DEGENERATING PRODUCTS OF FLAG VARIETIES AND APPLICATIONS TO THE BREUIL–MÉZARD CONJECTURE

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Abstract. We consider closed subschemes in the affine grassmannian obtained by degenerating e-fold products of flag varieties, embedded via a tuple of dominant cocharacters. For $G = \mathrm{GL}_2$, and cocharacters small relative to the characteristic, we relate the cycles of these degenerations to the representation theory of G. We then show that these degenerations smoothly model the geometry of (the special fibre of) low weight crystalline subspaces inside the Emerton–Gee stack classifying p-adic representations of the Galois group of a finite extension of $\mathbb{Q}_p$. As an application we prove new cases of the Breuil–Mézard conjecture in dimension two.

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1. Introduction

Overview. Let K be a finite extension of $\mathbb{Q}_p$ with residue field k and let $\mathcal{X}_d$ denote the Emerton–Gee stack classifying d-dimensional p-adic representations of

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G_K. Inside $\mathcal{X}_d$ there are closed substacks $\mathcal{X}_d^{\mu,\tau}$ classifying potentially crystalline representations of type (μ,τ), for μ and τ respectively Hodge and inertial types. When μ is regular (i.e. consists of distinct integers) these closed substacks have maximal dimension and the Breuil–Mézard conjecture [BM02, EG14, EG19] predicts the existence of top dimensional cycles $\mathcal{C}_\lambda$ in the special fibre $\overline{\mathcal{X}}_d$ such that

$$[\overline{\mathcal{X}}_d^{\mu,\tau}] = \sum_\lambda m(\lambda,\mu,\tau)\mathcal{C}_\lambda \tag{1.1}$$

where

- λ runs over irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_d(k)$.
- $m(\lambda,\mu,\tau)$ denotes the multiplicity with which λ appears in an explicit $\mathbb{F}$-representation $V_{\mu,\tau}$ of $\mathrm{GL}_d(k)$ attached to μ and τ.

(there is also a version of the conjecture for substacks of potentially semistable representations; the conjecture has the same shape but with altered $V_{\mu,\tau}$). These identities have been verified in only a small number of cases:

(1) When $K = \mathbb{Q}_p$ and $d = 2$, using the p-adic Langlands correspondence. See [Kis09a, Paš15, HT15, San14, Tun21].
(2) When $d = 2$ and $\mu = (1,0)$, as consequence of certain modularity lifting theorems. See [GK14].
(3) When K is unramified over $\mathbb{Q}_p$, d is arbitrary, and both p and τ are *generic* relative to μ. See [LLHLM20]. Again modularity lifting technique play an important role.

In this paper we construct Breuil–Mézard identities in a fourth setting: we are interested in the two dimensional case where $\tau = 1$ (i.e. we consider only crystalline rather than potentially crystalline representations) and μ is bounded so that the representation theory of $\mathrm{GL}_2(k)$ in the conjecture behaves as it does in characteristic zero. We do this by constructing analogous identities involving certain degenerations of products of flag varieties embedded in the affine grassmannian, and then relating the geometry of these degenerations to the geometry of the $\overline{\mathcal{X}}_d^{\lambda,1}$.

Main result. First we describe a bound on the Hodge types considered above, which is natural in the sense that the $\mathrm{GL}_d(k)$-representation theory appearing in the above conjecture changes markedly once the bound is passed. Recall that a Hodge type μ consists of a d-tuple of integers

$$\mu_\kappa = (\mu_{\kappa,1} \geq \ldots \geq \mu_{\kappa,d})$$

for each embedding $\kappa : K \to \overline{\mathbb{Q}}_p$. If one assumes that

$$\sum_{\kappa|_k=\kappa_0} \mu_{\kappa,1} - \mu_{\kappa,d} \leq e + p - 1 \tag{1.2}$$

for each embedding $\kappa_0 : k \hookrightarrow \overline{\mathbb{F}}_p$ then:

- $V_\mu = V_{\mu,1}$ is a tensor product over the embeddings κ of representations of highest weight μ_κ and the Jordan–Holder factors of this tensor product are computed in characteristic p just as they are in characteristic zero.
- Each Jordan–Holder factor λ of V_μ can be written as $V_{\widetilde{\lambda}}$ for some Hodge type $\widetilde{\lambda}$ uniquely determined up to an ordering of the embeddings κ.

In particular, the cycles $\mathcal{C}_\lambda$ appearing in (1.4) for these small μ are uniquely determined by the conjectured identity for $\mu = \widetilde{\lambda}$; one has $[\overline{\mathcal{X}}_d^{\widetilde{\lambda}}] = \mathcal{C}_\lambda$. Thus, the following theorem establishes cases of the conjecture:

Theorem 1.3. *Assume that $d = 2$, that μ is regular, and that*

$$\sum_{\kappa|_k = \kappa_0} \mu_{\kappa,1} - \mu_{\kappa,d} \leq p$$

for each embedding $\kappa_0 : k \to \overline{\mathbb{F}}_p$. Then

$$[\overline{\mathcal{X}}_2^{\mu,1}] \leq \sum_\lambda m(\lambda,\mu,1)[\overline{\mathcal{X}}_2^{\widetilde{\lambda},1}] \tag{1.4}$$

with equality if $p \neq 2$.

There are some comments to make before we discuss what goes into the proof. Firstly, the theorem has two clear limitations: the assumption that $d = 2$ and the fact that the bound on μ is worse than that in (1.2) (we have no expectation whatever that the methods in this paper apply beyond (1.2)).

As we explain in more detail below, there are two main difficulties to proving the theorem; namely relating $\overline{\mathcal{X}}_2^{\mu,1}$ with certain local models constructed in the affine grassmannian and proving Breuil–Mézard identities for these local models. The first difficulty can be overcome without any assumption on d but is where the unnatural[1] bound on μ arises. On the other hand, the natural bound (1.2) appears very clearly when considering cycle identities for our local models. However, due to our very direct approach, our calculation of these identities rely heavily upon the representation theory of the $V_\mu's$ being extremely explicit; something which is not the case for $d \geq 3$.

Regarding the inequality appearing in the theorem: The methods we develop in this paper, unlike the approaches (1)-(3) to the Breuil–Mézard conjecture mentioned previously, are purely local. However they are only sufficient to prove the inequality $\leq$.[2] Fortunately, global methods are effective for establishing the opposite inequality $\geq$ (at least in dimension two). In particular, when $d = 2$ and $p \neq 2$, the inequality $\geq$ is known without any assumptions on K, μ or τ. See [EG19, §8.6.5].

Finally, taking $\widetilde{\lambda} = \mu$ shows that the cycle $[\overline{X}_2^{\widetilde{\lambda}}]$ is independent of the choice of "lift" $\widetilde{\lambda}$ of λ. We also show that each of these cycles consists of a single irreducible component (probably with multiplicity one, but we do not prove this).

Method. The proof of Theorem 1.3, and the content of this paper, is divided into two parts:

Part 1: Breuil–Mézard identities in the affine grassmannian. Let $\mathcal{O}$ be a complete discrete valuation ring with residue field $\mathbb{F}$ and fix distinct $\pi_1, \ldots, \pi_e$ in its maximal

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[1]Unnatural in the same way that Fontaine–Laffaille bounds are unnatural; they arise from simplifying coincidences appearing only in the p-adic Hodge theory.

[2]Roughly speaking, we are able to construct an equality of cycles $[\overline{\mathcal{X}}_2'^{\mu,1}] = \sum_\lambda m(\lambda,\mu,\tau)[\overline{\mathcal{X}}_2'^{\widetilde{\lambda},1}]$ with $\overline{X}_2^{\mu,1} \subset \overline{X}_2'^{\mu,1}$. However, proving these inclusions are equalities seems difficult for $\mu \neq \widetilde{\lambda}$

ideal. We consider the (ind)-scheme Gr over $\mathcal{O}$ classifying rank d projective $A[[u]]$-submodules

$$\prod_i (u-\pi_i)^a A[[u]]^d \subset \mathcal{E} \subset A[[u]]^d$$

for some $a \geq 0$ (at least for points valued in a ring A complete with respect to the maximal ideal of $\mathcal{O}$). The generic fibre Gr° of Gr then decomposes as a product $\prod_{i=1}^e \mathrm{Gr}_i^\circ$ where Gr_i classifies those $\mathcal{E} \in \mathrm{Gr}$ with $(u-\pi_i)^a A[[u]]^d \subset \mathcal{E}$ for some $a \geq 0$. For each dominant cocharacter μ_i there is a closed embedding

$$G/P_{\mu_i} \to \mathrm{Gr}_i$$

and for $\mu = (\mu_1, \ldots, \mu_e)$ we consider M_μ equal to the closure in Gr of $\prod_i (G/P_{\mu_i})^\circ \hookrightarrow \mathrm{Gr}^\circ$. Note that the definition of these "local models" is very similar to those appearing in e.g [PR03] except that the flag varieties G/P_{μ_i} here replace the larger Schubert varieties $\mathrm{Gr}_{i,\leq\mu_i}$ usually considered (in particular, if each μ_i is miniscule so that $G/P_{\mu_i} = \mathrm{Gr}_{i,\leq\mu_i}$ then these local models coincide).

If we view each μ_i as an increasing tuple $(\mu_{i,1} \geq \ldots \geq \mu_{i,d})$ then our main result concerning the M_μ's is:

Theorem 1.5. *Assume that each μ_i is regular (i.e. consists of distinct integers), that $d = 2$, and that if $\mathbb{F}$ has characteristic $p > 0$ then*

$$\sum_i \mu_{i,1} - \mu_{i,d} \leq p + e - 1$$

Then there exists an equality of e-dimensional cycles in the special fibre $\overline{\mathrm{Gr}}$

$$[\overline{M}_\mu] = \sum_\lambda m(\lambda,\mu)[\overline{M}_{\widetilde{\lambda}}]$$

where

- *the sum runs over pairs $\lambda = (\lambda_1 > \lambda_2)$ for which $\det^{\lambda_2} \otimes \mathrm{Sym}^{\lambda_1-\lambda_2-1}\mathbb{F}^2$ appears as a Jordan–Holder factor of*

$$\bigotimes_{i=1}^e \det^{\mu_{i,2}} \otimes \mathrm{Sym}^{\mu_{i,1}-\mu_{i,2}-1}\mathbb{F}^2$$

  with multiplicity $m(\lambda,\mu)$.
- *$\widetilde{\lambda} = (\lambda, \rho, \ldots, \rho)$ with $\rho = (1,0)$.*

The main idea is to argue by an induction on the "ramification degree" e. We handle the initial case $e = 2$ by a direct computation; using appropriate big open cells we compute an affine cover of M_μ from which we obtain a description of its special fibre (when $e = 2$ the the Jordan–Holder factors of $V_{\mu_1} \otimes V_{\mu_2}$ are indexed by an explicit collection of $\lambda \leq \mu_1 + \mu_2$ and all appear with multiplicity one, correspondingly $\overline{M}_\mu$ is a reduced union of the $\overline{M}_{\widetilde{\lambda}}$ for such λ).

For the inductive step we give an interpretation of $[\overline{M}_\mu]$ as an e-fold intersection of classes of the partial local models obtained as the closures of

$$\mathrm{Gr}^\circ_{1,\leq\mu_1} \times \ldots \mathrm{Gr}^\circ_{i-1,\leq\mu_{i-1}} \times (G/P_{\mu_i})^\circ \times \mathrm{Gr}^\circ_{i,\leq\mu_{i+1}} \times \ldots \times \mathrm{Gr}^\circ_{e,\leq\mu_e}$$

in Gr. Making this precise requires some work since arbitrary intersection products are not well defined in $\overline{\mathrm{Gr}}$ (because $\overline{\mathrm{Gr}}$ is not smooth). We address this by giving an interpretation of $G/P_{\mu_i} \hookrightarrow \mathrm{Gr}_{i,\leq\mu_i}$ as a degeneracy locus of sections of a vector bundle. This allows us describe the desired intersection product as an e-fold intersection of Chern classes.

In practice, things are slightly more complicated. It is necessary to pull-back the situation to certain convolution grassmannians. Additionally, it is convenient to consider the intersection products as occuring in a Grothedieck group of coherent sheaves.

Remark 1.6. It would be interesting to know, say for $\mathbb{F}$ of characteristic zero, whether this result can be refined to an equivalence between a suitable derived category of coherent sheaves on Gr and the category of algebraic representations of GL_2.

Part 2: From the affine grassmannian to the Emerton–Gee stack. The second part of this paper aims to relate $\overline{\mathcal{X}}_d$ and the identities from Theorem 1.5. The basic strategy is to study the geometry of $\mathcal{X}_d$ via a resolution

$$Y_d \to \mathcal{X}_d$$

with Y_d an appropriate stack of Breuil–Kisin modules (i.e. a projective $W(k)[[u]]$-module equipped with a semilinear endomorphism φ). A local version of this construction was first made in [Kis09b] (with $\mathcal{X}_d$ replaced by Spec of a deformation ring) and its globalisation to stacks first appeared in [PR09], before being built upon in [EG19].

In our case, we take Y_d as the stack classifying pairs $(\mathfrak{M}, \sigma)$ with $\mathfrak{M}$ a rank d Breuil–Kisin module and σ a φ-equivariant action of G_K on $\mathfrak{M} \otimes_{W(k)[[u]]} A_{\mathrm{inf}}$ satisfying a “crystalline” condition. Inside Y_d there are $\mathbb{Z}_p$-flat closed substacks Y_d^μ whose $\mathcal{O}$-valued points correspond to Breuil–Kisin modules associated to crystalline representations of Hodge type μ whenever $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_p$. Then $\mathcal{X}_d^{\mu,1}$ is, by definition, the scheme theoretic image of the morphism $Y_d^\mu \to \mathcal{X}_d$.

To relate Y_d to the affine grassmannian we use the following diagram:

$$Y_d \xleftarrow{\Gamma} \widetilde{Y}_d \xrightarrow{\Psi} \prod_{\kappa_0 : k \to \overline{\mathbb{F}}_p} \mathrm{Gr} \tag{1.7}$$

Here the κ_0-th Gr is defined relative to the elements $\kappa(\pi)$ for $\kappa : K \to \overline{\mathbb{Q}}_p$ extending κ_0 and $\widetilde{Y}_d$ classifies Breuil–Kisin modules in Y_d together with a choice of basis (to stay in the world of finite type stacks this basis is taken modulo u^N for $N >> 0$). Using this choice of basis, the morphism Ψ takes a Breuil–Kisin module $\mathfrak{M}$ to the relative position of $\mathfrak{M}$ and its image of Frobenius. The morphism Γ forgets this choice of basis.

There are two key calculations which allow us to access Y_d^μ. Firstly, we show, for small μ, that the special fibre of the preimage $\widetilde{Y}_d^\mu$ of Y_d^μ in $\widetilde{Y}_d$ is mapped into the special fibre $\prod_{\kappa_0} \overline{M}_{\mu_{\kappa_0}}$ for $\mu_{\kappa_0} = (\mu_\kappa)_{\kappa|_k = \kappa_0}$. This is based on calculations, originally made in [GLS14, GLS15], which determine the shape of the image of Frobenius for crystalline Breuil–Kisin modules, at least up to some power of p controlled by the size of μ. This is done in Section 13 and is where the bound

$$\sum_{\kappa|_k = \kappa_0} \mu_{\kappa,1} - \mu_{\kappa,2} \leq p$$

appears.

The second key calculation controls the fibres of Ψ over $\prod_{\kappa_0} \overline{M}_{\mu_{\kappa_0}}$. For this note that $\widetilde{Y}_d \to Y_d$ is clearly a torsor for a certain smooth group scheme $\mathcal{G}$ acting on the choice of basis. We show that for bounded μ the morphism

$$\Psi : \Psi^{-1}(\prod_{\kappa_0} \overline{M}_{\kappa_0}) \to \prod_{\kappa_0} \overline{M}_{\kappa_0} \tag{1.8}$$

is also a $\mathcal{G}$-torsor (but for a second action of $\mathcal{G}$ on $\widetilde{Y}_d$). If in the previous morphism we replaced Y_d by the moduli stack Z_d of Breuil–Kisin modules (without any Galois action) then this assertion would be very easy. In fact Ψ would be a $\mathcal{G}$-torsor over all of $\prod_{\kappa_0} \overline{\mathrm{Gr}}$, see Proposition 2.4. Thus Ψ is a $\mathcal{G}$-torsor only on the loci over which $Y_d \to Z_d$ is an isomorphism, and our assertion regarding (1.8) comes down to the following: any mod p Breuil–Kisin module $\mathfrak{M}$ whose position relative to its image of Frobenius defines a point in $\prod_{\kappa_0} \overline{M}_{\kappa_0}$ admits a unique G_K-action σ making $(\mathfrak{M}, \sigma)$ into a point of Y_d. This is done by combining Proposition 12.5 and Lemma 14.4, and uses (1.2).[3]

By construction $\dim Y_d^\mu = \dim \prod_{\kappa_0} M_{\kappa_0}$ and so the above two paragraphs imply that

$$\dim \Psi^{-1}(\prod_{\kappa_0} \overline{M}_{\kappa_0}) = \dim \overline{\widetilde{Y}}{}_d^\mu$$

where $\widetilde{Y}_d^\mu$ denotes the preimage of Y_d^μ in $\widetilde{Y}_d$ and $\overline{\widetilde{Y}}{}_d^\mu$ denotes its special fibre. Since we also have $\overline{\widetilde{Y}}{}_d^\mu \subset \Psi^{-1}(\prod_{\kappa_0} \overline{M}_{\kappa_0})$ it follows that $[\overline{\widetilde{Y}}{}_d^\mu] \leq [\Psi^{-1}(\prod_{\kappa_0} \overline{M}_{\mu_{\kappa_0}})]$ as cycles. Since Ψ is smooth over $\prod_{\kappa_0} \overline{M}_{\mu_{\kappa_0}}$, Theorem 1.5 implies an analogous cycle identity for the $[\Psi^{-1}(\prod_{\kappa_0} \overline{M}_{\mu_{\kappa_0}})]$ when $d = 2$. Furthermore, when $\mu = \widetilde{\lambda}$ we show that $\Psi^{-1}(\prod_{\kappa_0} \overline{M}_{\widetilde{\lambda}_{\kappa_0}})$ is irreducible and generically reduced, and hence that $[\overline{\widetilde{Y}}{}^{\widetilde{\lambda}}] = [\Psi^{-1}(\prod_{\kappa_0} \overline{M}_{\widetilde{\lambda}_{\kappa_0}})]$. Descending from $\widetilde{Y}_2$ to Y_2 we conclude that

Theorem 1.9. *With notation as in Theorem 1.3 one has*

$$[\overline{Y}_2^\mu] \leq \sum_\lambda m(\lambda, \mu, 1)[\overline{Y}_2^{\widetilde{\lambda}}]$$

as cycles in the special fibre $\overline{Y}_2$.

We emphasise that this lifting of the Breuil–Mézard identities to the Kisin resolution Y_d requires bounding the Hodge type; we expect Theorem 1.9 to hold for μ satisfying (1.2) but it will fail as soon as one passes this bound.

Finally Theorem 1.3 is obtained by pushing forward the identity from Theorem 1.9 along the proper morphism $Y_2 \to \mathcal{X}_2$ (in practice we only do this after localising at a finite type point in $\overline{\mathcal{X}}_2$; this returns us to the world of schemes and allows us to avoid making things like "$Y_2^\mu \to \mathcal{X}_2^\mu$ is an isomorphism on the generic fibre" precise).

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[3]Actually a slight strengthening of (1.2) is used to avoid certain difficult cases where extensions of the cyclotomic character by the trivial character appear.

Part 1. Cycles in the affine grassmannian

2. Affine grassmannians

Let $\mathcal{O}$ be a complete discrete valuation ring with residue field $\mathbb{F}$, maximal ideal $\mathfrak{m}$, and fraction field E. Fix pairwise distinct $\pi_1, \ldots, \pi_e \in \mathfrak{m}$. We then write Gr for the ind-scheme over $\mathcal{O}$ whose A-points classify rank d projective $A[u]$-submodules in $A[u]^d$ satisfying

$$\prod (u - \pi_i)^a A[u]^d \subset \mathcal{E} \subset A[u]^d$$

for some $a \geq 0$. For $i = 1, \ldots, e$ we write Gr_i for the closed subfunctor classifying $\mathcal{E} \in \mathrm{Gr}(A)$ with $(u - \pi_i)^a A[u]^d \subset \mathcal{E}$ for some $a \geq 0$. The Beauville-Laszlo glueing lemma gives:

Lemma 2.1. *If A is $\mathfrak{m}$-adically complete then $\mathcal{E} \mapsto \mathcal{E} \otimes_{A[u]} A[[u]]$ identifies $\mathrm{Gr}(A)$ with the set of rank d projective $A[[u]]$-submodules of $A[[u]]^d$ satisfying $\prod (u - \pi_i)^a A[[u]]^d \subset \mathcal{E} \subset A[[u]]^d$ for some $a \geq 0$. Similarly, for Gr_i.*

Note that the automorphism $u - \pi_i \mapsto u$ identifies Gr_i with Gr when $e = 1$ and $\pi_1 = 0$. Therefore, the previous lemma allows us to write $\mathrm{Gr}_i = LG/L^+G$ where $G = \mathrm{GL}_d$ and LG and L^+G are the group schemes defined respectively by $A \mapsto G(A((u)))$ and $A \mapsto G(A[[u]])$.

Construction 2.2. A tuple of non-negative integers $\mu_i = (\mu_{i,1}, \ldots, \mu_{i,d})$ is dominant if each $\mu_{i,j} \geq \mu_{i,j+1}$. For such μ and a fixed i we write $\mathcal{E}_{i,\mu} \in \mathrm{Gr}_i$ for the element represented by

$$\begin{pmatrix} u^{\mu_{i,1}} & & \\ & \ddots & \\ & & u^{\mu_{1,d}} \end{pmatrix} \in LG$$

(thus $\mathcal{E}_{i,\mu}$ is generated by $(u - \pi_i)^{\mu_j} e_j$ where e_j denotes the standard basis of $A[u]^d$). We write $\mathrm{Gr}_{i,\mu}$ for the L^+G-orbit of $\mathcal{E}_{i,\mu}$ and $\mathrm{Gr}_{i,\leq\mu}$ for its reduced closure. We can also consider the orbit of $\mathcal{E}_{i,\mu}$ under $G \subset L^+G$. Since the stabiliser of $\mathcal{E}_{i,\mu}$ in G is a parabolic subgroup P_μ the orbit map $G \to \mathrm{Gr}_i$ factors through a proper monomorphism (i.e. a closed immersion) $G/P_\mu \to \mathrm{Gr}_i$.

Recall that for any (ind-)scheme X over $\mathcal{O}$ we write $\overline{X}$ for the special fibre and X° for the generic fibre.

Lemma 2.3. $\mathrm{Gr}^\circ \cong \prod_i \mathrm{Gr}_i^\circ$ *with the identification given by* $(\mathcal{E}_1, \ldots, \mathcal{E}_e) \mapsto \bigcap \mathcal{E}_i$.

Proof. This follows from the fact that $E[u]/\prod (u - \pi_i)^a \cong \prod E[u]/(u - \pi_i)^a$ since the $u - \pi_i$ are pairwise coprime. □

Definition 2.4. Now suppose that $\mu = (\mu_1, \ldots, \mu_e)$ with each μ_j a dominant tuple of non-negative integers. Define the *Schubert variety* $\mathrm{Gr}_{\leq\mu}$ as the closure in Gr of

$$\prod \mathrm{Gr}^\circ_{i,\leq\mu_i} \hookrightarrow \prod \mathrm{Gr}_i^\circ \cong \mathrm{Gr}^\circ$$

We also define *local models* M_μ as the closure in Gr of

$$\prod_{i=1}^{e} (G/P_{\mu_i})^\circ \hookrightarrow \prod \mathrm{Gr}_i^\circ \cong \mathrm{Gr}^\circ$$

Remark 2.5. If μ_i is miniscule then $\mathrm{Gr}_{i,\leq\mu_i} = G/P_{\mu_i}$ and so, if each μ_i is miniscule then $M_\mu = \mathrm{Gr}_{\leq\mu}$.

The following description will be useful:

Proposition 2.6. *The underlying reduced subscheme $\overline{\mathrm{Gr}}_{\leq\mu,\mathrm{red}}$ of $\overline{\mathrm{Gr}}_{\leq\mu}$ identifies with the reduced closure $\overline{\mathrm{Gr}}_{1,\leq\mu_1+\ldots+\mu_e}$ of the L^+G-orbit of $\mathcal{E}_{\mu_1+\ldots+\mu_e}$.*

It follows (as a special case) of [Lev16, 1.0.1] that $\overline{\mathrm{Gr}}_{\leq\mu}$ is itself reduced. However, the simpler assertion in the proposition is sufficient for our purposes.

Proof. This follows by the standard construction of Demazure resolutions for Schubert varieties: writing each $\mu_i = \sum \mu_{ij}$ with μ_{ij} miniscule allows us to construct a surjection $\widetilde{X} \to \mathrm{Gr}_{\leq\mu}$ with $\widetilde{X}$ smooth of the same dimension as $\mathrm{Gr}_{\leq\mu}$. Furthermore, over the special fibre this morphism surjects onto the L^+G-orbit closure of $\mathcal{E}_{\mu_1+\ldots+\mu_e}$. As a consequence the scheme theoretic image identifies $\overline{\mathrm{Gr}}_{\leq\mu,\mathrm{red}}$ with this orbit closure as desired. □

3. Grothendieck groups

For X a Noetherian scheme over $\mathcal{O}$ we write $K_0(X)$ for the Grothendieck group of coherent sheaves on X and $K^0(X)$ for the Grothendieck group of locally free coherent sheaves on X. The tensor product makes $K^0(X)$ into a ring and $K_0(X)$ into a module over $K^0(X)$. For a sheaf $\mathcal{F}$ we write $[\mathcal{F}]$ for the class of $\mathcal{F}$ in $K_0(X)$ or $K^0(X)$.

Construction 3.1 (Cycles)**.** For $k \geq 0$ we write $F^kK_0(X)$ for the subgroup of $K_0(X)$ generated by the classes of those sheaves with support of dimension $\leq k$. Then there is a homomorphism

$$F^kK_0(X) \to Z_k(X)$$

onto the group $Z_k(X)$ of k-dimensional cycles which sends the class of $\mathcal{F}$ onto the k-dimensional part of the support of $\mathcal{F}$. For a closed subscheme $Z \hookrightarrow X$ we write $[Z]$ for the class in $K_0(X)$ of the pushforward of the structure sheaf on Z. If $\dim Z \leq k$ then we also write $[Z]$ for its image in $Z_k(X)$.

Construction 3.2 (Functoriality)**.** The formation of K_0 is compatible with proper pushforward: if $f : X \to Y$ is a proper morphism then there are homomorphisms $f_* : K_0(X) \to K_0(Y)$ given by

$$\mathcal{F} \mapsto \sum_{i\geq 0}(-1)^i[R^if_*\mathcal{F}]$$

If instead f is flat then the pull-back of coherent sheaves is exact and so there is also a pull-back homomorphism $f^* : K_0(Y) \to K_0(X)$. This is a homomorphism of modules over K^0.

More generally, the pull-back is defined for any perfect morphism (in particular for any regular closed immersion). If f is a regular closed immersion then $f_*\mathcal{O}_X$ admits a resolution $\mathcal{E}_\bullet$ by locally free coherent sheaves on Y and

$$f^*[\mathcal{F}] = \sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^Y(\mathcal{O}_X,\mathcal{F})]$$

where $\mathrm{Tor}_i^Y(\mathcal{O}_X,\mathcal{F})$ is the i-th homology of the complex $\mathcal{E}_\bullet \otimes_{\mathcal{O}_Y} \mathcal{F}$, cf. [Ful98, 15.1.8].

Construction 3.3 (Insensitivity to nilpotents)**.** If X_{red} denotes the underlying reduced of X then pushforward along $X_{\mathrm{red}} \to X$ induces an identification $K_0(X_{\mathrm{red}}) = K_0(X)$.

Construction 3.4 (Specialisation)**.** Assume that X is $\mathcal{O}$-flat. Recall we write X° and $\overline{X}$ respectively for the generic and special fibre. Then the inclusion $i : \overline{X} \to X$ is a regular closed immersion and so the pull-back $i^* : K_0(X) \to K_0(\overline{X})$ is defined. Likewise there is a pull-back along the open immersion $j : X^\circ \to X$. In [71, X. App. 7.3] the existence of a *specialisation* morphism

$$\sigma : K_0(X^\circ) \to K_0(\overline{X})$$

is established, fitting into the commutative diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "j^*"] \arrow[d, "i^*"] & K_0(X^\circ) \arrow[dl, "\sigma"] \\ K_0(\overline{X}) & \end{tikzcd}$$

Furthermore, specialisation is compatible with the action of $K^0(X)$ in the sense that for $\mathcal{E} \in K^0(X)$ and $\mathcal{F} \in K_0(X^\circ)$

$$\sigma(j^*\mathcal{E} \cdot \mathcal{F}) = i^*\mathcal{E} \cdot \sigma(\mathcal{F})$$

Specialisation is also compatible with proper pushforward and pullback along a perfect morphism: if $f : X \to Y$ is proper then

$$\begin{tikzcd} K_0(X^\circ) \arrow[r, "f^\circ_*"] \arrow[d, "\sigma"] & K_0(Y^\circ) \arrow[d, "\sigma"] \\ K_0(\overline{X}) \arrow[r, "\overline{f}_*"] & K_0(\overline{Y}) \end{tikzcd}$$

commutes. Similarly for pull-back when f is perfect.

Lemma 3.5. *Suppose that Z is an $\mathcal{O}$-flat closed subscheme of X. Then*

$$\sigma([Z^\circ]) = [\overline{Z}]$$

Proof. Since $j : X^\circ \to X$ is flat we have $j^*([Z]) = [Z^\circ]$. Therefore it is enough to show that $i^*[Z] = [\overline{Z}]$. If ϖ is a uniformiser of $\mathcal{O}$ then $\mathcal{O}_X \xrightarrow{\varpi} \mathcal{O}_X$ is a resolution of $i_*\mathcal{O}_{\overline{X}}$. Since Z is $\mathcal{O}$-flat this resolution has homology concentrated in degree 0 after applying $\otimes_{\mathcal{O}_X} \mathcal{O}_Z$; hence $i^*[Z] = [\overline{Z}]$ as desired. □

Corollary 3.6. *Suppose that $f : X \to Y$ is a proper morphism between $\mathcal{O}$-flat Noetherian schemes which is an isomorphism on the generic fibre. Let $Z \subset X$ be an $\mathcal{O}$-flat closed subscheme. Then the class in $K_0(Y)$ of the special fibre of the closure of $Z^\circ \cong f(Z^\circ)$ equals $f_*[\overline{Z}]$.*

Proof. Lemma 3.5 implies $[\overline{Z}]$ is the specialisation of the class of $[Z^\circ]$. Since specialisation is compatible with proper-pushforward is follows that $f_*[\overline{Z}]$ is the specialisation of the pushforward of $[Z^\circ]$ to Y°. But f is an isomorphism on the generic fibre so applying Lemma 3.5 again shows this latter specialisation is the class of the special fibre of the closure of $Z^\circ \cong f(Z^\circ)$ in Y. □

4. Main result for our local models

In the following we state our main theorem regarding the affine grassmannian. For this we assume that $d = 2$ and that $\mu = (\mu_1, \dots, \mu_e)$ with each μ_i *regular*, i.e. that $\mu_{i,1} > \mu_{i,2}$ for each i. For such μ_i we can then define

$$V_{\mu_i} = \det^{\mu_{i,2}} \otimes_{\mathbb{F}} \mathrm{Sym}^{\mu_{i,1}-\mu_{i,2}-1}\, \mathbb{F}^2$$

viewed as an $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathbb{F})$. These representations are irreducible if $\mathbb{F}$ has characteristic zero and if $\mathbb{F}$ has characteristic p then they are irreducible if and only if $\mu_{i,1} - \mu_{i,2} \leq p$.

Theorem 4.1. *Assume that the characteristic of $\mathbb{F}$, if positive, is $> \sum_{i=1}^{e}(\mu_{i,1} - \mu_{i,2} - 1)$. Then*

$$[\overline{M}_\mu] = \sum_\lambda m(\lambda, \mu)[\overline{M}_{\widetilde{\lambda}}]$$

in $K_0(\overline{\mathrm{Gr}}_{\leq \mu})$, where

- *the sum runs over pairs $\lambda = (\lambda_1 > \lambda_2)$ for which V_λ appears as a Jordan–Holder factor of $V_{\mu_1} \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} V_{\mu_e}$.*
- *$m(\lambda, \mu)$ denotes the multiplicity of V_λ in $V_{\mu_1} \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} V_{\mu_e}$.*
- *$\widetilde{\lambda}$ is the e-tuple $(\lambda, \rho, \dots, \rho)$ where $\rho = (1, 0)$.*

Note that here $[\overline{M}_{\widetilde{\lambda}}]$ is viewed as a class in $K_0(\overline{\mathrm{Gr}}_{\leq \mu})$ via Proposition 2.6 and Construction 3.3.

Proof. Arguing by induction on e it suffices to prove the identity

$$[\overline{M}_\mu] = \sum_\lambda [\overline{M}_{(\lambda, \rho, \mu_3, \dots, \mu_e)}] \tag{4.2}$$

with the sum running over pairs $\lambda = (\lambda_1 > \lambda_2)$ for which V_λ appears as a Jordan–Holder factor of $V_{\mu_1} \otimes_{\mathbb{F}} V_{\mu_2}$ (note that by Lemma 5.2 below each Jordan–Holder factor of $V_{\mu_1} \otimes_{\mathbb{F}} V_{\mu_2}$ always appears with multiplicity one). The proof of (4.2) will eventually be given in Section 7. □

Since $\overline{M}_\mu$ and each $\overline{M}_{\widetilde{\lambda}}$ are e-dimensional we can take the image of the above identity in the group of e-dimensional cycles on $\overline{\mathrm{Gr}}_{\leq \mu}$. We obtain:

Corollary 4.3. *With notation as in Theorem 4.1*

$$[\overline{M}_\mu] = \sum_\lambda m(\lambda, \mu)[\overline{M}_{\widetilde{\lambda}}]$$

in $Z_e(\overline{\mathrm{Gr}}_{\leq \mu})$.

5. Big cell computations

In this section we prove Theorem 4.1 in the case $e = 2$. In fact we prove more, namely we show:

Proposition 5.1. *Assume that $e = d = 2$ and that the characteristic of $\mathbb{F}$, if positive, is*

$$> \mu_{1,1} - \mu_{1,2} - 1 + \mu_{2,1} - \mu_{2,2} - 1$$

Then $\overline{M}_\mu$ is reduced and

$$\overline{M}_\mu = \bigcup_\lambda \overline{M}_{(\lambda, \rho)}$$

where the union runs over those pairs $\lambda = (\lambda_1 > \lambda_2)$ for which V_λ appears as a Jordan–Holder factor of $V_{\mu_1} \otimes_{\mathbb{F}} V_{\mu_2}$.

For simplicity we assume $\mu_1 = (a,0), \mu_2 = (b,0)$ with $a \geq b$ (one easily reduces to this case by twisting). The Jordan–Holder factors of $V_{\mu_1} \otimes_{\mathbb{F}} V_{\mu_2}$ can easily be computed:

Lemma 5.2. *In the Grothendieck group of $\mathbb{F}$-representations $[V_{\mu_1} \otimes V_{\mu_2}] = \sum_{r=0}^{b-1} [V_{\lambda_r}]$ where $\lambda_r = (a+b-1-r, r)$.*

Proof. For this consider the exact sequences

$$0 \to \det \otimes_{\mathbb{F}} \mathrm{Sym}^{a-2}\,\mathbb{F}^2 \otimes_{\mathbb{F}} \mathrm{Sym}^{b-2}\,\mathbb{F}^2 \to \mathrm{Sym}^{a-1}\,\mathbb{F}^2 \otimes_{\mathbb{F}} \mathrm{Sym}^{b-1}\,\mathbb{F}^2 \to \mathrm{Sym}^{a+b-2}\,\mathbb{F}^2 \to 0$$

in which the right hand surjection given by $f \otimes g \mapsto fg$ for homogeneous polynomials f and g in X and Y, and the left hand injection given by $f \otimes g \mapsto Xf \otimes Yg - Yf \otimes Xg$. Arguing inductively this shows that $[\mathrm{Sym}^{a-1}\,\mathbb{F}^2 \otimes_{\mathbb{F}} \mathrm{Sym}^{b-1}\,\mathbb{F}^2] = \sum_{r=0}^{b-1} [\det^r \otimes \mathrm{Sym}^{a+b-2-2r}\,\mathbb{F}^2]$ as required. $\square$

To relate this description with the geometry of $\overline{M}_\mu$ we directly compute an open cover of $\overline{M}_\mu$.

Lemma 5.3. *For $a,b,r \in \mathbb{Z}_{\geq 0}$ with $b \geq r$ let $U_{a,b,r}$ denote the affine $\mathcal{O}$-scheme classifying matrices of the form*

$$U = \begin{pmatrix} (u-\pi_1)^a (u-\pi_2)^{b-r} + F & G \\ H & (u-\pi_2)^r + J \end{pmatrix} \in \mathrm{Mat}(A[u])$$

with

- $F, G \in A[u]$ *of degree* $< a+b-r$
- $H, J \in A[u]$ *of degree* $< r$
- $\det U = (u-\pi_1)^a (u-\pi_2)^b$

and consider the morphism

$$U_{a,b,r} \to \mathrm{Gr}$$

sending $U \mapsto \mathcal{E}_U$ where $\mathcal{E}_U \subset A[u]^2$ is the submodule generated by $(e_1, e_2)U$ (as usual e_1, e_2 denote the standard basis of $A[u]^2$). Then these are open immersions.

Proof. It is enough to show $U_{a,b,r} \to \mathrm{Gr}$ is a monomorphism and formally smooth. We leave the calculation showing the map is a monomorphism to the reader. For formal smoothness, suppose A is local, $I \subset A$ is a square-zero ideal, and $\mathcal{E} \in \mathrm{Gr}(A)$ is generated by $(e_1, e_2)X$ with X modulo I in $U_{a,b,r}$. Thus, we can write X as

$$X = U + Y$$

for some $U \in U_{a,b,r}$ and some $Y \in \mathrm{Mat}(I[u])$. If $\Lambda = \left(\begin{smallmatrix} u^{a+b-r} & 0 \\ 0 & u^r \end{smallmatrix}\right)$ we can assume $Y = \sum_{i=0}^n \Lambda u^i Y_i$ with $Y_i \in \mathrm{Mat}(I)$. As I is square-zero $X^{(1)} := X(1 - \Lambda u^n Y_n) = U + Y - \Lambda u^n U Y_n$. Any $U \in U_{a,b,r}$ can be written as Λ+(lower order terms). Therefore $X^{(1)} = U^{(1)} + \sum_{i=0}^{n-1} \Lambda u^i Y_i^{(1)}$ for some $U^{(1)} \in U_{a,b,r}$ and $Y^{(1)} \in \mathrm{Mat}(I)$. Since $\mathcal{E}$ is also generated by $(e_1, e_2)X^{(1)}$, inducting on n shows that $\mathcal{E}$ can be generated by $(e_1, e_2)X^{(n)}$ with $X^{(n)} \in U_{a,b,r}$. This shows formal smoothness. $\square$

Our computation of $M_\mu \cap U_{a,b,r}$ is based on properties of the following polynomials.

Lemma 5.4. *For $r = 0, \dots, b$ there exists a unique pair of polynomials*

$$X^{(r)} = \sum_{i=0}^{b-r} X_i^{(r)} (u - \pi_2)^i \in \mathcal{O}[u], \qquad Y^{(r)} = \sum_{i=0}^{r-1} Y_i^{(r)} (u - \pi_2)^i \in \mathcal{O}[u]$$

satisfying:

(1) $X_{b-r}^{(r)} = 1$
(2) $(u - \pi_1)^a X^{(r)} \equiv Y^{(r)}$ *modulo* $(u - \pi_2)^b$

Furthermore, $X_i^{(r)} \in (\pi_2 - \pi_1)^{i+1}\mathcal{O}$ and if $r \neq 0$ then $X_0^{(r)} \in (\pi_2 - \pi_1)\mathcal{O}^\times$ and $Y_i^{(r)} \in (\pi_2 - \pi_1)^{a-i+1}\mathcal{O}^\times$.

Proof. Write $(u - \pi_1)^a = \sum \eta_i (u - \pi_2)^i$, so that $\eta_i = \binom{a}{i}(\pi_2 - \pi_1)^{a-i}$ and

$$(u - \pi_1)^a X^{(r)} = \sum_n (u - \pi_2)^n \left(\sum_{i+j=n} \eta_i X_j^{(r)} \right)$$

We need to show that there are unique $X_j^{(r)} \in \mathcal{O}$ such that

$$\sum_{i+j=n} \eta_i X_j^{(r)} = 0$$

for $n = r, \dots, b-1$ and $X_{b-r}^{(r)} = 1$. In other words, we need to show that the following system of $b - r$ linear equations has a unique solution over $\mathcal{O}$:

$$\underbrace{\begin{pmatrix} \eta_r & \dots & \eta_{b-1} \\ \vdots & & \vdots \\ \eta_{2r-b+1} & \dots & \eta_r \end{pmatrix}}_{M_r = [\eta_{r+i-j}]_{ij}} \begin{pmatrix} X_{b-r-1}^{(r)} \\ \vdots \\ X_0^{(r)} \end{pmatrix} = - \begin{pmatrix} \eta_{r-1} \\ \vdots \\ \eta_{2r-b} \end{pmatrix}$$

(here we use the convention that $\eta_i = 0$ for $i < 0$). We have

$$\begin{aligned} \det(M_r) &= \sum_\sigma \operatorname{sgn}(\sigma) \prod_{i=1}^{b-r-1} \eta_{r+i-\sigma(i)} \\ &= (\pi_2 - \pi_1)^{(b-r)(a-r)} \sum_\sigma \operatorname{sgn}(\sigma) \prod_{i=1}^{b-r-1} \binom{a}{r + i - \sigma(i)} \\ &= (\pi_2 - \pi_1)^{(b-r)(a-r)} \det\left[\binom{a}{r+i-j} \right]_{ij} \\ &= (\pi_2 - \pi_1)^{(b-r)(a-r)} \prod_{i=1}^{b-r} \prod_{j=1}^{r} \prod_{k=1}^{a-r} \frac{i+j+k-1}{i+j+k-2} \end{aligned}$$

(for the last equality we use [Kra99, 2.3]). Since $(a + b - 2)!$ is invertible in $\mathcal{O}$ it follows that there is a unique solution in $\mathcal{O}$. Cramer's rule gives

$$\det(M_r) X_i^{(r)} = \begin{vmatrix} \eta_r & \dots & \eta_{r+i-1} & -\eta_{r-1} & \eta_{r+i+1} & \dots & \eta_{b-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \eta_{2r-b+1} & \dots & \eta_{2r-b+i} & -\eta_{2r-b} & \eta_{2r-b+i+2} & \dots & \eta_r \end{vmatrix}$$

and so, since the right hand determinant is an integer multiple of $(\pi_2 - \pi_1)^{(b-r)(a-r)+i+1}$, we conclude that $X_i^{(r)}$ is the product of $(\pi_2 - \pi_1)^{i+1}$ with an element of $\mathcal{O}$. Note also that when $i = 0$ and $r \neq 0$ the right hand determinant has the same shape as $\det(M_r)$ and so can computed by the same formula; thus $X_0^{(r)} \in (\pi_2 - \pi_1)\mathcal{O}^\times$.

Finally, $Y_i^{(r)} = X_0^{(r)}\eta_i + \ldots + X_i^{(r)}\eta_0$ with the j-th term in this sum an $\mathcal{O}$-multiple of $(\pi_2 - \pi_1)^{a+j+1-(i-j)}$ and an $\mathcal{O}^\times$ multiple when $j = 0$; this proves the last part of the lemma. □

Proof of Proposition 5.1. We begin by describing morphisms into $U_{a,b,r}$ using Lemma 5.4. For $r = 1, \ldots, b-1$ we write $\Theta_{a,b,r} : \operatorname{Spec} \frac{\mathcal{O}[g,z_1,z_2]}{z_1 z_2 - (\pi_2 - \pi_1)} \to U_{a,b,r}$ for the map defined by

$$(g, z_1, z_2) \mapsto \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (u-\pi_1)^a X^{(r)} & z_2(\pi_2-\pi_1)^{a-r+1}(u-\pi_1)^a \frac{X^{(r+1)}}{Y_r^{(r+1)}} \\ z_1 \frac{Y^{(r)}}{(\pi_2-\pi_1)^{a-r+2}} & \frac{Y^{(r+1)}}{Y_r^{(r+1)}} \end{pmatrix}$$

The assertion that the image of this morphism is contained in $U_{a,b,r}$ has two non-obvious parts. Firstly, one needs to check that the matrices in the image of this morphism have determinant $(u-\pi_1)^a(u-\pi_2)^b$; this follows from part (2) of Lemma 5.4. Secondly, one needs to check that the matrices in the image are $\mathcal{O}$-integral; this follows from the last part of Lemma 5.4. We also define $\Theta_{a,b,b} : \operatorname{Spec} \mathcal{O}[g, z_1] \to U_{a,b,b}$ and $\Theta_{a,b,0} : \operatorname{Spec} \mathcal{O}[g, z_2] \to U_{a,b,0}$ by

$$\Theta_{a,b,b}(g, z_1) = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (u-\pi_1)^a & 0 \\ z_1 Y^{(b)} & (u-\pi_2)^b \end{pmatrix}$$

and

$$\Theta_{a,b,0}(g, z_2) = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (u-\pi_1)^a(u-\pi_2)^b & z_2(\pi_2-\pi_1)^{a+1}(u-\pi_1)^a \frac{X^{(1)}}{Y_0^{(1)}} \\ 0 & \frac{Y^{(1)}}{Y_0^{(1)}} \end{pmatrix}$$

It is clear that both $\Theta_{a,b,0}$ and $\Theta_{a,b,b}$ takes values in $U_{a,b,0}$ and $U_{a,b,b}$ as claimed.

Next we give concrete descriptions of the points of M_μ. If $\mathcal{E} \in M_\mu$ has restriction $\mathcal{E}^\circ$ to the generic fibre then Lemma 2.3 and the definition of M_μ implies that $\mathcal{E}^\circ = g_1^\circ \mathcal{E}_{1,\mu_1} \cap g_2^\circ \mathcal{E}_{2,\mu_2}$ for some $g_i^\circ \in G^\circ$. If $\mathcal{E}$ corresponds to an $\mathcal{O}'$-valued point of M_μ, for $\mathcal{O}'$ the ring of integers in a finite extension E' or E, then the valuative criterion of properness shows that this description is valid integrally, i.e. that

$$\mathcal{E} = g_1 \mathcal{E}_{1,\mu_1} \cap g_2 \mathcal{E}_{2,\mu_2}$$

for some $g_1, g_2 \in G$. We begin by considering those $\mathcal{O}'$-points obtained by taking $g_i = \left(\begin{smallmatrix} 1 & \alpha_i \\ 0 & 1 \end{smallmatrix}\right)$. Translating $\mathcal{E}$ by an element of $\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$ we can assume $\alpha_1 = 0$ and write $\alpha_2 = \alpha$. Then the intersection consists of elements $Xe_1 + Ye_2$ with $X, Y \in \mathcal{O}'[u]$ satisfying

$$X \equiv 0 \text{ modulo } (u-\pi_1)^a, \qquad X - \alpha Y \equiv 0 \text{ modulo } (u-\pi_2)^b$$

Property (2) from Lemma 5.4 implies that $\Theta_{a,b,r}(0, z_1, z_2)$ is contained in $\mathcal{E}_{1,\mu_1} \cap \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_{2,\mu_2}$ for $\alpha = (\pi_2-\pi_1)^{a-r+2} z_1^{-1} = (\pi_2-\pi_1)^{a-r+1} z_1$. In fact this is an equality as can be seen by comparing determinants. On the other hand, if additionally $\mathcal{E}_{1,\mu_1} \cap \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_{2,\mu_2} = \mathcal{E}_U$ for some $U \in U_{a,b,r}$ and $r = 1, \ldots, b-1$ then the uniqueness from Lemma 5.4 implies that

$$U = \begin{pmatrix} (u-\pi_1)^a X^{(r)} & \alpha(u-\pi_1)^a \frac{X^{(r+1)}}{Y_r^{(r+1)}} \\ \alpha^{-1} Y^{(r)} & \frac{Y^{(r+1)}}{Y_r^{(r+1)}} \end{pmatrix}$$

with this matrix $\mathcal{O}'$-integral. Using the last part of Lemma 5.4 it follows that

$$(a-r+1)v_{\mathcal{O}}(\pi_2-\pi_1) \le v_{\mathcal{O}}(\alpha) \le (a-r+2)v_{\mathcal{O}}(\pi_2-\pi_1)$$

Therefore the $\mathcal{O}'$-points obtained via $\left(\begin{smallmatrix} 1 & \alpha_1 \\ 0 & 1 \end{smallmatrix}\right)\mathcal{E}_{1,\mu_1} \cap \left(\begin{smallmatrix} 1 & \alpha_2 \\ 0 & 1 \end{smallmatrix}\right)\mathcal{E}_{2,\mu_2}$ with α as above are in bijection with those of $\mathrm{Spec}\, \frac{\mathcal{O}[g,z_1,z_2]}{z_1 z_2 - (\pi_2 - \pi_1)}$ via Θ_r. Similarly, those $\mathcal{O}'$-points with $v_{\mathcal{O}}(\alpha) \geq (a+1)v_{\mathcal{O}}(\pi_2 - \pi_1)$ are in bijection with those of $\mathrm{Spec}\, \mathcal{O}[g, z_2]$ via Θ_0 and likewise for those with $v_{\mathcal{O}}(\alpha) \leq (a-b+2)v_{\mathcal{O}}(\pi_2 - \pi_1)$ and Θ_b. The upshot is that $\mathrm{Im}\,\Theta_{a,b,r}(\mathcal{O}') = (M_\mu \cap U_{a,b,r})(\mathcal{O}')$ and

$$\bigcup_r \mathrm{Im}\,\Theta_{a,b,r}(\mathcal{O}')$$

consists of all $\mathcal{O}'$ points corresponding to $\mathcal{E} = g_1\mathcal{E}_{1,\mu_1} \cap g_2\mathcal{E}_{2,\mu_2}$ with $g_1, g_2 \in \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$.

Since the source of each $\Theta_{a,b,r}$ is reduced and $\mathcal{O}$-flat the previous paragraph implies that $\Theta_{a,b,r}$ factors through $M_\mu \cap U_{a,b,r}$. It is obvious from their definitions that each $\Theta_{a,b,r}$ is a monomorphism. In particular, they are unramified and so Zariski locally on the source can be expressed as $f \circ g$ with f a closed immersion and g etale [Liu02, 4.11]. Since $M_\mu \cap U_{a,b,r}$ is reduced of relative dimension two over $\mathcal{O}$ the same is true of the source of g. Therefore f is a closed immersion between reduced schemes of the same dimension and so is an open immersion. We conclude that $\Theta_{a,b,r}$ is etale and so, being a monomorphism, is an open immersion. Writing $\omega = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we then claim that

$$M_\mu = \bigcup_r (\mathrm{Im}\,\Theta_{a,b,r} \cup \omega\, \mathrm{Im}\,\Theta_{a,b,r})$$

It suffices to check this on $\mathcal{O}'$-valued points and we've already seen that $\bigcup_r \mathrm{Im}\,\Theta_{a,b,r}$ contains all the points corresponding to $g_1\mathcal{E}_{1,\mu_1} \cap g_2\mathcal{E}_{2,\mu_2}$ with $g_i \in \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$. The remaining points will instead be obtained by taking g_1 or g_2 equal to ω. However, $\omega\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\mathcal{E}_{i,\mu_i} = \left(\begin{smallmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{smallmatrix}\right)\mathcal{E}_{i,\mu_i}$ for $\alpha \neq 0$ so either $\{g_1, g_2\} = \{1, \omega\}$ or $\omega\mathcal{E}$ is contained in $\bigcup_r \mathrm{Im}\,\Theta_{a,b,r}$. But clearly $\mathcal{E}_{1,\mu_1} \cap \omega\mathcal{E}_{2,\mu_2} = \Theta_{a,b,b}(0,0)$. Thus, the claim holds.

To finish the proof we compute $\Theta_{a,b,r}$ on the special fibre. For $r = 1, \ldots, b-1$ the last part of Lemma 5.4 shows this special fibre $\overline{\Theta}_{a,b,r}$ is given by

$$(g, z_1, z_2) \mapsto \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^{a+b-r} & z_2 u^{a+b-r-1} \\ z_1 u^{r-1} & u^r \end{pmatrix}$$

Similarly $\overline{\Theta}_{a,b,0}$ is given by $(g, z_2) \mapsto \left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} u^{a+b} & z_2 u^{a+b-1} \\ 0 & 1 \end{smallmatrix}\right)$ while $\overline{\Theta}_{a,b,b}$ is given by $(g, z_1) \mapsto \left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} u^{a} & 0 \\ z_1 u^{b-1} & u^b \end{smallmatrix}\right)$. In particular, we see that

$$\mathrm{Im}\,\overline{\Theta}_{a,b,r}|_{z_1=0} = \mathrm{Im}\,\overline{\Theta}_{a+b-2r,1,0}(r), \qquad \mathrm{Im}\,\overline{\Theta}_{a,b,r}|_{z_2=0} = \mathrm{Im}\,\overline{\Theta}_{a+b-2r+1,1,1}(r-1)$$

where $\Theta_{a+b-2r,1,0}(r)$ denotes the translate of $\Theta_{a+b-2r,1,0}$ by $\left(\begin{smallmatrix} u^r & 0 \\ 0 & u^r \end{smallmatrix}\right)$. On the other hand, the discussion above applied with μ replaced by $\widetilde{\lambda}_r = (\lambda_r, \rho)$ for $\lambda_r = (a+b-1-r, r)$ shows that

$$\overline{M}_{\widetilde{\lambda}_r} = \mathrm{Im}\,\overline{\Theta}_{a+b-2r,1,0}(r) \cup \mathrm{Im}\,\overline{\Theta}_{a+b-2r,1,1}(r) \cup \omega\,\mathrm{Im}\,\overline{\Theta}_{a+b-2r,1,0}(r) \cup \omega\,\mathrm{Im}\,\overline{\Theta}_{a+b-2r,1,1}(r)$$

It follows that

$$\overline{M}_\mu = \bigcup_{r=0}^{b-1} \overline{M}_{\widetilde{\lambda}_r}$$

Since each $\mathrm{Im}\,\overline{\Theta}_{a,b,r}$ is clearly reduced this finishes the proof. $\square$

6. Degeneracy loci

To prove (4.2) we would like to give an interpretation of $\overline{M}_\mu$ as an e-fold intersection of the closed subschemes

$$\{\mathcal{E} \in \overline{\mathrm{Gr}}_{\leq\mu} \mid \mathcal{E} \subset g\mathcal{E}_{i,\mu_i} \text{ for some } g \in G\} \tag{6.1}$$

However, since $\overline{\mathrm{Gr}}_{\leq\mu}$ is not smooth, intersection products are not defined for arbitrary closed subschemes. On K_0 the intersection product corresponds to the action of K^0 and so, to produce such an e-fold intersection, we need to show that the structure sheaves of (6.1) can be resolved by finite complexes of vector bundles on $\overline{\mathrm{Gr}}_{\leq\mu}$.

In this section we show that such resolutions exist, at least on the generic fibre.

Proposition 6.2. *Assume that $d = 2$. For each i there exists a finite complex $\mathcal{C}_{\mu_i}$ of vector bundles on $\mathrm{Gr}_{i,\leq\mu_i}$ such that the image in $K_0(\prod_{i=1}^e \mathrm{Gr}_{i,\leq\mu_i})$ of*

$$p_1^*[\mathcal{C}_{\mu_1}] \cdot \ldots \cdot p_j^*[\mathcal{C}_{\mu_j}]$$

(where $p_n : \prod_{i=1}^e \mathrm{Gr}_{i,\mu_i} \to \mathrm{Gr}_{n,\leq\mu_n}$ is the projection) equals the class of

$$\prod_{i=1}^{j} (G/P_{\mu_i}) \times_{\mathcal{O}} \prod_{i=j+1}^{e} \mathrm{Gr}_{i,\leq\mu_i}$$

Proof. We begin by resolving the structure sheaf of the closed subscheme G/P_{μ_i} in $\mathrm{Gr}_{i,\leq\mu_i}$. This is based on the following:

Claim. *If $\mu_i = (a,b)$ and $\mathcal{E} \in \mathrm{Gr}_{i,\leq\mu_i}$ corresponds to an A-valued point then $\mathcal{E} \in G/P_{\mu_i}$ if and only if the images of $u^b e_1$ and $u^b e_2$ in $A[u]^2/\mathcal{E}$ are A-linearly dependent*

Proof of Claim. If $\mathcal{E} \in G/P_{\mu_i}$ then $\mathcal{E}$ is generated by $u^a g(e_1), u^b g(e_2)$ for some $g \in G$ and so $u^b g(e_2) = 0$ in $A[u]^2/\mathcal{E}$. For the converse, after replacing $\mathcal{E}$ by $g\mathcal{E}$ for some $g \in G$, we can suppose that $u^b e_2 \in \mathcal{E}$. We can also assume that A is local so that $\mathcal{E}$ is generated by

$$(e_1, e_2)\begin{pmatrix} (u-\pi_i)^b \alpha & 0 \\ 0 & (u-\pi_i)^b \end{pmatrix}$$

for some $\alpha \in A[u]$. Since $\mathcal{E} \in \mathrm{Gr}_{i,\leq\mu}$ this matrix has determinant equal to a unit multiple of $(u-\pi_i)^{a+b}$, i.e. α is a unit multiple of $(u-\pi_i)^a$. Hence $\mathcal{E} = \mathcal{E}_{i,\mu_i} \in G/P_{\mu_i}$. □

Next write $\mathcal{V}$ for the vector bundle given by $\mathcal{E} \mapsto A[u]^2/\mathcal{E}$ and $\mathcal{V}^*$ for its linear dual. Then the claim implies that the structure sheaf of G/P_{μ_i} identifies with the cokernel of the morphism

$$\epsilon : \bigwedge^2 \mathcal{V}^* \to \mathcal{O}_{\mathrm{Gr}_{i,\leq\mu_i}}$$

given by evaluation along $u^b e_1 \wedge u^b e_2$ (here we view $u^b e_1, u^b e_2$, or rather their images, as global sections of $\mathcal{V}$). The morphism ϵ appears as the first term of the Eagon–Northcott complex which we denote by $\mathcal{C}_{\mu_i}$. This is a finite complex of vector bundles which is a resolution of $\operatorname{coker} \epsilon$ when the corresponding ideal sheaf has depth $\geq (a-b) - 2 + 1 = a - b - 1$ [Eis95, Theorem A2.10]. In our case G/P_{μ_i} has codimension $a-b-1$ in $\mathrm{Gr}_{i,\leq\mu_i}$ and $\mathrm{Gr}_{i,\leq\mu_i}$ is Cohen–Macauley, cf. for example [Lev16, 1.0.1]. Therefore, these conditions are satisfied and $\mathcal{C}_{\mu_i}$ is a resolution of the structure sheaf of $(G/P_{\mu_i})^\circ$.

As a consequence $p_1^*[\mathcal{C}_{\mu_1}]\cdot\ldots\cdot p_j^*[\mathcal{C}_{\mu_j}]$ then corresponds to the class of the j-fold tensor product of the $\mathcal{C}_{\mu_i}$'s. By the previous paragraph this tensor product is resolution of the j-fold tensor products of the structure sheaves of the pull-backs of $G/P_{\mu_i}\hookrightarrow \mathrm{Gr}_{i,\leq\mu_i}$, i.e. is a resolution for the structure sheaf of $\prod_{i=1}^j (G/P_{\mu_i})\times_{\mathcal{O}}\prod_{i=j+1}^e \mathrm{Gr}_{i,\leq\mu_i}$ as desired. □

7. Proof of the main result

Now we finish the proof of Theorem 4.1. Recall this requires proving the identity

$$[\overline{M}_\mu]=\sum_\lambda [\overline{M}_{(\lambda,\rho,\mu_3,\ldots,\mu_e)}]$$

in $K_0(\overline{\mathrm{Gr}}_{\leq\mu})$ from (4.2). We will do this by combining the results from Section 6 and Section 5.

Step 1. The first step is give interpretations of the terms in (4.2) in terms of the complexes from Section 6. Ideally this would be done by pulling back the complexes $\mathcal{C}_{\mu_i}$ along a morphism $\mathrm{Gr}_{\leq\mu}\to\mathrm{Gr}_{i,\leq\mu_i}$. However, such morphisms only exist on the generic fibre. To address this we consider convolution versions of $\mathrm{Gr}_{\leq\mu}$: let

$$\mathrm{Gr}^{\mathrm{con}}_{\leq\mu_1,\ldots,\mu_e}\subset\mathrm{Gr}_{\leq\mu}\times_{\mathcal{O}}\prod_{i=1}^e\mathrm{Gr}_{i,\leq\mu_i}$$

be the closed locus consisting of $(\mathcal{E},\mathcal{E}_i)$ with $\mathcal{E}\subset\mathcal{E}_i$. The proof of Lemma 2.3 shows that the projection $F:\mathrm{Gr}^{\mathrm{con}}_{\leq\mu_1,\ldots,\mu_e}\to\mathrm{Gr}_{\leq\mu}$ is an isomorphism on the generic fibre. As a consequence we deduce:

Lemma 7.1. *Let $\mathcal{C}^{\mathrm{con}}_{\mu_i}$ denote the pullback of the complexes $\mathcal{C}_{\mu_i}$ along $\mathrm{Gr}^{\mathrm{con}}_{\leq\mu_1,\ldots,\mu_e}\to\mathrm{Gr}_{i,\leq\mu_i}$ and write $\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_i}$ for their restriction to the special fibre. Then the image of*

$$[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_1}]\cdot\ldots\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_e}]$$

in $K_0(\overline{\mathrm{Gr}}^{\mathrm{con}}_{\leq\mu_1,\ldots,\mu_e})$ equals the class of the special fibre of the closure of $\prod_{i=1}^e(G/P_{\mu_i})^\circ$ in $\mathrm{Gr}^{\mathrm{con}}_{\leq\mu_1,\ldots,\mu_e}$.

Proof. Lemma 3.5 implies the the class of the special fibre of the closure is the specialisation of the class of $\prod_{i=1}^e(G/P_{\mu_i})^\circ$. By Corollary 6.2 this latter class can be expressed as $[\mathcal{C}^{\mathrm{con},\circ}_{\mu_1}]\cdot\ldots\cdot[\mathcal{C}^{\mathrm{con},\circ}_{\mu_e}]$ where $\mathcal{C}^{\mathrm{con},\circ}_{\mu_i}$ is the restriction of $\mathcal{C}^{\mathrm{con}}_{\mu_i}$ to the generic fibre. Since the specialisation map is compatible with the action of $K^0(\mathrm{Gr}^{\mathrm{con}}_{\leq\mu_1,\ldots,\mu_e})$ the lemma follows. □

Combining Lemma 7.1 and the fact that $F:\mathrm{Gr}^{\mathrm{con}}_{\leq\mu_1,\ldots,\mu_e}\to\mathrm{Gr}_{\leq\mu}$ is an isomorphism on the generic fibre with Corollary 3.5 gives:

$$[\overline{M}_\mu]=F_*\left([\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_1}]\cdot\ldots\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_e}]\right)\tag{7.2}$$

in $K_0(\overline{\mathrm{Gr}}_{\leq\mu})$. Note that the discussion here is equally valid with μ_1,μ_2 replaced by λ,ρ.

Step 2. Next we consider the following the factorisation of F

$$\mathrm{Gr}^{\mathrm{con}}_{\leq \mu_1,\ldots,\mu_e} \xrightarrow{F_{\mu_1,\mu_2}} \mathrm{Gr}^{\mathrm{con}}_{\leq \mu_3,\ldots,\mu_e} \to \mathrm{Gr}_{\leq \mu}$$

where $\mathrm{Gr}^{\mathrm{con}}_{\leq \mu_3,\ldots,\mu_e}$ is the closed locus of $(\mathcal{E},\mathcal{E}_i) \in \mathrm{Gr}_{\leq\mu} \times_{\mathcal{O}} \prod_{i=3}^e \mathrm{Gr}_{i,\leq\mu_i}$ defined by $\mathcal{E} \subset \mathcal{E}_i$. Each of the complexes $\mathcal{C}^{\mathrm{con}}_{\mu_i}$ for $i = 3,\ldots,e$ is obtained as a pull-back of a complex on $\mathrm{Gr}^{\mathrm{con}}_{\leq \mu_3,\ldots,\mu_e}$ and so the projection formula gives

$$F_{\mu_1,\mu_2,*}\left([\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_1}]\cdot\ldots\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_e}]\right) = F_{\mu_1,\mu_2,*}\left([\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_1}]\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_2}]\right)\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_3}]\cdot\ldots\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_e}]$$

(here we abusively write $\mathcal{C}^{\mathrm{con}}_{\mu_i}$ for the pullback to both $\mathrm{Gr}^{\mathrm{con}}_{\leq \mu_1,\ldots,\mu_e}$ and $\mathrm{Gr}^{\mathrm{con}}_{\leq \mu_3,\ldots,\mu_e}$ of $\mathcal{C}_{\mu_i}$). As with the previous step, this discussion is equally valid when μ_1,μ_2 is replaced by λ,ρ. Using (7.2) we reduce (4.2) to the assertion that

$$F_{\mu_1,\mu_2,*}([\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_1}]\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_2}]) = \sum_\lambda F_{\lambda,\rho,*}\left([\overline{\mathcal{C}}^{\mathrm{con}}_{\lambda}]\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\rho}]\right) \tag{7.3}$$

Step 3. Next we show that (7.3) can be obtained as the pushforward of a similar identity along the projection

$$F_{12} : \mathrm{Gr}^{\mathrm{con}}_{\leq(\mu_1,\mu_2),\mu_3,\ldots,\mu_e} \to \mathrm{Gr}^{\mathrm{con}}_{\leq\mu_3,\ldots,\mu_e}$$

where $\mathrm{Gr}^{\mathrm{con}}_{\leq(\mu_1,\mu_2),\mu_3,\ldots,\mu_e} \subset \mathrm{Gr}_{\leq\mu} \times_{\mathcal{O}} \mathrm{Gr}_{\leq(\mu_1,\mu_2)} \times_{\mathcal{O}} \prod_{i=3}^e \mathrm{Gr}_{i,\leq\mu_i}$ is the closed locus consisting of $(\mathcal{E},\mathcal{E}_{12},\mathcal{E}_i)$ with $\mathcal{E} \subset \mathcal{E}_{12}$ and $\mathcal{E} \subset \mathcal{E}_i$.

Note that once again this projection is an isomorphism on the generic fibre. Therefore we can define $M^{\mathrm{con}}_{\mu_1,\mu_2}$ as the closure of $(G/P_{\mu_1})^\circ \times_E (G/P_{\mu_2})^\circ \times_E \prod_{i=3}^e \mathrm{Gr}^\circ_{i,\leq\mu_i}$ in $\mathrm{Gr}^{\mathrm{con}}_{\leq(\mu_1,\mu_2),\mu_3,\ldots,\mu_e}$. We claim that

$$F_{\mu_1,\mu_2,*}\left([\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_1}]\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_2}]\right) = F_{12,*}[\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2}]$$

To see this note that an argument identical to that used to prove Lemma 7.1 identifies $[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_1}]\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_2}]$ with the class of the special fibre of the closure of $(G/P_{\mu_1})^\circ \times_E (G/P_{\mu_2})^\circ \times_E \prod_{i=3}^e \mathrm{Gr}^\circ_{i,\leq\mu_i}$ in $\mathrm{Gr}^{\mathrm{con}}_{\leq\mu_1,\ldots,\mu_e}$. Since F_{μ_1,μ_2} is an isomorphism on the generic fibre Corollary 3.5 identifies $F_{\mu_1,\mu_2,*}\left([\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_1}]\cdot[\overline{\mathcal{C}}^{\mathrm{con}}_{\mu_2}]\right)$ with the class of the special fibre of closure of $(G/P_{\mu_1})^\circ \times_E (G/P_{\mu_2})^\circ \times_E \prod_{i=3}^e \mathrm{Gr}^\circ_{i,\leq\mu_i}$ now taken in $\mathrm{Gr}^{\mathrm{con}}_{\leq\mu_3,\ldots,\mu_e}$. In the same way this equals the pushforward of $[\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2}]$ along F_{12} which proves our claim.

We are therefore reduced to the following proposition, which is proved using the results from Section 5.

Proposition 7.4. *$\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2}$ is generically reduced and topologically*

$$\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2} = \bigcup_\lambda \overline{M}^{\mathrm{con}}_{\lambda,\rho}$$

with λ running over pairs (λ_1,λ_2) as in Proposition 5.1.

Proof. First we show that $\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2}$ is generically reduced. For this we consider the pushforward of $[\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2}] \in K_0(\overline{\mathrm{Gr}}^{\mathrm{con}}_{\leq(\mu_1,\mu_2),\mu_3,\ldots,\mu_e})$ along the projection

$$\mathrm{Gr}^{\mathrm{con}}_{\leq(\mu_1,\mu_2),\mu_3,\ldots,\mu_e} \to \mathrm{Gr}_{\leq(\mu_1,\mu_2)} \tag{7.5}$$

As we have used many times above, this pushforward is the specialisation of the pushforward of the class of the generic fibre. On the generic fibre (7.5) is just the projection $\prod_i \mathrm{Gr}^\circ_{i,\leq\mu_i} \to \mathrm{Gr}^\circ_{1,\leq\mu_1} \times_E \mathrm{Gr}^\circ_{2,\leq\mu_2}$ and so the pushforward of $M^{\mathrm{con},\circ}_{\mu_1,\mu_2}$ is an integer multiple of $M^\circ_{(\mu_1,\mu_2)}$. Therefore the pushforward of $[\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2}]$ is an integer multiple of $[\overline{M}_{(\mu_1,\mu_2)}]$. But Proposition 5.1 implies every component of $\overline{M}_{(\mu_1,\mu_2)}$ has multiplicity one so the same must be true of $\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2}$.

For the topological part of the proposition we first note that $M^{\mathrm{con}}_{\mu_1,\mu_2} \to \mathrm{Gr}_{\leq(\mu_1,\mu_2)}$ factors through $M_{(\mu_1,\mu_2)}$ since this is case on the generic fibre. Therefore Proposition 5.1 gives

$$\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2} = \bigcup_{\lambda} \overline{M}^{(\lambda)}$$

where $\overline{M}^{(\lambda)}$ is the locus of $\overline{M}^{\mathrm{con}}_{\mu_1,\mu_2}$ consisting of $(\overline{\mathcal{E}}, \overline{\mathcal{E}}_{12}, \overline{\mathcal{E}}_i)$ with $\overline{\mathcal{E}}_{12} \in \overline{M}_{(\lambda,\rho)}$. We will be done if we can show that the underlying topological spaces of $\overline{M}^{(\lambda)}$ and $\overline{M}^{\mathrm{con}}_{\lambda,\rho}$ are equal.

The first paragraph implies that $\overline{M}^{(\lambda)}$ is irreducible. Replacing μ_1, μ_2 with λ, ρ, we see the same is also true for $\overline{M}^{\mathrm{con}}_{\lambda,\rho}$. Therefore it suffices to produce an open in

$$\overline{\mathcal{M}} \coloneqq \text{preimage of } \overline{M}_{(\lambda,\rho)} \text{ in } \overline{\mathrm{Gr}}^{\mathrm{con}}_{\leq(\mu_1,\mu_2),\mu_3,\ldots,\mu_e}$$

contained in both $\overline{M}^{(\lambda)}$ and $\overline{M}^{\mathrm{con}}_{\lambda,\rho}$. We will take the locus $\overline{U}$ classifying $(\overline{\mathcal{E}}, \overline{\mathcal{E}}_{12}, \overline{\mathcal{E}}_i) \in \overline{\mathcal{M}}$ with $\overline{\mathcal{E}}$ in the open Schubert cell $\overline{\mathrm{Gr}}_{1,\lambda+\rho+\mu_3+\ldots+\mu_e}$ (note that $\overline{\mathcal{M}} \subset \overline{\mathrm{Gr}}_{\leq(\lambda,\rho),\mu_3,\ldots,\mu_e}$ so $\overline{U}$ is indeed open). Then $\overline{U}$ is the locus on which $\overline{\mathrm{Gr}}^{\mathrm{con}}_{\leq(\lambda,\rho),\mu_3,\ldots,\mu_e} \to \overline{\mathrm{Gr}}_{\leq\mu}$ restricts to an isomorphism. More generally, if $(\overline{\mathcal{E}}, \overline{\mathcal{E}}_{12}, \overline{\mathcal{E}}_i) \in \overline{U}$ then there exists a unique sequence

$$\overline{\mathcal{E}} = \overline{\mathcal{E}}'_e \subset \overline{\mathcal{E}}'_{e-1} \subset \ldots \subset \overline{\mathcal{E}}'_2$$

with $\overline{\mathcal{E}}_i \in \overline{\mathrm{Gr}}_{1,\leq\lambda+\rho+\mu_3+\ldots+\mu_{i-2}}$ and one can write $\overline{\mathcal{E}}'_{i+1} = \overline{g}_i \left(\begin{smallmatrix} u^{\mu_{i,1}} & 0 \\ 0 & u^{\mu_{i,2}} \end{smallmatrix}\right) \overline{\mathcal{E}}'_i$ for some $\overline{g}_i \in L^+G$. In particular, $\overline{\mathcal{E}}'_2 = \overline{\mathcal{E}}_{12}$. Since $M_{(\lambda_1,\lambda_2)}$ is $\mathcal{O}$-flat and reduced we can lift $\overline{\mathcal{E}}_{12}$ to $\mathcal{E}'_2 \in M_{(\lambda_1,\lambda_2)}$ cf. [Bar20, 4.1.2]. Lifting the $\overline{g}_i$'s to g_i and defining $\mathcal{E}'_{i+1} = g_i \left(\begin{smallmatrix} (u-\pi_i)^{\mu_{i,1}} & 0 \\ 0 & (u-\pi_i)^{\mu_{i,2}} \end{smallmatrix}\right) \mathcal{E}'_i$ produces a lift $\mathcal{E} = \mathcal{E}'_e \in \mathrm{Gr}_{\leq\mu}$ of $\overline{\mathcal{E}}$. If the point $(\overline{\mathcal{E}}, \overline{\mathcal{E}}_{12}, \overline{\mathcal{E}}_i)$ corresponds to a closed point then we can choose the lift $\mathcal{E}$ to correspond to an $\mathcal{O}'$-valued point for $\mathcal{O}'$ the ring of integers in a finite extension of E, cf. for example [BM02, 5.1.1]. The valuative criterion for properness then produces a unique $\mathcal{O}'$-valued point $(\mathcal{E}, \mathcal{E}_{12}, \mathcal{E}_i) \in \mathrm{Gr}^{\mathrm{con}}_{\leq(\mu_1,\mu_2),\mu_3,\ldots,\mu_e}$. Since $\mathcal{E}_{12} = \mathcal{E}'_2 \in M_{(\mu_1,\mu_2)}$ this point is contained in $M^{\mathrm{con}}_{\mu_1,\mu_2}$. As the induced point on the special fibre will be $(\overline{\mathcal{E}}, \overline{\mathcal{E}}_{12}, \overline{\mathcal{E}}_i)$ it follows that $\overline{U} \subset \overline{M}^{(\lambda)}$. The same argument with μ_1, μ_2 replaced by λ, ρ shows $\overline{U} \subset \overline{M}^{\mathrm{con}}_{\lambda,\rho}$ also, so we are done. □

8. Fundamental cycles

The calculations made in Section 5 when $e = 2$ show that $M_{\widetilde{\lambda}}$ is irreducible. In fact this is true in general and can be seen directly:

Proposition 8.1. *Suppose $\widetilde{\lambda} = (\lambda, \rho, \ldots, \rho)$. Then $M_{\widetilde{\lambda}}$ is irreducible and generically reduced (note this requires no assumption on λ).*

Proof. First consider the morphism

$$f : \widetilde{X} \to \mathrm{Gr}_{\leq\widetilde{\lambda}}$$

where $\widetilde{X}$ is the $\mathcal{O}$-scheme classifying sequences $\mathcal{E}_e \subset \ldots \subset \mathcal{E}_1$ with $\mathcal{E}_1 \in \mathrm{Gr}_{1,\leq\lambda_1}$ and $(u - \pi_i)\mathcal{E}_{i-1} \subset \mathcal{E}_i \subset \mathcal{E}_{i-1}$ with $\mathcal{E}_{i-1}/\mathcal{E}_i$ of rank one. Then f is an isomorphism on the generic fibre. Furthermore the restriction of f to the closed subscheme

X consisting of $\mathcal{E}_e \subset \ldots \subset \mathcal{E}_1$ with $\mathcal{E}_1 \in G/P_\lambda$ induces, on the generic fibre an identification $X^\circ = M^\circ_{\widetilde{\lambda}}$. Since X is clearly $\mathcal{O}$-smooth (being a successive extension of flag varieties) the restriction of f to X factors through $M_{\widetilde{\lambda}}$. Lemma 3.6 gives

$$f_*[\overline{X}] = [\overline{M}_{\widetilde{\lambda}}]$$

Since X is irreducible so is $\overline{X}$. Therefore the same is true of $\overline{M}_{\widetilde{\lambda}}$. On the other hand $\overline{f}$ is also an isomorphism over the open locus of $\overline{\mathrm{Gr}}_{\leq \widetilde{\lambda}}$ given by the L^+G-orbit of $\mathcal{E}_{1,\lambda+\rho+\ldots+\rho}$. Since this locus clearly intersects $\overline{M}_{\widetilde{\lambda}}$ non-trivially it follows that the restriction of $\overline{f}$ to $\overline{X}$ is generically an isomorphism. Thus $\overline{M}_{\widetilde{\lambda}}$ is generically reduced. □

Part 2. **Cycles in moduli of Galois representations**

9. Setup

We fix a finite extension K of $\mathbb{Q}_p$ with residue field k of degree f over $\mathbb{F}_p$. Let C denote the completed algebraic closure of K, with ring of integers $\mathcal{O}_C$, and fix a compatible system π^{1/p^∞} of p-th power roots of a fixed choice of uniformiser $\pi \in K$ in $\mathcal{O}_C$. Set $K_\infty = K(\pi^{1/p^\infty})$. Write $E(u) \in W(k)[u]$ for the minimial polynomial of π. Thus $E(u)$ is Eisenstein of degree equal to the ramification degree e of K over $\mathbb{Q}_p$.

We also fix another finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. As in Part 1 this plays the role of our coefficient field. We assume that E contains a Galois closure of K so that $\mathrm{Hom}_{\mathbb{Q}_p}(K,E)$ consists of ef elements.

Notation 9.1. For any $\mathbb{Z}_p$-algebra A we write $\mathfrak{S}_A = (W(k) \otimes_{\mathbb{Z}_p} A)[[u]]$. This comes equipped with the A-linear endomorphism φ which on $W(k)$ acts as the lift of the p-th power map on k and sends $u \mapsto u^p$. We also consider

$$A_{\mathrm{inf},A} = \varprojlim_a \varprojlim_i (W(\mathcal{O}_{C^\flat})/p^a \otimes_{\mathbb{Z}_p} A)/u^i$$

where $\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p$ and $u = [(\pi, \pi^{1/p}, \pi^{1/p^2}, \ldots)] \in W(\mathcal{O}_{C^\flat})$. We view $A_{\mathrm{inf},A}$ as an $\mathfrak{S}_A$-algebra via u. Note that the lift of Frobenius on $W(\mathcal{O}_{C^\flat})$ induces a Frobenius φ on $A_{\mathrm{inf},A}$ which is compatible with that on $\mathfrak{S}_A$. The natural G_K-action on $\mathcal{O}_C$ also induces a continuous (for the (u,p)-adic topology) G_K-action on $A_{\mathrm{inf},A}$ commuting with φ. Write

$$W(C^\flat)_A = \varprojlim_a A_{\mathrm{inf},A}[\tfrac{1}{u}]/p^a$$

If A is topologically of finite type (i.e. $A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is of finite type) then $\mathfrak{S}_A \to A_{\mathrm{inf},A}$ is faithfully flat (in particular injective) [EG19, 2.2.13].

We also fix a compatible system $(1, \epsilon_1, \epsilon_2, \ldots)$ of p-th power roots of unity in $\mathcal{O}_C$ which we view as an element of $\mathcal{O}_{C^\flat}$. We write $\mu = [\epsilon] - 1 \in A_{\mathrm{inf},A}$.

Notation 9.2. Since E contains a Galois closure of K we can fix an indexing κ_{ij} of $\mathrm{Hom}_{\mathbb{Q}_p}(K,E)$ with $1 \leq i \leq f$ and $1 \leq j \leq e$. We do this so that

- $\kappa_{ij}|_{W(k)} = \kappa_{i'j'}|_{W(k)}$ if and only if $i = i'$
- $\kappa_{ij}|_{W(k)} \circ \varphi = \kappa_{i-1j}|_{W(k)}$.

Given any $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$-module M we then write $M_i \subset M$ for the subset on which the $W(k)$ action equals the action through $W(k) \xrightarrow{\kappa_{ij}} \mathcal{O}$ (for any, equivalently all,

j). Then $M = \prod_{i=1}^f M_i$. We repeatedly apply this construction when M is an $\mathfrak{S}_A$-module for some $\mathcal{O}$-algebra A. In this case M_i is a module over $\mathfrak{S}_{A,i} \cong A[[u]]$.

If M° is a $W(k)\otimes_{\mathbb{Z}_p} E$-module then we similarly write $M_{ij} \subset M$ for the subset on which the K-action equals the action through $K \xrightarrow{\kappa_{ij}} E$. In this setting we also have $M = \prod_{ij} M_{ij}$. Note however that this product decomposition need not descend to an integral decomposition because the ij-th idempotent in $W(k)\otimes_{\mathbb{Z}_p} E$ is not always contained in $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}$.

Notation 9.3. We write $\pi_{ij} = \kappa_{ij}(\pi)$ and $E_{ij}(u) = u - \pi_{ij} \in \mathcal{O}[u]$. We also write $E_{ij}(u)$ for the element of $\mathfrak{S}_{\mathcal{O}}$ corresponding to $(1,\ldots,1,u-\pi_{ij},1,\ldots,1)$ under the identification $\mathfrak{S}_{\mathcal{O}} = \prod_{i=1}^f \mathfrak{S}_{\mathcal{O},i} \cong \prod_{i=1}^f A[[u]]$. Thus $E(u) = \prod_{ij} E_{ij}(u)$. For $i = 1,\ldots,f$ set

$$E_i(u) = \prod_{j=1}^e E_{ij}(u)$$

viewed as either an element of $\mathfrak{S}_{\mathcal{O}}$ or $\mathcal{O}[u]$.

Notation 9.4. We specialise the results from Part 1 to the current setup. Thus for each i we write $\mathrm{Gr}^{(i)}$ for the affine grassmannian over $\mathcal{O}$ defined in Section 2 with $\pi_1,\ldots,\pi_e$ equal to $\kappa_{i1}(\pi),\ldots,\kappa_{ie}(\pi)$.

10. Breuil–Kisin modules

Let A be a p-adically complete $\mathcal{O}$-algebra. Then a *Breuil–Kisin module* $\mathfrak{M}$ over A is a finite projective $\mathfrak{S}_A$-module equipped with an $\mathfrak{S}_A$-linear homomorphism

$$\varphi_{\mathfrak{M}} = \varphi : \mathfrak{M}\otimes_{\varphi,\mathfrak{S}_A}\mathfrak{S}_A \to \mathfrak{S}_A$$

whose cokernel is killed by a power of $E(u)$. We say $\mathfrak{M}$ has height $\leq h$ if the cokernel is killed by $E(u)^h$. We write $\mathfrak{M}^\varphi$ for the image of $\varphi_{\mathfrak{M}}$ and $\varphi(\mathfrak{M})$ for the image of the composite $\mathfrak{M} \xrightarrow{m\mapsto m\otimes 1} \mathfrak{M}\otimes_{\varphi,\mathfrak{S}_A}\mathfrak{S}_A \to \mathfrak{S}$. Thus $\varphi(\mathfrak{M})$ is an $\varphi(\mathfrak{S}_A)$-submodule of $\mathfrak{M}^\varphi$ which generates $\mathfrak{M}^\varphi$ over $\mathfrak{S}_A$.

Definition 10.1. For any p-adically complete $\mathcal{O}$-algebra A write $Z_d^{\leq h}(A)$ for the groupoid of rank d Breuil–Kisin modules over A with height $\leq h$. Morphisms are $\mathfrak{S}_A$-linear isomorphisms compatible with the Frobenius. With pull-backs defined by base-change these categories form an fpqc stack over $\operatorname{Spf}\mathcal{O}$.

For $N \geq 0$ we let $\mathcal{G}_N$ denote the group scheme over $\mathcal{O}$ defined by $A \mapsto \mathrm{GL}_d(\mathfrak{S}_A/u^N)$. Then we can form the $\mathcal{G}_N$-torsor $\widetilde{Z}_d^{\leq h,N}$ over $Z_d^{\leq h}$ with $\widetilde{Z}_d^{\leq h,N}(A)$ the groupoid of pairs $(\mathfrak{M},\beta)$ with $\mathfrak{M}\in Z_d^{\leq h}(A)$ and $\beta = (\beta_1,\ldots,\beta_d)$ an $\mathfrak{S}_A$-basis of $\mathfrak{M}$. Morphisms are morphisms in $Z_d^{\leq h}(A)$ which identify the bases modulo u^N. This fits into the diagram

$$Z_d^{\leq h} \xleftarrow{\Gamma} \widetilde{Z}_d^{\leq h,N} \xrightarrow{\Psi} \prod_{i=1}^f \mathrm{Gr}^{(i)}$$

(recall Notation 9.4 for the definition of $\mathrm{Gr}^{(i)}$ which, here, is viewed as a formal scheme over $\operatorname{Spf}\mathcal{O}$) in which Γ forgets the choice of basis and in which Ψ is given by

$$(\mathfrak{M},\beta) \mapsto \big(\varphi(\beta)(E(u)^h\mathfrak{M}_i)\big)_{i=1,\ldots,f}$$

where $\varphi(\beta)$ is interpreted as an identification $\mathfrak{M}^\varphi \cong \mathfrak{S}_A^d = \prod_{i=1}^f A[[u]]^d$. The construction of this diagram goes back to [PR09].

Remark 10.2. Here is an explicit interpretation of Ψ: If $(\mathfrak{M},\beta) \in \widetilde{Z}_d^{\leq h,N}(A)$ then there is a matrix C with entries in $\mathfrak{S}_A$ such that $\varphi(\beta) = \beta C$. Therefore $E(u)^h\mathfrak{M}$ is generated by $\varphi(\beta)E(u)^hC^{-1}$ (note that $\mathfrak{M}$ having height $\leq h$ implies $E(u)^hC^{-1}$ is also a matrix with entries in $\mathfrak{S}_A$). Under the identification $\mathfrak{S}_A = \prod_{i=1}^f A[[u]]$ we can write $C = (C_i)_i$. Then $\Psi(\mathfrak{M},\beta) = (\mathcal{E}_i)_i$ with $\mathcal{E}_i \subset A[[u]]^d$ the submodule generated by $E(u)^hC_i^{-1}$.

Proposition 10.3. *There exists a second action of $\mathcal{G}_N$ on $\widetilde{Z}_d^{\leq h,N}$ such that if $E(u)^h$ divides $u^{(p-1)N-1}$ in $\mathfrak{S}_A$ for any $\mathcal{O}/\pi^a$-algebra then $\Psi \times_{\mathcal{O}} \mathcal{O}/\pi^a$ is a $\mathcal{G}_N$-torsor.*

A precise lower bound for N relative to n, h and a can be given cf. [EG21, 5.2.6].

Proof. First we define an action of $\mathrm{GL}_d(\mathfrak{S}_A)$: For $(\mathfrak{M},\beta) \in \widetilde{Z}^{\leq h,N}(A)$ with $\varphi(\beta) = \beta C$ and $g \in \mathrm{GL}_d(\mathfrak{S}_A)$ define

$$g \cdot (\mathfrak{M},\beta) = (\mathfrak{M}_g,\beta)$$

where $\mathfrak{M}_g \in Z^{\leq h}(A)$ is equal $\mathfrak{M}$ as an $\mathfrak{S}_A$-module with Frobenius φ_g given by $\varphi_g(\beta) = \beta g C$. We show this induces an action of $\mathcal{G}_N$ below.

Consider $g \in \mathrm{GL}_d(\mathfrak{S}_A)$ with A an $\mathcal{O}/\pi^a$-algebra. We claim that $(\mathfrak{M},\beta) \cong (\mathfrak{M}_g,\beta)$ if and only if $g \equiv 1$ modulo u^N. For the if direction, any such isomorphism is given by a matrix h with respect to β. Since this morphism is φ-equivariant it satisfies $C\varphi(h) = hgC$ and since it respects β modulo u^N we have $h \equiv 1$ modulo u^N. Therefore

$$g = h^{-1}C\varphi(h)C^{-1} = 1 + u^NC_1 + u^NC_2u^{(p-1)N}C^{-1}$$

for some $C_1, C_2 \in \mathrm{Mat}(\mathfrak{S}_A)$. Since $E(u)^h$ divides $u^{(p-1)N}$ and $E(u)^hC^{-1} \in \mathrm{Mat}(\mathfrak{S}_A)$ the if direction follows.

For the only if direction we show that if $h : \mathfrak{M} \to \mathfrak{M}_g$ is the identity map then $H := \sum_{n=0}^{\infty}(\varphi_g^n \circ h \circ \varphi^{-n} - \varphi_g^{n-1} \circ h \circ \varphi^{-n+1})$ converges to a φ-equivariant map $\equiv h$ modulo u^N. Since $\varphi^n(\beta) = \beta C\varphi(C)\dots\varphi^{n-1}(C)$ we see that relative to β the map $\varphi_g^n \circ h \circ \varphi^{-n}$ is given $J_nI_n^{-1}$ where

$$J_n = (gC)\varphi(gC)\dots\varphi^n(gC), \qquad I_n = C\varphi(C)\dots\varphi^{n-1}(C)$$

Therefore we just need to show that

$$J_nI_n^{-1} - J_{n-1}I_{n-1}^{-1} = J_{n-1}\left(\varphi^n(g) - 1\right)I_{n-1}^{-1} \in u^N\,\mathrm{Mat}(\mathfrak{S}_A)$$

and converges u-adically to zero. Since $g \equiv 1$ modulo u^N we can write $\varphi^n(g) - 1 = u^{N+(p^n-1)N}g'$. Therefore it suffices to show that $u^{(p^n-1)N}I_{n-1}^{-1} \in \mathrm{Mat}(\mathfrak{S}_A)$ converges u-adically to zero. Since $\varphi^i(E(u)^h)$ divides $u^{((p-1)N-1)p^i}$ and

$$E(u)^h\varphi(E(u))^h\dots\varphi^{n-1}(E(u)^h)I_{n-1}^{-1} \in \mathrm{Mat}(\mathfrak{S}_A)$$

the claim follows from the observation that $(p^n-1)N - ((p-1)N-1)(1+\ldots+p^{n-1}) = 1 + \ldots + p^{n-1}$ is ≥ 0 and $\to \infty$.

This shows that $g\cdot(\mathfrak{M},\beta)$ induces an action of $\mathcal{G}_N$ on $\widetilde{Z}^{\leq h,N} \otimes_{\mathcal{O}} \mathcal{O}/\pi^a$ with trivial stabilisers. From the explicit description given in Remark 10.2 we see that the fibres of Ψ are precisely the orbits in $\widetilde{Z}^{\leq h,N} \otimes_{\mathcal{O}} \mathcal{O}/\pi^a$ under this $\mathcal{G}_N$-action, which finishes the proof. ☐

Corollary 10.4. *$\widetilde{Z}_d^{\leq h,N} \times_{\mathcal{O}} \mathcal{O}/\pi^n$ is a finite type $\mathcal{O}$-scheme for $N >> 0$ and $Z_d^{\leq h}$ is a p-adic formal algebraic stack (in the sense of* [EG19, A7]*) of finite type over* $\mathrm{Spf}\,\mathcal{O}$*.*

Proof. The first part follows since we've just seen that $\widetilde{Z}_d^{\leq h,N} \times_{\mathcal{O}} \mathcal{O}/\pi^n$ is a torsor for a finite type group scheme over a finite type $\mathcal{O}/\pi^n$-scheme. The second part follows from the first and the definition of a p-adic formal algebraic stack. □

Corollary 10.5. *Suppose that $N >> 0$ relative to a as in Proposition 10.3 and that for $i = 1,\dots,f$ closed subschemes $V_i \subset \mathrm{Gr}^{(i)} \times_{\mathcal{O}} \mathcal{O}/\pi^a$ are given which are stable under the natural action of the group scheme $A \mapsto \mathrm{GL}_d(A[[u^p]]/u^N)$. Then there exists a closed substack $Z_V \hookrightarrow Z_d^{\leq h}$ fitting into a diagram*

$$\begin{tikzcd} & \widetilde{Z}_V \arrow[dl, "\Gamma"'] \arrow[dr, "\Psi"] & \\ Z_V & & \prod_{i=1}^f V_i \end{tikzcd}$$

with both Γ and Ψ $\mathcal{G}_N$-torsors (for the two different $\mathcal{G}_N$-actions on $\widetilde{Z}_d^{\leq h,N}$).

Proof. Let $\widetilde{Z}_V$ denote the pull-back of $\prod_{i=1}^f V_i$ along Ψ. The fact that each V_i is stable under the action of $A \mapsto \mathrm{GL}_d(A[[u^p]]/u^N)$ implies that $\widetilde{Z}_V$ is stable under the action $g(\mathfrak{M},\beta) = (\mathfrak{M},\beta g)$ of $\mathcal{G}_N$ on $\widetilde{Z}_d^{\leq h,N}$ (this can be easily seen using the description of Ψ from Remark 10.2). Since Γ is a $\mathcal{G}_N$-torsor for this action $\widetilde{Z}_V$ descends to a closed subscheme Z_V of $Z_d^{\leq h}$ as claimed. □

11. Crystalline Breuil–Kisin modules

If A is a p-adically complete $\mathcal{O}$-algebra which is of topologically finite type (i.e. $A \otimes_{\mathcal{O}} \mathbb{F}$ is of finite type over $\mathbb{F}$) then a *crystalline Breuil–Kisin module over A* is a pair $(\mathfrak{M},\sigma)$ with $\mathfrak{M}$ a Breuil–Kisin module over A and σ a continuous φ-equivariant $A_{\mathrm{inf},A}$-semilinear action of G_K on $\mathfrak{M} \otimes_{\mathfrak{S}_A} A_{\mathrm{inf},A}$ satisfying

$$(\sigma - 1)(m) \in \mathfrak{M} \otimes_{\mathfrak{S}_A} [\pi^\flat]\varphi^{-1}(\mu)A_{\mathrm{inf},A}, \qquad (\sigma_\infty - 1)(m) = 0$$

for every $m \in \mathfrak{M}$ and every $\sigma \in G_K, \sigma_\infty \in G_{K_\infty}$.

Definition 11.1. Write $Y_d^{\leq h}(A)$ for the groupoid consisting of rank d crystalline Breuil–Kisin modules over A with height $\leq h$.

One can attach a Hodge type to crystalline Breuil–Kisin modules (at least over coefficient rings which are $\mathcal{O}$-flat): For $n \in \mathbb{Z}$ one defines $\mathrm{Fil}^n(\mathfrak{M}^\varphi) = \mathfrak{M}^\varphi \cap E(u)^n\mathfrak{M}$ and equips the finite projective $\mathcal{O}_K \otimes_{\mathbb{Z}_p} A$-module $\mathfrak{M}^\varphi \otimes_{\mathfrak{S}_A} \mathfrak{S}_A/E(u)$ with the filtration whose n-th filtered piece is the image of $\mathrm{Fil}^n(\mathfrak{M}^\varphi)$. The graded pieces of this filtration become $A[\frac{1}{p}]$-projective after inverting p. This allows us to say that $(\mathfrak{M},\sigma)$ has Hodge type $\mu = (\mu_{ij})$ if n appears in μ_{ij} with multiplicity equal to the $A[\frac{1}{p}]$-rank of the ij-th part of $\mathrm{gr}^n(\mathfrak{M}^\varphi \otimes_{\mathfrak{S}_A} \mathfrak{S}_A[\frac{1}{p}]/E(u))$. The following motivates these constructions:

Theorem 11.2. *If $(\mathfrak{M},\sigma) \in Y_d^{\leq h}(A)$ with A a finite flat $\mathcal{O}$-algebra then*

$$V = (\mathfrak{M} \otimes_{\mathfrak{S}_A} W(C^\flat)_A)^{\varphi=1}$$

equipped with the G_K-action induced by σ is a crystalline representation of G_K on a finite projective A-module. Furthermore, the Hodge type of $(\mathfrak{M},\sigma)$ coincides with that attached to V via the filtered module $D_{\mathrm{crys}}(V)_K := (V \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^{G_K}$ with n-th filtered piece given by $(V \otimes_{\mathbb{Z}_p} t^n B_{\mathrm{dR}}^+)^{G_K}$.

Proof. The theorem as stated is taken from [Bar20, 2.1.12], but the result originates from a combination of ideas appearing in [Kis06, GLS14, Oze14]. □

The next result shows that fixing the Hodge type is closed condition on crystalline Breuil–Kisin modules (note that this is not the case without the crystalline condition).

Proposition 11.3. *There exists a limit preserving p-adic algebraic formal stack $Y_d^{\leq h}$ of topologically finite type over $\mathcal{O}$ whose groupoid of A-valued points is canonically equivalent to $Y_d^{\leq h}(A)$.*

For each Hodge type μ there exists a unique $\mathcal{O}$-flat closed substack Y_d^μ of $Y_d^{\leq h}$ with the property that the full subcategory $Y_d^\mu(A)$ of $Y_d^{\leq \mu}(A)$ consists of all crystalline Breuil–Kisin modules with Hodge type μ.

Proof. The first part follows from [EG19, §4.5]. There algebraic stacks $\mathcal{C}^a_{\pi^\flat,s,d,h}$ over $\operatorname{Spec}\mathcal{O}/\pi^a$ are constructed [EG19, 4.5.8] with $\pi^\flat = (\pi, \pi^{1/p}, \dots)$ and s some sufficiently large integer and in the proof of [EG19, 4.5.15] it is explained how $Y_\mu^{\leq h} \times_{\mathcal{O}} \mathcal{O}/\pi^a$ can be realised as a closed substack of $\mathcal{C}^a_{\pi^\flat,s,d,h}$. In the notation of loc. cit. the morphism $Y_d^{\leq h} \to Z_d^{\leq h}$ is obtained as the restriction of a morphism $\mathcal{C}^a_{\pi^\flat,s,d,h} \to \mathcal{C}^a_{\pi^\flat,d,h} = Z_d^{\leq h} \times_{\mathcal{O}} \mathcal{O}/\pi^a$. The second part is [EG19, 4.8.2]. □

We conclude with a useful lemma giving a description of the points of Y_d^μ valued in a finite local $\mathbb{F}$-algebra:

Lemma 11.4. *Suppose that $(\overline{\mathfrak{M}}, \overline{\sigma})$ corresponds to an $\overline{A}$-valued point of $\overline{Y}_d^\mu$ for $\overline{A}$ some finite local $\mathbb{F}$-algebra. Then there exists a local finite flat $\mathcal{O}$-algebra A with $\overline{A} = A \otimes_{\mathcal{O}} \mathbb{F}$ and an A-valued point $(\mathfrak{M}, \sigma)$ of Y_d^μ with special fibre $(\overline{\mathfrak{M}}, \overline{\sigma})$.*

Proof. Let $\mathbb{F}'/\mathbb{F}$ be a finite extension and write $R_{\overline{\rho}}$ for the framed $\mathcal{O}$-deformation ring corresponding to some $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbb{F}')$. In [Bar20, 2.2.11] a projective $R_{\overline{\rho}}$-scheme $\mathcal{L}_{\overline{\rho}}^{\leq h}$ is constructed with A'-points, for A' any p-adically complete $\mathcal{O}$-algebra, classifying pairs $(\mathfrak{M}, \rho)$ with ρ a framed deformation of $\overline{\rho}$ to A' and $\mathfrak{M} \in Z_d^{\leq h}(A')$ satisfying

$$\mathfrak{M} \otimes_{\mathfrak{S}_A} W(C^\flat)_A = \rho \otimes_A W(C^\flat)_A$$

so that φ (induced semilinearly from that on $\mathfrak{M}$) is the identity on $\overline{\rho}$ and the G_K-action (induced semilinearly from that on ρ) satisfies

$$(\sigma - 1)(m) \in \mathfrak{M} \otimes_{\mathfrak{S}_A} [\pi^\flat]\varphi^{-1}(\mu) A_{\mathrm{inf},A}, \qquad (\sigma_\infty - 1)(m) = 0$$

for every $\sigma \in G_K, \sigma_\infty \in G_{K_\infty}$ and $m \in \mathfrak{M}$. The morphism $\mathcal{L}_{\overline{\rho}}^{\leq h} \to Y_d^{\leq h}$ given by $(\mathfrak{M}, \rho) \mapsto (\mathfrak{M}, \sigma)$ with σ the G_K-action induced by ρ is easily seen to be formally smooth. Therefore, the preimage $\mathcal{L}_{\overline{\rho}}^\mu$ of Y_d^μ in $\mathcal{L}_{\overline{\rho}}^{\leq h}$ is $\mathcal{O}$-flat. The map $\mathcal{L}_{\overline{\rho}}^{\leq h} \to \operatorname{Spec} R_{\overline{\rho}}$ becomes a closed immersion after inverting p [Bar20, 2.2.14]. Theorem 11.2 therefore implies that $\mathcal{L}_{\overline{\rho}}^\mu[\frac{1}{p}] = \operatorname{Spec} R_{\overline{\rho}}^\mu$ where $R_{\overline{\rho}}^\mu$ is the reduced $\mathcal{O}$-flat quotient of $R_{\overline{\rho}}$ classifying crystalline representations of Hodge type μ [Kis08, 3.3.8]. In particular, $\mathcal{L}_{\overline{\rho}}^\mu$ is reduced.

Now apply the above construction with $\overline{\rho} = (\overline{\mathfrak{M}} \otimes_{\mathfrak{S}_{\overline{A}}} W(C^\flat)_{\overline{A}})^{\varphi=1} \otimes_{\overline{A}} \mathbb{F}'$ where $\mathbb{F}'$ denotes the residue field of $\overline{A}$. Then $(\overline{\mathfrak{M}}, \overline{\sigma})$ induces an $\overline{A}$-valued point of $\mathcal{L}_{\overline{\rho}}^\mu$. Applying [Bar20, 4.1.2] to the local ring of $\mathcal{L}_{\overline{\rho}}^\mu$ at this points produces a finite flat

$\mathcal{O}$-algebra A with $A \to \overline{A}$ and an A-valued point of $\mathcal{L}^{\leq\mu}_{\overline{\rho}}$ pulling back to our $\overline{A}$-valued point. The image of this A-valued point in Y^μ_d then corresponds to $(\mathfrak{M}, \sigma)$ as desired. $\square$

Corollary 11.5. *Y^μ_d has relative dimension $\sum_{ij} \dim G/P_{\mu_{ij}}$ over $\mathcal{O}$.*

Proof. We saw in the proof of Lemma 11.4 that $\mathcal{L}^\mu_{\overline{\rho}} \to Y^\mu_d$ is formally smooth with fibres classifying framings of the corresponding Galois representation, and so of relative dimension d^2. Hence Y^μ_d has dimension (in the sense of e.g. [Sta17, 0DRE]) $\dim \mathcal{L}^\mu_{\overline{\rho}} - d^2$ at the image of the closed point of $\mathcal{L}^\mu_{\overline{\rho}}$. Since $\mathcal{L}^\mu_{\overline{\rho}}[\frac{1}{p}] = \operatorname{Spec} R^\mu_{\overline{\rho}}$ it follows from [Kis08, 3.3.8] that $\mathcal{L}^\mu_{\overline{\rho}}$ has relative dimension $d^2 + \sum_{ij} \dim G/P_{\mu_{ij}}$ over $\mathcal{O}$. $\square$

12. Naive Galois actions

In this section we consider the morphism $Y^{\leq h}_d \to Z^{\leq h}_d$ which forgets the Galois action, and show it is an isomorphism over certain closed substacks in the special fibre of $Z^{\leq h}_d$.

Construction 12.1. The aim is to establish conditions which allow the following "naive" crystalline G_K-action on $(\mathfrak{M}, \beta) \in \widetilde{Z}^{\leq h,N}_d(A)$ to be perturbed into one which is φ-equivariant. Let $\sigma_{\mathrm{naive},\beta}$ denote the continuous $A_{\mathrm{inf},A}$-semilinear action of G_K on $\mathfrak{M} \otimes_{\mathfrak{S}_A} A_{\mathrm{inf},A}$ obtained from the coordinate-wise action on $A^d_{\mathrm{inf},A}$ via the identification

$$\mathfrak{M}^\varphi \otimes_{\mathfrak{S}_A} A_{\mathrm{inf},A} \cong A^d_{\mathrm{inf},A}$$

induced by $\varphi(\beta)$. Thus, $\sigma_{\mathrm{naive},\beta}$ is uniquely determined as the semilinear G_K-action fixing $\varphi(\beta)$.

Of course $\sigma_{\mathrm{naive},\beta}$ will not usually be φ-equivariant.

Proposition 12.2. *Let $r = (r_{ij})$ be a tuple of positive integers with $r_{ij} \leq h$ and $\sum_j r_{ij} \leq e + p - 1$ for each i. Then there exists a closed substack $\overline{Z}^{r,\sigma}_d$ of $\overline{Z}^{\leq h}_d$ whose $\overline{A}$-points, for any finite type $\mathbb{F}$-algebra $\overline{A}$, consists of those $\mathfrak{M} \in Z^{\leq h}_d(\overline{A})$ satisfying the following: if β is an $\mathfrak{S}_{\overline{A}}$-basis of $\mathfrak{M}$ then*

(1) *$(\sigma_{\mathrm{naive},\beta} - 1)(\mathfrak{M}) \subset \mathfrak{M} \otimes_{\mathfrak{S}_A} [\pi^\flat]\varphi^{-1}(\mu) A_{\mathrm{inf},\overline{A}}$ for every $\sigma \in G_K$.*
(2) *$u^{\sum_j r_{ij}} \mathfrak{M}_i \subset \mathfrak{M}^\varphi_i$ for each i.*

Proof. We first define, for each i, a closed subscheme $\mathrm{Gr}^{(i)}_{a_i,\sigma}$ of $\mathrm{Gr}^{(i)}$ whose A-points, for any p-adically complete $\mathcal{O}$-algebra of topologically finite type, classify submodules $\mathcal{E} \subset A[[u]]^d$ satisfying

(1) For every $\sigma \in G_K$ we have $(\sigma - 1)(\mathcal{E}) \subset \mathcal{E} \otimes_{A[[u]]} [\pi^\flat]\varphi^{-1}(\mu) A_{\mathrm{inf},A,i}$ where the action of G_K on $A^d_{\mathrm{inf},A,i}$ is coordinate-wise.
(2) $\prod_j E_{ij}(u)^{r_{ij}} \mathcal{E} \subset E(u)^h A[[u]]^d$.

(in the first bullet point recall that $A_{\mathrm{inf},A,i}$ is an $\mathfrak{S}_{A,i} = A[[u]]$-module). It is clear that condition (2) is closed. That the same is true of (1) follows from an application of [EG19, B29].

Next we claim that $\mathrm{Gr}^{(i)}_{a_i,\sigma}(A)$ is stable under the action of $\mathrm{GL}_d(A[[u^p]])$ whenever A is a finite type $\mathbb{F}$-algebra. This is clear for the second condition defining $\mathrm{Gr}^{(i)}_{a_i,\sigma}$. For the first condition recall that the image of μ in $\mathcal{O}_{C^\flat}$ generates the same ideal as $u^{ep/(p-1)}$ (cf. for example [Fon94, 5.1.2]) and so, since A is an $\mathbb{F}$-algebra, the first condition defining $\mathrm{Gr}^{(i)}_{a_i,\sigma}$ reads

$$(\sigma-1)(\mathcal{E}) \subset u^{\frac{e+p-1}{p-1}}\mathcal{E}\otimes_{A[[u]]} A_{\mathrm{inf},A,i}$$

If $\mathcal{E}$ is generated by $(e_1,\ldots,e_d)C$ then this is equivalent to asking that $C^{-1}\sigma(C)-1 \in u^{\frac{e+p-1}{p-1}}\mathrm{Mat}(A_{\mathrm{inf},A,i})$. If $g\in \mathrm{GL}_d(A[[u^p]])$ then

$$(gC)^{-1}\sigma(gC)-1 = C^{-1}\sigma(C)-1+C^{-1}(g^{-1}\sigma(g)-1)\sigma(C)$$

Since $u^{e+p-1}A[[u]]^d\subset u^{a_i}A[[u]]^d\subset\mathcal{E}$ we have $u^{e+p-1}C^{-1}\in\mathrm{Mat}(A[[u]])$. Therefore it suffices to show that $g^{-1}\sigma(g)-1\in u^{\frac{p(e+p-1)}{p-1}}\mathrm{Mat}(A_{\mathrm{inf},A,i})$. This follows from [Bar20, 3.2.11] which says that

$$\sigma(u^n)-u^n\in u^{n+ep^{v_p(n)}/(p-1)}\mathcal{O}_{C^\flat} \tag{12.3}$$

for any $n\geq 0$.

The claim from the previous paragraph allows us to apply Corollary 10.5 with $V_i=\overline{\mathrm{Gr}}^{(i)}_{a_i,\sigma}$. We obtain closed substacks $\overline{Z}^{a,\sigma}_d$ of $Z^{\leq h}_d$. We conclude by showing these $Z^{a,\sigma}_d$ satisfy the required properties. By construction, if $\mathfrak{M}\in\overline{Z}^{a,\sigma}_d(A)$ admits an $\mathfrak{S}_A$-basis β then $\Phi(\mathfrak{M},\beta)$ is contained in $\prod_{i=1}^f\mathrm{Gr}^{(i)}_{a_i,\sigma}$. Therefore, the first condition defining $\mathrm{Gr}^{(i)}_{a_i,\sigma}$ implies

$$(\sigma_{\mathrm{naive},\beta}-1)(E(u)^h\mathfrak{M})\subset\mathfrak{M}\otimes_{\mathfrak{S}_A}[\pi^\flat]\varphi^{-1}(\mu)E(u)^hA_{\mathrm{inf},A}$$

Thus $(\sigma_{\mathrm{naive},\beta}-1)(\mathfrak{M})\subset\mathfrak{M}\otimes_{\mathfrak{S}_A}[\pi^\flat]\varphi^{-1}(\mu)A_{\mathrm{inf},A}+(\frac{\sigma(E(u))^h}{E(u)^h}-1)\mathfrak{M}$. Since $pA=0$ we have $E(u)=u^e$ and so (12.3) implies $(\frac{\sigma(E(u))^h}{E(u)^h}-1)\mathfrak{M}\subset u^{\frac{ep}{p-1}}\mathfrak{M}\subset u^{\frac{e+p-1}{p-1}}\mathfrak{M}$. It follows that the first condition from the proposition is satisfied. The second condition defining $\mathrm{Gr}^{(i)}_{a_i,\sigma}$ implies $E(u)^h\mathfrak{M}_i\subset E(u)^h\prod_j E_{ij}(u)^{-r_{ij}}\mathfrak{M}_i^\varphi$, i.e. that $\prod_j E_{ij}(u)^{r_{ij}}\mathfrak{M}_i=u^{\sum_j r_{ij}}\mathfrak{M}_i\subset\mathfrak{M}_i^\varphi$. □

Remark 12.4. The first condition defining $\mathrm{Gr}^{(i)}_{r,\sigma}$ can be expressed as $\nabla_\sigma(\mathcal{E})\subset\frac{[\pi^\flat]}{E(u)}\mathcal{E}$ for each $\sigma\in G_K$ and $\nabla_\sigma=\frac{\sigma-1}{\mu}$ (here we use that $\frac{\mu}{\varphi^{-1}(\mu)}$ generates the same ideal of A_{inf} as $E(u)$). After possibly replacing the compatible system of primitive p-th power roots of unity ϵ we can choose $\sigma\in G_K$ so that $\sigma(u)/u=\epsilon$. Then

$$\nabla_\sigma(u^i)=u^i\left(\frac{\frac{\sigma(u)}{u}-1}{[\epsilon]-1}\right)=u^i\left(\frac{[\epsilon^i]-1}{[\epsilon]-1}\right)=u^i(1+[\epsilon]+\ldots+[\epsilon]^{i-1})$$

Thus $\nabla_\sigma=u\nabla_q$ where ∇_q is the q-derivation for $q=[\epsilon]$. In particular $\nabla_\sigma\equiv u\frac{d}{du}$ modulo $[\epsilon]-1$. This illustrates the close relationship between the $\mathrm{Gr}^{(i)}_{r,\sigma}$ defined here and the "monodromy condition" appearing in [LLHLM20].

Proposition 12.5. *Assume that at least one of the inequalities* $\sum_j r_{ij}\leq e+p-1$ *is strict. Then*

$$Y^{\leq h}_d\times_{Z^{\leq h}_d}\overline{Z}^{r,\sigma}_d\to\overline{Z}^{r,\sigma}_d$$

is an isomorphism.

Proof. By Lemma 15.1 is enough to show that any local finite type $\mathbb{F}$-algebra $\overline{A}$ and any $\mathfrak{M}\in\overline{Z}^{r,\sigma}_d(\overline{A})$ there exists a unique action σ of G_K making $(\mathfrak{M},\sigma)$ into an object of $Y_d^{\leq h}$. Existence implies essential surjectivity on A-valued points and fully-faithfullness follows from the uniqueness.

Write $\mathrm{Hom}(\mathfrak{M},\mathfrak{M})$ for the $\mathfrak{S}_A$-module of $\mathfrak{S}_A$-linear endomorphisms of $\mathfrak{M}$ and equip $\mathrm{Hom}(\mathfrak{M},\mathfrak{M})$ with the Frobenius φ_{Hom} given by $h \mapsto \varphi \circ h \circ \varphi^{-1}$. The bounds on the a_i imply:

Claim. *Set $\mathcal{H} := \mathrm{Hom}(\mathfrak{M},\mathfrak{M}) \otimes [\pi^\flat]\varphi^{-1}(\mu)A_{\mathrm{inf},\overline{A}}$. Then $\mathcal{H}$ is φ_{Hom}-stable and φ_{Hom} is topologically nilpotent on $\mathcal{H}$.*

Proof. Since $u^{a_i}\mathfrak{M}_i \subset \mathfrak{M}_i^\varphi$ it follows that

$$\varphi_{\mathrm{Hom}}(\mathrm{Hom}(\mathfrak{M},\mathfrak{M})_i) \subset u^{-a_i}\,\mathrm{Hom}(\mathfrak{M},\mathfrak{M})_{i-1}$$

Therefore $\varphi_{\mathrm{Hom}}(\mathcal{H}_i) \subset u^{e+p-1-a_i}\mathcal{H}_{i-1}$ and so, as $a_i \leq e+p-1$, it follows that $\mathcal{H}$ is φ_{Hom}-stable. Since the inequality is strict at least once we have $\varphi_{\mathrm{Hom}}(\mathcal{H}_i) \subset u\mathcal{H}_{i-1}$ for at least one i. In particular φ_{Hom} is topologically nilpotent. □

Since $\overline{A}$ is local we can choose an $\mathfrak{S}_{\overline{A}}$-basis β. The claim then implies that the limit

$$\sigma - 1 := \lim_{n\to\infty} \varphi_{\mathrm{Hom}}^n(\sigma_{\mathrm{naive},\beta} - 1) \in \mathcal{H}$$

converges with σ defining a φ-equivariant crystalline G_K-action as required. For uniqueness, if σ' is a second crystalline G_K-action then $\sigma'-\sigma$ is a φ_{Hom}-fixed element of $\mathcal{H}$. The topological nilpotence of φ_{Hom} therefore implies $\sigma = \sigma'$. □

13. Comparison with local models

From now on we fix $h = p$.

Theorem 13.1. *Assume that $\mu_{ij} \subset [0, r_{ij}]$ so that for each i*

$$\sum_j r_{ij} \leq \frac{p-1}{\nu_i} + 1$$

where ν_i is the maximum π-adic valuation for $\pi_{ij} - \pi_{ij'}$ for $j \neq j'$. Then

$$\overline{Y}_d^\mu \times_{Z_d^{\leq p}} \widetilde{Z}_d^{\leq p,N} \xrightarrow{\Phi} \prod_{i=1}^f \mathrm{Gr}$$

factors through $\prod_{i=1}^f \overline{M}_{\mu_i^*}$ for $\mu_i^* = (p-\mu_{i1},\ldots,p-\mu_{ie})$.

Remark 13.2. If K is tamely ramified, i.e. if e is not divisible by p, then $\pi_{ij} - \pi_{ij'}$ generates the same ideal of $\mathcal{O}_K$ as π when $j \neq j'$. This can be seen by considering the π-adic valuation of $\frac{d}{du}E(u)|_{u=\pi_{ij}} = \prod_{i'j'\neq ij}(\pi_{ij} - \pi_{i'j'})$.

The following proposition is the key technical result which goes into the proof of the theorem. It is a reworking of techniques originally developed in [GLS14, GLS15]. Recall $E_{ij}(u)$ is defined in Notation 9.3.

Proposition 13.3. *Let A be a finite flat $\mathcal{O}$-algebra and suppose $(\mathfrak{M},\sigma) \in Y_d^\mu(A)$ with β a $\mathfrak{S}_A$-basis of $\mathfrak{M}$. Define $M_{ij} = \mathfrak{M}_i^\varphi / E_{ij}(u)$ and choose a section s of the surjection $\varphi(\mathfrak{M})_i \to \mathfrak{M}_i^\varphi \to M_{ij}$. Then there exists a filtration $\mathrm{Fil}_{ij}^\bullet$ on M_{ij} by A-submodules with p-torsionfree graded pieces such that*

$$\sum_{n=0}^p E_{ij}(u)^{p-n}\,\mathrm{Fil}_{ij}^n + \mathfrak{M}_{\mathrm{err},ij}^\varphi = \mathfrak{M}_i^\varphi \cap E_{ij}(u)^p \mathfrak{M}_i + \mathfrak{M}_{\mathrm{err},ij}^\varphi$$

when Fil_{ij}^n is viewed as a submodule of $\mathfrak{M}_i^\varphi$ via s and $\mathfrak{M}_{\mathrm{err},ij}^\varphi := \sum_{l=1}^{p-1} \pi^{p-l} E_{ij}(u)^l \mathfrak{M}^\varphi$.

Note this is valid without any smallness assumption on μ.

Proof. First we define the filtration $\mathrm{Fil}^\bullet_{ij}$. Recall that we equipped $M := \mathfrak{M}^\varphi \otimes_{\mathfrak{S}_A} /E(u)$ with the filtration whose n-th piece is the image of $\mathfrak{M}^\varphi \cap E(u)^n \mathfrak{M}$. Then $D_K := M[\frac{1}{p}]$ is a filtered $K \otimes_{\mathbb{Z}_p} A$-module and can be written as $\prod_{ij} D_{K,ij}$ with each $D_{K,ij}$ a filtered A-module. The projection $D_K \to D_{K,ij}$ is obtained by base-change along the map $\mathfrak{S} \to \mathcal{O}_K$ given by $u \mapsto \pi_{ij} = \kappa_{ij}(\pi)$. Therefore the kernel of $\mathfrak{M}^\varphi \to D_K \to D_{K,ij}$ is $E_{ij}(u)\mathfrak{M}^\varphi$. This means M_{ij} can be viewed as a submodule of $D_{K,ij}$ and $M_{ij}[\frac{1}{p}] = D_{K,ij}$. Define

$$\mathrm{Fil}^n_{ij} = \mathrm{Fil}^n(D_{K,ij}) \cap M_{ij}$$

The filtered pieces of $D_{K,ij}$ are p-saturated so the graded pieces of $\mathrm{Fil}^\bullet_{ij}$ are p-torsionfree. This also shows that $\mathrm{Fil}^n_{ij}[\frac{1}{p}] = \mathrm{Fil}^n(D_{K,ij})$ which proves Corollary 13.4 below.

Next we use:

Claim. *For each $x \in \varphi(\mathfrak{M})_i$ with image in Fil^n_{ij} there exists $x_1, \dots, x_{p-1} \in \mathfrak{M}^\varphi_i$ such that*

$$x + E_{ij}(u)\pi^{p-1}x_1 + \ldots + E_{ij}(u)^{p-1}\pi x_{p-1} \in \mathfrak{M}^\varphi_i \cap E_{ij}(u)^{\min\{n,p\}}\mathfrak{M}_i$$

Proof of Claim. This follows from results in [Bar19, §5]. In [Bar19, 5.2.4] it is shown that $x_1, \dots, x_{p-1} \in \mathfrak{M}^\varphi$ exist so that

$$x + E_{ij}(u)\pi^{p-1}x_1 + \ldots + E_{ij}(u)^{p-1}\pi x_{p-1} \in \mathrm{Fil}^{\{p,0,\dots,0\}} \cap \mathfrak{M}^\varphi$$

(note that in loc. cit. i is taken equal to 1 but this plays no role in the proof). In [Bar19, 5.1.3] it is shown that $\mathrm{Fil}^{\{p,0,\dots,0} \cap \mathfrak{M}^\varphi = \mathfrak{M}^\varphi \cap E_i(u)^p$. Replacing x_n by the i-th part in $\mathfrak{M}^\varphi_i$ we obtain

$$x + E_{ij}(u)\pi^{p-1}x_1 + \ldots + E_{ij}(u)^{p-1}\pi x_{p-1} \in \mathfrak{M}^\varphi_i \cap E_{ij}(u)^{\min\{n,p\}}\mathfrak{M}_i$$

as desired. ☐

Applying the claim to $x = s(\overline{x})$ for $\overline{x} \in \mathrm{Fil}^n_{ij}$ shows that

$$\underbrace{\sum_{n=0}^{m} E_{ij}(u)^{m-n}\,\mathrm{Fil}^n_{ij}}_{:=Y_m} + \mathfrak{M}^\varphi_{\mathrm{err},ij} \subset \mathfrak{M}^\varphi_i \cap E_{ij}(u)^m\mathfrak{M}_i + \mathfrak{M}^\varphi_{\mathrm{err},ij}$$

for any $0 \leq m \leq p$. We prove the opposite inclusion by induction on m. When $m = 0$ this is clear since both sides equal $\mathfrak{M}^\varphi_i$. For $m > 0$ note that the image of $\mathfrak{M}^\varphi \cap E_{ij}(u)\mathfrak{M}_i$ in M_{ij} is contained in the image of Y_m and so these images are equal. Thus, if $x \in \mathfrak{M}^\varphi_i \cap E_{ij}(u)^m\mathfrak{M}_i$ then there exists $x' \in Y_m$ so that

$$x - x' \in E_{ij}(u)\mathfrak{M}^\varphi_i \cap \left(\mathfrak{M}^\varphi_i \cap E_{ij}(u)^m\mathfrak{M}_i + \mathfrak{M}^\varphi_{\mathrm{err},ij}\right)$$

Thus $x - x'$ can be written as $y + y_{\mathrm{err}}$ with $y_{\mathrm{err}} \in \mathfrak{M}^\varphi_{\mathrm{err},ij}$ and

$$y \in E_{ij}(u)\mathfrak{M}^\varphi_i \cap E_{ij}(u)^m\mathfrak{M}_i = E_{ij}(u)\left(\mathfrak{M}^\varphi_i \cap E_{ij}(u)^{m-1}\mathfrak{M}_i\right)$$

The inductive hypothesis therefore gives that $x - x' \in E_{ij}(u)Y_{m-1} + \mathfrak{M}^\varphi_{\mathrm{err},ij}$. Since $E_{ij}(u)Y_{m-1} \subset Y_m$ it follows that $x \in Y_m$ as desired. ☐

Corollary 13.4. *The graded pieces of $\mathrm{Fil}^\bullet_{ij}$ become $A[\frac{1}{p}]$-projective after inverting p and $\mathrm{Fil}^\bullet_{ij}[\frac{1}{p}]$ has type μ_{ij} (i.e. the n-th graded piece has dimension equal to the multiplicity of n in μ_{ij}).*

Proof of Theorem 13.1. By Corollary 15.2 it suffices to prove the factorisation on the level of $\overline{A}$-valued points for $\overline{A}$ any finite local $\mathbb{F}$-algebra. Let $(\overline{\mathfrak{M}},\overline{\sigma},\overline{\beta})$ be such a point. Applying Lemma 11.4 we obtain a local finite flat $\mathcal{O}$-algebra A with a map $A\rightarrow\overline{A}$ and $(\mathfrak{M},\sigma)\in Y^{\mu}_d(A)$ lifting $(\overline{\mathfrak{M}},\overline{\sigma})$. Additionally choose an $\mathfrak{S}_A$-basis β lifting $\overline{\beta}$. We will be done if we can show that the special fibre of $(\mathfrak{M},\sigma,\beta)$ maps into $\prod_{i=1}^f\overline{M}_{\mu_i}$. We can assume that $\overline{A}=A\otimes_{\mathcal{O}}\mathbb{F}$.

Applying Proposition 13.3 for each ij we obtain filtrations $\mathrm{Fil}^{\bullet}_{ij}$. Set $\mathcal{E}_{ij}:=\sum_{n=0}^pE_{ij}(u)^{p-n}\,\mathrm{Fil}^n_{ij}$. We first show that $\mathcal{E}_i:=\cap_j\mathcal{E}_{ij}$ and $E(u)^p\mathfrak{M}_i=\cap_j(\mathfrak{M}^{\varphi}\cap E_{ij}(u)^p\mathfrak{M})$ have the same image in $\overline{\mathfrak{M}}^{\varphi}=\mathfrak{M}^{\varphi}\otimes_{\mathcal{O}}\mathbb{F}$. Suppose $z\in\cap_j(\mathfrak{M}^{\varphi}\cap E_{ij}(u)^p\mathfrak{M})$. Proposition 13.3 ensures that for each j there is an $m_{ij}\in\mathfrak{M}^{\varphi}_{\mathrm{err},ij}$ such that $z-m_{ij}\in\mathcal{E}_{ij}$. We claim there then exists $m_i\in\pi\mathfrak{M}^{\varphi}$ such that

$$m_i\equiv m_{ij}\mod E_{ij}(u)^{r_{ij}}\mathfrak{M}^{\varphi}$$

for each j. Since $\mathfrak{M}^{\varphi}_i$ is $A[[u]]$-free this claim follows from Lemma 13.5 below. Since $E_{ij}(u)^{r_{ij}}\mathfrak{M}^{\varphi}\subset\mathcal{E}_{ij}$ the claim implies $z-m_i\in\mathcal{E}_{ij}$ for each j. Hence the image of z in $\overline{\mathfrak{M}}^{\varphi}$ is contained in the image of $\mathcal{E}_i$. By symmetry the same argument shows that the image in $\overline{\mathfrak{M}}^{\varphi}_i$ of each $z\in\mathcal{E}_i$ is contained in the image of $\cap_j(\mathfrak{M}^{\varphi}\cap E_{ij}(u)^p\mathfrak{M})$

Next we show that $\mathcal{E}_i$ is $A[[u]]$-projective. This is equivalent to $\mathfrak{M}^{\varphi}_i/\mathcal{E}_i$ being A-projective. From the definitions we see that $\mathfrak{M}^{\varphi}_i/\mathcal{E}_i$ is p-torsionfree. Therefore, by [Sta17, 00ML], it suffices to show that $\overline{\mathfrak{M}}^{\varphi}_i/\overline{\mathcal{E}}_i$ is $A\otimes_{\mathcal{O}}\mathbb{F}$-projective. But we saw in the previous paragraph that $\overline{\mathcal{E}}_i=E(u)^p\overline{\mathfrak{M}}_i$ which is $\overline{A}$-projective, and so $\overline{\mathfrak{M}}^{\varphi}_i/\overline{\mathcal{E}}_i$ is $\overline{A}$-projective. Thus $\mathcal{E}_i$ defines an A-valued point of $\mathrm{Gr}^{(i)}$ whose special fibre is $E(u)^p\overline{\mathfrak{M}}_i$, i.e. the i-th component of $\Phi(\overline{\mathfrak{M}},\overline{\sigma},\overline{\beta})$.

We will therefore be done if the generic fibre of $\mathcal{E}_i$ is contained in the generic fibre of $M_{\mu_i^*}$. Since $\mathcal{E}_i=\cap_j\mathcal{E}_{ij}$ it suffices to show that $\mathcal{E}_{ij}[\frac{1}{p}]$ corresponds to an $A[\frac{1}{p}]$-valued point of the closed subscheme $G/P_{p-\mu_{ij}}$ in Gr_j. In Corollary 13.4 we saw that $\mathrm{Fil}^{\bullet}_{ij}[\frac{1}{p}]$ has n-th graded piece of rank equal to the multiplicity of n in μ_{ij}. Hence, we can choose a basis $(f_1,\dots,f_d)$ of $D_{K,ij}=M_{ij}[\frac{1}{p}]$ adapted to the filtration, i.e. so that $\mathrm{Fil}^n_{ij}[\frac{1}{p}]$ is generated by those f_l with $\mu_{ij,l}\geq n$. If we view f_l as elements of $\mathfrak{M}^{\varphi}_i\cong A[[u]]^d$ via s then since $\mathcal{E}_{ij}[\frac{1}{p}]=\sum E_{ij}(u)^{p-n}\,\mathrm{Fil}^n_{ij}[\frac{1}{p}]$ it follows that $\mathcal{E}_{ij}[\frac{1}{p}]$ is generated by $E_{ij}(u)^{p-\mu_{ij,l}}f_l$ and so $\mathcal{E}_{ij}[\frac{1}{p}]\in G/P_{p-\mu_{ij}}$ as required. □

Lemma 13.5. *Suppose that $m_1,\dots,m_e\in\mathcal{O}[u]$ can each be written as*

$$m_j=\sum_{l=1}^pE_{ij}(u)^{p-l}\pi^{p-l}m_{j,l}$$

with $m_{j,l}\in\mathcal{O}$. Suppose that r_j are positive integers with

$$r_1+\ldots+r_e\leq\frac{p-1}{\nu}+1$$

for ν equal the maximum π-adic valuation of $\pi_{ij}-\pi_{ij'}$ for $1\leq j<j'\leq e$. Then there exists $m\in\pi\mathcal{O}[u]$ with $m\equiv m_j$ modulo $E_{ij}(u)^{r_j}$ for each j.

Proof. One immediately reduces to the case $m_2=\ldots=m_e=0$. Suppose that for $j=2,\dots,e$ there are polynomials $X_j\in E[u]$ with $(u-\pi_{ij})^{r_j}X_j\equiv1$ modulo $(u-\pi_{i1})^{r_1}$. Define $\widetilde{m}$ as the polynomial obtained by truncating $m_1\prod_{i=2}^eX_j$ up to (but not including) degree r_1 when viewed as a polynomial in $(u-\pi_{i1})$. If

$m = \widetilde{m}\prod_{j=2}^e (u-\pi_{ij})^{r_j}$ then $m \equiv 0$ modulo $(u-\pi_{ij})^{r_j}$ for $j = 2,\dots,e$ and $m \equiv m_1$ modulo $(u-\pi_{i1})^{r_1}$. Therefore, we will be done if there are X_j so that $m \in \pi\mathcal{O}[u]$.

The existence of the X_j follows from Lemma 5.4 but we can also give an explicit formula:

$$X_j = \sum_{n=0}^{r_1-1} \binom{r_j - 1 + j}{r_j - 1} \frac{(u-\pi_{i1})^n}{(\pi_{i1}-\pi_{ij})^{n+r_j}}$$

(that this has the desired property follows from the formal identity $\frac{1}{(1-y)^r} = \sum_{n=0}^\infty \binom{r-1+i}{r-1} y^i$). Therefore the coefficient of $(u-\pi_{i1})^n$ in m_1 has valuation $\geq p-n$ while the coefficient of $(u-\pi_{i1})^n$ in X_j has valuation $\geq -(n+r_j)\nu_1$ for ν_1 the maximum of the valuations of $\pi_{i1}-\pi_{ij}$. Since $\nu_1 \geq 1$ we have $p-n \geq p-n\nu_1$ and as such the coefficient of $(u-\pi_{i1})^n$ in $m_1\prod_{j=2}^e X_j$ has valuation

$$\geq p - (r_2 + \ldots + r_e + n)\nu_1$$

We will be done if $p-(r_2+\ldots+r_e+n)\nu_1 \geq 1$ for all $n = 0,\dots,r_1-1$, i.e. if $p-(r_1+\ldots+r_e-1)\nu_1 \geq 1$. Equivalently, if $r_1+\ldots+r_e \leq \frac{p-1}{\nu_1}+1$. Since $\nu_1 \leq \nu$ this finishes the proof. □

14. Breuil–Mézard identities

In this section we produce identities of cycles in $Y_2^{\leq h}$ and from this deduce analogous identities in spaces of two dimension p-adic Galois representations of G_K. As with cycles on a scheme, a k-dimensional cycle on a Noetherian algebraic stack $\mathcal{X}$ is a $\mathbb{Z}$-linear combination of irreducible k-dimensional closed subsets of the associated topological space $|\mathcal{X}|$. Associated to any closed substack $\mathcal{Z} \subset \mathcal{X}$ with $|\mathcal{Z}|$ equidimensional of dimension k is the cycle

$$[\mathcal{Z}] = \sum_Z m_{\mathcal{Z},Z} Z$$

Here $m_{\mathcal{Z},Z}$ denotes the multiplicity of Z in $\mathcal{Z}$ which is defined as the multiplicity of $f^{-1}(Z)$ in $f^{-1}(\mathcal{Z})$ for some (equivalently any) smooth surjection $f: X \to \mathcal{X}$ from a scheme X, cf. [Sta17, 0DR4]. In particular, to prove an identity

$$[\mathcal{Z}] = \sum m_i[\mathcal{Z}_i]$$

of k-dimensional cycles in $\mathcal{X}$ it suffices to find an f as above with relative dimension n so that $[f^{-1}(\mathcal{Z})] = \sum m_i[f^{-1}(\mathcal{Z}_i)]$ as $k+n$-dimensional cycles in X.

To state our result consider a variant of the representation theory considered in Section 4. For $\lambda = (\lambda_1,\dots,\lambda_f)$ with each $\lambda_i = (\lambda_{i,1} > \lambda_{i,2})$ regular define

$$V_{\lambda,\mathbb{F}} = \bigotimes_{i=1}^f \left(\det^{\lambda_{i,2}} \otimes \mathrm{Sym}^{\lambda_{i,1}-\lambda_{i,2}-1} k^2\right) \otimes_{\kappa_i} \mathbb{F}$$

viewed as a representation of $\mathrm{GL}_2(k)$. Recall that each $V_{\lambda,\mathbb{F}}$ is irreducible if and only if $\lambda_{i,1}-\lambda_{i,2} \leq p$ for each i, cf. [Her09, Corollary 3.17].

Theorem 14.1. *Assume that $d=2$ and that $\mu_{ij} \subset [0, r_{ij}]$ is regular with*

$$\sum_j r_{ij} \leq p$$

Then, in the group of ef-dimensional cycles on $Y_2^{\leq p}$, we have

$$[\overline{Y}_2^{\mu}] \leq \sum_\lambda m(\mu,\lambda)[\overline{Y}_2^{\widetilde{\lambda}}]$$

where:

(1) *the sum runs over tuples* $\lambda = (\lambda_1, \ldots, \lambda_f)$ *with* $\lambda_i = (\lambda_{i,1} > \lambda_{i,2})$ *for which* $V_{\lambda,\mathbb{F}}$ *is a Jordan–Holder factor of* $V_{\mu_1,\mathbb{F}} \otimes \ldots \otimes V_{\mu_e,\mathbb{F}}$ *with* $\mu_i = (\mu_{i1}, \ldots, \mu_{ie})$.
(2) $m(\mu, \lambda)$ *is the multiplicity with which* $V_{\lambda,\mathbb{F}}$ *appears.*
(3) $\widetilde{\lambda}$ *is the Hodge type with* $\widetilde{\lambda}_{i1} = \lambda_i$ *and* $\widetilde{\lambda}_{ij} = \rho = (1,0)$ *for* $j \neq 1$.

Proof. Since each μ_{ij} is regular we have $r_{ij} \geq 1$ and so the bound $\sum_j r_{ij} \leq p$ implies either $e < p$ or $e = p$ and each $r_{ij} = 1$. In the latter case $\mu = \widetilde{\lambda}$ for $\lambda = (1,0)$ and the theorem asserts only that $[\overline{Y}_2^{\widetilde{\lambda}}] = [\overline{Y}_2^{\widetilde{\lambda}}]$; thus we can assume that $e < p$ and K is tamely ramified over $\mathbb{Q}_p$. Remark 13.2 therefore ensures we can apply Theorem 13.1. We can also assume that $e > 1$, since when $e = 1$ we have $\mu = \widetilde{\lambda}$ and so the theorem is again trivial. Therefore we can also apply Proposition 12.5.

We begin by applying Theorem 4.1 to $\prod_{i=1}^f \overline{M}_{\mu_i^*}$ in $\prod_{i=1}^f \mathrm{Gr}^{(i)}$. Our bound on $\sum_j r_{ij}$ implies that the bound from Theorem 4.1 is satisfied for μ_i^*. Therefore, in either $K_0(\prod_{i=1}^f \mathrm{Gr}^{(i)}_{\leq \mu_i})$ or the group ef-dimensional cycles, we have

$$[\prod_{i=1}^f \overline{M}_{\mu_i^*}] = \sum_{\lambda'} m'(\lambda', \mu)[\prod_{i=1}^f \overline{M}_{\widetilde{\lambda}'_i}]$$

where the sum runs over those $\lambda' = (\lambda'_1, \ldots, \lambda'_f)$ for which $V_{\lambda'_i}$ appears in $V_{p-\mu_{i1}} \otimes \ldots \otimes V_{p-\mu_{ie}}$ for each i and, if $m_i(\lambda', \mu)$ is the multiplicity with which $V_{\lambda'_i}$ appears, then $m'(\lambda', \mu) = \prod_i m_i(\lambda', \mu)$.

Note that $V_{\lambda'_i}$ appears in $V_{p-\mu_{i1}} \otimes \ldots \otimes V_{p-\mu_{ie}}$ for each i if and only if $V_{p-\lambda'_i}$ appears in $V_{\mu_{i1}} \otimes \ldots \otimes V_{\mu_{ie}}$ for each i, and the multiplicities are the same. Therefore we can rewrite the above identity as

$$[\prod_{i=1}^f \overline{M}_{\mu_i^*}] = \sum_{\lambda} m(\lambda, \mu)[\prod_{i=1}^f \overline{M}_{\widetilde{\lambda}_i^*}]$$

with the sum running over λ as in the theorem and $m(\lambda, \mu)$ as in the theorem.

Next choose $N >> 0$ as in Proposition 10.3 so that $\widetilde{Z}_2^{\leq p,N} \xrightarrow{\Psi} \prod_{i=1}^f \mathrm{Gr}^{(i)}$ is a $\mathcal{G}_N$-torsor. In particular, this morphism is flat and so pulling-back the previous identity gives

$$[\widetilde{Z}_2^{\leq p,N} \times_{\prod \mathrm{Gr}^{(i)}} \prod \overline{M}_{\mu_i^*}] = \sum_{\lambda} m(\lambda, \mu)[\widetilde{Z}_2^{\leq p,N} \times_{\prod \mathrm{Gr}^{(i)}} \prod \overline{M}_{\widetilde{\lambda}_i^*}] \tag{14.2}$$

We view this as occurring in either $K_0(\widetilde{Z}_2^{\leq h,N})$ or in the group of $ef + \dim \mathcal{G}_N$-dimensional cycles. Lemma 14.4 below shows that each of the closed subschemes appearing in this identity are contained inside $\overline{Z}^{r,\sigma} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}$ where $\overline{Z}^{r,\sigma}$ is the closed substack from Proposition 12.2. As established in the first paragraph, we can apply Proposition 12.5 and view (14.2) as an identity either in $K_0(Y_2^{\leq p} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N})$ or in the group of $ef + \dim \mathcal{G}_N$-dimensional cycles on $Y_2^{\leq p} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}$.

Via the identification provided by Proposition 12.5, Theorem 13.1 implies that the preimage of $\overline{Y}_2^{\mu}$ in $Y_2^{\leq p} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}$ is contained in $\widetilde{Z}^{\leq p,N} \times_{\prod \mathrm{Gr}^{(i)}} \prod \overline{M}_{\mu_i^*}$ and so

$$[\overline{Y}_2^{\mu} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}] \leq [\widetilde{Z}_2^{\leq p,N} \times_{\prod \mathrm{Gr}^{(i)}} \prod \overline{M}_{\mu_i^*}]$$

In the case $\mu = \widetilde{\lambda}$ this is an equality since $\widetilde{Z}_2^{\leq p,N} \times_{\prod \mathrm{Gr}^{(i)}} \prod \overline{M}_{\widetilde{\lambda}_i^*}$ is irreducible and generically reduced (as follows from Proposition 8.1 and the fact that Ψ is smooth with irreducible fibres). We conclude that

$$[\overline{Y}_2^{\mu} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}] \leq \sum_{\lambda} m(\lambda,\mu)[\overline{Y}_2^{\widetilde{\lambda}} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}] \tag{14.3}$$

Since $Y_2^{\leq p} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N} \to Y_2^{\leq p}$ is smooth of relative dimension $\mathcal{G}_N$ we deduce the identity from the theorem. □

Lemma 14.4. *Let $\mathrm{Gr}_{r,\sigma}^{(i)}$ be as in the proof of Proposition 12.2 for $r = (r_{ij})$ and $h = p$. Then $M_{\mu_i^*} \subset \mathrm{Gr}_{r,\sigma}^{(i)}$.*

Proof. The subscheme $\mathrm{Gr}_{r,\sigma}^{(i)}$ is defined by two closed conditions. The second condition is that $\mathcal{E} \subset \prod_{ij} E_{ij}(u)^{p-r_{ij}} A[[u]]^d$. This condition is satisfied for all $\mathcal{E} \in \mathrm{Gr}_{\leq \mu_i^*}^{(i)}$ since any such $\mathcal{E}$ satisfies

$$\prod_j E_j(u)^p A[[u]]^d \subset \mathcal{E} \subset \prod_j E_{ij}(u)^{p-r_{ij}} A[[u]]^d$$

For the first closed condition we use that $M_{\mu_i^*}$ is reduced to reduce to consideration of $\mathcal{E} \in M_{\mu_i^*}$ corresponding to an $\mathcal{O}'$-valued point with $\mathcal{O}'$ the ring of integers in a finite extension of E. Any such $\mathcal{E}$ can be written as $\cap_j \mathcal{E}_j$ for $\mathcal{E}_j \in G/P_{\mu_{ij}^*} \hookrightarrow \mathrm{Gr}_{j,\leq \mu_{ij}}$ (cf. the second paragraph in the proof of Proposition 5.1). Each $\mathcal{E}_j$ is generated by $E_{ij}(u)^{p-\mu_{ij,l}} g(e_l)$ for e_l the standard basis of $\mathcal{O}'[[u]]^d$ and some $g \in G$. For the coordinate-wise action of G_K on $A_{\mathrm{inf},\mathcal{O}',d}$ we have

$$(\sigma - 1)(E_{ij}(u)^{p-\mu_{ij,l}} g(e_l)) = (\sigma - 1)(E_{ij}(u)^{p-\mu_{ij,l}}) g(e_l)$$

Since $(\sigma - 1)(E_{ij}(u)^{p-\mu_{ij,l}}) \in [\pi^\flat]\mu A_{\mathrm{inf},\mathcal{O}',i} \subset [\pi^\flat]\varphi^{-1}(\mu) A_{\mathrm{inf},\mathcal{O}',i}$ it follows that

$$(\sigma - 1)(\mathcal{E}_j) \subset \mathcal{E}_j \otimes_{A[[u]]} [\pi^\flat]\varphi^{-1}(\mu) A_{\mathrm{inf},\mathcal{O}',i}$$

for each j. Thus the same is true of $\mathcal{E}$ which finishes the proof. □

Next we give applications to the special fibre of deformation rings. Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be continuous. As in the proof of Lemma 11.4 let $R_{\overline{\rho}}$ denote the corresponding $\mathcal{O}$-framed deformation ring and $R_{\overline{\rho}}^{\mu}$ the reduced $\mathcal{O}$-flat quotient classifying crystalline deformations of Hodge type μ.

Theorem 14.5. *With notation as in Theorem 14.1 we have*

$$[\mathrm{Spec}\, \overline{R}_{\overline{\rho}}^{\mu}] \leq \sum_{\lambda} m(\lambda,\mu)[\mathrm{Spec}\, \overline{R}_{\overline{\rho}}^{\widetilde{\lambda}}]$$

as ef-dimensional cycles in $\mathrm{Spec}\, \overline{R}_{\overline{\rho}}$. If $p \neq 2$ then this is furthermore an equality.

Proof. Let $\mathcal{L}_{\overline{\rho}}^{\leq p}$ and $\mathcal{L}_{\overline{\rho}}^{\mu}$ be as in the proof of Lemma 11.4. Additionally consider

$$\widetilde{\mathcal{L}}_{\overline{\rho}}^{\leq p} := \mathcal{L}_{\overline{\rho}}^{\leq p} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}$$

for some $N >> 0$ and $\widetilde{\mathcal{L}}_2^{\mu} := \mathcal{L}_{\overline{\rho}}^{\mu} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}$ Pulling back (14.3) along the formally smooth map $\widetilde{\mathcal{L}}_{\overline{\rho}}^{\leq p} \to Y_2^{\leq p} \times_{Z_2^{\leq p}} \widetilde{Z}_2^{\leq p,N}$ gives

$$[\widetilde{\mathcal{L}}_{\overline{\rho}}^{\mu} \times_{\mathcal{O}} \mathbb{F}] \leq \sum_{\lambda} m(\lambda,\mu)[\widetilde{\mathcal{L}}_{\overline{\rho}}^{\widetilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}]$$

in $K_0(\widetilde{\mathcal{L}}^{\leq p}_{\overline{\rho}})$. We claim this implies that

$$[\overline{\mathcal{L}}^{\mu}_{\overline{\rho}}] \leq \sum_{\lambda} m(\lambda,\rho)[\overline{\mathcal{L}}^{\widetilde{\lambda}}_{\overline{\rho}}]$$

in $K_0(\mathcal{L}^{\leq p}_{\overline{\rho}})$. Certainly the first identity implies $\overline{\mathcal{L}}^{\mu}_{\overline{\rho}} \subset \bigcup_\lambda \overline{\mathcal{L}}^{\widetilde{\lambda}}_{\overline{\rho}}$ topologically since $\widetilde{\mathcal{L}}^{\mu}_{\overline{\rho}} \times_{\mathcal{O}} \mathbb{F}$ is the preimage of $\overline{\mathcal{L}}^{\mu}_{\overline{\rho}}$ under $\widetilde{\mathcal{L}}^{\leq p}_{\overline{\rho}} \to \mathcal{L}^{\leq p}_{\overline{\rho}}$ (and likewise with μ replaced by $\widetilde{\lambda}$). Since $\widetilde{\mathcal{L}}^{\leq p}_{\overline{\rho}} \to \mathcal{L}^{\leq p}_{\overline{\rho}}$ is smooth the multiplicities coincide also.

Finally, we note that $\mathcal{L}^{\mu}_{\overline{\rho}} \to \operatorname{Spec} R^{\mu}_{\overline{\rho}}$ is an isomorphism after inverting p and so Lemma 3.6 implies that the pushforward of $[\overline{\mathcal{L}}^{\mu}_{\overline{\rho}}]$ equals $[\operatorname{Spec} \overline{R}^{\mu}_{\overline{\rho}}]$. Therefore, pushing forward the previous identity gives

$$[\operatorname{Spec} \overline{R}^{\mu}_{\overline{\rho}}] \leq \sum_{\lambda} m(\lambda,\mu)[\operatorname{Spec} \overline{R}^{\widetilde{\lambda}}_{\overline{\rho}}]$$

(either as cycles or as classes in $K_0(\operatorname{Spec} R_{\overline{\rho}})$). Finally, if $p \neq 2$ then global methods show, without any assumptions on μ, that $[\operatorname{Spec} \overline{R}^{\mu}_{\overline{\rho}}] \geq \sum_\lambda m(\lambda,\mu)[\operatorname{Spec} \overline{R}^{\widetilde{\lambda}}_{\overline{\rho}}]$, cf. [EG19, 8.6.6]. Therefore, we have the equality as claimed. □

Corollary 14.6. *Let $\mathcal{X}_2$ denote the Emerton–Gee stack classifying two dimensional p-adic representations of G_K and let $\mathcal{X}_2^{\mu}$ denote the closed substack classifying crystalline representations of Hodge type μ. Then, with notation as in Theorem 14.1,*

$$[\overline{\mathcal{X}}^{\mu}_2] \leq \sum_{\lambda} m(\lambda,\mu)[\overline{\mathcal{X}}^{\widetilde{\lambda}}_2]$$

with equality if $p \neq 2$.

Proof. This follows directly from Theorem 14.5, cf. the discussion in [EG19, 8.3]. □

15. Miscellany

Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks of finite type over a field k and let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks.

Lemma 15.1. *If, for A any local finite k-algebra, the induced functor $\mathcal{X}(A) \to \mathcal{Y}(A)$:*

1. *(1) is fully faithful then f is a monomorphism (which by our definition implies being representable by algebraic spaces and separated).*
2. *(2) is an equivalence then $\mathcal{X} \to \mathcal{Y}$ is an isomorphism.*

Proof. First we prove (2) under the additional assumption that f is representable by algebraic spaces. Then, by choosing a smooth surjection $U \to \mathcal{Y}$ with U an algebraic space, we can assume that $\mathcal{X}$ and $\mathcal{Y}$ are algebraic spaces. With this reduction the argument given in [LLHLM20, 7.2.4] goes through with schemes replaced by algebraic spaces. Indeed, by [Sta17, 0APP] this morphism is smooth and quasi-finite, and hence étale. By [Sta17, 05W1] the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an open immersion. Since it is surjective on finite type points it is an isomorphism and so $\mathcal{X} \to \mathcal{Y}$ is a monomorphism. By [Sta17, 05W5] it is an open immersion, and so an isomorphism, again by surjectivity on finite type points.

Now we prove (1). By [Sta17, 04XS] the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by algebraic spaces. Full faithfullness of f on A-valued points implies that

Δ_f is an equivalence on such points. Therefore the first paragraph implies Δ_f is an isomorphism. From [Sta17, 04ZZ] we obtain (1).

To finish the proof of (2) note that by (1) we have f representable by algebraic spaces. □

Corollary 15.2. *Suppose $\mathcal{Z}$ is a closed substack of $\mathcal{Y}$ and that for every morphism* $\operatorname{Spec} A \to \mathcal{X}$*, with A any local finite k-algebra, the composite* $\operatorname{Spec} A \to \mathcal{X} \to \mathcal{Y}$ *factors through $\mathcal{Z}$. Then $\mathcal{X} \to \mathcal{Y}$ factors through $\mathcal{Z}$.*

Proof. This follow since by Lemma 15.1 the map $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is an isomorphism. □

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