive hypothesis. Thus we can assume $f|_{V_1}$ is injective. Set $W_1 = f(V_1)$; again since $f|_{V_1}$ is G_K-equivariant $W_1 \subset W$ is G_K-stable.

Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & V_1 & \to & V & \to & V/V_1 & \to & 0 \\ & & \downarrow & & \downarrow f & & \downarrow & & \\ 0 & \to & W_1 & \to & W & \to & W/W_1 & \to & 0 \end{array}$$

with the rows exact in $\mathrm{Rep}(G_K)$. As W/W_1 is cyclotomic-free by Lemma 2.1.2, $V/V_1 \to W/W_1$ is G_K-equivariant. Thus $f(\sigma(v)) - \sigma(f(v)) \in W_1$ for all $v \in V$ and $\sigma \in G_K$, and so $f(V) \subset W$ is G_K-stable. As $f(V)$ is cyclotomic-free we may assume f is surjective. This implies $V/V_1 \to W/W_1$ is surjective, and so is a G_K-isomorphism since V/V_1 is irreducible.

If we identify $V/V_{n-1} = W/W_{n-1}$ and $V_{n-1} = W_{n-1}$ via the outer arrows of the above diagram, the map f shows that the two horizontal rows in the diagram define the same class in $\mathrm{Ext}^1_{\mathrm{Rep}(G_{K_\infty})}(V/V_{n-1}, V_{n-1}) = H^1(G_{K_\infty}, \mathrm{Hom}(V/V_{n-1}, V_{n-1}))$. By Lemma 2.3.5 these two rows define the same class in $\mathrm{Ext}^1_{\mathrm{Rep}(G_K)}(V/V_{n-1}, V_{n-1})$. Hence there exists a G_K-equivariant $h : V \to W$ fitting into the diagram as f does. Thus $h - f$ describes a G_{K_∞}-equivariant map $V/V_1 \to W_1$. By induction $h - f$ is G_K-equivariant and so f is G_K-equivariant also. □

2.5. **Stable lattices.** Fix a finite $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. We conclude the section by using the above to show that certain G_{K_∞}-stable lattices inside E-representations of G_K are G_K-stable.

If G is a topological group acting linearly and continuously on a topological abelian group M then let $Z^1(G, M)$ denote the group of continuous 1-cocycles $G \to M$, and let $B^1(G, M) \subset Z^1(G, M)$ denote the group of 1-coboundaries.

Lemma 2.5.1. *Let V, W be continuous representations of G_K on finite free $\mathcal{O}$-modules. Assume $\overline{W} = W \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$ is cyclotomic-free, that $\overline{V} = V \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$ is irreducible, and that $\overline{V} \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p(1)$ is not a Jordan–Holder factor of $\overline{W}$.*

If $c \in Z^1(G_K, \mathrm{Hom}(V, W)[\frac{1}{p}])$ is such that $c(\sigma) \in \mathrm{Hom}(V, W)$ for all $\sigma \in G_{K_\infty}$ then $c \in Z^1(G_K, \mathrm{Hom}(V, W))$.

Proof. To prove the lemma it suffices to do so with V and W replaced by $V \otimes_{\mathcal{O}} \mathcal{O}^{\mathrm{ur}}$ and $W \otimes_{\mathcal{O}} \mathcal{O}^{\mathrm{ur}}$, where $\mathcal{O}^{\mathrm{ur}}$ denotes the ring of integers inside the maximal unramified extension of E. Thus we may assume that $\mathbb{F} = \overline{\mathbb{F}}_p$.

Put $H = \mathrm{Hom}(V, W)$ (the representation of $\mathcal{O}$-linear homomorphisms) and $\overline{H} = \mathrm{Hom}(\overline{V}, \overline{W}) = H \otimes_{\mathcal{O}} \mathbb{F}$. Lemma 2.3.5 implies $H^0(G_K, \overline{H}) = H^0(G_{K_\infty}, \overline{H})$ and $H^1(G_K, \overline{H}) \to H^1(G_{K_\infty}, \overline{H})$ is injective.

Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}$. We claim there exists $\mu \in \mathfrak{m}$ such that $\mu c \in Z^1(G_K, H)$. Recall from [Tat76, Proposition 2.3] that the map $H^1(G_K, H) \to H^1(G_K, H[\frac{1}{p}])$, coming from $H \hookrightarrow H[\frac{1}{p}]$, induces an isomorphism $H^1(G_K, H)[\frac{1}{p}] \cong H^1(G_K, H[\frac{1}{p}])$. It follows that there exists a $\mu' \in \mathfrak{m}$ so that the class of $\mu' c$ inside $H^1(G_K, H[\frac{1}{p}])$ is represented by $\widetilde{c} \in Z^1(G_K, H)$. Thus $\mu' c - \widetilde{c} \in B^1(G_K, H[\frac{1}{p}])$. Clearly our claim holds for cocycles in $B^1(G_K, H[\frac{1}{p}])$, so the claim holds in general.

To prove the lemma assume $c \notin H^1(G_K, H)$. By the previous paragraph there is a $\mu \in \mathfrak{m}$ so that $\mu c \in Z^1(G_K, H)$. Since $E/\mathbb{Q}_p$ is finitely ramified we can assume that μ is such that $v_p(\mu)$ is minimal. Since $c(\sigma) \in H$ for $\sigma \in G_{K_\infty}$ the image $\overline{\mu c}$ of μc in $Z^1(G_K, \overline{H})$ must vanish on G_{K_∞}. Injectivity of $H^1(G_K, \overline{H}) \to H^1(G_{K_\infty}, \overline{H})$ therefore implies $\overline{\mu c} \in B^1(G_K, \overline{H})$, so $\overline{\mu c} = (\sigma - 1)\overline{h}$ for some $\overline{h} \in \overline{H}$. As $\overline{\mu c}|_{G_{K_\infty}} = 0$, $\overline{h} \in H^0(G_{K_\infty}, \overline{H})$ and so $\overline{h} \in H^0(G_K, \overline{H})$. Thus $\overline{\mu c} = 0$ on G_K which contradicts the minimality of $v_p(\mu)$. □

Corollary 2.5.2. *Let V and W be as in Lemma 2.5.1, and*

$$0 \to W \to Z \to V \to 0 \tag{2.5.3}$$

be an exact sequence of G_{K_∞}-representations. If the G_{K_∞}-action on $Z[\frac{1}{p}]$ extends to a continuous G_K-action so that (2.5.3) *becomes G_K-equivariant after inverting p, then Z is G_K-stable inside $Z[\frac{1}{p}]$.*

Proof. In the usual way, choosing an $\mathcal{O}$-module splitting of (2.5.3) produces a continuous 1-cocycle $c : G_{K_\infty} \to \mathrm{Hom}(V,W)$ in $Z^1(G_{K_\infty}, \mathrm{Hom}(V,W))$. The extension of (2.5.3) to G_K after inverting p produces an extension of c to an element of $Z^1(G_K, \mathrm{Hom}(V,W)[\frac{1}{p}])$. To show Z is G_K-stable it suffices to show this extension of c lies in $Z^1(G_K, \mathrm{Hom}(V,W))$, and this follows from Lemma 2.5.1. □

3. Breuil–Kisin modules

In this section we recall the theory of Breuil–Kisin modules and their relationship to crystalline Galois representations.

Throughout let k denote a finite field of characteristic p. Write $K_0 = W(k)[\frac{1}{p}]$ and let K be a totally ramified extension of K_0 of degree e. Let C denote the completion of a fixed algebraic closure of K, with ring of integers $\mathcal{O}_C$.

3.1. **Breuil–Kisin modules.** Let $\mathfrak{S} = W(k)[[u]]$. We equip this ring with the $\mathbb{Z}_p$-linear endomorphism φ which acts on $W(k)$ by the Witt vector Frobenius and which sends $u \mapsto u^p$. Fix a uniformiser $\pi \in K$ and let $E(u) \in \mathfrak{S}$ denote the minimal polynomial of π over K_0.

Definition 3.1.1. A Breuil–Kisin module M is a finitely generated $\mathfrak{S}$-module equipped with an isomorphism

$$\varphi_M : M \otimes_{\varphi,\mathfrak{S}} \mathfrak{S}[\tfrac{1}{E}] \cong M[\tfrac{1}{E}]$$

When there is no risk of confusion we write φ in place of φ_M. We can identify φ_M with the semilinear map $M \to M[\frac{1}{E}]$ given by $m \mapsto \varphi_M(m \otimes 1)$. Denote the category of Breuil–Kisin modules by $\mathrm{Mod}_K^{\mathrm{BK}}$.

We now recall the connection between Breuil–Kisin modules and crystalline representations. As in the previous section choose a compatible system π^{1/p^n} of p^n-th roots of π in C, and let $K_\infty = K(\pi^{1/p^\infty})$. Let $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$ where $\mathcal{O}_{C^\flat} = \varprojlim \mathcal{O}_C/p$ with transition maps given by $x \mapsto x^p$. Our choice of π^{1/p^n} defines an element $\pi^\flat \in \mathcal{O}_{C^\flat}$ and we embed $\mathfrak{S} \to A_{\mathrm{inf}}$ via $\sum a_i u^i \mapsto \sum a_i [\pi^\flat]^i$. This inclusion is compatible with φ on $\mathfrak{S}$ and the Witt vector Frobenius on A_{inf}. Note that the G_K-action on $\mathcal{O}_C/p$ induces a G_K-action on A_{inf}, and via this action the image of $\mathfrak{S} \to A_{\mathrm{inf}}$ is G_{K_∞}-stable

Lemma 3.1.2. *There is an exact functor $M \mapsto T(M) = (M \otimes_{\mathfrak{S}} W(C^\flat))^{\varphi=1}$ from $\mathrm{Mod}_K^{\mathrm{BK}}$ to the category of finitely generated $\mathbb{Z}_p$-modules equipped with a continuous $\mathbb{Z}_p$-linear action of G_{K_∞}. Further, there are φ, G_{K_∞}-equivariant identifications*

$$M \otimes_{\mathfrak{S}} W(C^\flat) \cong T(M) \otimes_{\mathbb{Z}_p} W(C^\flat)$$

which are functorial in M.

Proof. This is proven in [Fon90]. We refer to [Bar20a, Proposition 4.1.5] and [Bar20a, Construction 4.2.3] for more details. □

In the following theorem a crystalline $\mathbb{Z}_p$-lattice is a G_K-stable $\mathbb{Z}_p$-lattice inside a crystalline (in the sense of [Fon94]) $\mathbb{Q}_p$-representation of G_K.

Theorem 3.1.3 (Kisin)**.** *There is a fully faithful functor $T \mapsto M(T)$ from the category of crystalline $\mathbb{Z}_p$-lattices into the category of Breuil–Kisin modules finite free over $\mathfrak{S}$. The module $M(T)$ is characterised up to isomorphism by $T(M(T)) \cong T$ as G_{K_∞}-representations.*

Proof. The functor $T \mapsto M(T)$ was first constructed by Kisin in [Kis06]. The formulation we give here is taken from [BMS18, Theorem 4.4]. ☐

3.2. **Coefficients.** In practice it is sometimes necessary to consider crystalline representations valued in extensions of $\mathbb{Z}_p$. Kisin's construction can be suitably adapted to allow this, provided the coefficient ring is finite over $\mathbb{Z}_p$.

Notation 3.2.1. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. Assume that $K_0 \subset E$.

By a crystalline $\mathcal{O}$-lattice we mean an $\mathcal{O}$-lattice inside a G_K-representation on an E-vector space, which is crystalline when viewed as a representation on a $\mathbb{Q}_p$-vector space. By functoriality, if T is a crystalline $\mathcal{O}$-lattice then $M(T)$ is a Breuil–Kisin module with $\mathcal{O}$-action as defined below.

Definition 3.2.2. A Breuil–Kisin module with $\mathcal{O}$-action is a pair (M, ι) where $M \in \mathrm{Mod}_K^{\mathrm{BK}}$ and ι is a $\mathbb{Z}_p$-algebra homomorphism $\iota : \mathcal{O} \to \mathrm{End}_{\mathrm{BK}}(M)$. Equivalently a Breuil–Kisin module with $\mathcal{O}$-action is a finitely generated $\mathfrak{S}_{\mathcal{O}} \coloneqq \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$-module equipped with an isomorphism

$$M \otimes_{\varphi \otimes 1, \mathfrak{S}_{\mathcal{O}}} \mathfrak{S}_{\mathcal{O}}[\tfrac{1}{E}] \cong M[\tfrac{1}{E}]$$

Let $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ denote the category of Breuil–Kisin modules with $\mathcal{O}$-action.

Construction 3.2.3. Our assumption that $K_0 \subset E$ has the following consequence. Since $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ is a finite $\mathfrak{S}$-module it is u-adically complete, and so the inclusion $\mathcal{O}[u] \to \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ given by $\sum a_i u^i \mapsto \sum u^i \otimes a_i$ extends to $\mathcal{O}[[u]] \to \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$. In this way we view $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ as an $\mathcal{O}[[u]]$-module. The map

$$(\sum a_i u^i) \otimes b \mapsto (\sum \tau(a_i) b u^i)_\tau$$

then describes an isomorphism of $\mathcal{O}[[u]]$-algebras $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O} \to \prod_\tau \mathcal{O}[[u]]$, the product running over $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mathbb{F})$ (we abusively write τ also for its extension to an embedding $\tau : W(k) \to \mathcal{O}$). Let $\widetilde{e}_\tau \in \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ be the idempotent corresponding to τ. As $\widetilde{e}_\tau$ is determined by the property $(a \otimes 1)\widetilde{e}_\tau = (1 \otimes \tau(a))\widetilde{e}_\tau$ for $a \in W(k)$, the map $\varphi \otimes 1$ sends

$$\widetilde{e}_{\tau \circ \varphi} \mapsto \widetilde{e}_\tau$$

If $M \in \mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ we set $M_\tau = \widetilde{e}_\tau M$ which we view as an $\mathcal{O}[[u]]$-algebra. By the above φ_M restricts to a map

$$M_{\tau \circ \varphi} \otimes_{\varphi, \mathcal{O}[[u]]} \mathcal{O}[[u]] \to M_\tau[\tfrac{1}{\tau(E)}] \tag{3.2.4}$$

which becomes an isomorphism after inverting $\tau(E)$. Here φ on $\mathcal{O}[[u]]$ is that induced by $\varphi \otimes 1$ on $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$, i.e. is given by $\sum a_i u^i \mapsto \sum a_i u^{ip}$.

Corollary 3.2.5. *If $M \in \mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ is free as an $\mathfrak{S}$-module then M is free over $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$.*

Proof. If M is free over $\mathfrak{S}$ then the $\mathcal{O}$-module M/uM is free over $W(k)$, is torsion-free, and is therefore free over $\mathcal{O}$. By Nakayama's lemma, any lift of an $\mathcal{O}$-basis of M/uM generates M; there is therefore a surjection $F \to M$ where F is $\mathcal{O}[[u]]$-free of $\mathfrak{S}$-rank equal to that of M. As surjective maps between free-modules of the same rank are isomorphisms, M is free over $\mathcal{O}[[u]]$. By (3.2.4) each M_τ has the same $\mathcal{O}[[u]]$-rank which proves M is free over $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O} = \prod_\tau \mathcal{O}[[u]]$. □

3.3. **Strong divisibility.** We now explain what it means for a p-torsion Breuil–Kisin module to be strongly divisible. After a result of Gee–Liu–Savitt (Theorem 3.3.9) strong divisibility is closely related to the reduction modulo p of crystalline representations.

Note that since $E \equiv u^e$ modulo p, a p-torsion Breuil–Kisin module is a finitely generated $k[[u]]$-module M equipped with an isomorphism $M \otimes_{\varphi, k[[u]]} k((u)) \to M[\frac{1}{u}]$.

From now on all Breuil–Kisin modules will be considered with $\mathcal{O}$-action.

Definition 3.3.1. Let $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ be the full sub-category of $M \in \mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ which are free modules over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$.

If $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ set M^φ equal to the image of $M \otimes_{\varphi, k[[u]]} k[[u]] \xrightarrow{\varphi} M[\frac{1}{u}]$. Equivalently M^φ is the $k[[u]]$-sub-module of $M[\frac{1}{u}]$ generated by the $k[[u^p]]$-module $\varphi(M)$

Construction 3.3.2. Equip M^φ with the filtration $F^i M^\varphi = M^\varphi \cap u^i M$. Similarly define a filtration on M by $F^i M = \{m \in M \mid \varphi(m) \in u^i M\}$. Note that the semi-linear map $\varphi : M \to M^\varphi$ is compatible with these filtrations.

Set $M_k^\varphi = M^\varphi / u M^\varphi$ and $M_k = M/uM$. These are both $k \otimes_{\mathbb{F}_p} \mathbb{F}$-modules and we equip both with the quotient filtration coming from M^φ and M respectively. In other words $F^i M_k^\varphi$ equals the image of $F^i M^\varphi$ under $M^\varphi \to M_k^\varphi$, and likewise $F^i M_k$ is the image of $F^i M$ under $M \to M_k$.

The filtration on M_k is by $k \otimes_{\mathbb{F}_p} \mathbb{F}$-sub-modules. Thus, as in Construction 3.2.3, there are decompositions $M_k = \prod_{\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mathbb{F})} M_{k,\tau}$ of filtered modules. Likewise $M_k^\varphi = \prod_\tau M_{k,\tau}^\varphi$. Each of $M_{k,\tau}$ and $M_{k,\tau}^\varphi$ is a filtered $\mathbb{F}$-vector space.

Remark 3.3.3. Note that the map $\varphi : M \to M^\varphi$ induces a semi-linear map $M_k \to M_k^\varphi$, which is compatible with filtrations. This latter map is a bijection but not necessarily an isomorphism of filtered modules.

Lemma 3.3.4. *The map $M_k \to M_k^\varphi$ is a semi-linear isomorphism of filtered modules if and only if for each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mathbb{F})$ there exist an $\mathbb{F}[[u]]$-basis (f_i) of M_τ and integers (r_i) such that $(u^{r_i} f_i)$ is an $\mathbb{F}[[u^p]]$-basis of $\varphi(M)_\tau = \varphi(M_{\tau \circ \varphi})$.*

Proof. This is [Bar20a, Lemma 5.3.4]. □

Definition 3.3.5. For $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mathbb{F})$ define $\mathrm{Weight}_\tau(M)$ to be the multiset of integers which contains i with multiplicity

$$\dim_{\mathbb{F}} \mathrm{gr}^i(M_k^\varphi)$$

Definition 3.3.6 (Strong divisibility)**.** $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ is strongly divisible if $\mathrm{Weight}_\tau(M) \subset [0, p]$ for each τ and if $M_k \to M_k^\varphi$ is a semi-linear isomorphism of filtered modules. Let $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ denote full subcategory of strongly divisible Breuil–Kisin modules.

Remark 3.3.7. Recall that any matrix $X \in \mathrm{GL}_n(\mathbb{F}((u)))$ can be written uniquely as $A\,\mathrm{diag}(u^{r_i})B$ for some $r_i \in \mathbb{Z}$ and $A, B \in \mathrm{GL}_n(\mathbb{F}[[u]])$. If $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and if we choose $\mathbb{F}[[u]]$-bases of $M_{\tau\circ\varphi}$ and M_τ then the with respect to these bases $\varphi : M_{\tau\circ\varphi} \to M_\tau$ may be represented by a matrix

$$A\,\mathrm{diag}(u^{r_i})B$$

with $A, B \in \mathrm{GL}_n(\mathbb{F}[[u]])$. Then $\{r_i\} = \mathrm{Weight}_\tau(M)$. Moreover the conditions of Lemma 3.3.4 are equivalent to asking that $B \in \mathrm{GL}_n(\mathbb{F}[[u^p]])$. In particular $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ if and only if $r_i \in [0,p]$ and $B \in \mathrm{GL}_n(\mathbb{F}[[u^p]])$.

Proposition 3.3.8. *Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$.*

(1) If $N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ then M and $P \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. Further, $\mathrm{Weight}_\tau(N) = \mathrm{Weight}_\tau(M) \cup \mathrm{Weight}_\tau(P)$.

(2) If $M, P \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ then $N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ if and only if the map $N \to P$ is a strict[2] map of filtered modules.

Proof. This is [Bar20a, Proposition 5.4.7]. □

Recall that if V is a crystalline representation of G_K on an E-vector space and $D_{\mathrm{crys}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K}$ is the associated filtered φ-module, then $D_{\mathrm{crys}}(V)$ is a free $K_0 \otimes_{\mathbb{Q}_p} E$-module, and so $D_{\mathrm{crys}}(V)_K = \prod_\tau D_{\mathrm{crys}}(V)_{K,\tau}$ as filtered modules. Each $D_{\mathrm{crys}}(V)_{K,\tau}$ is a $K \otimes_{K_0} E$-module; the τ-th Hodge–Tate weights of V are the elements of a multiset $\mathrm{HT}_\tau(V)$ defined by the condition that $i \in \mathrm{HT}_\tau(V)$ appears with multiplicity equal to

$$\dim_E \mathrm{gr}^i(D_{\mathrm{crys}}(V)_{K,\tau})$$

Thus $\mathrm{HT}_\tau(V)$ contains $e \dim_E V$ integers and our normalisations are such that the Hodge–Tate weight of the cyclotomic character is -1 (or rather e copies of -1).

The following theorem relates $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ to reductions of crystalline representations. Here we must assume that $K = K_0$ and that if $p = 2$ then π is chosen so that $K_\infty \cap K(\mu_{p^\infty}) = K$. That such π exist follows from [Wan17, Lemma 2.1].

Theorem 3.3.9 (Gee–Liu–Savitt, Wang)**.** *Assume K and π are as in the previous paragraph. Let T be a crystalline $\mathcal{O}$-lattice and let $V = T \otimes_{\mathcal{O}} E$. If $\mathrm{HT}_\tau(V) \subset [0,p]$ for each τ then $\overline{M} := M(T) \otimes_{\mathcal{O}} \mathbb{F} \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and $\mathrm{Weight}_\tau(\overline{M}) = \mathrm{HT}_\tau(V)$.*

Proof. When $p > 2$ this follows by reducing the description of $M(T)$ given in [GLS14, Theorem 4.22] modulo a uniformizer of $\mathcal{O}$. The case $p = 2$ is proven in [Wan17, Theorem 4.2].[3] □

4. Strongly divisible extensions

We maintain the notation from the previous subsection. Our aim here is to compute dimensions of the space of extensions of strongly divisible Breuil–Kisin modules.

[2]Recall that a map $f : M \to N$ of filtered modules is strict if $f(F^iM) = F^iN \cap f(M)$ for every $i \in \mathbb{Z}$.

[3]When using results from [GLS14, Wan17] it is important to keep track of normalisations. In both references Hodge–Tate weights are normalised to be the opposite of ours. Also Breuil–Kisin modules are attached contravariantly to crystalline representations; from the Breuil–Kisin module $\mathfrak{M}$ associated to T in [GLS14, Wan17] one recovers $M(T)$ as the dual of $\mathfrak{M}$ (for the dual of a Breuil–Kisin module see the construction at the start of Subsection 4.1)

Throughout we shall use the following construction. If $M,N \in \mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ define a Breuil–Kisin module $\mathrm{Hom}(M,N)^{\mathcal{O}}$ with underlying module $\mathrm{Hom}_{\mathfrak{S}_{\mathcal{O}}}(M,N)$ and with Frobenius given by $f \mapsto \varphi_N \circ f \circ \varphi_M^{-1}$ (see [Bar20a, Construction 4.2.5 and 4.3.3]).

4.1. **Cohomology and ext groups.** If $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ let $H^i(M)$ denote the cohomology of the complex

$$M \xrightarrow{\varphi-1} M[\tfrac{1}{u}]$$

The $H^i(M)$ are $\mathbb{F}$-vector spaces.

Construction 4.1.1. If $P,M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ then there is an $\mathbb{F}$-linear isomorphism

$$H^1(\mathrm{Hom}(P,M)^{\mathcal{O}}) \to \mathrm{Ext}^1_{\mathbb{F}}(P,M) \tag{4.1.2}$$

into the first Yoneda extension group in the exact category $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$. This map sends a class represented by $f \in \mathrm{Hom}(P,M)^{\mathcal{O}}[\frac{1}{u}]$ onto a class represented by an extension $0 \to M \to N_f \to P \to 0$ in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ where N_f is the Breuil–Kisin module with underlying $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$-module $M \oplus P$ and with Frobenius given by $(\varphi_M + f \circ \varphi_P, \varphi_P)$. This map is injective and functorial in P and M, in particular it is a map of $\mathbb{F}$-vector spaces. Since every extension in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ of P by M splits as a $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$-module (4.1.2) is surjective, and so an isomorphism.

If $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ let $H^i_{\mathrm{SD}}(M)$ denote the cohomology of the complex

$$F^0 M \xrightarrow{\varphi-1} M$$

Then $H^0_{\mathrm{SD}}(M) = H^0(M)$. The inclusion $M \to M[\frac{1}{u}]$ induces a map $H^1_{\mathrm{SD}}(M) \to H^1(M)$. If $m \in M$ can be written as $\varphi(m') - m'$ with $m' \in M$ then $\varphi(m') \in M$ and so $m' \in F^0 M$; therefore $H^1_{\mathrm{SD}}(M) \to H^1(M)$ is injective. Let

$$\mathrm{Ext}^1_{\mathrm{SD}}(P,M) \subset \mathrm{Ext}^1_{\mathbb{F}}(P,M)$$

denote the image of $H^1_{\mathrm{SD}}(\mathrm{Hom(P,M)}^{\mathcal{O}})$ under (4.1.2).

Lemma 4.1.3. *Let $0 \to M \to N \to P \to 0$ be an extension in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and suppose that $M,P \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. Then $N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ if and only if the class of this extension lies in $\mathrm{Ext}^1_{\mathrm{SD}}(P,M)$.*

Proof. As in Construction 4.1.1, we may assume $N = N_f$ for $f \in \mathrm{Hom}(P,M)^{\mathcal{O}}[\frac{1}{u}]$. Thus, as a module $N = M \bigoplus P$ and $\varphi_N = (\varphi_M + f \circ \varphi_P, \varphi_P)$. Proposition 3.3.8 implies $N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ if and only if $N \to P$ is strict as a map of filtered modules. We have to show that if $f \in \mathrm{Hom}(P,M)^{\mathcal{O}}$ then $N \to P$ is strict, and conversely that $N \to P$ being strict implies the existence of $g \in \mathrm{Hom}(P,M)^{\mathcal{O}}$ satisfying $f + \varphi(g) - g \in \mathrm{Hom}(P,M)^{\mathcal{O}}$.

The map $N \to P$ is strict if and only if for every $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k,\mathbb{F})$ and every $z \in F^i P_{\tau\circ\varphi}$, there exists $(m,z) \in N_{\tau\circ\varphi}$ such that

$$\varphi((m,z)) = (\varphi_M(m) + f(\varphi_P(z)), \varphi_P(z)) \in u^i N_\tau$$

If $f \in \mathrm{Hom}(P,M)^{\mathcal{O}}$ then $f(\varphi_P(z)) \in u^i M_\tau$, since $\varphi_P(z) \in u^i P_\tau$, and so we can take $m = 0$. This shows $f \in \mathrm{Hom}(P,M)^{\mathcal{O}}$ implies $N \to P$ is strict.

For the converse, since $P \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$, Lemma 3.3.4 implies that, for each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k,\mathbb{F})$, there exists a basis (z_i) of P_τ and integers r_i such that $u^{r_i} z_i$ forms a basis of $\varphi_P(P)_\tau$. If $N \to P$ is strict then, as in the previous paragraph, we may choose $m_i \in M_{\tau\circ\varphi}$ such that $\varphi_M(m_i) + f(u^{r_i}z_i) \in u^{r_i}M_\tau$. Since the $u^{r_i}z_i$ form an $\mathbb{F}[[u^p]]$-basis of $\varphi_P(P_{\tau\circ\varphi})$, the $n_i := \varphi_P^{-1}(u^{r_i}z_i)$ form an $\mathbb{F}[[u]]$-basis of $P_{\tau\circ\varphi}$. Define $g \in \mathrm{Hom}(P,M)^{\mathcal{O}}$ by asserting that on $P_{\tau\circ\varphi}$ this map sends $n_i \mapsto m_i$. Then $f + \varphi(g) - g$ sends

$$u^{r_i}z_i \mapsto f(u^{r_i}z_i) + \varphi_M \circ g \circ \varphi_P^{-1}(u^{r_i}z_i) - g(z_i) = f(u^{r_i}z_i) + \varphi_M(m_i) + g(u^{r_i}z_i) \in u^{r_i}M_\tau$$

Thus $f + \varphi(g) - g \in \mathrm{Hom}(P,M)^{\mathcal{O}}$. □

4.2. **Dimension calculations.** We now compute the dimensions of H^1_{SD}. Our proof will use that for $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ there are exact sequences

$$0 \to \mathrm{gr}^{i-p}(M) \xrightarrow{u} \mathrm{gr}^i(M) \to \mathrm{gr}^i(M_k) \to 0$$

(here $\mathrm{gr}^i(N) = F^iN/F^{i+1}N$ for any filtered module N). The exactness of this sequence follows from the observation that $F^iM \cap uM = u(F^{i-p}M)$.

Lemma 4.2.1. *Let M be an object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$. Then both $H^1_{\mathrm{SD}}(M)$ and $H^0(M)$ are finite dimensional over $\mathbb{F}$ and*

$$\chi(M) - \chi(uM) = \sum_{i \notin p\mathbb{Z}_{\leq 0} \cup \mathbb{Z}_{\geq 0}} \dim_{\mathbb{F}} \mathrm{gr}^i(M_k)$$

when $\chi(M) := \dim_{\mathbb{F}} H^1_{\mathrm{SD}}(M) - \dim_{\mathbb{F}} H^0(M)$.

Proof. As $H^0(M) \subset T(M)$ finiteness of $H^0(M)$ is clear. For the rest of the proof consider the inclusion $uM \to M$. It induces the following commutative diagram, whose rows are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^0(uM) & \longrightarrow & F^0M & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & uM & \longrightarrow & M & \longrightarrow & M_k & \longrightarrow & 0
\end{array}$$

The snake lemma yields a long exact sequence

$$0 \to H^0(uM) \to H^0(M) \to \ker\alpha \to H^1_{\mathrm{SD}}(uM) \to H^1_{\mathrm{SD}}(M) \to \operatorname{coker}\alpha \to 0$$

Provided we have finiteness of the $H^1_{\mathrm{SD}}(uM)$ and $H^1_{\mathrm{SD}}(M)$, consideration of the alternating sums of the dimensions in this long exact sequence gives that $\chi(N) - \chi(uN) = \dim_{\mathbb{F}} \operatorname{coker}\alpha - \dim_{\mathbb{F}} \ker\alpha$, which is equal to the $\mathbb{F}$-dimension of M_k minus the $\mathbb{F}$-dimension of Q_1. We claim that non-canonically

$$Q_1 = \bigoplus_{i \in p\mathbb{Z}_{\leq 0} \cup \mathbb{Z}_{\geq 1}} \mathrm{gr}^i(M_k) \tag{4.2.2}$$

as an $\mathbb{F}$-vector space. This will imply the second part of the lemma.

To verify (4.2.2) choose a splitting (as $\mathbb{F}$-vector spaces) of the exact sequence $0 \to F^1M \to F^0M \to \mathrm{gr}^0(M) \to 0$. Then we can write $F^0M = F^1M \oplus \mathrm{gr}^0(M)$. Observe that $F^0(uM) = (uM) \cap F^1M$ and that this is the kernel of $F^1M \to F^1M_k$. Therefore

$$F^0M/F^0(uM) = F^1M_k \oplus \mathrm{gr}^0(M)$$

Choosing splitting's of $0 \to F^{i+1}M_k \to F^iM_k \to \mathrm{gr}^i(M_k) \to 0$ allows us to identify the first term of the above sum with $\bigoplus_{i \in \mathbb{Z}_{\geq 1}} \mathrm{gr}^i(M_k)$. For the second term: splitting the exact sequences $0 \to \mathrm{gr}^{i-p}(M) \to \mathrm{gr}^i(M) \to \mathrm{gr}^i(M_k) \to 0$ described at the beginning of the subsection shows that $\mathrm{gr}^0(M) = \bigoplus_{i \in p\mathbb{Z}_{\leq 0}} \mathrm{gr}^i(M_k)$. This verifies (4.2.2).

It remains to prove that $H^1_{\mathrm{SD}}(M)$ is finite. Observe that, except for $H^1_{\mathrm{SD}}(M)$ and $H^1_{\mathrm{SD}}(uM)$, all the terms in the long exact sequence above are finite. Therefore finiteness of $H^1_{\mathrm{SD}}(M)$ can be deduced from finiteness of $H^1_{\mathrm{SD}}(u^n M)$ for large enough n. In fact $H^1_{\mathrm{SD}}(u^n M)$ will vanish for n large enough, as we now show. We need the following lemma, whose proof is straightforward.

Lemma 4.2.3. *Multiplication by u describes a bijection $F^{i+1-p}(M) \to F^i(uM)$.*

Continuing with the proof of Lemma 4.2.1, Lemma 4.2.3 implies that $F^1 N = N$ when $N = u^n M$ and n is large enough. In this case to show $H^1_{\mathrm{SD}}(N) = 0$ it suffices to show $\varphi - 1$ is surjective as a map $N \to N$. To do this note that for any $x \in N$ we have $\varphi(x) \in uN$ because $F^1 N = N$; thus $\sum_{i\geq 0} \varphi^i(-x)$ converges to an element $y \in N$ which satisfies $\varphi(y) - y = x$. This shows surjectivity of $\varphi - 1$ and completes the proof. □

Corollary 4.2.4. *Let M be an object of $\mathrm{Mod}^{\mathrm{BK}}_k(\mathcal{O})$ and assume that $\mathrm{gr}^i(M_k) = 0$ for $i < -p$. Then*

$$\chi(M) = \sum_{i<0} \dim_{\mathbb{F}} \mathrm{gr}^i(M_k)$$

(with χ as in Lemma 4.2.1).

Proof. Lemma 4.2.3 implies $\dim_{\mathbb{F}} \mathrm{gr}^i((uM)_k) = \dim_{\mathbb{F}} \mathrm{gr}^{i+1-p}(M_k)$. Note also that $H^0(u^n M) = 0$ for large enough n. Thus Lemma 4.2.1 shows (without using that $\mathrm{gr}^i(M_k) = 0$ for $i < -p$)

$$\chi(M) = \sum_{n\geq 0} \Big(\sum_{i \notin p\mathbb{Z}_{\leq 0} \cup \mathbb{Z}_{\geq 0}} \dim_{\mathbb{F}} \mathrm{gr}^{i+n(1-p)}(M_k) \Big)$$

Since $\mathrm{gr}^i(M_k) = 0$ for $i < -p$ the inner sum for $n = 0$ counts the dimensions of $\mathrm{gr}^i(M_k)$ for $i < 0$ and $\neq -p$. The inner sum for $n = 1$ counts the dimension of $\mathrm{gr}^{-p}(M_k)$. The remaining inner sums are all zero, which finishes the proof. □

Proposition 4.2.5. *If P and M are objects of $\mathrm{Mod}^{\mathrm{SD}}_k(\mathcal{O})$ then*

$$\dim_{\mathbb{F}} \mathrm{Ext}^1_{\mathrm{SD}}(P, M) - \dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{BK}}(P, M) = \sum_{\tau} \mathrm{Card}(\{ i - j < 0 \mid i \in \mathrm{Weight}_\tau(M), j \in \mathrm{Weight}_\tau(P) \})$$

Proof. First we show

$$\{ i - j \mid i \in \mathrm{Weight}_\tau(M), j \in \mathrm{Weight}_\tau(P) \} = \mathrm{Weight}_\tau(\mathrm{Hom}(P, M)^{\mathcal{O}})$$

To see this choose a basis (m_i) of M_τ such that $(u^{r_i} m_i)$ is a basis of $\varphi(M)_\tau$. The integers r_i are the elements of $\mathrm{Weight}_\tau(M)$. Likewise choose a basis (p_j) of P_τ such that $(u^{s_j} p_j)$ are a basis of $\varphi(P)_\tau$. One checks that if f_{ij} is the element of $\mathrm{Hom}(P, M)^{\mathcal{O}}$ which is zero everywhere except that it maps $p_j \mapsto m_i$ then the f_{ij} form a basis of $\mathrm{Hom}(P, M)^{\mathcal{O}}_\tau$ and $u^{r_i - s_j} f_{ij}$ forms a basis of $\varphi(\mathrm{Hom}(P, M)^{\mathcal{O}})_\tau$. Now appeal to the comment made after Lemma 2.2.1.

The previous paragraph shows that $\mathrm{Hom}(P, M)^{\mathcal{O}}$ satisfies the equivalent conditions of Lemma 2.2.1 and so $\dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(P, M)^{\mathcal{O}}_k) = \dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(P, M)^{\mathcal{O},\varphi}_k)$. Since $\mathrm{Weight}(\mathrm{Hom}(P, M)^{\mathcal{O}}) \subset [-p, p]$ it follows that $\mathrm{gr}^i(\mathrm{Hom}(P, M)^{\mathcal{O}}_k) = 0$ for $i < -p$. Thus Corollary 4.2.4 applies with $M = \mathrm{Hom}(P, M)^{\mathcal{O}}$. Using Construction 4.1.1 to identify $\mathrm{Ext}^1_{\mathrm{SD}}(P, M)$ and $H^1_{\mathrm{SD}}(\mathrm{Hom}(P, M)^{\mathcal{O}})$ the result follows. □

Remark 4.2.6. This proposition should be compared with the number of possible extensions described in [GLS14, Theorem 7.9].

5. Lifting extensions

In this section we show how to produce crystalline lifts of some exact sequences in $\mathrm{Mod}^{\mathrm{SD}}_k(\mathcal{O})$. We maintain the notation from the previous two sections.

5.1. **Isogeny categories of Breuil–Kisin modules.** When attempting to produce lifts of extensions in $\mathrm{Mod}^{\mathrm{SD}}_k(\mathcal{O})$ the functor $T \mapsto M(T)$ not being exact causes problems. However, as the following lemma shows, this problem disappears after inverting p.

Lemma 5.1.1. *If $0 \to T_1 \to T \to T_2 \to 0$ is an exact sequence of crystalline $\mathbb{Z}_p$-lattices then $0 \to M(T_1)[\frac{1}{p}] \to M(T)[\frac{1}{p}] \to M(T_2)[\frac{1}{p}] \to 0$ is an exact sequence of $\mathfrak{S}[\frac{1}{p}]$-modules.*

Proof. For this we must use that the functor in Theorem 3.1.3 satisfies an additional property; namely if T is a crystalline $\mathbb{Z}_p$-lattice inside $V = T[\frac{1}{p}]$ then there exists a φ-equivariant identification

$$M(T) \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\lambda}] \cong D_{\mathrm{crys}}(V) \otimes_{K_0} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\lambda}]$$

This is a consequence of [Kis06, Lemma 1.2.6]. Here $\mathcal{O}^{\mathrm{rig}} \subset K_0[[u]]$ is the subring of power series which converge on the open unit disk, and $\lambda \in \mathcal{O}^{\mathrm{rig}}$ is the convergent product $\prod_{n=0}^{\infty} \varphi^n(E(u)/E(0))$. Since $V \mapsto D_{\mathrm{crys}}(V)$ is an exact functor it suffices to show that $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ is faithfully flat over $\mathfrak{S}[\frac{1}{p}]$. Since $\mathfrak{S}[\frac{1}{p}]$ is a principal ideal domain faithful flatness follows because if $f \in \mathfrak{S}[\frac{1}{p}]$ is not a unit then f is not a unit in $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ (if it was then f must have either no zeroes on the open unit disk, or infinitely many, at the zeroes of λ). □

Consider the category $\mathrm{Mod}^{\mathrm{BK\text{-}iso}}_K(\mathcal{O})$ of $\mathfrak{S}_E := \mathfrak{S} \otimes_{\mathbb{Z}_p} E$-modules M equipped with isomorphisms $\varphi_M : M \otimes_{\varphi,\mathfrak{S}_E} \mathfrak{S}_E[\frac{1}{E}] \cong M[\frac{1}{E}]$, such that φ-equivariantly $M \cong M^\circ[\frac{1}{p}]$ for some $M^\circ \in \mathrm{Mod}^{\mathrm{BK}}_K(\mathcal{O})$. This category can be identified with the isogeny category of $\mathrm{Mod}^{\mathrm{BK}}_K(\mathcal{O})$, in particular it is abelian.

By [BMS18, Proposition 4.3] every object of $\mathrm{Mod}^{\mathrm{BK\text{-}iso}}_K(\mathcal{O})$ is free as an $\mathfrak{S}$-module; arguing as in Corollary 3.2.5 we see they are free as $\mathfrak{S} \otimes_{\mathbb{Z}_p} E$-modules.

Corollary 5.1.2. *The functor $T \mapsto M(T)$ induces an exact fully faithful functor $V \mapsto M(V)$ from the category of crystalline E-representations to* $\mathrm{Mod}^{\mathrm{BK\text{-}iso}}_K(\mathcal{O})$.

Proof. This follows from Lemma 5.1.1, and the fact that $T \mapsto M(T)$ is fully faithful. □

5.2. **More ext groups.** As in Subsection 4.1, if $M \in \mathrm{Mod}^{\mathrm{BK}}_K(\mathcal{O})$ we can define $H^1(M)$ as the cokernel of $M \xrightarrow{\varphi-1} M[\frac{1}{E}]$. Arguing as in Construction 4.1.1, if $P \in \mathrm{Mod}^{\mathrm{BK}}_K(\mathcal{O})$ there is a functorial inclusion

$$H^1(\mathrm{Hom}(P,M)^{\mathcal{O}}) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}}(P,M)$$

where $\mathrm{Ext}^1_{\mathcal{O}}(P,M)$ denotes the Yoneda extension group in the abelian category $\mathrm{Mod}^{\mathrm{BK}}_K(\mathcal{O})$. In particular this is a map of $\mathcal{O}$-modules. This map is surjective if P is projective as an $\mathfrak{S}_{\mathcal{O}}$-module (for then every extension of P by M splits as an $\mathfrak{S}_{\mathcal{O}}$-module).

Suppose P and M are free $\mathfrak{S}_{\mathcal{O}}$-modules, so that $\overline{P} = P \otimes_{\mathcal{O}} \mathbb{F}$ and $\overline{M} = M \otimes_{\mathcal{O}} \mathbb{F}$ are objects of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$. There is a map

$$\mathrm{Ext}^1_{\mathcal{O}}(P, M) = H^1(\mathrm{Hom}(P, M)^{\mathcal{O}}) \to H^1(\mathrm{Hom}(\overline{P}, \overline{M})^{\mathcal{O}}) = \mathrm{Ext}^1_{\mathbb{F}}(\overline{P}, \overline{M}) \tag{5.2.1}$$

induced by $\mathrm{Hom}(P, M)^{\mathcal{O}} \to \mathrm{Hom}(\overline{P}, \overline{M})^{\mathcal{O}}$. On the level of exact sequences this map is given by tensoring with $\mathbb{F}$ over $\mathcal{O}$. Via $\mathrm{Hom}(P, M)^{\mathcal{O}} \to \mathrm{Hom}(\overline{P}, \overline{M})^{\mathcal{O}}$ we can identify $\mathrm{Hom}(P, M)^{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F} = \mathrm{Hom}(\overline{P}, \overline{M})^{\mathcal{O}}$. Therefore, since

$$\mathrm{Hom}(P, M)^{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F} \to \mathrm{Hom}(P, M)^{\mathcal{O}}[\tfrac{1}{E}] \otimes_{\mathcal{O}} \mathbb{F} \to H^1(\mathrm{Hom}(P, M)^{\mathcal{O}}) \otimes_{\mathcal{O}} \mathbb{F} \to 0$$

is exact, it follows that via (5.2.1) we can identify $\mathrm{Ext}^1_{\mathcal{O}}(P, M) \otimes_{\mathcal{O}} \mathbb{F} = \mathrm{Ext}^1_{\mathbb{F}}(\overline{P}, \overline{M})$.

Analogously, for $M \in \mathrm{Mod}_K^{\text{BK-iso}}(\mathcal{O})$ we define $H^1(M)$ as the cokernel of $\varphi - 1 : M \to M[\frac{1}{E}]$. Just as in Construction 4.1.1, if $P \in \mathrm{Mod}_K^{\text{BK-iso}}(\mathcal{O})$ there are inclusions

$$H^1(\mathrm{Hom}(P, M)^E) \hookrightarrow \mathrm{Ext}^1_E(P, M)$$

where $\mathrm{Ext}^1_E(P, M)$ denotes the Yoneda extension group in $\mathrm{Mod}_K^{\text{BK-iso}}(\mathcal{O})$. Since P is free as an $\mathfrak{S}_E$-module this inclusion is an isomorphism. Choose $P^\circ, M^\circ \in \mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ such that $P^\circ \otimes_{\mathcal{O}} E = P, M^\circ \otimes_{\mathcal{O}} E = M$. Suppose P° and M° are free over $\mathfrak{S}_{\mathcal{O}}$, then the inclusion $\mathrm{Hom}(P, M)^{\mathcal{O}} \hookrightarrow \mathrm{Hom}(P, M)^E$ induces maps

(5.2.2)
$$\mathrm{Ext}^1_{\mathcal{O}}(P^\circ, M^\circ) = H^1(\mathrm{Hom}(P^\circ, M^\circ)^{\mathcal{O}}) \to H^1(\mathrm{Hom}(P, M)^E) = \mathrm{Ext}^1_E(P, M)$$

On the level of exact sequences this map is given by applying $\otimes_{\mathcal{O}} E$. Similarly to above (5.2.2) induces identifications $\mathrm{Ext}^1_{\mathcal{O}}(P^\circ, M^\circ) \otimes_{\mathcal{O}} E = \mathrm{Ext}^1_E(P, M)$.

5.3. **Lifting extensions.** Our aim is to prove the following.

Proposition 5.3.1. *Suppose $K = K_0$ and if $p = 2$ suppose further that π is chosen so that $K_\infty \cap K(\mu_{p^\infty}) = K$. Let T_2, T_1 be crystalline $\mathcal{O}$-lattices with Hodge–Tate weights contained in $[0, p]$. Set $\overline{T}_i = T_i \otimes_{\mathcal{O}} \mathbb{F}$. Assume $\overline{T}_1$ is cyclotomic-free, that $\overline{T}_2$ is irreducible, and that $\overline{T}_2 \otimes_{\mathbb{F}} \mathbb{F}(1)$ is not a Jordan–Holder factor of $\overline{T}_1$.*

Set $\overline{M}_i = M(T_i) \otimes_{\mathcal{O}} \mathbb{F}$ and suppose

$$0 \to \overline{M}_1 \to \overline{M} \to \overline{M}_2 \to 0 \tag{5.3.2}$$

is an exact sequence in $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. Then there exists a crystalline extension $0 \to T_1 \to T \to T_2 \to 0$ such that $0 \to M(T_1) \to M(T) \to M(T_2) \to 0$ is exact and recovers (5.3.2) after applying $\otimes_{\mathcal{O}} \mathbb{F}$.

Proof. Let $V_i = T_i \otimes_{\mathcal{O}} E$ and let $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1) \subset \mathrm{Ext}^1(V_2, V_1)$ denote the subset whose elements are represented by exact sequences $0 \to V_1 \to V \to V_2 \to 0$ with V crystalline. Under the usual identification $\mathrm{Ext}^1(V_2, V_1) = H^1(G_K, \mathrm{Hom}(V_2, V_1))$, the subspace $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1)$ identifies with $H^1_f(G_K, \mathrm{Hom}(V, W))$, i.e. with the kernel

$$\ker\Big(H^1(G_K, \mathrm{Hom}(V_2, V_1)) \to H^1(G_K, \mathrm{Hom}(V_2, V_1) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})\Big)$$

Since the Hodge–Tate weights of $\mathrm{Hom}(V_2, V_1)$ are equal to $i - j$ where i is a weight of V_1 and j a weight of V_2, [Nek93, Proposition 1.24] implies $H^1_f(G_K, \mathrm{Hom}(V_2, V_1))$ has E-dimension

$$\sum_{\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mathbb{F})} \mathrm{Card}(\{i - j < 0 \mid i \in \mathrm{HT}_\tau(V_1), j \in \mathrm{HT}_\tau(V_2)\}) + \dim_E \mathrm{Hom}_{E[G_K]}(V_2, V_1)$$

Now consider the diagram

$$\begin{array}{ccc} & \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1)) & \xrightarrow{\beta} \mathrm{Ext}^1_{\mathbb{F}}(\overline{M}_2, \overline{M}_1) \\ & \downarrow\alpha & \\ \mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1) \xrightarrow{\gamma} & \mathrm{Ext}^1_E(M(V_2), M(V_1)) & \end{array}$$

where the maps α and β are those described in (5.2.2) and (5.2.1) respectively. The map γ is obtained by applying $V \mapsto M(V)$ (from Corollary 5.1.2) to exact sequences representing classes in $\mathrm{Ext}_{\mathrm{crys}}(V_2, V_1)$. This makes sense since $V \mapsto M(V)$ is exact. Exactness of $M \mapsto M(V)$ also implies that this functor preserves pushouts and pullbacks; thus γ is E-linear. Since $V \mapsto M(V)$ is fully faithful we see that γ is injective.

Let Θ' denote the image of γ and let Θ denote the preimage of Θ' under α. If $0 \to M(T_1) \to M^\circ \to M(T_2) \to 0$ represents a class in Θ then by definition $M^\circ \otimes_{\mathcal{O}} E = M(V)$ where V is a crystalline E-representation fitting into an extension $0 \to V_1 \to V \to V_2 \to 0$. Thus $T = T(M^\circ)$ is a G_{K_∞}-stable $\mathcal{O}$-lattice inside V which, since $M \mapsto T(M)$ is exact, sits in a G_{K_∞}-equivariant exact sequence $0 \to T_1 \to T \to T_2 \to 0$. Our assumption on $\overline{T}_2$ and $\overline{T}_1$ allows us to apply Corollary 2.5.2; thus T is a G_K-stable lattice in V and so $M^\circ = M(T)$. Theorem 3.3.9 therefore implies β maps every element of Θ into $\mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$.

The map β can be identified with the reduction map

$$\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1)) \to \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1) \otimes_{\mathcal{O}} \mathbb{F}$$

Since $\alpha x \in \Theta$ for any $\alpha \in \mathcal{O}$ implies $x \in \Theta$, the cokernel of $\Theta \subset \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))$ is free over $\mathcal{O}$ and so $\Theta \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))$ induces an inclusion $\Theta \otimes_{\mathcal{O}} \mathbb{F} \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1)) \otimes_{\mathcal{O}} \mathbb{F}$. As β maps every element of Θ into $\mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$, we have

$$\Theta \otimes_{\mathcal{O}} \mathbb{F} \hookrightarrow \mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$$

On the other hand, since α is given by inverting p, the image of α is an $\mathcal{O}$-lattice inside $\mathrm{Ext}^1_E(M(V_2), M(V_1))$ and its kernel equals the torsion subgroup $\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))_{\mathrm{tors}}$. Thus, we can decompose Θ as

$$\Theta_{\mathrm{free}} \oplus \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))_{\mathrm{tors}}$$

where Θ_{free} is a free $\mathcal{O}$-module of rank equal to the E-dimension of $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1)$. Using Lemma 5.3.4 below, and the formula we stated above for the dimension of $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1) = H^1_f(G_K, \mathrm{Hom}(V_2, V_1))$, we deduce

$$\begin{aligned} &\dim_{\mathbb{F}}(\Theta \otimes_{\mathcal{O}} \mathbb{F}) - \dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{BK}}(\overline{M}_2, \overline{M}_1) = \\ &\qquad \sum_\tau \mathrm{Card}(\{i - j < 0 \mid i \in \mathrm{HT}_\tau(V_1), j \in \mathrm{HT}_\tau(V_2)\}) \end{aligned}$$

By Theorem 3.3.9 we have $\mathrm{HT}_\tau(V_i) = \mathrm{Weight}_\tau(\overline{M}_i)$. Therefore, by Proposition 4.2.5, each of $\mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$ and $\Theta \otimes_{\mathcal{O}} \mathbb{F}$ have the same $\mathbb{F}$-dimension. Hence

$$\Theta \otimes_{\mathcal{O}} \mathbb{F} = \mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$$

which shows that any extension $0 \to \overline{M}_1 \to M \to \overline{M}_2 \to 0$ which represents a class in $\mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_1, \overline{M}_2)$ arises as the reduction of $0 \to M(T_1) \to M(T) \to M(T_2) \to 0$ for some crystalline extension $0 \to T_1 \to T \to T_2 \to 0$. $\square$

Remark 5.3.3. We emphasise that Proposition 5.3.1 does not assert that, for T_1, T_2 as above, any exact sequence of crystalline representations $0 \to T_1 \to T \to T_2 \to 0$ remains exact after applying $T \mapsto M(T)$.

Lemma 5.3.4. *If T_1 and T_2 are crystalline $\mathcal{O}$-lattices then*

$$\begin{aligned}\dim_{\mathbb{F}}(\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))_{\mathrm{tors}} \otimes_{\mathcal{O}} \mathbb{F}) &= \dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{BK}}(\overline{M}_2, \overline{M}_1) \\ &\quad - \dim_E \mathrm{Hom}_{E[G_K]}(V_2, V_1)\end{aligned}$$

Proof. The $\mathbb{F}$-dimension of $\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))_{\mathrm{tors}} \otimes_{\mathcal{O}} \mathbb{F}$ equals the $\mathbb{F}$-dimension of the ϖ-torsion subgroup of $\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))$. To compute this latter group consider the exact sequence $0 \to M(T_1) \xrightarrow{\varpi} M(T_1) \to \overline{M}_1 \to 0$ in $\mathrm{Mod}^{\mathrm{BK}}_K(\mathcal{O})$; the associated long exact sequence reads

$$\begin{aligned}0 \to \mathrm{Hom}_{\mathrm{BK}}(M(T_2), M(T_1)) &\xrightarrow{\varpi} \mathrm{Hom}_{\mathrm{BK}}(M(T_2), M(T_1)) \to \mathrm{Hom}_{\mathrm{BK}}(M(T_2), \overline{M}_1) \\ &\to \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1)) \xrightarrow{\varpi} \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))\end{aligned}$$

Identifying $\mathrm{Hom}_{\mathrm{BK}}(M(T_2), \overline{M}_1) = \mathrm{Hom}_{\mathrm{BK}}(\overline{M}_2, \overline{M}_1)$ we see that the $\mathbb{F}$-dimension of the ϖ-torsion subgroup of $\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))$ equals

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{BK}}(\overline{M}_2, \overline{M}_1) - \dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{BK}}(M(T_2), M(T_1))/\varpi$$

By full faithfulness of $T \mapsto M(T)$, $\mathrm{Hom}_{\mathrm{BK}}(M(T_2), M(T_1)) = \mathrm{Hom}_{\mathcal{O}[G_K]}(T_2, T_1)$. Since $\mathrm{Hom}_{\mathcal{O}[G_K]}(T_2, T_1)$ is $\mathcal{O}$-free and equals $\mathrm{Hom}_{E[G_K]}(V_2, V_1)$ after inverting p the lemma follows. ☐

6. Lifting irreducibles

In this section we study simple objects of $\mathrm{Mod}^{\mathrm{SD}}_k(\mathcal{O})$ in low dimensions and show they arise from crystalline representations.

6.1. **Rank ones.**

Recall from Construction 3.2.3 how $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ is made into an $\mathcal{O}[[u]]$-algebra. In this way we view $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ as an $\mathbb{F}[[u]]$-algebra. Let $e_\tau \in k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ denote the image of the idempotent $\widetilde{e}_\tau \in \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ defined in Construction 3.2.3.

Lemma 6.1.1. *After possibly enlarging $\mathbb{F}$, every $M \in \mathrm{Mod}^{\mathrm{BK}}_k(\mathcal{O})$ of rank one over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ is isomorphic to a Breuil–Kisin module*

$$N = k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}, \qquad \varphi_N(1) = x \sum_{\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mathbb{F})} u^{r_\theta} e_\theta \tag{6.1.2}$$

for some $r_\theta \in \mathbb{Z}$ and some $x \in \mathbb{F}^\times$.

Proof. For any $f \in 1 + u\mathbb{F}[[u]]$ and $n \geq 0$ it is easy to check the equation $\varphi^n(z) = fz$ has a solution in $\mathbb{F}[[u]]^\times$ (here φ on $\mathbb{F}[[u]]$ denotes the Frobenius $\sum a_i u^i \mapsto \sum a_i u^{ip}$). Thus, after possibly enlarging $\mathbb{F}$, if $f \in \mathbb{F}[[u]]^\times$ there exists $z \in \mathbb{F}[[u]]^\times$ and $x \in \mathbb{F}$ such that $\varphi^n(z) = x^n fz$.

Now consider the statement of the lemma. For each τ choose a generator x_τ of M_τ over $\mathbb{F}[[u]]$. There are integers r_τ and $f_\tau \in \mathbb{F}[[u]]^\times$ such that $\varphi(x_{\tau\circ\varphi}) = u^{r_\tau} f_\tau e_\tau$. Recall that $\varphi_M : M_{\tau\circ\varphi} \to M_\tau[\frac{1}{u}]$ is semi-linear for the endomorphism φ of $\mathbb{F}[[u]]$ described in the previous paragraph. Choose $z \in \mathbb{F}[[u]]^\times$ and $x \in \mathbb{F}^\times$ so that

$$\frac{\varphi^{[k:\mathbb{F}_p]}(z)}{z} = x^{[k:\mathbb{F}_p]} \left(f_\tau \varphi(f_{\tau\circ\varphi^{-1}}) \dots \varphi^{[k:\mathbb{F}_p]-1}(f_{\tau\circ\varphi^{-[k:\mathbb{F}_p]-1}}) \right)^{-1} \tag{6.1.3}$$

Set $y_{-1} = zx_{\tau\circ\varphi^{-1}}$ and for $i > 1$ set

$$y_{-i} = \frac{\varphi^{i-1}(z)}{x^{i-1}} f_{\tau\circ\varphi^{-i}} \varphi(f_{\tau\circ\varphi^{-i+1}}) \dots \varphi^{i-2}(f_{\tau\circ\varphi^{-2}}) x_{\tau\circ\varphi^{-i}}$$

Then y_{-i} generates $M_{\theta\circ\varphi^{-i}}$. By construction $\varphi(y_{-i+1}) = xy_{-i}$ for $2 \leq i \leq [k:\mathbb{F}_p]$, also $\varphi(y_{-[k:\mathbb{F}_p]}) = xy_{-1}$ because x and z satisfy (6.1.3). Thus the map $M \to N$ sending y_{-i} onto $e_{\tau\circ\varphi^{-i}}$ is an isomorphism of Breuil–Kisin modules. □

Proposition 6.1.4. *Assume $K = K_0$ and $N \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ is as in (6.1.2). Then there exists a rank one crystalline $\mathcal{O}$-lattice T with τ-Hodge–Tate weight r_τ and such that $M(T) \otimes_{\mathcal{O}} \mathbb{F} \cong N$.*

Proof. This is proven in [GLS14, Lemma 6.3]. □

6.2. **Induction and restriction.** Let L/K be the unramified extension corresponding to a finite extension l/k, and let $L_\infty = K_\infty L$. Set $\mathfrak{S}_L = W(l)[[u]]$. Extension of scalars along the inclusion $f : \mathfrak{S} \to \mathfrak{S}_L$ describes a functor

$$f^* : \mathrm{Mod}_K^{\mathrm{BK}} \to \mathrm{Mod}_L^{\mathrm{BK}}$$

For $M \in \mathrm{Mod}_K^{\mathrm{BK}}$ the module $f^*M = M \otimes_{\mathfrak{S}} \mathfrak{S}_L$ is made into a Breuil–Kisin module via the semilinear map $m \otimes s \mapsto \varphi_M(m) \otimes \varphi(s)$. Similarly, restriction of scalars along f induces a functor

$$f_* : \mathrm{Mod}_L^{\mathrm{BK}} \to \mathrm{Mod}_K^{\mathrm{BK}}$$

If $M \in \mathrm{Mod}_L^{\mathrm{BK}}$ we equip f_*M with the obvious semilinear map $m \mapsto \varphi_M(m)$.

Lemma 6.2.1. *Let $N \in \mathrm{Mod}_L^{\mathrm{BK}}$ and $M \in \mathrm{Mod}_K^{\mathrm{BK}}$. Then there are functorial identifications $\mathrm{Hom}_{\mathrm{BK}}(M, f_*N) \cong \mathrm{Hom}_{\mathrm{BK}}(f^*M, N)$, $T(f^*M) \cong T(M)|_{G_{L_\infty}}$, and $T(f_*N) \cong \mathrm{Ind}_{L_\infty}^{K_\infty} T(N)$ making the following diagram commute.*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{BK}}(M, f_*N) & \longrightarrow & \mathrm{Hom}_{\mathrm{BK}}(f^*M, N) \\ \downarrow T & & \downarrow T \\ \mathrm{Hom}_{G_{K_\infty}}(T(M), \mathrm{Ind}_{L_\infty}^{K_\infty} T(N)) & \xrightarrow{(\mathrm{Frob})} & \mathrm{Hom}_{G_{L_\infty}}(T(M)|_{G_{L_\infty}}, T(N)) \end{array}$$

The lower horizontal arrow is given by Frobenius reciprocity.

Proof. This is [Bar20a, Lemma 6.2.4]. □

By functoriallity f_* and f^* induce functors on $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ and $\mathrm{Mod}_L^{\mathrm{BK}}(\mathcal{O})$. The following lemma shows how they also preserve strong divisibility.

Lemma 6.2.2. *Assume $k \subset l \subset \mathbb{F}$.*

(1) *If $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ then $f^*M \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ and for each $\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l, \mathbb{F})$ we have*

$$\mathrm{Weight}_\theta(f^*M) = \mathrm{Weight}_{\theta|_k}(M)$$

(2) *If $N \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ then $f_*N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and*

$$\mathrm{Weight}_\tau(f_*N) = \bigcup_{\theta|_k = \tau} \mathrm{Weight}_\theta(N)$$

Proof. This is [Bar20a, Lemma 6.2.6]. □

6.3. **Approximation by rank ones.** For this subsection fix an irreducible $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. Irreducible here means the only non-trivial sub-Brueil–Kisin module $M' \subset M$ in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ with torsion-free cokernel is M itself. Let L/K be the unramified extension of degree $\dim_{\mathbb{F}} T(M)$ with residue extension l/k, and let $f : \mathfrak{S} \to \mathfrak{S}_L$ be as in Subsection 6.2.

Proposition 6.3.1. *After enlarging $\mathbb{F}$ there exists a rank one $N \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ and an inclusion $M \to f_* N$ whose image contains $u(f_*N)$.*

Further, if for $\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l, \mathbb{F})$ we set $\delta_\theta = 0$ if the image of M contains N_θ, and $\delta_\theta = 1$ otherwise, then

$$r_\theta, r_\theta + p\delta_{\theta\circ\varphi} - \delta_\theta \in \mathrm{Weight}_{\theta|_k}(M)$$

where $\{r_\theta\} = \mathrm{Weight}_\theta(N)$.

The construction is identical to that given in [Bar20a, §6.3]. For the convenience of the reader we repeat the arguments. After Lemma 6.1.2 we may describe N explicitly as

$$N = k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}, \qquad \varphi_N(1) = x \sum u^{r_\theta} e_\theta$$

for some $x \in \mathbb{F}^\times$, with the sum running over $\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l, \mathbb{F})$.

Proof. First we show that M being irreducible implies $T(M)$ is irreducible as a G_{K_∞}-representation. This follows from the next lemma.

Lemma 6.3.2. *Let $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and let $0 \to T_1 \to T(M) \to T_2 \to 0$ be an exact sequence of G_{K_∞}-representations. Then there exists an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ which recovers the first exact sequence after applying $M \mapsto T(M)$.*

Proof. Since $T(M) \otimes_{\mathbb{Z}_p} W(C^\flat) = M \otimes_{\mathfrak{S}} W(C^\flat)$ we obtain a surjection $M \otimes_{\mathfrak{S}} W(C^\flat) \to T_2 \otimes_{\mathbb{Z}_p} W(C^\flat)$. Let M_2 be the image of M under this surjection. Then $M_2 \otimes_{\mathfrak{S}} W(C^\flat) = T_2 \otimes_{\mathbb{Z}_p} W(C^\flat)$ so $T(M_2) = T_2$. Take M_1 to be the kernel of $M \to M_2$. □

Enlarging $\mathbb{F}$ if necessary we can suppose $l \subset \mathbb{F}$ and that $T(M) \cong \mathrm{Ind}_{L_\infty}^{K_\infty} \chi$ (using Lemma 2.3.1 and Lemma 2.2.1) for a character χ. Frobenius reciprocity gives a non-zero map $T(M)|_{G_{L_\infty}} \to \chi$. Lemma 6.2.1 implies $T(M)|_{G_{L_\infty}} \cong T(f^*M)$, so Lemma 6.3.2 produces a surjection $f^*M \to N$ for a rank one $N \in \mathrm{Mod}_l^{\mathrm{BK}}(\mathcal{O})$ with $T(N) = \chi$. Lemma 6.2.1 then implies there exists a map

$$M \to f_* N$$

which, after applying T, induces the isomorphism $T(M) \cong \mathrm{Ind}_{L_\infty}^{K_\infty} \chi$. In other words $M \to f_*N$ becomes an isomorphism after inverting u; and so is injective.

Lemma 6.2.2 and Proposition 3.3.8 imply $N \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ and $\mathrm{Weight}_\theta(N) \subset \mathrm{Weight}_\theta(f^*M) = \mathrm{Weight}_{\theta|_k}(M)$.

Now let us assume that N is as in Lemma 6.1.2, so that N is generated over $\mathbb{F}[[u]]$ by e_θ satisfying $\varphi_N(e_{\theta\circ\varphi}) = xu^{r_\theta} e_\theta$. Since $M \to f_*N$ is an isomorphism after inverting u we may consider the smallest integer $\delta_\theta \geq 0$ satisfying $u^{\delta_\theta} e_\theta \in M$. We shall show that $\delta_\theta \in [0,1]$ which shows δ_θ is equal to the δ_θ defined in the proposition. Let $P \in \mathrm{Mod}_l^{\mathrm{BK}}(\mathcal{O})$ be the sub-Breuil–Kisin module of N generated by the $u^{\delta_\theta} e_\theta$. The map $P \to f^*M$ given by $u^{\delta_\theta} e_\theta \mapsto e_\theta(u^{\delta_\theta} e_\theta \otimes 1)$ is φ-equivariant and has u-torsion-free cokernel, by definition of the δ_θ. Therefore Proposition 3.3.8

implies $P \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ and $\mathrm{Weight}_\theta(P) \subset \mathrm{Weight}_\theta(f^*M) = \mathrm{Weight}_{\theta|_k}(M)$. Since $\mathrm{Weight}_\theta(P) = \{r_\theta + p\delta_{\theta\circ\varphi} - \delta_\theta\}$ it just remains to prove $\delta_\theta \in [0,1]$.

Since $r_\theta, r_\theta + p\delta_{\theta\circ\varphi} - \delta_\theta$ are both contained in $\mathrm{Weight}_{\theta|_k}(M)$ both are in $[0,p]$. Thus $p\delta_{\theta\circ\varphi} - \delta_\theta \leq p$ and so

$$(p^{[l:\mathbb{F}_p]} - 1)\delta_\theta = \sum_{i=1}^{[l:\mathbb{F}_p]} p^{i-1}(p\delta_{\theta\circ\varphi^i} - \delta_{\theta\circ\varphi^{i-1}}) \leq p(p^{[l:\mathbb{F}_p]} - 1)/(p-1)$$

which shows $\delta_\theta \in [0,1]$ unless $p = 2$, in which case we deduce $\delta_\theta \in [0,2]$. If $p = 2$ and $\delta_{\theta\circ\varphi} = 2$ then as $r_\theta + p\delta_{\theta\circ\varphi} - \delta_\theta \in [0,p]$ it follows that $r_\theta = 0$ and $\delta_\theta = 2$. Thus if $\delta_\theta \notin [0,1]$ for some θ then $r_\theta = 0$ for all θ; this implies $T(N)$ is an unramified character and so $\mathrm{Ind}_{L_\infty}^{K_\infty} T(N)$ is not irreducible, a contradiction. □

Remark 6.3.3. The surjection $f^*M \to N$ can be recovered from the inclusion $M \to f_*N$ explicitly. On $(f^*M)_\theta = M_{\theta|_k}$ it is given by $\sum_{\theta'|_k = \theta|_k} \alpha_{\theta'} e_{\theta'} \mapsto \alpha_\theta e_\theta$. In particular, for every θ there exist $\alpha_{\theta'} \in \mathbb{F}[[u]]$ such that

$$e_\theta + \sum_{\theta' \neq \theta} \alpha_{\theta'} e_{\theta'} \in M$$

6.4. **Low dimensional cases.** For weights in the Fontaine–Laffaille range $[0,p-1]$ the inclusion $M \subset f_*N$ from Proposition 6.3.1 is an equality:

Lemma 6.4.1. *Suppose $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ is irreducible and $\mathrm{Weight}_\tau(M) \subset [0,p-1]$ for every τ. Then $M = f_*N$ for a rank one $N \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$*

Proof. With notation as in Proposition 6.3.1, if $\delta_{\theta\circ\varphi} = 1$ and $\delta_\theta = 0$ then $r_\theta + p \in \mathrm{Weight}_{\theta|_k}(M) \subset [0,p-1]$ which is impossible. Therefore either all $\delta_\theta = 0$ (in which case $M = f_*N$) or all $\delta_\theta = 1$. In the later case, since $r_\theta + p - 1 \in [0,p-1]$ it follows that $r_\theta = 0$ for all θ, contradicting the irreducibility of $T(f_*N)$. □

On the other hand the previous lemma does not hold for every irreducible $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. In [Bar20a, §6.4] an example is given when $K = \mathbb{Q}_p$ and $\dim_{\mathbb{F}_p} T(M) = 5$. We conclude this section by showing this is the lowest dimensional counterexample. We do this by an explicit calculation.

Proposition 6.4.2. *Suppose $k = \mathbb{F}_p$. Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ be irreducible with $\dim_{\mathbb{F}} T(M) \leq 4$. After possibly enlarging $\mathbb{F}$, $M \cong f_*N$ for a rank one $N \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$.*

Proof. We only consider the case $\dim_{\mathbb{F}} T(M) = 4$, the two and three dimensional cases being much easier. We put ourselves in the situation of Proposition 6.3.1.

First suppose $e_\theta \notin M$ for all θ (i.e. $\delta_\theta = 1$). Then $r_\theta + p - 1 \in [0,p]$ so $r_\theta \in [0,1]$. Fix $\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l,\mathbb{F})$ and set $r := (r_{\theta\circ\varphi^3}, r_{\theta\circ\varphi^2}, r_{\theta\circ\varphi}, r_\theta)$. By replacing θ with $\theta\circ\varphi^i$ we may assume r is one of the following:

$$(0,0,0,0), \quad (1,1,1,1), \quad (1,0,1,0), \quad (0,0,1,1)$$

Note that in the first three cases f_*N is not irreducible (consider the sub-Breuil–Kisin module generated by $e_\theta + e_{\theta\circ\varphi}^2$ and $e_{\theta\circ\varphi} + e_{\theta\circ\varphi^3}$), so only the last case is possible. Since $u(f_*N) \subset M$, Remark 6.3.3 implies there exist $\alpha, \beta, \gamma \in \mathbb{F}$, not all zero, such that $\gamma e_{\theta\circ\varphi^3} + \beta e_{\theta\circ\varphi^2} + \alpha e_{\theta\circ\varphi} + e_\theta \in M$. Applying φ gives that $x(\gamma e_{\theta\circ\varphi^2} + \beta u e_{\theta\circ\varphi} + \alpha u e_\theta + e_{\theta\circ\varphi^3}) \in M$ and so $e_{\theta\circ\varphi^3} + \gamma e_{\theta\circ\varphi^2} \in M$. Applying φ again gives $e_{\theta\circ\varphi^2} + \gamma u e_{\theta\circ\varphi} \in M$, so $e_{\theta\circ\varphi^2} \in M$, a contradiction.

Now suppose $e_\theta \in M$ and $e_{\theta\circ\varphi} \notin M$ (i.e. $\delta_\theta = 0, \delta_{\theta\circ\varphi} = 1$). The requirement of Remark 6.3.3 restricts the possible image of $M/u(f_*N) \hookrightarrow (f_*N)/u(f_*N)$; it must be an $\mathbb{F}$-subspace $V \subset (f_*N)/u(f_*N)$ not containing $e_{\theta\circ\varphi}$ and such that each projection $V \to (f_*N)/u(f_*N) \xrightarrow{\text{proj}} \mathbb{F}e_{\theta\circ\varphi^i}$ is surjective. One easily checks any V satisfying these properties is one of the following, for some $\alpha, \beta \in \mathbb{F}^\times$:

(1) $V = \mathbb{F}(e_{\theta\circ\varphi^3} + \alpha e_{\theta\circ\varphi^2} + \beta e_{\theta\circ\varphi}) + \mathbb{F}e_\theta$,
(2) $V = \mathbb{F}(e_{\theta\circ\varphi^3} + \alpha e_{\theta\circ\varphi}) + \mathbb{F}(e_{\theta\circ\varphi^2} + \beta e_{\theta\circ\varphi}) + \mathbb{F}e_\theta$,
(3) $V = \mathbb{F}e_{\theta\circ\varphi^3} + \mathbb{F}(e_{\theta\circ\varphi^2} + \alpha e_{\theta\circ\varphi}) + \mathbb{F}e_\theta$,
(4) $V = \mathbb{F}(e_{\theta\circ\varphi^3} + \alpha e_{\theta\circ\varphi}) + \mathbb{F}e_{\theta\circ\varphi^2} + \mathbb{F}e_\theta$.

In each of (1)-(4) we write $e_{\theta\circ\varphi^i}$ when we mean the image of $e_{\theta\circ\varphi^i}$ in $(f_*N)/u(f_*N)$. To ease notation we now write e_i in place of $e_{\theta\circ\varphi^i}$, likewise we write $r_i = r_{\theta\circ\varphi^i}$ and $\delta_i = \delta_{\theta\circ\varphi^i}$. We now show each of (1)-(4) cannot occur.

- In case (1) the elements $e_3+\alpha e_2+\beta e_1$, ue_2, ue_1 and e_0 of M form an $\mathbb{F}[[u]]$-basis. Note then that $\delta_0 = 0, \delta_1 = \delta_2 = \delta_3 = 1$ and so $r_0 = 0, r_1, r_2 \in [0,1]$ and $r_3 > 0$. Further, since $\varphi(e_3 + \alpha e_2 + \beta e_1) = x(u^{r_2}e_2 + u^{r_1}\alpha e_1 + \beta e_0) \in M$ we must have $r_1 = r_2 = 1$. Therefore, with respect to the above basis, the matrix of φ_M is
$$x\begin{pmatrix} 0 & 0 & 0 & u^{r_3} \\ 1 & 0 & 0 & -\alpha u^{r_3-1} \\ \alpha & u^p & 0 & -\beta u^{r_3-1} \\ \beta & 0 & u^p & 0 \end{pmatrix} = x\begin{pmatrix} 0 & 0 & 0 & \frac{1}{\beta} \\ 0 & -\frac{\beta}{\alpha} & 1 & \frac{\beta-\alpha^2}{\alpha\beta} \\ \alpha\beta & \beta u & 0 & -u \\ 0 & -\frac{1}{\alpha} & 0 & \frac{1}{\alpha\beta} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u^p & 0 & 0 \\ 0 & 0 & u^{p+1} & 0 \\ 0 & 0 & 0 & u^{r_3-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{u^p}{\beta} & 0 \\ 0 & 1 & \frac{-\alpha^2+\beta}{\alpha\beta} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{u^{p+r_3-1}}{\alpha\beta} & 1 \end{pmatrix}^{-1}$$
Remark 3.3.7 implies $p+1$ is a weight of M, so M is not strongly divisible.

- In case (2) the elements $e_3 + \alpha e_1, e_2 + \beta e_1, ue_1, e_0$ form a basis of M over $\mathbb{F}[[u]]$. In this case $\delta_3 = \delta_2 = \delta_1 = 1$ and $\delta_0 = 0$ so again $r_0 = 0$, $r_1, r_2 \in [0,1]$, and $r_3 > 0$. Since $\varphi(e_3 + \alpha e_1) \in M$ we deduce $r_2 = 1$. Similarly $r_1 = 1$ because $\varphi(e_2 + \beta e_1) \in M$. With respect to this basis the matrix of φ_M is given by
$$x\begin{pmatrix} 0 & 0 & 0 & u^{r_3} \\ u & 0 & 0 & 0 \\ -\beta & 1 & 0 & -\alpha u^{r_3-1} \\ \alpha & \beta & u^p & 0 \end{pmatrix} = x\begin{pmatrix} 0 & 0 & 0 & \frac{1}{\alpha} \\ 0 & 0 & \frac{\alpha}{\alpha+\beta^2} & \frac{\beta}{\alpha+\beta^2} \\ -\alpha\beta & -(\alpha+\beta^2) & -\beta u & u \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u^{p+1} & 0 \\ 0 & 0 & 0 & u^{r_3} \end{pmatrix} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} & -\frac{u^p}{\alpha+\beta^2} & -\frac{\alpha\beta u^{r_3-1}}{\alpha+\beta^2} \\ 0 & 1 & -\frac{\beta u^p}{\alpha+\beta^2} & \frac{\alpha^2 u^{r_3-1}}{\alpha+\beta^2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$
if $\alpha + \beta^2 \neq 0$, or
$$x\begin{pmatrix} 0 & 0 & 0 & u^{r_3} \\ u & 0 & 0 & 0 \\ -\beta & 1 & 0 & -\alpha u^{r_3-1} \\ \alpha & \beta & u^p & 0 \end{pmatrix} = x\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\beta^2} \\ 0 & \beta & 0 & \frac{u}{\beta} \\ \beta^3 & 0 & -\beta u & u \\ 0 & 0 & \frac{1}{\beta^2} & -\frac{1}{\beta^3} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u^{p+1} & 0 \\ 0 & 0 & 0 & u^{r_3-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\beta} & 0 & 0 \\ 0 & 1 & -\frac{u^p}{\beta} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{u^{p-r_3+1}}{\beta^3} & 1 \end{pmatrix}^{-1}$$
if $\alpha + \beta^2 = 0$. Again $p+1$ is a weight of M so M is not strongly divisible.

- In case (3) the elements $e_3, e_2+\alpha e_1, ue_1, e_0$ form a $\mathbb{F}[[u]]$-basis of M. Since $\delta_0 = 0$ and $\delta_1 = \delta_2 = 1$ we have $r_0 = 0$, $r_1 \in [0,1]$ and $r_2 > 0$. Further, since $\varphi(e_2+\alpha e_1) \in M$ we must have $r_1 = 1$. With respect to this basis the matrix of φ_M is given by
$$x\begin{pmatrix} 0 & 0 & 0 & u^{r_3} \\ u^{r_2} & 0 & 0 & 0 \\ -\alpha u^{r_2-1} & 1 & 0 & 0 \\ 0 & \alpha & u^p & 0 \end{pmatrix} = x\begin{pmatrix} 0 & 0 & -\frac{1}{\alpha} & \frac{1}{\alpha^2} \\ 0 & 0 & 0 & \frac{1}{\alpha} \\ 0 & -\alpha^2 & -\alpha u & u \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} u^{r_2-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u^{p+1} & 0 \\ 0 & 0 & 0 & u^{r_3} \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{u^{p-r_2+1}}{\alpha^2} & 0 \\ 0 & 1 & -\frac{u^p}{\alpha} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$
Once more M is not strongly divisible.

- In case (4) the elements $e_3+\alpha e_1, e_2, ue_1, e_0$ form an $\mathbb{F}[[u]]$-basis of M. Since $\delta_0 = \delta_2 = 0$ and $\delta_1 = \delta_3 = 1$ we have $r_0 = r_2 = 0$ and $r_1, r_3 > 0$. With respect to this basis the matrix of φ_M is given by

$$x\begin{pmatrix} 0 & 0 & 0 & u^{r_3} \\ 1 & 0 & 0 & 0 \\ 0 & u^{r_1-1} & 0 & -\alpha u^{r_3-1} \\ \alpha & 0 & u^p & 0 \end{pmatrix} = x\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -a & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u^{r_1-1} & 0 & 0 \\ 0 & 0 & u^p & 0 \\ 0 & 0 & 0 & u^{r_3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha u^{r_3-r_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

If M is strongly divisible then the rightmost matrix must be an element of $\mathrm{GL}_n(\mathbb{F}[[u^p]])$. Therefore $p \mid r_3 - r_1$; as both lie in $[1, p]$ we must have $r_3 = r_1$. However then $f_* N$ is not irreducible (consider the sub-Breuil–Kisin module generated by $e_3 + e_1$ and $e_2 + e_0$).

We conclude that $e_\theta \in M$ for every θ. In other words, $M = f_* N$. □

6.5. **Crystalline liftings.** We deduce the following.

Corollary 6.5.1. *Assume $K = K_0$. Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ be irreducible and assume that one of the following holds.*

1. *M has rank one over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$.*
2. *$\mathrm{Weight}_\tau(M) \subset [0, p-1]$ for every $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mathbb{F})$.*
3. *$k = \mathbb{F}_p$ and $\dim_{\mathbb{F}} T(M) \leq 4$.*

Then, after possibly extending $\mathbb{F}$, there exists a crystalline $\mathcal{O}$-lattice T such that $M(T) \otimes_{\mathcal{O}} \mathbb{F} \cong M$ and such that $\mathrm{HT}_\tau(T) = \mathrm{Weight}_\tau(M)$. Further, T is induced over an unramified extension from a crystalline character.

Proof. If M is an in (1) then this follows from Proposition 6.1.4. If M is as in (1) or (2) then by Lemma 6.4.1 and Proposition 6.4.2, after possibly enlarging $\mathbb{F}$, we have that $M \cong f_* N$ where N is a rank one Breuil–Kisin module over L. Using Proposition 6.1.4 again, there exists a crystalline character $\chi : G_L \to \mathcal{O}^\times$ with associated Breuil–Kisin module $M(\chi)$ satisfying $N = M(\chi) \otimes_{\mathcal{O}} \mathbb{F}$ and $\mathrm{HT}_\theta(\chi) = \mathrm{Weight}_\theta(N)$. Since L/K is unramified, $\mathrm{Ind}_L^K \chi$ is again crystalline and

$$\mathrm{HT}_\tau(\mathrm{Ind}_L^K \chi) = \bigcup_{\theta|_k = \tau} \mathrm{HT}_\theta(\chi) = \bigcup_{\theta|_k = \tau} \mathrm{Weight}_\theta(N) = \mathrm{Weight}_\tau(M)$$

cf. [GHS18, Corollary 7.1.2]. Since $T(f_* M(\chi)) = \mathrm{Ind}_{L_\infty}^{K_\infty} \chi$, which equals the restriction to G_{K_∞} of $\mathrm{Ind}_L^K \chi$, we deduce that $f_* M(\chi) = M(\mathrm{Ind}_L^K \chi)$, and so

$$M(\mathrm{Ind}_L^K \chi) \otimes_{\mathcal{O}} \mathbb{F} = f_*\Big(M(\chi) \otimes_{\mathcal{O}} \mathbb{F}\Big) \cong M$$

Therefore we can take $T = \mathrm{Ind}_L^K \chi$.

□

7. Potentially diagonalisable lifts

We use the notation of potential diagonalisability as described in [BLGGT14] (the definition is given in the paragraph before [BLGGT14, Lemma 1.4.1]). In practice we shall only use the following elementary observations regarding potential diagonalisability.

- If a crystalline $\mathcal{O}$-lattice T admits a G_K-stable filtration $F^i T$ whose graded pieces are free $\mathcal{O}$-modules then T is potentially diagonalisable if and only if $\mathrm{gr}(T) := \bigoplus \mathrm{gr}^i(T)$ is potentially diagonalisable, cf. property (7) in the list preceding [BLGGT14, Lemma 1.4.1].
- If a crystalline $\mathcal{O}$-lattice T is induced from a character then T is potentially diagonalisable. This is because if T is induced over an extension L/K then $T|_{G_L}$ admits a G_L-stable filtration whose graded pieces are characters

and so $T|_{G_L}$ is potentially diagonalisable by the previous bullet point (cf. [BLGGT14, Lemma 1.4.3]). Thus T is potentially diagonalisable also.

7.1. **Obvious lifts.** The previous two bullet points imply that, with the following definition, obvious[4] crystalline $\mathcal{O}$-lattices are potentially diagonalisable.

Definition 7.1.1. A crystalline $\mathcal{O}$-lattice is obvious if it admits a filtration F^iT by G_K-stable submodules such that each graded piece $\mathrm{gr}^i(T)$ is irreducible after inverting p and induced from a crystalline character over an unramified extension of K.

Theorem 7.1.2. *Assume $K = K_0$ and π is chosen so that $K_\infty \cap K(\mu_{p^\infty}) = K$. Let $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and assume $T(\overline{M})$ is cyclotomic-free (as in Definition 2.2.2) and that one of the following conditions is satisfied.*

- *Every irreducible subquotient of $\overline{M}$ is of rank one.*
- $\mathrm{Weight}_\tau(\overline{M}) \subset [0, p-1]$ *for every* $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mathbb{F})$.
- *$k = \mathbb{F}_p$ and every irreducible subquotient of $\overline{M}$ has rank ≤ 4.*

Then, after possibly enlarging $\mathbb{F}$, there exists an obvious crystalline $\mathcal{O}$-lattice T such that $M(T) \otimes_{\mathcal{O}} \mathbb{F} \cong \overline{M}$.

Proof. We argue by induction on the length of $T(\overline{M})$. If $T(\overline{M})$ is irreducible then $\overline{M}$ is irreducible and the theorem follows from Corollary 6.5.1. If $T(\overline{M})$ has length > 1 then there is an exact sequence $0 \to \overline{T}_1 \to T(\overline{M}) \to \overline{T}_2 \to 0$ with neither $\overline{T}_1, \overline{T}_2 = 0$. Since $T(\overline{M})$ is cyclotomic-free we can assume $\overline{T}_1$ is cyclotomic-free, $\overline{T}_2$ is irreducible, and $\overline{T}_2 \otimes \overline{\mathbb{F}}(1)$ is not a Jordan–Holder factor of $\overline{T}_1$. Lemma 6.3.2 gives an exact sequence $0 \to \overline{M}_2 \to \overline{M} \to \overline{M}_1 \to 0$ in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ with $T(\overline{M}_i) = T_i$. Since $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ so are the $\overline{M}_i$ by Proposition 3.3.8. Proposition 3.3.8 also implies $\mathrm{Weight}_\tau(\overline{M}_1) \cup \mathrm{Weight}_\tau(\overline{M}_2) = \mathrm{Weight}_\tau(\overline{M})$. Thus both $\overline{M}_i$ satisfy the conditions of the theorem. Our inductive hypothesis provides obvious crystalline $\mathcal{O}$-lattices T_i such that $M(T_i) \otimes_{\mathcal{O}} \mathbb{F} = \overline{M}_i$. This puts us in the situation of Proposition 5.3.1, and this proposition provides us with a crystalline $\mathcal{O}$-lattice T as desired. □

As in the introduction, a crystalline lift of a representation $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ such that $\rho \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong \overline{\rho}$. We say ρ is an obvious crystalline lift if ρ is an obvious crystalline $\mathcal{O}$-lattice.

Corollary 7.1.3. *Suppose $K = K_0$. Suppose $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is continuous, cyclotomic-free, and has one-dimensional Jordan–Holder factors. Then $\overline{\rho}$ admits a crystalline lift ρ with $\mathrm{HT}_\tau(\rho) \subset [0, p]$ for each τ, if and only if $\overline{\rho}$ admits an obvious crystalline lift ρ' with $\mathrm{HT}_\tau(\rho) = \mathrm{HT}_\tau(\rho')$.*

Proof. Choose π so that $K_\infty \cap K(\mu_{p^\infty}) = K$. Suppose such a ρ exists. Choosing our coefficient field E sufficiently large we may suppose ρ factors through $\mathrm{GL}_n(\mathcal{O})$. Theorem 3.3.9 implies $M(\rho) \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and $\mathrm{Weight}_\tau(M) = \mathrm{HT}_\tau(\rho)$. Put $\overline{M} = M(\rho) \otimes_{\mathcal{O}} \mathbb{F}$. Then $T(\overline{M}) \cong \overline{\rho}$ as G_{K_∞}-representations, so $T(\overline{M})$ is cyclotomic-free as a G_{K_∞}-representation. Theorem 7.1.2 implies there exists a potentially diagonalisable ρ' with $M(\rho') \otimes_{\mathcal{O}} \mathbb{F} \cong \overline{M}$ and with $\mathrm{HT}_\tau(\rho') = \mathrm{Weight}_\tau(\overline{M})$. Thus $\rho' \otimes_{\mathcal{O}} \mathbb{F} \cong \overline{\rho}$ as G_{K_∞}-representations. By Theorem 2.1.3 they are isomorphic as G_K-representations and we are done. □

[4]Our terminology comes from [GHS18, Subsection 7.1] where the notion of an obvious weight is introduced.

If we assume $K = \mathbb{Q}_p$ we can also prove:

Corollary 7.1.4. *Let $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be continuous and cyclotomic-free, with $n \leq 5$. Then $\overline{\rho}$ admits a crystalline lift ρ with $\mathrm{HT}(\rho) \subset [0,p]$ for each τ, if and only if $\overline{\rho}$ admits an obvious crystalline lift ρ' with $\mathrm{HT}(\rho) = \mathrm{HT}(\rho')$.*

Proof. When every Jordan–Holder factor of $\overline{\rho}$ has dimension ≤ 4 this follows from Theorem 7.1.2 by the argument used in the above corollary. If $\overline{\rho}$ is irreducible this fact follows from the main result of [Bar20a], cf. Theorem 7.2.1. □

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