Yet another variant of Curry's paradox
(after Restall)

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May 24, 2009

The setup

A naïve set theory with membership ($\in$) and subset ($\subseteq$) predicates that satisfy the natural deduction rules below. This is enough to reduce the theory to triviality, assuming only that the ambient logic admits contraction.

The rules

$$\dfrac{\phi(t)}{t \in \{x \mid \phi(x)\}}\,[\in I] \qquad \dfrac{t \in \{x \mid \phi(x)\}}{\phi(t)}\,[\in E] \qquad \dfrac{x \in A \quad A \subseteq B}{x \in B}\,[\subseteq E] \qquad \dfrac{\begin{matrix}[x \in A]\\ \vdots \\ x \in B\end{matrix}}{A \subseteq B}\,[\subseteq I]$$

The proof

Fix some arbitrary γ and Δ, and let $P = \{x \mid \{y \mid x \in x\} \subseteq \Delta\}$. Now we can deduce $\gamma \in \Delta$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x \in \{y \mid P \in P\}]^*}{P \in P}\,[\in E]}{\{y \mid P \in P\} \subseteq \Delta}\,[\in E] \quad [x \in \{y \mid P \in P\}]^*}{x \in \Delta}\,[\subseteq E]}{\{y \mid P \in P\} \subseteq \Delta}\,[\subseteq I]^*}{P \in P}\,[\in I]}{\gamma \in \{y \mid P \in P\}}\,[\in I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{[x \in \{y \mid P \in P\}]^\dagger}{P \in P}\,[\in E]}{\{y \mid P \in P\} \subseteq \Delta}\,[\in E] \quad [x \in \{y \mid P \in P\}]^\dagger}{x \in \Delta}\,[\subseteq E]}{\{y \mid P \in P\} \subseteq \Delta}\,[\subseteq I]^\dagger}{\gamma \in \Delta}\,[\subseteq E]$$