PDEs assignment

Robin Evers

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1. Draw the characteristics for the following equations and write the general solution.

$$u_t + 7u_x = 0 \tag{1}$$
$$u_t - 2u_x + 3u = 0. \tag{2}$$

Sol. To find characteristics, we solve the ODE

$$\begin{aligned} x'(t) &= 7 \\ x(0) &= x_0. \end{aligned}$$

This gives us the characteristic curves $x(t)$ defined by

$$x(t) = x_0 + 7t.$$

On these curves we have

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= (u(x(t),t))_t + (x(t))_t(u(x(t)),t)_x \\ &= (u(x(t),t))_t + 7(u(x(t)),t)_x \\ &= 0. \end{aligned}$$

This means that solutions $u(x(t),t)$ to (1) are constant on these characteristic curves $x(t)$, hence the general solution to (1) is given by

$$u(x(t),t) = u(x_0,0) = u(x(t) - 7t, 0).$$

Sol. To find characteristics, we solve the ODE

$$\begin{aligned} x'(t) &= -2 \\ x(0) &= x_0. \end{aligned}$$

This gives us the characteristic curves $x(t)$ defined by

$$x(t) = x_0 - 2t.$$

On these curves we have

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= (u(x(t),t))_t + (x(t))_t(u(x(t)),t)_x \\ &= (u(x(t),t))_t - 2(u(x(t)),t)_x \\ &= -3u(x(t),t). \end{aligned}$$

Hence

$$\frac{d}{dt}u(x(t),t) + 3u(x(t),t) = 0.$$

This gives us the ODE

$$\begin{aligned} u_t + 3u &= 0 \\ u(x(0),0) &= u_0(x_0). \end{aligned}$$

We multiply the ODE by e^{3t} to get

$$\frac{d}{dt}ue^{3t} = u_te^{3t} + 3ue^{3t} = 0.$$

This means that ue^{3t} is constant on $x(t)$, so

$$e^{3t}u(x(t),t) = e^0u(x(0),0) = u_0(x_0)$$

hence the general solution to (2) is given by

$$u(x(t),t) = e^{-3t}u_0(x_0) = e^{-3t}u_0(x(t) + 2t).$$

2. Let g be the gravitational constant and $h : \mathbf{R} \times \mathbf{R}^+$ be the height of a fluid, define $\phi = gh$. Further, let $v : \mathbf{R} \times \mathbf{R}^+$ be the horizontal velocity. The shallow water equations are given by

$$\begin{aligned} \phi_t + (v\phi)_x &= 0 \\ v_t + \left(\frac{v^2}{2} + \phi\right)_x &= 0. \end{aligned} \tag{3}$$

Let $\eta = \eta(\phi, v)$ be an entropy for the shallow water equations. Prove that it must satisfy

$$\frac{\partial^2 \eta}{\partial v^2} = \phi \frac{\partial^2 \eta}{\partial \phi^2}. \tag{4}$$

Sol. We start by rewriting the shallow water equations as in (4) to one vectorial equation

$$\partial_t \vec{u} + \partial_x \vec{f}(\vec{u}) = 0$$

where $\vec{u} = (\phi, v)$ and $\vec{f} = (v\phi, \frac{v^2}{2} + \phi)$. The Jacobian of f is given by

$$D\vec{f} = \begin{pmatrix} \frac{\partial f_1}{\partial \phi} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} v & \phi \\ 1 & v \end{pmatrix}.$$

Let $\eta = \eta(\phi, v)$ be an entropy for the shallow water equations, the matrix of second derivatives of η is given by

$$D^2\eta = \begin{pmatrix} \frac{\partial^2 \eta}{\partial \phi^2} & \frac{\partial^2 \eta}{\partial \phi \partial v} \\ \frac{\partial^2 \eta}{\partial v \partial \phi} & \frac{\partial^2 \eta}{\partial v^2} \end{pmatrix}.$$

Since η is an entropy for the shallow water equations, we have

$$(D\vec{f})^T D^2\eta = D^2\eta D\vec{f}.$$

This gives us four equations, the equation given by the second row, first column is

$$\phi \frac{\partial^2 \eta}{\partial \phi^2} + v \frac{\partial^2 \eta}{\partial v \partial \phi} = v \frac{\partial^2 \eta}{\partial v \partial \phi} + \frac{\partial^2 \eta}{\partial v^2}.$$

By subtracting $v \frac{\partial^2 \eta}{\partial v \partial \phi}$ from both sides we find the requested equality for our entropy η

3. The Barotropic compressible Euler equations are given when the internal energy of the system is constant. The equations are given by conservation of mass and momentum

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + \left(\rho v^2 + p\right)_x &= 0, \end{aligned} \tag{5}$$

with $p = p(\rho)$ for a smooth function p. Assume $p'(\rho) > 0$ then show that

$$\eta(\rho, \rho v) = \frac{1}{2}\rho v^2 + P(\rho) \tag{6}$$

is an entropy when $P''(\rho) = \frac{p'(\rho)}{\rho}$ for $\rho > 0$. What is the associated entropy flux?

Sol. We start by rewriting the Euler equations as in (5) to one vectorial equation

$$\partial_t \vec{u} + \partial_x \vec{f}(\vec{u}) = 0$$

where $\vec{u} = (\rho, \rho v)$ and $\vec{f}(y, z) = (z, \frac{z^2}{y} + p(y))$. The Jacobian of f is given by

$$D\vec{f} = \begin{pmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p'(y) - (\frac{z}{y})^2 & \frac{2z}{y} \end{pmatrix}.$$

We write η as a function of variables y, z as $\eta(y, z) = \frac{z^2}{2y} + P(y)$. The Jacobian of η is given by

$$D\eta = \begin{pmatrix} \frac{\eta}{\partial y}, & \frac{\partial \eta}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{-z^2}{2y^2} + P'(y), & \frac{z}{y} \end{pmatrix}.$$

A pair (η, q) is an entropy/entropy flux pair associated with (5) if $Dq = D\eta D\vec{f}$. If η as defined in (6) is an entropy, we find that the entropy flux q must satisfy

$$\begin{aligned} Dq = D\eta D\vec{f} &= \begin{pmatrix} \frac{-z^2}{2y^2} + P'(y), & \frac{z}{y} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ p'(y) - (\frac{z}{y})^2 & \frac{2z}{y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{zp'(y)}{y} - (\frac{z}{y})^3, & \frac{3z^2}{2y^2} + P'(y). \end{pmatrix} \end{aligned}$$

Hence the partial derivative of q to z has to be

$$\frac{\partial q}{\partial z} = \frac{3z^2}{2y^2} + P'(y) = \frac{3}{2y^2} z^2 + P'(y),$$

which means q has to be of the form

$$q = \frac{1}{2y^2} z^3 + P'(y) z + C(y).$$

where C is independent of z. We derive this to y to find

$$\frac{\partial q}{\partial y} = -\left(\frac{z}{y}\right)^3 + P''(y) z + C'(y).$$

This equals the expression we found for Dq if $P''(y) = \frac{p'(y)}{y}$ and $C'(y) = 0$. Therefore $\eta(\rho, \rho v)$ is an entropy when $P''(\rho) = \frac{p'(\rho)}{\rho}$ for $\rho > 0$, the associated entropy flux is given by

$$q = \frac{(\rho v)^3}{2\rho^2} + P'(\rho)\rho v + C = \frac{\rho v^3}{2} + P'(\rho)\rho v + C,$$

where C is independent of ρ and v.

4. Consider the following partial differential equation (PDE)

$$u_t + u_{xxx} + 6uu_x = 0, u = u(x,t) \tag{7}$$

known as the Korteweg-de Vries equation (or KdV) and which describes water waves in shallow waters.

(a) Consider a scaling transformation, on both the independent and dependent variables, of the form

$$S : (t, x, u) \mapsto (T, X, U) = (at, bx, cu) \tag{8}$$

where a, b and c are nonzero constants. Find conditions for the parameters a, b and c such that the transformation S is a symmetry transformation of the KdV equation. Use the obtained symmetry to deduce that if $u = f(x,t)$ is a solution of the KdV equation then $u = \epsilon^2 f(\epsilon x, \epsilon^3 t)$ is also a solution for all nonzero ϵ.

Sol. The PDE given in (6) admits the symmetry if, given a solution $u = f(x,t)$, the transformed function

$$\tilde{u} = \tilde{f}(\tilde{x}, \tilde{t}) = cf(bx, at)$$

is also a solution of the same PDE, i.e.

$$\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + 6\tilde{u}\tilde{u}_{\tilde{x}} = 0. \tag{9}$$

To find a, b, c for which this is true, we first calculate the transformed derivatives

$$\frac{\partial}{\partial \tilde{t}} \tilde{u}(\tilde{x}, \tilde{u}) = \frac{dt}{d\tilde{t}} \frac{\partial}{\partial t} cu(x,t) = \frac{c}{a} \frac{\partial}{\partial t} u(x,t),$$

$$\frac{\partial}{\partial \tilde{x}} \tilde{u}(\tilde{x}, \tilde{u}) = \frac{dx}{d\tilde{x}} \frac{\partial}{\partial x} cu(x,t) = \frac{c}{b} \frac{\partial}{\partial x} u(x,t),$$

and similarly

$$\frac{\partial^2}{\partial \tilde{x}^2} \tilde{u}(\tilde{x}, \tilde{t}) = \frac{c}{b^2} \frac{\partial^2}{\partial x} u(x,t),$$

$$\frac{\partial^3}{\partial \tilde{x}^2} \tilde{u}(\tilde{x}, \tilde{t}) = \frac{c}{b^3} \frac{\partial^3}{\partial x} u(x,t).$$

Hence, if we substitute to the left hand side of (8) we obtain

$$\frac{c}{a} \frac{\partial}{\partial t} u(x,t) + \frac{c}{b^3} \frac{\partial^3}{\partial x} u(x,t) + \frac{6c^2}{b} u(x,t) \frac{\partial}{\partial x} u(x,t) = 0$$

when

$$\frac{c}{a} = \frac{c}{b^3} = \frac{c^2}{b}.$$

Since we assume a, b, c nonnegative, this holds when $a = b^3$ and $c = b^2$. Let us fix $b = \epsilon$ for any nonnegative ϵ, then

$$\tilde{u} = cf(bx, at) = \epsilon^2 f(\epsilon x, \epsilon^3)$$

is also a solution to (6).

(b)