Geometry in the Langlands program

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Part 1: Modularity and elliptic curves

Problem

For integers a, b consider the equation

$$E : Y^2 = X^3 + aX + b$$

If $4a^3 + 27b^2 \neq 0$ then this equation defines an *elliptic curve*.
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Example

For $E : Y^2 = X^3 + 4X$ we have

ℓ	2	3	5	7	11	13	17	19
$a_\ell(E)$	0	0	-2	0	0	6	2	0

Definition (of a completely different kind of object)

A modular form of weight $k \geq 1$ and level $N \geq 1$ is a function

$$f : \mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \to \mathbb{C}$$

such that

- f is holomorphic and satisfies a growth condition as $z \to i\infty$.
- For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \equiv 1$ modulo N one has

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$$

Every such f can be written as

$$f(z) = a_0(f) + a_1(f)q + a_2(f)q^2 + \dots, \qquad q = e^{2\pi iz}$$

with $a_n(f) \in \mathbb{C}$.

For fixed k and N these functions form a finite dimensional $\mathbb{C}$-vector space whose elements can be computed explicitly.

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For $N = 32$ and $k = 2$ this vector space is 8-dimensional and contains a unique (up to scaling) element

$$f = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25} - 10q^{29} - 2q^{37} + \dots$$

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Slogan: *All elliptic curves over* $\mathbb{Q}$ *are modular.*

It is now understood that this theorem is just the tip of the iceberg. The progam initiated by Robert Langlands in the 1970's predicts that the slogan should be

Everything[1] *in arithmetic is modular*

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To explain this we need to make the slogan more precise, and for this we need Galois representations.

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Part 2: Modularity and Galois representations

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$$0 = \sigma(P(x_1, \ldots, x_n)) = P(\sigma(x_1), \ldots, \sigma(x_n))$$

for $\sigma \in G_{\mathbb{Q}}$.

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$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{Q}_p)$$

where p is prime number and $\mathbb{Q}_p$ is the ring with elements $\sum_{i=-N}^{\infty} a_i p^i$.

Galois representations attached to elliptic curves

For $X = E$, an elliptic curve over $\mathbb{Q}$, this process produces Galois representations

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So our original slogan can be replaced with:

Galois representations attached to elliptic curves over $\mathbb{Q}$ are modular

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Theorem (Kisin, 2008)

Suppose $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p)$ is a representation coming from arithmetic. Then there exists a modular form f such that

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However, essentially nothing is known if one considers higher dimensional Galois representations or if $\mathbb{Q}$ is replaced by a finite extension like $\mathbb{Q}(\sqrt{D})$.

Part 3: Moduli spaces of Galois representations

Geometric strategy for proving modularity

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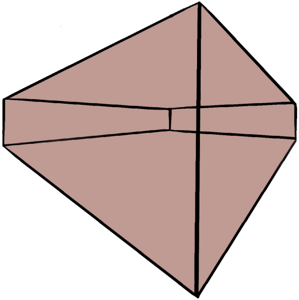

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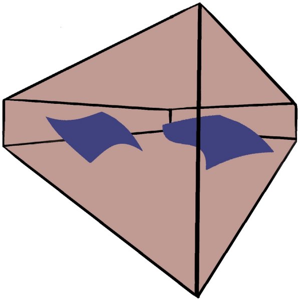

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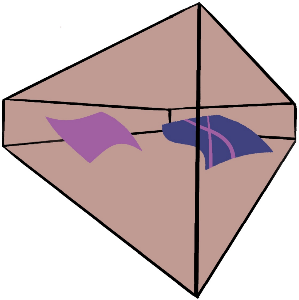

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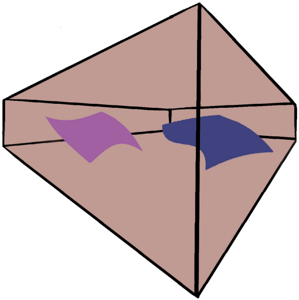

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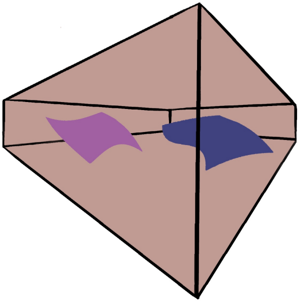

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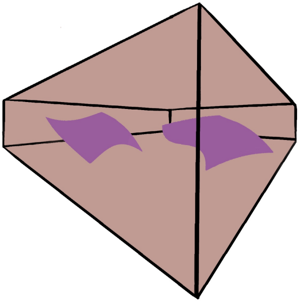

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Choose a (prime) number ℓ and consider the group generated by two elements σ, ψ with the relation

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The entries of any such matrices define a point in

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This is the zero set of a 4 polynomial equations. For example, for $l = 2$ the equations are:

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x_1y_1 + x_2y_3 &= y_1^2x_1 + x_1y_2y_3 + x_3y_1y_2 + x_3y_2y_4 \\
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Component matching in the $\ell \neq p$

There is a version of component matching for $\mathcal{M}_2$ which relates its geometry to the ℓ-adic behaviour of modular forms.
The most general statement in this direction was proven by Jack Shotton.

The key obstruction to proving the desired component matching is therefore contained in $\ell = p$ situation.

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There is a version of component matching for $\mathcal{M}_2$ which relates its geometry to the ℓ-adic behaviour of modular forms.
The most general statement in this direction was proven by Jack Shotton.

The key obstruction to proving the desired component matching is therefore contained in $\ell = p$ situation. Unfortunately, the geometric spaces controlling the restriction of $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p)$ to the subgroup

$$G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$$

cannot be described so easily.

The following conjecture encapsulates what is required for component matching when $\ell = p$.

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The Breuil–Mézard conjecture

Let $\mathcal{X}$ denote the moduli space of n-dimensional p-adic representations of G_K for $K/\mathbb{Q}_p$ a finite extension. Let $\mathcal{X}^\mu \subset \mathcal{X}$ denote the subspace of crystalline representations of weight μ. Then, as algebraic cycles

$$0 = \sum n_\mu [\mathcal{X}^\mu \otimes_{\mathbb{Z}_p} \mathbb{F}_p]$$

whenever $0 = \sum n_\mu [V_\mu]$ in the Grothendieck group of $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(k)$ where k denotes the residue field of K and V_μ is the algebraic representation of highest weight μ.

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This conjecture was proven by Kisin for $n = 2$ and $K = \mathbb{Q}_p$ and this was crucial step which enabled his modularity result. See Kisin's 2010 ICM talk for more details.

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Theorem (B., 2021)

The Breuil–Mézard conjecture is true when $n = 2$ and any K provided one considers weights contained in the range $[0, p]$.

Thank you for your time.
Are there any questions?