POTENTIAL DIAGONALISABILITY OF PSEUDO-BARSOTTI–TATE REPRESENTATIONS

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Abstract. Previous work of Kisin and Gee proves potential diagonalisability of two dimensional Barsotti–Tate representations of the Galois group of a finite extension $K/\mathbb{Q}_p$. In this paper we build upon their work by relaxing the Barsotti–Tate condition to one we call pseudo-Barsotti–Tate (which means that for certain embeddings $\kappa : K \to \overline{\mathbb{Q}}_p$ we allow the κ-Hodge–Tate weights to be contained in $[0, p]$ rather than $[0, 1]$). As an application we are able to obtain modularity lifting theorems for global representations which are pseudo-Barsotti–Tate above p.

Contents

1. Introduction

The motivation for this paper is to prove modularity lifting results for representations which are *pseudo-Barsotti–Tate* at places above p. This term is taken from [GLS15] and describes crystalline representations ρ of a finite extension $K/\mathbb{Q}_p$ whose Hodge–Tate weights are as follows. Let k denote the residue field of K. Then, for each embedding $\kappa : k \to \overline{\mathbb{F}}_p$ there exist an indexing $\kappa_0, \dots, \kappa_{e-1}$ of those embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$ which coincide with κ on k such that the κ_0-Hodge–Tate weights of ρ are contained in the interval $[0, p]$ and the κ_j-Hodge–Tate weights for $j = 1, \dots, e-1$ are contained in $[0, 1]$.

By combining the main result of this paper (cf. Theorem 1.0.2) with Theorem B of [BLGGT14] one immediately deduces the following (we refer to *loc. cit.* for any unfamiliar terminology):

Corollary 1.0.1. *Let F be an imaginary CM field with maximal totally real subfield F^+ and let c denote the non-trivial element of* $\mathrm{Gal}(F/F^+)$*. Suppose that $p \geq 5$ and that F does not contain a fifth root of unity. Let*

$$r : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

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be a continuous irreducible representation and let $\overline{r}$ *denote the semi-simplification of the reduction of* r. *Also let*

$$\mu : G_{F^+} \to \overline{\mathbb{Q}}_p^\times$$

be a continuous character. Suppose that (r,μ) *satisfy the following properties:*

(1) $r^c \cong r^\vee \mu$ *and* $\mu(c_v) = -1$ *for all* $v \mid \infty$.
(2) r *ramifies at finitely many primes.*
(3) $r|_{G_{F_v}}$ *is pseudo-Barsotti–Tate, had distinct Hodge–Tate weights, and has strongly cyclotomic-free reduction (in the sense of Notation 5.1.2) at all places* $v \mid p$.
(4) *The restriction* $\overline{r}|_{G_{F(\zeta_p)}}$ *is irreducible.*
(5) *There is a regular algebraic cuspidal polarised automorphic representation* (π,χ) *of* $\mathrm{GL}_n(\mathbb{A}_F)$ *such that*

$$(\overline{r},\overline{\mu}) \cong (\overline{r}_{p,\iota}(\pi), \overline{r}_{p,\iota}(\chi)\overline{\chi}_{\mathrm{cyc}}^{-1})$$

and π *is* ι*-ordinary.*

Then (r,μ) *is automorphic.*

To achieve this let $K/\mathbb{Q}_p$ be a finite extension and $\mathbb{F}$ a finite field of characteristic p. Let $V_{\mathbb{F}}$ be a continuous representation of G_K on a finite dimensional $\mathbb{F}$-vector space and $R^\square_{V_{\mathbb{F}}}$ the universal framed deformation ring. In [Kis08] it is shown there exists a reduced quotient $R^\square_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$ of $R^\square_{V_{\mathbb{F}}}[\frac{1}{p}]$ characterised by the property that, for $E/\mathbb{Q}_p$ a finite extension, $x : R^\square_{V_{\mathbb{F}}} \to E$ factors through $R^\square_{V_{\mathbb{F}}}$ if and only if the G_K-representation obtained by specialising the universal deformation over $R^\square_{V_{\mathbb{F}}}$ along x is crystalline with Hodge type $\mathbf{v}$. We then prove:

Theorem 1.0.2. *Assume that the Hodge type* $\mathbf{v}$ *is pseudo-Barsotti–Tate and* $V_{\mathbb{F}}$ *is strongly cyclotomic-free (again in the sense of Notation 5.1.2). Then every component of* $\mathrm{Spec}(R^\square_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$ *is potentially diagonalisable in the sense of* [BLGGT14].

More precisely, we show that if every Jordan–Holder factor of $V_{\mathbb{F}}$ is one dimensional, which can always be arranged by replacing K by a finite unramified extension and by enlarging the coefficient field $\mathbb{F}$, then every component of $\mathrm{Spec}(R^\square_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$ contains a $\overline{\mathbb{Q}}_p$-point corresponding to a crystalline $\overline{\mathbb{Q}}_p$-representation whose Jordan–Holder factors are all one-dimensional. Using [BLGGT14, 1.4.3], this easily implies potential diagonalisability as in the theorem.

When the Hodge type is Barsotti–Tate (that is when each Hodge–Tate weight is contained in $[0,1]$) then potential diagonalisability was previously known for two dimensional $V_{\mathbb{F}}$, cf. [Kis09, Gee06] as well as [GK14, 3.4.1].[1] When $K/\mathbb{Q}_p$ is unramified then pseudo-Barsotti–Tate representations are simply those with Hodge–Tate weights contained in $[0,p]$, and for such representations potential diagonalisability (in any dimension) was proven by the author in [Bar19]. In fact, many of the arguments in this paper recover those from [Bar19] in the unramified situation (cf. Section 6 for a some remarks regarding the differences). Finally, we mention the paper [GL14] which preceded [Bar19] and proved potential diagonalisability in the

[1]Thus, Theorem 1.0.2 is new even for Barsotti–Tate representations of dimension > 2. However, as the main application of potential diagonalisability is to modularity lifting theorems, the utility of these results is currently somewhat limited since Barsotti–Tate (or pseudo-Barsotti–Tate representations when $K/\mathbb{Q}_p$ is not unramified) representations in dimension > 2 never have distinct Hodge–Tate weights.

unramified situation for weights contained in $[0, p-1]$, using Fontaine–Laffaille theory.

Our approach to Theorem 1.0.2 is based on Kisin's original method from [Kis09] for analysing Barsotti–Tate deformation rings, though even in this Barsotti–Tate situation we offer a new point of view. Kisin's idea is to build a projective scheme over $\mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}})$ as a certain moduli space of Breuil–Kisin modules. After inverting p this moduli space identifies with a closed subspace of $\mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}}[\frac{1}{p}])$. On the other hand, the special fibres can be very different. As it turns out, this difference works to our advantage; typically the geometry of these moduli spaces of Breuil–Kisin modules is better behaved.

In this paper we consider a variant of this idea. We begin with the projective morphism $\mathcal{L}^{\leq h} \to \mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}})$ parametrising Breuil–Kisin modules of height $\leq h$, as constructed in [Kis08]. Then, for a fixed Hodge type $\mathbf{v}$ whose grading is concentrated in degrees $[0,h]$, we construct a second projective morphism $\widetilde{\mathcal{L}}^{\mathbf{v}} \to \mathcal{L}^{\leq h}$ by specifying, for a given Breuil–Kisin module, a filtration on the image of its Frobenius. The composite $\widetilde{\mathcal{L}}^{\mathbf{v}} \to \mathcal{L}^{\leq h} \to \mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}})$ is an isomorphism over $\mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$, which allows us to define $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ as the closure of $\mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$ in $\widetilde{\mathcal{L}}^{\mathbf{v}}$. The motivation for this construction comes from the techniques used in [PR03, §5] to analyse the local models which appear prominently in Kisin's approach to Barsotti–Tate deformations. These results suggest that, at least when $\mathbf{v}$ is Barsotti–Tate, the completed local rings of $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ at closed points should be formally smooth over $\mathcal{O}$, for $\mathcal{O}$ the ring of integers in a sufficiently large finite extension of $\mathbb{Q}_p$. The majority of this paper is devoted to proving this is the case, and that it remains the case when $\mathbf{v}$ is pseudo-Barsotti–Tate.

Since $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ is $\mathcal{O}$-flat and is such that $\widetilde{\mathcal{L}}^{\mathrm{loc}}[\frac{1}{p}] = \mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$, its dimension is computed by [Kis08]. It therefore suffices to show that the tangent space of $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$ has $\mathbb{F}$-dimension $\leq$ this dimension. To do this we observe that, in the usual way, there is a map from the tangent space of $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$ into an extension group computed in a category $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ whose objects are pairs $(\mathfrak{M}, \mathcal{F}_\bullet)$ where

- $\mathfrak{M}$ is a finite projective $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$-module equipped with a homomorphism $\mathfrak{M} \otimes_{k[[u]],\varphi} k[[u]] \to \mathfrak{M}$ with cokernel killed by u^{e+p-1}.
- $\mathcal{F}_\bullet$ is a filtration $u^{e-1}\mathfrak{M} = \mathcal{F}_e \subset \ldots \subset \mathcal{F}_0 = \mathfrak{M}$ by $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$-submodules.

The dimension of the kernel of this map can be computed (this is where the strong cyclotomic-free assumption is used) so it remains to describe the dimension of the image. To do this we show that if B is any finite $\mathbb{F}$-algebra then the B-valued points of $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ correspond to objects of $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ which, when viewed as modules over $\mathbb{F}_p[[u]]$, are such that

- There exists an $\mathbb{F}_p[[u]]$-basis (e_i) of $\mathcal{F}_1$ and integers $r_i \in [0,p]$ such that $u^{r_i} e_i$ is an $\mathbb{F}_p[[u^p]]$-basis of $\varphi(\mathfrak{M})$.

This is done by first observing that, since $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ is $\mathcal{O}$-flat, any B-point lifts to a C-point for some finite flat $\mathcal{O}$-algebra C. Such a C-point corresponds to the Breuil–Kisin module associated to a crystalline representation with C-coefficients. By viewing this crystalline representation as an $\mathcal{O}$-lattice it suffices to construct an appropriate basis of any Breuil–Kisin modules associated to $\mathcal{O}$-crystalline representations, which we do following the strategy of [GLS15]. We write $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ for the full subcategory of $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ whose objects satisfy this third bullet point. Using results previously proved in [Bar19] we show that $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ is an exact subcategory of $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ so that extensions groups computed in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ are contained inside those computed in $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$. By computing the dimensions of these extension groups we deduce the formal smoothness of the completed local rings of the $\widetilde{\mathcal{L}}^{\mathrm{loc}}$.

We conclude that each $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ is normal, and so the irreducible components of $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ coincide with the connected components of $\mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$. Thus, to prove potential diagonalisability it suffices to show that any Breuil–Kisin module associated to an $\mathcal{O}$-lattice inside a crystalline representation is congruent to a Breuil–Kisin module associated to an $\mathcal{O}$-lattice inside a crystalline representations whose Jordan–Holder factors are all 1-dimensional. To do this we use the already obtained formal smoothness to prove our second main theorem.

Theorem 1.0.3. *Assume $V_{\mathbb{F}}$ is strongly cyclotomic-free and has every Jordan–Holder factor one-dimensional. Then there exists a pseudo-Barsotti–Tate representation with reduction $V_{\mathbb{F}}$ if and only if there exists an $\mathfrak{M} \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ such that $V_{\mathbb{F}} \cong (\mathfrak{M} \otimes_{\mathbb{F}_p} C^{\flat})^{\varphi=1}$ as G_{K_∞}-representations.*

The Hodge types of the pseudo-Barsotti–Tate representations lifting $V_{\mathbb{F}}$ can be recovered easily from the grading on $\mathcal{F}^{\bullet}$, as well as from the modules $\mathfrak{M}^{\varphi} \cap u^i\mathcal{F}_1/\mathfrak{M}^{\varphi} \cap u^{i+1}\mathcal{F}_1$.

1.1. Notation. Let k be a finite extension of $\mathbb{F}_p$ of degree f, and let $K_0 = W(k)[\frac{1}{p}]$. Fix a totally ramified extension K of K_0 of degree e and fix a uniformiser $\pi \in K$. Let $E(u) \in W(k)[u]$ denote the minimal polynomial of π over K_0. We also fix a compatible system π^{1/p^∞} of p-th power roots of π inside a completed algebraic closure C of K and set $K_\infty = K(\pi^{1/p^\infty})$. When $p = 2$ we additionally choose π as in the following lemma, which is taken from [Wan17, 2.1]. This assumption is only used in the proof of Proposition 2.2.6.

Lemma 1.1.1. *If $p = 2$ then there exists a uniformiser $\pi \in K$ so that $K_\infty \cap K(\mu_{p^\infty}) = K$; here μ_{p^∞} denotes the group of p-th power roots of unity in C.*

Let $\mathcal{O}_{C^\flat}$ denote the inverse limit of the system $\mathcal{O}_C/p \leftarrow \mathcal{O}_C/p \leftarrow \dots$ whose transition maps are given by $x \mapsto x^p$. This is an integral domain of characteristic p, whose field of fractions $C^\flat$ is algebraically closed. The action of G_K on $\mathcal{O}_C/p$ induces an action on $\mathcal{O}_{C^\flat}$ and $C^\flat$, and hence on the $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$ and $W(C^\flat)$. The compatible system π^{1/p^∞} gives rise to an element $\pi^\flat \in \mathcal{O}_{C^\flat}$. Via this choice we embed $\mathfrak{S} = W(k)[[u]] \to A_{\mathrm{inf}}$ by $u \mapsto [\pi^\flat]$. This embedding is φ-equivariant when $\mathfrak{S}$ is equipped with the Frobenius which on $W(k)$ is the Witt vector Frobenius and which sends $u \mapsto u^p$. It is also G_{K_∞}-equivariant when $\mathfrak{S}$ is equipped with the trivial G_{K_∞}-action. The embedding $\mathfrak{S} \to A_{\mathrm{inf}}$ extends to a φ, G_{K_∞}-equivariant embedding $\mathcal{O}_{\mathcal{E}} \to W(C^\flat)$, where $\mathcal{O}_{\mathcal{E}}$ denotes the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$.

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2. Filtrations on the image of Frobenius

2.1. Rational filtrations. Let V be a crystalline representation of G_K. Associated to V is the K_0-vector space

$$D_{\mathrm{crys}}(V) = D = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K}$$

Via the inclusion $B_{\mathrm{crys}} \otimes_{K_0} K \to B_{\mathrm{dR}}$ we identify

$$D_K = D \otimes_{K_0} K = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$$

and equip this K-vector space with the filtration given by $\mathrm{Fil}(D_K) = (V \otimes_{\mathbb{Q}_p} t^i B_{\mathrm{dR}}^+)^{G_K}$. Assume that $\mathrm{Fil}^0(D_K) = D_K$. By combining [Kis10, 1.2.1] and [Kis06, 1.2.6] we functorially associate to V an $\mathfrak{S}[\frac{1}{p}]$-module $\mathfrak{M}$ equipped with a homomorphism $\varphi : \mathfrak{M} \otimes_{\mathfrak{S}[\frac{1}{p}],\varphi} \mathfrak{S} \to \mathfrak{M}$ whose cokernel is killed by a power of $E(u)$, and which has the following properties:

(1) There is a functorial φ-equivariant isomorphism

$$\mathfrak{M}^\varphi \otimes_{\mathfrak{S}[\frac{1}{p}]} \mathcal{O}_{[0,r)} \cong D \otimes_{K_0} \mathcal{O}_{[0,r)}$$

where $\mathfrak{M}^\varphi$ denotes the image of φ and $\mathcal{O}_{[0,r)}$ denotes the subring of $K_0[[u]]$ consisting of series which converge on the open disk of radius r for any $r \in (|\pi|, |\pi|^{1/p})$.

(2) Tensoring (1) along the map $\mathcal{O}_{[0,r)} \to K$ given by $u \mapsto \pi$ produces a surjection $\mathfrak{M}^\varphi \otimes_{\mathfrak{S}[\frac{1}{p}]} \mathcal{O}^{[0,r)} \to D_K$. The image of $\mathfrak{M}^\varphi \cap E(u)^i \mathfrak{M}$ under this map equals $\mathrm{Fil}^i(D_K)$.

(3) There exists a φ, G_{K_∞}-equivariant identification

$$\mathfrak{M} \otimes_{\mathfrak{S}[\frac{1}{p}]} W(C^\flat)[\tfrac{1}{p}] \cong V \otimes_{\mathbb{Q}_p} W(C^\flat)[\tfrac{1}{p}]$$

Here the left hand side is equipped with a G_{K_∞}-action by $W(C^\flat)[\frac{1}{p}]$-semilinearly extending the trivial G_{K_∞}-action on $\mathfrak{M}$ (note this make sense since G_{K_∞} acts trivially on $\mathfrak{S}[\frac{1}{p}] \hookrightarrow W(C^\flat)[\frac{1}{p}]$). The right hand side is equipped the $W(C^\flat)[\frac{1}{p}]$-semilinear extension of the trivial Frobenius on V.

Suppose further that V is a finite free B-module with B a finite $\mathbb{Q}_p$-algebra and that the G_K action on V is B-linear. Then functoriality of $V \mapsto \mathfrak{M}$ implies that $\mathfrak{M}$ is an $\mathfrak{S} \otimes_{\mathbb{Z}_p} B$-module and that both φ and the isomorphism from (1) are B-linear. If B is an algebra over a sufficiently large extension E of $\mathbb{Q}_p$ then, as is explained in Notation 2.1.1 below, there is extra structure on the filtered module D_K. The aim of this section is to refine (2) to account for this. We will do this using ideas from [GLS15].

Notation 2.1.1. Fix a finite extension E of $\mathbb{Q}_p$ containing a Galois closure of K. Let $\mathcal{O}$ denote the ring of integers in E. Label the elements of $\mathrm{Hom}_{\mathbb{Q}_p}(K, E)$ as follows:

- Fix $\kappa_0 \in \mathrm{Hom}_{\mathbb{Q}_p}(K_0, E)$ and for $i \in [1, f-1]$ set $\kappa_i = \kappa_{i-1} \circ \varphi$.
- Then choose an indexing κ_{ij} of $\mathrm{Hom}_{\mathbb{Q}_p}(K, E)$ for $i \in [0, f-1], j \in [0, e-1]$ such that $\kappa_{ij}|_{K_0} = \kappa_i$.

There is an isomorphism $K \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{ij} E$ given by $a \otimes b \mapsto (\kappa_{ij}(a)b)_{ij}$. Thus, any $K \otimes_{\mathbb{Q}_p} E$-module M can be written as $M = \prod_{ij} M_{ij}$ with M_{ij} equal to the basechange of M along the projection $K \otimes_{\mathbb{Q}_p} E \to E$ onto the ij-th factor. As a submodule $M_{ij} \subset M$ is characterised by the property that $(a \otimes 1)m = (1 \otimes \kappa_{ij}(a))m$ for every $a \in K$.

From now on assume B is a finite E-algebra. Then D is a finite free $K_0 \otimes_{\mathbb{Q}_p} B$-module and so D_K is a finite free $K \otimes_{\mathbb{Q}_p} B$-module whose filtration is by $K \otimes_{\mathbb{Q}_p} B$-submodules. Thus, D_K decomposes as a product $\prod_{ij} D_{K,ij}$ of filtered B-modules and we can consider the surjection:

$$f_{ij} : \mathfrak{M}^{\varphi} \otimes_{\mathfrak{S}} \mathcal{O}_{[0,r)} \xrightarrow{(2)} D_K \to D_{K,ij}$$

Concretely this map is obtained as follows. The inclusion $\mathcal{O}_{[0,r)} \subset K_0[[u]]$ produces an inclusion $\mathcal{O}_{[0,r)} \otimes_{\mathbb{Q}_p} B \to K_0[[u]] \otimes_{\mathbb{Q}_p} B \cong \prod_i B[[u]]$ which sends $(\sum_{i\geq 0} a_i u^i) \otimes b \mapsto (\sum_{i\geq 0} \kappa_i(a) b u^i)_i$. Then f_{ij} is obtained by composing with the isomorphism in (1) and then tensoring $\mathfrak{M}^{\varphi} \otimes_{\mathfrak{S}} \mathcal{O}_{[0,r)}$ along the map $\mathcal{O}_{[0,r)} \otimes_{\mathbb{Q}_p} B \to B$ given by $(\sum_{i\geq 0} a_i u^i) \otimes b \mapsto b(\kappa_{ij}(\sum_{i\geq 0} a_i \pi^i))$. In particular, the kernel of f_{ij} equals $E_{ij}(u)\mathfrak{M}^{\varphi} \otimes_{\mathbb{Z}_p} \mathcal{O}_{[0,r)}$ where $E_{ij}(u) \in \mathcal{O}_{[0,r)} \otimes_{\mathbb{Q}_p} B$ is the element corresponding to

$$(1, \ldots, \underbrace{(u - \kappa_{ij}(\pi))}_{i\text{-th position}}, \ldots, 1) \in \prod_i B[[u]]$$

In fact, each $E_{ij}(u)$ is contained in $W(k)[u] \otimes_{\mathbb{Z}_p} \mathcal{O}$. This follows because the idempotents in $K_0 \otimes_{\mathbb{Q}_p} E$ are contained in $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$ (which is only the case because $W(k)$ is unramified over $\mathbb{Z}_p$).

Proposition 2.1.2. *For integers n_{ij} set*

$$\mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi}) = \mathfrak{M}^{\varphi} \cap (\prod_{ij} E_{ij}(u)^{n_{ij}})\mathfrak{M}$$

Then the image of $\mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi})$ under f_{ij} equals $\mathrm{Fil}^{n_{ij}}(D_{K,ij})$.

Before giving the proof we record an important corollary.

Corollary 2.1.3. *Fix (i',j') and set $n'_{ij} = n_{ij}$ for $(i,j) \neq (i',j')$ and $n_{i'j'}+1$. Then $\mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi})/\mathrm{Fil}^{\{n'_{ij}\}}(\mathfrak{M}^{\varphi})$ is B-projective of rank*

$$\sum_{i=0}^{n_{i'j'}} \mathrm{rank}_B(\mathrm{gr}^i(D_{K,i'j'}))$$

Proof. We begin by recalling that each $\mathrm{gr}^i(D_{K,ij})$ is B-projective, cf. [Kis09, 1.3.4]. Next we fit $\mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi})/\mathrm{Fil}^{\{n'_{ij}\}}(\mathfrak{M}^{\varphi})$ into an exact sequence

$$0 \to \frac{\mathrm{Fil}^{\{n''_{ij}\}}(\mathfrak{M}^{\varphi})}{\mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi})} \xrightarrow{E_{i'j'}(u)} \frac{\mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi})}{\mathrm{Fil}^{\{n'_{ij}\}}(\mathfrak{M}^{\varphi})} \xrightarrow{f_{i'j'}} \mathrm{gr}^{n_{i'j'}}(D_{K,i'j'}) \to 0$$

where $n''_{ij} = n_{ij}$ for $(i,j) \neq (i',j')$ and $n''_{i'j'} = n_{i'j'} - 1$. Proposition 2.1.2 implies this sequence is exact on the right. Exactness in the middle and on the left follows from the fact that the kernel of $f_{i'j'}$, when restricted to $\mathfrak{M}^{\varphi}$, equals $E_{i'j'}(u)\mathfrak{M}^{\varphi}$. Since we have assumed the grading on D_K is concentrated in positive degrees, the corollary will follow provided we can show $\mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi})/\mathrm{Fil}^{\{n'_{ij}\}}(\mathfrak{M}^{\varphi}) = 0$ when $n_{i'j'}$ is sufficiently small. This is clear from the definition. $\square$

Proof of Proposition 2.1.2. Consider a collection of subgroups $F^{\{n_{ij}\}}$ of $\mathfrak{M}^{\varphi}$. We claim the following conditions uniquely determine the $F^{\{n_{ij}\}}$.

(1) Set $n'_{i'j'} = n_{i'j'} - 1$ and $n'_{ij} = n_{ij}$ for $(i,j) \neq (i',j')$. Then $E_{i'j'}(u)F^{\{n'_{ij}\}} \subset F^{\{n_{ij}\}}$.
(2) With n'_{ij} as in (1), if $x \in \mathfrak{M}^{\varphi}$ and $E_{i'j'}(u)x \in F^{\{n_{ij}\}}$ then $x \in F^{\{n'_{ij}\}}$.
(3) If $n_{ij} \leq 0$ for all ij then $F^{\{n_{ij}\}} = \mathfrak{M}^{\varphi}$.

To see this suppose $G^{\{n_{ij}\}}$ is another such sequence. By symmetry, it suffices to show $G^{\{n_{ij}\}} \subset F^{\{n_{ij}\}}$. We do this by increasing induction on $n = \max\{n_{ij}\}$. If $n \leq 0$ there is nothing to prove. If not take $x \in G^{\{n_{ij}\}}$. Using (1) for G we have $(\prod_{ij} E_{ij}(u))x \in G^{\{n_{ij}-1\}}$. The inductive hypothesis gives $(\prod_{ij} E_{ij}(u))x \in F^{\{n_{ij}-1\}}$. Using (2) we deduce $x \in F^{\{n_{ij}\}}$.

It is clear that the $\mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi})$ satisfy these three properties. It therefore suffices to define a collection of subgroups $F^{\{n_{ij}\}}$ which satisfy (1)-(3) and which have $f_{ij}(F^{\{n_{ij}\}}) = \mathrm{Fil}^{n_{ij}}(D_{K,ij})$ for each i,j. To do this we follow the construction of [GLS15, 2.1.7]. Let S denote the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. Then $S[\frac{1}{p}]$ identifies as a subring of $K_0[[u]]$ containing elements of the form $\sum_{i\geq 0} a_i \frac{u^i}{e(i)!}$; here $e(i) = \lfloor \frac{i}{e} \rfloor$ and $a_i \in K_0$ is a sequence converging p-adically to zero. In particular, $S[\frac{1}{p}]$ contains $\mathcal{O}_{[0,r)}$ and the evaluation map $u \mapsto \pi$ on $\mathcal{O}_{[0,r)}$ extends to a surjection $S[\frac{1}{p}] \to K$. Hence, we obtain surjections

$$f_{ij} : \mathcal{D} := D \otimes_{K_0} S[\tfrac{1}{p}] \to D_K \to D_{K,ij}$$

extending those defined above. The φ on $\mathcal{O}_{[0,r)}$ also extends uniquely to $\varphi : S[\frac{1}{p}] \to S[\frac{1}{p}]$. Finally, we equip $S[\frac{1}{p}]$ with a K_0-linear derivative $\partial = -u\frac{d}{du}$. This allows us to define an operator $\mathcal{N}$ on $\mathcal{D}$ by $\mathcal{N}(a \otimes s) = a \otimes \partial(s)$ for $a \in D, s \in S[\frac{1}{p}]$, as well as:

- $\mathrm{Fil}^{\{n_{ij}\}}(\mathcal{D}) = \mathcal{D}$ whenever every n_{ij} is ≤ 0.
- For general n_{ij}, inductively define $\mathrm{Fil}^{\{n_{ij}\}}(\mathcal{D})$ as the set of $x \in \mathcal{D}$ with $f_{ij}(x) \in \mathrm{Fil}^{n_{ij}}(D_{K,ij})$ for every i,j and $\mathcal{N}(x) \in \mathrm{Fil}^{\{n_{ij}-1\}}(\mathcal{D})$.

Defining $F^{\{n_{ij}\}} = \mathfrak{M}^{\varphi} \cap \mathrm{Fil}^{\{n_{ij}\}}(\mathcal{D})$, it is immediate that condition (3) above is satisfied. Conditions (1) and (2) follow from 6. and 8. of [GLS15, 2.1.9].

It remains to show that f_{ij} induces a surjection $F^{\{n_{ij}\}} \to \mathrm{Fil}^{n_{ij}}(D_{K,ij})$. For this we use that, for each $r \geq 1$, every $s \in S[\frac{1}{p}]$ can be written as $s_1 + s_2$ with $s_1 \in K_0[u]$ and $s_2 \in \mathrm{Fil}^r(S)[\frac{1}{p}]$ where $\mathrm{Fil}^r(S)$ equals the closure of the ideal of S generated by $E(u)^i/i!$ for $i \geq r$. Since $\mathcal{D} = \mathfrak{M}^{\varphi} \otimes_{\mathfrak{S}[\frac{1}{p}]} S[\frac{1}{p}]$ it follows that every $x \in \mathrm{Fil}^{\{n_{ij}\}}(\mathcal{D})$ can be written as $x_1 + x_2$ with $x_1 \in \mathfrak{M}^{\varphi}[\frac{1}{p}]$ and $x_2 \in \mathrm{Fil}^r(S)[\frac{1}{p}]\mathcal{D}$. Using 5. of [GLS15, 2.1.9], we deduce that r may be chosen sufficiently large that $\mathrm{Fil}^r(S)[\frac{1}{p}]\mathcal{D} \subset \mathrm{Fil}^{\{n_{ij}\}}(\mathcal{D})$. Since $f_{ij}(\mathrm{Fil}^r(S)) = 0$ for each ij, we conclude that for each $x \in \mathrm{Fil}^{\{n_{ij}\}}(\mathcal{D})$ there exists an $x_1 \in \mathrm{Fil}^{\{n_{ij}\}}(\mathfrak{M}^{\varphi})$ with $f_{ij}(x_1) = f_{ij}(x)$. Thus, the surjectivity of f_{ij} follows from the surjectivity of $f_{ij} : \mathrm{Fil}^{\{n_{ij}\}}(\mathcal{D}) \to \mathrm{Fil}^{n_{ij}}(D_{K,ij})$ which is established in 4. of [GLS15, 2.1.9]. $\square$

2.2. Integral filtrations.

Notation 2.2.1. Maintain the notation from Section 2.1. To each G_{K_∞}-stable $\mathbb{Z}_p$-lattice $V^\circ \subset V$ there corresponds a finite free $\mathfrak{S}$-submodule $\mathfrak{M}^\circ \subset \mathfrak{M}$ such that the Frobenius on $\mathfrak{M}$ restricts to a map $\mathfrak{M}^\circ \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \to \mathfrak{M}$ with cokernel killed by a power of $E(u)$ and such that the identification from (3) restricts to an identification

$$\mathfrak{M}^\circ \otimes_{\mathfrak{S}} W(C^\flat) \cong V^\circ \otimes_{\mathbb{Z}_p} W(C^\flat) \tag{2.2.2}$$

Moreover, the association $V \mapsto \mathfrak{M}^\circ$ defines a bijection between G_K-stable $\mathbb{Z}_p$-lattices $V^\circ \subset V$ and $\mathfrak{M}^\circ \subset \mathfrak{M}$ as above (cf. [Kis06, 2.1.15]). For a finite $\mathbb{Z}_p$-subalgebra $C \subset B$

with $C[\frac{1}{p}] = B$, the functoriality of this bijection means it restricts to a bijection between G_{K_∞}-stable C-submodules in $V_C^\circ \subset V$ with $V_C^\circ[\frac{1}{p}] = V$ and submodules $\mathfrak{M}_C^\circ \subset \mathfrak{M}$ as above which are additionally stable under the C-action on $\mathfrak{M}$. Such $\mathfrak{M}_C^\circ$ are therefore modules over $\mathfrak{S}\otimes_{\mathbb{Z}_p} C$. While these $\mathfrak{M}_C^\circ$ will be finite free over $\mathfrak{S}$ they need not be finite free, or even finite projective, over $\mathfrak{S}\otimes_{\mathbb{Z}_p} C$. One exception to this is when $C = \mathcal{O}$, cf. [GLS14].

In this section we fix an $\mathfrak{M}^\circ \subset \mathfrak{M}$ corresponding to a G_K-stable $\mathcal{O}$-lattice $V^\circ \subset V$. Our aim is then to describe the intersection of $\mathfrak{M}^\circ$ with the first few steps of the filtration defined in Section 2.1. More precisely, we let $E_0(u) = \prod_i E_{i0}(u) \in \mathfrak{S}\otimes_{\mathbb{Z}_p}\mathcal{O}$ and consider

$$(\mathfrak{M}^\circ)^\varphi \cap E_0(u)^r \mathfrak{M}^\circ[\tfrac{1}{p}]$$

for $r \in [0,p]$. If we write $\mathrm{Fil}^r(\mathcal{D})$ for $\mathrm{Fil}^{\{n_{ij}\}}(\mathcal{D})$, where $n_{ij} = 0$ if $j \neq 0$ and $n_{i0} = r$, then this can be written as $\mathrm{Fil}^r(\mathcal{D}) \cap (\mathfrak{M}^\circ)^\varphi$ by the proof of Proposition 2.1.2.

For the following lemma we recall that under the isomorphism $\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathcal{O} \cong \prod_i \mathcal{O}[[u]]$ the element $E_0(u)$ corresponds to $(u-\kappa_{i0}(\pi))_i$. Thus, if we write $\pi^\circ \in \mathfrak{S}\otimes_{\mathbb{Z}_p}\mathcal{O}$ for the element corresponding to $(\kappa_{i0}(\pi))_i$, then $E_0(u) = u - \pi^\circ$.

Lemma 2.2.3. *For $x \in \mathcal{D}$ and $i \geq 0$, inductively define*

$$x^{(i)} = \sum_{l=0}^{i-1} \frac{H(u)^l}{l!}\mathcal{N}^l(x^{(i-1)}) \in \mathcal{D}$$

where $H(u) = \frac{E_0(u)}{\pi^\circ} = \frac{u-\pi^\circ}{\pi^\circ}$. If f_0 denotes the surjection $\prod_i f_{i0} : \mathcal{D} \to \prod_i D_{K,i0}$ and $f_0(x) \in \prod_i \mathrm{Fil}^r(D_{K,i0})$ then $x^{(i)} \in \mathrm{Fil}^{\delta_i}(\mathcal{D})$ where $\delta_i = \min\{i,r\}$.

Proof. Induct on i. When $i=0$ there is nothing to prove. For general i, since $f_0(x) = f_0(x^{(i)})$, it suffices to show $\mathcal{N}(x^{(i)}) \in \mathrm{Fil}^{\delta_i - 1}(\mathcal{D})$. Since $\delta_{i-1} \geq \delta_i - 1$ we may instead show $\mathcal{N}(x^{(i)}) \in \mathrm{Fil}^{\delta_{i-1}}(\mathcal{D})$. We compute

$$\begin{aligned}\mathcal{N}(x^{(i)}) &= \sum_{l=0}^{i-1}\left(\frac{H(u)^{l-1}\partial(H(u))}{(l-1)!} + \frac{H(u)^l}{l!}\mathcal{N}^{l+1}(x^{i-1})\right) \\ &= \underbrace{\frac{H(u)^{i-1}}{(i-1)!}\mathcal{N}^i(x^{(i-1)})}_{(a)} + \sum_{l=1}^{i-1}\underbrace{(1+\partial(H(u)))\frac{H(u)^{l-1}}{(l-1)!}\mathcal{N}^l(x^{(i-1)})}_{(b)}\end{aligned}$$

(here ∂ denotes the E-linear extension to $S\otimes_{\mathbb{Z}_p} E$ of $\partial = -u\frac{d}{du}$ on S). It follows from 6. of [GLS15, 2.1.9] that if $x \in \mathrm{Fil}^r(\mathcal{D})$ then $H(u)^l x \in \mathrm{Fil}^{r+l}(\mathcal{D})$. From this we deduce (a) is contained in $\mathrm{Fil}^{i-1}(\mathcal{D}) \subset \mathrm{Fil}^{\delta_{i-1}}(\mathcal{D})$. The inductive hypothesis implies $x^{(i-1)} \in \mathrm{Fil}^{\delta_{i-1}}(\mathcal{D})$ and so $\mathcal{N}^l(x^{(i-1)}) \in \mathrm{Fil}^{\delta_{i-1}-l}(\mathcal{D})$. Since $1+\partial(H(u)) = -H(u)$, each (b) term is contained $\mathrm{Fil}^{\delta_{i-1}-l+l}(\mathcal{D}) = \mathrm{Fil}^{\delta_{i-1}}(\mathcal{D})$ also. $\square$

Lemma 2.2.4. *For $x \in \mathcal{D}$, $x^{(i)} - x$ can be written as a $\mathbb{Z}$-linear combination of terms of the form $\frac{H(u)^a}{a'!}\mathcal{N}^b(x)$ for $a,b \geq 1$ and $1 \leq a' \leq a$.*

Proof. Arguing by induction on i, it suffices to show, for $a,b \geq 1$ and $1 \leq a' \leq a$, that

$$\frac{H(u)^l}{l!}\mathcal{N}^l(\frac{H(u)^a}{a'}\mathcal{N}^b(x)) = \sum_{k=0}^{l}\binom{l}{k}\frac{H(u)^l\partial^k(H(u)^a)}{l!a'!}\mathcal{N}^{l-k+b}(x)$$

is a $\mathbb{Z}$-linear combination of terms as in the lemma. This will follow from the claim that $\frac{\partial^k(H(u)^a)}{a!}$ is a $\mathbb{Z}$-linear combination of terms of the form $\frac{H(u)^{a'}}{a'!}$ for $1 \le a' \le a$. To see this note that $\partial(H(u)^a) = aH(u)^{a-1}(-1-H(u))$ and so $\partial^k(H(u)^a)/a!$ equals

$$\begin{aligned}\frac{1}{(a-1)!}\partial^{k-1}(H(u)(-1-H(u))) &= \frac{1}{(a-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}\partial^j(H(u)^{a-1})\partial^{k-1-j}(-1-H(u)) \\ &= \frac{1}{(a-1)!}(-1-H(u))\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{k-1-j}\partial^j(H(u)^{a-1})\end{aligned}$$

(for the second equality we've used that $\partial^n(H(u)) = (-1)^n(-1-H(u))$ for $n > 0$). The claim then follows by induction on k. □

In the following we view $\mathfrak{M}^\circ$ simply as a module over $\mathbb{Z}_p[[u]]$. Set $M_0 = f_0(\mathfrak{M}^{\circ,\varphi})$. Proposition 2.1.2 implies that M_0 is an $\mathcal{O}$-lattice, and so also a $\mathbb{Z}_p$-lattice, inside $\prod_i D_{K,i0}$. We equip M_0 with the filtration $\mathrm{Fil}^i(M_0) = M_0 \cap \prod_i \mathrm{Fil}^i(D_{K,i0})$.

Lemma 2.2.5. *There exists a $\mathbb{Z}_p[[u^p]]$-basis (e_i) of $\varphi(\mathfrak{M})$ and integers (r_i) such that $\mathrm{Fil}^n(M_0)$ if generated over $\mathbb{Z}_p$ by those $f_0(e_i)$ with $r_i \ge n$.*

Proof. The graded pieces of the filtration on M_0 are, by construction, p-torsionfree. Thus, there exists a $\mathbb{Z}_p$-basis (d_i) of M_0 and integers (r_i) such that $\mathrm{Fil}^n(M_0)$ is generated over $\mathbb{Z}_p$ by the d_i's for which $r_i \ge n$. If $(\hat{e}_i)$ is any $\mathbb{Z}_p[[u^p]]$-basis of $\varphi(\mathfrak{M}^\circ)$ then $(f_0(\hat{e}_i))$ is a $\mathbb{Z}_p$-basis of M_0. Hence, there exists $A \in \mathrm{GL}(\mathbb{Z}_p)$ such that $(d_i) = (f_0(\hat{e}_i))A$. Since the Frobenius on $\mathfrak{M}^\circ$ is $\mathbb{Z}_p$-linear we see that $(e_i) = (\hat{e}_i)A$ is another $\mathbb{Z}_p[[u^p]]$-basis of $\varphi(\mathfrak{M}^\circ)$ and, since f_0 is $\mathbb{Z}_p$-linear, we have $(f_0(e_i)) = d_i$. □

Proposition 2.2.6. *For any $a, b \ge 1$ and any $1 \le a' \le a$ one can write $\frac{H(u)^a}{a'!}\mathcal{N}^b(e_i)$ as $x_1 + x_2$ with $x_2 \in H(u)^p\mathcal{D}$ and $x_1 \in \pi^\circ(\mathfrak{M}^\circ)^\varphi$.*

Proof. Using [GLS14, 4.11] when $p > 2$, and the variant described in [Wan17, §4] when $p = 2$,[2] implies that $\mathcal{N}^b(e_i)$ can be written as a linear combination of the e_j with entries in $\mathcal{I}_b = \sum_{m=0}^b p^{b-m}u^{pm}S' \subset S$. Here $S' = W(k)[[u^p, \frac{u^{ep}}{p}]][\frac{1}{p}] \cap S$. It therefore suffices to show that if $x \in \mathcal{I}_b$ then $\frac{H(u)^a}{a'!}x \in S \otimes_{\mathbb{Z}_p} E$ can be written as $x_1 + x_2$ with $x_1 \in \pi^\circ(\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O})$ and $x_2 \in H(u)^p(S \otimes_{\mathbb{Z}_p} E)$.

Lemma 2.3.9 of [GLS15] shows that $x \in \mathcal{I}_b$ can, when viewed as an element of $\prod_i E[[u]]$, be written as $(\sum_{n\ge 0} x_n^{(i)}\left(\frac{u-\kappa_{i0}(\pi)}{\kappa_{i0}(\pi)}\right)^n)_i$ so that $\pi^{p+(b-1)\min\{e,p\}}$ divides $x_n^{(i)}$ in $\mathcal{O}_K$ for $0 \le n < p$. We point out that this lemma is stated only for $b \le p$. However, the proof goes through unchanged for any $b \ge 1$. It follows that the image of $\frac{H(u)^a}{a'!}x$ in $\prod_i E[[u]]$ can be written as

$$\frac{1}{a'!}\left(\sum_{a+n<p} x_n^{(i)}\left(\frac{u-\kappa_{i0}(\pi)}{\kappa_{i0}(\pi)}\right)^{a+n}\right)_i + \left(\sum_{a+n\ge p} x_n^{(i)}\left(\frac{u-\kappa_{i0}(\pi)}{\kappa_{i0}(\pi)}\right)^{a+n}\right)_i$$

The first term is contained in $\pi^\circ(\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O})$ and the second in $H(u)^p(S \otimes_{\mathbb{Z}_p} E)$, so the claim follows. □

[2]This is where our assumption that π is chosen as in Lemma 1.1.1 is used.

Corollary 2.2.7. *Suppose the* (e_i) *and* (r_i) *are as in Lemma 2.2.5. Then there exists a* $\mathbb{Z}_p[[u]]$*-basis* $(\hat{e}_i)$ *of* $(\mathfrak{M}^\circ)^\varphi$ *such that* $\hat{e}_i - e_i \in \pi^\circ(\mathfrak{M}^\circ)^\varphi$ *and such that*

$$(\mathfrak{M}^\circ)^\varphi \cap E_0(u)^r \mathfrak{M}^\circ[\tfrac{1}{p}] = \bigoplus_i E_0(u)^{\max\{r-r_i,0\}} \mathbb{Z}_p[[u]]\hat{e}_i$$

for $0 \leq r \leq p$*. In particular,* $(\mathfrak{M}^\circ)^\varphi \cap E_0(u)^r \mathfrak{M}^\circ[\frac{1}{p}] = (\mathfrak{M}^\circ)^\varphi \cap E_0(u)^r \mathfrak{M}^\circ$ *for* $0 \leq r \leq p$*.*

Proof. Lemma 2.2.3 implies $e_i^{(r_i)} \in \mathrm{Fil}^{r_i}(\mathcal{D})$. Combining Lemma 2.2.4 and Proposition 2.2.6 allows us to write $e_i^{(r_i)} = \hat{e}_i + H(u)^p f_i$ for some $f_i \in \mathcal{D}$ and some $\hat{e}_i$ with $\hat{e}_i - e_i \in \pi^\circ(\mathfrak{M}^\circ)^\varphi$ and $f_0(\hat{e}_i) = f_0(e_i)$. The fact that $\hat{e}_i - e_i \in \pi^\circ(\mathfrak{M}^\circ)^\varphi$ implies the $\hat{e}_i$ form a $\mathbb{Z}_p[[u]]$-basis of $(\mathfrak{M}^\circ)^\varphi$. Since

$$\hat{e}_i \in (\mathfrak{M}^\circ)^\varphi \cap \mathrm{Fil}^{r_i}(\mathcal{D}) = (\mathfrak{M}^\circ)^\varphi \cap E_0(u)^{r_i}\mathfrak{M}^\circ[\tfrac{1}{p}]$$

we also have that $X_r := \bigoplus_i E_0(u)^{\max\{r-r_i,0\}}\mathbb{Z}_p[[u]]\hat{e}_i$ is contained in $Y_r = \mathfrak{M}^{\circ,\varphi} \cap E_0(u)^r\mathfrak{M}^\circ[\frac{1}{p}]$. We want to show $X_r = Y_r$; for this note that $f_0 : Y_r \to \prod_i \mathrm{Fil}^r(D_{K,i0})$ is surjective with kernel $E_0(u)Y_{r-1}$. Since $f_0(\hat{e}_i) = f_0(e_i)$, $f_0 : X_r \to \prod_i \mathrm{Fil}^r(D_{K,i0})$ is surjective. Thus, if $X_{r-1} = Y_{r-1}$ then, because $E_0(u)X_{r-1} \subset X_r$, it would follow that $X_r = Y_r$. The result therefore follows by induction since we know $X_0 = Y_0$. □

3. Moduli of Breuil–Kisin modules

3.1. A Demazure resolution. In this section we modify a construction from [Kis08] to produce a moduli space of Breuil–Kisin modules over Galois deformation rings.

Construction 3.1.1. Let V be a finitely generated $\mathbb{Z}_p$-algebra equipped with a continuous $\mathbb{Z}_p$-linear action of G_{K_∞}. The results of [Fon90] associate to V an etale φ-module. That is, a finitely generated $\mathcal{O}_\mathcal{E}$-module M equipped with an isomorphism $\varphi : M \otimes_{\mathcal{O}_\mathcal{E},\varphi} \mathcal{O}_\mathcal{E} \to M$ and a φ, G_{K_∞}-equivariant isomorphism

$$V \otimes_{\mathbb{Z}_p} W(C^\flat) \cong M \otimes_{\mathcal{O}_\mathcal{E}} W(C^\flat)$$

Recall that $\mathcal{O}_\mathcal{E}$ is defined in Section 1.1. The association $V \mapsto M$ is functorial and so if V is an A-algebra with G_{K_∞}-equivariant A-action then M is naturally an $\mathcal{O}_{\mathcal{E},A} = \mathcal{O}_\mathcal{E} \otimes_{\mathbb{Z}_p} A$-module.

Let A be a complete Noetherian local ring with residue field a finite extension of $\mathbb{F}_p$, and V_A a finite free A-module equipped with a continuous A-linear action of G_K. associates to $V_A|_{G_{K_\infty}}$ an $\mathcal{O}_{\mathcal{E},A}$-module M_A.

Definition 3.1.2. For any A-algebra B set $V_B = V_A \otimes_A B$. Suppose the homomorphism $A \to B$ factors as $A \to A' \to B$ with A' finite as a $\mathbb{Z}_p$-module; we can then define $M_B = M_{A'} \otimes_{A'} B$ where $M_{A'}$ is the $\mathcal{O}_{\mathcal{E},A'}$-module corresponding to $V_{A'}$ as in Construction 3.1.1. The $\mathcal{O}_{\mathcal{E},B}$-module M_B is independent of A'. Following [Kis08, 1.1] we define, for $h \geq 0$ and B as above, the set $\mathcal{L}^{\leq h}(V_B)$ of projective $\mathfrak{S}_B = \mathfrak{S} \otimes_{\mathbb{Z}_p} B$-submodules $\mathfrak{M}_B \subset M_B$ such that

(1) $\mathfrak{M}_B \otimes_\mathfrak{S} \mathcal{O}_\mathcal{E} = M_B$.
(2) If $\mathfrak{M}_B^\varphi$ denotes the image of $\mathfrak{M}_B \otimes_{\mathfrak{S},\varphi} \mathfrak{S}$ under $M_B \otimes_{\mathcal{O}_\mathcal{E},\varphi} \mathcal{O}_\mathcal{E} \to M_B$ then $E(u)^h\mathfrak{M}_B \subset \mathfrak{M}_B^\varphi \subset \mathfrak{M}_B$.

If B' is a B-algebra and $\mathfrak{M}_B \in \mathcal{L}^{\le h}(V_B)$ then $\mathfrak{M}_{B'} = \mathfrak{M}_B \otimes_B B' \in \mathcal{L}^{\le h}(V_{B'})$. Thus, $B \mapsto \mathcal{L}^{\le h}(V_B)$ is functor on the category of A-algebras which factor through a $\mathbb{Z}_p$-finite quotient of A.

Proposition 3.1.3. *Assume that A is Artinian. Then the functor $B \mapsto \mathcal{L}^{\le h}(V_B)$ on A-algebras is represented by a projective A-scheme $\mathcal{L}^{\le h}$ which is functorial in A. Moreover, $\mathcal{L}^{\le h}$ is equipped with a very ample line bundle which is likewise functorial in A.*

Proof. This is [Kis08, 1.3]. □

We are going to produce a variant of this construction. Let $E/\mathbb{Q}_p$ be as in Notation 2.1.1 and assume that A is an $\mathcal{O}$-algebra. Then, for each $j \in [0, e-1]$, choose integers $h_j \ge 0$. If $h \ge \max\{h_j\}$ define $\mathcal{L}^{\le h_j}(V_B) \subset \mathcal{L}^{\le h}(V_B)$ to be the subset containing those $\mathfrak{M}_B \in \mathcal{L}^{\le h}(V_B)$ with

$$\left(\prod_{ij} E_{ij}(u)^{h_j}\right)\mathfrak{M}_B \subset \mathfrak{M}_B^{\varphi} \subset \mathfrak{M}_B \tag{3.1.4}$$

Recall here that $E_{ij}(u) \in \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ is the element defined in the paragraph before Proposition 2.1.2.

Lemma 3.1.5. *Assume that A is Artinian. The functor $\mathcal{L}^{\le h_{ij}}(V_B)$ on A-algebras B is represented by a closed subscheme $\mathcal{L}^{\le h_{ij}} \subset \mathcal{L}^{\le h}$. Again $\mathcal{L}^{\le h_j}$ is functorial in A and is equipped with a functorial very ample line bundle.*

Proof. The argument given in [Kis09, 1.2.2] shows that if $\mathfrak{M}_B \in \mathcal{L}^{\le h}(V_B)$ then $\mathfrak{M}_B/\mathfrak{M}_B^{\varphi}$ is finite projective as a B-module. Thus (3.1.4) describes a closed subscheme of $\mathcal{L}^{\le h}$. The very ample line bundle on $\mathcal{L}^{\le h_j}$ is obtained by restricting that on $\mathcal{L}^{\le h}$. □

To refine this construction further we fix a Hodge type $\mathbf{v}$, i.e. a finite free $K \otimes_{\mathbb{Q}_p} E$-module $D(\mathbf{v})$ equipped with a decreasing, separated, and exhaustive filtration by $K \otimes_{\mathbb{Q}_p} E$-submodules. Assume that for $i \in [0, f-1]$ and $j \in [0, e-1]$ the grading on $D(\mathbf{v})_{ij}$ is concentrated in degrees $[0, h_j]$. We say a crystalline representation V_B on a finite free B-module, with B a finite E-algebra, has Hodge type $\mathbf{v}$ if the associated filtered φ-module $D = D_{\mathrm{crys}}(V_B)$ is such that

$$\mathrm{gr}^n(D_K) \cong \mathrm{gr}^n(D(\mathbf{v})) \otimes_E B$$

as $K \otimes_{\mathbb{Q}_p} B$-modules for every $n \in \mathbb{Z}$.

Definition 3.1.6. For any A-algebra B which factors through a $\mathbb{Z}_p$-finite quotient of A define $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_B)$ to be the set whose elements are pairs $(\mathfrak{M}_B, \mathcal{F}^\bullet)$ with $\mathfrak{M}_B \in \mathcal{L}^{\le h_j}(V_B)$ and with $\mathcal{F}^\bullet$ a collection of $\mathfrak{S}_B$-submodules of $\mathfrak{M}_B$:

$$\left(\prod_{ij} E_{ij}(u)^{h_j}\right)\mathfrak{M}_B = \mathcal{F}^{\sum_j h_j} \subset \ldots \subset \mathcal{F}^0 = \mathfrak{M}_B^{\varphi}$$

such that, if $n = k + \sum_{j=0}^{l-1} h_j$ and $0 \le k < h_l$, then:

- $(\prod_i E_{il}(u))\mathcal{F}^{n-1} \subset \mathcal{F}^n$.
- $\mathcal{F}^{n-1}/\mathcal{F}^n$ is B-projective of rank $\sum_i \sum_{m=0}^{k} \dim_E(\mathrm{gr}^m(D(\mathbf{v})_{il}))$.

Clearly $B \mapsto \widetilde{\mathcal{L}}^{\mathbf{v}}(V_B)$ is functorial in B.

Corollary 3.1.7. *The functor $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_B)$ on A-algebras B, with $\mathfrak{m}_A^i B = 0$ for some i, is represented by a projective A-scheme $\widetilde{\mathcal{L}}^{\mathbf{v}}$.*

Proof. If A is Artinian then $\widetilde{\mathcal{L}}^{\mathbf{v}}$ is obtained as a closed subscheme of a flag variety over $\mathcal{L}^{\leq h_j}$ classifying filtrations on the locally free coherent sheaf $\mathcal{F} : B \mapsto \mathfrak{M}_B^{\varphi}/(\prod_{ij} E_{ij}(u)^{h_j})\mathfrak{M}_B$. The pull-back of the ample line bundle on $\mathcal{L}^{\leq h_j}$ gives an ample line bundle on $\widetilde{\mathcal{L}}^{\mathbf{v}}$.

Both $\widetilde{\mathcal{L}}^{\mathbf{v}}$ and the ample line bundle are functorial in A. Therefore, if we drop the assumption that A is Artinian, the previous paragraph applied to $A/\mathfrak{m}_A^i$ for $i \geq 1$ produces a compatible system of projective $A/\mathfrak{m}_A^i$-schemes together with a compatible system of ample line bundles. By formal GAGA [Gro61, 5.4.5] (see also [Sta17, Tag 089A]) the resulting formal scheme is obtained by completing the projective A-scheme $\widetilde{\mathcal{L}}^{\mathbf{v}}$ along $\mathfrak{m}_A$. □

Proposition 3.1.8. *Suppose that $A \to C$ is a homomorphism with C finite flat over $\mathcal{O}$. Then*

1. *Elements of $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_C)$ correspond bijectively to morphisms $\mathrm{Spec}(C) \to \widetilde{\mathcal{L}}^{\mathbf{v}}$ of A-schemes.*
2. *If $C = \mathcal{O}$ and V_C is crystalline of Hodge type $\mathbf{v}$ then $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_C)$ contains precisely one element $(\mathfrak{M}_C, \mathcal{F}_C^{\bullet})$.*

Proof. For (1), a morphism $\mathrm{Spec}(C) \to \widetilde{\mathcal{L}}^{\mathbf{v}}$ of A-schemes induces morphisms $\mathrm{Spec}(C/\mathfrak{m}_A^i C) \to \widetilde{\mathcal{L}}^{\mathbf{v}}$ and so a compatible system of elements in $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_{C/\mathfrak{m}_A^i C})$. Since C is $\mathcal{O}$-finite it is $\mathfrak{m}_A C$-adically complete, and so the inverse limit of this system produces an element of $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_C)$. Conversely, any element of $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_C)$ induces a compatible system of A-morphisms $\mathrm{Spec}(C/\mathfrak{m}_A^i C) \to \widetilde{\mathcal{L}}^{\mathbf{v}}$, and hence a morphism $\mathrm{Spec}(C) \to \widetilde{\mathcal{L}}^{\mathbf{v}}$.

For (2), combining [Kis06, 2.1.12] and [Kis10, 1.2.1] we deduce that $\mathcal{L}^{\leq h}(V_C)$ contains precisely one element $\mathfrak{M}$. Also $\mathfrak{M}[\frac{1}{p}]$ equals the $\mathfrak{S}[\frac{1}{p}]$-module denoted $\mathfrak{M}$ in Section 2.1 when V is taken equal to $V_C[\frac{1}{p}]$. Using Proposition 2.1.2 we deduce that $(\prod_{ij} E_{ij}(u)^{h_j})\mathfrak{M}[\frac{1}{p}] \subset \mathfrak{M}^{\varphi}[\frac{1}{p}]$ and so, since $\mathfrak{M}/\mathfrak{M}^{\varphi}$ is $\mathcal{O}$-flat, we deduce $(\prod_{ij} E_{ij}(u)^{h_j})\mathfrak{M} \subset \mathfrak{M}^{\varphi}$. Thus $\mathfrak{M} \in \mathcal{L}^{\leq h_j}(V_C)$. For $n = k + \sum_{j=0}^{l-1} h_j$ with $0 \leq k < h_l$ set

$$\mathcal{F}^n = \mathfrak{M} \cap \left(\prod_i E_{i0}(u)^{h_0} E_{i1}(u)^{h_1} \dots E_{i(l-1)}(u)^{h_{l-1}} E_{il}(u)^k \right) \mathfrak{M}[\tfrac{1}{p}]$$

Clearly this filtration satisfies the first bullet point of Definition 3.1.6 and is such that $\mathcal{F}^{n-1}/\mathcal{F}^n$ flat over $\mathcal{O}$. From Corollary 2.1.3 we deduce that it also satisfies the second bullet point of Definition 3.1.6. Thus $(\mathfrak{M}, \mathcal{F}^{\bullet}) \in \widetilde{\mathcal{L}}^{\mathbf{v}}(V_C)$. Finally, for uniqueness, suppose $(\mathfrak{M}, \mathcal{G}^{\bullet})$ is another element of $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_C)$. By induction on n we show $\mathcal{G}^n \subset \mathcal{F}^n$; certainly this is true for $n = 0$ and if $\mathcal{G}^{n-1} \subset \mathcal{F}^{n-1}$ then $x \in \mathcal{G}^n$ implies $(\prod_i E_{ij}(u))x \in \mathcal{G}^{n-1} \subset \mathcal{F}^{n-1}$ for an appropriate j, it follows that $x \in \mathcal{F}^n$. Since $\mathcal{F}^0 = \mathcal{G}^0$ we therefore have maps $\mathcal{G}^0/\mathcal{G}^n \to \mathcal{F}^0/\mathcal{F}^n$ which, being surjections between projective $\mathcal{O}$-modules of the same rank, are isomorphisms. We conclude $\mathcal{F}^n = \mathcal{G}^n$ which completes the proof. □

By [Kis08, 2.7.7] there exists a quotient $A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$ of $A[\frac{1}{p}]$ such that, for any finite E-algebra B, a map $A[\frac{1}{p}] \to B$ of E-algebras factors through $A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$ if and only if V_B is crystalline of Hodge type $\mathbf{v}$. Since $A[\frac{1}{p}]$ is Jacobson, this condition determines $A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$ uniquely if we further ask that this quotient be reduced.

Corollary 3.1.9. *The closed immersion* $\mathrm{Spec}(A[\frac{1}{p}])^{\mathrm{cr},\mathbf{v}} \to \mathrm{Spec}(A[\frac{1}{p}])$ *factors through a closed immersion* $\mathrm{Spec}(A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}) \to \widetilde{\mathcal{L}}^{\mathbf{v}}[\frac{1}{p}]$.

Proof. Let $A^{\mathrm{cr},\mathbf{v}}$ be a quotient of A so that $A^{\mathrm{cr},\mathbf{v}}[\frac{1}{p}] = A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$. It suffices to show that $\widetilde{\mathcal{L}}^{\mathbf{v}} \otimes_A A^{\mathrm{cr},\mathbf{v}} \to A^{\mathrm{cr},\mathbf{v}}$ becomes an isomorphism after inverting p. Proposition 3.1.8 implies this map induces an injection on C-valued points for any A-algebra C which is finite and flat over $\mathbb{Z}_p$. The arguments of [Kis08, 1.6.4] then imply this morphism becomes a closed immersion after inverting p. If follows that $\widetilde{\mathcal{L}}^{\mathbf{v}} \otimes_A A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}} = \mathrm{Spec}(A')$ for some quotient $A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}} \to A'$.

Proposition 3.1.8 also implies that $\widetilde{\mathcal{L}}^{\mathbf{v}} \otimes_A A^{\mathrm{cr},\mathbf{v}} \to A^{\mathrm{cr},\mathbf{v}}$ induces a bijection on $\mathcal{O}_{E'}$-valued points for E'/E a finite extension. From this we deduce that the kernel of $A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}} \to A'$ contains every maximal ideal of $A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$. As $A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$ is reduced and Jacobson this kernel is zero. ☐

3.2. Pseudo-Barsotti–Tate. In this paper we shall apply the previous construction when $\mathbf{v}$ is pseudo-Barsotti–Tate in the following sense.

Definition 3.2.1. A Hodge type $\mathbf{v}$ is pseudo-Barsotti–Tate if

- The grading on $D(\mathbf{v})_{i0}$ is concentrated in degrees $[0,p]$.
- The grading on $D(\mathbf{v})_{ij}$ for $j \in [1,e-1]$ is concentrated in degrees $[0,1]$.

In this case the integers h_j above may be chosen so that $h_0 = p$ and $h_j = 1$ for $j \in [1,e-1]$. Thus, elements of $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_B)$ consist of a Breuil–Kisin module $\mathfrak{M}$ and a filtration

$$\left(\prod_i E_{i0}(u)^p E_{i1}(u) \dots E_{ie-1}(u)\right)\mathfrak{M} = \mathcal{F}^{p+e-1} \subset \dots \subset \mathcal{F}^0 = \mathfrak{M}^{\varphi}$$

Via Corollary 3.1.9 we identify $\mathrm{Spec}(A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$ as a closed subscheme of $\widetilde{\mathcal{L}}^{\mathbf{v}}[\frac{1}{p}]$ and define $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ to be its closure inside $\widetilde{\mathcal{L}}^{\mathbf{v}}$. The following lemma provides the key to controlling $\widetilde{\mathcal{L}}^{\mathrm{loc}}$.

Lemma 3.2.2. *Suppose B is a finite local $\mathbb{F}$-algebra and $(\mathfrak{M},\mathcal{F}^\bullet)$ corresponds to a B-valued point of $\widetilde{\mathcal{L}}^{\mathrm{loc}}$. Then:*

(1) *Under the identification $\mathfrak{M} \otimes_{k[[u]]} C^\flat = V_B \otimes_{\mathbb{F}_p} C^\flat$, the $C^\flat$-semilinear extension of the G_K-action on V_B is such that*
$$(\sigma - 1)(m) \in u^{\frac{p+e-1}{p-1}} \mathfrak{M} \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$$
for all $m \in \mathfrak{M}$ and $\sigma \in G_K$.

(2) *There exists an $\mathbb{F}_p[[u^p]]$-basis (e_i) of $\varphi(\mathfrak{M})$ and integers $r_i \in [0,p]$ such that $\mathcal{F}^n$ is generated over $\mathbb{F}_p[[u]]$ by the $(u^{\max\{n-r_i,0\}}e_i)$ for $n \in [0,p]$. In particular, $\mathcal{F}^n = \mathfrak{M}^{\varphi} \cap u^{p-n}\mathcal{F}^p$.*

Proof. The B-valued point corresponds to a local homomorphism $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x} \to B$ for some closed point $x \in \widetilde{\mathcal{L}}^{\mathrm{loc}}$. By construction $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x}$ is reduced and $\mathbb{Z}_p$-flat. Since it is the localisation of a finite type algebra over a complete local Noetherian ring it is also Nagata (cf. [Sta17, Tag 032E]). Thus [Bar19, 4.1.2] implies that $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x} \to B$ factors through a quotient $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x} \to C$ with C finite flat over $\mathbb{Z}_p$. We conclude

$(\mathfrak{M}, \mathcal{F}^\bullet) = (\mathfrak{M}_C, \mathcal{F}_C^\bullet) \otimes_C B$ for $(\mathfrak{M}_C, \mathcal{F}_C^\bullet) \in \widetilde{\mathcal{L}}^{\mathbf{v}}(V_C)$ for V_C a G_K-stable lattice inside a crystalline representation of Hodge type $\mathbf{v}$.

For (1) we use [Bar19, 2.1.12]. This asserts that, under the identification $\mathfrak{M}_C \otimes_{\mathfrak{S}} W(C^\flat) = V_C \otimes_{\mathbb{Z}_p} W(C^\flat)$, the $W(C^\flat)$-semilinear extension of the G_K-action on V_C is such that

$$(\sigma - 1)(m) \in [\pi^\flat]\varphi^{-1}(\mu)\mathfrak{M}_C \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$$

for every $m \in \mathfrak{M}_C$ and $\sigma \in G_K$. Here $\mu = [\epsilon] - 1 \in A_{\mathrm{inf}}$ where $\epsilon \in \mathcal{O}_{C^\flat}$ corresponds to a fixed choice of p-th power roots of unity in $C^\flat$. A standard calculation shows that the image of μ in $\mathcal{O}_{C^\flat}$ is contained in $u^{\frac{ep}{p-1}}\mathcal{O}_{C^\flat}$. Thus, the image of $[\pi^\flat]\varphi^{-1}(\mu)$ in $\mathcal{O}_{C^\flat}$ is contained in $u^{\frac{e+p-1}{p-1}}\mathcal{O}_{C^\flat}$ and (1) follows. Finally, (2) follows immediately from Corollary 2.2.7. □

4. Strong divisibility

4.1. Preliminaries.

Definition 4.1.1. Write $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ for the category whose objects are pairs $(\mathfrak{M}, \mathcal{F}_\bullet)$ with $\mathfrak{M}$ a finite projective $\mathfrak{S}_{\mathbb{F}} = k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$-module equipped with a homomorphism $\varphi : \mathfrak{M} \otimes_{k[[u]],\varphi} k[[u]] \to \mathfrak{M}$ whose cokernel is killed by u^{p+e-1}, and $\mathcal{F}_\bullet$ a filtration

$$u^{e-1}\mathfrak{M} = \mathcal{F}_e \subset \ldots \subset \mathcal{F}_0 = \mathfrak{M}$$

such that $u\mathcal{F}^{i-1} \subset \mathcal{F}^i$ for $i \geq 2$. Morphisms are φ-equivariant maps of $\mathfrak{S}_{\mathbb{F}}$-modules which respect the filtrations. Let $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}} \subset \mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ be the full subcategory whose objects are pairs $(\mathfrak{M}, \mathcal{F}_\bullet) \in \mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ for which there exists an $\mathbb{F}_p[[u]]$-basis (e_i) of $\mathcal{F}_1$ with integers $r_i \in [0,p]$ so that $(u^{r_i}e_i)$ is an $\mathbb{F}_p[[u^p]]$-basis of $\varphi(\mathfrak{M})$. Observe that if $(\mathfrak{M}, \mathcal{F}_\bullet) \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ then $u^p\mathcal{F}_1 \subset \mathfrak{M}^\varphi \subset \mathcal{F}_1$.

For $(\mathfrak{M}, \mathcal{F}_\bullet) \in \mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ define

- $\mathrm{Fil}^i(\mathfrak{M}^\varphi) = \mathfrak{M}^\varphi \cap u^i\mathcal{F}_1$.
- $\mathrm{Fil}^i(\overline{\mathfrak{M}})$ equal to the image of $\mathrm{Fil}^i(\mathfrak{M}^\varphi)$ in $\overline{\mathfrak{M}} = \mathfrak{M}^\varphi/u\mathfrak{M}^\varphi$.
- $\mathrm{Fil}^i(\mathfrak{M}) = \{m \in \mathfrak{M} \mid \varphi(m) \in u^i\mathfrak{M}\}$.

Since φ restricts to an automorphism of $k \otimes_{\mathbb{F}_p} \mathbb{F}$ the composite $\mathfrak{M} \xrightarrow{\varphi} \mathfrak{M}^\varphi \to \overline{\mathfrak{M}}$ is surjective. It also maps $\mathrm{Fil}^i(\mathfrak{M})$ into $\mathrm{Fil}^i(\overline{\mathfrak{M}})$.

Lemma 4.1.2. $(\mathfrak{M}, \mathcal{F}_\bullet) \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ *if and only if the image of* $\mathrm{Fil}^i(\mathfrak{M})$ *in* $\overline{\mathfrak{M}}$ *identifies with* $\mathrm{Fil}^i(\overline{\mathfrak{M}})$.

Proof. In this paper we just need the *only if* implication, so we leave the converse to the reader (argue as in [Bar18a, 5.3.4]). Choose an $\mathbb{F}_p[[u]]$-basis (e_i) of $\mathcal{F}_1$ so that $(u^{r_i}e_i)$ generate $\varphi(\mathfrak{M})$ over $\mathbb{F}_p[[u^p]]$. Then $\mathrm{Fil}^n(\mathfrak{M}^\varphi)$ is generated over $\mathbb{F}_p[[u]]$ by $u^{\min\{n,r_i\}}e_i$. Thus, $\mathrm{Fil}^n(\overline{\mathfrak{M}})$ is generated by the image of $u^{r_i}e_i$ in $\overline{\mathfrak{M}}$ for those i with $r_i = n$. Since $\varphi^{-1}(u^{r_i}e_i) \in \mathrm{Fil}^i(\mathfrak{M})$ we conclude the map $\mathrm{Fil}^i(\mathfrak{M}) \to \mathrm{Fil}^i(\overline{\mathfrak{M}})$ is surjective. □

To motivate these constructions suppose V is a G_K-stable lattice inside a crystalline representation and $(\mathfrak{M}^\circ, \widetilde{\mathcal{F}}^\bullet)$ is the unique element inside $\widetilde{\mathcal{L}}^{\mathbf{v}}(V)$. Then, if $(\mathfrak{M}, \mathcal{F}^\bullet) = (\mathfrak{M}^\circ, \widetilde{\mathcal{F}}^\bullet) \otimes_{\mathcal{O}} \mathbb{F}$, $\mathcal{F}^n \subset u^p\mathfrak{M}$ for $n \geq p$ (indeed $u^{p+e-1-n}\mathcal{F}^n \subset \mathcal{F}^{e+p-1} = u^{p+e-1}\mathfrak{M}$), so we can define $\mathcal{F}_j = u^{-p}\mathcal{F}^{p+j-1}$ for $j \geq 1$ to obtain an object $(\mathfrak{M}, \mathcal{F}_\bullet) \in \mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$. Moreover:

Proposition 4.1.3. *$(\mathfrak{M},\mathcal{F}_\bullet)$ is an object of* $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ *and*

- $\dim_{\mathbb{F}} \mathrm{gr}^n(\overline{\mathfrak{M}}) = \sum_{i\in[0,f-1]} \dim_E \mathrm{gr}^n(D(\mathbf{v})_{i0})$ *for* $n\in[0,p]$.
- $\dim_{\mathbb{F}} \mathcal{F}_j/\mathcal{F}_{j+1} = \sum_{i\in[0,f-1]} \dim_E \mathrm{gr}^0(D(\mathbf{v})_{ij})$ *for* $j\in[1,e-1]$.

Proof. Part (2) of Lemma 3.2.2 implies $(\mathfrak{M},\mathcal{F}_\bullet)\in \mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$. It also implies that $\mathcal{F}^n = \mathrm{Fil}^n(\mathfrak{M}^\varphi)$, and so

$$\dim_{\mathbb{F}} \mathrm{gr}^n(\mathfrak{M}^\varphi) = \sum_{i\in[0,f-1]}\sum_{m=0}^{n} \dim_{\mathbb{F}} \mathrm{gr}^m(D(\mathbf{v})_{i0})$$

since $(\mathfrak{M},\mathcal{F}^\bullet)\in\widetilde{\mathcal{L}}^{\mathbf{v}}(V\otimes_{\mathcal{O}}\mathbb{F})$. The first bullet point therefore follows from consideration of the exact sequence

$$0\to \mathrm{gr}^{n-1}(\mathfrak{M}^\varphi)\xrightarrow{u}\mathrm{gr}^n(\mathfrak{M}^\varphi)\to \mathrm{gr}^n(\overline{\mathfrak{M}})\to 0$$

The second bullet point follows since multiplication by u^p induces an identification $\mathcal{F}_j/\mathcal{F}_{j+1}\to\mathcal{F}^{p+j-1}/\mathcal{F}^{p+j}$. □

4.2. Extension groups. We would like to compute extension groups computed in $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$. To make sense of this we must equip $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ with the structure of an exact category. To do this we first equip $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{BK}}$ with the structure of an exact category by asserting that a sequence

$$0\to(M,\mathcal{E}_\bullet)\to(N,\mathcal{F}_\bullet)\to(P,\mathcal{G}_\bullet)\to 0$$

in $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{BK}}$ is exact if it induces exact sequences $0\to\mathcal{E}_i\to\mathcal{F}_i\to\mathcal{G}_i\to 0$ for each i. That this makes $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{BK}}$ into an exact category comes down to showing that if

$$0\to(\mathfrak{M},\mathcal{E}_\bullet)\xrightarrow{g}(\mathfrak{N},\mathcal{F}_\bullet)\to(\mathfrak{P},\mathcal{G}_\bullet)\to 0$$

is an exact sequence in $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{BK}}$ then: (i) pull-backs along any morphism $(\mathfrak{Z},\mathcal{Z}_\bullet)\to(\mathfrak{P},\mathcal{G}^\bullet)$ exist and fit into an exact sequence

$$0\to(\mathfrak{M},\mathcal{G}_\bullet)\to(\mathfrak{Z},\mathcal{Z}_\bullet)\times_{(\mathfrak{P},\mathcal{G}_\bullet)}(\mathfrak{N},\mathcal{F}_\bullet)\to(\mathfrak{Z},\mathcal{Z}_\bullet)\to 0$$

and (ii) pushouts along any morphism $f:(\mathfrak{M},\mathcal{E}_\bullet)\to(\mathfrak{Z},\mathcal{Z}_\bullet)$ exist and fit into an exact sequence

$$0\to(\mathfrak{M},\mathcal{E}_\bullet)\to(\mathfrak{Z},\mathcal{Z}_\bullet)\coprod_{(\mathfrak{M},\mathcal{E}_\bullet)}(\mathfrak{N},\mathcal{F}_\bullet)\to(\mathfrak{P},\mathcal{G}_\bullet)\to 0$$

Both these statements are easy to check. For example, the pushout in (ii) is constructed, as a filtered module, as the cokernel of $(f,-g):(M,\mathcal{E}_\bullet)\to(Z,\mathcal{Z}_\bullet)\oplus(N,\mathcal{F}_\bullet)$. It is equipped with the unique Frobenius making this surjection φ-equivariant. Likewise, the pullback is obtained as a kernel of a similar such map.

Lemma 4.2.1. *Suppose that* $0\to(\mathfrak{M},\mathcal{E}_\bullet)\xrightarrow{g}(\mathfrak{N},\mathcal{F}_\bullet)\to(\mathfrak{P},\mathcal{G}_\bullet)\to 0$ *is an exact sequence in* $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{BK}}$ *with* $(\mathfrak{N},\mathcal{F}_\bullet)\in\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$. *Then* $(\mathfrak{M},\mathcal{E}_\bullet)$ *and* $(\mathfrak{P},\mathcal{G}_\bullet)$ *are objects of* $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$.

Proof. This follows from Lemma 4.3.1 in the next section applied with $B=k\otimes_{\mathbb{F}_p}\mathbb{F}$ and with the pair $(\mathfrak{M},\mathfrak{N})$ equal to $(\varphi(\mathfrak{N}),\mathcal{F}_1)$. □

A sequence in $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ is exact if it is exact in $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{BK}}$. Lemma 4.2.1 then implies this makes $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ also into an exact category. Thus, we can consider the extension groups $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P},\mathfrak{M})$ computed in $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$ as subgroups of the extension groups computed in $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{BK}}$.

4.3. The image of Frobenius. Let B be a ring, $\mathfrak{M}$ a finite projective $B[[u^p]]$-module, and $\mathfrak{N}$ a finite projective $B[[u]]$-module with $u^n\mathfrak{N} \subset \mathfrak{M}^\varphi \subset u^{-n}\mathfrak{N}$ for some $n \in \mathbb{Z}$ and $\mathfrak{M}^\varphi := \mathfrak{M} \otimes_{B[[u^p]]} B[[u]]$. Define

- $\mathrm{Fil}^i(\mathfrak{M}^\varphi) = \mathfrak{M}^\varphi \cap u^i\mathfrak{N}$.
- $\mathrm{Fil}^i(\overline{\mathfrak{M}})$ as the image of $\mathrm{Fil}^i(\mathfrak{M}^\varphi)$ in $\overline{\mathfrak{M}} = \mathfrak{M}/u^p\mathfrak{M} = \mathfrak{M}^\varphi/u\mathfrak{M}^\varphi$.

For two other such pairs $(\mathfrak{M}_j, \mathfrak{N}_j)$ with $j = 1,2$ we consider an exact sequence $0 \to (\mathfrak{M}_1, \mathfrak{N}_1) \to (\mathfrak{M}, \mathfrak{N}) \to (\mathfrak{M}_2, \mathfrak{N}_2) \to 0$, i.e. an exact sequence $0 \to \mathfrak{M}_1 \to \mathfrak{M} \to \mathfrak{M}_2 \to 0$ of $B[[u^p]]$-modules which induces an exact sequence $0 \to \mathfrak{N}_1 \to \mathfrak{N} \to \mathfrak{N}_2 \to 0$ after inverting u.

Lemma 4.3.1. *Suppose that B is a finite $\mathbb{F}_p$-algebra and that there exists an $\mathbb{F}_p[[u]]$-basis of $\mathfrak{N}$ and integers $r_i \in [0,p]$ such that $(u^{r_i}e_i)$ is an $\mathbb{F}_p[[u^p]]$-basis of $\mathfrak{M}$. Then*

1. *There exists an $\mathbb{F}_p[[u]]$-basis (e_i) of $\mathfrak{N}_2$ and integers r_i such that $(u^{r_i}e_i)$ is an $\mathbb{F}_p[[u^p]]$-basis of $\mathfrak{M}_2$. Likewise with $(\mathfrak{M}_2, \mathfrak{N}_2)$ replaced by $(\mathfrak{M}_1, \mathfrak{N}_1)$.*
2. *The induced sequence $0 \to \mathrm{gr}^i(\overline{\mathfrak{M}}_1) \to \mathrm{gr}^i(\overline{\mathfrak{M}}) \to \mathrm{gr}^i(\overline{\mathfrak{M}}_2) \to 0$ of B-modules is exact for each i.*
3. *$\mathrm{gr}^i(\overline{\mathfrak{M}})$ is B-projective for each i.*

When such a basis exists we say $(\mathfrak{M}, \mathfrak{N})$ is strongly divisible.

Proof. For (1) and (2) we can assume $B = \mathbb{F}_p$. Then both these results are proved in [Bar18a], at least when $\mathfrak{N}$ is a Breuil–Kisin module over $\mathbb{F}_p[[u]]$ and $\mathfrak{M} = \varphi(\mathfrak{N})$. To prove the general case we show that we can always put ourselves in this situation.

Since $\mathfrak{N}_i$ has $\mathbb{F}_p[[u]]$-rank equal to the $\mathbb{F}_p[[u^p]]$-rank of $\mathfrak{M}_i$, we can always choose isomorphisms $\mathfrak{N}_i \otimes_{\mathbb{F}_p[[u]],\varphi} \mathbb{F}_p[[u^p]] \to \mathfrak{M}_i$ where φ on $\mathbb{F}_p[[u]]$ is the p-th power map. Likewise, we can choose isomorphisms $\mathfrak{N} \otimes_{\mathbb{F}_p[[u]],\varphi} \mathbb{F}_p[[u^p]] \to \mathfrak{M}$ so that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{N}_1 \otimes_{\mathbb{F}_p[[u]],\varphi} \mathbb{F}_p[[u^p]] & \to & \mathfrak{N} \otimes_{\mathbb{F}_p[[u]],\varphi} \mathbb{F}_p[[u^p]] & \to & \mathfrak{N}_2 \otimes_{\mathbb{F}_p[[u]],\varphi} \mathbb{F}_p[[u^p]] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{M}_1 & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{M}_2 & \longrightarrow & 0 \end{array}$$

In the notation of [Bar18a, 5.3.1], this produces an exact sequence $0 \to \mathfrak{N}_1 \to \mathfrak{N} \to \mathfrak{N}_2 \to 0$ in $\mathrm{Mod}^{\mathrm{BK}}_{\mathbb{F}_p}$. We see also that the strong divisibility of $(\mathfrak{M}, \mathfrak{N})$, or $(\mathfrak{M}_i, \mathfrak{N}_i)$, is equivalent to $\mathfrak{N}$, or $\mathfrak{N}_i$, being objects of $\mathrm{Mod}^{\mathrm{SD}}_{\mathbb{F}_p}$ as defined in [Bar18a, 5.3.7]. Thus, (1) and (2) follow from [Bar18a, 5.4.6].

Now we turn to (3). For any finite B-module N we can view $(\mathfrak{M} \otimes_B N, \mathfrak{N} \otimes_B N)$ as a pair as above. If $(\mathfrak{M}, \mathfrak{N})$ is strongly divisible and N is B-free then $(\mathfrak{M} \otimes_B N, \mathfrak{N} \otimes_B N)$ is also strongly divisible. By writing N as a quotient of a finite free B-module and using (1) we see the same is true for any N. Furthermore, there are induced maps

$$\mathrm{gr}^i(\overline{\mathfrak{M}}) \otimes_B N \to \mathrm{gr}^i(\overline{\mathfrak{M}} \otimes_B N)$$

When N is free as a B-module then it is easy to see this map is an isomorphism. For general N, (2) implies both the source and target are right exact in N, and so, by choosing a presentation of N by finite free B-modules and using the five lemma, we deduce the map is an isomorphism in general. Since (2) actually implies the target of this map is exact in N, the same is true of the source; hence $\mathrm{gr}^i(\overline{\mathfrak{M}})$ is B-projective. $\square$

Lemma 4.3.2. *Suppose that $(\mathfrak{M}_i, \mathfrak{N}_i)$ are strongly divisible for $i = 1, 2$. Then $(\mathfrak{M}, \mathfrak{N})$ is strongly divisible if and only if there exists an $\mathbb{F}_p[[u^p]]$-splitting of $\mathfrak{M} \to \mathfrak{M}_2$ which, after basechanging to $B((u))$, maps $\mathfrak{N}_2$ into $\mathfrak{N}$.*

Proof. It is easy to check that if such a splitting exists then $(\mathfrak{M}, \mathfrak{N})$ is strongly divisible so we focus on the converse. If $(\mathfrak{M}, \mathfrak{N})$ is strongly divisible then it follows from the above that $\mathrm{gr}^i(\overline{\mathfrak{M}}) \to \mathrm{gr}^i(\overline{\mathfrak{M}}_2)$ is a surjective map of B-modules and that $\mathrm{gr}^i(\overline{\mathfrak{M}}_2)$ is B-projective. By repeatedly choosing B-linear splittings of these surjections we may produce a B-linear splitting $\overline{s}$ of $\overline{\mathfrak{M}} \to \overline{\mathfrak{M}}_2$ which maps $\mathrm{Fil}^i(\overline{\mathfrak{M}}_2)$ into $\mathrm{Fil}^i(\overline{\mathfrak{M}})$ for each i.

Set

$$\mathrm{Fil}^i(\mathfrak{M}) = \mathfrak{M} \cap u^i \mathfrak{N}$$

Then the image of $\mathrm{Fil}^i(\mathfrak{M})$ in $\overline{\mathfrak{M}} = \mathfrak{M}^\varphi / u\mathfrak{M}^\varphi = \mathfrak{M}/u^p\mathfrak{M}$ is contained in $\mathrm{Fil}^i(\overline{\mathfrak{M}})$. Just as in Lemma 4.1.2, if $(\mathfrak{M}, \mathfrak{N})$ is strongly divisible this inclusion is an equality. Thus, applying $\otimes_B B[[u^p]]$ to $\overline{s}$ produces a $B[[u^p]]$-linear splitting s of

$$\mathfrak{M} = \overline{\mathfrak{M}} \otimes_B B[[u^p]] \to \mathfrak{M}_2 = \overline{\mathfrak{M}}_2 \otimes_B B[[u^p]]$$

such that if $m \in \mathrm{Fil}^i(\mathfrak{M}_2)$ then $s(m) + u^p m' \in \mathrm{Fil}^i(\mathfrak{M})$ for some $m \in \mathfrak{M}$. If $i \in [0, p]$ then $u^p m' \in \mathrm{Fil}^i(\mathfrak{M})$ and so s maps $\mathrm{Fil}^i(\mathfrak{M}_2)$ into $\mathrm{Fil}^i(\mathfrak{M})$ for $i \in [0, p]$. To complete the proof we must show s maps $\mathfrak{N}_2$ into $\mathfrak{N}$. For this choose an $\mathbb{F}_p[[u]]$-basis of $\mathfrak{N}_2$ such that $(u^{r_i} e_i)$ is an $\mathbb{F}_p[[u^p]]$-basis of $\mathfrak{M}_2$. Then $u^{r_i} e_i \in \mathrm{Fil}^{r_i}(\mathfrak{M}_2)$ we have $u^{r_i} s(e_i) \in \mathrm{Fil}^i(\mathfrak{M})$ and so $s(e_i) \in \mathfrak{N}$. ☐

4.4. Strongly divisible extensions.

Notation 4.4.1. For any pair $\mathfrak{P}$ and $\mathfrak{M}$ of $\mathfrak{S}_{\mathbb{F}}$-modules write $\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ for the $\mathfrak{S}_{\mathbb{F}}$-module of $\mathfrak{S}_{\mathbb{F}}$-linear maps $\mathfrak{P} \to \mathfrak{M}$. If $(\mathfrak{P}, \mathcal{G}_\bullet)$ and $(\mathfrak{M}, \mathcal{E}_\bullet) \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ equip $\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ with the Frobenius given by $f \mapsto \varphi_{\mathfrak{M}} \circ f \circ \varphi_{\mathfrak{P}}^{-1}$. More precisely, we equip it with the isomorphism

$$\begin{aligned} \mathrm{Hom}(\mathfrak{P}, \mathfrak{M}) \otimes_{k[[u]],\varphi} k((u)) &\cong \mathrm{Hom}_{k((u))}(\mathfrak{P} \otimes_{k[[u]],\varphi} k((u)), \mathfrak{M} \otimes_{k[[u]],\varphi} k((u))) \\ &\xrightarrow{\alpha} \mathrm{Hom}_{k((u))}(\mathfrak{P}[\tfrac{1}{u}], \mathfrak{M}[\tfrac{1}{u}]) \\ &\cong \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})[\tfrac{1}{u}] \end{aligned}$$

where α sends $f \mapsto \varphi_{\mathfrak{M}} \circ f \circ \varphi_{\mathfrak{P}}^{-1}$ and the $\cong$'s are the canonical maps, both of which are isomorphisms due to the flatness of $\varphi : k[[u]] \to k((u))$ and $k[[u]] \to k((u))$.

For the rest of this section fix objects $(\mathfrak{P}, \mathcal{G}_\bullet)$ and $(\mathfrak{M}, \mathcal{E}_\bullet)$ in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$. We shall compute $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M})$ as the cohomology of an explicit complex. Define[3]

$$\mathcal{C}^1_{\mathrm{SD}} \subset \left(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})[\tfrac{1}{u}]\right)^{e-2} \times \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})[\tfrac{1}{u}]$$

as the subset consisting of $(g_2, \ldots, g_{e-1}, g_e) \in \left(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})[\tfrac{1}{u}]\right)^{e-2} \times \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})[\tfrac{1}{u}]$ satisfying

$$g_i(\mathcal{G}_i) \subset \mathcal{E}_{i-1}, \text{ and } g_i(u\mathcal{G}_{i-1}) \subset \mathcal{E}_i \tag{4.4.2}$$

[3]The product is written like to accommodate the case $e = 1$.

for $i \geq 2$. This fits into the following complex of $\mathbb{F}$-vector spaces.

$$\mathcal{C}_{\mathrm{SD}} : \prod_{i=2}^{e-1} \mathrm{Hom}(\mathcal{G}_i, \mathcal{E}_i) \times \mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})) \xrightarrow{d_{\mathrm{SD}}} \mathcal{C}^1_{\mathrm{SD}}$$
$$(h_2, \ldots, h_{e-1}, h_e) \mapsto (h_2 - \varphi(h_e), h_3 - h_2, \ldots, h_e - h_{e-1})$$

Here $\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ is defined just as it is for objects of $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$, by

$$\{h \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M}) \mid \varphi(h) \text{ maps } \mathcal{G}_1 \text{ into } \mathcal{E}_1\}$$

When $e = 1$, $\mathcal{C}^1_{\mathrm{SD}} = \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ and $\mathcal{C}_{\mathrm{SD}}$ recovers the complex described in the paragraph before 4.1.3 of [Bar18b].

Proposition 4.4.3. *There are isomorphisms*

$$H^i(\mathcal{C}_{\mathrm{SD}}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M})$$

for $i = 0, 1$.

Proof. When $i = 0$ this is clear, so we focus on the case $i = 1$. First, we define a map

$$\mathcal{C}^1_{\mathrm{SD}} \to \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M}) \tag{4.4.4}$$

For $g = (g_2, \ldots, g_e) \in \mathcal{C}^1_{\mathrm{SD}}$ let $\mathfrak{N}_g$ denote the $\mathfrak{S}_{\mathbb{F}}$-module $\mathfrak{M} \oplus \mathfrak{P}$ and with Frobenius

$$\varphi((m, z)) = (\varphi_M(m) + g^0(\varphi_P(z)), \varphi_P(z))$$

where $g^0 = -\sum_{j=2}^{e-1} g_j - g_e$. Furthermore, we set

$$\mathcal{F}_i = \left\{ (a + g^0(b) + \sum_{j=2}^{i} g_j(b), b) \mid a \in \mathcal{E}_i, b \in \mathcal{G}_i \right\}$$

for $1 \leq i \leq e-1$, as well as $\mathcal{F}_e = u^{e-1}\mathfrak{N}_g$ and $\mathcal{F}_0 = \mathfrak{N}_g$. With the $\mathcal{F}_i$ defined like this we have $u\mathcal{F}_{i-1} \subset \mathcal{F}_i \subset \mathcal{F}_{i-1}$ for $i = 2, \ldots, e$ if and only if (4.4.2) is satisfied by g_i. Similarly, $\mathcal{F}_1 \subset \mathfrak{N}_g$ if and only if $g^0(\mathcal{G}_1) \subset \mathfrak{M}$; but this is implied by (4.4.2) since $u^{i-1}\mathcal{G}_1 \subset \mathcal{G}_i$ and $\mathcal{E}_{i-1} \subset u^{i-1}\mathfrak{M}$. Thus, we obtain an exact sequence

$$0 \to (\mathfrak{M}, \mathcal{E}_\bullet) \to (\mathfrak{N}_g, \mathcal{F}_\bullet) \to (\mathfrak{P}, \mathcal{G}_\bullet) \to 0$$

in $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$. To obtain a map as claimed we must show $(N_g, \mathcal{F}_\bullet) \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$. By Proposition 4.3.2, it suffices to construct a $\varphi(\mathfrak{S}_{\mathbb{F}})$-linear splitting of $\varphi(\mathfrak{N}_g) \to \varphi(P)$ which also splits $\mathcal{F}_1 \to \mathcal{G}_1$. The map $\varphi(P) \to \varphi(\mathfrak{N}_g)$ given by $z \mapsto (g^0(z), z)$ is such a splitting.

Next we show this map is surjective. If $0 \to (\mathfrak{M}, \mathcal{E}_\bullet) \to (\mathfrak{N}, \mathcal{F}_\bullet) \to (\mathfrak{P}, \mathcal{G}_\bullet) \to 0$ is an exact sequence in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ then we can choose $\mathfrak{S}_{\mathbb{F}}$-linear splittings $s_i : \mathcal{G}_i \to \mathcal{F}_i$ for $i = 2, \ldots e-1$. Further, using Proposition 4.3.2, we can choose a splitting $s_1 = s_e : \varphi(\mathfrak{P}) \to \varphi(\mathfrak{N})$ which maps $\mathcal{G}_1$ onto $\mathcal{F}_1$. Set

$$g = (s_2 - s_1, s_3 - s_2, \ldots, s_{e-1} - s_{e-2}, s_e - s_{e-1})$$

As $\mathcal{F}_i = \{a + s_i(b) \mid a \in \mathcal{E}_i \text{ and } b \in \mathcal{G}_i\}$, the fact that $u\mathcal{F}_{i-1} \subset \mathcal{F}_i \subset \mathcal{F}_{i-1}$ implies $(s_i - s_{i-1})(\mathcal{G}_i) \subset \mathcal{E}_{i-1}$ and $(s_i - s_{i-1})(u\mathcal{G}_{i-1}) \subset \mathcal{E}_i$. Hence $g \in \mathcal{C}^1_{\mathrm{SD}}$. It is easy to check that the map $(\mathfrak{N}, \mathcal{F}_\bullet) \to (\mathfrak{N}_g, \mathcal{F}_\bullet)$, given by $n \mapsto (n - \varphi_{\mathfrak{M}}^{-1} \circ s_1 \circ \varphi_{\mathfrak{P}}(\overline{n}), \overline{n})$ where $\overline{n}$ denotes the image of n in $\mathfrak{P}$, is φ-equivariant, respects the $\mathcal{F}_\bullet$'s, and so shows that both represent the same class in $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P}, \mathfrak{M})$.

Finally, an element $g = (g_i) \in \mathcal{C}^1_{\mathrm{SD}}$ is in the kernel of (4.4.4) if and only if there exists a morphism $\mathfrak{N}_g \to \mathfrak{N}_0$ in $\mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ which induces the identity on $\mathfrak{N}$ and $\mathfrak{P}$. Every such map can be written as

$$(m, z) \mapsto (m + h_1(z), z)$$

for some $h_1 \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$. The φ-equivariance of this map is equivalent to asking that $g^0 = (\varphi - 1)(h_1)$. That it respects the $\mathcal{F}_\bullet$ is equivalent to asking that $g^0 + \sum_{j=2}^{i} g_j + h_1 \in \mathrm{Hom}(\mathcal{G}_i, \mathcal{E}_i)$ for $1 \leq i \leq e-1$. If there exists $(h_2, \dots, h_{e-1}, h_e) \in \prod_{i=2}^{e-1} \mathrm{Hom}(\mathcal{G}_i, \mathcal{E}_i) \times \mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ such that

$$(g_i) = (h_2 - \varphi(h_0), h_3 - h_2, \dots, h_e - h_{e-1}) = d_{\mathrm{SD}}(h_2, \dots, h_e)$$

then we can take $h_1 = h_e$. Conversely, if an h_1 exists then $\varphi(h_1) \in \mathrm{Hom}(\mathcal{G}_1, \mathcal{E}_1)$ so $h_1 \in \mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$. If we set $h_i = g^0 + \sum_{j=2}^{i} g_j + h_1 \in \mathrm{Hom}(\mathcal{G}_i, \mathcal{E}_i)$ for $i \geq 2$ then $h_i - h_{i-1} = g_i$ for $i = 3, \dots, e$ and $h_2 - \varphi(h_1) = g_2$. Thus g is zero in $H^1(\mathcal{C}_{\mathrm{SD}})$. □

Now we compute the dimension of $H^1(\mathcal{C}_{\mathrm{SD}})$.

Lemma 4.4.5. *Assume that every $h \in \varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ is such that $h(\mathcal{G}_i) \subset \mathcal{E}_i$ for $i \geq 2$. Assume also that $\varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})) \subset u\,\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$. Then $H^0(\mathcal{C}_{\mathrm{SD}}) = 0$ and $H^1(\mathcal{C}_{\mathrm{SD}})$ is $\mathbb{F}$-finite of dimension*

$$\sum_{i=2}^{e} \dim_{\mathbb{F}} \mathrm{Hom}(\mathcal{G}_i / u\mathcal{G}_{i-1}, \mathcal{E}_{i-1}/\mathcal{E}_i)$$

Proof. The assumption that $\varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})) \subset u\,\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ implies $\varphi^n(h) \to 0$ u-adically for every $h \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$. From this we deduce $\varphi - 1$ is an $\mathbb{F}$-linear automorphism of $\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$. Injectivity is clear and, for surjectivity, if $h \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ then $\varphi - 1$ sends $-\sum_{n \geq 0} \varphi^n(h)$ onto h. From injectivity of $\varphi - 1$ we deduce that $H^0(\mathcal{C}_{\mathrm{SD}}) = 0$.

For $j \geq -1$ define $\mathcal{C}^{-j}_{\mathrm{SD}} \subset \mathcal{C}^1_{\mathrm{SD}}$ to be the set consisting of $(h_i) \in \mathcal{C}^1_{\mathrm{SD}}$ such that $h_i(\mathcal{G}_{i+j'}) \subset \mathcal{E}_{i+j'}$ for all $i \geq 2$ and all $0 \leq j' \leq j$. Define $d_{\mathrm{SD}} : \mathcal{C}^{-j-1}_{\mathrm{SD}} \to \mathcal{C}^{-j}_{\mathrm{SD}}$ via the formula

$$d_{\mathrm{SD}}(h_2, \dots, h_e) = (h_2 - \varphi(h_e), h_2 - h_3, \dots, h_e - h_{e-1})$$

The assumption that every element of $\varphi(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}))$ maps $\mathcal{G}_i$ into $\mathcal{E}_i$ for each $i \geq 2$ implies that the image of this map is indeed contained in $\mathcal{C}^{-j}_{\mathrm{SD}}$. Write $\mathcal{C}_{\mathrm{SD},j}$ for this complex. Furthermore, define maps

$$\mathcal{C}^{-j}_{\mathrm{SD}} \to \prod_{i=2}^{e-1} \mathrm{Hom}(\mathcal{G}_i / u\mathcal{G}_{i-1}, \mathcal{E}_{i-1}/\mathcal{E}_i) \times \mathrm{Hom}(\mathcal{G}_e / u\mathcal{G}_{e-1}, \mathcal{E}_{e-1}/\mathcal{E}_e)$$
$$(h_i) \mapsto (-1)^{j+1}(\underbrace{0, \dots, 0}_{j+1}, \overline{h}_2, \dots, \overline{h}_{e-j-1})$$

where $\overline{h}_i$ denotes the image of h_i in $\mathrm{Hom}(\mathcal{G}_{i+j+1}/u\mathcal{G}_{i+j}, \mathcal{E}_{i+j}/\mathcal{E}_{i+j+1})$. Let $\mathcal{H}_j$ denote the image of this map. A short computation shows that the following diagram commutes, and has exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}^{-j-1}_{\mathrm{SD}} \arrow[r] & \mathcal{C}^{-j}_{\mathrm{SD}} \arrow[r] & \mathcal{H}_j \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{C}^{-j-2}_{\mathrm{SD}} \arrow[r] \arrow[u, "d_{\mathrm{SD}}"] & \mathcal{C}^{-j-1}_{\mathrm{SD}} \arrow[r] \arrow[u, "d_{\mathrm{SD}}"] & \mathcal{H}_{j+1} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

By considering the associated long exact sequence we deduce that $H^1(\mathcal{C}_{\mathrm{SD},j})$ is finite if and only if $H^1(\mathcal{C}_{\mathrm{SD},j+1})$ is. Since $H^0(\mathcal{C}_{\mathrm{SD},j}) = 0$, if $H^1(\mathcal{C}_{\mathrm{SD},j})$ is $\mathbb{F}$-finite then we also have:

$$\dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD,j}}) = \dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD},j+1}) + \dim_{\mathbb{F}} \mathcal{H}_j - \dim_{\mathbb{F}} \mathcal{H}_{j+1}$$

It is easy to see $\mathcal{H}_{-1} = \prod_{i=2}^{e} \mathrm{Hom}(\mathcal{G}_i/u\mathcal{G}_{i-1}, \mathcal{E}_{i-1}/\mathcal{E}_i)$ and that $\mathcal{H}_j = 0$ for $j \geq e-2$. Our assumptions on $\varphi(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$ imply $\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) = \mathrm{Hom}(\mathfrak{P},\mathfrak{M})$, and so $\mathcal{C}^0_{\mathrm{SD}} = \prod_{i=2}^{e-1} \mathrm{Hom}(\mathcal{G}_i,\mathcal{E}_i) \times \mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$. Thus $\mathcal{C}_{\mathrm{SD},-1} = \mathcal{C}_{\mathrm{SD}}$. To complete the proof if suffices to show $H^1(\mathcal{C}_{\mathrm{SD},j}) = 0$ for sufficiently large j.

For this, note that if $j \geq e-2$ then $\mathcal{C}^{-j}_{\mathrm{SD}}$ consists of those $(h_i) \in \mathcal{C}^1_{\mathrm{SD}}$ with $h_i(\mathcal{G}_{i'}) \subset \mathcal{E}_{i'}$ for all $i' \geq i$. If (h_i) is such an element choose $h'_e \in \mathrm{Hom}(\mathfrak{P},\mathfrak{M})$ so that $(\varphi - 1)(h'_e) = h_2 + \ldots + h_e$. This is possible by the first paragraph. Then define

$$h'_2 = h_2 + \varphi(h'_e),\ h'_3 = h_3 + h'_2, \ldots,\ h'_{e-1} = h_{e-1} + h'_{e-2}$$

Using that $\varphi(h'_e)$ maps $\mathcal{G}_i$ into $\mathcal{E}_i$ for every i we deduce $(h'_i) \in \mathcal{C}^{-j}_{\mathrm{SD}} = \mathcal{C}^{-j-1}_{\mathrm{SD}}$ and that $d_{\mathrm{SD}}((h'_i)) = (h_i)$. This completes the proof. □

Lemma 4.4.6. *After replacing $(\mathfrak{M},\mathcal{E}_\bullet)$ with $(u^n\mathfrak{M}, u^n\mathcal{E}_\bullet) \in \mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ for n sufficiently large, the conditions of Lemma 4.4.5 are satisfied.*

Proof. For $N \geq 0$ is sufficiently large then $u^N \mathrm{Hom}(\mathcal{G}_1,\mathcal{E}_1)$ is contained in both $u\,\mathrm{Hom}(\mathfrak{P},\mathfrak{M})$ and $\mathrm{Hom}(\mathcal{G}_i,\mathcal{E}_i)$ for each i. For any $n \in \mathbb{Z}$ we have

$$u^n \,\mathrm{Fil}^{i-(p-1)n}(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) = \mathrm{Fil}^i(\mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M})) \tag{4.4.7}$$

and so, since $\mathrm{Fil}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) = \mathrm{Hom}(\mathfrak{P},\mathfrak{M})$ for sufficiently small i, we have $\mathrm{Fil}^N(\mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M})) = \mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M})$ for sufficiently large n. It follows that if $h \in \mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M})$ then $\varphi(h) \in u^N \mathrm{Hom}(\mathcal{G}_1,u^n\mathcal{E}_1)$, and so also in $u\,\mathrm{Hom}(\mathfrak{P},u^n\mathfrak{M})$ and $\mathrm{Hom}(\mathcal{G}_i,u^n\mathcal{E}_i)$ for each i. □

In the following proposition we set $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k = \mathrm{Hom}(\mathfrak{P},\mathfrak{M}) \otimes_{k[[u]]} k$ and let $\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$ denote the image of $\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$ in $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k$.

Proposition 4.4.8. *The cohomology of $\mathcal{C}_{\mathrm{SD}}$ is $\mathbb{F}$-finite and, if we set $\chi(\mathfrak{P},\mathfrak{M}) := \dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD}}) - \dim_{\mathbb{F}} H^0(\mathcal{C}_{\mathrm{SD}})$, then*

$$\chi(\mathfrak{P},\mathfrak{M}) = \dim_{\mathbb{F}} \frac{\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k}{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)} + \sum_{i=2}^{e} \dim_{\mathbb{F}} \mathrm{Hom}(\mathcal{G}_i/u\mathcal{G}_{i-1}, \mathcal{E}_{i-1}/\mathcal{E}_i)$$

whenever $\mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) = 0$ for all $i < p$.

Proof. First, note that $H^0(\mathcal{C}_{\mathrm{SD}})$ equals $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})^{\varphi=1}$ and this is always $\mathbb{F}$-finite since it is contained in the finite dimensional $\mathbb{F}$-vector space $(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}) \otimes_{k[[u]]} C^\flat)^{\varphi=1}$. If we replace $(\mathfrak{M},\mathcal{E}_\bullet)$ by $(u^n\mathfrak{M},u^n\mathcal{E}_\bullet) \in \mathrm{Mod}^{\mathrm{BK}}_{\mathcal{F}}$ in the definition of $\mathcal{C}_{\mathrm{SD}}$ we obtain another complex which we denote $\mathcal{C}_{\mathrm{SD}}(n)$. Taking $n = 1$ we obtain an exact sequence

$$0 \to \mathcal{C}_{\mathrm{SD}}(1) \to \mathcal{C}_{\mathrm{SD}} \to \mathcal{Q} \to 0$$

of complexes, whose associated long exact sequence reads

$$\begin{aligned} 0 \to H^0(\mathcal{C}_{\mathrm{SD}}(1)) \to H^0(\mathcal{C}_{\mathrm{SD}}) \to H^0(\mathcal{Q}) \\ \to H^1(\mathcal{C}_{\mathrm{SD}}(1)) \to H^1(\mathcal{C}_{\mathrm{SD}}) \to H^1(\mathcal{Q}) \to 0 \end{aligned} \tag{4.4.9}$$

Note that $\mathcal{Q}$ is a two term complex $\mathcal{Q}^0 \xrightarrow{\gamma} \mathcal{Q}^1$ and the $\mathcal{Q}^i$ can be described explicitly. It is easy to see that $\mathcal{C}_{\mathrm{SD}}(1)^1 = u\mathcal{C}^1_{\mathrm{SD}}$, and so $\mathcal{Q}_1 \cong \mathcal{C}^1_{\mathrm{SD}}/u\mathcal{C}^1_{\mathrm{SD}}$. On the other hand,

$$\mathcal{Q}^0 \cong \prod_{i=2}^{e-1} \mathrm{Hom}(\mathcal{G}_i, \mathcal{E}_i/u\mathcal{E}_i) \times \frac{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))}{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},u\mathfrak{M}))}$$

We claim

$$\frac{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))}{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},u\mathfrak{M}))} \cong \bigoplus_{i\in p\mathbb{Z}_{\leq 0}\cup\mathbb{Z}_{\geq 1}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

as $\mathbb{F}$-vector spaces. In particular, we claim both $\mathcal{Q}^0$ and $\mathcal{Q}^1$ are $\mathbb{F}$-finite and so the same is true for the cohomology of $\mathcal{Q}$. Together Lemma 4.4.5 and Lemma 4.4.6 imply $H^1(\mathcal{C}_{\mathrm{SD}}(n))$ is finite for large n. From (4.4.9) we deduce finiteness of $H^1(\mathcal{C}_{\mathrm{SD}})$.

To verify the claim first choose an $\mathbb{F}$-linear splitting of $0 \to \mathrm{Fil}^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to \mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to 0$ to write

$$\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \cong \mathrm{Fil}^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \oplus \mathrm{gr}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$$

Note that $\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},u\mathfrak{M})) = u\,\mathrm{Hom}(\mathfrak{P},\mathfrak{M}) \cap \mathrm{Fil}^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$, which is the kernel of the surjection $\mathrm{Fil}^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to \mathrm{Fil}^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$. Therefore,

$$\frac{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))}{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},u\mathfrak{M}))} \cong \mathrm{gr}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \oplus \mathrm{Fil}^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

Splitting $0 \to \mathrm{Fil}^{i+1}(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \to \mathrm{Fil}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \to \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \to 0$ for $i \geq 1$ allows us to write

$$\mathrm{Fil}^1(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \cong \bigoplus_{i\in\mathbb{Z}_{\geq 1}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

There are also exact sequences $0 \to \mathrm{gr}^{i-p}(\mathrm{Hom}(\mathfrak{P},\mathfrak{N})) \xrightarrow{u} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \to \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) \to 0$ and, by choosing $\mathbb{F}$-linear splitting, we can identify

$$\mathrm{gr}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \cong \bigoplus_{i\in p\mathbb{Z}_{\leq 0}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

The claim follows.

To finish the proof note that (4.4.9) implies

$$\begin{aligned}\chi(\mathfrak{P},\mathfrak{M}) &= \chi(\mathfrak{P},u\mathfrak{M}) + \dim_{\mathbb{F}} H^1(\mathcal{C}_{\mathrm{SD}}) - \dim_{\mathbb{F}} H^0(\mathcal{C}_{\mathrm{SD}}) \\ &= \chi(\mathfrak{P},u\mathfrak{M}) + \dim_{\mathbb{F}} \mathcal{Q}^1 - \dim_{\mathbb{F}} \mathcal{Q}^0\end{aligned}$$

Since $\mathcal{C}^1_{\mathrm{SD}}$ is an $\mathbb{F}[[u]]$-lattice inside $\prod_{i=2}^{e} \mathrm{Hom}(\mathfrak{P},\mathfrak{M})[\frac{1}{u}]$, $\mathcal{Q}^1$ has $\mathbb{F}$-dimension equal to $(e-1)r$ where r is the $\mathbb{F}[[u]]$-rank of $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})$. The above description of $\mathcal{Q}^0$ shows it has $\mathbb{F}$-dimension $(e-2)r + \sum_{i\notin p\mathbb{Z}_{\leq 0}\cup\mathbb{Z}_{\geq 1}} \dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$. Since $r = \sum_i \dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$ it follows that

$$\chi(\mathfrak{P},\mathfrak{M}) = \chi(\mathfrak{P},u\mathfrak{M}) + \sum_{\substack{i<0\\ p\nmid i}} \dim_{\mathbb{F}} \mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$$

Using (4.4.7) and the assumption that $\mathrm{gr}^i(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k) = 0$ for $i < -p$ we deduce

$$\begin{aligned}\chi(\mathfrak{P},\mathfrak{M}) &= \chi(\mathfrak{P},u^n\mathfrak{M}) + \sum_{m=0}^{n-1}\left(\sum_{\substack{i<0\\ p\nmid i}} \dim_{\mathbb{F}} \mathrm{gr}^{i-(p-1)m}(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)\right)\\ &= \chi(\mathfrak{P},u^n\mathfrak{M}) + \sum_{i<0}\dim_{\mathbb{F}}\mathrm{gr}^i(\mathrm{Hom})(\mathfrak{P},\mathfrak{M})_k\\ &= \chi(\mathfrak{P},u^n\mathfrak{M}) + \dim_{\mathbb{F}} \frac{\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k}{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)}\end{aligned}$$

for $n > 2$. Combining Lemma 4.4.5 and Lemma 4.4.6 shows that, for large n, $\chi(\mathfrak{P},u^n\mathfrak{M}) = \sum_{i=2}^e \dim_{\mathbb{F}}\mathrm{Hom}(\mathcal{G}_i/u\mathcal{G}_{i-1},\mathcal{E}_{i-1}/\mathcal{E}_i)$. The proposition follows. □

Corollary 4.4.10. *Suppose that* $(\mathfrak{M},\mathcal{E}_\bullet)$ *and* $(\mathfrak{P},\mathcal{G}_\bullet)$ *arise, as in Section 4.1, from crystalline representations of Hodge type* $\mathbf{v}$ *and* $\mathbf{w}$ *respectively. Then*

$$\dim_{\mathbb{F}}\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{P},\mathfrak{M}) = \dim_{\mathbb{F}}\mathrm{Hom}_{\mathrm{SD}}(\mathfrak{P},\mathfrak{M}) + \dim_E \frac{\mathrm{Hom}(D(\mathbf{w}),D(\mathbf{v}))}{\mathrm{Fil}^0(\mathrm{Hom}(D(\mathbf{w}),D(\mathbf{v})))}$$

where $\mathrm{Hom}(D(\mathbf{w}),D(\mathbf{v}))$ *denotes the module of* $K\otimes_{\mathbb{Q}_p}E$*-homomorphisms* $D(\mathbf{w})\to D(\mathbf{v})$ *and* $\mathrm{Fil}^0(\mathrm{Hom}(D(\mathbf{w}),D(\mathbf{v})))$ *denotes the submodule consisting of* $h : D(\mathbf{w})\to D(\mathbf{z})$ *which map* $\mathrm{Fil}^l(D(\mathbf{w}))$ *into* $\mathrm{Fil}^l(D(\mathbf{v}))$ *for every* $l\in\mathbb{Z}$*.*

Proof. An element of $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})$ is contained in $\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M}))$ if and only if it maps $\mathrm{Fil}^i(\mathfrak{P})$ onto $\mathrm{Fil}^i(\mathfrak{M})$ for all $i\in\mathbb{Z}$. Thus, an element of $\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k$ is in $\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)$ if and only if it maps the image of $\mathrm{Fil}^i(\mathfrak{P})$ in $\mathfrak{P}\otimes_{k[[u]]}k$ onto the image of $\mathrm{Fil}^i(\mathfrak{M})$ in $\mathfrak{M}\otimes_{k[[u]]}k$. Therefore Lemma 4.1.2 implies that restriction

$$\frac{\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k}{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)} \to \bigoplus_i \mathrm{Hom}(\mathrm{Fil}^i(\overline{\mathfrak{P}}),\mathrm{gr}^{i-1}(\overline{\mathfrak{M}}))$$

defines an isomorphism. Using Proposition 4.1.3 we deduce

$$\begin{aligned}\dim_{\mathbb{F}}\frac{\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k}{\mathrm{Fil}^0(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})_k)} + \sum_{i=2}^e \dim_{\mathbb{F}}\mathrm{Hom}(\mathcal{G}_i/u\mathcal{G}_{i-1},\mathcal{E}_{i-1}/\mathcal{E}_i) =\\ \sum_{i\in[0,f-1]}\dim_E\mathrm{Hom}(\mathrm{Fil}^n(D(\mathbf{w})_{i0}),\mathrm{gr}^{n-1}(D(\mathbf{v})_{i0}))+\\ \sum_{i\in[0,f-1]}\sum_{j\in[1,e-1]}\dim_E\mathrm{Hom}(\mathrm{Fil}^1(D(\mathbf{w})_{ij}),\mathrm{gr}^0(D(\mathbf{v})_{ij})\end{aligned}$$

This equals the E-dimension of $\mathrm{Hom}(D(\mathbf{w}),D(\mathbf{v}))/\mathrm{Fil}^0(\mathrm{Hom}(D(\mathbf{w}),D(\mathbf{v})))$. □

5. Applications to deformation rings

5.1. Main results.

Definition 5.1.1. We say that an object $(\mathfrak{M},\mathcal{F}_\bullet)\in\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ has a pseudo-Barsotti–Tate crystalline lift if there exists a G_K-stable $\mathcal{O}$-lattice V inside a crystalline representation with pseudo-Barsotti–Tate Hodge type such that $(\mathfrak{M},\mathcal{F}_\bullet)$ is obtained from the Breuil–Kisin module $\mathfrak{M}^\circ$ associated to V as in Section 4.1.

Notation 5.1.2. Let $\mathbb{F}$ be a finite field of characteristic p. A continuous representation $V_{\mathbb{F}}$ of $G\in\{G_K,G_{K_\infty}\}$ on a finite dimensional $\mathbb{F}$-vector space is strongly cyclotomic-free if:

- Every Jordan–Holder factor of $V_{\mathbb{F}}$ is absolutely irreducible.
- If V is such a Jordan–Holder factor and $V \otimes_{\mathbb{F}} \mathbb{F}(1)$ is unramified then no twist of $V \otimes_{\mathbb{F}} \mathbb{F}(1)$ by an unramified character is also a Jordan–Holder factor of $V_{\mathbb{F}}$.

Here $\mathbb{F}(1)$ denotes the one dimensional $\mathbb{F}$-vector space on which G acts by the cyclotomic character.

Irreducible $\mathbb{F}$-representations of G_K are also irreducible as G_{K_∞}-representations (cf. [Bar18a, 2.2]). Thus, a G_K-representation is strongly cyclotomic-free if and only if it is when viewed as a G_{K_∞}-representation, since any G_K-composition series is also a G_{K_∞}-composition series.

Theorem 5.1.3. *Assume that $\mathbb{F}$ is the residue field of the extension $E/\mathbb{Q}_p$ defined in Notation 2.1.1 and that $(\mathfrak{M}, \mathcal{F}_\bullet) \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ is such that the G_{K_∞}-representation $V_{\mathbb{F}} = (\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1}$ is strongly cyclotomic-free and has every Jordan–Holder factor one dimensional over $\mathbb{F}$. Then*

1. *There exists a pseudo-Barsotti–Tate crystalline lift $(V, \mathfrak{M}^\circ)$ of $(\mathfrak{M}, \mathcal{F}_\bullet)$.*
2. *This lift may be chosen so that every Jordan–Holder factor of $V[\frac{1}{p}]$ is one dimensional over E.*
3. *If $(W, \mathfrak{N}^\circ)$ is another lift then $W \otimes_{\mathcal{O}} \mathbb{F} \cong V \otimes_{\mathcal{O}} \mathbb{F}$ as G_K-representations.*

For our second main theorem we fix a continuous representation $V_{\mathbb{F}}$ of G_K on a finite dimensional $\mathbb{F}$-vector space and complete local Noetherian $\mathcal{O}$-algebra $R^\square_{V_{\mathbb{F}}}$ representing the functor sending a complete local Noetherian $\mathcal{O}$-algebra onto the set of framed deformations of $V_{\mathbb{F}}$ over that $\mathcal{O}$-algebra. As described before Corollary 3.1.9, for any Hodge type $\mathbf{v}$ there exists a quotient $R^\square_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$ of $R^\square_{V_{\mathbb{F}}}[\frac{1}{p}]$ parametrising deformations of $V_{\mathbb{F}}$ which are crystalline of Hodge type $\mathbf{v}$.

Theorem 5.1.4. *Assume that $\mathbf{v}$ is pseudo-Barsotti–Tate and that $V_{\mathbb{F}}$ is cyclotomic-free. Then every component of $\mathrm{Spec}(R^\square_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$ is potentially diagonalisable in the sense of* [BLGGT14, §1.4].[4]

The proofs of both these theorems will be given in Section 5.4

5.2. Cyclotomic-free.

Definition 5.2.1. Let (Z, W) be a pair of continuous representations of G_K on finite dimensional $\mathbb{F}$-vector spaces. We will consider pairs such that whenever $H \subset \mathrm{Hom}(Z, W)$ is a G_K-stable subspace all of whose Jordan–Holder factors are unramified twists of the inverse of the cyclotomic character, there exists a composition series $0 = H_n \subset \ldots \subset H_0 = H \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ such that

- The G_K-action on $H_i/H_{i+1} \otimes_{\overline{\mathbb{F}}} \overline{\mathbb{F}}(1)$ is not trivial for any i.
- For each i, $H_i/H_{i+1} \otimes_{\overline{\mathbb{F}}} \overline{\mathbb{F}}(1)$ is not a Jordan–Holder factor of H_{i+1}.

Note this second condition is automatic unless the mod p cyclotomic character is unramified.

To describe the relationship to strong cyclotomic-freeness, let $V_{\mathbb{F}}$ be an representation of G_K on an $\mathbb{F}$-vector space and let K^{ur} denote the maximal unramified

[4] That is, every $\overline{\mathbb{Q}}_p$-point of this component is potentially diagonalisable.

extension of K. Set I equal to $G_{K^{\mathrm{ur}}}$. Then, for any continuous $\mathbb{F}$-representation $V_{\mathbb{F}}$ with absolutely irreducible Jordan–Holder factors,

$$\begin{aligned} &V_{\mathbb{F}} \text{ has an unramified Jordan–Holder factor} \\ \Leftrightarrow\ &V_{\mathbb{F}}|_I \text{ has a trivial Jordan–Holder factor} \end{aligned} \tag{5.2.2}$$

To prove this equivalence we may assume $V_{\mathbb{F}}$ is absolutely irreducible. We may also argue after extending scalars to an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. As every irreducible $\overline{\mathbb{F}}$-representation is induced from a character $\psi : H \to \mathbb{F}^\times$ with H equal to G_L for L/K finite and unramified, cf. [Bar18a, §2], the claim follows easily.

Lemma 5.2.3. *With notation as above:*

1. *A necessary and sufficient condition for $V_{\mathbb{F}}$ to be strongly cyclotomic-free is: if $V_{\mathbb{F}}|_I$ has a Jordan–Holder factor on which I acts by the inverse of the cyclotomic-character then it has no Jordan–Holder factor on which I acts trivially.*
2. *If V_1 and V_2 are subquotients of $V_{\mathbb{F}}$ then (V_1, V_2) are as in Definition 5.2.1.*

Proof. The first statement follows immediately from (5.2.2). For the second it suffices to show $\mathrm{Hom}(V_1 \otimes_{\mathbb{F}} \mathbb{F}(1), V_2)$ has no irreducible Jordan–Holder factor on which G_K acts by an unramified character, and this follows from (1) and (5.2.2). □

Notation 5.2.4. Consider the following situation. Suppose $\mathfrak{M}$ and $\mathfrak{P}$ are finite projective $\mathfrak{S}_{\mathbb{F}}$-modules equipped with homomorphisms

$$\mathfrak{M} \otimes_{k[[u]],\varphi} k[[u]] \to \mathfrak{M}, \quad \mathfrak{P} \otimes_{k[[u]],\varphi} k[[u]] \to \mathfrak{P}$$

whose cokernels are killed by u^{e+p-1}. Assume also that $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ is equipped with a continuous φ-equivariant $\mathbb{F}$-linear G_K-action such that $(\sigma - 1)(m) = 0$ for $m \in \mathfrak{M}$ and $\sigma \in G_{K_\infty}$, and such that

$$(\sigma - 1)(m) \in u^{\frac{p+e-1}{p-1}} \mathfrak{M} \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$$

for $\sigma \in G_K$ and $m \in \mathfrak{M}$. Assume likewise when $\mathfrak{M}$ is replaced by $\mathfrak{P}$. Since these G_K-actions are φ-equivariant, we obtain continuous G_K-actions on the finite dimensional $\mathbb{F}_p$-vector spaces

$$W = (\mathfrak{M} \otimes_{k[[u]]} C^\flat)^{\varphi=1}, \quad Z = (\mathfrak{P} \otimes_{k[[u]]} C^\flat)^{\varphi=1}$$

Furthermore, if $\mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ is as in the previous section, then $\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}) \otimes_{k[[u]]} C^\flat$ is equipped with a φ-equivariant G_K-action via $\sigma(h) = \sigma \circ h \circ \sigma^{-1}$. This G_K-action also has the property that $(\sigma - 1)(h) = 0$ for $h \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})_k$ and $\sigma \in G_{K_\infty}$ and

$$(\sigma - 1)(h) \in u^{\frac{p+e-1}{p-1}} \mathrm{Hom}(\mathfrak{P}, \mathfrak{M}) \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$$

for $h \in \mathrm{Hom}(\mathfrak{P}, \mathfrak{M})$ and $\sigma \in G_K$. Note, the induced G_K-action on

$$(\mathrm{Hom}(\mathfrak{P}, \mathfrak{M}) \otimes_{k[[u]]} C^\flat)^{\varphi=1} = \mathrm{Hom}(Z, W)$$

is the usual one, i.e. is given by $\sigma(h) = \sigma \circ h \circ \sigma^{-1}$.

The relevance of Definition 5.2.1 comes from the following lemma.

Lemma 5.2.5. *Suppose that $\sigma \mapsto h_\sigma$ is a continuous 1-cocycle of G_K valued in $\mathcal{H} = u^{\frac{p+e-1}{p-1}} \mathrm{Hom}(\mathfrak{P}, \mathfrak{M}) \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$. If*

1. *$(\varphi - 1)(h_\sigma) = 0$ for every $\sigma \in G_K$.*
2. *$h_\sigma = 0$ for $\sigma \in G_{K_\infty}$.*

(3) The pair (Z,W) are as in Definition 5.2.1.

Then $h_\sigma = 0$ for all $\sigma \in G_K$.

Proof. As $u^{p+e-1}\mathfrak{P} \subset \mathfrak{P}^\varphi$ we have $\varphi(\mathrm{Hom}(\mathfrak{P},\mathfrak{M})) \subset u^{-p-e+1}\mathrm{Hom}(\mathfrak{P},\mathfrak{M})$, and so $\mathcal{H}$ is φ-stable. The surjection $\mathcal{O}_{C^\flat} \to \overline{k}$ admits a section, and via this section we can φ-equivariantly view $\mathcal{H} \otimes_{\mathcal{O}_{C^\flat}} \overline{k} \subset \mathcal{H}$. If $h \in \mathcal{H}$ is φ-equivariant then the image of h in $\mathcal{H} \otimes_{\mathcal{O}_{C^\flat}} \overline{k}$ must be non-zero, otherwise $\varphi^n(h) \to 0$. It follows that

$$\mathcal{H}^{\varphi=1} = (\mathcal{H} \otimes_{\mathcal{O}_{C^\flat}} \overline{k})^{\varphi=1}$$

Thus, any $h \in \mathcal{H}^{\varphi=1}$ can be written as $u^{\frac{p+e-1}{p-1}} \sum \alpha_i h_i$ for $\alpha_i \in \overline{k}$ and $h_i \in \mathrm{Hom}(\mathfrak{P},\mathfrak{M})$. Since $\mathcal{H} \subset \mathrm{Hom}(\mathfrak{P},\mathfrak{M}) \otimes_{k[[u]]} C^\flat$ is G_K-stable, $\mathcal{H}^{\varphi=1} \subset \mathrm{Hom}(Z,W)$ is a G_K-stable subspace. For $\sigma \in G_K$ we have $\sigma(u^{\frac{p+e-1}{p-1}}) = \chi(\sigma)u^{\frac{e+p-1}{p-1}}$ for a character χ which equals the inverse of the cyclotomic character on the inertia subgroup $I_K \subset G_K$. It follows that $I_K \cap G_{K_\infty}$ acts on $\mathcal{H}^{\varphi=1}$ by the inverse of the cyclotomic character. Using e.g. [Bar18b, 2.2] we deduce that every G_K-Jordan–Holder factor of $\mathcal{H}^{\varphi=1} \otimes \mathbb{F}(1)$ is unramified.

Assumption (3) implies $\mathcal{H}^{\varphi=1} \otimes_{\mathbb{F}} \overline{\mathbb{F}}(1)$ has no trivial Jordan–Holder factors and that $\mathcal{H}^{\varphi=1} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is cyclotomic-free in the sense of [Bar18b, 2.1.1]. Thus, [Bar18b, 2.3.5] applied with $V = \overline{\mathbb{F}}$ and $W = \mathcal{H}^{\varphi=1} \otimes \overline{\mathbb{F}}$ shows that any G_K-cocycle valued in $\mathcal{H}^{\varphi=1}$ which vanishes on G_{K_∞} also vanishes on G_K. Indeed, *loc. cit.* applied with $i=1$ implies the restriction map $H^1(G_K, \mathcal{H}^{\varphi=1}) \to H^1(G_{K_\infty}, \mathcal{H}^{\varphi=1})$ is injective, and when applied with $i=0$ it implies $\mathcal{H}^{\varphi=1,G_K} = \mathcal{H}^{\varphi=1,G_{K_\infty}}$. The lemma follows. □

5.3. Smoothness. For this section we shall apply the constructions of Section 3.1 with A equal to $R^\square_{V_{\mathbb{F}}}$ and V_A equal to the universal deformation on A. Let $\mathbf{v}$ be a Hodge type which is pseudo-Barsotti–Tate. Then Section 3.2 produces a projective A-scheme $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ so that $\widetilde{\mathcal{L}}^{\mathrm{loc}}[\frac{1}{p}] = \mathrm{Spec}(R^\square_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr}\,\mathbf{v}})$.

Lemma 5.3.1. *Let $x \in \widetilde{\mathcal{L}}^{\mathrm{loc}}$ be a closed point corresponding to $(\mathfrak{M},\mathcal{F}^\bullet) \in \widetilde{\mathcal{L}}^{\mathbf{v}}(V_{\mathbb{F}'})$ for a finite extension $\mathbb{F}'/\mathbb{F}$. Then there exists an $\mathbb{F}'$-linear map*

$$\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x}(\mathbb{F}'[\epsilon]) \to \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M}) \tag{5.3.2}$$

where $(\mathfrak{M},\mathcal{F}_\bullet)$ is an object of $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ as in Proposition 3.2.2.

Proof. Lemma 3.2.2 implies that any element of $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x}(\mathbb{F}'[\epsilon])$ gives rise to an object $(\mathfrak{M}_{\mathbb{F}'[\epsilon]},\mathcal{G}_\bullet) \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$. Since this object is flat as an $\mathbb{F}'[\epsilon]$-module, the sequence

$$0 \to (\mathfrak{M},\mathcal{F}_\bullet) \xrightarrow{\epsilon} (\mathfrak{M}_{\mathbb{F}'[\epsilon]},\mathcal{G}_\bullet) \to (\mathfrak{M},\mathcal{F}_\bullet) \to 0$$

is exact. Thus we obtain a class in $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M})$. We leave the verification that this map is $\mathbb{F}'$-linear to the reader. □

Proposition 5.3.3. *Suppose that $(V_{\mathbb{F}},V_{\mathbb{F}})$ is as in Definition 5.2.1. Then for any closed point $x \in \widetilde{\mathcal{L}}^{\mathrm{loc}}$ with residue field $\mathbb{F}'$,*

$$\dim_{\mathbb{F}'} \mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x}(\mathbb{F}'[\epsilon]) \le d^2 + \dim_E \frac{\mathrm{ad}\,D(\mathbf{v})}{\mathrm{Fil}^0(\mathrm{ad}\,D(\mathbf{v}))}$$

where $\mathrm{ad}\,D(\mathbf{v})$ denotes the filtered module of $K \otimes_{\mathbb{Q}_p} E$-linear endomorphisms of $D(\mathbf{v})$ and d equals the $\mathbb{F}'$-dimension of $V_{\mathbb{F}'}$. Furthermore, if this inequality is an equality then (5.3.2) is surjective.

Proof. We can assume that $\mathbb{F}' = \mathbb{F}$. By Corollary 4.4.10

$$\dim_E \frac{\operatorname{ad} D(\mathbf{v})}{\operatorname{Fil}^0(\operatorname{ad} D(\mathbf{v}))} = \dim_{\mathbb{F}} \operatorname{Ext}^1_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M}) - \dim_{\mathbb{F}} \operatorname{Hom}_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M})$$

It therefore suffices to show that the kernel of (5.3.2) has $\mathbb{F}$-dimension

$$\leq d^2 - \dim_{\mathbb{F}} \operatorname{Hom}_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M})$$

We will prove this by first showing that the kernel of (5.3.2) is contained in the kernel of the composite

$$\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x}(\mathbb{F}[\epsilon]) \to A(\mathbb{F}[\epsilon]) \to \operatorname{Ext}^1(V_{\mathbb{F}}, V_{\mathbb{F}})$$

Here the last map is that sending $A \to \mathbb{F}[\epsilon]$ onto the exact sequence $0 \to V_{\mathbb{F}} \xrightarrow{\epsilon} V_A \otimes_A \mathbb{F}[\epsilon] \to V_{\mathbb{F}} \to 0$.

If $(\mathfrak{M}_{\mathbb{F}[\epsilon]}, \mathcal{G}^\bullet) \in \widetilde{\mathcal{L}}^{\mathbf{v}}(V_{\mathbb{F}[\epsilon]})$ corresponds to an element in the kernel of (5.3.2) then the surjection $\mathfrak{M}_{\mathbb{F}[\epsilon]} \to \mathfrak{M}$ admits a φ-equivariant $\mathfrak{S}_{\mathbb{F}}$-linear splitting s. Since $\mathfrak{M}_{\mathbb{F}[\epsilon]} \to \mathfrak{M}$ becomes G_K-equivariant after applying $\otimes_{k[[u]]} C^\flat$ it follows that $(\sigma - 1)(s) := \sigma \circ s \circ \sigma^{-1} - s$ is an element of $\operatorname{Hom}(\mathfrak{M},\mathfrak{M}) \otimes_{k[[u]]} C^\flat$. Using that the G_K-action on $\mathfrak{M} \otimes_{k[[u]]} C^\flat$ satisfies (1) in Proposition 3.2.2, we see further that $(\sigma - 1)(s) \in \operatorname{Hom}(\mathfrak{M},\mathfrak{M}) \otimes_{k[[u]]} u^{\frac{e+p-1}{p-1}} \mathcal{O}_{C^\flat}$ for each $\sigma \in G_K$. Lemma 5.2.5 then implies $(\sigma - 1)(s) = 0$ and so s induces a G_K-equivariant splitting of $V_{\mathbb{F}[\epsilon]} \to V_{\mathbb{F}}$. This proves the claim.

We are reduced to proving the kernel of the above composite has dimension

$$\leq d^2 - \operatorname{Hom}_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M})$$

Applying Lemma 5.2.5 to $\sigma \mapsto (\sigma - 1)(h)$ for $h \in \operatorname{Hom}_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M})$ shows that $\operatorname{Hom}_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M})$ is contained in $\operatorname{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K}$. We claim that the kernel of the first map in the above composite is a torsor for $\operatorname{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K} / \operatorname{Hom}_{\mathrm{SD}}(\mathfrak{M},\mathfrak{M})$. The kernel of the second map is clearly a torsor for $\operatorname{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}}) / \operatorname{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K}$ and so proving this claim will complete the argument.

Let $X \subset \widetilde{\mathcal{L}}^{\mathbf{v}}(V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon])$ denote the subset whose elements correspond to elements in the kernel of $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x}(\mathbb{F}[\epsilon]) \to A(\mathbb{F}[\epsilon])$. In other words, it is the subset of $(\mathfrak{M}_{\mathbb{F}[\epsilon]}, \mathcal{G}^\bullet)$ with $(\mathfrak{M}_{\mathbb{F}[\epsilon]}, \mathcal{G}^\bullet) \otimes_{\mathbb{F}[\epsilon]} \mathbb{F} = (\mathfrak{M}, \mathcal{F}^\bullet)$. Any $h \in \operatorname{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K}$ produces an automorphism $a + b\epsilon \mapsto a + h(b)\epsilon$ of $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$. By viewing this as an automorphism of $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon] \otimes_{\mathbb{F}_p} C^\flat$, we obtain an action of $\operatorname{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K}$ on X. This action is transitive since any two elements of X are isomorphic as objects of $\operatorname{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ by a morphism inducing the identity modulo ϵ and, by Proposition 5.2.5, this isomorphism induces an automorphism of $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ which, being the identity modulo ϵ, comes from some $h \in \operatorname{Hom}(V_{\mathbb{F}}, V_{\mathbb{F}})^{G_K}$. Finally we note that $\operatorname{Hom}_{\mathrm{SD}}(\mathfrak{M}_{\mathbb{F}}, \mathfrak{M}_{\mathbb{F}})$ is the stabliser of any point of X under this action. □

Corollary 5.3.4. *Assume that* $(V_{\mathbb{F}}, V_{\mathbb{F}})$ *is as in Definition 5.2.1. For each closed point* $x \in \widetilde{\mathcal{L}}^{\mathrm{loc}}$ *the map of local rings* $\mathcal{O} \to \mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x}$ *is formally smooth in the sense of* [Sta17, Tag 0DYG]. *Furthermore, the maps* (5.3.2) *are surjective.*

Proof. After [Sta17, Tag 07PM], it suffices to show that $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x} \otimes_{\mathcal{O}} \mathbb{F}$ is regular. In [Kis08, 3.3.8] it is shown that $A[\frac{1}{p}]$ has dimension

$$d^2 + \dim_E \frac{\operatorname{ad} D(\mathbf{v})}{\operatorname{Fil}^0(\operatorname{ad} D(\mathbf{v}))}$$

Since $\mathrm{Spec}(A[\frac{1}{p}]) = \widetilde{\mathcal{L}}^{\mathrm{loc}}[\frac{1}{p}]$ and $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ is $\mathcal{O}$-flat it follows that $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$ has the same dimension. It therefore suffices to show that the tangent space of $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x} \otimes_{\mathcal{O}} \mathbb{F}$ has $\mathbb{F}'$-dimension $\leq$ this dimension (where $\mathbb{F}'$ denotes the residue field of at x) and this follows from Proposition 5.3.3. It also follows that this inequality is in fact an equality, from which we deduce surjectivity of the (5.3.2). □

5.4. Proof of the main theorems.

Proof of Theorem 5.1.3. We begin with (3). If $(\mathfrak{M}, \mathcal{F}_\bullet)$ admits two crystalline lifts $(V, \mathfrak{M}^\circ)$ and $(W, \mathfrak{N}^\circ)$ then there is a φ-equivariant isomorphism $h : \mathfrak{M}^\circ \otimes_{\mathcal{O}} \mathbb{F} \to \mathfrak{N}^\circ \otimes_{\mathcal{O}} \mathbb{F}$. The G_K-actions on V and W induce $C^\flat$-semilinear G_K-actions on $(\mathfrak{M}^\circ \otimes_{\mathcal{O}} \mathbb{F}) \otimes_{\mathbb{F}_p} C^\flat$ and $(\mathfrak{N}^\circ \otimes_{\mathcal{O}} \mathbb{F}) \otimes_{\mathbb{F}_p} C^\flat$ satisfying the conditions from Notation 5.2.4. Therefore, by viewing h as an element of $\mathrm{Hom}(\mathfrak{M}^\circ \otimes_{\mathcal{O}} \mathbb{F}, \mathfrak{N}^\circ \otimes_{\mathcal{O}} \mathbb{F})$, we obtain a cocycle $\sigma \mapsto (\sigma - 1)(h)$ valued in $\mathrm{Hom}(\mathfrak{M}^\circ \otimes_{\mathcal{O}} \mathbb{F}, \mathfrak{N}^\circ \otimes_{\mathcal{O}} \mathbb{F}) \otimes_{k[[u]]} u^{\frac{e+p-1}{p-1}} \mathcal{O}_{C^\flat}$. Lemma 5.2.3 implies $(V_{\mathbb{F}}, V_{\mathbb{F}})$ is a pair as in Definition 5.2.1, and so Lemma 5.2.5 implies h is G_K-equivariant. Since $W \otimes_{\mathcal{O}} \mathbb{F} = ((\mathfrak{N}^\circ \otimes_{\mathcal{O}} \mathbb{F}) \otimes_{\mathbb{F}_p} C^\flat)^{\varphi=1}$, and similarly for $V \otimes_{\mathcal{O}} \mathbb{F}$, we deduce (3).

To prove (1) and (2) we induct on the dimension of $V_{\mathbb{F}}$. This is where we use the assumption that every Jordan–Holder factor of $V_{\mathbb{F}}$ is one-dimensional.

Step 1. Suppose $V_{\mathbb{F}}$ is one-dimensional over $\mathbb{F}$. It follows from [GLS14, 6.2] that there exists an $\mathfrak{S}_{\mathbb{F}}$-generator $\overline{e}$ of $\mathfrak{M}$ so that $\varphi(\overline{e}) = \overline{\alpha}(u^{r_i})_i \overline{e}$ for some $\overline{\alpha} \in (k \otimes_{\mathbb{F}_p} \mathbb{F})^\times$ and some $r_i \geq 0$. Here we view $(u^{r_i})_i$ as an element of $\mathfrak{S}_{\mathbb{F}}$ via the identification $\mathfrak{S}_{\mathbb{F}} \cong \prod_i \mathbb{F}[[u]]$. Moreover each $\mathcal{F}_j$ is generated by $(u^{s_i^{(j)}})_i \overline{e}$ for integers $s_i^{(1)} \leq \ldots \leq s_i^{(e-1)} = e - 1$ such that $s_i^{(1)} \leq r_i \leq s_i^{(1)} + p$ and $s_i^{(j+1)} \leq s_i^{(j)} + 1$.

To construct a crystalline lift begin by choosing any collection of integers t_{ij} and $\alpha \in (W(k) \otimes_{\mathbb{Z}_p} \mathcal{O})^\times$. Then we can define $\mathfrak{M}^\circ = \mathfrak{S}_{\mathcal{O}} \cdot e$ with Frobenius given by

$$\varphi(e) = \alpha \left(\prod_{ij} E_{ij}(u)^{t_{ij}} \right) e$$

We claim there exists a one dimensional crystalline $\mathcal{O}$-lattice V such that $\mathfrak{M}^\circ$ is the Breuil–Kisin module associated to V. Using that the relationship between crystalline representations and Breuil–Kisin modules is compatible with tensor products, together with [GLS15, 2.2.3], we can reduce this claim to the case where each $t_{ij} = 0$. It is then a straightforward exercise to check that $\mathfrak{M}^\circ$ is the Breuil–Kisin module associated to the unramified character given by $\sigma \mapsto \sigma(\beta)/\beta \in \mathcal{O}^\times$ where $\beta \in (W(C^\flat) \otimes_{\mathbb{Z}_p} \mathcal{O})^\times$ is such that $\varphi(\beta)\alpha = \beta$. Using [GLS15, 2.2.3] we see that V is pseudo-Barsotti–Tate if and only if $t_{i0} \in [0, p]$ and $t_{ij} \in [0, 1]$ for $j \in [1, e - 1]$.

Let us now construct the object of $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ obtained from this $\mathfrak{M}^\circ$. For $j \in [0, e-1]$ we have $\mathcal{F}^{p+j} = (\mathfrak{M}^\circ)^\varphi \cap (\prod_i E_{i0}(u)^p E_{i1}(u) \ldots E_{ij}(u)) \mathfrak{M}^\circ$ which is therefore generated by

$$\left(\prod_i E_{i0}(u)^p E_{i1}(u) \ldots E_{ij}(u) E_{ij+1}(u)^{t_{ij+1}} \ldots E_{ie-1}(u)^{t_{ie-1}} \right) e$$

Thus, $\mathfrak{M}^\circ \otimes_{\mathcal{O}} \mathbb{F}$ is generated by an element $\overline{e}$ with $\varphi(\overline{e}) = \overline{\alpha}(u^{t_{i0} + \ldots + t_{ie-1}})_i \overline{e}$ and $\mathcal{F}_j = u^{-p}(\mathcal{F}^{p+j-1} \otimes_{\mathcal{O}} \mathbb{F})$ is generated by $(u^{j-1+t_{ij}+\ldots+t_{ie-1}})_i \overline{e}$. for $j \in [1, e - 1]$.

Now return to $(\mathfrak{M}, \mathcal{F}_\bullet)$ as in the first paragraph. For $j \in [1, e - 1]$ define t_{ij} by $1 - t_{ij} = s_i^{(j+1)} - s_i^{(j)}$. Then $s_i^{(j)} = j - 1 + t_{ij} + \ldots + t_{ie-1}$ and $t_{ij} \in [0, 1]$. Also define

$t_{i0} \in [0,p]$ by $r_i = t_{i0} + t_{i1} + \ldots + t_{ie-1}$. If $\mathfrak{M}^\circ$ is as above for these t_{ij}'s and for some $\alpha \in \mathfrak{S}_{\mathcal{O}}^\times$ lifting $\overline{\alpha}$ then we see that $(V, \mathfrak{M}^\circ)$ is a crystalline lift of $(\mathfrak{M}, \mathcal{F}_\bullet)$.

Step 2. Now we treat the inductive step. If $V_{\mathbb{F}}$ has dimension > 1 then we can fit it into a G_{K_∞}-equivariant exact sequence $0 \to W_{\mathbb{F}} \to V_{\mathbb{F}} \to Z_{\mathbb{F}} \to 0$ with neither $W_{\mathbb{F}}$ or $Z_{\mathbb{F}}$ zero. Using [Bar18a, 5.1.3] we obtain a φ-equivariant exact sequence of $\mathfrak{S}_{\mathbb{F}}$-modules $0 \to \mathfrak{W} \to \mathfrak{M} \to \mathfrak{Z} \to 0$ which recovers $0 \to W_{\mathbb{F}} \to V_{\mathbb{F}} \to Z_{\mathbb{F}} \to 0$ after applying $\otimes_{k[[u]]} C^\flat$ and taking φ-invariants. The filtration $\mathcal{F}_\bullet$ on $\mathfrak{M}$ induces filtrations $\mathcal{G}_\bullet$ and $\mathcal{E}_\bullet$ on $\mathfrak{Z}$ and $\mathfrak{W}$ respectively so that we obtain

$$0 \to (\mathfrak{W}, \mathcal{E}_\bullet) \to (\mathfrak{M}, \mathcal{F}_\bullet) \to (\mathfrak{Z}, \mathcal{G}_\bullet) \to 0$$

which, in view of Lemma 4.2.1, is an exact sequence in $\mathrm{Mod}_{\mathcal{F}}^{\mathrm{SD}}$. Hence it defines a class in $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{Z}, \mathfrak{W})$. Pulling back along $(\mathfrak{Z} \oplus \mathfrak{W}, \mathcal{G}_\bullet \oplus \mathcal{E}_\bullet) \to (\mathfrak{Z}, \mathcal{G}_\bullet)$ and pushing forward along $(\mathfrak{W}, \mathcal{E}_\bullet) \to (\mathfrak{Z} \oplus \mathfrak{W}, \mathcal{G}_\bullet \oplus \mathcal{E}_\bullet)$ produces a class $\beta \in \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{Z} \oplus \mathfrak{W}, \mathfrak{Z} \oplus \mathfrak{W})$. Note that from this class we can recover our original class in $\mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{Z}, \mathfrak{W})$ by pushing forward along $(\mathfrak{Z} \oplus \mathfrak{W}, \mathcal{G}_\bullet \oplus \mathcal{E}_\bullet) \to (\mathfrak{Z}, \mathcal{G}_\bullet)$ and pulling back along $(\mathfrak{W}, \mathcal{E}_\bullet) \to (\mathfrak{Z} \oplus \mathfrak{W}, \mathcal{G}_\bullet \oplus \mathcal{E}_\bullet)$.

Now we use our inductive hypothesis to produce pseudo-Barsotti–Tate crystalline lifts $(W, \mathfrak{W}^\circ)$ and $(Z, \mathfrak{Z}^\circ)$ of $(\mathfrak{W}, \mathcal{E}_\bullet)$ and $(\mathfrak{Z}, \mathcal{G}_\bullet)$ respectively. We may assume further that each Jordan–Holder factor of $W[\frac{1}{p}]$ and $Z[\frac{1}{p}]$ is one-dimensional. Let $\mathbf{w}$ be the Hodge type of W and $\mathbf{z}$ the Hodge type of Z. Apply the constructions of Section 3.2 and Section 5.3 to the residual representation $Z_{\mathbb{F}} \oplus W_{\mathbb{F}}$ (now viewed as G_K-representations, the G_K-action coming from our crystalline lifts) with Hodge type $\mathbf{z} \oplus \mathbf{w}$. Then $(W \oplus Z, \mathfrak{W}^\circ \oplus \mathfrak{Z}^\circ)$ corresponds to a morphism $x^\circ : \mathrm{Spec}(\mathcal{O}) \to \widetilde{\mathcal{L}}^{\mathrm{loc}}$ and $(\mathfrak{Z} \oplus \mathfrak{W}, \mathcal{G}_\bullet \oplus \mathcal{E}_\bullet)$ corresponds to the composite $x : \mathrm{Spec}(\mathbb{F}) \to \mathrm{Spec}(\mathcal{O}) \to \widetilde{\mathcal{L}}^{\mathbf{v}}$. By Corollary 5.3.4 there exists a tangent vector in $\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x} \to \mathbb{F}[\epsilon]$ whose image under (5.3.2) equals $\beta \in \mathrm{Ext}^1_{\mathrm{SD}}(\mathfrak{Z} \oplus \mathfrak{W}, \mathfrak{Z} \oplus \mathfrak{W})$. As $\mathcal{O} \to \mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x}$ is formally smooth over $\mathcal{O}$ this tangent vector lifts to a morphism $\mathcal{O}\mathcal{O}_{\widetilde{\mathcal{L}}^{\mathrm{loc}},x} \to \mathcal{O}[\epsilon]$, cf. [Sta17, Tag 07NJ]. This $\mathcal{O}[\epsilon]$-valued point corresponds to a φ-equivariant exact sequence

$$0 \to \mathfrak{Z}^\circ \oplus \mathfrak{W}^\circ \to \mathfrak{M}'^\circ \to \mathfrak{Z}^\circ \oplus \mathfrak{W}^\circ \to 0$$

of $\mathfrak{S}_{\mathcal{O}}$-modules which, after applying $\otimes_{\mathbb{Z}_p} W(C^\flat)$ and taking φ-invariants produces a G_K-equivariant sequence of crystalline $\mathcal{O}$-lattices. Pushing forward and pulling back as in the previous paragraph we obtain a φ-equivariant exact sequence

$$0 \to \mathfrak{W}^\circ \to \mathfrak{M}^\circ \to \mathfrak{Z}^\circ \to 0$$

which induces the exact sequence $0 \to (\mathfrak{W}, \mathcal{E}_\bullet) \to (\mathfrak{M}, \mathcal{F}_\bullet) \to (\mathfrak{Z}, \mathcal{G}_\bullet) \to 0$ in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ after applying $\otimes_{\mathcal{O}} \mathbb{F}$. It also produces an exact sequence $0 \to W \to V \to Z \to 0$ of crystalline $\mathcal{O}$-lattices. We see that $(V, \mathfrak{M}^\circ)$ is a crystalline lift of $(\mathfrak{M}, \mathcal{F}_\bullet$ as required by (1) and (2). $\square$

Proof of Theorem 5.1.4. With notation as in Section 3.2, Corollary 3.1.9 implies that the irreducible components of $\mathrm{Spec}(R^{\square}_{V_{\mathbb{F}}}[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}})$ coincide with those of $\widetilde{\mathcal{L}}^{\mathrm{loc}}[\frac{1}{p}]$. Using Corollary 5.3.4 we see that $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ is normal and therefore the irreducible components of $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ coincide with those of $\widetilde{\mathcal{L}}^{\mathrm{loc}}[\frac{1}{p}]$. In view of (1) in [BLGGT14, 1.4.3], it suffices to show that for any $\mathcal{O}$-valued point of $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ there exists a second $\mathcal{O}$-valued point with the same special fibre and whose generic fibre corresponds to a crystalline representation on an E-vector space, all of whose Jordan–Holder factors are one dimensional. The special fibre of the first $\mathcal{O}$-valued point corresponds to an element of $\widetilde{\mathcal{L}}^{\mathbf{v}}(V_{\mathbb{F}})$, and so an object in $\mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$; the second $\mathcal{O}$-valued point is then constructed by applying Theorem 5.1.3 to this object. □

6. Towards a moduli interpretation of $\widetilde{\mathcal{L}}^{\mathrm{loc}}$

We would like a description of $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ that does not resort to flat closure. We are not able to do this in general, but are able to for p-torsion coefficients, i.e. for $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$. Since the main purpose of this section is to orient the reader, and since what we say here has no bearing on the main results of this paper, we only sketch the arguments.

6.1. The situation when $K/\mathbb{Q}_p$ is unramified.

Definition 6.1.1. With notation as in Definition 3.1.2, define $\mathcal{L}^{\leq h}_{\mathrm{cr}}(V_B) \subset \mathcal{L}^{\leq h}(V_B)$ as the subset consisting of $\mathfrak{M}$ such that, under the identification $\mathfrak{M} \otimes_{\mathbb{Z}_p} W(C^\flat) \cong V_B \otimes_{\mathbb{Z}_p} W(C^\flat)$, the $W(C^\flat)$-semilinear extension of the G_K-action on V_B satisfies

$$(\sigma - 1)(m) \in \mathfrak{M} \otimes_{\mathfrak{S}} [\pi^\flat]\varphi^{-1}(\mu) A_{\mathrm{inf}}$$

for every $\sigma \in G_K$ and $m \in \mathfrak{M}$. These subsets are clearly functorial in B.

In [Bar19, 2.2.9] it is shown that the functor $B \mapsto \mathcal{L}^{\leq h}_{\mathrm{cr}}(V_B)$ is represented by a closed subscheme of the $\mathcal{L}^{\leq h}$ from Proposition 3.1.3. It follows that for any local Noetherian $\mathbb{Z}_p$-algebra A with finite residue field $\mathbb{F}$ and for any finite free A-module equipped with a continuous A-linear action of G_K there exists a projective A-scheme $\mathcal{L}^{\leq h}_{\mathrm{cr}}$ representing $B \mapsto \mathcal{L}^{\leq h}_{\mathrm{cr}}(V_B)$ on A-algebras with $\mathfrak{m}_A^i B = 0$ for some i. The following then summarises the main results of [Bar19].

Theorem 6.1.2. *Assume that $K/\mathbb{Q}_p$ is unramified and that $V_A \otimes_A \mathbb{F}$ is strongly cyclotomic-free. Then:*

1. *Let $\mathcal{L}^\circ \subset \mathcal{L}^{\leq p}_{\mathrm{cr}}$ denote the closure in $\mathcal{L}^{\leq p}_{\mathrm{cr}}$ of $\mathcal{L}^{\leq p}_{\mathrm{cr}}[\frac{1}{p}]$. Then $\mathcal{L}^\circ$ is a union of connected components of $\mathcal{L}^{\leq p}_{\mathrm{cr}}$. If every Jordan–Holder factor of $V_A \otimes_A \mathbb{F}$ is one dimensional then $\mathcal{L}^\circ = \mathcal{L}^{\leq p}_{\mathrm{cr}}$.*
2. *The completed local rings of $\mathcal{L}^\circ$ at closed point are formally smooth $\mathbb{Z}_p$-algebras.*
3. *Each component of $\mathcal{L}^\circ$ can be labelled by a Hodge type and if $\mathcal{L}^{\mathbf{v}}$ denotes the union of those components labelled with the Hodge type $\mathbf{v}$ then the scheme-theoretic image of the projective map $\mathcal{L}^{\mathbf{v}} \to \mathrm{Spec}(A)$ corresponds to the $\mathbb{Z}_p$-flat and reduced quotient $A^{\mathrm{cr},\mathbf{v}}$ of A with $A^{\mathrm{cr},\mathbf{v}}[\frac{1}{p}] = A[\frac{1}{p}]^{\mathrm{cr},\mathbf{v}}$.*

Proof. Part (1) follows from [Bar19, 4.4.5], and (2) follows from [Bar19, 4.4.4]. For part (3) consider [Bar19, 4.2.11 and 4.2.12]. □

The following example indicates how (3) fails once we drop the assumption that $K/\mathbb{Q}_p$ is unramified and explains one reason why it is necessary to rigidify the functor $B \mapsto \mathcal{L}^{\leq h}(V_B)$ by adding filtrations as in the definition of $\widetilde{\mathcal{L}}^{\mathbf{v}}$.

Example 6.1.3. Suppose $K/\mathbb{Q}_p$ is totally ramified of degree $e = 2$, and let $\mathfrak{M}$ denotes the finite free $\mathfrak{S}_{\mathbb{F}} = \mathbb{F}[[u]]$-module generated by e with $\varphi(e) = ue$. Then $\mathfrak{M}$ can be upgraded to an object $(\mathfrak{M}, \mathcal{F}_\bullet) \in \mathrm{Mod}^{\mathrm{SD}}_{\mathcal{F}}$ in more than one way: we can set $\mathcal{F}_1$ equal to $\mathfrak{M}$ or $u\mathfrak{M}$. In view of Theorem 5.1.3, these two choices realises $\mathfrak{M}$ as the reduction of a Breuil–Kisin module coming from a crystalline representation with two different Hodge types.

6.2. The ramified situation. A natural idea to produce a moduli interpretation for $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ would be to impose the Galois condition from Definition 6.1.1 on elements of $\widetilde{\mathcal{L}}^{\mathbf{v}}$ as well as a strong divisibility condition in the sense of (2) in Lemma 3.2.2. Unfortunately we do not know how to do this. Strong divisibility is only a valid condition for p-torsion coefficients, and it is difficult to show it describes a closed condition. Instead we consider the following lemma:

Lemma 6.2.1. *Suppose $(\mathfrak{M}, \mathcal{F}^\bullet)$ is as in Lemma 3.2.2. Then, as well as conditions (1) and (2), the following property is satisfied:*

(3) The $C^\flat$-semilinear action of the G_K action on V_B is such that
$$(\sigma - 1)(m) \in u^{-p}\mathcal{F}^p \otimes_{k[[u]]} u^{\frac{ep}{p-1}} \mathcal{O}_{C^\flat}$$
for every $m \in u^{-p}\mathcal{F}^p$.

Proof. By (2) of Lemma 3.2.2 we can choose an $\mathbb{F}_p[[u]]$-basis (e_i) of $u^{-p}\mathcal{F}^p$ and integers $r_i \in [0,p]$ so that $u^{r_i}e_i$ generate $\varphi(\mathfrak{M})$. It suffices to prove (3) for $m = \alpha e_i$ for some $\alpha \in \mathbb{F}_p[[u]]$. Now
$$(\sigma-1)(\alpha e_i) = \underbrace{(\sigma-1)(\alpha u^{-r_i})u^{r_i}e_i}_{(a)} + \underbrace{\sigma(\alpha u^{-r_i})(\sigma-1)(u^{r_i}e_i)}_{(b)}$$
Since $u^{r_i}e_i \in \varphi(\mathfrak{M})$ we have $(\sigma-1)(u^{r_i}e_i) \in \varphi(\mathfrak{M}) \otimes_{k[[u^p]]} u^{\frac{p(p+e-1)}{p-1}} \mathcal{O}_{C^\flat}$. As $u^p\mathfrak{M}^\varphi \subset \mathcal{F}_p$ it follows that (b) is contained in $u^{-p}\mathcal{F}^p \otimes_{k[[u]]} u^{\frac{ep}{p-1}}\mathcal{O}_{C^\flat}$. Using [Bar19, 3.2.11] we see that $(\sigma-1)(\alpha u^{-r_i}) \in u^{\frac{ep}{p-1}-r_i}\mathcal{O}_{C^\flat}$ so $(a) \in u^{-p}\mathcal{F}^p \otimes_{k[[u]]} u^{\frac{ep}{p-1}}\mathcal{O}_{C^\flat}$ also. □

Definition 6.2.2. For any A-algebra B which factors through an $\mathbb{F}$-finite quotient of A define $\widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}1}(V_B) \subset \widetilde{\mathcal{L}}^{\mathbf{v}}(V_B)$ as the subset containing those $(\mathfrak{M}, \mathcal{F}^\bullet)$ such that:

(1) The $C^\flat$-semilinear extension of the G_K-action on V_B is such that
$$(\sigma-1)(m) \in \mathfrak{M} \otimes_{k[[u]]} u^{\frac{e+p-1}{p-1}} \mathcal{O}_{C^\flat}$$
for all $m \in \mathfrak{M}$.

(2) Furthermore, this G_K-action is such that
$$(\sigma-1)(m) \in u^{-p}\mathcal{F}^p \otimes_{k[[u]]} u^{\frac{ep}{p-1}} \mathcal{O}_{C^\flat}$$
for every $m \in u^{-p}\mathcal{F}^p$.

(3) For $i \in [0,p]$ we have $\mathfrak{M}^\varphi \cap u^{i-p}\mathcal{F}^p = \mathcal{F}^i$.

Proposition 6.2.3. *The functor $B \mapsto \widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}(V_B)$ is represented by a closed subscheme $\widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}} \hookrightarrow \widetilde{\mathcal{L}}^{\mathbf{v}} \otimes_{\mathcal{O}} \mathbb{F}$ containing $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$. If $A = R^\square_{V_\mathbb{F}}$ as described before Theorem 5.1.4 and $(V_\mathbb{F}, V_\mathbb{F})$ is a pair as in Definition 5.2.1 then $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$ is a union of connected components of $\widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}$. If $V_\mathbb{F}$ is strongly cyclotomic free and every Jordan–Holder factor is one-dimensional then $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F} = \widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}$.*

Sketch of proof. Arguing as in [Bar19, 2.2.8] it is easy to see that conditions (1) and (2) from Definition 6.2.2 define closed subschemes of $\widetilde{\mathcal{L}}^{\mathbf{v}}$. To see that (3) defines a further closed subscheme is more difficult; by assumption $\mathfrak{M}^\varphi/\mathcal{F}^i$ is B-projective so the locus where $\mathfrak{M}^\varphi \cap u^{i-p}\mathcal{F}^p \subset \mathcal{F}^i$ is indeed closed. On the other hand it is not clear that $\mathfrak{M}^\varphi/(\mathfrak{M}^\varphi \cap u^{i-p}\mathcal{F}^p)$ is projective (and in general it won't be without conditions (1) and (2) being satisfied). To obtain this projectivity one has to show that if $(\mathfrak{M}, \mathcal{F}^\bullet) \in \widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}(V_B)$ then $(\varphi(\mathfrak{M}), u^{-p}\mathcal{F}^p)$ is strongly divisible in the sense of Section 4.3. When $e = 1$ this was proven in [Bar19, 3.3.1] and the proof in the general case is very similar; since the argument is quite long we do not give it here. Once one knows $(\varphi(\mathfrak{M}), u^{-p}\mathcal{F}^p)$ is strongly divisible one deduces projectivity of $\mathfrak{M}^\varphi/(\mathfrak{M}^\varphi \cap u^{i-p}\mathcal{F}^p)$ by adapting the argument in [Bar19, 4.2.3]. From this we obtain the desired closed subscheme. In view of Lemma 6.2.1, a B-valued point of $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$ factors through $\widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}$ whenever B is finite over $\mathbb{F}$. Since each of these schemes is projective over $A \otimes_{\mathcal{O}} \mathbb{F}$ one deduces that $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$ is contained in $\widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}$.

When $A = R^{\square}_{V_{\mathbb{F}}}$ and $(V_{\mathbb{F}}, V_{\mathbb{F}})$ is as in Definition 5.2.1 then precisely the same arguments as used in Section 5.3, together with the statement made in the previous paragraph that elements of $\widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}(V_B)$ are strongly divisible, produce a bound on the dimension of the tangent spaces of $\widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}$ at closed points. If such a closed point lies on an irreducible component which meets $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$ then the dimension of this irreducible component must be $\geq$ this bound (since the dimension of $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$ equals this bound). It follows that the completed local ring at this point is formally smooth over $\mathbb{F}$, that this irreducible component is a connected component, and that this component equals a component of $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F}$.

For the last statement, applying Theorem 5.1.3 to Breuil–Kisin modules arising from closed points of $\widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}$ shows that any such point is contained in $\widetilde{\mathcal{L}}^{\mathrm{loc}}$ and so $\widetilde{\mathcal{L}}^{\mathrm{loc}} \otimes_{\mathcal{O}} \mathbb{F} = \widetilde{\mathcal{L}}^{\mathbf{v}}_{\mathrm{cr}}$. □

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