Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 1, pp. 179–183

# DECOMPOSITION OF REPRESENTATIONS INTO TENSOR PRODUCTS

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**1.** In this paper some generalizations of the classical theorem which classifies the irreducible representations of the direct product of two finite groups in terms of those of the factors will be discussed. The first generalization consists in expanding the class of groups considered.

THEOREM 1. *Let $G_1$, $G_2$ be locally compact totally disconnected groups and let $G = G_1 \times G_2$.*

(1.1) *If $\pi_i$ is an admissible irreducible representation of $G_i$, $i = 1, 2$, then $\pi_1 \otimes \pi_2$ is an admissible irreducible representation of $G$.*

(1.2) *If $\pi$ is an admissible irreducible representation of $G$, then there exist admissible irreducible representations $\pi_i$ of $G_i$ such that $\pi \simeq \pi_1 \otimes \pi_2$. The classes of the $\pi_i$ are determined by that of $\pi$.*

We recall some notation. For a locally compact totally disconnected group $G$, the *Hecke algebra* $H(G)$ of $G$ is the convolution algebra of locally constant complex valued functions on $G$ with compact support. For a compact open subgroup $K$ of $G$, let $e_K$ be the function $(\text{meas } K)^{-1} \cdot \text{ch}_K$, where $\text{ch}_K$ is the characteristic function of $K$ and meas is the Haar measure on $G$ which has been used to define convolution in $H(G)$. Then $e_K$ is an idempotent of $H(G)$. The subalgebra $e_K H(G) e_K$ of $H(G)$ will be denoted $H(G, K)$. A smooth $G$-module $W$ is in a natural way an $H(G)$-module, and for every compact open subgroup $K \subset G$ the space $W^K = e_K W$ is an $H(G, K)$-module.

Before proving Theorem 1 we state an *Irreducibility Criterion. A smooth $G$-module $W$ is irreducible if and only if $W^K$ is an irreducible $H(G, K)$-module for all compact open subgroups $K$ of $G$.*

PROOF. This follows from the fact that if $U$ is an $H(G, K)$-submodule of $W^K$, then $(H(G) \cdot U)^K = U$. $\square$

Remark that in applying the irreducibility criterion it is sufficient to check that $W^K$ is an irreducible $H(G, K)$-module for a set of $K$ which forms a neighborhood base of the identity in $G$.

COROLLARY. *Let $K$ be a compact open subgroup of $G$ such that $H(G, K)$ is commutative, and let $W$ be an admissible irreducible $G$-module. Then* $\dim W^K \leqq 1$.

*AMS (MOS) subject classifications* (1970). Primary 22E55.

PROOF OF THEOREM 1. It is straightforward that

(i) $H(G_1 \times G_2) \simeq H(G_1) \otimes H(G_2)$,

(ii) $H(G_1 \times G_2, K_1 \times K_2) \simeq H(G_1, K_1) \otimes H(G_2, K_2)$ and

(iii) $(W_1 \otimes W_2)^{K_1 \times K_2} \simeq W_1^{K_1} \otimes W_2^{K_2}$

for every pair of compact open subgroups $K_i$ of $G_i$ and every pair of smooth $G_i$-modules $W_i$.

Assertion (1.1) follows from (iii) and the irreducibility criterion.

Conversely, let $W$ be an admissible irreducible $G$-module. Let $K = K_1 \times K_2$, where $K_i$ is a compact open subgroup of $G_i$, $i = 1, 2$, be such that $W^K \neq 0$. The space $W^K$ is finite dimensional, so by the corollary on p. 94 of **[2]** there exist irreducible $H(G_i, K_i)$-modules $W_i^{K_i}$ and an $H(G, K)$ isomorphism $a_K$ from $W^K$ to $W_1^{K_1} \otimes W_2^{K_2}$. Similar remarks apply to every open subgroup $K' = K_1' \times K_2'$ of $K$. There exist $H(G_i, K_i)$-maps $b_i = b_i(K, K')$: $W_i^{K_i} \to W_i^{K_i'}$ such that the following diagram is commutative.

$$\begin{tikzcd} W^K \arrow[r, "a_K"] \arrow[d, hook, "\text{incl.}"] & W_1^{K_1} \otimes W_2^{K_2} \arrow[d, "b_1 \otimes b_2"] \\ W^{K'} \arrow[r, "a_{K'}"] & W_1^{K_1'} \otimes W_2^{K_2'} \end{tikzcd}$$

Moreover, the maps $b_i(K, K')$ can be chosen for every pair of compact open subgroups $K, K'$ of this type in such a way as to form an inductive system. Then $W \simeq W_1 \otimes W_2$, where $W_i = \text{ind}\lim_{K_i} W_i^{K_i}$, and $W_i$ is an admissible irreducible representation of $G_i$, $i = 1, 2$.

The class of $W_i$ is determined by that of $W$, for the restriction of $W$ to $G_i$ is $W_i$-isotypic. □

An analysis of the proof of Theorem 1 reveals that the groups $G$ and $G_i$ enter only through their Hecke algebras. This leads one to define an *idempotented algebra* $(A, E)$ to be an algebra $A$ with a directed family of idempotents $E$ such that $A = \bigcup_{e \in E} eAe$. An *admissible module* $W$ for $(A, E)$ is an $A$-module $W$ which is *nondegenerate* in the sense that $AW = W$ and is such that $\dim eW$ is finite for all $e \in E$. The tensor product of two idempotented algebras is naturally idempotented. The proof of Theorem 1 is readily adapted to establish a similar theorem about the admissible irreducible modules of the tensor product of two idempotented algebras.

**2.** The study of the representations of adelic groups, which are infinite restricted products of groups, requires the notion of restricted tensor product of vector spaces which was introduced in **[4]**.

Let $\{W_v \mid v \in V\}$ be a family of vector spaces. Let $V_0$ be a finite subset of $V$. For each $v \in V \backslash V_0$, let $x_v$ be a nonzero vector in $W_v$. For each finite subset $S$ of $V$ containing $V_0$, let $V_S = \bigotimes_{v \in S} W_v$; and if $S \subset S'$, let $f_S$: $W_S \to W_{S'}$ be defined by $\bigotimes_{v \in S} w_v \mapsto \bigotimes_{v \in S} w_v \bigotimes_{v \in S' \backslash S} x_v$. Then $W = \bigotimes_{x_v} W_v$, the *restricted tensor product* of the $W_v$ with respect to the $x_v$, is defined by $W = \text{ind}\lim_S W_S$. The space $W$ is spanned by elements written in the form $w = \bigotimes w_v$, where $w_v = x_v$ for almost all $v \in V$.

The ordinary constructions with finite tensor products extend easily to restricted tensor products.

(1) Given linear maps $B_v\colon W_v \to W_v$ such that $B_v x_v = x_v$ for almost all $v \in V$, then one can define $B = \bigotimes B_v\colon W \to W$ by $B(\bigotimes w_v) = \bigotimes B_v w_v$.

(2) Given a family of algebras $\{A_v \mid v \in V\}$ and given nonzero idempotents $e_v \in A_v$ for almost all $v$, then $A = \bigotimes_{e_v} A_v$ is an algebra in the obvious way.

(3) If $W_v$ is an $A_v$-module for each $v \in V$ such that $e_v \cdot x_v = x_v$ for almost all $v$, then $\bigotimes_{x_v} W_v$ is an $A$-module. The isomorphism class of $W$ depends on $\{x_v\}$. However, if $\{x_v'\}$ is another collection of nonzero vectors such that $x_v$ and $x_v'$ lie on the same line in $W_v$ for almost all $v$, then the $A$-modules $\bigotimes_{x_v} W_v$ and $\bigotimes_{x_v'} W_v$ are isomorphic.

EXAMPLE 1. The polynomial ring in an infinite number of variables $C[X_1, X_2, \ldots]$ is isomorphic to $\bigotimes_{e_i} C[X_i]$, where $e_i$ is the identity element of $C[X_i]$.

EXAMPLE 2. Let $G = \prod'_{K_v} G_v$ be the restricted product of locally compact totally disconnected groups $G_v$, restricted with respect to the compact open subgroups $K_v$. Then $G$ itself is locally compact and totally disconnected, and $H(G)$ is isomorphic to $\bigotimes_{e_{K_v}} H(G_v)$.

For each $v \in V$ let $W_v$ be an admissible $G_v$-module. Assume that $\dim W_v^{K_v} = 1$ for almost all $v$. Choosing for almost all $v$ a nonzero vector $x_v \in W_v^{K_v}$, we may form the $G$-module $W = \bigotimes_{x_v} W_v$. The isomorphism class of $W$ is in fact independent of the choice of $x_v \in W_v^{K_v}$ and will be called the tensor product of the representations $W_v$. One sees that $W$ is admissible, and that it is irreducible if and only if each $W_v$ is. The admissible irreducible representations of $G$ isomorphic to ones constructed in this way are said to be *factorizable.*

THEOREM 2. *Suppose that $H(G_v, K_v)$ is commutative for almost all $v$. Then every admissible irreducible representation $W$ of $G$ is factorizable, $W \simeq \bigotimes W_v$. The isomorphism classes of the factors $W_v$ are determined by that of $W$. For almost all $v$, $\dim W_v^{K_v} = 1$.*

PROOF. One first factorizes the finite dimensional spaces $W^{K'}$ for compact open subgroups $K' = \prod K_v'$ of $G$, then continues as in the proof of Theorem 1. □

**3.** Let $G$ be a connected reductive algebraic group over a global field $F$. Let $A$ be the adele ring of $F$, and let $V$ be the set of places of $F$. The adelic group $G(A)$ is isomorphic to a restricted product $\prod'_{K_v} G(F_v)$, where the subgroups $K_v$ are defined for all finite $v$ and are certain maximal compact subgroups of $G(F_v)$. For almost all finite $v \in V$, $G(F_v)$ is a quasi-split group over $F_v$, and $K_v$ is a special maximal compact subgroup. For these places $v$, $H(G(F_v), K_v)$ is commutative. See [5]. So the function field case of the following theorem, whose meaning has yet to be explained in the number field case, is a special case of Theorem 2.

THEOREM 3. *Every admissible irreducible representation of $G(A)$ is factorizable. The factors are unique up to equivalence.*

Let $F$ be a number field. Then the class of admissible representations of $G(A)$ has yet to be defined. For each archimedean place $v \in V$, let $K_v$ be a maximal compact subgroup of $G(F_v)$, and let $\mathfrak{g}_v$ be the real Lie algebra of $G(F_v)$. Let $K_\infty = \prod_{\text{arch } v} K_v$, $K = \prod_{\text{all } v} K_v$, and $G_\infty = \prod_{\text{arch } v} G(F_v)$. Let $\mathfrak{g}_\infty$ be the real Lie algebra of $G_\infty$. Let $A_f$ be the ring of finite adeles of $F$.

DEFINITION. An *admissible* $G(A)$-module $W$ is a vector space $W$ which is both a $(\mathfrak{g}_\infty, K_\infty)$-module and a smooth $G(A_f)$-module such that

(1) the action of $G(A_f)$ commutes with the action of $\mathfrak{g}_\infty$ and $K_\infty$, and

(2) for each isomorphism class $\gamma$ of continuous irreducible representations of $K$, the $\gamma$-isotypic component of $W$ has finite dimension.

In Theorem 3, the factors at the archimedean places $v$ are to be admissible $(\mathfrak{g}_v, K_v)$-modules. The proof when $F$ is a number field is the same as that when it is a function field once an idempotented algebra is found for each archimedean place $v$ whose admissible modules are the same as admissible $(\mathfrak{g}_v, K_v)$-modules.

Let $G$ be a Lie group, and let $K$ be a compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the real Lie algebras of $G$ and $K$. Let $U(\mathfrak{g}_C)$ and $U(\mathfrak{k}_C)$ be the universal enveloping algebras of the complexified Lie algebras. Define the *Hecke algebra* $H(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$ to be the algebra of left and right $K$-finite distributions on $G$ with support in $K$. It contains the algebra $A_K$ of $K$-finite measures on $K$ viewed as distributions on $G$. The map $(X, \mu) \to X * \mu$ from $U(\mathfrak{g}_C) \times A_K$ to the space of distributions on $G$ induces a vector space isomorphism of $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{k}_C)} A_K$ with $H(\mathfrak{g}, K)$. With the set $E$ of central idempotents of $A_K$, $H(\mathfrak{g}, K)$ is an idempotented algebra.

Let $(\pi, W)$ be a $(\mathfrak{g}, K)$-module. By means of the formula $X \otimes \mu \cdot w = \pi(X)\pi(\mu)w$ for $X \in U(\mathfrak{g}_C)$, $\mu \in A_K$, and $w \in W$, the space $W$ becomes a nondegenerate $H(\mathfrak{g}, K)$-module. Moreover, it is not difficult to verify that this construction establishes an isomorphism between the categories of admissible $(\mathfrak{g}, K)$-modules and of admissible $(H(\mathfrak{g}, K), E)$-modules.

**4.** In practice, a more analytic theory than that described above is needed as well.

Let $\{H_v \mid v \in V\}$ be a family of Hilbert spaces. For almost all $v \in V$, let $x_v$ be a unit vector in $H_v$. The *Hilbert restricted product* $H = \hat{\bigotimes}_{x_v} H_v$ is a Hilbert space which can be conveniently described by giving an orthonormal basis. Let $P_v$ be an orthonormal basis for $H_v$ for each $v \in V$ which extends $\{x_v\}$ for almost all $v$. The set of symbols $\hat{\otimes} h_v$ such that $h_v \in P_v$ for all $v$ and $h_v = x_v$ for almost all $v$ is an orthonormal basis for $H$. Constructions analogous to those described above with reference to the ordinary restricted tensor product are available in the Hilbert space context.

For the following theorem see [3] and [1].

THEOREM 4. *Let $G(A)$ be as in Theorem 3. Let $\pi$ be a continuous irreducible unitary Hilbert space representation of $G(A)$. Then*

(1) *There exist continuous irreducible unitary Hilbert space representations $\pi_v$ of $G(F_v)$, almost all of which are unramified, such that $\pi \simeq \hat{\otimes}\pi_v$. The factors $\pi_v$ are unique up to isomorphism.*

(2) *For each isomorphism class $\gamma$ of continuous irreducible representations of $K$, the $\gamma$-isotypic component of $\pi$ has finite dimension.*

(3) *The space of $K$-finite vectors of $\pi$ is in a natural way an admissible irreducible $G(A)$-module $\pi^K$. Let $\pi^K \simeq \otimes \pi_v^K$ be the factorization of $\pi^K$ given by Theorem 3. Then $\pi_v^K$ is isomorphic as an admissible $G(F_v)$-module to the space of $K_v$-finite vectors of $\pi_v$.*

## REFERENCES

1. I. N. Bernshtein, *All reductive p-adic groups are tame*, Functional Anal. Appl. **8** (1974), 91–93.

2. N. Bourbaki, *Modules et anneaux semi-simples*, Hermann, Paris, 1958.

3. I. M. Gelfand, M. I. Graev and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, W. B. Saunders Co., Philadelphia, 1969.

4. H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer-Verlag, Berlin and New York, 1970.

5. J. Tits, *Reductive groups over local fields*, these PROCEEDINGS, part 1, pp. 29–69.

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