POTENTIAL DIAGONALISABILITY OF PSEUDO-BARSOTTI–TATE REPRESENTATIONS

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Abstract. Previous work of Kisin and Gee proves potential diagonalisability of two dimensional Barsotti–Tate representations of the Galois group of a finite extension $K/\mathbb{Q}_p$. In this paper we build upon their work by relaxing the Barsotti–Tate condition to one we call pseudo-Barsotti–Tate (which means that for certain embeddings $\kappa : K \to \overline{\mathbb{Q}}_p$ we allow the κ-Hodge–Tate weights to be contained in $[0, p]$ rather than $[0, 1]$).

Contents

1. Introduction

1.1. **Overview.** Following [BLGGT14, §1.4], a potentially crystalline representation of G_K is potentially diagonalisable if, after restricting to $G_{K'}$ for some finite K'/K, it is contained in the same irreducible component of a crystalline deformation ring as a direct sum of characters. In [BLGGT14] automorphy lifting theorems are proved for global representations which are potentially diagonalisable at places above p.

Unfortunately, potential diagonalisability has been established in only a small number of cases. If $K/\mathbb{Q}_p$ is unramified then crystalline representations with Hodge type in the Fontaine–Laffaille range are known to be potentially diagonalisable, cf. [BLGGT14] and [GL14] for the extended Fontaine–Laffaille range. It is also known for ramified K and Barsotti–Tate Hodge types (i.e. those concentrated in degrees $[0, 1]$) by results in [Kis09] and [Gee06].

In [Bar20b] the author extended these results when $K/\mathbb{Q}_p$ is unramified to Hodge types concentrated in degrees $[0, p]$ (also a mild cyclotomic-freeness assumption is required). The aim of this paper is to show how the methods of *loc. cit.* can also be applied when K ramifies. The following are the precise assumptions we require:

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Definition 1.1.1. Let k denote the residue field of K and choose an indexing κ_{ij} for the embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$ so that $\kappa_{ij}|_k = \kappa_{i'j'}|_k$ if and only if $i = i'$. Also let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$.

(1) A Hodge type $\mu = (\mu_\kappa)_{\kappa: K \to \overline{\mathbb{Q}}_p}$ is pseudo-Barsotti–Tate if there exists such an indexing κ_{ij} so that $\mu_{\kappa_{ij}} \subset [0, h_j]$ with $h_1 = p$ and $h_2 = \ldots = h_e = 1$.

(2) A continuous representation $V_{\mathbb{F}}$ of G_K on an $\mathbb{F}$-vector space is cyclotomic-free if there exists an unramified extension K'/K such that every Jordan–Holder factor V of $V_{\mathbb{F}}|_{G_{K'}}$ is one-dimensional, and if V is unramified then $V \otimes \mathbb{F}(-1)$ is not a Jordan–Holder factor of $V_{\mathbb{F}}|_{G_{K'}}$.

Thus, being pseudo–Barsotti–Tate is somewhere between being Barsotti–Tate and being concentrated in degrees $[0, p]$. The cyclotomic-freeness condition avoids possible extensions of the inverse of the cyclotomic character by the trivial representations. For example, the only non-cyclotomic-free two dimensional representations are of the form $\psi \otimes \begin{pmatrix} 1 & * \\ & \chi_{\mathrm{cyc}}^{-1} \end{pmatrix}$ for some unramified character ψ. Note also that cyclotomic-freeness depends only on the representations semi-simplifaction.

Theorem 1.1.2. *Every crystalline representation V of G_K with pseudo-Barsotti–Tate Hodge type and cyclotomic-free residual representation is potentially diagonalisable.*

1.2. **Method.** The typical method for establishing potential diagonalisability is to replace V by $V|_{G'_K}$ for K'/K a sufficiently large unramified extension so that the residual representation becomes a successive extension of one-dimensional representations (such a K' always exists after possibly extending the coefficient field). While V may not itself be ordinary (that is, have every Jordan–Holder factor one-dimensional) one aims to produce an ordinary V' lying on the same irreducible component in the crystalline deformation ring.

For $K/\mathbb{Q}_p$ unramified and Hodge types in the Fontaine–Laffaille range the key input which enables this approach is the observation that the deformation rings in question are formally smooth over $\mathbb{Z}_p$. Unfortunately, this is not the case once one leaves the Fontaine–Laffaille range. In [Bar20b] we addressed this by considering instead Kisin's "resolution" by moduli of Breuil–Kisin modules:

$$\mathcal{L}^{\mu}_{R_{V_{\mathbb{F}}}} \to \operatorname{Spec} R^{\mu}_{V_{\mathbb{F}}}$$

The key calculation was that for μ concentrated in degrees $[0, p]$, $\mathcal{L}^{\mu}$ is formally smooth over $\mathbb{Z}_p$. As this morphism becomes an isomorphism after inverting p, potentialy diagonalisability can then be established by arguing as in the previous paragraph, but with V and V' replaced with points in $\mathcal{L}^{\mu}_{R_{V_{\mathbb{F}}}}$, i.e. after replacing V and V' by their corresponding Breuil–Kisin modules.

When $K/\mathbb{Q}_p$ ramifies the situation is worse still. Even in the case of Barsotti–Tate μ considered in [Kis09] the $\mathcal{L}^{\mu}_{R_{V_{\mathbb{F}}}}$ have normal special fibres but need not be smooth. This normality is sufficient to establish potential diagonalisability in some cases via an explicit construction of paths between points in these spaces. However, this involves some laborious computations. The key idea in this paper is recover smoothness by replacing $\mathcal{L}^{\mu}_{R_{V_{\mathbb{F}}}}$ by a futher "Demazure" type resolution

$$\mathcal{L}^{\mu,\mathrm{conv}}_{R_{V_{\mathbb{F}}}} \to \mathcal{L}^{\mu}_{R_{V_{\mathbb{F}}}}$$

classifying Breuil–Kisin modules together with a specific filtration $\mathcal{F}^\bullet$ on the image of its Frobenius. The key technical result is then:

Theorem 1.2.1. *Assume that μ is pseudo–Barsotti–Tate and $V_{\mathbb{F}}$ is cyclotomic-free (in fact a weaker condition suffices here). Then*

$$\mathcal{L}^{\mu,\mathrm{conv}}_{R_{V_{\mathbb{F}}}} \to \operatorname{Spec} R^{\mu}_{V_{\mathbb{F}}}$$

becomes an isomorphism after inverting p and the local rings of $\mathcal{L}^{\mu,\mathrm{conv}}_{R_{V_{\mathbb{F}}}}$ are formally smooth over $\mathbb{Z}_p$ at closed points.

Once we have this theorem potential diagonalisability follows by an essentially identical argument to that employed in [Bar20b].

The proof of Theorem 1.2.1 is based on a tangent space calculation; we show that at any closed point the tangent space of the special fibre is $\leq$ the dimension of generic fibre. Since the generic fibre identifies with the generic fibre of $\operatorname{Spec} R^{\mu}_{V_{\mathbb{F}}}$ the latter value is well-known. To bound the mod p tangent space we observe that since $\mathcal{L}^{\mu,\mathrm{conv}}_{R_{V_{\mathbb{F}}}}$ is $\mathbb{Z}_p$-flat by definition any mod p tangent vector is induced from an A-valued point for A some finite flat $\mathbb{Z}_p$-algebra. Such an A-valued point corresponds to a filtered Breuil–Kisin module attached to a crystalline representation on a finite A-module. Forgetting the A-action we use a generalisation to dimensions > 2 of a result from [GLS15] to ensure that the reduction modulo p of this filtered Breuil–Kisin module is of a specific form (this is where the restriction to pseudo-Barsotti–Tate Hodge types is crucial). Computing the possible extensions of filtered Breuil–Kisin modules of this specific form produces the desired bound.

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2. Notation

2.1. **General conventions.** Throughout we let K denote a finite extension of $\mathbb{Q}_p$ with residue field a degree f extension k of $\mathbb{F}_p$. Let e denote the ramification degree of K over $\mathbb{Q}_p$ and fix a uniformiser $\pi \in K$. Let G_K denote the absolute Galois group of K. We write $E(u) \in W(k)[u]$ for the minimal polynomial of π over $W(k)$. This is a degree e polynomial with $E(u) \equiv u^e$ modulo π. We also fix a compatible system π^{1/p^∞} of p-th power roots of π inside a completed algebraic closure C of K. We set $K_\infty = K(\pi^{1/p^\infty})$. When $p = 2$ we additionally require that π be as in the following lemma (when $p > 2$ this condition is automatic).

Lemma 2.1.1. *If $p = 2$ then there exists a uniformiser $\pi \in K$ so that $K_\infty \cap K(\mu_{p^\infty}) = K$; here μ_{p^∞} denotes the group of p-th power roots of unity in C.*

Proof. See [Wan17, 2.1]. ☐

2.2. **Coefficients.** We also fix a finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. These play the role of coefficient rings. We assume that E contains a Galois closure of K so that there are ef distinct embeddings $K \hookrightarrow E$. It will be convenient to choose an indexing κ_{ij} of these embeddings by $1 \leq i \leq f$ and $1 \leq j \leq e$ as in Definition 1.1.1 so that

$$\kappa_{ij}|_k = \kappa_{i'j'}|_k \Leftrightarrow i = i'$$

There is an isomorphism $K \otimes_{\mathbb{Q}_p} E \cong \prod_{ij} E$ given by $a \otimes b \mapsto (\kappa_{ij}(a)b)_{ij}$. This allows us to decompose any $K \otimes_{\mathbb{Q}_p} E$-module M as $\prod_{ij} M_{ij}$ where M_{ij} is the submodule of M on which K acts via κ_{ij}.

We emphasise that the identification $K \otimes_{\mathbb{Q}_p} E \cong \prod_{ij} E$ does not descend to $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}$ because the idempotents in $K \otimes_{\mathbb{Q}_p} E$ involve non-integral terms (the only exception being when $K = K_0$). However, we do have a similar decomposition $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O} \cong \prod_i \mathcal{O}$ given by $a \otimes b \mapsto (\kappa_i(a)b)_i$ where $\kappa_i = \kappa_{ij}|_{W(k)}$ (which by construction is independent of j). Thus, every $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$-module M can be decomposed as $M = \prod_i M_i$ where M_i denotes the submodule of M on which $W(k)$ acts through κ_i.

In particular this allows us to refine the construction of $E(u) \in W(k)[u]$ as follows. Define $E_{ij}(u) \in W(k)[u] \otimes_{\mathbb{Z}_p} \mathcal{O}$ as the element corresponding to

$$(\dots 0, \underbrace{u - \kappa_{ij}(\pi)}_{i-th \text{ position}}, 0 \dots) \in \prod_i \mathcal{O}[u]$$

under the identification $W(k)[u] \otimes_{\mathbb{Z}_p} \mathcal{O} \cong \prod_i \mathcal{O}[u]$. We also set $E_j(u) = \prod_{i=1}^f E_{ij}(u)$. Note that $\prod_{ij} E_{ij}(u) = \prod_j E_j(u) = E(u)$.

2.3. **Filtered modules.** A filtered module M over a ring A is a finite A-module equipped with a filtration

$$\dots \subset \mathrm{Fil}^{i+1}(M) \subset \mathrm{Fil}^i(M) \subset \dots$$

by A-submodules of M with A-projective graded pieces and with $\mathrm{Fil}^n(M) = 0$ for $n >> 0$ and $\mathrm{Fil}^n(M) = M$ for $n << 0$. If λ is a multiset of integers then we say M has type λ is $\mathrm{gr}^n(M)$ has constant rank equal to the multiplicity of n in λ. If M' is another filtered A-module write $\mathrm{Hom}_{\mathrm{Fil}}(M, M')$ for the module of A-linear homomorphisms $M \to M'$ equipped with the filtration

$$\mathrm{Fil}^n(\mathrm{Hom}_{\mathrm{Fil}}(M, M')) = \{x : M \to M' \mid x(\mathrm{Fil}^n(M)) \subset \mathrm{Fil}^n(M')\}$$

If M' has type λ' then

$$d(M, M') = \mathrm{rank}_A \frac{\mathrm{Hom}_{\mathrm{Fil}}(M, M')}{\mathrm{Fil}^0(\mathrm{Hom}_{\mathrm{Fil}}(M, M'))}$$

is equal to the $\sum_{x \in \lambda} \mathrm{Card}(\{x' \in \lambda' \mid x > x'\})$. In particular, it depends only on λ and λ' and we write $d(M, M') = d(\lambda, \lambda')$.

2.4. **Hodge types.** A Hodge type μ is a tuple (μ_{ij}) indexed by $1 \leq i \leq f, 1 \leq j \leq e$ of multisets of integers (all of the same cardinality). The decompositions from Section 2.2 allows us to produce, from either of the following two sets of data,

(1) A tuple $D_1, \dots, D_e$ of filtered $k \otimes_{\mathbb{F}_p} \mathbb{F}$-modules.
(2) A filtered $K \otimes_{\mathbb{Q}_p} E$-module.

a tuple of filtered vector spaces indexed by $1 \leq i \leq f, 1 \leq j \leq e$. We say that objects as in either (1) or (2) have Hodge type μ if the ij-th filtered vector space has type μ_{ij}. We also write

$$d(\mu, \mu') = \sum_{ij} d(\mu_{ij}, \mu'_{ij})$$

2.5. **Period rings.** Let $\mathcal{O}_{C^\flat}$ denote the inverse limit of the system

$$\mathcal{O}_C/p \leftarrow \mathcal{O}_C/p \leftarrow \dots$$

whose transition maps are given by $x \mapsto x^p$. This is a domain in characteristic p equipped with an action of G_K induced by that on $\mathcal{O}_C/p$. Its field of fractions $C^\flat$ is algebraically closed (and identifies non-canonically with the completed algebraic closure of $k((u))$). Hence $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$ and $W(C^\flat)$ admit G_K-actions as well as the Witt vector Frobenius. The compatible system π^{1/p^∞} gives rise to an element $\pi^\flat \in \mathcal{O}_{C^\flat}$. Via this choice we embed $\mathfrak{S} = W(k)[[u]] \to A_{\mathrm{inf}}$ by $u \mapsto [\pi^\flat]$. This embedding is φ-equivariant when $\mathfrak{S}$ is equipped with the Frobenius which on $W(k)$ is the Witt vector Frobenius and which sends $u \mapsto u^p$. It is also G_{K_∞}-equivariant when $\mathfrak{S}$ is equipped with the trivial G_{K_∞}-action.

2.6. **Crystalline representations.** A continuous representation of G_K on a finite dimensional E-vector space V is crystalline if $D_{\mathrm{crys}}(V) \coloneqq (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K}$ has K_0-dimension equal to the $\mathbb{Q}_p$-dimension of V. In this case $D_{\mathrm{crys}}(V)$ is a finite free $K_0 \otimes_{\mathbb{Q}_p} E$-module of rank equal to the E-dimension of V. We write $D_{\mathrm{crys},K}(V) = D_{\mathrm{crys}}(V) \otimes_{K_0} K$ which is equipped with the filtration given by

$$\mathrm{Fil}^n D_{\mathrm{crys},K}(V) = (V \otimes_{\mathbb{Q}_p} t^i B_{\mathrm{dR}}^+)^{G_K}$$

and we say V has Hodge type μ is $D_{\mathrm{crys},K}(V)$ has Hodge type μ. Note that our normalisations are such that the cyclotomic-character has Hodge type -1.

3. Moduli of Breuil–Kisin modules

3.1. **Basic definitions.** For any $\mathcal{O}$-algebra A set $\mathfrak{S}_A = \mathfrak{S} \otimes_{\mathbb{Z}_p} A$ and write φ for the A-linear extension of φ on $\mathfrak{S}$. Recall also the elements $E_j(u) = \prod_{i=1}^f E_{ij}(u)$ in $W(k)[u] \otimes_{\mathbb{Z}_p} \mathcal{O} \subset \mathfrak{S}_{\mathcal{O}}$ from Section 2.2. In this paper only the case of A finite over $\mathcal{O}$ will be relevant.

Definition 3.1.1. Consider integers $h_j \geq 0$ for $j = 1, \dots, e$. A Breuil–Kisin module over A of height $\leq h_j$ is a finite projective $\mathfrak{S}_A$-module $\mathfrak{M}$ equipped with an $\mathfrak{S}_A$-linear homomorphism

$$\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \to \mathfrak{M}$$

with cokernel killed by $\prod_{j=1}^e E_j(u)^{h_j}$.

For any such $\mathfrak{M}$ write $\mathfrak{M}^\varphi$ for the image of this homomorphism and $\varphi(\mathfrak{M})$ for the image of the composite $\mathfrak{M} \to \mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \to \mathfrak{M}$ with the first map given by $m \mapsto m \otimes 1$. Note that $\varphi(\mathfrak{M})$ is a $\varphi(\mathfrak{S}_A)$-module which generates $\mathfrak{M}^\varphi$ over $\mathfrak{S}_A$.

Lemma 3.1.2. *If A is $\mathcal{O}$-finite and $\mathfrak{M}$ is a Breuil–Kisin module over A of height $\leq h_j$ then both $\mathfrak{M}/\mathfrak{M}^\varphi$ and $\mathfrak{M}^\varphi / \prod E_j(u)^{h_j} \mathfrak{M}$ are finite projective A-modules.*

Proof. From the exact sequence

$$0 \to \mathfrak{M}^\varphi / (\prod_{i=1}^e E_j(u)^{h_j})\mathfrak{M} \to \mathfrak{M} / (\prod_{i=1}^e E_j(u)^{h_j})\mathfrak{M} \to \mathfrak{M}/\mathfrak{M}^\varphi \to 0$$

it is enough to consider $\mathfrak{M}/\mathfrak{M}^\varphi$. That this is finite projective over A is proven in [CEGS19, 3.4.1]. ☐

3.2. **Recalling a construction of Kisin.** Now fix a continuous representation of G_K on a finite dimension $\mathbb{F}$-vector space $V_{\mathbb{F}}$ together with a choice of $\mathbb{F}$-basis and let $R = R_{V_{\mathbb{F}}} = R^{\square}_{V_{\mathbb{F}}} \otimes_{W(k)} \mathcal{O}$ denote the corresponding $\mathcal{O}$-framed deformation ring. Write V_R for the universal deformation and for any homomorphism $\alpha : R \to A$ write $V_A = V_\alpha = V_R \otimes_{\alpha,R} A$.

Construction 3.2.1. For each $h_j \geq 0$ and each Artin $\mathcal{O}$-algebra with finite residue field set $\mathcal{L}^{\leq h_j}(A)$ equal to the set pairs $(\mathfrak{M}, \alpha)$ where $\alpha : R \to A$ is a homomorphism and $\mathfrak{M}$ is a G_{K_∞}-stable $\mathfrak{S}_A$-submodule of $V_A \otimes_{\mathbb{Z}_p} W(C^\flat)$ with

$$\mathfrak{M} \otimes_{\mathfrak{S}} W(C^\flat) = V_A \otimes_{\mathbb{Z}_p} W(C^\flat) \tag{3.2.2}$$

and for which the semilinear extension of the trivial Frobenius on V_A makes $\mathfrak{M}_A$ into a Breuil–Kisin module over A of height $\leq h_j$. Base-change along A make this into a functor on Artin $\mathcal{O}$-algebras.

For any Hodge type μ let R^μ denote the unique $\mathcal{O}$-flat reduced quotient of R with the property that a homomorphism $R \to B$ into a finite E-algebra factors through R^μ if and only if V_B is crystalline of Hodge type μ. The existence of such a quotient is the main result of [Kis08]. If we assume μ is concentrated in degrees $[0, h_j]$ then have the following:

Proposition 3.2.3 (Kisin)**.** *The functor $A \mapsto \mathcal{L}^{\leq h_j}(A)$ is represented by a scheme $\mathcal{L}^{\leq h_j}_R$ and the morphism $\Theta : \mathcal{L}^{\leq h_j}_R \to \operatorname{Spec} R$ given by $(\mathfrak{M}, \alpha) \mapsto \alpha$ is projective. Furthermore $\Theta[\frac{1}{p}]$ is a closed immersion and $\operatorname{Spec} R^\mu \to \operatorname{Spec} R$ factors through the scheme-theoretic image of Θ.*

Proof. When each $h_j = h$ then this follows from [Kis08] (in particular see 1.5.1 and 1.6.4 of *loc. cit.*). The construction of $\mathcal{L}^{\leq h}_R$ also shows that for any $h_j \leq h$ both $A \mapsto (\prod_{i=1}^e E_j(u)^{h_j})\mathfrak{M}/E(u)^h\mathfrak{M}$ and $A \mapsto \mathfrak{M}/\mathfrak{M}^\varphi$ extend to shaves of $\mathcal{O}_{\mathcal{L}^{\leq h}} \otimes_{\mathbb{Z}_p} \mathfrak{S}$-modules which are coherent as $\mathcal{O}_{\mathcal{L}^{\leq h}_R}$-modules. Lemma 3.1.2 shows they are also locally free. As a consequence, the locus of $\mathcal{L}^{\leq h}_R$ over which $(\prod_{i=1}^e E_j(u)^{h_j})\mathfrak{M} \subset \mathfrak{M}^\varphi$ is closed. This is precisely $\mathcal{L}^{\leq h_j}_R$. It remains only to show that if μ is concentrated in degrees $[0, h_j]$ then it factors through $\mathcal{L}^{\leq h_j}_R$. This follows from part (4) of Lemma 5.1.2 which is proven in Section 5.1. □

An easy limit argument shows that the description of the A-points of $\mathcal{L}^{\leq h_j}_R$ is valid whenever A is a finite $\mathcal{O}$-algebra (i.e. not necessarily Artinian).

3.3. **A convolution variant.** We now produce a variant of $\mathcal{L}^{\leq h_j}_R$. For any Artin R-algebra A set $\mathcal{L}^{\leq h_j, \mathrm{conv}}(A)$ equal to the set of triples $(\mathfrak{M}, \alpha, \mathcal{F})$ for which $(\mathfrak{M}, \alpha) \in \mathcal{L}^{\leq h_j}(A)$ and $\mathcal{F}$ is a sequence of $\mathfrak{S}_A$-submodules

$$\left(\prod_{j=1}^e E_j(u)^{h_j}\right)\mathfrak{M} = \mathcal{F}^e \subset \ldots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = \mathfrak{M}^\varphi \tag{3.3.1}$$

with $E_j(u)^{h_j}\mathcal{F}^j \subset \mathcal{F}^j \subset \mathcal{F}^{j-1}$ and $\mathcal{F}^{i-1}/\mathcal{F}^i$ finite projective over A for each j.

Proposition 3.3.2. *The functor $A \mapsto \mathcal{L}^{\leq h_j, \mathrm{conv}}(A)$ is represented by a scheme $\mathcal{L}^{\leq h_j, \mathrm{conv}}_R$. Furthermore, the morphism $\mathcal{L}^{\leq h_j, \mathrm{conv}}_R \to \mathcal{L}^{\leq h_j}_R$ given by $(\mathfrak{M}, \alpha, \mathcal{F}) \mapsto (\mathfrak{M}, \alpha)$ is projective and, after inverting p, becomes a closed immersion through which the reduced subscheme of $\mathcal{L}^{\leq h_j}_R[\frac{1}{p}]$ factors.*

Proof. The representability of $\mathcal{L}^{\leq h_j,\mathrm{conv}}$ follows from the observation made in the proof of Proposition 3.2.3 that $A \mapsto \mathfrak{M}^\varphi/(\prod E_j(u)^{h_j})\mathfrak{M}$ extends to a coherent locally free sheaf on $\mathcal{L}_R^{\leq h_j}$. Indeed, this shows that $\mathcal{L}^{\leq h_j,\mathrm{conv}}$ can be constructed as a succession of extensions of Grassmannians over $\mathcal{L}_R^{\leq h_j}$.

To show that $\mathcal{L}_R^{\leq h_j,\mathrm{conv}} \to \mathcal{L}_R^{\leq h_j}$ becomes a closed immersion after inverting p it suffices, since this morphism is proper, to show the induced map on B-valued points is injective for any finite E-algebra B. As explained in e.g. the first paragraph of the proof of [Kis08, 1.6.4], any B-valued point is induced from an A-valued point for some finite flat $\mathcal{O}$-algebra A. Thus, it suffices to show that for any such A and any A-valued point $(\mathfrak{M},\alpha)$ of $\mathcal{L}_R^{\leq h_j}$ there exists at most one filtration $\mathcal{F}$ as in (3.3.1). Since $\mathfrak{S}_E$ is a principal ideal domain and the $E_j(u)$ are pairwise coprime

$$\mathfrak{S}_E/(\prod E_j(u)^{h_j}) \cong \prod \mathfrak{S}_E/E_j(u)^{h_j}$$

From this it is immediate that after inverting p a unique such filtration $\mathcal{F}'$ exists. Then $\mathcal{F}^i \subset \mathcal{F}'^i$ and $\mathcal{F}^i[\frac{1}{p}] = \mathcal{F}'$. This implies $\mathcal{F}^i = \mathcal{F}'^i \cap \mathfrak{M}^\varphi$, since $\mathfrak{M}^\varphi/\mathcal{F}^i$ is p-torsionfree, and so $\mathcal{F}$ is unique as claimed.

Note that if A is the ring of integers in a finite extension of E then $\mathcal{F}'^i \cap \mathfrak{M}^\varphi$ defines a filtration as in (3.3.1) because the graded pieces, being p-torsionfree, are A-projective (this need not be the case for more general A). This shows that $\mathcal{L}_R^{\leq h_j,\mathrm{conv}} \to \mathcal{L}_R^{\leq h_j}$ induces a bijection on A-valued points for such A. It follows from e.g. [LLHLM20, 7.2.5] that $\mathcal{L}_R^{\leq h_j,\mathrm{conv}} \to \mathcal{L}_R^{\leq h_j}$ induces an isomorphism on underlying reduced subschemes. □

Remark 3.3.3. In [Bar20b, 2.2.16] it is explained how $\mathcal{L}_R^\mu \to \operatorname{Spec} R^\mu$ induces a bijection on $\mathcal{O}'$-valued points when $\mathcal{O}'$ is the ring of integers inside a finite extension E'/E. The last paragraph of the previous proof shows that the same is true with $\mathcal{L}_R^\mu$ replaced by $\mathcal{L}_R^{\mu,\mathrm{conv}}$. We emphasise that this is not true when $\mathcal{O}'$ is an arbitrary finite flat $\mathcal{O}$-algebra (otherwise $\mathcal{L}_R^{\mu,\mathrm{conv}} \to \operatorname{Spec} R^\mu$ would be an isomorphism).

Since $\operatorname{Spec} R^\mu[\frac{1}{p}]$ is reduced Proposition 3.3.2 implies that it can be viewed as a closed subscheme of $\mathcal{L}_R^{\leq h_j,\mathrm{conv}}[\frac{1}{p}]$. This allows us to make the following definition.

Definition 3.3.4. For any Hodge type μ concentrated in degrees $[0,h_j]$ define $\mathcal{L}_R^{\mu,\mathrm{conv}}$ as the closure of $\operatorname{Spec} R^\mu[\frac{1}{p}]$ in $\mathcal{L}_R^{\leq h_j,\mathrm{conv}}$

The main object of this paper is to describe the local geometry of $\mathcal{L}_R^{\mu,\mathrm{conv}}$ under the assumptions from Definition 1.1.1.

Theorem 3.3.5 (Main theorem)**.** *Assume that*

- *μ is pseudo-Barsotti–Tate, i.e. that μ is concentrated in degrees $[0,h_j]$ for $h_1 = p$ and $h_2 = \ldots = h_e = 1$*
- *For any G_{K_∞}-stable submodule $V \subset V_{\mathbb{F}}$ which is unramified there exists no G_{K_∞}-equivariant quotient $V_{\mathbb{F}} \to W$ with $W \cong V \otimes \mathbb{F}(-1)$.*

Then the local rings of $\mathcal{L}^{\mu,\mathrm{conv}}$ at closed points are formally smooth over $\mathbb{Z}_p$.

In Lemma 6.2.6 we explain why condition (2) is satisfied whenever $V_{\mathbb{F}}$ is cyclotomic-free in the sense of Definition 1.1.1.

Proof granting the results of Section 6.1. Let $x \in \mathcal{L}_R^{\mu,\mathrm{conv}}$ any closed point. Enlarging $\mathbb{F}$ if necessary we can assume that x is an $\mathbb{F}$-valued point. We show in Proposition 6.3.1 below that the tangent space of $\mathcal{L}_R^{\mu,\mathrm{conv}} \otimes_{\mathcal{O}} \mathbb{F}$ at x has dimension

$$\leq d^2 + d(\mu,\mu)$$

(recall $d(\mu,\mu)$ is the value described in Section 5.3.1). On the other hand, in [Kis08, 3.3.8] it is shown that $R^\mu[\frac{1}{p}]$ is equidimensional of dimension $d^2 + d(\mu,\mu)$. The same is therefore true of $\mathcal{L}_R^{\mu,\mathrm{conv}}[\frac{1}{p}]$ and so, by flatness, also for $\mathcal{L}_R^{\mu,\mathrm{conv}} \otimes_{\mathcal{O}} \mathbb{F}$. This shows that the above inequality is an equality and that the local rings of $\mathcal{L}_R^{\mu,\mathrm{conv}} \otimes_{\mathcal{O}} \mathbb{F}$ at closed points are regular. Since the local rings of $\mathcal{L}_R^{\mu,\mathrm{conv}}$ are $\mathbb{Z}_p$-flat by definition it follows from [Sta17, 07NQ] that they are formally smooth over $\mathbb{Z}_p$. ∎

Remark 3.3.6. One could also consider the closure of $\operatorname{Spec} R^\mu[\frac{1}{p}]$ in $\mathcal{L}^{\leq h_j}$. However the schemes obtained in this way typically fail to be regular. For example, it is shown in [Kis09] that when μ is concentrated in degrees $[0,1]$ these spaces are smoothly equivalent to certain local models defined in [PR09] which, while normal, are not necessarily smooth.

4. Strongly divisible extensions

4.1. **Categories of mod p Breuil–Kisin modules.** Let $\mathrm{Mod}_{\mathcal{F}}$ denote the category whose objects are pairs $(\mathfrak{M},\mathcal{F})$ with $\mathfrak{M}$ a Breuil–Kisin module over $\mathbb{F}$ of any height ≥ 0 and $\mathcal{F}$ a sequence of $\mathfrak{S}_{\mathbb{F}}$-submodules

$$u^{e+p-1}\mathfrak{M} = \mathcal{F}^e \subset \ldots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = \mathfrak{M}^\varphi$$

satisfying $u^p\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^0$ and $u\mathcal{F}^{i-1} \subset \mathcal{F}^i \subset \mathcal{F}^{i-1}$ for $i = 2,\ldots,e$. Morphisms $\mathrm{Hom}_{\mathcal{F}}(\mathfrak{P},\mathfrak{M})$ in this category are φ-equivariant maps of $\mathfrak{S}_{\mathbb{F}}$-modules respecting the filtrations.

Definition 4.1.1. Let $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ denote the full subcategory of $\mathrm{Mod}_{\mathcal{F}}$ consisting of those $(\mathfrak{M},\mathcal{F})$ for which there exists an $\mathbb{F}_p[[u^p]]$-basis (e_i) of $\varphi(\mathfrak{M})$ and integers $r_i \in [0,p]$ such that $\mathcal{F}^1$ is generated by $(u^{r_i}e_i)$.

Remark 4.1.2. When $e = 1$ the category $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ is precisely the category denoted in $\mathrm{Mod}_k^{\mathrm{SD}}$ and studied in [Bar20a].

The following is another interpretation of $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$. For $(\mathfrak{M},\mathcal{F}) \in \mathrm{Mod}_{\mathcal{F}}$ define

- $\mathrm{Fil}^i(\mathfrak{M}^\varphi) = \mathfrak{M}^\varphi \cap u^i\mathcal{F}^1$.
- $\mathrm{Fil}^i(\overline{\mathfrak{M}})$ equal to the image of $\mathrm{Fil}^i(\mathfrak{M}^\varphi)$ in $\overline{\mathfrak{M}} = \mathfrak{M}^\varphi/u\mathfrak{M}^\varphi$.
- $F^i(\mathfrak{M}) = \varphi(\mathfrak{M}) \cap u^i\mathcal{F}^1$

Lemma 4.1.3. $(\mathfrak{M},\mathcal{F}) \in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ *if and only if image of the composite*

$$F^i(\mathfrak{M}) \to \mathfrak{M}^\varphi \to \overline{\mathfrak{M}}$$

is $\mathrm{Fil}^n(\overline{\mathfrak{M}})$ *for all* n.

Proof. If $(\mathfrak{M},\mathcal{F}) \in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ then choose (e_i) and (r_i) as in Definition 4.1.1. We see that $\mathrm{Fil}^n(\mathfrak{M}^\varphi)$ is generated over $\mathbb{F}_p[[u]]$ by $u^{\min\{n+r_i,0\}}e_i$. Therefore $\mathrm{Fil}^n(\overline{\mathfrak{M}})$ is generated by the images of those e_i with $n + r_i \leq 0$. This shows that the image of the composite in the lemma surjects onto $\mathrm{Fil}^n(\overline{\mathfrak{M}})$.

For the converse, choose an $\mathbb{F}_p$-basis of $\overline{\mathfrak{M}}$ adapted to the filtration, i.e. choose a basis $(\overline{e}_i)$ and integers (r_i) so that $\mathrm{Fil}^n(\overline{\mathfrak{M}})$ is generated by those $\overline{e}_i$ for which $n + r_i \leq 0$. By assumption we can find $e_i \in \varphi(\mathfrak{M}) \cap u^{r_i}\mathcal{F}^1$ whose image in $\overline{\mathfrak{M}}$ is $\overline{e}_i$. Clearly any such choice of e_i produces an $\mathbb{F}_p[[u^p]]$-basis of $\varphi(\mathfrak{M})$. To see that $\mathcal{F}^1$ is generated by $u^{r_i}e_i$ we argue by (decreasing) induction on the smallest integer n such that $u^n x \in \mathfrak{M}^\varphi$. Since $u^n x \in \mathrm{Fil}^n(\mathfrak{M}^\varphi)$ we can write

$$u^n x \equiv \sum_{r_i + n \leq 0} \alpha_i e_i \mod u\mathfrak{M}^\varphi$$

and so $x \equiv \sum_{r_i \leq -n} \alpha_i u^{-n} e_i \mod u^{1-n}\mathfrak{M}^\varphi$. Using the inductive hypothesis we deduce x can be expressed as an $\mathbb{F}_p[[u]]$-linear combination of the $u^{r_i}e_i$ as claimed. □

4.2. **Properties of exact sequences in** $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$**.** We say that a sequence of morphisms

$$0 \to (\mathfrak{M}, \mathcal{E}) \to (\mathfrak{N}, \mathcal{F}) \to (\mathfrak{P}, \mathcal{G}) \to 0$$

in $\mathrm{Mod}_{\mathcal{F}}$ is exact if the induced sequences $0 \to \mathcal{E}^i \to \mathcal{F}^i \to \mathcal{G}^i \to 0$ are exact for all i when viewed as sequences of $\mathfrak{S}_{\mathbb{F}}$-modules.

Proposition 4.2.1. *Suppose* $(\mathfrak{N}, \mathcal{F}) \in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$.

1. *Then* $(\mathfrak{M}, \mathcal{E})$ *and* $(\mathfrak{P}, \mathcal{G}) \in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$.
2. *The induced sequences* $0 \to \mathrm{gr}^i(\overline{\mathfrak{M}}) \to \mathrm{gr}^i(\overline{\mathfrak{N}}) \to \mathrm{gr}^i(\overline{\mathfrak{P}}) \to 0$ *are exact for each* i.
3. *There exists an* $\mathfrak{S}_{\mathbb{F}}$*-linear splitting* $s : \mathfrak{P} \to \mathfrak{N}$ *such that* $s(\mathcal{G}^1) \subset \mathcal{F}^1$ *and* $s(\varphi(\mathfrak{P})) \subset \varphi(\mathfrak{N})$.

Parts (1) and (2) of this proposition were proved in [Bar20a] in the case $e = 1$. For the general case we observe that the condition in Definition 4.1.1 only refers to the relative positions of $\mathcal{F}^1$ and $\varphi(\mathfrak{M})$; in particular it is a condition on the image of the Frobenius morphism rather than the morphism itself.

Proof. To reduce to the case $e = 1$ we produce a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & u^{-p}\mathcal{E}^1 \otimes_\varphi \varphi(\mathfrak{S}_{\mathbb{F}}) & \to & u^{-p}\mathcal{F}^1 \otimes_\varphi \varphi(\mathfrak{S}_{\mathbb{F}}) & \to & u^{-p}\mathcal{G}^1 \otimes_\varphi \varphi(\mathfrak{S}_{\mathbb{F}}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varphi(\mathfrak{M}) & \longrightarrow & \varphi(\mathfrak{N}) & \longrightarrow & \varphi(\mathfrak{P}) & \longrightarrow & 0 \end{array}$$

with the vertical arrows being isomorphisms of $\varphi(\mathfrak{S}_{\mathbb{F}})$-modules. As the vertical arrows go between projective $\varphi(\mathfrak{S}_{\mathbb{F}})$-modules of the same rank the outer arrows can be chosen arbitrarily and, after choosing an $\varphi(\mathfrak{S}_{\mathbb{F}})$-linear splitting of the top exact sequence, this determines the central vertical arrow. In the language of [Bar20a], this makes $0 \to u^{-p}\mathcal{E}^1 \to u^{-p}\mathcal{F}^1 \to u^{-p}\mathcal{G}^i \to 0$ into an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}$ with $u^{-p}\mathcal{F}^1$ strongly divisible (as in [Bar20a, 5.2.9]). Applying [Bar20a, 5.4.6] we deduce (1) and (2).

Now we address (3). Note that (2) ensures we can choose a $k \otimes_{\mathbb{F}_p} \mathbb{F}$-linear splitting $\overline{s}$ of $\overline{\mathfrak{N}} \to \overline{\mathfrak{P}}$ mapping $\mathrm{Fil}^i(\overline{\mathfrak{P}})$ into $\mathrm{Fil}^i(\overline{\mathfrak{N}})$ by choosing successive splittings of the surjections $\mathrm{gr}^i(\overline{\mathfrak{N}}) \to \mathrm{gr}^i(\overline{\mathfrak{P}})$. From $\overline{s}$ we obtain a splitting

$$s : \varphi(\mathfrak{P}) = \overline{\mathfrak{P}} \otimes_{k \otimes_{\mathbb{F}_p} \mathbb{F}} \varphi(\mathfrak{S}_{\mathbb{F}}) \to \overline{\mathfrak{N}} \otimes_{k \otimes_{\mathbb{F}_p} \mathbb{F}} \varphi(\mathfrak{S}_{\mathbb{F}}) = \varphi(\mathfrak{N})$$

of $\varphi(\mathfrak{N}) \to \varphi(\mathfrak{P})$ which we claim maps $\mathcal{G}^1$ into $\mathcal{F}^1$. For this we first show that

$$s(F^n(\mathfrak{P})) \subset F^n(\mathfrak{N})$$

for $n \leq p$ As $\overline{s}$ is compatible with the filtrations on $\overline{\mathfrak{P}}$ and $\overline{\mathfrak{N}}$ it follows that the image of $F^n(\mathfrak{P}) := \varphi(\mathfrak{P}) \cap u^n \mathcal{G}^1$ under s in $\overline{\mathfrak{N}}$ is contained in $\mathrm{Fil}^i(\overline{\mathfrak{N}})$. Lemma 4.1.3 therefore implies that if $x \in F^n(\mathfrak{P})$ then $s(x) = x_1 + u^p x_2$ for $x_1 \in F^n(\mathfrak{N})$ and $x_2 \in \varphi(\mathfrak{N})$. If $n \leq p$ then $u^p x_2 \in F^n(\mathfrak{N})$ and so $s(x) \in F^n(\mathfrak{N})$ also. This establishes the displayed inclusion above. To show $s(\mathcal{G}^1) \subset \mathcal{F}^1$ note that by (1) we have $\mathfrak{P} \in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ and so there is a basis (e_i) of $\varphi(\mathfrak{P})$ as in Definition 4.1.1. Since $e_i \in u^{-r_i}\mathcal{G}^1$ we have $s(e_i) \in u^{-r_i}\mathcal{F}^1$ and so

$$s(u^{r_i} e_i) = u^{r_i} s(e_i) \in \mathcal{F}^1$$

which finishes the proof. □

4.3. **Ext group via an explicit complex.** For $\mathfrak{M} = (\mathfrak{M}, \mathcal{E})$ and $\mathfrak{P} = (\mathfrak{P}, \mathcal{G})$ in $\mathrm{Mod}_{\mathcal{F}}$ consider the first Yoneda extension group $\mathrm{Ext}^1_{\mathcal{F}}(\mathfrak{P}, \mathfrak{M})$ in $\mathrm{Mod}_{\mathcal{F}}$, i.e. the set of exact sequences

$$0 \to \mathfrak{M} \to (\mathfrak{N}, \mathcal{F}) \to \mathfrak{P} \to 0, \qquad (\mathfrak{N}, \mathcal{F}) \in \mathrm{Mod}_{\mathcal{F}}$$

as considered in the previous section, modulo the equivalence relation identifying two sequences if and only if there exists a morphism α in $\mathrm{Mod}_{\mathcal{F}}$ making the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M} & \longrightarrow & (\mathfrak{N}, \mathcal{F}) & \longrightarrow & \mathfrak{P} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{M} & \longrightarrow & (\mathfrak{N}', \mathcal{F}') & \longrightarrow & \mathfrak{P} & \longrightarrow & 0 \end{array}$$

We define $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M}) \subset \mathrm{Ext}^1_{\mathcal{F}}(\mathfrak{P}, \mathfrak{M})$ as the subset consisting of those classes which can be represented by exact sequences as above with $(\mathfrak{N}, \mathcal{F}) \in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$. Proposition 4.2.1 implies that $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M})$ is empty unless $\mathfrak{P}, \mathfrak{M} \in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$.

Notation 4.3.1. In what follows, for any pair of $\mathfrak{S}_{\mathbb{F}}$-modules, we write $\mathrm{Hom}(-,-)$ for the set of $\mathfrak{S}_{\mathbb{F}}$-linear maps. In the special case of $\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ we equip this set with the Frobenius given by $\varphi(h) = \varphi_{\mathfrak{M}} \circ h \circ \varphi_{\mathfrak{P}}^{-1}$.

For $\mathfrak{M}$ and $\mathfrak{P}$ as above (we emphasis that for this definition we do not require $\mathfrak{M}, \mathfrak{P} \in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$) consider the submodule

$$\mathcal{C}^1_{\mathrm{SD}} \subset \prod_1^{e-1} \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})[\tfrac{1}{u}]$$

consisting of those $(g_1, \ldots, g_{e-1})$ satisfying $g_i(\mathcal{G}^{i+1}) \subset \mathcal{E}^i$ and $g_i(u\mathcal{G}^i) \subset \mathcal{E}^{i+1}$ for each i. This fits into a two-term complex

$$\mathcal{C}_{\mathrm{SD}} : F^0(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})) \times \prod_{i=\min\{e,2\}}^{e-1} \mathrm{Hom}(\mathcal{G}^i, \mathcal{E}^i) \xrightarrow{d_{\mathrm{SD}}} \mathcal{C}^1_{\mathrm{SD}}$$
$$(h_1, \ldots, h_{e-1}) \mapsto (h_2 - h_1, h_3 - h_2, \ldots, \varphi^{-1}(h_1) - h_{e-1})$$

where, as for objects in $\mathrm{Mod}_{\mathcal{F}}$, we write $F^i(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ for the set of $h \in \varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ mapping $\mathcal{G}^1$ into $u^i\mathcal{E}^1$

Remark 4.3.2. When $e = 1$ the above formula does not strictly make sense. Instead d_{SD} should be defined as $d_{\mathrm{SD}}(h_1) = \varphi^{-1}(h_1) - h_1$. In this case we recover the complex considered in [Bar18, 4.1].

. The following lemma motivates the construction of $\mathcal{C}_{\mathrm{SD}}$.

Lemma 4.3.3. $H^0(\mathcal{C}_{\mathrm{SD}}) = \mathrm{Hom}_{\mathcal{F}}(\mathfrak{P}, \mathfrak{M})$ *and there is an injection*

$$\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M}) \to H^1(\mathcal{C}_{\mathrm{SD}})$$

(in fact it is a bijection if $\mathfrak{P}, \mathfrak{M} \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ *we but only require injectivity for our applications).*

Proof. The first assertion is easy so we focus on the second. To construct the injection begin by considering an exact sequence $0 \to \mathfrak{M} \to (\mathfrak{N}, \mathcal{F}) \to \mathfrak{P} \to 0$ in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$. For each i we can choose $\mathfrak{S}_{\mathbb{F}}$-linear splittings s_i of $\mathcal{F}^i \to \mathcal{G}^i$ for $i = 1, \ldots, e-1$. Proposition 4.2.1 allows us to assume that s_1 maps $\varphi(\mathfrak{P})$ into $\varphi(\mathfrak{N})$. As such $\varphi^{-1}(s_1)$ maps $\mathfrak{P}$ into $\mathfrak{M}$ and so also $\mathcal{G}^e$ into $\mathcal{F}^e$. It is then immediate that

$$g = (s_2 - s_1, s_3 - s_2, \ldots, \varphi^{-1}(s_1) - s_{e-1})$$

defines an element in $\mathcal{C}^1_{\mathrm{SD}}$. Suppose that s_i' is another choice of splittings with corresponding $g' \in \mathcal{C}^1_{\mathrm{SD}}$. Then $s_i' - s_i \in \mathrm{Hom}(\mathcal{G}^i, \mathcal{E}^i)$ and $s_1 - s_1' \in F^0(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})))$. This shows that $g - g'$ is contained in the image of d_{SD} so we obtain a well-defined element of $H^1(\mathcal{C}_{\mathrm{SD}})$.

Now suppose $0 \to \mathfrak{M} \to (\mathfrak{N}, \mathcal{F}) \to \mathfrak{P} \to 0$ and $0 \to \mathfrak{M} \to (\mathfrak{N}', \mathcal{F}') \to \mathfrak{P} \to 0$ are exact sequences mapping, by the construction from the previous subsection, into the same element $H^1(\mathcal{C}_{\mathrm{SD}})$. Then each exact sequence admits splittings s_i and s_i' so that

$$(s_2 - s_1, s_3 - s_2, \ldots, \varphi^{-1}(s_1) - s_{e-1}) = (s_2' - s_1', s_3' - s_2', \ldots, \varphi^{-1}(s_1') - s_{e-1}')$$

Equivalently

$$s_1 - s_1' = s_2 - s_2' = \ldots = s_{e-1} - s_{e-1}' = \varphi^{-1}(s_1) - \varphi^{-1}(s_1')$$

For $n \in \mathfrak{N}$ write $\overline{n}$ for its image in $\mathfrak{P}$ and consider the map $\alpha : \mathfrak{N} \to \mathfrak{N}'$ given by

$$n \mapsto n - s_1(\overline{n}) + s_1'(\overline{n})$$

Note this makes sense because $n - s_1(\overline{n}) \in \mathcal{E}^1$ and so can be viewed as an element of $\mathcal{F}'^1$. The fact that $s_1 - s_1' = \varphi^{-1}(s_1) - \varphi^{-1}(s_1')$ shows this map is φ-equivariant. The fact that $s_1 - s_1' = s_i - s_i'$ implies $\mathcal{F}^i$ is mapped into $\mathcal{F}'^i$. Therefore α is a morphism in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ which shows our two exact sequences define the same element in $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M})$. As a consequence the construction from the first paragraph produces an injection $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M}) \to H^1(\mathcal{C}_{\mathrm{SD}})$ as desired. □

4.4. **Dimension calculations.** For $\mathfrak{M}$, $\mathfrak{P} \in \mathrm{Mod}_{\mathcal{F}}$ write

$$\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})_k := \varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))/u^p$$

and let $F^i(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})_k)$ denote the image of $F^i(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ in $\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})_k$.

Proposition 4.4.1. *The cohomology of* $\mathcal{C}_{\mathrm{SD}}$ *is* $\mathbb{F}$*-finite and if*

$$\chi(\mathfrak{P}, \mathfrak{M}) := \dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD}}) - \dim_{\mathbb{F}} H^0(\mathcal{C}_{\mathrm{SD}})$$

then

$$\chi(\mathfrak{P}, \mathfrak{M}) = \dim_{\mathbb{F}} \frac{\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})_k}{F^0(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})_k)} + \sum_{i=1}^{e-1} \dim_{\mathbb{F}} \mathrm{Hom}(\mathcal{G}^{i+1}/u\mathcal{G}^i, \mathcal{E}^i/\mathcal{E}^{i+1})$$

provided that $\mathrm{gr}(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})_k) = 0$ *for* $i < -p$.

We begin by proving the proposition under the following assumptions: (i) every $h \in \varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ maps $\mathcal{G}^i$ into $\mathcal{E}^i$ for every i and (ii) $\varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})) \subset u\,\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$.

Lemma 4.4.2. *Proposition 4.4.1 holds under assumptions (i) and (ii).*

Proof. Assumption (ii) implies that u-adically $\varphi^n(h) \to 0$ for every $h \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$. From this we deduce $\varphi - 1$ is an $\mathbb{F}$-linear automorphism of $\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$. Injectivity is clear and, for surjectivity, if $h \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ then $\varphi - 1$ sends $-\sum_{n \geq 0} \varphi^n(h)$ onto h. From injectivity of $\varphi - 1$ we deduce that $H^0(\mathcal{C}_{\mathrm{SD}}) = 0$.

Since $H^0(\mathcal{C}_{\mathrm{SD}}) = 0$ the map $d_{\mathrm{SD}} : \mathcal{C}^0_{\mathrm{SD}} \to \mathcal{C}^1_{\mathrm{SD}}$ is injective. However, assumption (i) also allows us to view $\mathcal{C}^0_{\mathrm{SD}} \subset \mathcal{C}^1_{\mathrm{SD}}$ via obvious map $(h_i) \mapsto (h_i)$, Furthermore, the cokernel of this second inclusion naturally identifies with

$$\mathcal{H}^1 := \prod_{i=1}^{e-1} \mathrm{Hom}(\mathcal{G}^{i+1}/u\mathcal{G}_i, \mathcal{E}^i/\mathcal{E}^{i+1})$$

To relate the cokernel of d_{SD} with $\mathcal{H}^1$ we refine $\mathcal{C}^0_{\mathrm{SD}} \subset \mathcal{C}^1_{\mathrm{SD}}$ to a sequence

$$\ldots \subset \mathcal{C}^{-1}_{\mathrm{SD}} \subset \mathcal{C}^0_{\mathrm{SD}} \subset \mathcal{C}^1_{\mathrm{SD}}$$

by defining $\mathcal{C}^j_{\mathrm{SD}} \subset \mathcal{C}^0_{\mathrm{SD}}$ as the subset consisting of those $(g_i) \in \mathcal{C}^0_{\mathrm{SD}}$ for which $g_i(\mathcal{G}^{i+j'}) \subset \mathcal{E}^{i+j'}$ for all $0 \leq j' \leq -j$. By assumption (i) $\varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ maps $\mathcal{G}_i$ into $\mathcal{E}_i$ for each i. This implies d_{SD} induces complexes:

$$\mathcal{C}_{\mathrm{SD},j} : \mathcal{C}^{j-1}_{\mathrm{SD}} \to \mathcal{C}^j_{\mathrm{SD}}$$

For $j \leq 0$ we can also define maps $\mathcal{C}^j_{\mathrm{SD}} \to \mathcal{H}^1$ by

$$(h_i) \mapsto (-1)^{-j+1}(\underbrace{0, \ldots, 0}_{-j+1}, \overline{h}_2, \ldots, \overline{h}_{e+j-1})$$

(here $\overline{h}_i$ denotes the image of h in $\mathrm{Hom}(\mathcal{G}^{i-j+1}/u\mathcal{G}^{i-j}, \mathcal{E}^{i-j}/\mathcal{E}^{i-j+1})$). A short computation shows that

$$\begin{tikzcd} \mathcal{C}^j_{\mathrm{SD}} \arrow[r] & \mathcal{H}_1 \\ \mathcal{C}^{j-1}_{\mathrm{SD}} \arrow[r] \arrow[u] & \mathcal{H}_1 \arrow[u] \end{tikzcd}$$

commutes for all j. If $\mathcal{H}^j$ denotes the image of $\mathcal{C}^j_{\mathrm{SD}} \to \mathcal{H}^1$ then the following diagram commutes and has exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}^{j-1}_{\mathrm{SD}} \arrow[r] & \mathcal{C}^j_{\mathrm{SD}} \arrow[r] & \mathcal{H}^j \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{C}^{j-2}_{\mathrm{SD}} \arrow[r] \arrow[u, "d_{\mathrm{SD}}"] & \mathcal{C}^{j-1}_{\mathrm{SD}} \arrow[r] \arrow[u, "d_{\mathrm{SD}}"] & \mathcal{H}^{j-1} \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

By considering the associated long exact sequence we deduce that $H^1(\mathcal{C}_{\mathrm{SD},j})$ is finite if and only if $H^1(\mathcal{C}_{\mathrm{SD},j+1})$ is. Since $H^0(\mathcal{C}_{\mathrm{SD},j}) = 0$ if $H^1(\mathcal{C}_{\mathrm{SD},j})$ is $\mathbb{F}$-finite then we also have:

$$\dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD,j}}) = \dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD},j+1}) + \dim_{\mathbb{F}} \mathcal{H}^j - \dim_{\mathbb{F}} \mathcal{H}^{j+1}$$

It is easy to see $\mathcal{H}^j = 0$ for $j \leq -e+2$. Therefore, since $\mathcal{C}^1_{\mathrm{SD},1} = \mathcal{C}^1_{\mathrm{SD}}$, the result will follow if $H^1(\mathcal{C}_{\mathrm{SD},j}) = 0$ for sufficiently small j.

For this, note that if $j \leq -e+2$ then $\mathcal{C}^j_{\mathrm{SD}}$ consists of those $(h_i) \in \mathcal{C}^1_{\mathrm{SD}}$ with $h_i(\mathcal{G}^{i'}) \subset \mathcal{E}^{i'}$ for all $i' \geq i$. In particular, if (h_i) is such an element then $h_i \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ for each i and so we can choose, by the first paragraph of the proof, $h'_1 \in \varphi(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$ so that $\varphi^{-1}(h'_1) - h'_1 = h_1 + h_2 + \ldots + h_{e-1}$. Then define

$$h'_2 = h_2 + h'_1,\ h'_3 = h_3 + h'_2, \ldots,\ h'_{e-1} = h_{e-1} + h'_{e-2}$$

Using that $\varphi(h'_e)$ maps $\mathcal{G}_i$ into $\mathcal{E}_i$ for every i we deduce $(h'_i) \in \mathcal{C}^j_{\mathrm{SD}} = \mathcal{C}^{j-1}_{\mathrm{SD}}$ and that $d_{\mathrm{SD}}((h'_i)) = (h_i)$. This completes the proof. □

Lemma 4.4.3. *After replacing* $(\mathfrak{M},\mathcal{E})$ *with* $(u^n\mathfrak{M}, u^n\mathcal{E}) \in \mathrm{Mod}_{\mathcal{F}}$ *for n sufficiently large conditions (i) and (ii) are satisfied.*

Proof. If $N \geq 0$ is sufficiently large then $u^N \mathrm{Hom}(\mathcal{G}^1,\mathcal{E}^1)$ is contained in both $u\,\mathrm{Hom}(\mathfrak{P},\mathfrak{M})$ and $\mathrm{Hom}(\mathcal{G}^i,\mathcal{E}^i)$ for each i. For any $n \in \mathbb{Z}$ we have

$$F^N(\mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M})) = u^{pn}F^{N-(p-1)n}(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \tag{4.4.4}$$

For sufficiently large n we have $F^{N-(p-1)n}(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) = \varphi(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$ and so $u^{pn}F^{N-(p-1)n}(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) = \varphi(\mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M}))$. This shows $\varphi(\mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M})) \subset u^N \mathrm{Hom}(\mathcal{G}_1,u^n\mathcal{E}_1)$ and so is contained in $u\,\mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M})$ and $\mathrm{Hom}(\mathcal{G}_i,u^n\mathcal{E}_i)$ for each i. □

Proof of Propsition 4.4.1. First, note that $H^0(\mathcal{C}_{\mathrm{SD}})$ is contained $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})^{\varphi=1}$ which is always $\mathbb{F}$-finite since it is contained in the finite dimensional $\mathbb{F}$-vector space $(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}) \otimes_{k[[u]]} C^\flat)^{\varphi=1}$. If we replace $(\mathfrak{M},\mathcal{E})$ by $(u^n\mathfrak{M},u^n\mathcal{E}) \in \mathrm{Mod}_{\mathcal{F}}$ in the definition of $\mathcal{C}_{\mathrm{SD}}$ we obtain another complex which we denote $\mathcal{C}_{\mathrm{SD}}(n)$. Taking $n = 1$ we obtain an exact sequence

$$0 \to \mathcal{C}_{\mathrm{SD}}(1) \to \mathcal{C}_{\mathrm{SD}} \to \mathcal{Q} \to 0$$

of complexes, whose associated long exact sequence reads

$$\begin{aligned} 0 \to H^0(\mathcal{C}_{\mathrm{SD}}(1)) \to H^0(\mathcal{C}_{\mathrm{SD}}) \to H^0(\mathcal{Q}) \\ \to H^1(\mathcal{C}_{\mathrm{SD}}(1)) \to H^1(\mathcal{C}_{\mathrm{SD}}) \to H^1(\mathcal{Q}) \to 0 \end{aligned} \tag{4.4.5}$$

Note that $\mathcal{Q}$ is a two term complex $\mathcal{Q}^0 \xrightarrow{\gamma} \mathcal{Q}^1$ and the $\mathcal{Q}^i$ can be described explicitly. It is easy to see that $\mathcal{C}_{\mathrm{SD}}(1)^1 = u\mathcal{C}^1_{\mathrm{SD}}$, and so $\mathcal{Q}_1 \cong \mathcal{C}^1_{\mathrm{SD}}/u\mathcal{C}^1_{\mathrm{SD}}$. On the other hand,

$$\mathcal{Q}^0 \cong \frac{F^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))}{F^0(\mathrm{Hom}(\mathfrak{P},u\mathfrak{M}))} \times \prod_{i=2}^{e-1} \mathrm{Hom}(\mathcal{G}^i,\mathcal{E}^i/u\mathcal{E}_i)$$

We claim

$$\frac{F^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))}{F^0(\mathrm{Hom}(\mathfrak{P},u\mathfrak{M}))} \cong \bigoplus_{i \in p\mathbb{Z}_{\leq 0} \cup \mathbb{Z}_{\geq 1}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

as $\mathbb{F}$-vector spaces. In particular, we claim both $\mathcal{Q}^0$ and $\mathcal{Q}^1$ are $\mathbb{F}$-finite and so the same is true for the cohomology of $\mathcal{Q}$. Together Lemma 4.4.2 and Lemma 4.4.3 imply $H^1(\mathcal{C}_{\mathrm{SD}}(n))$ is finite for large n. From (4.4.5) we deduce finiteness of $H^1(\mathcal{C}_{\mathrm{SD}})$.

To verify the claim first choose an $\mathbb{F}$-linear splitting of $0 \to F^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to F^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to 0$ and so write

$$F^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \cong F^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \oplus \mathrm{gr}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$$

Note that $F^0(\mathrm{Hom}(\mathfrak{P},u\mathfrak{M})) = u\,\mathrm{Hom}(\mathfrak{P},\mathfrak{M}) \cap F^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$, which is the kernel of the surjection $F^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to F^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$. Therefore,

$$\frac{F^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))}{F^0(\mathrm{Hom}(\mathfrak{P},u\mathfrak{M}))} \cong \mathrm{gr}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \oplus F^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

Splitting $0 \to F^{i+1}(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \to F^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \to \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \to 0$ for $i \geq 1$ allows us to write

$$F^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \cong \bigoplus_{i\in\mathbb{Z}_{\geq 1}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

There are also exact sequences $0 \to \mathrm{gr}^{i-p}(\mathrm{Hom}(\mathfrak{P},\mathfrak{N})) \xrightarrow{u} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \to 0$ and, by choosing $\mathbb{F}$-linear splitting, we can identify

$$\mathrm{gr}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \cong \bigoplus_{i\in p\mathbb{Z}_{\leq 0}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

The claim follows.

To finish the proof note that (4.4.5) implies

$$\begin{aligned}\chi(\mathfrak{P},\mathfrak{M}) &= \chi(\mathfrak{P},u\mathfrak{M}) + \dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD}}) - \dim_{\mathbb{F}} H^0(\mathcal{C}_{\mathrm{SD}}) \\ &= \chi(\mathfrak{P},u\mathfrak{M}) + \dim_{\mathbb{F}} \mathcal{Q}^1 - \dim_{\mathbb{F}} \mathcal{Q}^0\end{aligned}$$

Since $\mathcal{C}^1_{\mathrm{SD}}$ is an $\mathbb{F}[[u]]$-lattice inside $\prod_{i=1}^{e-1} \mathrm{Hom}(\mathfrak{P},\mathfrak{M})[\frac{1}{u}]$, $\mathcal{Q}^1$ has $\mathbb{F}$-dimension equal to $(e-1)r$ where r is the $\mathbb{F}[[u]]$-rank of $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})$. The above description of $\mathcal{Q}^0$ shows it has $\mathbb{F}$-dimension $(e-2)r + \sum_{i\notin p\mathbb{Z}_{\leq 0}\cup\mathbb{Z}_{\geq 1}} \dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$. Since $r = \sum_i \dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$ it follows that

$$\chi(\mathfrak{P},\mathfrak{M}) = \chi(\mathfrak{P},u\mathfrak{M}) + \sum_{\substack{i<0\\ p\nmid i}} \dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

Using (4.4.4) and the assumption that $\mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) = 0$ for $i < -p$ we deduce

$$\begin{aligned}\chi(\mathfrak{P},\mathfrak{M}) &= \chi(\mathfrak{P},u^n\mathfrak{M}) + \sum_{m=0}^{n-1}\left(\sum_{\substack{i<0\\ p\nmid i}} \dim_{\mathbb{F}} \mathrm{gr}^{i-(p-1)m}(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)\right) \\ &= \chi(\mathfrak{P},u^n\mathfrak{M}) + \sum_{i<0} \dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))_k \\ &= \chi(\mathfrak{P},u^n\mathfrak{M}) + \dim_{\mathbb{F}} \frac{\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k}{F^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)}\end{aligned}$$

for $n > 2$. Combining this with Lemma 4.4.2 and Lemma 4.4.3 gives the proposition. □

4.5. **Reformulating the previous result.** We now suppose that $\mathfrak{P}$ and $\mathfrak{M}$ are objects in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$. Attached to $\mathfrak{P}$ is an e-tuple of filtered $k\otimes_{\mathbb{F}_p}\mathbb{F}$-modules

$$(\overline{\mathfrak{P}}, \overline{\mathcal{G}}^1, \dots, \overline{\mathcal{G}}^{e-1})$$

where $\overline{\mathcal{G}}^i = \mathcal{G}^i/u\mathcal{G}^i$ equipped with the one step filtration with $\mathrm{Fil}^1(\overline{\mathcal{G}^i})$ equal the image of $\mathcal{G}^{i+1}$. Thus, we can attach to $\mathfrak{P}$ a Hodge type $\mu(\mathfrak{P})$. Likewise for $\mathfrak{M}$. In this context Proposition 4.4.1 can be reformulated as:

Proposition 4.5.1. $\dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD}}) = \dim_{\mathbb{F}} H^0(\mathcal{C}_{\mathrm{SD}}) + d(\mu(\mathfrak{P}),\mu(\mathfrak{M}))$

Proof. As both $\mathfrak{P}$ and $\mathfrak{M}$ are strongly divisible Lemma 4.1.3 implies that the filtered module $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k$ identifies with the filtered module $\mathrm{Hom}_{\mathrm{Fil}}(\overline{\mathfrak{P}},\overline{\mathfrak{M}})$ (as described in Section 2.3). Since the filtrations on $\overline{\mathfrak{P}}$ and $\overline{\mathfrak{M}}$ are concentrated in degrees $[0,p]$ this implies $\mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) = 0$ for $i < -p$. Thus Proposition 4.4.1 applies. Since

$$\mathrm{Hom}(\mathcal{G}^{i+1}/u\mathcal{G}^i, \mathcal{E}^i/\mathcal{E}^{i+1}) = \mathrm{Hom}(\mathrm{Fil}^1(\overline{\mathcal{G}}^i), \mathrm{gr}^0(\overline{\mathcal{E}}^i)) = \frac{\mathrm{Hom}(\overline{\mathcal{G}}^i, \overline{\mathcal{E}}^i)}{\mathrm{Fil}^0(\mathrm{Hom}(\overline{\mathcal{G}^i}, \overline{\mathcal{E}}^i))}$$

it follows that $\dim_{\mathbb{F}} \frac{\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k}{F^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)} + \sum_{i=1}^{e-1} \dim_{\mathbb{F}} \mathrm{Hom}(\mathcal{G}^{i+1}/u\mathcal{G}^i, \mathcal{E}^i/\mathcal{E}^{i+1})$ is precisely $d(\mu(\mathfrak{P}),\mu(\mathfrak{M}))$. $\square$

5. Crystalline vs. strong divisibility

5.1. **Filtrations on the image of Frobenius.** Fix $(\mathfrak{M},\alpha,\mathcal{F})$ corresponding to an $\mathcal{O}$-valued point of $\mathcal{L}_R^{\mu,\mathrm{conv}}$ with $\mathcal{L}_R^{\mu,\mathrm{conv}}$ as in Section 3. For the moment μ is any Hodge type concentrated in degrees $[0,h_j]$ with $h_j \geq 0$. Write $V = V_R \otimes_\alpha \mathcal{O}$. Then $V[\frac{1}{p}]$ is a crystalline representation of Hodge type μ and so, as described in Section 2.6 we have $D \coloneqq D_{\mathrm{crys}}(V[\frac{1}{p}])$ a finite free $K_0 \otimes_{\mathbb{Q}_p} E$-module and $D_K = D \otimes_{K_0} K$ a filtered $K \otimes_{\mathbb{Q}_p} E$-module of type μ.

Since $\mathfrak{M}$ and V satisfy (3.2.2) it follows that $\mathfrak{M}$ is the Breuil–Kisin module associated to V by Kisin as in e.g. [Kis10, 1.2.1] or [Kis06]. In particular there is φ-equivariant identification

$$\mathfrak{M}^\varphi \otimes_{\mathfrak{S}} S[\tfrac{1}{p}] \cong D \otimes_{K_0} S[\tfrac{1}{p}]$$

where S denotes the p-adic completion of the divided power envelope of $W(k)[u]$ relative to the ideal generated by $E(u)$, cf. [GLS14, 3.2]. Concretely S can be viewed as the subring of $K_0[[u]]$ consisting of series of the form

$$\sum_{i=0}^{\infty} a_i \frac{u^i}{e(i)!}, \qquad a_i \in W(k)$$

where $e(i)$ denotes the largest integer $\leq i/e$. This allows us to define a φ-equivariant diffential operator on $\mathfrak{M}^\varphi \otimes_{\mathfrak{S}} S$ by the formula

$$\mathcal{N}(d \otimes a) = d \otimes (-u\frac{d}{du}(a))$$

for $d \in D$ and $a \in S$.

Construction 5.1.1. For integers $n_{ij} \geq 0$ inductively define $S \otimes_{\mathbb{Z}_p} E$-submodules $\mathrm{Fil}^{\{n_{ij}\}}$ of $\mathfrak{M}^\varphi \otimes_{\mathfrak{S}} S$ by $\mathrm{Fil}^{\{n_{ij}\}} = \mathfrak{M}^\varphi \otimes_{\mathfrak{S}} S$ if every $n_{ij} \leq 0$ and

$$\mathrm{Fil}^{\{n_{ij}\}} = \{x \in \mathfrak{M}^\varphi \otimes_{\mathfrak{S}} S \mid f_{ij}(x) \in \mathrm{Fil}^{n_{ij}}(D_{K,ij}) \text{ for all } ij \text{ and } \mathcal{N}(x) \in \mathrm{Fil}^{\{n_{ij}-1\}}\}$$

otherwise.

Tensoring along the map $S \to K$ sending $u \mapsto \pi$ produces a surjection $f_\pi : \mathfrak{M}^\varphi \otimes_{\mathfrak{S}} S[\frac{1}{p}] \to D_K$. Let f_{ij} denote the composition of this surjection with the projection $D_K = \prod_{ij} D_{K,ij} \to D_{K,ij}$.

Lemma 5.1.2. *The* $\mathrm{Fil}^{\{n_{ij}\}}$ *enjoy the following properties*

(1) $f_{ij}(\mathrm{Fil}^{\{n_{ij}\}}) = \mathrm{Fil}^{n_{ij}}(D_{K,ij})$ *for each* ij.
(2) These are filtrations in the sense that $E_{ij}(u)\,\mathrm{Fil}^{\{n_{ij}-1_{i'j'}\}} \subset \mathrm{Fil}^{\{n_{ij}\}} \subset \mathrm{Fil}^{\{n_{ij}-1_{i'j'}\}}$ *for every* $i'j'$ *(here* $1_{i'j'}$ *denotes the tuple which is zero everywhere but in the* $i'j'$*-th position where it is* 1*).*
(3) $E_{i'j'}(u)x \in \mathrm{Fil}^{\{n_{ij}\}}$ *implies* $x \in \mathrm{Fil}^{\{n_{ij}-1_{i'j'}\}}$.

(4) $\mathrm{Fil}^{\{h_j\}} \cap \mathfrak{M}^{\varphi} = (\prod E_{ij}(u)^{h_j})\mathfrak{M}$.

Proof. Properties (1), (2) and (3) follow from [GLS15, 2.1.9]. Part (5) follows from [GLS15, (2.2.1)] and part (3) of [GLS15, 2.2.2]. Finally, for (4) note that (5) ensures $f_{ij}(\mathfrak{M}^{\varphi})$ is an $\mathcal{O}$-lattice in $D_{K,ij}$. □

Corollary 5.1.3. *We have*

$$\mathcal{F}^j = \mathrm{Fil}^{\{h_1,\dots,h_j,0,\dots,0\}} \cap \mathfrak{M}^{\varphi}$$

where $(h_1,\dots,h_j,0,\dots,0)$ indicates the tuple with $h_{j'}$ in the ij'-th position if $j' \leq j$ and 0 otherwise.

Proof. With $\mathcal{F}^j$ defined by the above equality the previous lemma shows that $E_j(u)^{h_j}\mathcal{F}^{j-1} \subset \mathcal{F}^j \subset \mathcal{F}^{j-1}$ and that $\mathcal{F}^e = (\prod E_j(u)^{h_j})\mathfrak{M}$. Since each $\mathrm{Fil}^{\{n_{ij}\}}$ is a $\mathbb{Q}_p$-vector space $\mathcal{F}^{j-1}/\mathcal{F}^j$ is $\mathcal{O}$-flat. The proof of Proposition 3.3.2 shows that these conditions uniquely determine $\mathcal{F}^j$. □

5.2. **Integral filtrations.** Our aim is to prove the following:

Proposition 5.2.1. *There exists a $\mathbb{Z}_p[[u]]$-basis (e_i) of $\mathfrak{M}^{\varphi}$ and integers $r_i \in [0,p]$ such that*

(1) *$\mathrm{Fil}^{\{p,0,\dots,0\}} \cap \mathfrak{M}^{\varphi}$ is generated by $(E_1(u)^{r_i}e_i)$*
(2) *$e_i = f_i + \pi g_i$ for some $f_i \in \varphi(\mathfrak{M})$ and $g_i \in \mathfrak{M}^{\varphi}$.*

Here, as in Corollary 5.1.3, $\{p,0,\dots,0\}$ indicates the tuple with p in the $i1$-th position for each i and 0 everywhere else. Before proving the proposition we need some preparations. First take any $x = x^{(0)} \in \mathfrak{M}^{\varphi} \otimes_{\mathfrak{S}} S$ and inductively define

$$x^{(i)} = \sum_{l=0}^{i-1} \frac{H(u)^l}{l!}\mathcal{N}^l(x^{(i-1)})$$

where $H(u) = \frac{E_1(u)}{E_1(0)}$.

Lemma 5.2.2. *If $f_{i1}(x) \in \mathrm{Fil}^n(D_{K,i1})$ for each i and $\delta_i = \min\{i,n\}$ then $x^{(i)} \in \mathrm{Fil}^{\{\delta_i,0,\dots,0\}}$.*

Proof. It suffices to show $\mathcal{N}(x^{(i)}) \in \mathrm{Fil}^{\{\delta_i-1,0,\dots,0\}}$. Since $\delta_{i-1} \geq \delta_i - 1$ we may instead show $\mathcal{N}(x^{(i)}) \in \mathrm{Fil}^{\{\delta_{i-1},0,\dots,0\}}$. Writing $\partial = -u\frac{d}{du}$ we compute

$$\begin{aligned}\mathcal{N}(x^{(i)}) &= \sum_{l=0}^{i-1}\left(\frac{H(u)^{l-1}\partial(H(u))}{(l-1)!} + \frac{H(u)^l}{l!}\mathcal{N}^{l+1}(x^{i-1})\right)\\ &= \underbrace{\frac{H(u)^{i-1}}{(i-1)!}\mathcal{N}^i(x^{(i-1)})}_{(a)} + \sum_{l=1}^{i-1}\underbrace{(1+\partial(H(u)))\frac{H(u)^{l-1}}{(l-1)!}\mathcal{N}^l(x^{(i-1)})}_{(b)}\end{aligned}$$

If $x \in \mathrm{Fil}^{\{r,0,\dots,0\}}$ then $H(u)^l x \in \mathrm{Fil}^{\{r+l,0,\dots,0\}}$. Therefore (a) is contained in $\mathrm{Fil}^{\{i-1,0,\dots,0\}} \subset \mathrm{Fil}^{\{\delta_{i-1},0,\dots,0\}}$. The inductive hypothesis implies $x^{(i-1)} \in \mathrm{Fil}^{\{\delta_{i-1},0,\dots,0\}}$ and so $\mathcal{N}^l(x^{(i-1)}) \in \mathrm{Fil}^{\{\delta_{i-1}-l,0,\dots,0\}}$. Since $1+\partial(H(u)) = -H(u)$, each (b) term is contained $\mathrm{Fil}^{\{\delta_{i-1}-l+l,0,\dots,0\}} = \mathrm{Fil}^{\{\delta_{i-1},0,\dots,0\}}$ also. □

To apply Lemma 5.2.2 in the proof of Proposition 5.2.1 we require some control of the denominators appearing in the operator $\mathcal{N}$. This is given by the following result. It is here that the particular choice of π from Section 2 when $p = 2$ is important.

Theorem 5.2.3 (Gee–Liu–Savitt,Wang)**.** *For each $x \in \varphi(\mathfrak{M})$ and $b \geq 1$ there exist $x_i \in \varphi(\mathfrak{M})$ and $a_i \in E$ such that*

$$\mathcal{N}^b(x) = \sum_{i=0}^{\infty} E_1(u)^i a_i x_i$$

with $\pi^{p-i} \mid a_i$ for $i = 0, \dots, p-1$.

Proof. If $\partial = -u\frac{d}{du}$ then $\partial(E_1(u)) = E_1(0) - E_1(u)$. Therefore

$$\mathcal{N}(E_1(u)^i a_i x_i) = E_1(u)^i a_i (1-i)\mathcal{N}(x_i) + iE_1(u)^{i-1}E_1(0)a_i x_i$$

Since $E_1(0)$ is divisible by π in $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$ this shows, by an easy induction, that if the statement holds for $b = 1$ then it holds for $b \geq 1$.

If $p > 2$ then [GLS14, 4.7] implies that $\mathcal{N}(x)$ is contained in $\mathfrak{M}^{\varphi} \otimes_{\mathfrak{S}} u^p S'$ for $S' = W(k)[[u^p, \frac{u^{ep}}{p}]][\frac{1}{p}] \cap S$ and $x \in \varphi(\mathfrak{M})$. When $p = 2$ the same is true by the results in [Wan17]. Therefore, the theorem will follow if every $s \in u^p S'$ can be written as

$$\sum_{i=0}^{\infty} E_1(u)^i a_i$$

for $a_i \in W(k) \otimes_{\mathbb{Z}_p} E$ with $\pi^{p-i} \mid a_i$ for $i = 0, \dots, p-1$. Here we are viewing s as an element of $K_0[[u]] \otimes_{\mathbb{Q}_p} E$ and so also as a tuple $(s_i) \in \prod_i E[[u]]$. Each s_i can be written as

$$\begin{aligned}\sum_{j=0}^{\infty} a_{ij}\frac{u^{p(j+1)}}{e(pj)!} &= \sum_{j=1}^{\infty} \frac{a_{ij}}{e(pj)!}\left(\sum_{l=0}^{p(j+1)} \binom{p(j+1)}{l}(u-\pi_{i1})^l \pi_{i1}^{p(j+1)-l}\right) \\ &= \sum_{l=0}^{\infty}(u-\pi_{i1})^l \underbrace{\left(\sum_{j+1 \geq l/p} \frac{a_{ij}}{e(pj)!}\binom{p(j+1)}{l}\pi_{i1}^{p(j+1)-l}\right)}_{a_l}\end{aligned}$$

for some $a_{ij} \in \mathcal{O}$ and $\pi_{i1} = \kappa_{i1}(\pi)$. We must show that the a_l term is divisible by π^{p-l} in $\mathcal{O}$ for $l = 0, \dots, p-1$. This follows from the observation that $\frac{\pi_{i1}^{pj}}{e(pj)!} \in \mathcal{O}$. $\square$

Corollary 5.2.4. *Suppose that $x \in \varphi(\mathfrak{M})$. Then for $i \leq p$ we can write*

$$x^{(i)} - x = \pi x' + H(u)^p x''$$

for $x' \in \mathfrak{M}^{\varphi}$ and $x'' \in \mathfrak{M}^{\varphi} \otimes_{\mathfrak{S}} S$.

Proof. Using Theorem 5.2.3 it suffices to show that $x^{(i)} - x$ can be written as a $\mathbb{Z}$-linear combination of terms $\frac{H(u)^a}{a'!}\mathcal{N}^b(x)$ for $a, b \geq 1$ and $1 \leq a' \leq a$. Arguing by induction it is enough to show

$$\frac{H(u)^l}{l!}\mathcal{N}^l(\frac{H(u)^a}{a'}\mathcal{N}^b(x)) = \sum_{k=0}^{l}\binom{l}{k}\frac{H(u)^l \partial^k(H(u)^a)}{l! a'!}\mathcal{N}^{l-k+b}(x)$$

is a $\mathbb{Z}$-linear combination of such terms. This follows from the claim that $\frac{\partial^k(H(u)^a)}{a!}$ is a $\mathbb{Z}$-linear combination of terms of the form $\frac{H(u)^{a'}}{a'!}$ for $1 \leq a' \leq a$. To see this note that $\partial(H(u)^a) = aH(u)^{a-1}(-1-H(u))$ and so $\partial^k(H(u)^a)/a!$ equals

$$\frac{1}{(a-1)!}\partial^{k-1}(H(u)(-1-H(u))) = \frac{1}{(a-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}\partial^j(H(u)^{a-1})\partial^{k-1-j}(-1-H(u))$$
$$= \frac{1}{(a-1)!}(-1-H(u))\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{k-1-j}\partial^j(H(u)^{a-1})$$

(for the second equality we've used that $\partial^n(H(u)) = (-1)^n(-1-H(u))$ for $n>0$). Inducting on k finishes the proof. □

Proof of Proposition 5.2.1. Set M_j equal to the image of $\mathfrak{M}^\varphi$ under $f_j := \prod_i f_{ij}$. This is a $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}$-lattice inside $\prod_i D_{K,ij}$ which we equip with the filtration

$$\mathrm{Fil}^n(M_j) = M_j \cap \prod_i \mathrm{Fil}^n(D_{K,ij})$$

Notice that the kernel of $\mathfrak{M}^\varphi \to M_1$ equals $E_1(u)\mathfrak{M}^\varphi$. Choose a $\mathbb{Z}_p$-basis $(\overline{e}_i)$ of M_1 adapted to the filtration, i.e. so that there exists integers r_i with $\mathrm{Fil}^n(M)$ is generated by those $\overline{e}_i$ with $n \leq r_i$. This is possible since the graded pieces of the filtration on M_1 are p-torsionfree by construction.

Note that f_1 induces a surjection $\varphi(\mathfrak{M}) \to \mathfrak{M}^\varphi \to M_1$ and so we can lift $(\overline{e}_i)$ to a $\mathbb{Z}_p[[u^p]]$-basis (f_i) of $\varphi(\mathfrak{M})$. Lemma 5.2.2 implies

$$f_i^{(p)} \in \mathrm{Fil}^{\{\min\{r_i,p\},0,\dots,0\}}$$

If $f_i^{(p)} = f_i + pf_i' + H(u)^p f_i''$ is as in Lemma 5.2.2 then

$$e_i = f_i + pf_i' \in \mathrm{Fil}^{\{\min\{r_i,p\},0,\dots,0\}} \cap \mathfrak{M}^\varphi$$

To finish the proof we show by induction on n that $\mathrm{Fil}^{\{n,0,\dots,0\}} \cap \mathfrak{M}^\varphi$ is equal to the submodule Y_n generated over $\mathbb{Z}_p[[u]]$ by $E_1(u)^{\max\{n-r_i,0\}}e_i$ whenever $n \leq p$. The case $n=0$ is clear so assume that assertion holds for $n-1$. Since the image of e_i in M equals that of f_i, the image of Y_n in M equals $\mathrm{Fil}^n(M)$ and so contains the image of $\mathrm{Fil}^{\{n,0,\dots,0\}} \cap \mathfrak{M}^\varphi$. Using (4) of Lemma 5.1.2 we see that the kernel of $\mathrm{Fil}^{\{n,0,\dots,0\}} \cap \mathfrak{M}^\varphi \to M$ equals $E_1(u)\,\mathrm{Fil}^{\{n-1,0,\dots,0\}} \cap \mathfrak{M}^\varphi$ which, by the inductive hypothesis, equals $E_1(u)Y_{n-1}$. Since $E_1(u)Y_{n-1} \subset Y_n$ we conclude that $Y_n = \mathrm{Fil}^{\{n,0,\dots,0\}} \cap \mathfrak{M}^\varphi$ as desired. □

5.3. **Application to strong divisibility.** Maintain the notation from above but assume additionally that μ is pseudo-Barsotti–Tate, i.e. that $h_1 = p$ and $h_2 = \dots = h_e = 1$.

Proposition 5.3.1. *If $(\mathfrak{M},\alpha,\mathcal{F})$ corresponds to an A-valued point of $\mathcal{L}_R^{\mu,\mathrm{conv}}$ for any finite flat $\mathcal{O}$-algebra A then*

$$(\mathfrak{M}_\mathbb{F},\mathcal{F}_\mathbb{F}) = (\mathfrak{M},\mathcal{F})\otimes_\mathcal{O}\mathbb{F}$$

is an object of $\mathrm{Mod}_\mathcal{F}^{\mathrm{SD}}$. If $A=\mathcal{O}$ then this object has Hodge type μ (in the sense of Section 4.5) .

Proof. The first part follows immediately by viewing $\mathfrak{M}$ as an $\mathfrak{S}_\mathcal{O}$-module rather than an $\mathfrak{S}_A$-module and V_α as an $\mathcal{O}$-representation rather than a representation on an A-module, and then applying Proposition 5.2.1.

For the second part, Proposition 5.2.1 and its proof show that the filtration on $\mathfrak{M}^\varphi/E_1(u)$ defined by the images of $\mathfrak{M}^\varphi \cap E_1(u)^n\mathcal{F}^1$ identifies with $M_1 :=$

$\prod_i f_{i1}(\mathfrak{M}^\varphi)$ with the filtration $\mathrm{Fil}^n(M_1) = M_1 \cap \prod_i \mathrm{Fil}^n(D_{K,i1})$. In particular, it has p-torsionfree graded pieces and is a filtered $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$-module of type $(\mu_{11}, \ldots, \mu_{f1})$. A consequence of the graded pieces being p-torsionfree is that the filtered module $\overline{\mathfrak{M}}_{\mathbb{F}}$ (in the sense of Section 4.1) is obtained by applying $\otimes_{\mathcal{O}} \mathbb{F}$ to $\mathfrak{M}^\varphi / E_1(u)$. This shows that $\overline{\mathfrak{M}}_{\mathbb{F}}$ has the same Hodge type.

For $j \geq 2$ the modules $\mathcal{F}^{j-1}/E_j(u)$ equipped with the the one step filtration given by the image of $\mathcal{F}^j$ also have p-torsionfree graded pieces (in this case this is immediate) and so $\overline{\mathcal{F}}^{j-1}_{\mathbb{F}}$ is similarly obtained by applying $\otimes_{\mathcal{O}} \mathbb{F}$ to $\mathcal{F}^{j-1}/E_j(u)$. Therefore it suffices to show $\mathcal{F}^{j-1}/E_j(u)$ has Hodge type $(\mu_{1j}, \ldots, \mu_{fj})$. But this follows from Lemma 5.1.2. More precisely, parts (1) and (3) imply $f_j := \prod_i f_{ij}$ defines an isomorphism

$$\mathrm{Fil}^{\{h_1, \ldots, h_j, 0, \ldots, 0\}} / E_j(u) \, \mathrm{Fil}^{\{h_1, \ldots, h_{j-1}, 0, \ldots, 0\}} \to \prod_i \mathrm{Fil}^1(D_{K,ij})$$

and part (4) implies that $f_j(\mathcal{F}^j)$ is an $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$-lattice inside $\prod_i D_{K,ij}$ and so also in $\prod_i \mathrm{Fil}^1(D_{K,ij})$ also. □

6. Tangent spaces

6.1. **Tangent space dimensions and extension groups.** For any $\mathbb{F}$-valued point $x \in \mathcal{L}_R^{\mu,\mathrm{conv}}$ corresponding to $(\mathfrak{M}_x, \alpha_x, \mathcal{F}_x)$ we write T_x for the tangent space of $\mathcal{L}_R^{\mu,\mathrm{conv}} \otimes_{\mathcal{O}} \mathbb{F}$ at x. In other words $T_x = \mathcal{L}_R^{\mu,\mathrm{conv}}(\mathbb{F}[\epsilon])$ where $\mathbb{F}[\epsilon]$ denotes the ring of dual numbers over $\mathbb{F}$. To understand these vector spaces observe that if $(\mathfrak{M}, \alpha, \mathcal{F}) \in T_x$ then, since $\mathfrak{M}$ and $\mathcal{F}^i$ are $\mathbb{F}[\epsilon]$-flat, $0 \to \mathfrak{M}_x \to \mathfrak{M} \xrightarrow{\epsilon} \mathfrak{M}_x \to 0$ is an exact sequence in $\mathrm{Mod}_{\mathcal{F}}$ (here we write $\mathfrak{M}_x$ in place of $(\mathfrak{M}_x, \mathcal{F}_x)$ and likewise for $\mathfrak{M}$). This construction produces an $\mathbb{F}$-linear homomorphism

$$T_x \to \mathrm{Ext}^1_{\mathcal{F}}(\mathfrak{M}_x, \mathfrak{M}_x)$$

into the Yoneda extension group in $\mathrm{Mod}_{\mathcal{F}}$.

Proposition 6.1.1. *If μ is pseudo-Barsotti–Tate then $T_x \to \mathrm{Ext}^1_{\mathcal{F}}(\mathfrak{M}_x, \mathfrak{M}_x)$ factors through $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{M}_x, \mathfrak{M}_x)$.*

Proof. Since $\mathcal{L}_R^{\mu,\mathrm{conv}}$ is $\mathcal{O}$-flat and $\mathcal{L}_R^{\mu,\mathrm{conv}}[\frac{1}{p}] = \mathrm{Spec}\, R^\mu[\frac{1}{p}]$ which is reduced it follows from [Bar20b, 4.1.2] that every $\overline{A}$-valued point of $\mathcal{L}_R^{\mu,\mathrm{conv}}$ valued in an Artin local ring with finite residue field is induced from an A-valued point for A some finite flat $\mathcal{O}$-algebra. The claim therefore follows by applying this with $\overline{A} = \mathbb{F}[\epsilon]$ and using the first part of Proposition 5.3.1. □

6.2. **Cyclotomic-freeness.** To describe the kernel of $T_x \to \mathrm{Ext}^1_{\mathcal{F}}(\mathfrak{M}_x, \mathfrak{M}_x)$ we will need to use the cyclotomic-freeness assumption. We will also need a second technical result from [GLS14]. It is very closely related to Theorem 5.2.3 (in fact it is the main ingredient going into the proof of Theorem 5.2.3).

Theorem 6.2.1 (Gee–Liu–Savitt)**.** *Suppose that A is a finite local $\mathcal{O}$-algebra and $(\mathfrak{M}, \alpha, \mathcal{F}) \in \mathcal{L}_R^{\mu,\mathrm{conv}}$ for any μ. Then the identification*

$$\mathfrak{M} \otimes_{\mathfrak{S}} W(C^\flat) = V_\alpha \otimes_{\mathbb{Z}_p} W(C^\flat)$$

is such that $(\sigma - 1)(m) \in \mathfrak{M} \otimes [\pi^\flat] \varphi^{-1} \mu A_{\mathrm{inf}}$ for all $\sigma \in G_K$ and all $m \in \mathfrak{M}$. Here $\mu = [\epsilon] - 1$ for some generator $\epsilon \in \mathbb{Z}_p(1)$.

Proof. We reduce to the case where A is $\mathcal{O}$-flat using [Bar20b, 4.1.2]. In this case the Theorem is one direction of [Bar20b, 2.1.12]. $\square$

When $pA = 0$ we have $\mathfrak{M} \otimes_{\mathfrak{S}} [\pi^\flat]\varphi^{-1}(\mu)A_{\mathrm{inf}} = \mathfrak{M} \otimes_{k[[u]]} u^{(e+p-1)/(p-1)}\mathcal{O}_{C^\flat}$ as follows from the well-known calculation that $\epsilon - 1$ generates the ideal $u^{ep/(p-1)}\mathcal{O}_{C^\flat}$ whenever ϵ is a compatible system of primitive p-th power roots of unity, cf. [Fon90, 5.2.1]. This motivates the following setup. Let $\mathfrak{M}_1, \mathfrak{M}_2$ be Breuil–Kisin modules over $\mathfrak{S}_{\mathbb{F}}$ satisfying

(1) $u^{e+p-1}\mathfrak{M}_i \subset \mathfrak{M}_i^{\varphi}$.
(2) Each $\mathfrak{M}_i \otimes_{k[[u]]} C^\flat$ is equipped with a φ-equivariant $C^\flat$-semilinear action of G_K for which
  (a) $(\sigma - 1)(m) \in \mathfrak{M}_i \otimes_{k[[u]]} u^{(e+p-1)/(p-1)}\mathcal{O}_{C^\flat}$ for $\sigma \in G_K$
  (b) $(\sigma - 1)(m) = 0$ for $\sigma \in G_{K_\infty}$
  whenever $m \in \mathfrak{M}_i$.

Proposition 6.2.2. *Let $\gamma \in \overline{k}$ be such that $\sigma(\gamma u^{(p+e-1)/(p-1)}) = \gamma u^{(p+e-1)/(p-1)}\chi_{\mathrm{cyc}}(\sigma)$ for all $\sigma \in G_{K_\infty}$. If there exists no φ-equivariant $\mathfrak{S}_A$-linear map*

$$\mathfrak{M}_1 \to \gamma u^{(p+e-1)/(p-1)}\mathfrak{M}_2$$

then any φ-equivariant $\mathfrak{S}_{\mathbb{F}}$-linear map $\mathfrak{M}_1 \to \mathfrak{M}_2$ becomes G_K-equivariant after extending scalars to $\mathcal{O}_{C^\flat}$.

We point out that such a γ exists because the character of G_{K_∞} defined by $\sigma(u^{(e+p-1)/(p-1)}) = \chi(\sigma)u^{(e+p-1)/(p-1)}$ is an unramified twist of the χ_{cyc}.

Proof. The G_K-action on $\mathrm{Hom}(\mathfrak{M}_1, \mathfrak{M}_2) \otimes_{k[[u]]} C^\flat$ given by $h \mapsto \sigma \circ h \circ \sigma^{-1}$ is φ-equivariant. We must show $(\sigma - 1)(H) = 0$ for all $\sigma \in G_K$ if $H \in \mathrm{Hom}(\mathfrak{M}, \mathfrak{N})$ satisfies $(\varphi - 1)(H) = 0$. Assumption (2a) implies $(\sigma - 1)(H) \in \mathcal{H} \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ for

$$\mathcal{H} = \mathrm{Hom}(\mathfrak{M}_1, \gamma u^{(p+e-1)/(p-1)}\mathfrak{M}_2)$$

Since the element γ satisfies $\varphi(\gamma)/\gamma \in k$ assumption (1) implies

$$(\gamma u^{(p+e-1)/(p-1)}\mathfrak{N})^{\varphi} \subset \gamma u^{p+e-1+(p+e-1)/(p-1)}\mathfrak{N}$$

It follows that $\mathcal{H}$ is φ-stable inside $\mathrm{Hom}(\mathfrak{M}, \mathfrak{N}) \otimes_k \overline{k}$. Since H is φ-equivariant $\sigma \mapsto (\sigma - 1)(H)$ defines a continuous 1-cocycle

$$G_K \to (\mathcal{H} \otimes_{k[[u]]} \mathcal{O}_{C^\flat})^{\varphi = 1}$$

which vanishes on G_{K_∞}. We will show any such cocycle is zero. First, since $\mathcal{H}$ is φ-stable it is easy to see that $V := (\mathcal{H} \otimes_k \overline{k})^{\varphi=1} = (\mathcal{H} \otimes_{k[[u]]} \mathcal{O}_{C^\flat})^{\varphi=1}$. By the choice of γ and the fact that G_{K_∞}-acts as the identity on $\mathrm{Hom}(\mathfrak{M}_1, \mathfrak{M}_2)$ it follows that the action of G_{K_∞} on $V \otimes_{\mathbb{F}} \mathbb{F}(-1)$ is unramified (i.e, is trivial on $G_{K_\infty} \cap I_K$ where $I_K \subset G_K$ denotes the inertia subgroup). We also claim that $V \otimes_{\mathbb{F}} \mathbb{F}(-1)$ contains no element invariant under G_{K_∞}. This follows because, K_∞ being totally ramified, the composite $G_{K_\infty} \to G_K \to G_k$ is surjective and so any such element would lie in $\mathcal{H}^{\varphi=1}$. However, by assumption no such element exists. The proposition then follows from Lemma 6.2.3 below. $\square$

Lemma 6.2.3. *Let V be a continuous representation of G_K on an $\mathbb{F}$-vector space such that $V \otimes_{\mathbb{F}} \mathbb{F}(-1)|_{G_{K_\infty}}$ is unramified and contains no G_{K_∞}-invariant element. Then any continuous 1-cocycle $G_K \to V$ which vanishes on G_{K_∞} is zero.*

Proof. Let $\widehat{K} = K_\infty(\mu_{p^\infty})$. Since the cyclotomic character is trivial on $G_{\widehat{K}}$ our first assumption implies that $G_{\widehat{K}}$ acts on V through the composite $G_{\widehat{K}} \hookrightarrow G_K \to G_k$. This composite is surjective and so our second assumption implies $V^{G_{\widehat{K}}} = 0$. Inflation-restriction therefore implies $H^1(G_K, V) \to H^1(G_{\widehat{K}}, V)$ is injective and so $H^1(G_K, V) \to H^1(G_{K_\infty}, V)$ is also. It follows that any 1-cocycle as in the lemma is a coboundary $\sigma \mapsto (\sigma - 1)(v)$ for some $v \in V^{G_{K_\infty}}$. However $V^{G_{K_\infty}} \subset V^{G_{\widehat{K}}}$ which we've just seen is zero. ☐

Lemma 6.2.4. *Suppose that for every unramified G_{K_∞}-submodule $V \subset V_{\mathbb{F}}$ there exists no G_{K_∞}-equivariant quotient $V_{\mathbb{F}} \to W$ with $V \cong W \otimes \mathbb{F}(1)$. Then there exists no non-zero φ-equivariant map*

$$\mathfrak{M}_x \to \gamma u^{(e+p-1)/(p-1)}\mathfrak{M}_x$$

Proof. First consider $\mathfrak{M}$ and $\mathfrak{N}$ with $u^{e+p-1}\mathfrak{M} \subset \mathfrak{M}^\varphi$ and $\mathfrak{N}^\varphi \subset u^{e+p-1}\mathfrak{N}$ and suppose $H : \mathfrak{M} \to \mathfrak{N}$ is non-zero and φ-equivariant. Then there is a non-zero induced G_{K_∞}-equivariant map

$$(\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1} \to (\mathfrak{N} \otimes_{k[[u]]} C^\flat)^{\varphi=1} \tag{6.2.5}$$

We claim that the action of G_{K_∞} on the image of this map is unramified after twisting by $\mathbb{F}(1)$. Applying this with $\mathfrak{M} = \mathfrak{M}_x$ and $\mathfrak{N} = \gamma u^{(e+p-1)/(p-1)}\mathfrak{M}_x$ gives the lemma since then $(\mathfrak{N} \otimes_{k[[u]]} C^\flat)^{\varphi=1} = V_{\mathbb{F}} \otimes \mathbb{F}(-1)$ and $(\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1} = V_{\mathbb{F}}$.

To establish the claim we can assume that H is injective and becomes surjective after inverting u so that (6.2.5) is an isomorphism. Choose bases of $\mathfrak{M}$ and $\mathfrak{N}$ so that their Frobenii are respectively represented by matrices A and B, and so that H is represented by C. The φ-equivariance of H implies $B\varphi(C)A^{-1} = C$. The fact that $u^{e+p-1}\mathfrak{M} \subset \mathfrak{M}^\varphi$ and $\mathfrak{N}^\varphi \subset u^{e+p-1}\mathfrak{N}$ implies $u^{-(e+p-1)}B$ and $u^{e+p-1}A^{-1}$ have coefficients in $\mathfrak{S}_{\mathbb{F}}$. Considering the u-adic valuation of determinants in the identity $u^{-(e+p-1)}B\varphi(C)u^{e+p-1}A^{-1} = C$ implies that $C, u^{-(e+p-1)}B$ and $u^{(e+p-1)}A$ are invertible over $\mathfrak{S}_{\mathbb{F}}$. In particular, the Frobenius on $\mathfrak{M}' := (\gamma u^{(e+p-1)/(p-1)})^{-1}\mathfrak{M}$ is a semilinear automorphism and so the G_{K_∞}-action on $(\mathfrak{M}' \otimes_{k[[u]]} C^\flat)^{\varphi=1}$ is unramified. The definition of γ gives that $(\mathfrak{M}' \otimes_{k[[u]]} C^\flat)^{\varphi=1} = (\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1} \otimes \mathbb{F}(1)$ as G_{K_∞}-representations, and the claim follows. ☐

Lemma 6.2.6. *The conclusions of Lemma 6.2.4 apply if $V_{\mathbb{F}}$ is cyclotomic-free.*

Proof. Restriction from G_K to G_{K_∞} is an equivalence between irreducible representations of either group (cf. [Bar18, 2.2.1]) which respects being unramified. Thus, if V and W as in Lemma 6.2.4 exist then there is an unramified G_K-Jordan–Holder factor V' of $V_{\mathbb{F}}$ for which $V' \otimes \mathbb{F}(1)$ is also a G_K-Jordan–Holder factor. Thus $V_{\mathbb{F}}$ is not cyclotomic-free. ☐

6.3. **Tangent space bounds.**

Proposition 6.3.1. *If μ is pseudo-Barsotti–Tate and there are no non-zero φ-equivariant maps*

$$\mathfrak{M}_x \to \gamma u^{(e+p-1)/(p-1)}\mathfrak{M}_x$$

then $\dim_{\mathbb{F}} T_x \leq d^2 + d(\mu, \mu)$.

Proof. By Proposition 6.1.1, Proposition 4.5.1, and Proposition 5.3.1 it suffices to show that the kernel of $T_x \to \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{M}_x, \mathfrak{M}_x)$ has $\mathbb{F}$-dimension

$$\leq d^2 - \mathrm{Hom}_{\mathcal{F}}(\mathfrak{M}_x, \mathfrak{M}_x)$$

For this suppose $(\mathfrak{M}_i, \alpha_i, \mathcal{F}_i)$ are in this kernel for $i = 1, 2$. Observe that there is no non-zero φ-equivariant map $H : \mathfrak{M}_1 \to \gamma u^{(e+p-1)/(p-1)}\mathfrak{M}_2$ as in Proposition 6.2.2. indeed, since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are both extensions of $\mathfrak{M}_x$ by itself any non-zero such H would induce a non-zero $\mathfrak{M}_x \to \gamma u^{(e+p-1)/(p-1)}\mathfrak{M}_x$.

The fact that $(\mathfrak{M}_i, \alpha_i, \mathcal{F}_i)$ are both in the kernel of $T_x \to \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{M}_x, \mathfrak{M}_x)$ implies the existence of a morphism $\alpha : \mathfrak{M}_1 \to \mathfrak{M}_2$ in $\mathrm{Mod}_{\mathcal{F}}$ which is the identity on $\mathfrak{M}_x$ viewed as either a submodule and a quotient. The previous paragraph combined with Proposition 6.2.2 shows this identifies with a G_K-equivariant map $V_{\alpha_1} \to V_{\alpha_2}$ after base-changing to $W(C^\flat)$ which induces the identity on $V_{\mathbb{F}}$ when viewed as either a submodule or a quotient.

In particular, it follows that the kernel of $T_x \to \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{M}_x, \mathfrak{M}_x)$ is contained in the kernel of the composite $T_x \to T \to \mathrm{Ext}^1(V_{\mathbb{F}}, V_{\mathbb{F}})$ where T denotes the tangent space of $R \otimes_{\mathcal{O}} \mathbb{F}$ at its closed point. Since the kernel of $T \to \mathrm{Ext}^1(V_{\mathbb{F}}, V_{\mathbb{F}})$ identifies with

$$\mathrm{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}}) / \mathrm{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K}$$

we are reduced to considering the kernel of $T_x \to T$. The group of G_K-equivariant automorphisms of $V_{\mathbb{F}} \oplus \epsilon V_{\mathbb{F}}$ which are the identity on $\epsilon V_{\mathbb{F}}$ and modulo ϵ act on this kernel. This group identifies with $\mathrm{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})$ via $h \mapsto a + b\epsilon \mapsto a + h(b)\epsilon$. The first paragraph shows that this action is transitive. Furthermore, the stabiliser of the zero element in T_x clearly identifies with those $h \in \mathrm{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K}$ inducing an endomorphism of $\mathfrak{M}_x$ which is a morphism in $\mathrm{Mod}_{\mathcal{F}}$. In this way we identify the kernel of $T_x \to T$ with

$$\mathrm{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K} / \mathrm{Hom}_{\mathcal{F}}(\mathfrak{M}_x, \mathfrak{M}_x)$$

(where $\mathrm{Hom}_{\mathcal{F}}(\mathfrak{M}_x, \mathfrak{M}_x)$) is viewed as a submodule of $\mathrm{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K}$ using Proposition 6.2.2). This gives the desired bound. □

Corollary 6.3.2. *If μ is pseudo-Barsotti–Tate and $V_{\mathbb{F}}$ is cyclotomic-free then the map $T_x \to \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{M}_x, \mathfrak{M}_x)$ from Proposition 6.1.1 is surjective.*

Proof. If $T_x \to \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{M}_x, \mathfrak{M}_x)$ is not surjective then the bound in Proposition 6.3.1 would be strict. However, the discussion in the proof of Theorem 3.3.5 illustrates that $\dim_{\mathbb{F}} T_x \geq d^2 + d(\mu, \mu)$. □

7. Applications

7.1. **Constructing crystalline lifts.** Given two Hodge types μ and μ' we write (μ, μ') for the Hodge type obtained by taking $\mu_{ij} \cup \mu'_{ij}$ for each ij. Note that if V and V' are crystalline representations of Hodge type μ and μ' respectively then $V \oplus V'$ has Hodge type (μ, μ').

Lemma 7.1.1. *Suppose that $(\mathfrak{M}, \alpha, \mathcal{F})$ and $(\mathfrak{M}', \alpha', \mathcal{F}')$ respectively correspond to $\mathcal{O}$-valued points in $\mathcal{L}^{\mu,\mathrm{conv}}_{R_{V_{\mathbb{F}}}}$ and $\mathcal{L}^{\mu',\mathrm{conv}}_{R_{V'_{\mathbb{F}}}}$ and consider an exact sequence*

$$0 \to (\mathfrak{M}, \mathcal{F}) \otimes_{\mathcal{O}} \mathbb{F} \to (\mathfrak{N}_{\mathbb{F}}, \mathcal{G}_{\mathbb{F}}) \to (\mathfrak{M}', \mathcal{F}') \otimes_{\mathcal{O}} \mathbb{F} \to 0 \tag{7.1.2}$$

in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$. Assume that μ is pseudo-Barsotti–Tate and $V_{\mathbb{F}} \oplus V'_{\mathbb{F}}$ is cyclotomic-free. Then there exists a G_K-equivariant exact sequence

$$0 \to V_{\mathbb{F}} \to W_{\mathbb{F}} \to V'_{\mathbb{F}} \to 0$$

which identifies φ, G_{K_∞}-equivariantly with (7.1.2) *after base-change to* $W(C^\flat)$*, and an* $\mathcal{O}$*-valued point* $(\mathfrak{N}, \beta, \mathcal{G}) \in \mathcal{L}^{(\mu,\mu'),\mathrm{conv}}_{R_{W_\mathbb{F}}}$ *with* $\mathfrak{N}$ *fitting into a* φ*-equivariant exact sequence of* $\mathfrak{S}_\mathcal{O}$*-modules*

$$0 \to \mathfrak{M} \to \mathfrak{N} \to \mathfrak{M}' \to 0$$

which recovers (7.1.2) *after applying* $\otimes_\mathcal{O} \mathbb{F}$.

Proof. The triple $(\mathfrak{M} \oplus \mathfrak{M}', \alpha \oplus \alpha', \mathcal{F} \oplus \mathcal{F}')$ can be viewed as an $\mathcal{O}$-valued point x of $\mathcal{L}^{(\mu,\mu'),\mathrm{conv}}_{R_{V_\mathbb{F} \oplus V'_\mathbb{F}}}$. Let $x_\mathbb{F}$ be the composite $\operatorname{Spec} \mathbb{F} \to \operatorname{Spec} \mathcal{O} \xrightarrow{x} \mathcal{L}^{(\mu,\mu'),\mathrm{conv}}_{R_{V_\mathbb{F} \oplus V'_\mathbb{F}}}$. From (7.1.2) we can construct an exact sequence

$$0 \to (\mathfrak{M} \oplus \mathfrak{M}, \mathcal{F} \oplus \mathcal{F}') \otimes_\mathcal{O} \mathbb{F} \to (\mathfrak{N}^*_\mathbb{F}, \mathcal{G}^*_\mathbb{F}) \to (\mathfrak{M} \oplus \mathfrak{M}, \mathcal{F} \oplus \mathcal{F}') \otimes_\mathcal{O} \mathbb{F} \to 0$$

recovering (7.1.2) after pulling back[1] along $(\mathfrak{M}', \mathcal{F}') \otimes_\mathcal{O} \mathbb{F} \to (\mathfrak{M} \oplus \mathfrak{M}', \mathcal{F} \oplus \mathcal{F}') \otimes_\mathcal{O} \mathbb{F}$ and then pushing out along $(\mathfrak{M} \oplus \mathfrak{M}', \mathcal{F} \oplus \mathcal{F}') \otimes_\mathcal{O} \mathbb{F} \to (\mathfrak{M}, \mathcal{F}) \otimes_\mathcal{O} \mathbb{F}$. Corollary 6.3.2 implies that this new exact sequence arises from a tangent vector t in $\mathcal{L}^{(\mu,\mu'),\mathrm{conv}}_{R_{V_\mathbb{F} \oplus V'_\mathbb{F}}}$ over the point $x_\mathbb{F}$.

As $V_\mathbb{F}$ and $V'_\mathbb{F}$ are cyclotomic-free the same is true of $V_\mathbb{F} \oplus V'_\mathbb{F}$. Therefore Theorem 3.3.5 applies and the completed local ring of $\mathcal{L}^{(\mu,\mu'),\mathrm{conv}}_{R_{V_\mathbb{F} \oplus V'_\mathbb{F}}}$ is formally smooth over $\mathbb{Z}_p$. Hence t can be lifted to an $\mathcal{O}[\epsilon]$-valued point (dual numbers over $\mathcal{O}$) inducing $\overline{x}$, cf. . Such a point gives rise to a φ-equivariant exact sequence

$$0 \to \mathfrak{M} \oplus \mathfrak{M}' \to \mathfrak{N}^* \to \mathfrak{M} \oplus \mathfrak{M}' \to 0$$

of $\mathfrak{S}_\mathcal{O}$-modules which φ, G_{K_∞}-equivariantly identifies with an exact sequence of crystalline G_K-representations $0 \to V_\alpha \oplus V_{\alpha'} \to W^* \to V_\alpha \oplus V_{\alpha'} \to 0$ after base-changing to $W(C^\flat)$. Pulling back along $\mathfrak{M}' \to \mathfrak{M} \oplus \mathfrak{M}'$ and then pushing out along $\mathfrak{M} \oplus \mathfrak{M}' \to \mathfrak{M}$ produces a φ-equivariant exact sequence

$$0 \to (\mathfrak{M}, \mathcal{F}) \to (\mathfrak{N}, \mathcal{G}) \to (\mathfrak{M}', \mathcal{F}') \to 0$$

and a G_K-equivariant exact sequence $0 \to V_\alpha \to W \to V_{\alpha'} \to 0$ of crystalline representations which φ, G_{K_∞}-equivariantly identify after base-changing to $W(C^\flat)$. Since the formation of the pullbacks and pushouts commutes with $\otimes_\mathcal{O}\mathbb{F}$ we recover (7.1.2) after basechanging to $\mathbb{F}$. Thus $W = R_{W_\mathbb{F}} \otimes_\beta \mathcal{O}$ for some β and $(\mathfrak{N}, \beta, \mathcal{G})$ defines an $\mathcal{O}$-point of $\mathcal{L}^{(\mu,\mu'),\mathrm{conv}}_{R_{W_\mathbb{F}}}$ as desired. □

Lemma 7.1.3. *Suppose that* $V_\mathbb{F}$ *is one-dimensional and that* $(\mathfrak{M}_\mathbb{F}, \alpha_\mathbb{F}, \mathcal{F}_\mathbb{F})$ *corresponds to an* $\mathbb{F}$*-valued* $x_\mathbb{F}$ *point of* $\mathcal{L}^{\leq h_j,\mathrm{conv}}_R$ *with* $(\mathfrak{M}, \mathcal{F}) \in \mathrm{Mod}^{\mathrm{SD}}_\mathcal{F}$ *with pseudo-Barsotti–Tate Hodge type* μ.[2] *Then there exists an* $\mathcal{O}$*-valued point* $(\mathfrak{M}, \alpha, \mathcal{F})$ *of* $\mathcal{L}^{\mu,\mathrm{conv}}_R$ *inducing* $x_\mathbb{F}$.

Proof. The correspondence between Breuil–Kisin modules over $\mathfrak{S}_\mathcal{O}$ of rank one and crystalline characters is an equivalence. Moreover, in the rank one case the Hodge type of any such character is easily described in terms of the associated Breuil–Kisin module, cf. [GLS15, 2.2.3]. It is therefore sufficient to observe that every rank one object in $\mathrm{Mod}_\mathcal{F}$ with a given Hodge type can be lifted to a rank

[1] Note that pullbacks and pushouts exist in $\mathrm{Mod}_\mathcal{F}$; the pushout of two morphisms $f : \mathfrak{M} \to \mathfrak{N}$ and $g : \mathfrak{M} \to \mathfrak{N}'$ is constructed as the cokernel of $(f, -g) : \mathfrak{M} \to \mathfrak{N} \oplus \mathfrak{N}'$. Similarly the pullback of $f : \mathfrak{M} \to \mathfrak{N}$ and $g : \mathfrak{M}' \to \mathfrak{N}$ is constructed as the kernel of $f - g : \mathfrak{M} \oplus \mathfrak{M}' \to \mathfrak{N}$. It follows from Proposition 4.2.1 that these construction respect the full subcategory $\mathrm{Mod}^{\mathrm{SD}}_\mathcal{F}$.

[2] We point out that every object of $\mathrm{Mod}_\mathcal{F}$ of rank one over $\mathfrak{S}_\mathbb{F}$ is contained in $\mathrm{Mod}^{\mathrm{SD}}_\mathcal{F}$.

one Breuil–Kisin module over $\mathcal{O}\mathfrak{S}_{\mathcal{O}}$ so that the Hodge type of the object in $\mathrm{Mod}_{\mathcal{F}}$ coincides with that of the corresponding crystalline character. This follows easily from the definitions. □

Corollary 7.1.4. *Continue to assume μ is pseudo-Barsotti–Tate and $V_{\mathbb{F}}$ is cyclotomic-free. Suppose that every Jordan–Holder factor of $V_{\mathbb{F}}$ is one-dimensional and $(\mathfrak{M}_{\mathbb{F}}, \alpha_{\mathbb{F}}, \mathcal{F}_{\mathbb{F}})$ corresponds to an $\mathbb{F}$-valued point of $\mathcal{L}_R^{\mu,\mathrm{conv}}$. Then there exists an $\mathcal{O}$-valued point $(\mathfrak{M}, \alpha, \mathcal{F})$ of $\mathcal{L}_R^{\mu,\mathrm{conv}}$ lying over $x_{\mathbb{F}}$ such that every Jordan–Holder factor of $V_\alpha[\frac{1}{p}]$ is one-dimensional.*

Proof. We induct on the dimension of $V_{\mathbb{F}}$. If $V_{\mathbb{F}}$ is one-dimensional there is nothing to prove. For the general case, any G_K-equivariant exact sequence $0 \to V_{\mathbb{F},1} \to V_{\mathbb{F}} \to V_{\mathbb{F},2} \to 0$ induces a unique φ-equivariant exact sequence $0 \to \mathfrak{M}_{\mathbb{F},1} \to \mathfrak{M}_{\mathbb{F}} \to \mathfrak{M}_{\mathbb{F},2} \to 0$ which recovers the sequence of G_K-representations φ, G_{K_∞}-equivariantly after base-change to $C^\flat$, cf. [Bar20a, 5.1.3]. By equipping $\mathfrak{M}_{\mathbb{F},i}$ with the appropriate filtrations we view this as a sequence in $\mathrm{Mod}_{\mathcal{F}}$. Using part (2) of Proposition 4.2.1 we see that if $\mathfrak{M}_{\mathbb{F},i}$ has Hodge type μ_i then $\mu = (\mu_1, \mu_2)$. Both $V_{\mathbb{F},i}$ are also cyclotomic-free and so our inductive hypothesis produces lifts of $\mathfrak{M}_{\mathbb{F},i}$. Using Lemma 7.1.1 we obtain such a lift for $V_{\mathbb{F}}$ also. □

7.2. **Potential diagonalisability.** Let V be a G_K-stable $\mathcal{O}$-lattice inside a crystalline representation of Hodge type μ and set $V_{\mathbb{F}} = V \otimes_{\mathcal{O}} \mathbb{F}$. Following [BLGGT14] we say V is diagonalisable if V lies on the same irreducible component of $\mathrm{Spec}\, R^\mu$ (equivalently the same irreducible component of $\mathrm{Spec}\, R^\mu[\frac{1}{p}]$) as an E'-valued point, for E'/E finite, which is a direct sum of characters. We say V is potentially diagonalisable if $V|_{G_{K'}}$ is diagonalisable for K'/K some finite extension.

Lemma 7.2.1. *If $V[\frac{1}{p}]$ lies in the same irreducible component of $\mathrm{Spec}\, R^\mu$ as an E'-valued point whose corresponding representation admits a G_K-stable filtration with one-dimensional graded pieces then V is potentially diagonalisable.*

Proof. See [GL14, 2.1.2]. □

Corollary 7.2.2. *If μ is pseudo-Barsotti–Tate and $V_{\mathbb{F}}$ is cyclotomic-free then V is potentially diagonalisable.*

Proof. We can replace V by $V|_{G_{K'}}$ for any finite extension K'/K. Since $V_{\mathbb{F}}$ is cyclotomic-free we can choose K' so that $V_{\mathbb{F}}|_{G_{K'}}$ has one-dimensional Jordan–Holder factors and is also cyclotomic-free. Let x be the $\mathcal{O}$-valued point of $\mathcal{L}_R^{\mu,\mathrm{conv}}$ corresponding to the $\mathcal{O}$-valued point of $\mathrm{Spec}\, R^\mu$ associated to V. Corollary 7.1.4 produces an $\mathcal{O}$-valued point x' such that (i) x and x' coincide on the closed point of $\mathrm{Spec}\, \mathcal{O}$, and (ii) $\mathrm{Spec}\, \mathcal{O} \xrightarrow{x'} \mathcal{L}_R^{\mu,\mathrm{conv}} \to \mathrm{Spec}\, R^\mu$ corresponds to a deformation V' with every Jordan–Holder factor of $V'[\frac{1}{p}]$ one-dimensional. Part (i) implies x and x' lie in the same connected component of $\mathcal{L}_R^{\mu,\mathrm{conv}}$, and so the same irreducible component in view of Theorem 3.3.5. Hence V and V' lie on the same irreducible component of $\mathrm{Spec}\, R^\mu$. As V' is potentially diagonalisable by Lemma 7.2.1 so is V. □

References

[Bar18] Robin Bartlett, *Potentially diagonalisable lifts with controlled Hodge–Tate weights*, Preprint, arXiv:1812.02042 (2018).

[Bar20a] ______, *Inertial and Hodge-Tate weights of crystalline representations*, Math. Ann. **376** (2020), no. 1-2, 645–681.

[Bar20b] ———, *On the irreducible components of some crystalline deformation rings*, Forum of Mathematics, Sigma **8** (2020), e22.

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609.

[CEGS19] Ana Caraiani, Matthew Emerton, Toby Gee, and David Savitt, *Moduli stacks of two-dimensional galois representations*, Preprint, arXiv:1908.07019 (2019).

[Fon90] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, 1990, pp. 249–309.

[Gee06] Toby Gee, *A modularity lifting theorem for weight two Hilbert modular forms*, Math. Res. Lett. **13** (2006), no. 5-6, 805–811.

[GL14] Hui Gao and Tong Liu, *A note on potential diagonalizability of crystalline representations*, Math. Ann. **360** (2014), no. 1-2, 481–487.

[GLS14] Toby Gee, Tong Liu, and David Savitt, *The Buzzard-Diamond-Jarvis conjecture for unitary groups*, J. Amer. Math. Soc. **27** (2014), no. 2, 389–435.

[GLS15] ———, *The weight part of Serre's conjecture for* $\mathrm{GL}(2)$, Forum Math. Pi **3** (2015), e2, 52.

[Kis06] Mark Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, 2006, pp. 459–496.

[Kis08] ———, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546.

[Kis09] ———, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180.

[Kis10] ———, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.

[LLHLM20] Daniel Le, Bao V. Le Hung, Brandon Levin, and Stefano Morra, *Local models for galois deformation rings and applications* (2020), available at `arXiv:2007.05398`.

[PR09] G. Pappas and M. Rapoport, *Φ-modules and coefficient spaces*, Mosc. Math. J. **9** (2009), no. 3, 625–663, back matter.

[Sta17] The Stacks Project Authors, *stacks project*, 2017.

[Wan17] Xiyuan Wang, *Weight elimination in two dimensions when* $p = 2$ (2017), available at `arXiv:1711.09035`.