Linear Logic without Units

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Contents

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Linear Logic Without Units: Abstract

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We study categorical models for the unitless fragment of multiplicative linear logic. We find that the appropriate notion of model is a special kind of promonoidal category. Since the theory of promonoidal categories has not been developed very thoroughly, at least in the published literature, we need to develop it here. The most natural way to do this – and the simplest, once the (substantial) groundwork has been laid – is to consider promonoidal categories as an instance of the general theory of pseudomonoids in a monoidal bicategory. Accordingly, we describe and explain the notions of monoidal bicategory and pseudomonoid therein.

The higher-dimensional nature of monoidal bicategories presents serious notational difficulties, since to use the natural analogue of the commutative diagrams used in ordinary category theory would require the use of three-dimensional diagrams. We therefore introduce a novel technical device, which we dub the *calculus of components*, that dramatically simplifies the business of reasoning about a certain class of algebraic structure internal to a monoidal bicategory. When viewed through this simplifying lens, the theory of pseudomonoids turns out to be essentially formally identical to the ordinary theory of monoidal categories – at least in the absence of permutative structure such as braiding or symmetry. We indicate how the calculus of components may be extended to cover structures that make use of the braiding in a braided monoidal bicategory, and use this to study braided pseudomonoids.

A higher-dimensional analogue of Cayley's theorem is proved, and used to deduce a novel characterisation of the unit of a promonoidal category. This, and the other preceding work, is then used to give two characterisations of the categories that model the unitless fragment of intuitionistic multiplicative linear logic. Finally we consider the non-intuitionistic case, where the second characterisation in particular takes a surprisingly simple form.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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I am indebted to many people for interesting and informative discussions over the past four years. There follows a list of some of the people who have influenced my thinking in varied and often immeasurable ways – in some cases through just one or two conversations, and in others through repeated contact or protracted corespondence. I apologise to all those whose names I have mistakenly omitted. Thanks, then (without, as the lawyers say, prejudice to the generality) to John Baez, Eugenia Cheng, Robin Cockett, Jeff Egger, Marcelo Fiore, Nicola Gambino, Richard Garner, Martin Hyland, Achim Jung, Joachim Kock, Tom Leinster, Don MacInnes, Craig Pastro, John Power, Luigi Santocanale, Peter Selinger, Harold Simmons, Paul Taylor, Todd Trimble, and Glynn Winskel.

A particular debt is owed to Dominic Hughes, who set me on the road that led to this thesis by innocently asking what the definition of 'unitless star-autonomous category' should be; and without whose unceasing encouragement and boundless optimism this work would never have reached the stage that it has.

I am also deeply indebted to Andrea Schalk, an unfailingly helpful and supportive supervisor whose seemingly inexhaustible patience I must have severely tested at times.

My friends and family have helped in more ways than I could list, and this thesis would not exist without them.

To Miranda

Chapter 1

Introduction

1.1 Linear logic without units

The starting point of this work is the question: what is a categorical model of the unitless fragment of multiplicative linear logic? The question has at least some intrinsic interest, and we shall see that a proper understanding of the natural answer demands some unexpectedly sophisticated mathematics. Also the category of proof nets (Girard, 1987) is an important part of the proof theory of linear logic, and proof nets do not give a natural interpretation of the units. (Several authors *have* considered extended notions of proof net that include units – see Trimble (1994); Blute et al. (1996); Straßburger and Lamarche (2004); Hughes (2005) – but it must be admitted that these extended proof nets are substantially more complicated, and none succeeds in giving a purely geometric normal form for proofs. As a symptom of this complication, it is an open question whether equivalence of MLL proofs can be decided in polynomial time; whereas Girard's proof nets immediately suggest a polynomial time algorithm to decide proof equivalence in the unitless fragment.)

On a more pragmatic note, linear logic and related systems have a number of applications to computer programming. One example is the linear logic programming developed by Miller (2004). Systems of linear types, and the closely related 'uniqueness types', are also increasingly important; the functional programming language Clean (Plasmeijer and van Eekelen, 2001) uses a system of uniqueness types to facilitate integration of effects such as input and output with purely-functional code. For many practical purposes, the unit objects (or unit types, in a type system) do not play an important role. What is more, they can create significant complication, illustrated by the remarkable fact that the provability problem for the unit-*only* fragment of multiplicative linear logic is NP-complete (Lincoln and Winkler, 1994). So it is reasonable to imagine that the unitless fragment of multiplicative linear logic will prove to be of practical importance.

The answer may appear at first glance to be trivial. After all it is well known, following Seely (1989), that the star-autonomous categories of Barr (1979) are the appropriate structures to model multiplicative linear logic, so surely one may simply describe some obvious notion of 'unitless star-autonomous category'? This superficially reasonable idea turns out to be too simple-minded to work. Consider the following proof:

$$\dfrac{\dfrac{}{p \vdash p}\,\mathrm{Ax} \qquad \dfrac{\dfrac{}{q \vdash q}\,\mathrm{Ax}}{\vdash q^{\perp} \mathbin{\wp} q}\,\perp\mathrm{R}}{p \vdash p \otimes (q^{\perp} \mathbin{\wp} q)}\,\otimes\mathrm{R}$$

Although this proof does not involve any units, it makes essential use of a sequent with an empty left-hand side, and so its interpretation in a star-autonomous category *does* necessarily involve the unit object. (A sequent of type $\vdash q^{\perp} \mathbin{\wp} q$ would be interpreted as an arrow $I \to Q \mathbin{\wp} Q^{\perp}$, where Q is the object that interprets the propositional variable q.)

So it is clear that the units cannot be dispensed with altogether: we need at least to have some way to interpret sequents that have an empty left- or right-hand side. Having thus isolated the difficulty, we begin by concentrating on the intuitionistic fragment. An intuitionistic sequent always has precisely one formula on the right, so only the left-hand side may be empty. Thus we need some analogue of 'arrow $I \to X$', without there actually being a unit object I. In other words, we need some functor $\mathbb{C} \to \mathrm{Set}$ (where $\mathbb{C}$ is the category in question) to play the role of the hom functor $\mathbb{C}(I, -)$. It turns out that structures of this sort have previously been studied, for a very different reason: Day (1970) considered what he at the time called 'premonoidal' categories, while studying monoidal structure on presheaf categories. Nowadays these structures are known as *promonoidal* instead,[1] which is the term we shall use.

A promonoidal category is something more general than a monoidal category: instead of having a tensor functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and a unit object $I \in \mathbb{C}$, it has a tensor profunctor $P : \mathbb{C} \times \mathbb{C} \longrightarrow\!\!\!\!\!\!+\;\, \mathbb{C}$ and a unit profunctor $J : 1 \longrightarrow\!\!\!\!\!\!+\;\, \mathbb{C}$. A profunctor $1 \longrightarrow\!\!\!\!\!\!+\;\, \mathbb{C}$ is precisely a functor $\mathbb{C} \to \mathrm{Set}$, which is what we are looking for. So we are interested in particular in the special case of those promonoidal categories whose tensor is an honest functor, but whose unit is a general profunctor.

[1] Indeed, the term *premonoidal* has since been reused to mean something altogether different (Power and Robinson, 1997).

1.2 The multicategory approach

It is worth briefly contemplating the road not taken (particularly since it looks more attractive at first, but turns out on examination to lead to the same destination). One might take the view that any sequent system – or at least any sequent system in which the cut rule is admissible, which is to say any sequent system that might reasonably be considered to describe a logic – may be interpreted in a multicategory. The presence of such rules as the left-tensor and left-unit rules of linear logic allows us to restrict our attention to representable multicategories (Hermida, 2000), which are essentially the same as monoidal categories. On this view, in order to model the unitless fragment we should be looking at multicategories in which only *non-empty* sequences of objects admit a representation. Such multicategories are *prima facie* more general than the promonoidal categories considered here, for the following reason. Although a multicategory does have, for example, a natural transformation of the form

$$\int^{A} \mathbb{C}(;A) \times \mathbb{C}(A,B;C) \to \mathbb{C}(B;C) \tag{1.2.1}$$

given by composition, this is not necessarily invertible (as it must be in a promonoidal category). Since there seems to be no intrinsic reason that we should demand invertibility here, the multicategory formulation looks like an improvement over the promonoidal one. But this is an illusion: in the cases that we are considering, there is also an implication connective, so we may suppose that for every object A there is a functor $A \multimap -$ and an isomorphism

$$\mathbb{C}(\vec{X}, A; B) \cong \mathbb{C}(\vec{X}; A \multimap B)$$

natural in the sequence $\vec{X}$ and the object B. This does force the transformation (1.2.1) to be invertible, since we now have a sequence of isomorphisms

$$\begin{aligned}\int^{A} \mathbb{C}(;A) \times \mathbb{C}(A,B;C) &\cong \int^{A} \mathbb{C}(;A) \times \mathbb{C}(A;B \multimap C)\\ &\cong \mathbb{C}(;B \multimap C)\\ &\cong \mathbb{C}(B;C).\end{aligned}$$

So we have arrived at the same destination by a different, and arguably more natural, route.

1.3 The study of pseudomonoids

Having established that we are engaged in the study of promonoidal categories, there is the immediate problem that not much has been written about them – certainly when compared with monoidal categories, which have been very well-studied. It seems clear that most of the known results about monoidal categories have analogies in the promonoidal setting, but it would be unaccountably tedious merely to 'translate' huge portions of the monoidal categories literature into the promonoidal setting. Better would be to find a general argument to the effect that such a translation is possible.

In fact there is nothing particularly special about promonoidal categories in an abstract sense. They are but one example of the general notion of *pseudomonoid* in a monoidal bicategory, and we expect (and shall prove) that much of what is known about monoidal categories in particular is actually true of pseudomonoids in general, when formulated appropriately. Furthermore, when considering structures internal to a monoidal bicategory there is nothing particularly special about pseudomonoids! The translation procedure can in fact be carried through for a substantial class of structures internal to a monoidal bicategory. So we arrive at a general translation result that has potential applications that go far beyond those we consider here.

The thesis may be approximately divided into three parts. The first part (Chapters 2 and 3) consists of background material: we review the basics of bicategories and monoidal bicategories, before going on to define pseudomonoids. The second part (Chapters 4–6) establishes the 'translation' mentioned above. Only in the third part (Chapters 8 and 9) do we finally return to the original question, and use all this machinery to study the models of unitless linear logic.

1.4 Prerequisites

We assume the basics of linear logic, category theory, and categorical proof theory. As far as linear logic and its categorical interpretation is concerned, these prerequisites are essentially contained in Girard (1987) and Seely (1989). Of category theory we assume a little more: say the contents of Mac Lane (1978). Some familiarity with the theory of profunctors (Lawvere, 1973; Bénabou, 2000) and promonoidal categories (Day, 1970) would be useful, but is not strictly assumed.

1.5 Other approaches

Others have recently considered the question of defining categorical models for the unitless fragment of multiplicative linear logic. A preprint of Lamarche and Straßburger (2005) gave a definition that, on examination, appeared weaker than the one developed in this work. Correspondence with the authors established that this difference was not intended, and the final version includes an additional axiom that makes the definition equivalent to ours. Došen and Petrić (2005) give a very different-looking definition just for the star-autonomous case, which is nevertheless again equivalent to ours.

Chapter 2

Bicategories

Monoidal and promonoidal categories are both instances of the general notion of a pseudomonoid in a monoidal bicategory. We shall need some results about promonoidal categories that are in fact generally true of pseudomonoids in any monoidal bicategory, and which we should therefore like to prove as such.

Unfortunately the literature on monoidal bicategories is still fairly sparse. The relevant definitions may be obtained by specialising to the one-object case those given for tricategories by Gordon et al. (1995), and this has been done explicitly in the unpublished dissertation of Carmody (1995). Also Day and Street (1997) and Baez and Neuchl (1996) have given explicit definitions for the important special case of Gray-categories. However there is no explicit, published account of the general notion.

Even the literature on plain bicategories is rather scanty, and although the situation is improving (see Lack, 2007, for example) many fundamental results have no published proof, and there is still a substantial gap between what is known to the experts and what has been written down. Neither is notation yet standardised. For these reasons, we give in this chapter a rapid but reasonably thorough account of the bicategory theory that we need, with the occasional digression.

2.1 Bicategories: basic definitions

Definition 2.1. A bicategory $\mathcal{B}$ consists of:

- a set[1] $|\mathcal{B}|$ of objects,
- for every pair A, B of objects, a hom-category $\mathcal{B}(A, B)$, whose objects are

[1] Possibly quite a large set: whereas most of ordinary category theory can be formalised in a "one universe" foundation, it's convenient to assume at least two Grothendieck universes for the purposes of bicategory theory. (We want to permit e.g. the bicategory of large categories.) We shall leave these considerations implicit, on the whole.

called *1-cells* or arrows, and whose morphisms are called *2-cells*

- for every object A, a selected 'identity' 1-cell $1_A \in \mathcal{B}(A,A)$,
- for every triple A,B,C of objects, a 'horizontal composition' functor

$$\circ : \mathcal{B}(B,C) \times \mathcal{B}(A,B) \to \mathcal{B}(A,C),$$

- for every pair A, B of objects, natural isomorphisms

$$\lambda : 1_B \circ - \Rightarrow - : \mathcal{B}(A,B) \to \mathcal{B}(A,B),$$
$$\rho : - \circ 1_A \Rightarrow - : \mathcal{B}(A,B) \to \mathcal{B}(A,B),$$

- for every four objects A,B,C,D, a natural isomorphism

$$\alpha : - \circ (- \circ -) \Rightarrow (- \circ -) \circ - : \mathcal{B}(C,D) \times \mathcal{B}(B,C) \times \mathcal{B}(A,B) \to \mathcal{B}(A,D)$$

subject to two coherence conditions: for all $f \in \mathcal{B}(A,B)$ and $h \in \mathcal{B}(B,C)$, the diagram

$$\begin{tikzcd}
h \circ (1_B \circ f) \arrow[rr, "\alpha_{h,1_B,f}"] \arrow[dr, "h \circ \lambda_f"'] & \arrow[d, phantom, "(\boldsymbol{\lambda\rho})"] & (h \circ 1_B) \circ f \arrow[dl, "\rho_h \circ f"] \\
& h \circ f, &
\end{tikzcd}$$

commutes in $\mathcal{B}(A,C)$, and for all $e \in \mathcal{B}(A,B)$, $f \in \mathcal{B}(B,C)$, $g \in \mathcal{B}(C,D)$, $h \in \mathcal{B}(D,E)$, the diagram

$$\begin{tikzcd}
h \circ \big(g \circ (f \circ e)\big) \arrow[r, "\alpha_{h,g,f\circ e}"] \arrow[d, "h \circ \alpha_{g,f,e}"'] & (h \circ g) \circ (f \circ e) \arrow[r, "\alpha_{h\circ g,f,e}"] \arrow[d, phantom, "(\boldsymbol{\alpha})"] & \big((h \circ g) \circ f\big) \circ e \\
h \circ \big((g \circ f) \circ e\big) \arrow[rr, "\alpha_{h,g\circ f,e}"'] & & \big(h \circ (g \circ f)\big) \circ e \arrow[u, "\alpha_{h,g,f} \circ e"']
\end{tikzcd}$$

commutes in $\mathcal{B}(A,E)$.

We shall often omit the subscript on identity 1-cells, writing just 1 rather than 1_A, when the object can be easily determined from the context. We also omit the 1-cell subscripts of α, λ and ρ from time to time.

Remark 2.2. A bicategory with just one object may be regarded as a monoidal category: the definition simply reduces to the usual definition of monoidal category, in that case. Furthermore, Mac Lane's coherence theorem for monoidal categories (Mac Lane, 1963) equally well applies to bicategories in general: the proof goes through essentially unchanged. Kelly (1964) shows – again for monoidal categories – that just two axioms, corresponding to our $(\boldsymbol{\alpha})$ and $(\boldsymbol{\lambda\rho})$, suffice for coherence; and that proof, too, applies equally well to bicategories in general.

This shows that, between any pair of functors built up from identities and composition, there is at most one natural transformation built from α, λ, ρ and their inverses. Therefore we shall not usually give names to the 'structural isomorphisms' α, λ, ρ, their inverses, and composites thereof. We shall instead use the symbol '$\cong$' as a generic label for a structural isomorphism. Since by coherence there is a unique such isomorphism of each type, this practice introduces no ambiguity.

Remark 2.3. There is another version of the coherence theorem, proven for the case of monoidal categories by Joyal and Street (1993, section 1), and explicitly for bicategories by Gurski (2006, chapter 2)[2]. Anticipating some definitions we have yet to make, this version says that the canonical functor from a free bicategory to the corresponding free 2-category is a biequivalence. This is a powerful result, which gives an honest justification for neglecting the structural isomorphisms in many circumstances.

Remark 2.4. We shall often describe 2-cells, and equations between them, using pasting diagrams. It's important to be clear about how such diagrams are to be interpreted, which we'll explain by reference to the following example. Let σ and τ be 2-cells that fit into the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\sigma"] & B \arrow[r, "g"] \arrow[d, "k"] \arrow[dr, phantom, "\swarrow\tau"] & C \arrow[d, "l"] \\
X \arrow[r, "m"'] & Y \arrow[r, "n"'] & Z
\end{tikzcd}$$

Firstly, notice that this diagram does not uniquely define a 2-cell. Instead it defines a *family* of 2-cells, one for each bracketing of the source and target edges, e.g. for

[2] Gurski's account is exceptionally thorough and largely self-contained. He does use the bicategorical Yoneda lemma without proof, and indeed that proof does not appear ever to have been published – presumably because the idea is straightforward, even if the details verge on overwhelming. We provide a proof as Prop. 2.22 below.

the bracketings $l \circ (g \circ f)$ and $(h \circ m) \circ n$, the pasting diagram describes the 2-cell

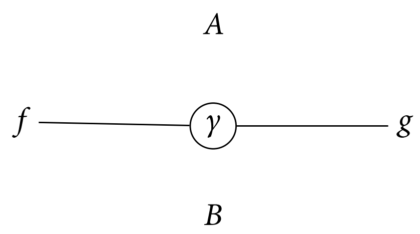
$$l \circ (g \circ f) \xrightarrow{\alpha_{l,g,f}} (l \circ g) \circ f \xrightarrow{\tau \circ f} (n \circ k) \circ f \xrightarrow{\alpha^{-1}_{n,k,f}} n \circ (k \circ f) \xrightarrow{n \circ \sigma} (h \circ m) \circ n.$$

This also demonstrates the second subtlety: the associativity $(n \circ k) \circ f \xrightarrow{\alpha^{-1}_{n,k,f}} n \circ (k \circ f)$ must be implicitly inserted between τ and σ.

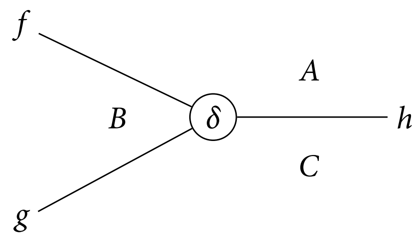
Observe also that an *equation* between pasting diagrams may be regarded as a family of equations, one for each bracketing of the source and target 1-cells. These equations are all equivalent, in the sense that each implies the others, so to prove such a family of equations it suffices to prove one of them.

These remarks are intended only to help the reader to understand what we mean when we draw a pasting diagram or an equation involving them. A rigorous treatment of bicategorical pasting is given by Verity (1992).

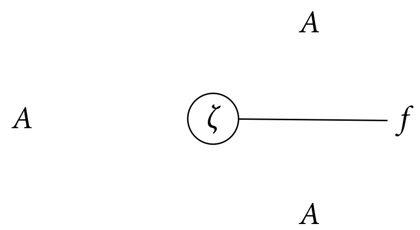
Remark 2.5. As well as pasting diagrams, one can also use *string diagrams* (Joyal and Street, 1991; Street, 1995) to represent and calculate with 2-cells in a bicategory. In a string diagram, an object is represented by a region of the plane, a 1-cell by a line, and a 2-cell by a node. For example, the 2-cell $\gamma : f \Rightarrow g : A \to B$ is drawn as



and a 2-cell $\delta : g \circ f \Rightarrow h$ would be drawn as



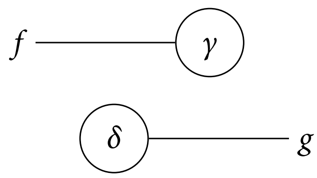
for $f : A \to B$, $g : B \to C$ and $h : A \to C$. Identity 1-cells are not drawn, so a 2-cell $\zeta : 1_A \Rightarrow f : A \to A$ is drawn



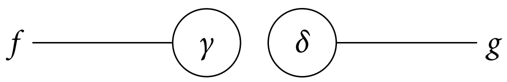
Composition of 2-cells corresponds to the pasting together of string diagrams along the appropriate edge: the orientation we have chosen for our diagrams has the unfortunate effect that horizontal composition is represented by vertical pasting, and vice versa. However it has the psychological advantage that the diagrams read from left to right.

The object labels may usually be omitted, since they can be inferred from the types of the 1-cells.

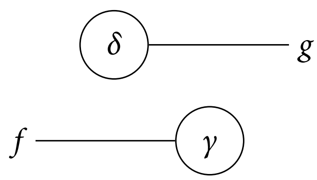
It is the case that geometric manipulations of string diagrams always correspond to allowable operations. For example, given 2-cells $\gamma : f \Rightarrow 1 : A \to A$ and $\delta : 1 \Rightarrow g : A \to A$, the diagram



can be deformed to



and then to



The corresponding sequence of pasting diagrams is

$$\begin{tikzcd}[column sep=large]
A \arrow[r, bend left, "f"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\gamma"] \arrow[rr, bend left=70, "f"] \arrow[rr, bend right=70, "g"'] \arrow[rr, phantom, bend left=40, "\cong"] \arrow[rr, phantom, bend right=40, "\cong"] & A \arrow[r, bend left, "1"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\delta"] & A
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
A \arrow[r, "1"] \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, bend left=35, "\Downarrow\gamma"] \arrow[r, phantom, bend right=35, "\Downarrow\delta"] & A
\end{tikzcd}$$

$$
\begin{tikzcd}
A \arrow[r, bend left, "1"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\delta"] \arrow[rr, bend left=80, "f"] \arrow[rr, bend right=80, "g"'] \arrow[rr, phantom, bend left=50, "\cong"] \arrow[rr, phantom, bend right=50, "\cong"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\gamma"] & A
\end{tikzcd}
$$

which is clearly much harder to follow. This example illustrates the way that string diagrams leave the unit constraints λ and ρ implicit, making powerful use of coherence. (Of course the string diagram formalism – in common with pasting diagrams – also leaves implicit the associator α, though that is not shown in this particular example. But the real power of string diagrams comes from the ease with which they handle the identities.)

Definition 2.6. For any bicategory $\mathcal{B}$ there is a bicategory $\mathcal{B}^{\mathrm{op}}$ obtained from $\mathcal{B}$ by reversing the direction of the 1-cells, so $\mathcal{B}^{\mathrm{op}}(A, B) = \mathcal{B}(B, A)$; and also a bicategory $\mathcal{B}^{\mathrm{co}}$ obtained by reversing the direction of the 2-cells, so $\mathcal{B}^{\mathrm{co}}(A, B) = \mathcal{B}(A, B)^{\mathrm{op}}$.

Definition 2.7. Bicategories $\mathcal{B}$ and $\mathcal{C}$ have a product formed in the obvious way: the set of objects is $|\mathcal{B}\times\mathcal{C}| = |\mathcal{B}|\times|\mathcal{C}|$, the hom-categories are $(\mathcal{B}\times\mathcal{C})(\langle A, B\rangle, \langle X, Y\rangle) = \mathcal{B}(A, X)\times\mathcal{C}(B, Y)$, and horizontal composition is defined pointwise. This also extends in the obvious way to the product of three of more bicategories.

Definition 2.8. Given bicategories $\mathcal{B}$ and $\mathcal{C}$, a *pseudo-functor* $F : \mathcal{B} \to \mathcal{C}$ consists of:

- for every object $A \in \mathcal{B}$, an object $FA \in \mathcal{C}$,
- for every pair A, B of objects of $\mathcal{B}$, a functor

$$F_{A,B} : \mathcal{B}(A, B) \to \mathcal{B}(FA, FB),$$

- for every $A \in \mathcal{B}$, an invertible 2-cell $F_A : 1_{FA} \Rightarrow F(1_A) : FA \to FA$,
- for every A and $B \in \mathcal{B}$, a natural isomorphism with components

$$F_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f),$$

such that for every $f : A \to B$ in $\mathcal{B}$, the diagrams below commute in the category

$\mathcal{C}(FA, FB)$,

$$\begin{tikzcd}
1_{FB} \circ F(f) \arrow[r, "F_B \circ F(f)"] \arrow[d, "\lambda_{F(f)}"'] & F(1_B) \circ F(f) \arrow[d, "F_{1_B,f}"] \\
F(f) \arrow[ur, phantom, "{[\boldsymbol{\lambda}]}"] & F(1_B \circ f) \arrow[l, "F(\lambda_f)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(f) \circ 1_{FA} \arrow[r, "F(f) \circ F_A"] \arrow[d, "\rho_{F(f)}"'] & F(f) \circ F(1_A) \arrow[d, "F_{f,1_A}"] \\
F(f) \arrow[ur, phantom, "{[\boldsymbol{\rho}]}"] & F(f \circ 1_A) \arrow[l, "F(\rho_f)"]
\end{tikzcd}$$

and for every $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathcal{B}$, the diagram

$$\begin{tikzcd}
F(h) \circ (F(g) \circ F(f)) \arrow[r, "\alpha_{Fh,Fg,Ff}"] \arrow[d, "F(h) \circ F_{g,f}"'] & (F(h) \circ F(g)) \circ F(f) \arrow[d, "F_{h,g} \circ F(f)"] \\
F(h) \circ F(g \circ f) \arrow[r, phantom, "{[\boldsymbol{\alpha}]}"] \arrow[d, "F_{h,(g \circ f)}"'] & F(h \circ g) \circ F(f) \arrow[d, "F_{(h \circ g),f}"] \\
F(h \circ (g \circ f)) \arrow[r, "F(\alpha_{h,g,f})"'] & F((h \circ g) \circ f)
\end{tikzcd}$$

commutes in the category $\mathcal{C}(FA, FD)$. A pseudo-functor is also known as a *homomorphism of bicategories*.

Remark 2.9. There is also a notion of *lax functor* between bicategories, defined as above, except that the 2-cells that appear in the definition need not be invertible. Lax functors have their uses – for example, a lax functor $1 \to \mathcal{B}$ is the same as a monad in $\mathcal{B}$ – but we have no need of them.

Definition 2.10. Given pseudo-functors $F, G : \mathcal{B} \to \mathcal{C}$, a *pseudo-natural transformation* $\gamma : F \Rightarrow G$ consists of:

- for every $A \in \mathcal{B}$, a 1-cell $\gamma_A : FA \to GA$,
- for every $f : A \to B$ in $\mathcal{B}$, an invertible 2-cell

$$\begin{tikzcd}
FA \arrow[r, "\gamma_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\gamma_B"'] \arrow[ur, phantom, "\nearrow \gamma_f"] & GB
\end{tikzcd}$$

such that this assignment is natural in f. Naturality amounts to asking that, for every 2-cell $\tau : f \Rightarrow g : A \to B$, we have

$$
\begin{tikzcd}
FA \arrow[r, "\gamma_A"] \arrow[d, "Ff"'] & GA \arrow[d, bend right, "Gf"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\overset{\Rightarrow}{G(\tau)}"] \\
FB \arrow[r, "\gamma_B"'] \arrow[ur, phantom, "\Nearrow \gamma_f"] & GB
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\gamma_A"] \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\overset{\Rightarrow}{F(\tau)}"] & GA \arrow[d, "Gg"] \\
FB \arrow[r, "\gamma_B"'] \arrow[ur, phantom, "\Nearrow \gamma_g"] & GB
\end{tikzcd}
$$

- These data must satisfy the *unit condition*: for every $A \in \mathcal{B}$ we have

$$
\begin{tikzcd}
FA \arrow[r, "\gamma_A"] \arrow[d, bend right, "1_{FA}"'] \arrow[d, bend left, "F(1_A)"] \arrow[d, phantom, "\overset{\Rightarrow}{F_A}"] & GA \arrow[d, "G(1_A)"] \\
FA \arrow[r, "\gamma_A"'] \arrow[ur, phantom, "\Nearrow \gamma_{1_A}"] & GA
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\gamma_A"] \arrow[d, "1_{FA}"'] & GA \arrow[d, bend right, "1_{GA}"'] \arrow[d, bend left, "G(1_A)"] \arrow[d, phantom, "\overset{\Rightarrow}{G_A}"] \\
FA \arrow[r, "\gamma_A"'] \arrow[ur, phantom, "\cong"] & GA
\end{tikzcd}
$$

- and the *composition condition*: for all $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{B}$, we have

$$
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[rr, bend right, "F(g \circ f)"'] \arrow[d, "\gamma_A"'] & FB \arrow[r, "Fg"] \arrow[d, phantom, "\Downarrow F_{g,f}"] & FC \arrow[d, "\gamma_C"] \\
GA \arrow[rr, "G(g \circ f)"'] & \Swarrow \gamma_{g \circ f} & GC
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[dd, "\gamma_A"'] \arrow[dr, phantom, "\Swarrow \gamma_f"] & FB \arrow[r, "Fg"] \arrow[d, "\gamma_B"] \arrow[dr, phantom, "\Swarrow \gamma_g"] & FC \arrow[dd, "\gamma_C"] \\
& GB \arrow[dr, "Gg"] \arrow[d, phantom, "\Downarrow G_{g,f}"] & \\
GA \arrow[ur, "Gf"] \arrow[rr, "G(g \circ f)"'] & & GC
\end{tikzcd}
$$

Remark 2.11. Of course there is such a thing as a lax natural transformation, where the 2-cells γ_f need not be invertible. (In Bénabou's original terminology, this would actually be an oplax transformation – his 2-cells point the other way – but the direction we use seems to be a more natural and useful choice: see Lack (2007, Section 5.7) for one piece of technical evidence for this assertion.) However, it should be noted that the collection of lax functors $\mathcal{B} \to \mathcal{C}$, lax transformations between them, and modifications (for which see below) between those does *not* form a bicategory, which shows that one must be careful what one laxifies. In any case, pseudo-functors and pseudo-natural transformations are all that we shall need.

Definition 2.12. Given pseudo-natural transformations $\gamma, \delta : F \Rightarrow G : \mathcal{B} \to \mathcal{C}$, a *modification* $m : \gamma \Rrightarrow \delta$ consists of: for every $A \in \mathcal{B}$, a 2-cell $m_A : \gamma_A \Rightarrow \delta_A$ such that for every $f : A \to B$ in $\mathcal{B}$, we have

$$
\begin{tikzcd}
FA \arrow[r, bend left, "\delta_A"] \arrow[r, bend right, "\gamma_A"'] \arrow[r, phantom, "\Uparrow m_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\nearrow \gamma_f"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\gamma_B"'] & GB
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "\delta_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\nearrow \delta_f"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, bend left, "\delta_B"] \arrow[r, bend right, "\gamma_B"'] \arrow[r, phantom, "\Uparrow m_B"] & GB
\end{tikzcd}
$$

Remark 2.13. Recall that, given a commutative diagram in an ordinary category, if one of the arrows is invertible, and is replaced by its inverse in such a way that the resulting diagram still has a single source and a single target object, then this resulting diagram also commutes. This fact has a two-dimensional analogue, as follows: a pasting equation may be viewed as a polyhedron, by gluing the two pasting diagrams together along their (common) boundary. If, in this polyhedron, one of the cells (faces) is invertible and is replaced by its inverse in such a way that there is still a unique way to decompose the resulting polyhedron as a pair of pasting diagrams, then this resulting pasting equation also holds. We shall sometimes use this implicitly, when it is more convenient to use such a variant of some equation.

The usual 2-categorical notions of adjunction, equivalence, monad, etc. may also be defined in a bicategory in the obvious way: a little of the theory of adjunctions is developed below. Since it will be useful almost immediately, we give here the definition of equivalence:

Definition 2.14. An *equivalence* from A to B in a bicategory consists of a pair of 1-cells $f : A \to B$ and $g : B \to A$, and a pair of invertible 2-cells $e : 1_A \Rightarrow g \circ f$ and $e' : 1_B \Rightarrow f \circ g$. We also say that the arrow f is an equivalence just when there exist g, e, e' as above.

In particular, every identity arrow 1_A is an equivalence, as is any 1-cell isomorphic to an identity. There is also an obvious notion of isomorphism, but it is not very useful in general. In particular, identity arrows in a bicategory are generally only equivalences and not isomorphisms, and an object of a bicategory need not even be isomorphic to itself.

Pseudo-natural transformations compose in the obvious way, as do modifications. Given any two bicategories $\mathcal{B}$ and $\mathcal{C}$, there is a *pseudo-functor bicategory*

Bicat$(\mathcal{B}, \mathcal{C})$ whose objects are pseudo-functors $\mathcal{B} \to \mathcal{C}$, whose 1-cells are pseudo-natural transformations and whose 2-cells are modifications.[3] We omit the routine verification that this is indeed a bicategory, but remark that coherence lifts from $\mathcal{C}$. It is significant that if $\mathcal{C}$ is in fact a 2-category (i.e. its associativity and unit isomorphisms are all identities) then **Bicat**$(\mathcal{B}, \mathcal{C})$ is also a 2-category.

A pseudo-functor $F : \mathcal{B} \to \mathcal{C}$ is said to be a *biequivalence* if it is a local equivalence and *biessentially surjective* – i.e. every object of $\mathcal{C}$ is equivalent to one of the form FA. If there is such a biequivalence then we say that $\mathcal{B}$ is *biequivalent* to $\mathcal{C}$. (There is a lot more that could be said about biequivalence, of course…)

For any bicategories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, there is a biequivalence

$$\mathbf{Bicat}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \approx \mathbf{Bicat}(\mathcal{A}, \mathbf{Bicat}(\mathcal{B}, \mathcal{C}))$$

defined in the natural way.

2.2 On the identities

It is a trivial observation that, say, a semigroup may have at most one unit: if i and j are both units then $ij = i$ – because j is a unit – and also $ij = j$ because i is a unit. So $i = j$. Thus the existence of a unit is a property of a semigroup, rather than being real additional structure. In higher dimensions, this phenomenon persists: for example, let A be an object of the bicategory $\mathcal{B}$, and let $1^\bullet : A \to A$ be equipped with natural isomorphisms $\lambda^\bullet$ and $\rho^\bullet$ making $1^\bullet$ act as an identity. Then there is an isomorphism

$$1^\bullet \xrightarrow{(\rho^\bullet_{1_A})^{-1}} 1_A \circ 1^\bullet \xrightarrow{\lambda_{1^\bullet}} 1_A,$$

so identities in a bicategory are unique up to isomorphism, which is as much as one could reasonably expect in the circumstances.

An important, if partially-submerged, theme of the present work is that the property-likeness of units has some interesting consequences. One consequence of this has been studied in detail by Kock (2006a); Joyal and Kock (2005); Kock (2006b); a different aspect is visible in the present work, especially in Section 8.3, where we find that a braided promonoidal category has a *canonical* unit, and identify a simple property that holds just when this unit exists.

In the immediate context, we can illustrate some of the phenomena as follows: the unit conditions in the definitions of pseudo-functor and pseudo-natural transformation are, in a sense, redundant. (This is not the case where lax functors or lax

[3] Note that this notation is inconsistent with that of Street (1980), which uses **Bicat**$(\mathcal{B}, \mathcal{C})$ to denote the bicategory whose objects are *lax* functors from $\mathcal{B}$ to $\mathcal{C}$.

natural transformations are concerned – the invertibility of our 2-cells is essential to the argument.) Although there is, to my knowledge, no written account available of the material of this section, in view of its elementary nature it is reasonable to suppose that it is known to experts in the field. (The expert whom I asked declined to comment on the question of how well-known these results are: make of that what you will.)

It will be convenient to introduce some temporary notation, that we use only in this section. Let $F : \mathcal{B} \to \mathcal{C}$ be a pseudo-functor. Then given a 1-cell $f : FA \to FB$ in $\mathcal{C}$, let us write ρ^F_f for the composite

$$f \circ F(1_A) \xrightarrow{f \circ F_A^{-1}} f \circ 1_{FA} \xrightarrow{\rho_f} f,$$

and λ^F_f for

$$F(1_A) \circ f \xrightarrow{F_A^{-1} \circ f} 1_{FA} \circ f \xrightarrow{\lambda_f} f.$$

Observe that ρ^F and λ^F inherit the coherence of ρ and λ, so that in particular the diagrams

$$\begin{tikzcd}
h \circ (k \circ F1) \arrow[rr, "\alpha"] \arrow[dr, "h \circ \rho^F_k"'] & & (h \circ k) \circ F1 \arrow[dl, "\rho^F_{h \circ k}"] \\
& h \circ k &
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
h \circ (F1 \circ k) \arrow[rr, "\alpha"] \arrow[dr, "h \circ \lambda^F_k"'] & & (h \circ F1) \circ k \arrow[dl, "\rho^F_h \circ k"] \\
& h \circ k &
\end{tikzcd}$$

commute for all suitably-typed 1-cells h and k.

Now we can demonstrate the promised redundancy. We shall start with the pseudo-natural transformations, since the situation there is very simple: the unit condition is quite redundant:

Proposition 2.15. *Given pseudo-functors $F, G : \mathcal{B} \to \mathcal{C}$ and the data of Definition 2.10, the composition condition implies the unit condition.*

Proof. Let γ be given as in Definition 2.10, and suppose it to satisfy the composition condition. Now, for every $f : A \to B$ in $\mathcal{B}$, we have the following diagram of 1-cells and 2-cells (with associativities left implicit):

$$\begin{tikzcd}
 & \gamma_B \circ Ff \circ F1 \arrow[r, "\gamma_f \circ F1"] \arrow[dl, "\gamma_B \circ F_{f,1}"'] \arrow[d, "\rho^F"] \arrow[dr, phantom, "\natural"] & Gf \circ \gamma_A \circ F1 \arrow[d, "\rho^F"'] \arrow[dr, "Gf \circ \gamma_1"] & \\
\gamma_B \circ F(f \circ 1) \arrow[r, "\gamma_B \circ F(\rho_f)"'] \arrow[dr, bend right, "\gamma_{f\circ 1}"'] \arrow[ur, phantom, "\gamma_B \circ [\rho]"'] & \gamma_B \circ Ff \arrow[r, "\gamma_f"'] \arrow[d, phantom, "\natural"] & Gf \circ \gamma_A \arrow[dr, phantom, "{[\rho] \circ \gamma_A}"] & Gf \circ G1 \circ \gamma_A \arrow[l, "\rho^G \circ \gamma_A"] \arrow[dll, bend left, "G_{f,1} \circ \gamma_A"] \\
 & G(f \circ 1) \circ \gamma_A \arrow[ur, "G(\rho_f) \circ \gamma_A"] & &
\end{tikzcd}$$

The marked cells commute for the reasons shown, and the outside commutes by the composition condition. Since $\gamma_f \circ F1$ is invertible, it follows that the unlabelled triangle commutes. By the observation above, about coherence of r^F and λ^F, this triangle is equivalent to

$$\begin{tikzcd}
Gf \circ \gamma_A \circ F1 \arrow[rr, "Gf \circ \gamma_1"] \arrow[dr, "Gf \circ \rho^{F\gamma_A}"'] & & Gf \circ G1 \circ \gamma_A \arrow[dl, "Gf \circ \lambda^F_{\gamma_A}"] \\
 & Gf \circ \gamma_A &
\end{tikzcd}$$

so, by letting $f = 1$, we conclude that

$$\begin{tikzcd}
\gamma_A \circ F1 \arrow[rr, "\gamma_1"] \arrow[dr, "\rho^F_{\gamma_A}"'] & & G1 \circ \gamma_A \arrow[dl, "\lambda^F_{\gamma_A}"] \\
 & \gamma_A &
\end{tikzcd}$$

commutes, which is equivalent to the unit condition. $\square$

For pseudo-functors, the situation is a little more subtle. The following notion will be useful, both here and later in Chapter 7.

Definition 2.16. A 1-cell $f : A \to B$ is (*representably*) *fully-faithful* if, for every object X, the functor $\mathcal{B}(X, f) : \mathcal{B}(X, A) \to \mathcal{B}(X, B)$ is fully faithful. Concretely, this means that, for every pair of arrows $h, k : X \to A$, every 2-cell

$$\begin{tikzcd}
 & A \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow"] & \\
X \arrow[ur, "h"] \arrow[dr, "k"'] & & B \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd} X \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\gamma"] & A \arrow[r, "f"] & B \end{tikzcd}$$

for some unique γ.

Dually, f is *co-fully-faithful* if, for every object X, the functor $\mathcal{B}(f, X) : \mathcal{B}(B, X) \to \mathcal{B}(A, X)$ is fully faithful.

Remark 2.17. Some remarks on the definition:

- It's easy to check that the fully-faithful 1-cells in Cat are precisely the full and faithful functors. On the other hand, the analogous property does not generally hold for enriched categories: the $\mathcal{V}$-fully-faithful functors do not always coincide with those 1-cells of $\mathcal{V}$-Cat that are representably fully-faithful.
- The co-fully-faithful 1-cells of Cat are characterised by Adámek et al. (2001), who call them 'lax epimorphisms'.
- Every equivalence is both fully-faithful and co-fully-faithful.

Lemma 2.18. *The conditions* $[\boldsymbol{\lambda}]$ *and* $[\boldsymbol{\rho}]$ *of Definition 2.8 are redundant, in the sense that each implies the other.*

Proof. Let $F : \mathcal{B} \to \mathcal{C}$ be a pseudo-functor, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be 1-cells in $\mathcal{B}$. Consider the following diagram in $\mathcal{C}(FA, FC)$:

$$\begin{tikzcd}
Fg \circ (F1 \circ Ff) \arrow[rr, "\alpha"] \arrow[dr, "Fg \circ \lambda^F"] \arrow[dd, "Fg \circ F_{1,f}"'] & & (Fg \circ F1) \circ Ff \arrow[dl, "\rho^F \circ Ff"'] \arrow[dd, "F_{g,1} \circ Ff"] \\
 & Fg \circ Ff \arrow[l, phantom, "Fg \circ [\boldsymbol{\lambda}]"] \arrow[r, phantom, "[\boldsymbol{\rho}] \circ Ff"] \arrow[dd, "F_{g,f}"] & \\
Fg \circ F(1 \circ f) \arrow[ur, "Fg \circ F(\lambda)"] \arrow[dd, "F_{g,1\circ f}"'] \arrow[r, phantom, "\natural"] & & F(g \circ 1) \circ Ff \arrow[ul, "F(\rho) \circ Ff"'] \arrow[dd, "F_{g\circ 1,f}"] \arrow[l, phantom, "\natural"] \\
 & F(g \circ f) & \\
F(g \circ (1 \circ f)) \arrow[ur, "F(g \circ \lambda)"] \arrow[rr, "F(\alpha)"'] & & F((g \circ 1) \circ f) \arrow[ul, "F(\rho \circ f)"']
\end{tikzcd}$$

The upper and lower triangles commute, as does the outside edge. Since all the 2-cells in the diagram are invertible, if $[\boldsymbol{\lambda}]$ holds then so does $[\boldsymbol{\rho}] \circ Ff$. Taking $f = 1$ and using the fact that $F1$ is co-fully-faithful, we conclude that $[\boldsymbol{\rho}]$ holds. In the other direction, if $[\boldsymbol{\rho}]$ holds then so does $Fg \circ [\boldsymbol{\lambda}]$, whence taking $g = 1$ and using the fact that $F1$ is fully-faithful, we conclude that $[\boldsymbol{\lambda}]$ holds. □

Lemma 2.19. *The invertible 2-cells $F_A : 1_{FA} \to F(1_A)$ of Definition 2.8 are uniquely determined by the other data.*

Proof. Let $F : \mathcal{B} \to \mathcal{C}$ be a pseudo-functor. Condition $[\boldsymbol{\lambda}]$ implies that, for every object $A \in \mathcal{B}$, the 2-cell $F_A \circ F(1_A)$ is equal to

$$1_{FA} \circ F(1_A) \xrightarrow{\lambda_{F(f)}} F(1_A) \xrightarrow{F(\lambda^{-1}_{1_A})} F(1_A \circ 1_A) \xrightarrow{F^{-1}_{1_A,1_A}} F(1_A) \circ F(1_A)$$

(just by taking $f = 1_A$). Since $F(1_A)$ is fully-faithful, this equation uniquely determines F_A. □

Proposition 2.20. *Let $F : \mathcal{B} \to \mathcal{C}$ be as in the definition of pseudo-functor, though without the 2-cells F_A. These data may be augmented to give a pseudo-functor F, in a unique way, if and only if $F(1_A)$ is both fully-faithful and co-fully-faithful for each object $A \in \mathcal{B}$, if and only if $F(1_A)$ is* either *fully-faithful* or *co-fully-faithful for each object $A \in \mathcal{B}$.*

Proof. If we have invertible 2-cells F_A, then each $F(1_A)$ is isomorphic to the identity, hence an equivalence, so in particular is fully-faithful and co-fully-faithful. For the converse, take an object $A \in \mathcal{B}$, and suppose that $F(1_A)$ is co-fully-faithful. Let $F_A : 1_{FA} \Rightarrow F(1_A)$ be the unique 2-cell for which $F_A \circ F(1_A)$ is equal to the composite

$$1_{FA} \circ F(1_A) \xrightarrow{\lambda} F(1_A) \xrightarrow{F(\lambda^{-1})} F(1_A \circ 1_A) \xrightarrow{F^{-1}_{1_A,1_A}} F(1_A) \circ F(1_A).$$

Since this F_A is invertible, $F(1_A)$ is isomorphic to the identity, hence also fully-faithful. For any $f : A \to B$, we have the following diagram in $\mathcal{B}(FA, FB)$ (with associativities left implicit):

$$\begin{tikzcd}
Ff \circ 1 \circ F1 \arrow[r, "Ff \circ \lambda"] & Ff \circ F1 \arrow[r, "Ff \circ F(\lambda^{-1})"] \arrow[ddr, bend right, "F_{f,1}"'] \arrow[dr, phantom, "\natural"] & Ff \circ F(1 \circ 1) \arrow[r, "Ff \circ F^{-1}_{1,1}"] \arrow[d, "F_{f,f\circ 1}"'] \arrow[dr, phantom, "{[\boldsymbol{\alpha}]}"] & Ff \circ F1 \circ F1 \arrow[d, "F_{f,1} \circ F1"] \\
& & F(f \circ 1 \circ 1) \arrow[r, "F^{-1}_{f\circ 1,1}"'] \arrow[d, "{F(f \circ \lambda) = F(\rho \circ 1)}"'] \arrow[dr, phantom, "\natural"] & F(f \circ 1) \circ F1 \arrow[d, "F(\rho) \circ F1"] \\
& & F(f \circ 1) \arrow[r, "F^{-1}_{f,1}"'] & Ff \circ F1
\end{tikzcd}$$

The top row is equal to $Ff \circ F_A \circ F1$ by definition, hence this diagram is equivalent to

$$\begin{tikzcd}
Ff \circ 1 \circ F1 \arrow[r, "Ff \circ F_A \circ F1"] \arrow[d, "Ff \circ \lambda_{F1}"'] & Ff \circ F1 \circ F1 \arrow[d, "F_{f,1} \circ F1"] \\
Ff \circ F1 & F(f \circ 1) \circ F1 \arrow[l, "F(\rho_f) \circ F1"]
\end{tikzcd}$$

and since $F1$ is co-fully-faithful, it follows that $[\boldsymbol{\lambda}]$ holds. By Lemma 2.18, it follows that condition $[\boldsymbol{\rho}]$ holds too.

If we start instead with the assumption that $F(1_A)$ is fully-faithful, the dual argument applies. Finally, the uniqueness is a consequence of Lemma 2.19. $\square$

2.3 The bicategorical Yoneda lemma

The Yoneda lemma for bicategories was first stated by Street (1980), in a long paper that states many basic results without proof. Since the proof, even if it is in some sense routine, is rather intricate, we give a detailed account here.

Definition 2.21. Let A be an object of the bicategory $\mathcal{B}$. The *covariant representable pseudo-functor determined by* A,

$$\mathcal{B}(A,-) : \mathcal{B} \to \mathrm{Cat},$$

is defined as follows. On an object $X \in \mathcal{B}$, the category $\mathcal{B}(A,X)$ is just the the hom-category of the same name. The action

$$\mathcal{B}(A,-)_{X,Y} : \mathcal{B}(X,Y) \to [\mathcal{B}(A,X), \mathcal{B}(A,Y)]$$

is defined to be the currying of the composition functor

$$\circ : \mathcal{B}(X,Y) \times \mathcal{B}(A,X) \to \mathcal{B}(A,Y).$$

For an object $X \in \mathcal{B}$, the unit isomorphism $\mathcal{B}(A,-)_X : 1_{\mathcal{B}(A,X)} \Rightarrow 1_X \circ -$ is defined to be λ^{-1}, and for a composable pair

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{B}$, the isomorphism $\mathcal{B}(A,-)_{g,f} : \mathcal{B}(A,g) \circ \mathcal{B}(A,f) \Rightarrow \mathcal{B}(A, g \circ f)$ is defined to be $\alpha_{g,f,-}$. I.e. for $x \in \mathcal{B}(A,X)$, the component $(\mathcal{B}(A,-)_{g,f})_x : g \circ (f \circ x) \to (g \circ f) \circ x$ is just $\alpha_{g,f,x}$.

By duality there is also a *contravariant representable pseudo-functor*, $\mathcal{B}(-,A) = \mathcal{B}^{\mathrm{op}}(A,-) : \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$.

Proposition 2.22. *For any bicategory $\mathcal{B}$, pseudo-functor $F : \mathcal{B} \to \mathrm{Cat}$ and object $A \in \mathcal{B}$, there is an equivalence of categories*

$$\psi : FA \simeq \mathbf{Bicat}(\mathcal{B}, \mathrm{Cat})(\mathcal{B}(A,-), F).$$

Proof. Fix F and A. We shall define a functor

$$\psi : FA \to \mathbf{Bicat}(\mathcal{B}, \mathrm{Cat})(\mathcal{B}(A,-), F),$$

and show that it is an equivalence. (This gets rather dizzying, so take a deep breath.) First we shall define ψ on objects: for any $a \in FA$, we need a pseudo-natural transformation $\psi(a) : \mathcal{B}(A,-) \Rightarrow F$. Thus for every object $X \in \mathcal{B}$, we must give a functor

$$\psi(a)_X : \mathcal{B}(A,X) \to FX.$$

This functor is defined as follows. On objects: for $f \in \mathcal{B}(A,X)$, let $\psi(a)_X(f) =$

$F(f)(a)$. On morphisms: for $\beta : f \Rightarrow g : A \to X$, let $\psi(a)_X(\beta) = F(\beta)_a$. (Note that $F(\beta)$ is a natural transformation $F(f) \Rightarrow F(g)$, whose component $F(\beta)_a$ is therefore indeed a map $F(f)(a) \to F(g)(a)$.)

Also, for every 1-cell $k : X \to Y$ in $\mathcal{B}$, we must give a natural isomorphism

$$\psi(a)_k : \psi(a)_Y \cdot \mathcal{B}(A, k) \Rightarrow F(k) \cdot \psi(a)_X.$$

For $x \in \mathcal{B}(A, X)$, we define the component

$$(\psi(a)_k)_x : F(k \circ x)(a) \to F(k)(F(x)(a))$$

to be $(F^{-1}_{k,x})_a$. The naturality of $F_{k,x}$ ensures that $\psi(a)_k$ is natural.

This completes the definition of ψ on objects, though we must check that $\psi(a)$ is indeed a pseudo-natural transformation. That follows from the pseudo-functoriality of F, as the reader may verify: for $\psi(a)$ to satisfy the unit condition is precisely for F to satisfy condition $[\boldsymbol{\lambda}]$, and for $\psi(a)$ to satisfy the composition condition is precisely for F to satisfy $[\boldsymbol{\alpha}]$.

Next we define ψ on morphisms. For each morphism $h : a \to b$ in FA, we must define a modification $\psi(h) : \psi(a) \Rrightarrow \psi(b)$, so for each $X \in \mathcal{B}$ we need a natural transformation $\psi(h)_X : \psi(a)_X \Rightarrow \psi(b)_X$, which means that for every $f : A \to X$ in $\mathcal{B}$ we require a map $(\psi(h)_X)_f : F(f)(a) \to F(f)(b)$ in the category FX. So we define $(\psi(h)_X)_f$ to be the map $F(f)(h)$. It's easy to check that this makes $\psi(h)_X$ into a natural transformation. To complete the definition of ψ, we must confirm that $\psi(h)$ is indeed a modification. This amounts to checking that, for every $f : A \to X$ and $k : X \to Y$ in $\mathcal{B}$, and every $h : a \to b$ in FA, the square

$$\begin{tikzcd}
F(kf)(a) \arrow[r, "(F^{-1}_{k,f})_a"] \arrow[d, "F(kf)(h)"'] & (Fk \cdot Ff)(a) \arrow[d, "(Fk \cdot Ff)(h)"] \\
F(kf)(b) \arrow[r, "(F^{-1}_{k,f})_b"'] & (Fk \cdot Ff)(b)
\end{tikzcd}$$

commutes, which is of course precisely the naturality of $F^{-1}_{k,f}$.

We have defined ψ, and need to check that it is indeed a functor. Consider the modification $\psi(1_a) : \psi(a) \Rrightarrow \psi(a)$: for $f : A \to X$ we have $(\psi(h)_X)_f = F(f)(1_a)$, and since $F(f)$ is a functor, this is equal to $1_{F(f)(a)}$ as required. For composition, a similar argument applies: given arrows $h : a \to b$ and $j : b \to c$ in FA, we have $(\psi(jh)_X)_f = F(f)(jh)$; and since $F(f)$ is a functor, this is equal to $F(f)(j) \cdot F(f)(h)$ as required.

It remains to show that ψ is an equivalence. We begin by exhibiting a local inverse, showing that ψ is full and faithful. Fix objects a and $b \in FA$. We shall define a function $\psi^{-1}_{a,b}$ from the set of modifications $\psi(a) \Rrightarrow \psi(b)$ to the set $FA(a, b)$, and show that it is inverse to $\psi_{a,b}$. The definition is as follows. For a modification $\mu : \psi(a) \Rrightarrow \psi(b)$, let $\psi^{-1}_{a,b}(\mu)$ be the composite

$$a = 1_{FA}(a) \xrightarrow{(F_A)_a} F(1_A)(a) \xrightarrow{(\mu_A)_{1_A}} F(1_A)(b) \xrightarrow{(F_A^{-1})_b} 1_{FA}(b) = b.$$

It is easy to check that, for any $h : a \to b$, we have $\psi^{-1}_{a,b}(\psi(h)) = h$: indeed it is immediate from the definition of ψ, and the naturality of F_A. The other direction is more interesting. Fix some $\mu : \psi(a) \Rrightarrow \psi(b)$, and take $X \in \mathcal{B}$ and $f : A \to X$. We wish to show that $(\psi(\psi^{-1}_{a,b}(\mu))_X)_f$ is equal to $(\mu_X)_f$. Consider the diagram

$$\begin{tikzcd}
& (Ff.1_{FA})(a) \arrow[dr, "Ff((F_A)_a)"] \arrow[d, phantom, "{[\rho]}"] & \\
F(f)(a) \arrow[ur, "="] \arrow[dr, "F(\rho_f^{-1})(a)"'] \arrow[ddd, "(\mu_X)_f"'] \arrow[ddr, phantom, "\natural_{\mu_X}"] & {} & (Ff.F1_A)(a) \arrow[ddd, "(Ff)((\mu_A)_{1_A})"] \arrow[ddl, phantom, "\S_\mu"] \\
& F(f.1_A)(a) \arrow[ur, "(F_{f,1_A})_a"'] \arrow[d, "(\mu_X)_{f.1_A}"] & \\
& F(f.1_A)(b) \arrow[dl, "F(\rho_f)(b)"'] \arrow[d, phantom, "{[\rho]}"] & \\
F(f)(b) & {} & (Ff.F1_A)(b) \arrow[ul, "(F_{f,1_A})_b"'] \arrow[dl, "Ff((F_A^{-1})_b)"] \\
& (Ff.1_{FA})(b) \arrow[ul, "="] &
\end{tikzcd}$$

whose regions commute for the reasons marked: $\natural_{\mu_X}$ means that the square commutes because μ_X is natural, and $\S_\mu$ means that the square commutes because μ is a modification.

The composite around the upper, right, and bottom edges is equal to $(\psi(\psi^{-1}_{a,b}(\mu))_X)_f$ by definition, which is therefore equal to $(\mu_X)_f$ as required. Thus ψ is indeed full and faithful. It remains only to show that ψ is essentially surjective on objects. Consider an arbitrary pseudo-natural transformation $\gamma : \mathcal{B}(A, -) \Rightarrow F$. We intend to show that $\psi(\gamma_A(1_A))$ is isomorphic to γ. For any $X \in \mathcal{B}$ and $f : A \to X$, we have an invertible 2-cell

$$\begin{tikzcd}
\mathcal{B}(A,A) \arrow[r, "\gamma_A"] \arrow[d, "\mathcal{B}(A,f)"'] \arrow[dr, phantom, "\nearrow \gamma_f"] & FA \arrow[d, "Ff"] \\
\mathcal{B}(A,X) \arrow[r, "\gamma_X"'] & FX
\end{tikzcd}$$

thus an isomorphism

$$\gamma_X(f) \xrightarrow{\gamma_X(\rho_f^{-1})} \gamma_X(f.1_A) \xrightarrow{(\gamma_f)_{1_A}} F(f)(\gamma_A(1_A)) = \psi(\gamma_A(1_A))_X(f).$$

This isomorphism is natural in f, since ρ_f and γ_f both are. So, for every $X \in \mathcal{B}$ we have defined a natural isomorphism $\gamma_X \Rightarrow \psi(\gamma_A(1_A))_X$. Finally it remains to check that this collection constitutes a modification. Take $k : X \to Y$: we need to check the commutativity of the diagram

$$\begin{tikzcd}
\gamma_Y(kf) \arrow[rr, "(\gamma_k)_f"] \arrow[d, "\gamma_Y(\rho_{kf}^{-1})"'] & & F(k)(\gamma_X(f)) \arrow[d, "Fk(\gamma_X(\rho_f^{-1}))"] \\
\gamma_Y((kf)1) \arrow[d, "(\gamma_{kf})_1"'] & & Fk(\gamma_X(f1)) \arrow[d, "Fk((\gamma_f)_1)"] \\
F(kf)(\gamma_A(1)) \arrow[rr, "(F_{k,f}^{-1})_{\gamma_A(1)}"'] & & (Fk \cdot Ff)(\gamma_A(1))
\end{tikzcd} \tag{2.3.1}$$

Since γ is pseudo-natural, we know that

$$\begin{tikzcd}
\mathcal{B}(A,A) \arrow[r, "\gamma_A"] \arrow[d, "\mathcal{B}(A,f)"'] \arrow[dr, phantom, "\nearrow \gamma_f"] & FA \arrow[d, "Ff"] \arrow[dd, bend left=60, "F(kf)"] \\
\mathcal{B}(A,X) \arrow[r, "\gamma_X"'] \arrow[d, "\mathcal{B}(A,k)"'] \arrow[dr, phantom, "\nearrow \gamma_k"] & FX \arrow[d, "Fk"] \\
\mathcal{B}(A,Y) \arrow[r, "\gamma_Y"'] & FY
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{B}(A,A) \arrow[r, "\gamma_A"] \arrow[dl, "\mathcal{B}(A,f)"'] \arrow[dd, "\mathcal{B}(A,kf)"] \arrow[ddr, phantom, "\nearrow \gamma_{kf}"] & FA \arrow[dd, "F(kf)"] \\
\mathcal{B}(A,X) \arrow[dr, "\mathcal{B}(A,k)"'] \arrow[r, phantom, "\Rightarrow \alpha_{k,f,-}"] & {} & \\
& \mathcal{B}(A,Y) \arrow[r, "\gamma_Y"'] & FY
\end{tikzcd}$$

hence in particular that the diagram

$$
\begin{tikzcd}
\gamma_Y(k(f1)) \arrow[r, "(\gamma_k)_{f1}"] \arrow[dd, "\gamma_Y(k)"'] & Fk(\gamma_X(f1)) \arrow[d, "Fk((\gamma_f)_1)"] \\
 & (Fk \cdot Ff)(\gamma_A(1)) \arrow[d, "(F_{k,f})_{\gamma_A(1)}"] \\
\gamma_Y((kf)1) \arrow[r, "(\gamma_{kf})_1"'] & F(kf)(\gamma_A(1))
\end{tikzcd}
\tag{2.3.2}
$$

commutes. Now we have

$$
\begin{tikzcd}
\gamma_Y(kf) \arrow[rr, "(\gamma_k)_f"] \arrow[d, "\gamma_Y(\rho^{-1}_{kf})"'] \arrow[dr, "\gamma_Y(k\rho^{-1}_f)"] & & F(k)(\gamma_X(f)) \arrow[d, "Fk(\gamma_X(\rho^{-1}_f))"] \\
\gamma_Y((kf)1) \arrow[r, "\gamma_Y(\alpha^{-1}_{k,f,1})"'] \arrow[d, "(\gamma_{kf})_1"'] & \gamma_Y(k(f1)) \arrow[r, "(\gamma_k)_{f1}"'] & Fk(\gamma_X(f1)) \arrow[d, "Fk((\gamma_f)_1)"] \\
F(kf)(\gamma_A(1)) \arrow[rr, "(F^{-1}_{k,f})_{\gamma_A(1)}"'] & & (Fk \cdot Ff)(\gamma_A(1))
\end{tikzcd}
$$

where the triangle commutes by coherence, the upper-right quadrilateral by naturality of γ_k and the lower region by (2.3.2). Thus diagram (2.3.1) does indeed commute, and we are done. □

Definition 2.23. For any bicategory $\mathcal{B}$, the *bicategorical Yoneda embedding*

$$Y : \mathcal{B}^{\mathrm{op}} \to \mathbf{Bicat}(\mathcal{B}, \mathrm{Cat})$$

is defined as follows. On objects $A \in \mathcal{B}$, we define $YA = \mathcal{B}(A, -)$; and on hom-categories $\mathcal{B}(A, B)$ we define the component

$$Y_{B,A} : \mathcal{B}^{\mathrm{op}}(A, B) = \mathcal{B}(B, A) \to \mathbf{Bicat}(\mathcal{B}, \mathrm{Cat})(\mathcal{B}(A, -), \mathcal{B}(B, -))$$

to be $\psi_A^{\mathcal{B}(B,-)}$.

Corollary 2.24. *The Yoneda embedding is locally an equivalence.*

Proof. Immediate from the definition, by Prop. 2.22. ☐

Remark 2.25. Since Cat is a 2-category, so is $\mathbf{Bicat}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat})$. And since the Yoneda embedding Y is locally an equivalence, any bicategory $\mathcal{B}$ is biequivalent to the full sub-bicategory of $\mathbf{Bicat}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat})$ determined by the objects YA for $A \in \mathcal{B}$, which is of course still a 2-category. Thus any bicategory is biequivalent to a 2-category. This is a simple coherence result, which can serve as a stepping-stone to more sophisticated coherence theorems (Gurski, 2006, Chapter 2).

2.4 Adjunctions

Definition 2.26. An *adjunction* in a bicategory $\mathcal{B}$ consists of 1-cells $f : A \to B$ and $g : B \to A$, and 2-cells $\eta : 1_A \Rightarrow gf$ and $\varepsilon : fg \Rightarrow 1_B$ with the property that

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\overset{\Rightarrow}{\eta}", near start] & B \arrow[d, "1_B"] \arrow[dl, "g" description] \\
A \arrow[r, "f"'] & B \arrow[ul, phantom, "\overset{\Rightarrow}{\varepsilon}", near start]
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "1_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "1_B"'] \arrow[dr, phantom, "\overset{\Leftarrow}{\varepsilon}", near start] & A \arrow[d, "1_A"] \arrow[dl, "f" description] \\
B \arrow[r, "g"'] & A \arrow[ul, phantom, "\overset{\Leftarrow}{\eta}", near start]
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "1_B"'] \arrow[dr, phantom, "\cong"] & A \arrow[d, "1_A"] \\
B \arrow[r, "g"'] & A
\end{tikzcd}
$$

We write $f \dashv g$ to indicate that there is such an adjunction, and say that f is *left adjoint* to g, and g is *right adjoint* to f. We also write $f \dashv g : A \to B$ to mean that $f \dashv g$ for $f : A \to B$ and $g : B \to A$.

Remark 2.27. There is an *identity adjunction* on every object A, which is just $1_A \dashv 1_A$, with the unit and counit being structural isomorphisms.

Also, adjunctions may be composed: given adjunctions $f \dashv g : A \to B$ and

$f' \dashv g' : B \to C$, there is a composite adjunction $f' \circ f \dashv g \circ g'$ with unit

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{\Rightarrow}{\eta}" near start] & B \arrow[r, "f'"] \arrow[dl, "g"] \arrow[d, "1"] \arrow[dr, phantom, "\overset{\Rightarrow}{\eta}" near start] & C \arrow[dl, "g'"] \\
A \arrow[ur, phantom, "\cong" near start] & B \arrow[l, "g"]
\end{tikzcd}
\quad \text{and counit} \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, phantom, "\overset{\Leftarrow}{\varepsilon}" near start] & B \arrow[r, "f'"] \arrow[d, "1"] \arrow[dr, phantom, "\overset{\Leftarrow}{\varepsilon'}" near start] & C \arrow[d, "1"] \\
 & B \arrow[ul, "g"] \arrow[ur, phantom, "\cong" near start] & B \arrow[ul, "g'"] \arrow[l, "g'"]
\end{tikzcd}$$

Indeed, every bicategory $\mathcal{B}$ has a *bicategory of adjunctions*, whose objects are the objects of $\mathcal{B}$ and whose 1-cells are adjunctions.

Remark 2.28. Pseudofunctors preserve adjunctions, in the following sense. If $F : \mathcal{B} \to \mathcal{C}$ is a pseudo-functor, and $f \dashv g : A \to B$ is an adjunction in $\mathcal{B}$, then there is an adjunction $Ff \dashv Fg$ in $\mathcal{C}$ with the following unit and counit:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "F1"] \arrow[d, bend right=60, "1"'] \arrow[d, phantom, "\overset{\Rightarrow}{F_A}", shift right=6ex] \arrow[dr, phantom, "\overset{\Rightarrow}{F\eta}" near start] & FB \arrow[dl, "Fg"] \\
FA &
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[dr, "Ff"'] & FB \arrow[l, "Fg"'] \arrow[d, "F1"'] \arrow[d, bend left=60, "1"] \arrow[d, phantom, "\overset{\Rightarrow}{F_B^{-1}}", shift left=6ex] \arrow[dl, phantom, "\overset{\Rightarrow}{F\varepsilon}" near start] \\
 & FB
\end{tikzcd}$$

2.4.1 Mates

The theory of mates (Kelly and Street, 1974, §2) is a useful tool for dealing with adjunctions in a bicategory. We shall give a brief overview here.

Definition 2.29. Let $f \dashv g : A \to B$ and $f' \dashv g' : A' \to B'$. Given a 2-cell

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\Nearrow \sigma"] & B \arrow[d, "k"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

we may form a 2-cell

$$\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow \tau"] & B \arrow[l, "g"'] \arrow[d, "k"] \\
A' & B' \arrow[l, "g'"]
\end{tikzcd}$$

as the pasting

$$\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, "f"] \arrow[dd, bend right=60, "h"'] & B \arrow[l, "g"'] \arrow[d, "1"] \arrow[dd, bend left=60, "k"] \arrow[dl, phantom, "\Rightarrow \varepsilon"] \\
A' \arrow[d, "1"'] \arrow[dr, "f'"] \arrow[r, phantom, "\Rightarrow \sigma"] & B \arrow[d, "k"] \\
A' \arrow[ur, phantom, "\Rightarrow \eta'"] & B'. \arrow[l, "g'"]
\end{tikzcd}$$

We say that τ is the *right mate* of σ. Conversely, given a 2-cell τ as above, we may form its *left mate* as

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] \arrow[dd, bend right=60, "h"'] \arrow[dr, phantom, "\Rightarrow \eta"] & B \arrow[d, "k"] \arrow[dl, "g"] \arrow[dd, bend left=60, "k"] \\
A \arrow[d, "h"'] \arrow[r, phantom, "\Rightarrow \tau"] & B' \arrow[d, "1"] \arrow[dl, "g'"'] \\
A' \arrow[r, "f'"'] & B' \arrow[ul, phantom, "\Rightarrow \varepsilon'"]
\end{tikzcd}$$

Proposition 2.30. *The 'left mate' and 'right mate' operations are mutually inverse, i.e. σ is the left mate of τ if and only if τ is the right mate of σ. In this case we say that σ* and τ are mates *(with respect to the adjunctions $f \dashv g$ and $f' \dashv g'$).*

Proof. This follows easily from the definition of adjunction and the coherence of structural isomorphisms. □

Matehood can be characterised in terms of either the units or counits of the adjunctions.

Proposition 2.31. *The 2-cells σ and τ are mates if and only if*

$$
\begin{tikzcd}
A \arrow[d, "h"] \arrow[dr, "f"] \arrow[dd, bend right=60, "h"'] \arrow[d, phantom, shift right=6, "\cong"] & \\
A' \arrow[d, "1"'] \arrow[dr, "f'"] \arrow[r, phantom, "\Rightarrow \sigma"] & B \arrow[d, "k"] \\
A' \arrow[r, phantom, shift left=3, "\Rightarrow \eta'"] & B' \arrow[l, "g'"]
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] \arrow[dd, bend right=60, "h"'] \arrow[d, phantom, shift right=6, "\cong"] \arrow[r, phantom, shift right=3, "\Rightarrow \eta"] & B \arrow[d, "k"] \arrow[dl, "g"] \\
A \arrow[d, "h"] \arrow[r, phantom, "\Rightarrow \tau"] & B' \arrow[dl, "g'"] \\
A' &
\end{tikzcd}
\tag{2.4.1}
$$

if and only if

$$
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, "f"'] \arrow[r, phantom, shift right=3, "\Rightarrow \varepsilon"] & B \arrow[l, "g"'] \arrow[d, "1"] \arrow[dd, bend left=60, "k"] \arrow[d, phantom, shift left=6, "\cong"] \\
A' \arrow[dr, "f"'] \arrow[r, phantom, "\Rightarrow \sigma"] & B \arrow[d, "k"] \\
& B'
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[dl, "g"'] \arrow[d, "k"] \arrow[dd, bend left=60, "k"] \arrow[d, phantom, shift left=6, "\cong"] \\
A \arrow[d, "h"'] \arrow[r, phantom, "\Rightarrow \tau"] & B' \arrow[dl, "g'"'] \arrow[d, "1"] \\
A' \arrow[r, "f'"'] \arrow[r, phantom, shift left=3, "\Rightarrow \varepsilon'"] & B'
\end{tikzcd}
\tag{2.4.2}
$$

Proof. By definition we have

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow \sigma"] & B \arrow[d, "k"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] \arrow[dd, bend right=60, "\underline{h}"'] \arrow[d, phantom, shift right=6, "\cong"] \arrow[r, phantom, shift right=3, "\Rightarrow \eta"] & B \arrow[d, "k"] \arrow[dl, "g"] \arrow[dd, bend left=60, "k"] \arrow[d, phantom, shift left=6, "\cong"] \\
A \arrow[d, "h"'] \arrow[r, phantom, "\Rightarrow \tau"] & B' \arrow[dl, "g'"'] \arrow[d, "1"] \\
A' \arrow[r, "f'"'] \arrow[r, phantom, shift left=3, "\Rightarrow \varepsilon'"] & B'
\end{tikzcd}
$$

Onto both sides, we paste the 2-cell

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[dr, "h"'] \arrow[r, phantom, shift right=3ex, "\cong"] & A' \arrow[d, "1"] \arrow[dr, "f'"] \arrow[dr, phantom, shift left=1ex, "\underset{\eta'}{\Rightarrow}"] & \\
& A' & B' \arrow[l, "g'"]
\end{tikzcd}$$

along the edge $A \xrightarrow{h} A' \xrightarrow{f'} B'$, then use coherence (on the right) to deduce (2.4.1).

Similarly we can deduce (2.4.2) by taking the equation displayed above and pasting the 2-cell

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[r, phantom, shift right=3ex, "\underset{\eta}{\Rightarrow}"] & B \arrow[d, "1"] \arrow[dr, "k"] \arrow[dr, phantom, shift right=1ex, "\cong"] & \\
& B \arrow[ul, "g"] \arrow[r, "k"'] & B'
\end{tikzcd}$$

onto both sides along the edge $A \xrightarrow{f} B \xrightarrow{k} B'$. □

Next we list some useful elementary properties of mates.

Proposition 2.32. *Mates have the following properties:*

1. *Mating is natural in h and k, i.e. if σ and τ are mates then so are*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"] \arrow[d, bend right=80, "h'"'] \arrow[d, phantom, shift right=4ex, "\underset{\gamma}{\Rightarrow}"] \arrow[dr, phantom, "\nearrow\sigma"] & B \arrow[d, "k"'] \arrow[d, bend left=80, "k'"] \arrow[d, phantom, shift left=4ex, "\underset{\delta}{\Rightarrow}"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[d, "h"] \arrow[d, bend right=80, "h'"'] \arrow[d, phantom, shift right=4ex, "\underset{\gamma}{\Rightarrow}"] \arrow[dr, phantom, "\searrow\tau"] & B \arrow[l, "g"'] \arrow[d, "k"'] \arrow[d, bend left=80, "k'"] \arrow[d, phantom, shift left=4ex, "\underset{\delta}{\Rightarrow}"] \\
A' & B' \arrow[l, "g'"]
\end{tikzcd}$$

   for all appropriately-typed 2-cells γ and δ.

2. *Mating preserves horizontal pasting: if σ_1 and τ_1 are mates with respect to the adjunctions $f_1 \dashv g_1$ and $f_1' \dashv g_1'$, and σ_2 and τ_2 are mates with respect to $f_2 \dashv g_2$*

and $f_2' \dashv g_2'$, then

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow \sigma_1"] & B \arrow[r, "f_2"] \arrow[d, "k"] \arrow[dr, phantom, "\nearrow \sigma_2"] & C \arrow[d, "n"] \\ A' \arrow[r, "f_1'"'] & B' \arrow[r, "f_2'"'] & C' \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\searrow \tau_1"] & B \arrow[l, "g_1"'] \arrow[d, "k"] \arrow[dr, phantom, "\searrow \tau_2"] & C \arrow[l, "g_2"'] \arrow[d, "n"] \\ A' & B' \arrow[l, "g_1'"] & C' \arrow[l, "g_2'"] \end{tikzcd}$$

are mates, with respect to the adjunctions $f_2 \circ f_1 \dashv g_1 \circ g_2$ and $f_2' \circ f_1' \dashv g_1' \circ g_2'$.

3. *Mating preserves vertical pasting: if σ and τ are mates with respect to $f \dashv g$ and $f' \dashv g'$, and σ' and τ' are mates with respect to $f' \dashv g'$ and $f'' \dashv g''$, then*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow \sigma"] & B \arrow[d, "k"] \\ A' \arrow[r, "f'"'] \arrow[d, "h'"'] \arrow[dr, phantom, "\nearrow \sigma'"] & B' \arrow[d, "k'"] \\ A'' \arrow[r, "f''"'] & B'' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\searrow \tau"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ A' \arrow[d, "h'"'] \arrow[dr, phantom, "\searrow \tau'"] & B' \arrow[l, "g'"] \arrow[d, "k'"] \\ A'' & B'' \arrow[l, "g''"] \end{tikzcd}$$

are mates with respect to $f \dashv g$ and $f'' \dashv g''$.

4. *For every adjunction $f \dashv g : A \to B$, the structural isomorphisms*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "1"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
A \arrow[d, "1"'] \arrow[dr, phantom, "\cong"] & B \arrow[l, "g"'] \arrow[d, "1"] \\
A & B \arrow[l, "g"]
\end{tikzcd}
$$

are mates.

We omit the (routine) verification of this proposition, which requires nothing more than the definitions of mate and adjunction, and the coherence of the unit isomorphisms. The next proposition is essential for some of our applications of mates in later chapters.

Proposition 2.33. *Let $\gamma : F \Rightarrow G : \mathcal{B} \to \mathcal{C}$ be a pseudo-natural transformation, and let $f \dashv g : A \to B$ be an adjunction in $\mathcal{B}$. Then*

$$
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\Nearrow \gamma_f^{-1}"] & FB \arrow[d, "\gamma_B"] \\
GA \arrow[r, "Gf"'] & GB
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
FA \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\Searrow \gamma_g"] & FB \arrow[l, "Fg"'] \arrow[d, "\gamma_B"] \\
GA & GB \arrow[l, "Gg"]
\end{tikzcd}
$$

are mates with respect to the adjunctions $Ff \dashv Fg$ and $Gf \dashv Gg$.

Proof. Consider the 2-cell

$$
\begin{tikzcd}
FA \arrow[r, "\gamma_A"] \arrow[dr, "Ff"] \arrow[dd, "F1"'] \arrow[dd, bend right=60, "1"'] & GA \arrow[dr, "Gf"] \arrow[d, phantom, "\Uparrow \gamma_f"] & \\
\arrow[r, phantom, "\overset{\Rightarrow}{F\eta}"] & FB \arrow[r, "\gamma_B"] \arrow[dl, "Fg"] \arrow[d, phantom, "\Nearrow \gamma_g"] & GB \arrow[dl, "Gg"] \\
FA \arrow[r, "\gamma_A"'] & GA &
\end{tikzcd}
$$

Since γ is pseudo-natural, by the naturality condition in Definition 2.10 this is equal to

$$
\begin{tikzcd}
FA \arrow[r, "\gamma_A"] \arrow[dd, "F1"] \arrow[dd, bend right=70, "1"'] \arrow[dd, phantom, bend right=35, "\underset{F_A}{\Rightarrow}"] \arrow[ddr, phantom, "\nearrow \gamma_{1_A}"] & GA \arrow[dd, "G1"'] \arrow[dr, "Gf"] & \\
& & GB \arrow[dl, "Gg"] \arrow[l, phantom, "\underset{G\eta}{\Rightarrow}"] \\
FA \arrow[r, "\gamma_A"'] & GA &
\end{tikzcd}
$$

which, by the unit condition, is equal to

$$
\begin{tikzcd}
FA \arrow[r, "\gamma_A"] \arrow[dd, "1"'] \arrow[ddr, phantom, "\cong"] & GA \arrow[dd, "G1"'] \arrow[dd, bend right=70, "1"'] \arrow[dd, phantom, bend right=35, "\underset{G_A}{\Rightarrow}"] \arrow[dr, "Gf"] & \\
& & GB \arrow[dl, "Gg"] \arrow[l, phantom, "\underset{G\eta}{\Rightarrow}"] \\
FA \arrow[r, "\gamma_A"'] & GA &
\end{tikzcd}
$$

If we paste γ_f^{-1} onto the top of this equation, and $\rho_{\gamma_A}^{-1}$ onto the bottom, then we get

$$
\begin{tikzcd}
FA \arrow[dr, "Ff"] \arrow[dd, "F1"] \arrow[dd, bend right=70, "1"'] \arrow[dd, phantom, bend right=35, "\underset{F_A}{\Rightarrow}"] \arrow[dddd, bend right=90, "\gamma_A"'] & \\
& FB \arrow[dl, "Fg"] \arrow[dd, "\gamma_B"] \arrow[l, phantom, "\underset{F\eta}{\Rightarrow}"] \\
FA \arrow[dd, "\gamma_A"'] \arrow[ddr, phantom, "\swarrow \gamma_g"] \arrow[dd, phantom, bend right=60, "\cong"'] & \\
& GB \arrow[dl, "Gg"] \\
GA &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[dr, "Ff"] \arrow[dd, "\gamma_A"] \arrow[dd, phantom, bend right=50, "\cong"'] \arrow[dddd, bend right=90, "\gamma_A"'] & \\
& FB \arrow[dd, "\gamma_B"] \arrow[dl, phantom, "\nearrow \gamma_f^{-1}"] \\
GA \arrow[dr, "Gf"] \arrow[dd, "G1"] \arrow[dd, bend right=70, "1"'] \arrow[dd, phantom, bend right=35, "\underset{G_A}{\Rightarrow}"] & \\
& FB \arrow[dl, "Fg"] \arrow[l, phantom, "\underset{G\eta}{\Rightarrow}"] \\
GA &
\end{tikzcd}
$$

Thus, by equation (2.4.1) and Remark 2.28, γ_f^{-1} and γ_g are indeed mates. □

A special case of mating that is sometimes useful occurs when the vertical 1-cells h and k are identities. In this case we may omit them entirely, and speak of the left mate, or right mate, of a 2-cell $f' \Rightarrow f$, for adjunctions $f \dashv g$, $f' \dashv g' : A \to B$. It should be clear what is meant by this: for example, to take the right mate of a 2-cell

$\gamma : f' \Rightarrow f$, one forms the composite

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] \arrow[dr, bend left, "f"] & B \arrow[d, "1"] \\
A \arrow[r, "f'"'] & B \arrow[ul, bend left, "f'"] \arrow[ul, phantom, "\Nearrow\gamma"]
\end{tikzcd}$$

(where the triangular cells contain unit isomorphisms), and takes its right mate, say

$$\begin{tikzcd}
A \arrow[d, "1"'] \arrow[dr, phantom, "\Searrow\tau"] & B \arrow[l, "g"'] \arrow[d, "1"] \\
A & B \arrow[l, "g'"]
\end{tikzcd}$$

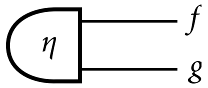
and then forms the composite

$$\begin{tikzcd}
A \arrow[d, "1"'] \arrow[dr, phantom, "\Searrow\tau"] & B \arrow[l, "g"'] \arrow[d, "1"] \arrow[dl, bend right=90, "g"', "\cong"] \arrow[dl, bend left=90, "g'", "\cong"'] \\
A & B \arrow[l, "g'"]
\end{tikzcd}$$

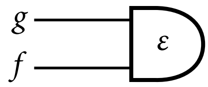
In general it is necessary to distinguish between left mate and right mate in this situation, because one cannot tell from the context which is intended. However, see Prop. 2.42 below for a case where they coincide.

2.4.2 Adjunctions and mates in terms of string diagrams

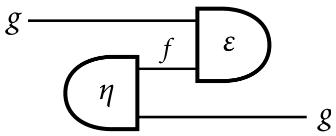
Adjunctions and mates have a particularly elegant string diagram representation. Let $f \dashv g : A \to B$ be an adjunction with unit η and counit ε. Then the unit and counit are drawn as



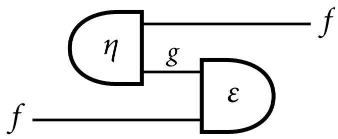
and



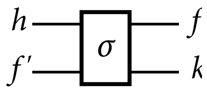
The adjunction axioms say precisely that



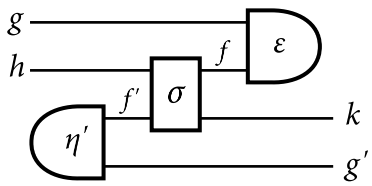
and



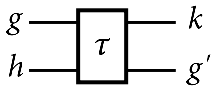
are both identities. Given a 2-cell σ:



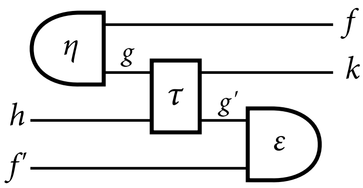
its right mate is



and given a 2-cell τ:



its left mate is



2.4.3 Adjoint pseudo-natural transformations

Now we turn our attention to adjoint pseudo-natural transformations, i.e. adjunctions in a pseudo-functor bicategory.

Proposition 2.34. *Let there be given an adjoint pair of pseudo-natural transformations*

$$\phi \dashv \gamma : F \Rightarrow G : \mathcal{B} \to \mathcal{C}$$

with unit $\eta : 1 \Rrightarrow \gamma\phi$ and counit $\varepsilon : \phi\gamma \Rrightarrow 1$. Then

1. *For every object $A \in \mathcal{B}$, there is an adjunction*

$$\phi_A \dashv \gamma_A$$

with unit η_A and counit ε_A.

2. *For every 1-cell $f : A \to B$ in $\mathcal{B}$, the 2-cells*

$$\begin{tikzcd}
FA \arrow[r, "\phi_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\nearrow \phi_f"] & GA \arrow[d, "G(f)"] \\
FB \arrow[r, "\phi_B"'] & GB
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
FA \arrow[d, "F(f)"'] \arrow[dr, phantom, "\searrow \gamma_f^{-1}"] & GA \arrow[l, "\gamma_A"'] \arrow[d, "G(f)"] \\
FB & GB \arrow[l, "\gamma_B"]
\end{tikzcd}$$

are mates with respect to the adjunctions $\phi_A \dashv \gamma_A$ and $\phi_B \dashv \gamma_B$.

Proof. The first part is immediate by definition, so let's consider the second. Since η is a modification, we know that for every $f : A \to B$,

$$\begin{tikzcd}
FA \arrow[r, "\phi_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\nearrow \phi_f"] & GA \arrow[r, "\gamma_A"] \arrow[d, "Gf"] \arrow[dr, phantom, "\nearrow \gamma_f"] & FA \arrow[d, "Ff"] \\
FB \arrow[r, "\phi_B"'] \arrow[rr, bend right, "1"'] & GB \arrow[r, "\gamma_B"'] \arrow[d, phantom, "\Uparrow \eta_B"] & FB \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
& GA \arrow[dr, "\gamma_A"] \arrow[d, phantom, "\Uparrow \eta_A"] & \\
FA \arrow[ur, "\phi_A"] \arrow[rr, "1"'] \arrow[d, "Ff"'] & {} \arrow[d, phantom, "\cong"] & FA \arrow[d, "Ff"] \\
FB \arrow[rr, "1"'] & {} & FB
\end{tikzcd}$$

Onto both sides of the equation, we paste the 2-cells γ_f^{-1} and $\lambda^{-1} : Ff \to 1 \circ Ff$, yielding the equation

$$\begin{tikzcd}
FA \arrow[r, "\phi_A"] \arrow[d, "Ff"] \arrow[dd, bend right=60, "Ff"'] \arrow[dr, phantom, "\nearrow \phi_f"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\phi_B"] \arrow[d, "1"] \arrow[l, phantom, "\cong"] \arrow[dr, phantom, "\Rightarrow \eta_B"] & GB \arrow[dl, "\gamma_B"] \\
FB &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[d, "1"] \arrow[dr, "\phi_A"] \arrow[dd, bend right=60, "Ff"'] \arrow[d, phantom, shift left=4ex, "\Rightarrow \eta_A"] & \\
FA \arrow[d, "Ff"] \arrow[l, phantom, "\cong"] \arrow[dr, phantom, "\searrow \gamma_f^{-1}"] & GA \arrow[l, "\gamma_A"] \arrow[d, "Gf"] \\
FB & GB \arrow[l, "\gamma_B"]
\end{tikzcd}$$

which, by (2.4.1), is what we require. □

Proposition 2.35. *Let there be given a pseudo-natural transformation*

$$\phi : F \Rightarrow G : \mathcal{B} \to \mathcal{C}.$$

To give a right adjoint $\gamma : G \Rightarrow F$ for ϕ is to give

- *for each $A \in \mathcal{B}$, a 1-cell $\gamma_A : GA \to FA$ and an adjunction $\phi_A \dashv \gamma_A$,*
- *such that for every $f : A \to B$ in $\mathcal{B}$, the mate of ϕ_f with respect to the adjunctions $\phi_A \dashv \gamma_A$ and $\phi_B \dashv \gamma_B$ is invertible.*

Proof. We have already shown (Prop. 2.34) that every right-adjoint pseudo-natural transformation γ has these properties, so suppose that we have a collection of adjunctions $\phi_A \dashv \gamma_A$ as in the statement, each with unit η_A and counit ε_A, say. For each 1-cell $f : A \to B$ in $\mathcal{B}$, define the 2-cell

$$\begin{tikzcd} GA \arrow[r, "\gamma_A"] \arrow[d, "Gf"'] \arrow[dr, phantom, "\nearrow \gamma_f"] & FA \arrow[d, "Ff"] \\ GB \arrow[r, "\gamma_B"'] & FB \end{tikzcd}$$

to be the inverse of the mate of ϕ_f. We shall show that these data constitute a pseudo-natural transformation γ, checking the naturality and composition conditions of Definition 2.10. This is essentially a matter of writing down the corresponding conditions for ϕ and taking mates.

- The naturality condition for ϕ states that

$$\begin{tikzcd} FA \arrow[r, "\phi_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\nearrow \phi_f"] & GA \arrow[d, bend right, "Gf"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\overset{\Rightarrow}{G(\tau)}"] \\ FB \arrow[r, "\phi_B"'] & GB \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\phi_A"] \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\overset{\Rightarrow}{F(\tau)}"] \arrow[dr, phantom, "\nearrow \phi_g"] & GA \arrow[d, "Gg"] \\ FB \arrow[r, "\phi_B"'] & GB \end{tikzcd}$$

Taking mates of both sides, by Prop. 2.32(1) we have

$$\begin{tikzcd}
FA \arrow[d, "Ff"'] \arrow[dr, phantom, "\Nearrow \gamma_f^{-1}"] & GA \arrow[l, "\gamma_A"'] \arrow[d, bend right, "Gf"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\overset{\Rightarrow}{G(\tau)}"] \\
FB & GB \arrow[l, "\gamma_B"]
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\overset{\Rightarrow}{F(\tau)}"] \arrow[dr, phantom, "\Nearrow \gamma_g^{-1}"] & GA \arrow[l, "\gamma_A"'] \arrow[d, "Gg"] \\
FB & GB \arrow[l, "\gamma_B"]
\end{tikzcd}$$

which may be rearranged into the naturality condition for γ.

- The composition condition for ϕ states that

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[rr, bend right, "F(g\circ f)"'] \arrow[d, "\phi_A"'] & FB \arrow[r, "Fg"] \arrow[d, phantom, pos=0.2, "\Downarrow F_{g,f}"] \arrow[d, phantom, pos=0.75, "\Swarrow \phi_{g\circ f}"] & FC \arrow[d, "\phi_C"] \\
GA \arrow[rr, "G(g\circ f)"'] & {} & GC
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[dd, "\phi_A"'] \arrow[dr, phantom, "\Swarrow \phi_f"] & FB \arrow[r, "Fg"] \arrow[d, "\phi_B"] \arrow[dr, phantom, "\Swarrow \phi_g"] & FC \arrow[dd, "\phi_C"] \\
{} & GB \arrow[dr, "Gg"] \arrow[d, phantom, "\Downarrow G_{g,f}"] & {} \\
GA \arrow[ur, "Gf"] \arrow[rr, "G(g\circ f)"'] & {} & GC
\end{tikzcd}$$

Taking mates, and using Prop. 2.32(1, 3), gives

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[rr, bend right, "F(g\circ f)"'] & FB \arrow[r, "Fg"] \arrow[d, phantom, pos=0.2, "\Downarrow F_{g,f}"] \arrow[d, phantom, pos=0.75, "\Searrow \gamma_{g\circ f}^{-1}"] & FC \\
GA \arrow[u, "\gamma_A"] \arrow[rr, "G(g\circ f)"'] & {} & GC \arrow[u, "\gamma_C"']
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[dr, phantom, "\Searrow \gamma_f^{-1}"] & FB \arrow[r, "Fg"] \arrow[d, "\gamma_B"] \arrow[dr, phantom, "\Searrow \gamma_g^{-1}"] & FC \\
{} & GB \arrow[dr, "Gg"] \arrow[d, phantom, "\Downarrow G_{g,f}"] & {} \\
GA \arrow[uu, "\gamma_A"] \arrow[ur, "Gf"] \arrow[rr, "G(g\circ f)"'] & {} & GC \arrow[uu, "\gamma_C"']
\end{tikzcd}$$

which may be rearranged into the composition condition for γ.

It remains to show that each of the collections η_A and ε_A constitutes a modification.

By equation (2.4.1), we know that

$$
\begin{tikzcd}
FA \arrow[d, "Ff"'] \arrow[dr, "\phi_A"] \arrow[dd, bend right=60, "Ff"'] & \\
FB \arrow[d, "1"'] \arrow[dr, "\phi_B"] \arrow[r, phantom, "\Rightarrow \phi_f"] \arrow[dr, phantom, shift right=3ex, "\Rightarrow \eta_B"] & GA \arrow[d, "Gf"] \\
FB & GB \arrow[l, "\gamma_B"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "\phi_A"] \arrow[d, "1"'] \arrow[dd, bend right=60, "Ff"'] \arrow[r, phantom, shift right=4ex, "\Rightarrow \eta_A"] & GA \arrow[dl, "\gamma_A"] \arrow[d, "Gf"] \\
FA \arrow[d, "Ff"'] \arrow[r, phantom, "\Rightarrow \gamma_f^{-1}"] & GB \arrow[dl, "\gamma_B"] \\
FB &
\end{tikzcd}
$$

This can be rearranged to give

$$
\begin{tikzcd}
FB \arrow[r, "Ff"] \arrow[dr, "\phi_B"] \arrow[dd, "1"'] & FA \arrow[dr, "\phi_A"] \arrow[d, phantom, "\swarrow \phi_f"] & \\
\arrow[r, phantom, "\Rightarrow \eta_B"] & GB \arrow[r, "Gf"'] \arrow[dl, "\gamma_A"] \arrow[d, phantom, "\searrow \gamma_f"] & GA \arrow[dl, "\gamma_A"] \\
FB & FA \arrow[l, "Ff"] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FB \arrow[dd, "1"'] \arrow[ddr, phantom, "\cong"] & FA \arrow[l, "Ff"'] \arrow[dd, "1"'] \arrow[dr, "\phi_A"] & \\
& \arrow[r, phantom, "\Rightarrow \eta_A"] & GA \arrow[dl, "\gamma_A"] \\
FB & FA \arrow[l, "Ff"] &
\end{tikzcd}
$$

which shows that η is a modification. Similarly, we may use equation (2.4.2) to show that ε is a modification. □

In the bicategory Cat, adjunctions $F \dashv G : \mathbb{C} \to \mathbb{D}$ are characterised by the existence of a natural isomorphism

$$\mathbb{D}(FA, B) \cong \mathbb{C}(A, GB),$$

natural in A and B. It is interesting to observe that a similar characterisation exists for adjunctions in an *arbitrary* bicategory, as the next proposition shows.

Proposition 2.36. *Let there be given 1-cells $f : A \to B$ and $g : B \to A$ in a bicategory $\mathcal{B}$. To give an adjunction $f \dashv g$ is to give, for every $A \xleftarrow{a} X \xrightarrow{b} B$, an isomorphism*

$$\phi_{a,b} : \mathcal{B}(X, B)(f \circ a, b) \cong \mathcal{B}(X, A)(a, g \circ b),$$

natural in the sense that:

- *for every $\sigma : a \Rightarrow a'$, $\tau : b \Rightarrow b'$, and $\zeta : f \circ a' \Rightarrow b$, we have*

$$
\begin{tikzcd}
& X \arrow[dl, bend right=50, "a"'] \arrow[dl, "a'"] \arrow[dr, "b"'] \arrow[dr, bend left=50, "b'"] \arrow[d, phantom, "\Rightarrow \atop \phi_{a',b}(\zeta)"] & \\
A \arrow[ur, phantom, "\Searrow \sigma"] & {} & B \arrow[ll, "g"] \arrow[ul, phantom, "\Nearrow \tau"]
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[dl, bend right=30, "a"'] \arrow[dr, bend left=30, "b'"] \arrow[d, phantom, "\Rightarrow \atop \phi_{a,b'}(\tau \cdot \zeta \cdot (f \circ \sigma))"] & \\
A & {} & B \arrow[ll, "g"]
\end{tikzcd}
$$

- *for every $k : Y \to X$ and $\zeta : fa \Rightarrow b$, we have*

$$
\begin{tikzcd}
& Y \arrow[d, "k"] & \\
& X \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "\Rightarrow \atop \phi_{a,b}(\zeta)"] & \\
A & {} & B \arrow[ll, "g"]
\end{tikzcd}
=
\begin{tikzcd}
& Y \arrow[dl, "k"'] \arrow[dr, "k"] & \\
X \arrow[d, "a"'] \arrow[rr, phantom, "\Rightarrow \atop \phi_{ak,bk}(\zeta \circ k)"] & & X \arrow[d, "b"] \\
A & & B \arrow[ll, "g"]
\end{tikzcd}
$$

Proof. By Yoneda, we know that to give an adjunction $f \dashv g$ is to give an adjunction $\mathcal{B}(-, f) \dashv \mathcal{B}(-, g)$. By Prop. 2.35, we know that to give such an adjunction is to give, for every $X \in \mathcal{B}$, an adjunction $\mathcal{B}(X, f) \dashv \mathcal{B}(X, g)$, collectively subject to condition (2.4.1). This is just an ordinary adjunction, which can therefore be given as a natural isomorphism

$$\mathcal{B}(X, B)(\mathcal{B}(X, f)(a), b) \cong \mathcal{B}(X, A)(a, \mathcal{B}(X, g)(b))$$

natural in $a \in \mathcal{B}(X, A)$ and $b \in \mathcal{B}(X, B)$, i.e. a natural isomorphism

$$\mathcal{B}(X, B)(f \circ a, b) \cong \mathcal{B}(X, A)(a, g \circ b).$$

This corresponds to the data in the statement of this Proposition, subject to our first naturality condition. We shall write the unit of this adjunction as

$$\eta_a : a \Rightarrow g \circ (f \circ a)$$

for $a : X \to A$.

It remains to show that the second naturality condition is satisfied just when

(2.4.1) is. Writing down the concrete interpretation of (2.4.1) in our setting, we find that it holds when, for every $a : X \to A$ and $k : Y \to X$, the diagram

$$\begin{tikzcd}
a \circ k \arrow[r, "\eta_{a \circ k}"] \arrow[d, "\eta_a \circ k"'] & g \circ (f \circ (a \circ k)) \arrow[d, "g \circ \alpha_{f,a,k}"] \\
(g \circ (f \circ a)) \circ k \arrow[r, "\alpha_{g,f \circ a,k}"'] & g \circ ((f \circ a) \circ k)
\end{tikzcd}$$

commutes in $\mathcal{B}(Y, B)$. In pictures, this says that

$$\begin{tikzcd}
 & Y \arrow[d, "k"] & \\
 & X \arrow[ddl, "a"'] \arrow[dr, "a"] \arrow[ddr, bend left=60, "b"] \arrow[dd, phantom, "\overset{\Rightarrow}{\eta_a}"'] & \\
 & & A \arrow[d, "f"] \arrow[u, phantom, "\Nearrow \zeta"', shift left=6] \\
A & & B \arrow[ll, "g"]
\end{tikzcd}
=
\begin{tikzcd}
 & Y \arrow[dl, "k"'] \arrow[dr, "k"] & \\
X \arrow[dd, "a"'] & & X \arrow[d, "a"] \arrow[dd, bend left=60, "b"] \\
 & \overset{\Rightarrow}{\eta_{a \circ k}} & A \arrow[d, "f"] \arrow[r, phantom, "\overset{\Rightarrow}{\zeta}"] & {} \\
A & & B \arrow[ll, "g"]
\end{tikzcd}$$

And since $\eta_a = \phi_{a,fa}(1_{fa})$ by definition, this is equivalent to the particular case of our second naturality condition with $b = f \circ a$ and $\zeta = 1_{f \circ a}$.

But this special case implies the general case, by the first naturality condition: for we have

$$
\begin{tikzcd}
 & Y \arrow[d, "k"] & \\
 & X \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "\overset{\Rightarrow}{\phi_{a,b}(\zeta)}"] & \\
A & {} & B \arrow[ll, "g"]
\end{tikzcd}
=
\begin{tikzcd}
 & Y \arrow[d, "k"] & \\
 & X \arrow[dl, "a"'] \arrow[dr, "a"] \arrow[ddr, bend left=60, "b"] \arrow[d, phantom, "\overset{\Rightarrow}{\eta_a}"] & \\
 & {} & A \arrow[d, "f"] \arrow[r, phantom, "\Nearrow\zeta"] & {} \\
A & & B \arrow[ll, "g"]
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
 & Y \arrow[dl, "k"'] \arrow[dr, "k"] & \\
X \arrow[d, "a"'] \arrow[r, phantom, "\overset{\Rightarrow}{\eta_{a\circ k}}"] & {} & X \arrow[d, "b"] \arrow[dd, bend left=60, "b"] \\
 & & A \arrow[d, "f"] \arrow[r, phantom, "\overset{\Rightarrow}{\zeta}"] & {} \\
A & & B \arrow[ll, "g"]
\end{tikzcd}
=
\begin{tikzcd}
 & Y \arrow[dl, "k"'] \arrow[dr, "k"] & \\
X \arrow[d, "a"'] \arrow[rr, phantom, "\overset{\Rightarrow}{\phi_{a\circ k, b\circ k}(\zeta\circ k)}"] & & X \arrow[d, "b"] \\
A & & B \arrow[ll, "g"]
\end{tikzcd}
$$

as required. □

Remark 2.37. In this section we have considered adjunctions *in* a bicategory. It is also possible to consider *pseudo-adjunctions between bicategories*: to exhibit $G : \mathcal{C} \to \mathcal{B}$ as right pseudo-adjoint to $F : \mathcal{B} \to \mathcal{C}$ is to give an equivalence $\mathcal{C}(FA, X) \simeq \mathcal{B}(A, GX)$ pseudo-natural in A and X. We do not pursue pseudo-adjunctions further here.

2.5 On equivalence

Recall the definition of equivalence (Definition2.14).

Definition 2.38. An *adjoint equivalence* is an adjunction whose unit and counit are invertible.

Remark 2.39. If $f \dashv g$ is an adjoint equivalence with unit η and counit ε, then $g \dashv f$ is an adjoint equivalence with unit ε^{-1} and counit η^{-1}.

In later chapters, particularly Chapter 6, we will often consider mates with respect to adjoint equivalences. Such mating has some special properties which are crucial for our applications.

Lemma 2.40. *If we have mates*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, "\nearrow\sigma"] & B' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "g"'] \arrow[d, "k"] \arrow[dl, phantom, "\searrow\tau"] \\ A' & B' \arrow[l, "g'"] \end{tikzcd}$$

with respect to adjoint equivalences *$f \dashv g$ and $f' \dashv g'$, then σ is invertible if and only if τ is.*

Proof. Immediate from the definition of mate. □

Lemma 2.41. *Given adjoint equivalences $f \dashv g : A \to B$ and $f' \dashv g' : A' \to B'$, and an invertible 2-cell*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, "\nearrow\sigma"] & B', \end{tikzcd}$$

the inverse of the right mate of σ is equal to the left mate of its inverse.

Proof. The right mate of σ is

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, "f"'] \arrow[dd, bend right=60, "h"', "\cong"] & B \arrow[l, "g"'] \arrow[d, "1"] \arrow[dd, bend left=60, "k", "\cong"'] \arrow[dl, phantom, "\Rightarrow \varepsilon"] \\ A' \arrow[d, "1"'] \arrow[dr, "f'"] \arrow[r, phantom, "\Rightarrow \sigma"] & B \arrow[d, "k"] \\ A' \arrow[ur, phantom, "\Rightarrow \eta'"] & B', \arrow[l, "g'"] \end{tikzcd}$$

whose inverse is

$$\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, "f"] \arrow[dd, bend right=60, "h"'] \arrow[r, phantom, ""] & B \arrow[l, "g"'] \arrow[d, "1"] \arrow[dd, bend left=60, "k"] \arrow[dl, phantom, "\overset{\Leftarrow}{\varepsilon^{-1}}" near start] \\
A' \arrow[l, phantom, "\cong"] \arrow[d, "1"'] \arrow[dr, "f'"] \arrow[r, phantom, "\overset{\Leftarrow}{\sigma^{-1}}"] & B \arrow[d, "k"] \arrow[r, phantom, "\cong"] \\
A' \arrow[ur, phantom, "\overset{\Leftarrow}{\eta'^{-1}}" near start] & B', \arrow[l, "g'"]
\end{tikzcd}$$

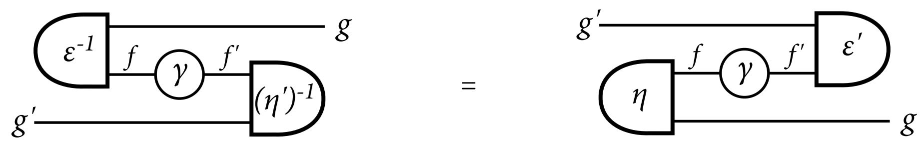
which is the left mate of σ^{-1}. □

Proposition 2.42. *Given adjoint equivalences $f \dashv g : A \to B$ and $f' \dashv g' : A \to B$, and an invertible 2-cell*

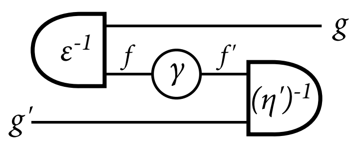
$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, phantom, "\Downarrow\gamma"] & B \arrow[l, bend left, "f'"]
\end{tikzcd}$$

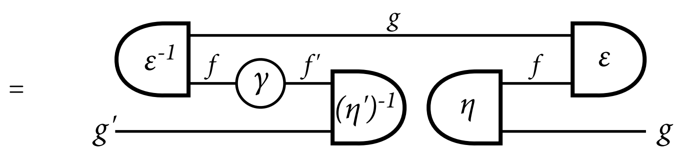
the left mate of γ is equal to its right mate.

Proof. The proof is surprisingly intricate, and rather difficult to follow unless string diagrams are used. (In fact, this proof is the reason that we have introduced string diagrams into this chapter.) In string diagram terms, what we have to prove is that



The proof is as follows:





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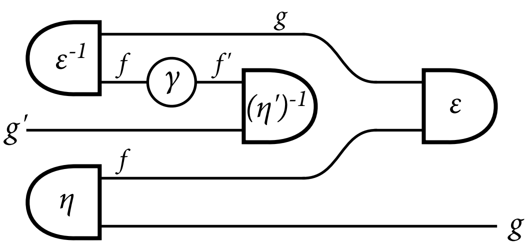

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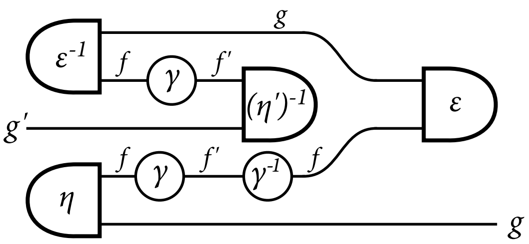

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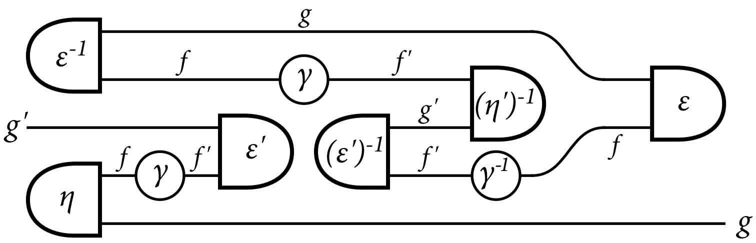

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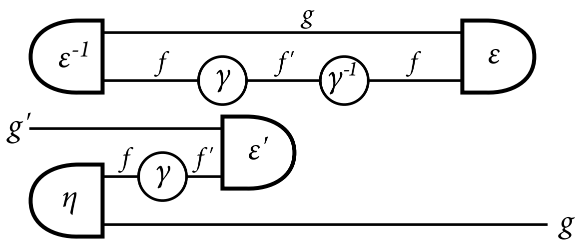

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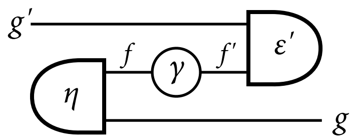


□

Proposition 2.43. *If there is an equivalence* (f, g, e, e') *from A to B, then there is an adjoint equivalence* $f \dashv g$ *with unit e.*

Proof. It is well known[4] that this is true in Cat. We shall use Yoneda to infer that it is therefore true in an arbitrary bicategory. Let there be given an equivalence (f, g, e, e') from A to B. This induces an equivalence from $\mathcal{B}(-, A)$ to $\mathcal{B}(-, B)$ in $\mathbf{Bicat}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat})$. Thus for every $X \in \mathcal{B}$ we have an adjoint equivalence in Cat from $\mathcal{B}(X, A)$ to $\mathcal{B}(X, B)$, with unit $\mathcal{B}(X, e)$. By Prop. 2.35 and Lemma 2.40 this induces

[4] And easy to prove using the ordinary Yoneda lemma.

an adjoint equivalence in **Bicat**$(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat})$, and Yoneda therefore yields the desired adjoint equivalence in $\mathcal{B}$. □

Remark 2.44. It is possible to give a more elementary proof of the preceding Proposition, by directly constructing a counit for the adjoint equivalence. This parallels the situation that one frequently encounters in ordinary category theory, where there is a choice between a concise, perspicuous Yoneda proof and an obscure but elementary equational one.

Remark 2.45. An adjunction in $\mathcal{B}$ is an adjunction in $\mathcal{B}^{\mathrm{co\,op}}$, with the unit and counit reversed. Thus in the situation of Prop. 2.43 there is also a (generally different) adjoint equivalence with counit e'.

Just as an ordinary natural transformation is invertible just when all its components are, so a modification is invertible just when all its components are. Furthermore a pseudo-natural transformation is an equivalence in **Bicat**$(\mathcal{B}, \mathcal{C})$ just when all its components are equivalences in their respective hom-categories. This latter fact, though unsurprising, is not altogether trivial to prove – though we have done the hard work already.

Proposition 2.46. *Let there be given a pseudo-natural transformation*

$$\gamma : F \Rightarrow G : \mathcal{B} \to \mathcal{C}.$$

This γ is an equivalence in **Bicat**$(\mathcal{B}, \mathcal{C})$ *just when for every $A \in \mathcal{B}$ the 1-cell $\gamma_A : FA \to GA$ is an equivalence.*

Proof. Suppose that for every A, the component $\gamma_A : FA \to GA$ is an equivalence. By Prop. 2.43 we may suppose that there is an arrow δ_A : and an adjoint equivalence $\gamma_A \dashv \delta_A$. Now the claim follows from Lemma 2.40 and Prop. 2.35. □

2.6 Normal pseudo-functors

Here we prove a useful coherence-type result about pseudo-functors. It is certainly well-known, but I am not aware of a published proof.

Definition 2.47. A pseudo-functor $F : \mathcal{B} \to \mathcal{C}$ is *normal* if $F_A : 1_{FA} \to F(1_A)$ is an identity map for every $A \in \mathcal{B}$.

Lemma 2.48. *Let $F : \mathcal{B} \to \mathcal{C}$ be a pseudo-functor, and let there be given, for all A, $B \in \mathcal{B}$, a functor $G_{A,B} : \mathcal{B}(A, B) \to \mathcal{C}(FA, FB)$ and a natural isomorphism $\phi_{A,B}$: $F_{A,B} \Rightarrow G_{A,B}$.*

Then G may be extended to a pseudo-functor that coincides with F on objects, such that ϕ becomes a pseudo-natural equivalence between F and G.

Proof. Define G like F on objects, and let its action on the hom-category $\mathcal{B}(A,B)$ be the functor $G_{A,B}$. For an object A, let G_A be the composite

$$1_{GA} = 1_{FA} \xrightarrow{F_A} F(1_A) \xrightarrow{(\phi_{A,A})_{1_A}} G(1_A),$$

and for a composable pair $A \xrightarrow{f} B \xrightarrow{g} C$ let $G_{g,f}$ be the composite

$$G(g)\circ G(f) \xrightarrow{(\phi^{-1}_{B,C})_g \circ (\phi^{-1}_{A,B})_f} F(g)\circ F(f) \xrightarrow{F_{g,f}} F(g\circ f) \xrightarrow{(\phi_{A,C})_{g\circ f}} G(g\circ f).$$

It is necessary to check that G is indeed a pseudo-functor. For example, for condition $[\boldsymbol{\lambda}]$ take an arrow $f : A \to B$. We have the diagram

$$\begin{tikzcd}
1\circ Gf \arrow[rr, "F_B\circ Gf"] \arrow[dr, "1\circ\phi_f^{-1}"] \arrow[ddd, swap, "\lambda_{Gf}"] \arrow[ddr, phantom, "\natural\lambda"] & & F1\circ Gf \arrow[rr, "\phi_1\circ Gf"] \arrow[dr, swap, "F1\circ\phi_f^{-1}"] & & G1\circ Gf \arrow[dl, swap, "\phi_1^{-1}\circ\phi_f^{-1}"] \arrow[ddd, "G_{1,f}"] \\
& 1\circ Ff \arrow[rr, swap, "F_B\circ Ff"] \arrow[d, swap, "\lambda_{Ff}"] \arrow[drr, phantom, "{[\boldsymbol{\lambda}]}"] & & F1\circ Ff \arrow[d, swap, "F_{1,f}"] & \\
& Ff \arrow[rr, swap, "F(\lambda_f)"] \arrow[dl, "\phi_f"] \arrow[drr, phantom, "\natural\phi"] & & F(1\circ f) \arrow[dr, "\phi_{1\circ f}"] & \\
Gf \arrow[rrrr, swap, "G(\lambda_f)"] & & & & G(1\circ f)
\end{tikzcd}$$

where we have omitted the object subscripts of ϕ. The regions commute for the marked reasons, or functoriality of composition, except for the rightmost region which commutes by definition of $G_{1,f}$. Thus the outside commutes, and G satisfies condition $[\boldsymbol{\lambda}]$. The other conditions may be checked similarly.

We make ϕ into a pseudo-natural transformation by defining ϕ_A to be the identity at $FA = GA$, for every object A. For an arrow $f : A \to B$, ϕ_f is defined to be the pasting

$$\begin{tikzcd}
FA \arrow[r, "\phi_A = 1"] \arrow[d, "Ff"'] \arrow[dr, bend left, "Gf"] \arrow[dr, bend right, "Ff"'] \arrow[dr, phantom, "\Nearrow (\phi_{A,B})_f"] & GA \arrow[d, "Gf"] \\
FA \arrow[r, "\phi_B = 1"'] & GB
\end{tikzcd}$$

It is easy to check that this constitutes a pseudo-natural transformation, and its components are identities (hence equivalences), so by Prop. 2.46 it is a pseudo-natural equivalence, as required. □

Proposition 2.49. *Every pseudo-functor F is equivalent to a normal pseudo-functor that agrees with F on objects, as well as on non-identity 1-cells and the 2-cells between them.*

Proof. Let there be given a pseudofunctor $F : \mathcal{B} \to \mathcal{C}$. We shall construct an equivalent normal pseudofunctor G. By Lemma 2.48 it suffices to do so for each hom-category separately. The definition of $G_{A,B}$ and $\phi_{A,B}$ is by cases, as follows.

- Given objects $A \neq B$, let $G_{A,B} = F_{A,B}$ and let $\phi_{A,B}$ be the identity.
- For each object A, let $G_{A,A}(1) = 1$, and let $G_{A,A}(f) = F_{A,A}(f)$ for $f \neq 1$.
- Given a 2-cell $\beta : f \Rightarrow g : A \to A$ with $f \neq 1 \neq g$, let $G_{A,A}(\beta) = \beta$.
- Given a 2-cell $\beta : 1 \Rightarrow f : A \to A$ with $f \neq 1$, let $G_{A,A}(\beta) = F(\beta) \cdot F_A$.
- Given a 2-cell $\gamma : f \Rightarrow 1 : A \to A$ with $f \neq 1$, let $G_{A,A}(\gamma) = F_A^{-1} \cdot F(\gamma)$.
- Given a 2-cell $\delta : 1 \Rightarrow 1 : A \to A$, let $G_{A,A}(\delta) = F_A^{-1} \cdot F(\delta) \cdot F_A$.
- Let $(\phi_{A,A})_1 : F(1_A) \to G(1_A) = 1_{GA} = 1_{FA}$ be F_A^{-1},
- and let $(\phi_{A,A})_f = 1$.

It is straightforward to check (four cases) that this makes $\phi_{A,A}$ a natural transformation.

Now we may extend G to a pseudo-functor using Lemma 2.48, in which G_A is defined to be $(\phi_{A,A})_1 \cdot F_A$. Since $(\phi_{A,A})_1 = F_A^{-1}$ by definition, G_A is the identity and so G is normal, as required. □

Chapter 3

Monoidal Bicategories

Definition 3.1. A monoidal bicategory $\mathcal{B}$ is a bicategory equipped with a unit object $\mathbb{I} \in \mathcal{B}$, a pseudo-functor

$$\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B},$$

pseudo-natural equivalences a, l and r with components

$$\begin{aligned} &a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C, \\ &l_A : \mathbb{I} \otimes A \to A, \\ &r_A : A \otimes \mathbb{I} \to A, \end{aligned}$$

and invertible modifications π, μ, L and R with components

$$\begin{tikzcd}
A \otimes \big(B \otimes (C \otimes D)\big) \arrow[r, "a_{A,B,C\otimes D}"] \arrow[dr, "A \otimes a_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{A\otimes B,C,D}"] \arrow[d, phantom, "\Downarrow \pi_{A,B,C,D}"] & \big((A \otimes B) \otimes C\big) \otimes D \\
& A \otimes \big((B \otimes C) \otimes D\big) \arrow[r, "a_{A,B\otimes C,D}"'] & \big(A \otimes (B \otimes C)\big) \otimes D \arrow[u, "a_{A,B,C} \otimes D"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (\mathbb{I} \otimes C) \arrow[rr, "a_{A,\mathbb{I},C}"] \arrow[dr, "A \otimes l_C"'] & \arrow[d, phantom, "\Rightarrow \mu_{A,C}"] & (A \otimes \mathbb{I}) \otimes C \arrow[dl, "r_A \otimes C"] \\
& A \otimes C &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{I} \otimes (B \otimes C) \arrow[rr, "a_{\mathbb{I},B,C}"] \arrow[dr, "l_{B\otimes C}"'] & \arrow[d, phantom, "\Rightarrow L_{B,C}"] & (\mathbb{I} \otimes B) \otimes C \arrow[dl, "l_B \otimes C"] \\
& B \otimes C &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes (B \otimes \mathbb{I}) \arrow[rr, "a_{A,B,\mathbb{I}}"] \arrow[dr, "A \otimes r_B"'] & \arrow[d, phantom, "\Rightarrow R_{A,B}"] & (A \otimes B) \otimes \mathbb{I} \arrow[dl, "r_{A\otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

such that for all A, B, C, D and E in $\mathcal{B}$, the condition shown in Fig. 3.1 holds, and for all A, B and C, the following two conditions hold:

$$
\begin{tikzcd}
 & & A\otimes(B\otimes C) \arrow[rr, "a_{A,B,C}"] & & (A\otimes B)\otimes C \\
A\otimes\big(\mathbb{I}\otimes(B\otimes C)\big) \arrow[urr, "A\otimes l_{B\otimes C}"] \arrow[rr, "a_{A,\mathbb{I},B\otimes C}"] \arrow[dr, "A\otimes a_{\mathbb{I},B,C}"'] & & (A\otimes\mathbb{I})\otimes(B\otimes C) \arrow[u, "r_A\otimes(B\otimes C)"'] \arrow[rr, "a_{A\otimes\mathbb{I},B,C}"] \arrow[ul, phantom, "\Searrow\ \mu_{A,B\otimes C}"] \arrow[urr, phantom, "\Searrow\ a_{r_A,B,C}"] \arrow[d, phantom, "\Downarrow\ \pi_{A,\mathbb{I},B,C}"] & & \big((A\otimes\mathbb{I})\otimes B\big)\otimes C \arrow[u, "(r_A\otimes B)\otimes C"'] \\
 & A\otimes\big((\mathbb{I}\otimes B)\otimes C\big) \arrow[rr, "a_{A,\mathbb{I}\otimes B,C}"'] & & \big(A\otimes(\mathbb{I}\otimes B)\big)\otimes C \arrow[ur, "a_{A,\mathbb{I},B}\otimes C"'] &
\end{tikzcd}
$$

is equal to

$$
\begin{tikzcd}
 & & A\otimes(B\otimes C) \arrow[rr, "a_{A,B,C}"] & & (A\otimes B)\otimes C & & \\
A\otimes(\mathbb{I}\otimes(B\otimes C)) \arrow[urr, "A\otimes l_{B\otimes C}"] \arrow[drr, "A\otimes a_{\mathbb{I},B,C}"'] \arrow[rr, phantom, "\Searrow\ A\otimes L_{B,C}"] & & & & & & ((A\otimes\mathbb{I})\otimes B)\otimes C \arrow[ull, "(r_A\otimes B)\otimes C"'] \arrow[ll, phantom, "\Rightarrow\ \mu_{A,B}\otimes C"'] \\
 & & A\otimes((\mathbb{I}\otimes B)\otimes C) \arrow[uu, "A\otimes(l_B\otimes C)"'] \arrow[rr, "a_{A,\mathbb{I}\otimes B,C}"'] \arrow[uurr, phantom, "\Searrow\ a_{A,l_B,C}"] & & (A\otimes(\mathbb{I}\otimes B))\otimes C \arrow[uu, "(A\otimes l_B)\otimes C"] \arrow[urr, "a_{A,\mathbb{I},B}\otimes C"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (A\otimes B)\otimes\big(C\otimes(D\otimes E)\big) \arrow[rr, "a"] \arrow[ddr, "(A\otimes B)\otimes a"] & & ((A\otimes B)\otimes C)\otimes(D\otimes E) \arrow[rr, "a"] & & \big(((A\otimes B)\otimes C)\otimes D\big)\otimes E & \\
 & & & & & & ((A\otimes(B\otimes C))\otimes D)\otimes E \arrow[ul, "(a\otimes D)\otimes E"'] \\
A\otimes(B\otimes(C\otimes(D\otimes E))) \arrow[uur, "a"] \arrow[ddr, "A\otimes(B\otimes a)"'] \arrow[rr, phantom, "\Downarrow a^{-1}"] & & (A\otimes B)\otimes\big((C\otimes D)\otimes E\big) \arrow[rr, "a"'] \arrow[ur, phantom, "\Downarrow\pi_{(A\otimes B),C,D,E}"] \arrow[dr, phantom, "\Searrow\pi_{A,B,(C\otimes D),E}"] & & \big((A\otimes B)\otimes(C\otimes D)\big)\otimes E \arrow[uur, "a\otimes E"] \arrow[rr, phantom, "\overset{\Rightarrow}{\pi_{A,B,C,D}\otimes E}"] & & \\
 & & & & & & (A\otimes((B\otimes C)\otimes D))\otimes E \arrow[uu, "a\otimes E"'] \\
 & A\otimes(B\otimes((C\otimes D)\otimes E)) \arrow[uur, "a"'] \arrow[rr, "A\otimes a"'] & & A\otimes((B\otimes(C\otimes D))\otimes E) \arrow[rr, "a"'] & & (A\otimes(B\otimes(C\otimes D)))\otimes E \arrow[uul, "a\otimes E"'] \arrow[ur, "(A\otimes a)\otimes E"'] &
\end{tikzcd}
$$

must be equal to

$$
\begin{tikzcd}
 & (A\otimes B)\otimes(C\otimes(D\otimes E)) \arrow[r, "a"] \arrow[d, phantom, "\Searrow\pi_{A,B,C,(D\otimes E)}"] & ((A\otimes B)\otimes C)\otimes(D\otimes E) \arrow[r, "a"] \arrow[dr, phantom, "\Searrow a"] & (((A\otimes B)\otimes C)\otimes D)\otimes E \\
 & & (A\otimes(B\otimes C))\otimes(D\otimes E) \arrow[u, "a\otimes(D\otimes E)"] \arrow[r, "a"] \arrow[ddr, phantom, "\Searrow\pi_{A,(B\otimes C),D,E}"] & ((A\otimes(B\otimes C))\otimes D)\otimes E \arrow[u, "(a\otimes D)\otimes E"'] \\
A\otimes(B\otimes(C\otimes(D\otimes E))) \arrow[uur, "a"] \arrow[r, "A\otimes a"] \arrow[ddr, "A\otimes(B\otimes a)"'] & A\otimes((B\otimes C)\otimes(D\otimes E)) \arrow[ur, "a"] \arrow[dr, "A\otimes a"'] \arrow[dd, phantom, "\Downarrow A\otimes\pi_{B,C,D,E}"] & & \\
 & & A\otimes(((B\otimes C)\otimes D)\otimes E) \arrow[r, "a"'] \arrow[dr, phantom, "\Searrow a"] & (A\otimes((B\otimes C)\otimes D))\otimes E \arrow[uu, "a\otimes E"'] \\
 & A\otimes(B\otimes((C\otimes D)\otimes E)) \arrow[r, "A\otimes a"'] & A\otimes((B\otimes(C\otimes D))\otimes E) \arrow[u, "A\otimes(a\otimes E)"'] \arrow[r, "a"'] & (A\otimes(B\otimes(C\otimes D)))\otimes E \arrow[u, "(A\otimes a)\otimes E"']
\end{tikzcd}
$$

Figure 3.1: The associativity axiom used in the definition of monoidal bicategory (sometimes called the *non-abelian 4-cocycle condition*).

and

$$\begin{tikzcd}
A\otimes(B\otimes C) \arrow[rr, "a_{A,B,C}"] \arrow[drr, phantom, "\Searrow a_{A,B,l_C}"] & & (A\otimes B)\otimes C \arrow[drr, phantom, "\Rightarrow \mu_{(A\otimes B),C}"] & & \\
A\otimes(B\otimes(\mathbb{I}\otimes C)) \arrow[u, "A\otimes(B\otimes l_C)"] \arrow[rr, "a_{A,B,\mathbb{I}\otimes C}"] \arrow[dr, "A\otimes a_{B,\mathbb{I},C}"'] & & (A\otimes B)\otimes(\mathbb{I}\otimes C) \arrow[u, "(A\otimes B)\otimes l_C"'] \arrow[rr, "a_{(A\otimes B),\mathbb{I},C}"'] \arrow[d, phantom, "\Downarrow \pi_{A,\mathbb{I},B,C}"] & & ((A\otimes B)\otimes\mathbb{I})\otimes C \arrow[ull, "r_{A\otimes B}\otimes C"'] \\
& A\otimes((B\otimes\mathbb{I})\otimes C) \arrow[rr, "a_{A,B\otimes\mathbb{I},C}"'] & & (A\otimes(B\otimes\mathbb{I}))\otimes C \arrow[ur, "a_{A,\mathbb{I},B}\otimes C"'] &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
& A\otimes(B\otimes C) \arrow[r, "a_{A,B,C}"] & (A\otimes B)\otimes C & \\
A\otimes(B\otimes(\mathbb{I}\otimes C)) \arrow[ur, "A\otimes(B\otimes l_C)"] \arrow[dr, "A\otimes a_{B,\mathbb{I},C}"'] \arrow[r, phantom, "\Searrow A\otimes\mu_{B,C}"] & {} \arrow[r, phantom, "\Searrow a_{A,r_B,C}"] & {} \arrow[r, phantom, "\overset{\Rightarrow}{R_{A,B}\otimes C}"] & ((A\otimes B)\otimes\mathbb{I})\otimes C \arrow[ul, "r_{A\otimes B}\otimes C"'] \\
& A\otimes((B\otimes\mathbb{I})\otimes C) \arrow[uu, "A\otimes(r_B\otimes C)"] \arrow[r, "a_{A,B\otimes\mathbb{I},C}"'] & (A\otimes(B\otimes\mathbb{I}))\otimes C \arrow[uu, "(A\otimes r_B)\otimes C"] \arrow[ur, "a_{A,B,\mathbb{I}}\otimes C"'] &
\end{tikzcd}$$

This is not *quite* the most general possible definition, since we have merely specified an object $\mathbb{I}$ rather than a pseudo-functor $1 \to \mathcal{B}$. But since every pseudo-functor is equivalent to a normal one, there is no essential loss of generality.

When we have occasion to refer explicitly to the equivalence-inverse of a, l or r, we shall denote it as a', l' or r'. Furthermore, we shall assume where necessary that we have *adjoint* equivalences $a \dashv a'$, $l \dashv l'$ etc.

When working in a monoidal bicategory, we extend our convention of not explicitly naming structural isomorphisms to the isomorphisms representing the pseudo-functoriality of tensor. Instead we mark them with the symbol ~. Note that there will usually be some implicit structural isomorphisms too: for example, given 1-cells $f : A \to B$ and $g : C \to D$, the diagram

$$\begin{tikzcd}
A\otimes B \arrow[r, "A\otimes g"] \arrow[d, "f\otimes B"'] \arrow[dr, phantom, "\sim"] & A\otimes D \arrow[d, "f\otimes D"] \\
C\otimes B \arrow[r, "C\otimes g"'] & C\otimes D
\end{tikzcd}$$

indicates the 2-cell

$$(f\otimes D)\circ(A\otimes g) \xrightarrow{\otimes_{(f,1),(1,g)}} (f\circ 1)\otimes(1\circ g) \xrightarrow{\rho_f\otimes\lambda_g} f.g$$

$$\xrightarrow{\lambda_f^{-1}\otimes\rho_g^{-1}} (1.f)\otimes(g.1) \xrightarrow{\otimes^{-1}_{(1,f),(g,1)}} (C\otimes g)\circ(f\otimes B).$$

Remark 3.2. Notice that the first unit equation – since all its cells are invertible –

allows $A \otimes L_{B,C}$ to be expressed in terms of π, μ and a. In particular $\mathbb{I} \otimes L_{B,C}$ may be so expressed which, since the pseudofunctor $\mathbb{I} \otimes -$ is equivalent, via l, to the identity, allows $L_{B,C}$ to be expressed in terms of π, μ, a and the implicit data that make l an equivalence. Furthermore it is not hard to see that the components $L_{A,B}$ thus defined constitute a modification. Therefore the modification L may be defined in terms of the other data. It is perhaps tempting to conclude that L is redundant, but there is a subtlety: the modification L so defined does not necessarily – at least as far as I can tell – satisfy the necessary equation. Thus we *could* suppress L (and R too, since its definition is symmetrical to that of L) from our definition, but only at the expense of introducing a new and complicated equation involving the other data.

3.1 Monoidal pseudo-functors and transformations

Definition 3.3. A *monoidal pseudofunctor* $F : \mathcal{B} \to \mathcal{C}$, between monoidal bicategories $\mathcal{B}$ and $\mathcal{C}$, consists of a pseudofunctor equipped with:

- a 1-cell $F^{\otimes}_{\mathbb{I}} : \mathbb{I} \to F\mathbb{I}$,
- a pseudo-natural transformation $F^{\otimes}$ with components

$$F^{\otimes}_{A,B} : FA \otimes FB \to F(A \otimes B)$$

- an invertible modification F^a with components

$$\begin{tikzcd}
FA \otimes (FB \otimes FC) \arrow[r, "a_{FA,FB,FC}"] \arrow[d, "FA \otimes F^{\otimes}_{B,C}"'] & (FA \otimes FB) \otimes FC \arrow[d, "F^{\otimes}_{A,B} \otimes FC"] \\
FA \otimes F(B \otimes C) \arrow[d, "F^{\otimes}_{A,B\otimes C}"'] \arrow[r, phantom, "\nearrow F^a_{A,B,C}"] & F(A \otimes B) \otimes FC \arrow[d, "F^{\otimes}_{A\otimes B,C}"] \\
F(A \otimes (B \otimes C)) \arrow[r, "F(a_{A,B,C})"'] & F((A \otimes B) \otimes C)
\end{tikzcd}$$

invertible modifications F^l and F^r with components

$$
\begin{tikzcd}
\mathbb{I}\otimes FA \arrow[r, "F^{\otimes}_{\mathbb{I}}\otimes FA"] \arrow[d, "l_{FA}"'] \arrow[dr, phantom, "\overset{\Rightarrow}{F^l_A}"] & F\mathbb{I}\otimes FA \arrow[d, "F^{\otimes}_{\mathbb{I},A}"] \\
FA & F(\mathbb{I}\otimes A) \arrow[l, "F(l_A)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA\otimes\mathbb{I} \arrow[r, "FA\otimes F^{\otimes}_{\mathbb{I}}"] \arrow[d, "r_{FA}"'] \arrow[dr, phantom, "\overset{\Rightarrow}{F^r_A}"] & FA\otimes F\mathbb{I} \arrow[d, "F^{\otimes}_{A,\mathbb{I}}"] \\
FA & F(A\otimes\mathbb{I}) \arrow[l, "F(r_A)"]
\end{tikzcd}
$$

satisfying the equation shown in Figs. 3.2 and 3.3, and also such that for all $A, B \in \mathcal{B}$ the pasting

$$
\begin{tikzcd}
FA\otimes(\mathbb{I}\otimes FB) \arrow[r, "a_{FA,\mathbb{I},FB}"] \arrow[d, "FA\otimes(F^{\otimes}_{\mathbb{I}}\otimes FB)"'] \arrow[dr, phantom, "\Nearrow a_{FA,F^{\otimes}_{\mathbb{I}},FB}"] \arrow[ddd, bend right=80, "FA\otimes l_{FB}"'] \arrow[ddd, phantom, bend right=40, "\overset{\Rightarrow}{FA\otimes F^l_B}"] & (FA\otimes\mathbb{I})\otimes FB \arrow[d, "(FA\otimes F^{\otimes}_{\mathbb{I}})\otimes FB"] & \\
FA\otimes(F\mathbb{I}\otimes FB) \arrow[r, "a_{FA,F\mathbb{I},FB}"'] \arrow[d, "FA\otimes F^{\otimes}_{\mathbb{I},B}"] & (FA\otimes F\mathbb{I})\otimes FB \arrow[r, "F^{\otimes}_{A,\mathbb{I}}\otimes FB"] \arrow[d, phantom, "\Nearrow F^a_{A,\mathbb{I},B}"] & F(A\otimes\mathbb{I})\otimes FB \arrow[d, "F^{\otimes}_{A\otimes\mathbb{I},B}"] \\
FA\otimes F(\mathbb{I}\otimes B) \arrow[r, "F^{\otimes}_{A,\mathbb{I},B}"] \arrow[d, "FA\otimes FB"'] \arrow[dr, phantom, "\Nearrow F^{\otimes}_{A,l_B}"] & F(A\otimes(\mathbb{I}\otimes B)) \arrow[r, "F(a_{A,\mathbb{I},B})"] \arrow[d, "F(A\otimes l_B)"'] \arrow[dr, phantom, "\overset{\Rightarrow}{F(\mu_{A,B})}"] & F((A\otimes\mathbb{I})\otimes B) \arrow[dl, "F(r_A\otimes B)"] \\
FA\otimes FB \arrow[r, "F^{\otimes}_{A,B}"'] & F(A\otimes B) &
\end{tikzcd}
$$

is equal to

$$
\begin{tikzcd}
FA\otimes(\mathbb{I}\otimes FB) \arrow[rr, "a_{FA,\mathbb{I},FB}"] \arrow[dr, "FA\otimes l_{FB}"'] & \overset{\Rightarrow}{\mu_{FA,FB}} & (FA\otimes\mathbb{I})\otimes FB \arrow[rr, "(FA\otimes F^{\otimes}_{\mathbb{I}})\otimes FB"] \arrow[dl, "r_{FA}\otimes FB"] & \overset{\Rightarrow}{F^r_A\otimes FB} & (FA\otimes F\mathbb{I})\otimes FB \arrow[d, "F^{\otimes}_{A,\mathbb{I}}\otimes FB"] \\
& FA\otimes FB \arrow[d, "F^{\otimes}_{A,B}"'] & & \Searrow F^{\otimes}_{r_A,B} & F(A\otimes\mathbb{I})\otimes FB \arrow[lll, "F(r_A)\otimes FB"'] \arrow[d, "F^{\otimes}_{A\otimes\mathbb{I},B}"] \\
& F(A\otimes B) & & & F((A\otimes\mathbb{I})\otimes B) \arrow[lll, "F(r_A\otimes B)"]
\end{tikzcd}
$$

Definition 3.4. A *strong* monoidal pseudofunctor is a monoidal pseudofunctor F for which the 1-cell $F^{\otimes}_{\mathbb{I}}$ is an equivalence, and the pseudo-natural transformation $F^{\otimes}$ is a pseudo-natural equivalence.

Remark 3.5. One might try to define a *strict* map of monoidal bicategories to be a monoidal strict functor F such that $F^{\otimes}_{I}$ and $F_{\otimes}$ are identities. However it's easy to see that, with this definition, the composite of two strict maps is not necessarily strict

$$\begin{tikzcd}[column sep=large]
FA \otimes (FB \otimes (FC \otimes FD)) \arrow[r, "FA \otimes (FB \otimes F^{\otimes}_{C,D})"] \arrow[d, "FA \otimes a_{FB,FC,FD}"'] \arrow[drr, phantom, "\swarrow FA \otimes F^{a}_{B,C,D}"] & FA \otimes (FB \otimes F(C \otimes D)) \arrow[r, "FA \otimes F^{\otimes}_{B,C\otimes D}"] & FA \otimes F(B \otimes (C \otimes D)) \arrow[r, "F^{\otimes}_{A,B\otimes(C\otimes D)}"] \arrow[d, "FA \otimes F(a_{B,C,D})"'] \arrow[dr, phantom, "\swarrow (F^{\otimes}_{A,a_{B,C,D}})^{-1}"] & F(A \otimes (B \otimes (C \otimes D))) \arrow[d, "F(A \otimes a_{B,C,D})"] \arrow[dr, bend left, "F(a_{A,B,C\otimes D})"] & \\
FA \otimes ((FB \otimes FC) \otimes FD) \arrow[r, "FA \otimes (F^{\otimes}_{B,C} \otimes FD)"] \arrow[d, "a_{FA,FB\otimes FC,FD}"'] \arrow[dr, phantom, "\swarrow \alpha_{FA,F^{\otimes}_{B,C},FD}"] & FA \otimes (F(B \otimes C) \otimes FD) \arrow[r, "FA \otimes F^{\otimes}_{B\otimes C,D}"] \arrow[d, "a_{FA,F(B\otimes C),FD}"] \arrow[drr, phantom, "\swarrow F^{a}_{A,B\otimes C,D}"] & FA \otimes F((B \otimes C) \otimes D) \arrow[r, "F^{\otimes}_{A,(B\otimes C)\otimes D}"] & F(A \otimes ((B \otimes C) \otimes D)) \arrow[d, "F(a_{A,B\otimes C,D})"'] \arrow[r, phantom, "\overset{\Leftarrow}{F(\pi_{A,B,C,D})}"] & F((A \otimes B) \otimes (C \otimes D)) \arrow[ddl, bend left, "F(a_{A,B,C\otimes D})"] \\
(FA \otimes (FB \otimes FC)) \otimes FD \arrow[r, "(FA \otimes F^{\otimes}_{B,C}) \otimes FD"'] \arrow[d, "a_{FA,FB,FC} \otimes FD"'] \arrow[drr, phantom, "\swarrow F^{a}_{A,B,C} \otimes FD"] & (FA \otimes F(B \otimes C)) \otimes FD \arrow[r, "F^{\otimes}_{A,B\otimes C} \otimes FD"'] & F(A \otimes (B \otimes C)) \otimes FD \arrow[r, "F^{\otimes}_{A\otimes(B\otimes C),D}"'] \arrow[d, "F(a_{A,B,C}) \otimes FD"'] \arrow[dr, phantom, "\swarrow (F^{\otimes}_{a_{A,B,C},D})^{-1}"] & F((A \otimes (B \otimes C)) \otimes D) \arrow[d, "F(a_{A,B,C} \otimes D)"] & \\
((FA \otimes FB) \otimes FC) \otimes FD \arrow[r, "(F^{\otimes}_{A,B} \otimes FC) \otimes FD"'] & (F(A \otimes B) \otimes FC) \otimes FD \arrow[r, "F^{\otimes}_{A\otimes B,C} \otimes FD"'] & F((A \otimes B) \otimes C) \otimes FD \arrow[r, "F^{\otimes}_{(A\otimes B)\otimes C,D}"'] & F(((A \otimes B) \otimes C) \otimes D) &
\end{tikzcd}$$

Figure 3.2: Left-hand side of an equation used in the definition of monoidal pseudo-functor: for all $A, B, C, D \in \mathcal{B}$, this pasting must be equal to the one shown in Fig. 3.3.

$$
\begin{tikzcd}[column sep=small]
& FA \otimes (FB \otimes (FC \otimes FD)) \arrow[r, "FA \otimes (FB \otimes F^{\otimes}_{C,D})"] \arrow[dl, bend right, "FA \otimes a_{FB,FC,FD}"'] \arrow[dd, "a_{FA,FB,FC\otimes FD}"] \arrow[dr, phantom, "\Swarrow a_{FA,FB,F(C\otimes D)}"] & FA \otimes (FB \otimes F(C \otimes D)) \arrow[r, "FA \otimes F^{\otimes}_{B,C\otimes D}"] \arrow[d, "a_{FA,FB,F(C\otimes D)}"] & FA \otimes F(B \otimes (C \otimes D)) \arrow[r, "F^{\otimes}_{A,B\otimes(C\otimes D)}"] \arrow[dd, phantom, "\Swarrow F^{a}_{A,B,C\otimes D}"] & F(A \otimes (B \otimes (C \otimes D))) \arrow[dd, "F(a_{A,B,C\otimes D})"] \\
FA \otimes ((FB \otimes FC) \otimes FD) \arrow[dd, "a_{FA,FB\otimes FC,FD}"'] & & (FA \otimes FB) \otimes F(C \otimes D) \arrow[dr, "F^{\otimes}_{A,B} \otimes F(C \otimes D)"] \arrow[dd, phantom, "\cong"] & & \\
\arrow[r, phantom, "\Leftarrow \pi_{FA,FB,FC,FD}"] & (FA \otimes FB) \otimes (FC \otimes FD) \arrow[ur, "(FA \otimes FB) \otimes F^{\otimes}_{C,D}"] \arrow[dr, "F^{\otimes}_{A,B} \otimes (FC \otimes FD)"'] \arrow[dd, "a_{FA\otimes FB,FC,FD}"] & & F(A \otimes B) \otimes F(C \otimes D) \arrow[r, "F^{\otimes}_{A\otimes B,C\otimes D}"] \arrow[dd, phantom, "\Swarrow F^{a}_{A\otimes B,C,D}"] & F((A \otimes B) \otimes (C \otimes D)) \arrow[dd, "F(a_{A\otimes B,C,D})"] \\
(FA \otimes (FB \otimes FC)) \otimes FD \arrow[dr, bend right, "a_{FA,FB,FC} \otimes FD"'] & \arrow[dr, phantom, "\Swarrow a_{F^{\otimes}_{A,B},FC,FD}"] & F(A \otimes B) \otimes (FC \otimes FD) \arrow[ur, "F(A \otimes B) \otimes F^{\otimes}_{C,D}"'] \arrow[d, "a_{F(A\otimes B),FC,FD}"] & & \\
& ((FA \otimes FB) \otimes FC) \otimes FD \arrow[r, "(F^{\otimes}_{A,B} \otimes FC) \otimes FD"'] & (F(A \otimes B) \otimes FC) \otimes FD \arrow[r, "F^{\otimes}_{A\otimes B,C} \otimes FD"'] & F((A \otimes B) \otimes C) \otimes FD \arrow[r, "F^{\otimes}_{(A\otimes B)\otimes C,D}"'] & F(((A \otimes B) \otimes C) \otimes D)
\end{tikzcd}
$$

Figure 3.3: Right-hand side of the equation: this pasting must be equal to the one shown in Fig. 3.2

(because the composite of two identities is not necessarily an identity). On the other hand, there certainly *is* a category whose objects are monoidal bicategories and whose arrows preserve all the structure on the nose. Moreover, these strict maps are in one-one correspondence with monoidal strict functors defined as above; but their composition is subtly different.

The moral of the story is that we cannot, in general, expect to be able to define strict maps merely as special pseudo-maps. That this is possible for bicategories is merely a low-dimensional quirk.

In a similar way, composition of monoidal pseudofunctors is not associative on the nose; again, the existence of a category (as opposed to a bi- or tricategory) of bicategories and pseudofunctors is a low-dimensional phenomenon that cannot be expected to persist at higher dimensions.

Definition 3.6. A *monoidal pseudo-natural transformation* $\gamma : F \Rightarrow G : \mathcal{B} \to \mathcal{C}$ between monoidal pseudo-functors F and G is a pseudo-natural transformation equipped with an invertible 2-cell

$$\gamma^{\otimes}_{\mathbb{I}} : \gamma_{\mathbb{I}} \circ F^{\otimes}_{\mathbb{I}} \Rightarrow G^{\otimes}_{\mathbb{I}}$$

and an invertible modification with components

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\gamma_A \otimes \gamma_B"'] \arrow[dr, phantom, "\swarrow \gamma^{\otimes}_{A,B}"] & F(A \otimes B) \arrow[d, "\gamma_{A\otimes B}"] \\
GA \otimes GB \arrow[r, "G^{\otimes}_{A,B}"'] & G(A \otimes B)
\end{tikzcd}$$

such that for all A, B, $C \in \mathcal{B}$, the pasting

$$\begin{tikzcd}
FA \otimes (FB \otimes FC) \arrow[r, "FA \otimes F^{\otimes}_{B,C}"] \arrow[d, "a_{FA,FB,FC}"'] & FA \otimes F(B \otimes C) \arrow[r, "F^{\otimes}_{A,B\otimes C}"] \arrow[d, phantom, "\swarrow F^{a}_{A,B,C}"] & F(A \otimes (B \otimes C)) \arrow[d, "F(a_{A,B,C})"] \\
(FA \otimes FB) \otimes FC \arrow[r, "F^{\otimes}_{A,B} \otimes FC"] \arrow[d, "(FA \otimes FB) \otimes \gamma_C"'] \arrow[dd, bend right=60, "\gamma_A \otimes (\gamma_B \otimes \gamma_C)"'] \arrow[dr, phantom, "\cong"] & F(A \otimes B) \otimes FC \arrow[r, "F^{\otimes}_{A\otimes B,C}"] \arrow[d, "F(A \otimes B) \otimes \gamma_C"'] \arrow[dd, bend left=40, "\gamma_{A\otimes B} \otimes \gamma_C"] \arrow[dr, phantom, "\swarrow \gamma^{\otimes}_{A\otimes B,C}"] & F((A \otimes B) \otimes C) \arrow[dd, "\gamma_{A\otimes B,C}"] \\
(FA \otimes FB) \otimes GC \arrow[r, "F^{\otimes}_{A,B} \otimes GC"'] \arrow[d, "(\gamma_A \otimes \gamma_B) \otimes GC"'] \arrow[dr, phantom, "\swarrow \gamma^{\otimes}_{A,B} \otimes GC"] & F(A \otimes B) \otimes GC \arrow[d, "\gamma_{A\otimes B} \otimes GC"'] & \\
(GA \otimes GB) \otimes GC \arrow[r, "G^{\otimes}_{A,B} \otimes GC"'] & G(A \otimes B) \otimes GC \arrow[r, "G^{\otimes}_{A\otimes B,C}"'] & G((A \otimes B) \otimes C)
\end{tikzcd}$$

is equal to

$$
\begin{tikzcd}
 & FA \otimes (FB \otimes FC) \arrow[r, "FA \otimes F^{\otimes}_{B,C}"] \arrow[d, "FA \otimes (\gamma_B \otimes \gamma_C)"] \arrow[dd, bend right=60, "\gamma_A \otimes (\gamma_B \otimes \gamma_C)"', "\cong"] \arrow[dddl, bend right, "a_{FA,FB,FC}"'] \arrow[dr, phantom, "\Swarrow FA \otimes \gamma^{\otimes}_{B,C}"] & FA \otimes F(B \otimes C) \arrow[r, "F^{\otimes}_{A,B\otimes C}"] \arrow[d, "FA \otimes \gamma_{B\otimes C}"', "\cong"] \arrow[dd, bend left=50, "\gamma_A \otimes \gamma_{B\otimes C}"] & F(A \otimes (B \otimes C)) \arrow[dd, "\gamma_{A\otimes(B\otimes C)}"] \arrow[dr, "F(a_{A,B,C})"] & \\
 & FA \otimes (GB \otimes GC) \arrow[r, "FA \otimes G^{\otimes}_{B,C}"'] \arrow[d, "\gamma_A \otimes (GB \otimes GC)"] \arrow[dr, phantom, "\cong"] & FA \otimes G(B \otimes C) \arrow[d, "\gamma_A \otimes G(B \otimes C)"'] \arrow[r, phantom, "\Swarrow \gamma^{\otimes}_{A,B\otimes C}"] & {} \arrow[r, phantom, "\Leftarrow \gamma_{\alpha_{A,B,C}}"] & F((A \otimes B) \otimes C) \arrow[ddl, "\gamma_{(A\otimes B)\otimes C}"] \\
{} \arrow[dr, phantom, "\Swarrow a_{\gamma_A,\gamma_B,\gamma_C}"] & GA \otimes (GB \otimes GC) \arrow[r, "GA \otimes G^{\otimes}_{B,C}"'] \arrow[d, "a_{GA,GB,GC}"] & GA \otimes G(B \otimes C) \arrow[r, "G^{\otimes}_{A,B\otimes C}"'] \arrow[dr, phantom, "\Swarrow G^{a}_{A,B,C}"] & G(A \otimes (B \otimes C)) \arrow[d, "G(a_{A,B,C})"'] & \\
(FA \otimes FB) \otimes FC \arrow[r, "(\gamma_A \otimes \gamma_B) \otimes \gamma_C"'] & (GA \otimes GB) \otimes GC \arrow[r, "G^{\otimes}_{A,B} \otimes GC"'] & G(A \otimes B) \otimes GC \arrow[r, "G^{\otimes}_{A\otimes B,C}"'] & G((A \otimes B) \otimes C) &
\end{tikzcd}
$$

and for all $A \in \mathcal{B}$, the pasting

$$
\begin{tikzcd}
 & FA \otimes \mathbb{I} \arrow[r, "FA \otimes F^{\otimes}_{\mathbb{I}}"] \arrow[dl, bend right, "\gamma_A \otimes \mathbb{I}"'] \arrow[d, "r_{FA}"'] \arrow[dr, phantom, "\Rightarrow F^{r}_{A}"] & FA \otimes F\mathbb{I} \arrow[d, "F^{\otimes}_{A,\mathbb{I}}"] \arrow[dr, bend left, "\gamma_A \otimes \gamma_{\mathbb{I}}"] & \\
GA \otimes \mathbb{I} \arrow[r, phantom, "\Rightarrow r_{\gamma_A}"] \arrow[dr, bend right, "r_{GA}"'] & FA \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\Nearrow \gamma_{r_A}"] & F(A \otimes \mathbb{I}) \arrow[l, "F(r_A)"] \arrow[d, "\gamma_{A\otimes\mathbb{I}}"] \arrow[r, phantom, "\Rightarrow \gamma^{\otimes}_{A,\mathbb{I}}"] & GA \otimes G\mathbb{I} \arrow[dl, bend left, "G^{\otimes}_{A,\mathbb{I}}"] \\
 & GA & G(A \otimes \mathbb{I}) \arrow[l, "G(r_A)"] &
\end{tikzcd}
$$

is equal to

$$
\begin{tikzcd}
FA \otimes \mathbb{I} \arrow[r, "FA \otimes F^{\otimes}_{\mathbb{I}}"] \arrow[d, "\gamma_A \otimes \mathbb{I}"'] \arrow[dr, phantom, "\cong"] & FA \otimes F\mathbb{I} \arrow[d, "\gamma_A \otimes F\mathbb{I}"'] \arrow[dr, "\gamma_A \otimes \gamma_{\mathbb{I}}"] \arrow[dr, phantom, "\cong"'] & \\
GA \otimes \mathbb{I} \arrow[r, "GA \otimes F^{\otimes}_{\mathbb{I}}"'] \arrow[rr, bend right=30, "GA \otimes G^{\otimes}_{\mathbb{I}}"', "\Uparrow GA \otimes \gamma^{\otimes}_{\mathbb{I}}"] \arrow[d, "r_{GA}"'] \arrow[drr, phantom, "\Rightarrow G^{r}_{A}"] & GA \otimes F\mathbb{I} \arrow[r, "GA \otimes \gamma_{\mathbb{I}}"'] & GA \otimes G\mathbb{I} \arrow[d, "G^{\otimes}_{A,\mathbb{I}}"] \\
GA & & G(A \otimes \mathbb{I}) \arrow[ll, "G(r_A)"]
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & \mathbb{I} \otimes FA \arrow[r, "F^{\otimes}_{\mathbb{I}} \otimes FA"] \arrow[dl, bend right, "\mathbb{I} \otimes \gamma_A"'] \arrow[d, "l_{FA}"'] \arrow[dr, phantom, "\Rightarrow F^l_A"] & F\mathbb{I} \otimes FA \arrow[d, "F^{\otimes}_{\mathbb{I},A}"] \arrow[dr, bend left, "\gamma_{\mathbb{I}} \otimes \gamma_A"] & \\
\mathbb{I} \otimes GA \arrow[r, phantom, "\Rightarrow l_{\gamma_A}"] \arrow[dr, bend right, "l_{GA}"'] & FA \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\Nearrow \gamma_{l_A}"] & F(\mathbb{I} \otimes A) \arrow[l, "F(l_A)"] \arrow[d, "\gamma_{\mathbb{I} \otimes A}"] \arrow[r, phantom, "\Rightarrow \gamma^{\otimes}_{\mathbb{I},A}"] & G\mathbb{I} \otimes GA \arrow[dl, bend left, "G^{\otimes}_{\mathbb{I},A}"] \\
 & GA & G(\mathbb{I} \otimes A) \arrow[l, "G(l_A)"] &
\end{tikzcd}
$$

is equal to

$$
\begin{tikzcd}
\mathbb{I} \otimes FA \arrow[r, "F^{\otimes}_{\mathbb{I}} \otimes FA"] \arrow[d, "\mathbb{I} \otimes \gamma_A"'] \arrow[dr, phantom, "\cong"] & F\mathbb{I} \otimes FA \arrow[d, "F\mathbb{I} \otimes \gamma_A"'] \arrow[dr, "\gamma_I \otimes \gamma_A"] \arrow[dr, phantom, "\cong"'] & \\
\mathbb{I} \otimes GA \arrow[r, "F^{\otimes}_{\mathbb{I}} \otimes GA"'] \arrow[rr, bend right, "G^{\otimes}_{\mathbb{I}} \otimes GA"'] \arrow[d, "l_{GA}"'] \arrow[drr, phantom, "\Rightarrow G^l_A"] & F\mathbb{I} \otimes GA \arrow[r, "\gamma_{\mathbb{I}} \otimes GA"'] \arrow[d, phantom, "\Uparrow \gamma^{\otimes}_{\mathbb{I}} \otimes GA"] & G\mathbb{I} \otimes GA \arrow[d, "G^{\otimes}_{\mathbb{I},A}"] \\
GA & & G(\mathbb{I} \otimes A) \arrow[ll, "G(l_A)"]
\end{tikzcd}
$$

Definition 3.7. A *monoidal modification* $m : \gamma \Rrightarrow \delta : F \Rightarrow G$, between monoidal pseudo-natural transformations F and G, is a modification with the property that

$$
\begin{tikzcd}
 & F\mathbb{I} \arrow[dd, "\delta_{\mathbb{I}}"'] \arrow[dd, bend left=80, "\gamma_{\mathbb{I}}"] \arrow[dd, phantom, "\Leftarrow m_{\mathbb{I}}", shift left=4ex] & & \\
\mathbb{I} \arrow[ur, "F^{\otimes}_{\mathbb{I}}"] \arrow[dr, "G^{\otimes}_{\mathbb{I}}"'] \arrow[r, phantom, "\swarrow \delta^{\otimes}_{\mathbb{I}}"] & & = & \gamma^{\otimes}_{\mathbb{I}} \\
 & G\mathbb{I} & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
FA \otimes FB \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\gamma_A \otimes \gamma_B"] \arrow[d, bend right=60, "\delta_A \otimes \delta_B"'] \arrow[d, phantom, bend right=30, "\Leftarrow\ m_A \otimes m_B"'] \arrow[dr, phantom, "\swarrow \gamma^{\otimes}_{A,B}"] & F(A \otimes B) \arrow[d, "\gamma_{A \otimes B}"] \\
GA \otimes GB \arrow[r, "G^{\otimes}_{A,B}"'] & G(A \otimes B)
\end{tikzcd}
=
\begin{tikzcd}
FA \otimes FB \arrow[r, "F^{\otimes}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] \arrow[dr, phantom, "\swarrow \delta^{\otimes}_{A,B}"] & F(A \otimes B) \arrow[d, "\delta_{A \otimes B}"'] \arrow[d, bend left=60, "\gamma_{A \otimes B}"] \arrow[d, phantom, bend left=30, "\Leftarrow\ m_{A \otimes B}"] \\
GA \otimes GB \arrow[r, "G^{\otimes}_{A,B}"'] & G(A \otimes B)
\end{tikzcd}
$$

3.2 On coherence

This section aims to give a concise overview of the available coherence results for monoidal bicategories. In a later chapter (4), we shall prove some further coherence results, which are then used to develop a framework that allows us to reason about pseudomonoids without becoming bogged down in coherence conditions. First, we survey the coherence results that do (and do not) exist in the literature. The existing coherence theorems apply to tricategories, of which monoidal bicategories are a special case: a tricategory with one object is essentially a monoidal bicategory, in just the same way that a bicategory with one object is essentially a monoidal category. We shall state the results as they apply to monoidal bicategories, since that is the situation of interest here.

The original coherence theorem (Gordon et al., 1995) implies that every monoidal bicategory is monoidally biequivalent to a Gray monoid:

Definition 3.8. A *Gray monoid* is a monoidal bicategory $\mathcal{B}$ in which:

- the underlying bicategory $\mathcal{B}$ is a 2-category,
- given composable pairs f, g and h, k of 1-cells, if either f or k is an identity then the structural 2-cell

  $$(f \otimes h) \circ (g \otimes k) \Rightarrow (f \circ g) \otimes (h \circ k)$$

  is an identity,
- the structural equivalences a, l and r are identities.

The second condition here is the most mysterious. It means that for every object $A \in \mathcal{B}$, the pseudofunctors $A \otimes -$ and $- \otimes A$ are 2-functors, and that furthermore the tensor product $f \otimes g$ of 1-cells $f : A \to C$ and $g : B \to D$ is equal to the composite

$$A \otimes B \xrightarrow{f \otimes B} C \otimes B \xrightarrow{C \otimes g} C \otimes D.$$

When working in a Gray monoid, it will often be convenient to decompose tensor products of 1-cells in this way. Then the only structural 2-cells are of the form

$$\begin{tikzcd} A \otimes B \arrow[r, "f \otimes B"] \arrow[d, "A \otimes g"'] \arrow[dr, phantom, "\sim"] & C \otimes B \arrow[d, "C \otimes g"] \\ A \otimes D \arrow[r, "f \otimes D"'] & C \otimes D, \end{tikzcd}$$

or composites thereof. We shall label them merely with the symbol ~ (as above), since there is no possible ambiguity.

This line of work is extended by Gurski (2006), who shows that every *free* monoidal bicategory is monoidally biequivalent to a free Gray monoid on the same generators. In particular, Gurski shows that the *canonical* monoidal pseudofunctor from the free monoidal bicategory to the free Gray monoid is a monoidal biequivalence. For our purposes, this theorem is not as useful as it sounds. For example, we should like to say that the free monoidal bicategory on a pseudomonoid object is canonically monoidally biequivalent to the free Gray monoid on a pseudomonoid object. I imagine this is true[1], but it does not follow from Gurski's theorem. The reason is that the theorem applies only to free constructions of a particular sort, in which the 1-cell generators are of the form $X \to Y$, where X and Y are 0-cell generators. A pseudomonoid can not be described in this form, since it requires a 'tensor' 1-cell $P : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$.

Gurski also provides a strictification (Grayification?) construction, showing explicitly how to construct, for any monoidal bicategory $\mathcal{B}$, a Gray monoid $\mathrm{Gr}(\mathcal{B})$ together with explicit monoidal biequivalences $e : \mathrm{Gr}(\mathcal{B}) \to \mathcal{B}$ and $f : \mathcal{B} \to \mathrm{Gr}(\mathcal{B})$. Furthermore, for any strong monoidal pseudofunctor $F : \mathcal{B} \to \mathcal{C}$, Gurski constructs a strict functor $\mathrm{Gr}F : \mathrm{Gr}(\mathcal{B}) \to \mathrm{Gr}(\mathcal{C})$ such that the diagrams

$$\begin{tikzcd} \mathrm{Gr}(\mathcal{B}) \arrow[r, "\mathrm{Gr}F"] \arrow[d, "e"'] & \mathrm{Gr}(\mathcal{C}) \arrow[d, "e"] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "f"'] & \mathcal{C} \arrow[d, "f"] \\ \mathrm{Gr}(\mathcal{B}) \arrow[r, "\mathrm{Gr}F"'] & \mathrm{Gr}(\mathcal{C}) \end{tikzcd}$$

[1] Let me be bold: I conjecture that the free monoidal bicategory on a pseudomonoid object is canonically monoidally biequivalent to the free *strict 2-monoidal 2-category* on a pseudomonoid object, where '2-monoidal' means that the tensor product is given by a 2-functor rather than a general pseudofunctor.

commute up to monoidal equivalence. We make essential use of this construction in Chapter 5.

3.3 Braided monoidal bicategories

The history of attempts to define the concept of braided monoidal bicategory highlights the difficulties inherent in even finding the correct definition. The first definition was given by Kapranov and Voevodsky (1994), with some errors and omissions. The errors and some omissions were pointed out by Carmody (1995) and Baez and Neuchl (1996) – presumably independently, since neither cites the other – though Baez and Neuchl go further than Carmody and give their own definition of braiding for a Gray monoid. This definition includes an additional axiom (our third axiom below), which they attribute to Breen (1994). Baez and Neuchl (1996) do not give any axioms relating the unit object to the braiding. This is mentioned in their Section 5(1) as an issue that remains to be resolved. Subsequently Crans (1998) noticed that this omission causes an error in Baez and Neuchl's 'center' construction, and showed how it could be fixed by adding six axioms for the unit. Day and Street (1997) also gave a definition of braiding with just one axiom for the unit, and despite the superficial differences this axiomatisation is equivalent to Crans's. (It turns out that four of Crans's six unit axioms are redundant.)

These definitions both take the unit to be strict – in terms of our definition below, they take the 1-cells $s_{A,\mathbb{I}}$ and $s_{\mathbb{I},A}$ and the 2-cells $U_{A|\mathbb{I}}$ and $U_{\mathbb{I}|A}$ to be *identities*. The justification for such a restriction is presumably the reasonable expectation that a coherence theorem for tetracategories would show every braided monoidal bicategory to be suitably equivalent to one with such a strict unit. Since an expectation, however reasonable, is not a proof, we have opted to eschew the modest simplification that such a restriction brings. Thus the definition below is conceivably the first that includes all the structure known to be necessary at its natural level of strictness, though we do not claim that any new insight was needed to formulate it. (It is not unimaginable that further axioms should yet be found wanting, though the results of Chapter 4 provide strong evidence that these axioms do suffice.)

Definition 3.9. A *braiding* for a monoidal bicategory $\mathcal{B}$ consists of a pseudo-natural equivalence s with 1-cell components

$$s_{A,B} : A \otimes B \to B \otimes A,$$

invertible modifications $S_{-|-,-}$ and $S_{-,-|-}$ with components

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "a"] \arrow[d, "A \otimes s_{B,C}"'] & (A \otimes B) \otimes C \arrow[r, "s_{A\otimes B,C}"] \arrow[d, phantom, "\swarrow S_{A,B|C}"] & C \otimes (A \otimes B) \arrow[d, "a"] \\
A \otimes (C \otimes B) \arrow[r, "a"'] & (A \otimes C) \otimes B \arrow[r, "s_{A,C} \otimes B"'] & (C \otimes A) \otimes B,
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a'"] \arrow[d, "s_{A,B} \otimes C"'] & A \otimes (B \otimes C) \arrow[r, "s_{A,B\otimes C}"] \arrow[d, phantom, "\nearrow S_{A|B,C}"] & (B \otimes C) \otimes A \arrow[d, "a'"] \\
(B \otimes A) \otimes C \arrow[r, "a'"'] & B \otimes (A \otimes C) \arrow[r, "B \otimes s_{A,C}"'] & B \otimes (C \otimes A),
\end{tikzcd}$$

and invertible modifications $U_{\mathbb{I}|-}$ and $U_{-|\mathbb{I}}$ with components

$$\begin{tikzcd}
\mathbb{I} \otimes A \arrow[rr, "s_{\mathbb{I},A}"] \arrow[dr, "l_A"'] & \arrow[d, phantom, "\Leftarrow U_{\mathbb{I}|A}"] & A \otimes \mathbb{I} \arrow[dl, "r_A"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbb{I} \arrow[rr, "s_{A,\mathbb{I}}"] \arrow[dr, "r_A"'] & \arrow[d, phantom, "\Rightarrow U_{A|\mathbb{I}}"] & \mathbb{I} \otimes A \arrow[dl, "l_A"] \\
& A, &
\end{tikzcd}$$

subject to various axioms. We have chosen to state these axioms in the Gray monoid setting, since the general versions are horribly unwieldy. This can be justified by the observation that the data above may be transported across a monoidal biequivalence: given a monoidal biequivalence (preferably a monoidal pseudo-adjoint biequivalence)

$$\mathcal{B} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

between monoidal bicategories, braiding data for $\mathcal{B}$, say, induces braiding data for $\mathcal{C}$ in a canonical way. For example, the 1-cell $s_{A,B} : A \otimes B \to B \otimes A$ in $\mathcal{C}$ is defined as

$$A \otimes B \simeq GFA \otimes GFB \simeq G(FA \otimes FB) \xrightarrow[G(s_{FA,FB})]{} G(FB \otimes FA) \simeq GFB \otimes GFA \simeq B \otimes A.$$

It is easy, though tedious, to check that all the braiding data may be transported in this way. Given this, and the axioms below, we may define a braided monoidal bicategory to be a monoidal bicategory $\mathcal{B}$ equipped with braiding data, such that these data satisfy the axioms when transported to $\mathrm{Gr}(\mathcal{B})$. This is unsatisfyingly indirect, and complicates the task of verifying these axioms of any particular monoidal bicategory, but seems preferable to the alternative. One should also verify that, given two Gray monoids $\mathcal{B}$ and $\mathcal{C}$ and a monoidal biequivalence as above, the braiding data for $\mathcal{B}$ satisfy the axioms if and only if the transported data satisfy the axioms in $\mathcal{C}$. I confess I have not explicitly done this, though it seems unlikely to be false.

In a more pragmatic (if less precise) vein, one could regard these axioms as being mere abbreviations of the general versions. In practice it is clear how to write down the equivalent axioms for a general monoidal bicategory, by inserting structural 2-cells where necessary.

The axioms follow. There are four axioms that relate the S modifications to each other:

$$
\begin{tikzcd}
A\otimes B\otimes C\otimes D \arrow[r, "s_{A\otimes B\otimes C,D}"] \arrow[d, "A\otimes B\otimes s_{C,D}"'] \arrow[dr, "A\otimes s_{B\otimes C,D}" description] \arrow[dr, phantom, "\Downarrow S_{A,B\otimes C|D}" near start, shift left=6ex] \arrow[dr, phantom, "\swarrow A\otimes S_{B,C|D}" near end, shift right=6ex] & D\otimes A\otimes B\otimes C \\
A\otimes B\otimes D\otimes C \arrow[r, "A\otimes s_{B,D}\otimes C"'] & A\otimes D\otimes B\otimes C \arrow[u, "s_{A,D}\otimes B\otimes C"']
\end{tikzcd}
=
\begin{tikzcd}
A\otimes B\otimes C\otimes D \arrow[r, "s_{A\otimes B\otimes C,D}"] \arrow[d, "A\otimes B\otimes s_{C,D}"'] \arrow[d, phantom, "\Downarrow S_{A\otimes B,C|D}", shift left=8ex] & D\otimes A\otimes B\otimes C \arrow[d, phantom, "\searrow S_{A,B|D}\otimes C", shift right=8ex] \\
A\otimes B\otimes D\otimes C \arrow[r, "A\otimes s_{B,D}\otimes C"'] \arrow[ur, "s_{A\otimes B,D}\otimes C" description] & A\otimes D\otimes B\otimes C \arrow[u, "s_{A,D}\otimes B\otimes C"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\otimes B\otimes C\otimes D \arrow[r, "s_{A\ B\otimes C\otimes D}"] \arrow[d, "s_{A,B}\otimes C\otimes D"'] \arrow[dr, "s_{A,B\otimes C}\otimes D" description] \arrow[dr, phantom, "\Uparrow S_{A|B\otimes C,D}" near start, shift left=6ex] \arrow[dr, phantom, "\nearrow S_{A|B,C}\otimes D" near end, shift right=6ex] & B\otimes C\otimes D\otimes A \\
B\otimes A\otimes C\otimes D \arrow[r, "B\otimes s_{A,C}\otimes D"'] & B\otimes C\otimes A\otimes D \arrow[u, "B\otimes C\otimes s_{A,D}"']
\end{tikzcd}
=
\begin{tikzcd}
A\otimes B\otimes C\otimes D \arrow[r, "s_{A\ B\otimes C\otimes D}"] \arrow[d, "s_{A,B}\otimes C\otimes D"'] \arrow[d, phantom, "\Uparrow S_{A|B,C\otimes D}", shift left=8ex] & B\otimes C\otimes D\otimes A \arrow[d, phantom, "\nwarrow B\otimes S_{A|C,D}", shift right=8ex] \\
B\otimes A\otimes C\otimes D \arrow[r, "B\otimes s_{A,C}\otimes D"'] \arrow[ur, "B\otimes s_{A,C\otimes D}" description] & B\otimes C\otimes A\otimes D \arrow[u, "B\otimes C\otimes s_{A,D}"']
\end{tikzcd}
$$

$$\begin{tikzcd}
A \otimes B \otimes C \otimes D \arrow[rrr, "A \otimes s_{B,C\otimes D}"] \arrow[ddd, "s_{A\otimes B,C} \otimes D"'] \arrow[dr, "A\otimes s_{B,C\otimes D}"] & & & A \otimes C \otimes D \otimes B \arrow[ddd, "s_{A,C\otimes D} \otimes B"] \arrow[dl, "s_{A,C}\otimes B\otimes D"'] \\
& \arrow[r, phantom, "\Uparrow A\otimes S_{B|C,D}"] & {} & \\
\arrow[r, phantom, "\Rightarrow S_{A,B|C}\otimes D"] & A \otimes C \otimes B \otimes D \arrow[r, phantom, "\sim"] \arrow[uurr, "A\otimes C\otimes s_{B,D}"] \arrow[dl, "s_{A,C}\otimes B\otimes D"'] & C \otimes A \otimes D \otimes B \arrow[dr, "C\otimes s_{A,D}\otimes B"] \arrow[r, phantom, "\Rightarrow S_{A|C,D}\otimes B"] & {} \\
C \otimes A \otimes B \otimes D \arrow[rrr, "C\otimes s_{A\otimes B,D}"'] \arrow[urr, "C\otimes A\otimes s_{B,D}"'] & & \arrow[u, phantom, "\Uparrow C\otimes S_{A,B|D}"] & C \otimes D \otimes A \otimes B
\end{tikzcd}$$

$$= \quad \begin{tikzcd}
A \otimes B \otimes C \otimes D \arrow[r, "A \otimes s_{B,C\otimes D}"] \arrow[d, "s_{A\otimes B,C} \otimes D"'] \arrow[dr, "s_{A\otimes B,C\otimes D}" description] \arrow[dr, phantom, shift left=4ex, "\Nearrow S_{A,B|C\otimes D}"] \arrow[dr, phantom, shift right=4ex, "\Nearrow S_{A\otimes B|C,D}"] & A \otimes C \otimes D \otimes B \arrow[d, "s_{A,C\otimes D} \otimes B"] \\
C \otimes A \otimes B \otimes D \arrow[r, "C \otimes s_{A\otimes B,D}"'] & C \otimes D \otimes A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "s_{A\otimes B,C}"] \arrow[dd, "s_{A,B\otimes C}"'] \arrow[dr, "A \otimes s_{B,C}"] & \arrow[d, phantom, "\Downarrow S_{A,B|C}"] & C \otimes A \otimes B \arrow[dd, "C \otimes s_{A,B}"] \\
\arrow[r, phantom, "\swarrow s_{A,s_{B,C}}"] & A \otimes C \otimes B \arrow[ur, "s_{A,C} \otimes B"] \arrow[dr, "s_{A,C\otimes B}"] \arrow[r, phantom, "\swarrow S_{A|C,B}"] & {} \\
B \otimes C \otimes A \arrow[rr, "s_{B,C} \otimes A"'] & & C \otimes B \otimes A
\end{tikzcd}
=
\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "s_{A\otimes B,C}"] \arrow[dd, "s_{A,B\otimes C}"'] \arrow[dr, "s_{A,B} \otimes C"] & & C \otimes A \otimes B \arrow[dd, "C \otimes s_{A,B}"] \arrow[dl, phantom, "\swarrow s^{-1}_{s_{A,B},C}"] \\
\arrow[r, phantom, "\Leftarrow S_{A|B,C}"] & B \otimes A \otimes C \arrow[dl, "B \otimes s_{A,C}"'] \arrow[dr, "s_{B\otimes A,C}"] \arrow[d, phantom, "\Leftarrow S_{B,A|C}"] & \\
B \otimes C \otimes A \arrow[rr, "s_{B,C} \otimes A"'] & {} & C \otimes B \otimes A
\end{tikzcd}$$

and two axioms that relate the U and S modifications: the 2-cells pictured below should each be equal to the identity on $s_{A,B}$:

$$\begin{tikzcd}
A \otimes \mathbb{I} \otimes B \arrow[rr, "s_{A\otimes\mathbb{I},B}"] \arrow[dd, "1"'] \arrow[dr, "A \otimes s_{\mathbb{I},B}"] & \arrow[d, phantom, "\Downarrow S_{A,\mathbb{I}|B}"] & B \otimes A \otimes \mathbb{I} \arrow[dd, "1"] \\
\arrow[r, phantom, "\Leftarrow A\otimes U_{\mathbb{I}|B}"] & A \otimes B \otimes \mathbb{I} \arrow[ur, "s_{A,B} \otimes \mathbb{I}"'] \arrow[dl, "1"] & \\
A \otimes B \arrow[rr, "s_{A,B}"'] & & B \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbb{I} \otimes B \arrow[rr, "s_{A,\mathbb{I}\otimes B}"] \arrow[dd, "1"'] \arrow[dr, "s_{A,\mathbb{I}} \otimes B"] & \arrow[d, phantom, "\Uparrow S_{A|\mathbb{I},B}"] & \mathbb{I} \otimes B \otimes A \arrow[dd, "1"] \\
\arrow[r, phantom, "\Rightarrow U_{A|\mathbb{I}}\otimes B"] & \mathbb{I} \otimes A \otimes B \arrow[ur, "\mathbb{I} \otimes s_{A,B}"'] \arrow[dl, "1"] & \\
A \otimes B \arrow[rr, "s_{A,B}"'] & & B \otimes A
\end{tikzcd}$$

Definition 3.10. A *braided monoidal bicategory* is a monoidal bicategory equipped with a braiding.

3.3.1 The duality of the definition

Since s is a pseudo-natural equivalence, it has an adjoint equivalence-inverse s'. We shall assume that such an s' has been chosen: by definition it has 1-cell components

$$s'_{A,B} : B \otimes A \to A \otimes B,$$

and by Prop. 2.34 each 2-cell component $s'_{f,g}$ is the inverse of the mate of $s_{f,g}$. Now we may define a pseudo-natural equivalence s^* with 1-cell components $s^*_{A,B} = s'_{A,B}$ and 2-cell components $s^*_{f,g} = s'_{g,f}$. The duality is expressed by the following:

Proposition 3.11. *The pseudo-natural transformation s^* can be made into a braiding in the following way.*

- $S^*_{A|B,C}$ *is defined to be the mate of* $S_{B,C|A}$ *with respect to the adjoint equivalences* $(s_{A,C} \otimes B) \circ a \circ (A \otimes s_{B,C})$ *and* $a \circ (s_{A\otimes B,C}) \circ a$,
- $S^*_{A,B|C}$ *is defined to be the mate of* $S_{C|A,B}$ *with respect to the adjoint equivalences* $a' \circ (s_{A,B\otimes C}) \circ a'$ *and* $(B \otimes s_{A,C}) \circ a' \circ (s_{A,B} \otimes C)$*; Note that, by Prop. 2.42, the left and right mates are equal in this case and the previous one.*
- $U^*_{\mathbb{I}|A}$ *is defined to be the right mate of* $U_{A|\mathbb{I}}$ *with respect to the adjoint equivalences* $s_{A,\mathbb{I}}$ *and* 1_A,
- $U^*_{A|\mathbb{I}}$ *is defined to be the left mate of* $U_{\mathbb{I}|A}$ *with respect to the adjoint equivalences* $s_{\mathbb{I},A}$ *and* 1_A.

Proof. The first two axioms for S are duals of each other: taking mates in one of them yields the other for S^*. The third axiom is self-dual, in that taking mates gives the corresponding equation for S^*. The fourth axiom is also self-dual, using the fact that the mate of $s_{A,s_{B,C}}$ is the inverse of $s^*_{s^*_{C,B},A}$. Finally, the two unit axioms are duals of each other. □

This symmetry can sometimes spare us a certain amount of repetition, in Section 6.5, for example.

3.3.2 The unit axioms

The first thing to say about the unit axioms is that we have here another instance of the phenomenon discussed in Remark 3.2: the unit axioms show that the 2-cells

$U_{\mathbb{I}|A}$ and $U_{A,\mathbb{I}}$ are definable in terms of the other data. However, the U cells defined in this way do not themselves necessarily satisfy the unit axioms! So these cells are redundant data in a sense, but if we were to eliminate them, we should instead have to impose rather unnatural-looking axioms involving $S_{A,\mathbb{I}|B}$ and $S_{A|\mathbb{I},B}$.

One could write down several other natural conditions on the unit cells. These turn out to be derivable from the two conditions we have. In particular:

Proposition 3.12. *In any braided Gray monoid, the 2-cells*

$$\begin{tikzcd}
\mathbb{I}\otimes A\otimes B \arrow[r, "\mathbb{I}\otimes s_{A,B}"] \arrow[d, "s_{\mathbb{I}\otimes A,B}"'] \arrow[dr, phantom, pos=0.3, "\overset{\Rightarrow}{S_{\mathbb{I},A|B}}"] & \mathbb{I}\otimes B\otimes A \arrow[d, "1"] \arrow[dl, "s_{\mathbb{I},B}\otimes A" description] \\
B\otimes\mathbb{I}\otimes A \arrow[r, "1"'] \arrow[ur, phantom, pos=0.7, "\overset{\Rightarrow}{U_{\mathbb{I}|B}\otimes A}"] & B\otimes A
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
A\otimes B\otimes\mathbb{I} \arrow[r, "s_{A,B}\otimes\mathbb{I}"] \arrow[d, "s_{A,B\otimes\mathbb{I}}"'] \arrow[dr, phantom, pos=0.3, "\overset{\Leftarrow}{S_{A|B,\mathbb{I}}}"] & B\otimes A\otimes\mathbb{I} \arrow[d, "1"] \arrow[dl, "B\otimes s_{A,\mathbb{I}}" description] \\
B\otimes\mathbb{I}\otimes A \arrow[r, "1"'] \arrow[ur, phantom, pos=0.7, "\overset{\Leftarrow}{B\otimes U_{A|\mathbb{I}}}"] & B\otimes A
\end{tikzcd}$$

are both identities.

Proof. By duality, it suffices to prove one of the two: we'll prove the second one. If we set $B = \mathbb{I}$ in our first axiom for S, we obtain

$$\begin{tikzcd}
A\otimes\mathbb{I}\otimes C\otimes D \arrow[r, "s_{A\otimes\mathbb{I}\otimes C,D}"] \arrow[d, "A\otimes\mathbb{I}\otimes s_{C,D}"'] \arrow[dr, "A\otimes s_{\mathbb{I}\otimes C,D}" description] \arrow[dr, phantom, pos=0.3, shift left=6ex, "\Downarrow S_{A,\mathbb{I}\otimes C|D}"] & D\otimes A\otimes\mathbb{I}\otimes C \\
A\otimes\mathbb{I}\otimes D\otimes C \arrow[r, "A\otimes s_{\mathbb{I},D}\otimes C"'] \arrow[ur, phantom, pos=0.3, "\swArrow A\otimes S_{\mathbb{I},C|D}"] & A\otimes D\otimes\mathbb{I}\otimes C \arrow[u, "s_{A,D}\otimes\mathbb{I}\otimes C"']
\end{tikzcd}
=
\begin{tikzcd}
A\otimes\mathbb{I}\otimes C\otimes D \arrow[r, "s_{A\otimes\mathbb{I}\otimes C,D}"] \arrow[d, "A\otimes\mathbb{I}\otimes s_{C,D}"'] \arrow[dr, phantom, pos=0.3, "\Downarrow S_{A\otimes\mathbb{I},C|D}"] & D\otimes A\otimes\mathbb{I}\otimes C \\
A\otimes\mathbb{I}\otimes D\otimes C \arrow[r, "A\otimes s_{\mathbb{I},D}\otimes C"'] \arrow[ur, "s_{A\otimes\mathbb{I},D}\otimes C" description] & A\otimes D\otimes\mathbb{I}\otimes C \arrow[u, "s_{A,D}\otimes\mathbb{I}\otimes C"'] \arrow[ul, phantom, pos=0.3, "\seArrow S_{A,\mathbb{I}|D}\otimes C"]
\end{tikzcd}$$

Cancelling the invertible 2-cell $S_{A,\mathbb{I}\otimes C|D} = S_{A\otimes\mathbb{I},C|D}$, we have

$$\begin{tikzcd}
A\otimes\mathbb{I}\otimes C\otimes D \arrow[d, "A\otimes\mathbb{I}\otimes s_{C,D}"'] \arrow[dr, "A\otimes s_{\mathbb{I}\otimes C,D}" description] & D\otimes A\otimes\mathbb{I}\otimes C \\
A\otimes\mathbb{I}\otimes D\otimes C \arrow[r, "A\otimes s_{\mathbb{I},D}\otimes C"'] \arrow[ur, phantom, pos=0.3, "\swArrow A\otimes S_{\mathbb{I},C|D}"] & A\otimes D\otimes\mathbb{I}\otimes C \arrow[u, "s_{A,D}\otimes\mathbb{I}\otimes C"']
\end{tikzcd}
=
\begin{tikzcd}
A\otimes\mathbb{I}\otimes C\otimes D \arrow[d, "A\otimes\mathbb{I}\otimes s_{C,D}"'] & D\otimes A\otimes\mathbb{I}\otimes C \\
A\otimes\mathbb{I}\otimes D\otimes C \arrow[r, "A\otimes s_{\mathbb{I},D}\otimes C"'] \arrow[ur, "s_{A\otimes\mathbb{I},D}\otimes C" description] & A\otimes D\otimes\mathbb{I}\otimes C \arrow[u, "s_{A,D}\otimes\mathbb{I}\otimes C"'] \arrow[ul, phantom, pos=0.3, "\seArrow S_{A,\mathbb{I}|D}\otimes C"]
\end{tikzcd}$$

Thus

$$
\begin{tikzcd}
A\otimes\mathbb{I}\otimes C\otimes D \arrow[d, "A\otimes\mathbb{I}\otimes s_{C,D}"'] \arrow[drr, "A\otimes s_{\mathbb{I}\otimes C,D}"] \arrow[dr, phantom, "\Swarrow A\otimes S_{\mathbb{I},C|D}"] & & D\otimes A\otimes\mathbb{I}\otimes C \arrow[drr, "1"] & & \\
A\otimes\mathbb{I}\otimes D\otimes C \arrow[rr, "A\otimes s_{\mathbb{I},D}\otimes C"'] \arrow[drr, "1"'] & & A\otimes D\otimes\mathbb{I}\otimes C \arrow[u, "s_{A,D}\otimes\mathbb{I}\otimes C"] \arrow[d, "1"] \arrow[dl, phantom, "\Swarrow A\otimes U_{\mathbb{I}|D}\otimes C"] & & D\otimes A\otimes C \\
& & A\otimes D\otimes C \arrow[urr, "s_{A,D}\otimes C"'] & &
\end{tikzcd}
$$

is equal to

$$
\begin{tikzcd}
A\otimes\mathbb{I}\otimes C\otimes D \arrow[d, "A\otimes\mathbb{I}\otimes s_{C,D}"'] & & D\otimes A\otimes\mathbb{I}\otimes C \arrow[drr, "1"] & & \\
A\otimes\mathbb{I}\otimes D\otimes C \arrow[urr, "s_{A\otimes\mathbb{I},D}\otimes C"] \arrow[rr, "A\otimes s_{\mathbb{I},D}\otimes C"'] \arrow[drr, "1"'] \arrow[ur, phantom, "\Searrow S_{A,\mathbb{I}|D}\otimes C"] & & A\otimes D\otimes\mathbb{I}\otimes C \arrow[u, "s_{A,D}\otimes\mathbb{I}\otimes C"] \arrow[d, "1"] \arrow[dl, phantom, "\Swarrow A\otimes U_{\mathbb{I}|D}\otimes C"] & & D\otimes A\otimes C \\
& & A\otimes D\otimes C \arrow[urr, "s_{A,D}\otimes C"'] & &
\end{tikzcd}
$$

which, by our first unit axiom, is the identity. Cancelling the equivalence $s_{A,D}\otimes C$, we find that

$$
\begin{tikzcd}
A\otimes\mathbb{I}\otimes C\otimes D \arrow[d, "A\otimes\mathbb{I}\otimes s_{C,D}"'] \arrow[dr, "A\otimes s_{\mathbb{I}\otimes C,D}"] \arrow[d, phantom, shift left=8ex, "\Swarrow A\otimes S_{\mathbb{I},C|D}"] & \\
A\otimes\mathbb{I}\otimes D\otimes C \arrow[r, "A\otimes s_{\mathbb{I},D}\otimes C"'] \arrow[dr, "1"'] & A\otimes D\otimes\mathbb{I}\otimes C \arrow[d, "1"] \arrow[d, phantom, shift right=8ex, "\Swarrow A\otimes U_{\mathbb{I}|D}\otimes C"] \\
& A\otimes D\otimes C
\end{tikzcd}
$$

is the identity, and setting $A = \mathbb{I}$ yields the claim. $\square$

Proposition 3.13. *In any braided Gray monoid, the equations*

$$
\begin{tikzcd}
A \otimes B \otimes \mathbb{I} \arrow[r, "A \otimes s_{B,\mathbb{I}}"] \arrow[d, "1"'] \arrow[dr, "s_{A\otimes B,\mathbb{I}}" description] & A \otimes \mathbb{I} \otimes B \arrow[d, "s_{A,\mathbb{I}} \otimes B"] \arrow[dl, phantom, "\Nearrow S_{A,B|\mathbb{I}}" near start] \\
A \otimes B \arrow[ur, phantom, "\overset{\Rightarrow}{U_{A\otimes B|\mathbb{I}}}" near start] & \mathbb{I} \otimes A \otimes B \arrow[l, "1"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \otimes B \otimes \mathbb{I} \arrow[r, "A \otimes s_{B,\mathbb{I}}"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{\Rightarrow}{A\otimes U_{B|\mathbb{I}}}" near start] & A \otimes \mathbb{I} \otimes B \arrow[d, "s_{A,\mathbb{I}} \otimes B"] \arrow[dl, "1" description] \\
A \otimes B & \mathbb{I} \otimes A \otimes B \arrow[l, "1"] \arrow[ul, phantom, "\Searrow U_{A|\mathbb{I}}\otimes B" near start]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{I} \otimes A \otimes B \arrow[r, "s_{\mathbb{I},A} \otimes B"] \arrow[d, "1"'] \arrow[dr, "s_{A\otimes B,\mathbb{I}}" description] & A \otimes \mathbb{I} \otimes B \arrow[d, "A \otimes s_{\mathbb{I},B}"] \arrow[dl, phantom, "\Swarrow S_{\mathbb{I}|A,B}" near start] \\
A \otimes B \arrow[ur, phantom, "\overset{\Leftarrow}{U_{\mathbb{I}|A\otimes B}}" near start] & A \otimes B \otimes \mathbb{I} \arrow[l, "1"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{I} \otimes A \otimes B \arrow[r, "s_{\mathbb{I},A} \otimes B"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{\Leftarrow}{U_{\mathbb{I}|A}\otimes B}" near start] & A \otimes \mathbb{I} \otimes B \arrow[d, "A \otimes s_{\mathbb{I},B}"] \arrow[dl, "1" description] \\
A \otimes B & A \otimes B \otimes \mathbb{I} \arrow[l, "1"] \arrow[ul, phantom, "\Nwarrow A\otimes U_{\mathbb{I}|B}" near start]
\end{tikzcd}
$$

hold.

Proof sketch. This is proved in a similar way to the preceding proposition, though the argument is lengthier. Start with the third axiom, setting $C = \mathbb{I}$. $\square$

Chapter 4

Some Coherence Results

In this chapter, we prove some new coherence results for Gray monoids. These results pave the way for the following chapter, which describes a simple but apparently novel technique for stating definitions and performing calculations in a monoidal bicategory.

At the time of writing, the sum of human knowledge about coherence for monoidal bicategories – and, more generally, for tricategories – is contained in the PhD dissertation of Gurski (2006), which builds on the pioneering work of Gordon et al. (1995). Nothing has yet been written about coherence for *braided* monoidal bicategories. It is gradually becoming clear that tricategories, and other such higher-dimensional structures, enjoy more coherence than they are generally credited with. One manifestation of this is that many diagrams of 2-cells commute in any Gray monoid. To be precise, we shall prove:

Theorem 4.1. *In the free Gray monoid generated by a multigraph, every diagram of 2-cells commutes.*

This strengthens Gurski's Theorem 10.2.2, which (specialised to monoidal bicategories) addresses the free Gray monoid generated by a mere graph, rather than a multigraph. It is apparently possible to strengthen it still further, but the statement here is adequate for our present purposes.

Our second result encompasses the braided case:

Theorem 4.2. *In the free braided Gray monoid generated by a multigraph, every diagram of 2-cells whose source and target are 'positive' 1-cells commutes.*

Here 'positive' means that the 1-cell is built without using s', the equivalence inverse of s. This restriction is made because it significantly simplifies the proof, yet remains sufficient for our applications in this work. The result does appear in fact to hold for all 2-cells, and we conjecture that a more complex application of the techniques of this chapter suffices to prove the general version.

In both cases, the method of proof is essentially that used by Mac Lane (1978) in his coherence theorem for monoidal categories, using term rewriting. This syntactic technique seems to permit a finer analysis of coherence than the powerful but blunt semantic methods employed by Gordon et al. (1995) and refined by Gurski (2006).

4.1 Non-braided Case

A multigraph consists of the data for a multicategory, but without the identity arrows or the composition operation. More formally, a multigraph G consists of:

- a set G_0 of objects,
- for each finite 'source' sequence $A_1, \ldots, A_n$ of objects and 'target' object B, a set $G(A_1, \ldots, A_n; B)$ of multiarrows.

A morphism of multigraphs $f : G \to H$ consists of an object function $f_0 : G_0 \to H_0$ together with a family of functions

$$f_{A_1,\ldots,A_n;B} : G(A_1, \ldots, A_n; B) \to H(f_0(A_1), \ldots, f_0(A_n); f_0(B)).$$

The category $\mathrm{GrayMon}_{\mathrm{str}}$ of Gray monoids and *strictly* structure-preserving maps has an obvious forgetful functor to the category of multigraphs, and clearly this forgetful functor can be furnished with a left adjoint F by the usual syntactic construction. Naively, then, the objects, 1-cells and 2-cells of the Gray category $F(G)$ are formal expressions built from the objects and multiarrows of G, quotiented by the smallest equivalence relation that makes the resulting structure into a Gray monoid, i.e. the equivalence relation generated by the axioms that define Gray monoid. But of course this description can be simplified, as follows.

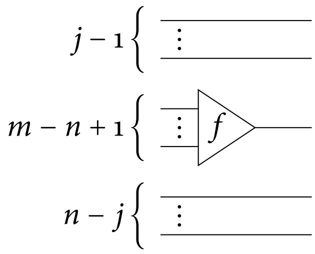
Since the objects of a Gray monoid form a monoid under tensor, an object of $F(G)$ can be represented by a finite sequence $\langle X_1, \ldots, X_n \rangle$ of objects of G; the tensor of two objects is their concatenation as sequences, and the unit object is represented by the empty sequence. Turning next to the 1-cells, notice that since the tensor of $f : A \to B$ and $g : C \to D$ is equal to $(B \otimes g)(f \otimes C)$, the tensor of 1-cells can be expressed in terms of the tensor of a 1-cell with an object, and composition. Therefore we need not regard the tensor of two 1-cells as a primitive operation, provided that we can take the tensor of a 1-cell with an object. Furthermore, every 1-cell is equal to a finite composite of *multiarrow 1-cells*, where a multiarrow 1-cell is obtained by tensoring a multiarrow of G with objects, on either side. More formally, a multiarrow 1-cell

$$\langle X_1, \ldots, X_m \rangle \to \langle Y_1, \ldots, Y_n \rangle$$

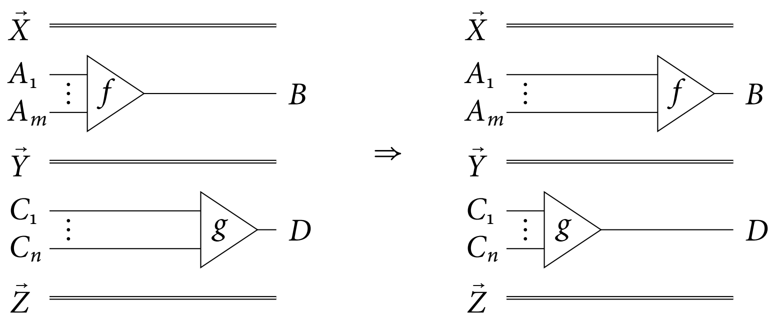
consists of a natural number $1 \leq j \leq n$ and a multiarrow

$$f \in G(X_j, \ldots, X_{j+m-n}; Y_j)$$

such that $\langle X_1, \ldots, X_{j-1} \rangle = \langle Y_1, \ldots, Y_{j-1} \rangle$ and $\langle X_{j+m-n+1}, \ldots, X_m \rangle = \langle Y_{j+1}, \ldots, Y_n \rangle$. This might be pictured as follows:



Similarly, the tensor of 2-cells can be expressed in terms of the tensor of a 1-cell with an object, and horizontal composition. In turn, horizontal composition can be expressed in terms of whiskering and vertical composition. Therefore our 2-cells may be built from *interchange cells*, where an interchange cell is either a *positive interchange cell* $b_{\vec{X},f,\vec{Y},g,\vec{Z}}$ like this:



or a *negative interchange cell* $b^{-1}_{\vec{X},f,\vec{Y},g,\vec{Z}}$ in the other direction. In case it is not clear from the picture, a positive interchange cell is a 2-cell

$$b_{\vec{X},f,\vec{Y},g,\vec{Z}} : (\vec{X} \otimes B \otimes \vec{Y} \otimes g \otimes \vec{Z}) \cdot (\vec{X} \otimes f \otimes \vec{Y} \otimes \vec{C} \otimes \vec{Z}) \Rightarrow (\vec{X} \otimes f \otimes \vec{Y} \otimes D \otimes \vec{Z}), \cdot (\vec{X} \otimes \vec{A} \otimes \vec{Y} \otimes g \otimes \vec{Z})$$

where $f \in G(\vec{A}; B)$ and $g \in G(\vec{C}; D)$. In context, we shall omit some of the subscripts and write just $b_{f,g}$. Now a *basic 2-cell* is the result of whiskering an interchange cell on both sides, by arbitrary 1-cells, and every 2-cell is a finite vertical composite of basic 2-cells.

To summarise, a 1-cell f may be canonically represented as a finite sequence of multiarrow cells. There is a basic 2-cell $f \Rightarrow g$ whenever g can be obtained from f by interchanging two adjacent (but non-interfering) multiarrows. (In particular, f

and g contain the same number of multiarrow cells.)

As explained above, we plan to use Mac Lane's technique for proving coherence. So the next step is to define a rewriting system on 1-cells, where each rewrite rule corresponds to a basic 2-cell. We shall show that this rewriting system is strongly normalising and locally confluent, hence that every 1-cell is isomorphic to one in normal form. Now, local confluence means that, for every 1-cell f, if we have rewriting steps $\gamma : f \Rightarrow f_1$ and $\delta : f \Rightarrow f_2$ then there are sequences of rewrites $\gamma^* : f_1 \Rightarrow g$ and $\delta^* : f_2 \Rightarrow g$ for some 1-cell g. The final requirement is to show that the corresponding diagram of 2-cells commutes:

$$\begin{tikzcd} f \arrow[r, "\gamma"] \arrow[d, "\delta"'] & f_1 \arrow[d, "\gamma^*"] \\ f_2 \arrow[r, "\delta^*"'] & g. \end{tikzcd}$$

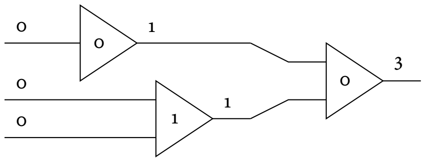
It will be convenient, of course, to prove this at the same time as we establish local confluence. This extended confluence property does not seem to have a standard name: we shall call it *local coherent confluence*. Notice that, in a strongly normalising system, local coherent confluence implies (*global*) *coherent confluence*: i.e. given any *sequences* of rewrites $\gamma : f \Rightarrow f_1$ and $\delta : f \Rightarrow f_2$, there are sequences of rewrites $\gamma^* : f_1 \Rightarrow g$ and $\delta^* : f_2 \Rightarrow g$ for some 1-cell g such that the corresponding diagram of 2-cells commutes.

In this, the non-braided case, the required rewriting system is very simple: we treat every 1-cell as a composite of multiarrow 1-cells, and a rewriting step simply corresponds to a positive interchange cell applied to a pair of consecutive multiarrow 1-cells.

Lemma 4.3. *This rewriting system is strongly normalising, i.e. every reduction path is finite.*

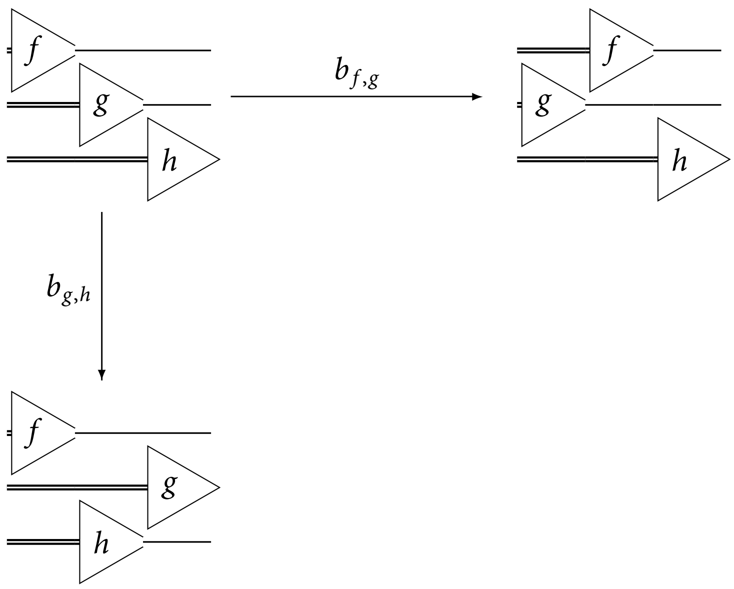
Proof. We shall define a weighting function that assigns a natural number to each 1-cell, so that each rewriting step strictly decreases the weight. The appropriate weighting function here is the *total prefix weight*, which is defined as follows. Recall that a 1-cell is represented as a sequence $f_1, \ldots, f_n$ of multiarrow 1-cells. If we imagine the 1-cell drawn as a diagram like those above, then each output wire (at the far right of the diagram, according to our convention above) is attached to a distinct tree of multiarrows. Define the *weight* of an output wire to be the number of multiarrows in the tree. So for each multiarrow, the weight of its output wire is 1+ the sum of the weights of its input wires.

A multiarrow 1-cell consists of a single multiarrow with a number of bare wires (objects) above and below it. Let us refer to the bare wires above as the 'prefix' of the multiarrow 1-cell. In the context of a 1-cell $f_1, \ldots, f_n$, define the *prefix weight* of a constituent multiarrow 1-cell f_i to be the sum of the weights of each wire in the prefix. For example, in the diagram below each wire is annotated with its weight, and each multiarrow is annotated with its prefix weight.

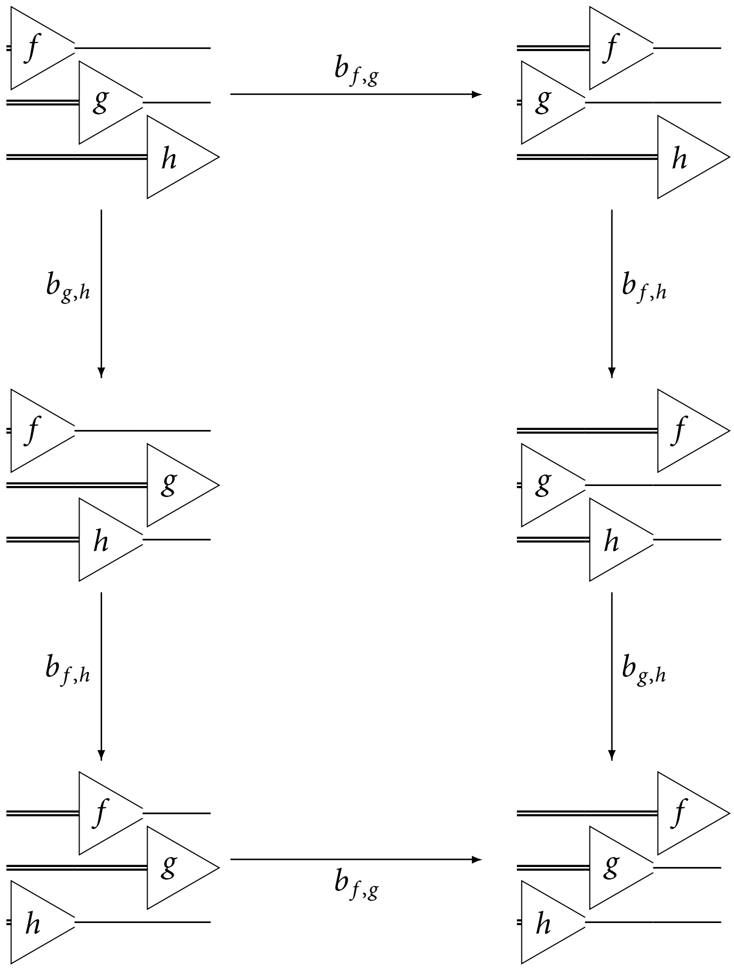


The total prefix weight of a 1-cell is simply the sum of the prefix weights of the constituent multiarrow cells, so the example above has a total prefix weight of 1. Clearly the target of a positive interchange cell must have a strictly smaller total prefix weight than the source. ☐

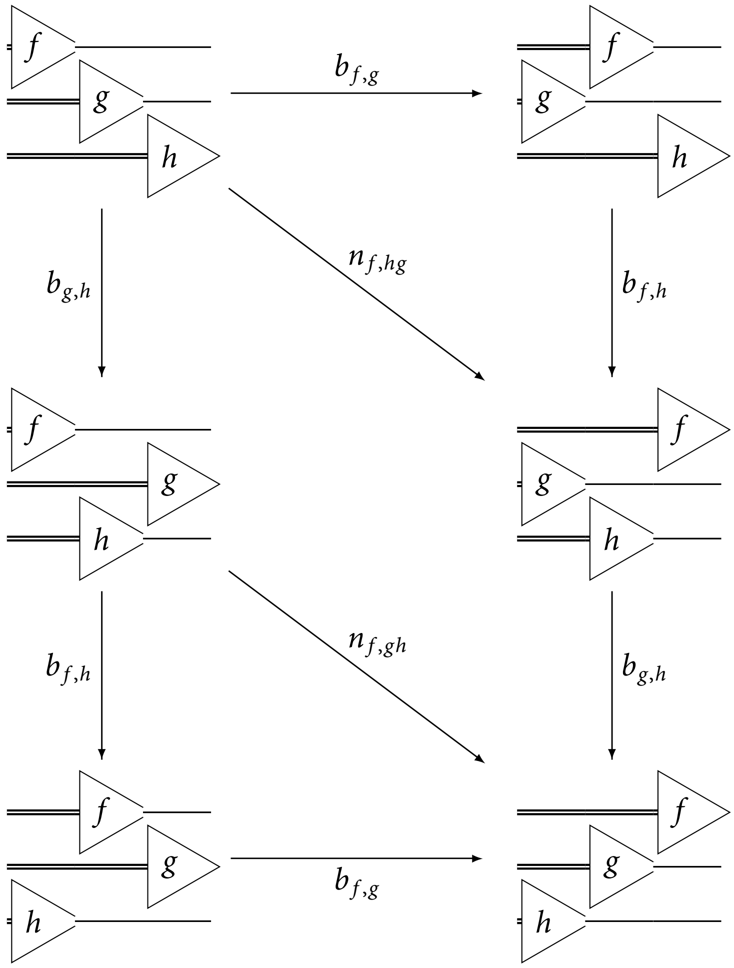
It remains to show confluence. Let $\vec{f}$ be a 1-cell, and suppose that we have two different interchanges γ and δ applied to $\vec{f}$. If γ and δ do not overlap, then the result of applying γ followed by δ is the same as the result of applying δ followed by γ, and the corresponding diagram of 2-cells commutes by naturality. If γ and δ do overlap, then we essentially have the following situation:



(where we have omitted from the diagram wires which are not connected to one of the three multiarrows that participate in these interchanges.) This may be completed as shown in the following diagram:



To see why the corresponding diagram of 2-cells must commute, fill in diagonals as follows:



where $n_{f,hg}$ denotes the (non-basic) interchange of f with the composite of g and h, and similarly $n_{f,gh}$. The triangles commute by definition, and the central quadrilateral is another instance of naturality of interchange.

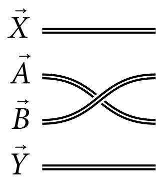
Therefore for every 1-cell f, there is a (necessarily unique) corresponding 1-cell f' in normal form, together with an invertible 2-cell $\gamma_f : f \Rightarrow f'$ built from interchange cells. Since the system is coherently confluent, and every 2-cell can be represented as a zig-zag of rewrites, γ_f is also unique. Hence there is a 2-cell connecting 1-cells f and g just when f and g have the same normal form, and this 2-cell is unique.

4.2 The braided case

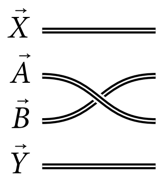
In the braided case the form of the argument is the same, though the details are a little more involved. The free braided Gray monoid $F_\beta(G)$ on a multigraph G can be described in a similar way to the free ordinary Gray monoid, with the addition of braiding data. Precisely:

- A 1-cell of $F_\beta(G)$ is the composite of a finite sequence of multiarrow cells and crossing cells, which we collectively term *basic 1-cells*. The multiarrow cells are as above and:

- A braid cell is either a positive crossing $\beta_{\vec{X}(\vec{A},\vec{B})\vec{Y}}$:

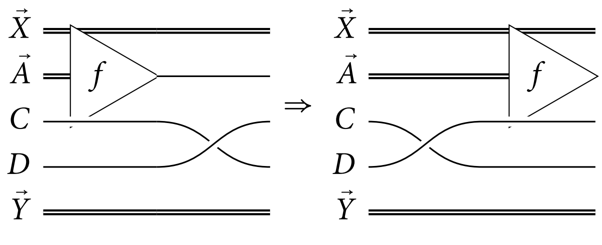


  or a negative crossing $\beta'_{\vec{X}(\vec{A},\vec{B})\vec{Y}}$:



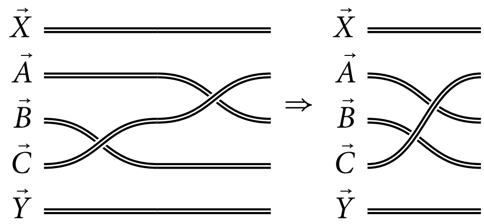
  As with the interchange cells, we shall typically omit the subscripts corresponding to wires that do not participate in the crossing, writing the above crossings as $\beta_{\vec{A},\vec{B}}$ and $\beta'_{\vec{A},\vec{B}}$.

- A 2-cell is a finite vertical composite of basic 2-cells, where a basic 2-cell is either a whiskered interchange cell as above, or else a whiskered unit cell or braiding cell: the latter two are described below. Note that crossings – as well as multiarrows – may participate in interchange; for example, there is an interchange cell $b_{f,\beta_{C,D}}$ as follows:

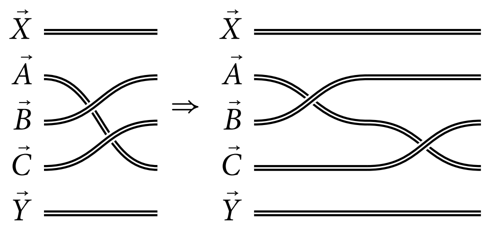


- A unit cell is either $U_{\vec{X}(\mathbb{I}|\vec{A})\vec{Y}} : \beta_{\vec{X}(\langle\rangle,\vec{A})\vec{Y}} \Rightarrow 1$ or $U_{\vec{X}(\vec{A}|\mathbb{I})\vec{Y}} : \beta_{\vec{X}(\vec{A},\langle\rangle)\vec{Y}} \Rightarrow 1$, or the inverse of one of these. In terms of our representation of a 1-cell as a finite sequence of basic 1-cells, the target of a unit cell is the empty sequence. So the sequence representing the target (of a whiskered unit cell) has one element fewer than the sequence representing the source.

- There are two types of braiding cell, overbraiding and underbraiding cells. An overbraiding cell has the form $B_{\vec{X}(\vec{A},\vec{B}|\vec{C})\vec{Y}}$:



  or the inverse $B^{-1}_{\vec{X}(\vec{A},\vec{B}|\vec{C})\vec{Y}}$. An underbraiding cell has the form $B_{\vec{X}(\vec{A}|\vec{B},\vec{C})\vec{Y}}$:

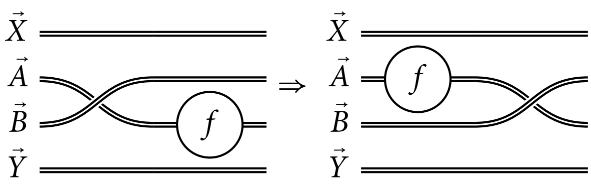


  or the inverse. As with the other types of cell, in context we shall usually write these simply as $B_{\vec{A},\vec{B}|\vec{C}}$ and $B_{\vec{A}|\vec{B},\vec{C}}$.

- We also need unit cells $U_{\vec{X}(\mathbb{I}|\vec{A})\vec{Y}} : \beta_{\vec{X}(\langle\rangle,\vec{A})\vec{Y}} \Rightarrow 1_{\vec{X}\otimes\vec{A}\otimes\vec{Y}}$ and $U_{\vec{X}(\vec{A}|\mathbb{I})\vec{Y}} : \beta_{\vec{X}(\vec{A},\langle\rangle)\vec{Y}} \Rightarrow 1_{\vec{X}\otimes\vec{A}\otimes\vec{Y}}$.

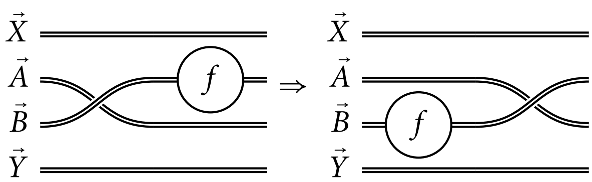
- Finally, there are the 2-cells that give the pseudonaturality of the crossings. We shall take the following as basic:

$N_{\vec{X}(f,\vec{B})\vec{Y}}$:



  and

$N_{\vec{X}(\vec{A},f)\vec{Y}}$:



  where f is a basic 1-cell, i.e. a multiarrow or a crossing. These also have inverses.

Recall that here we are considering only positive 1-cells, where all the crossings are positive. (By duality we could equally well take the case where all crossings are negative; it is mixing the two that introduces additional complexity.)

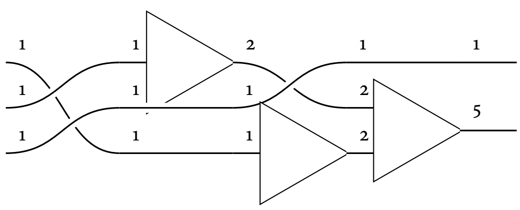
For rewriting rules, we shall essentially take the interchange, over- and under-braiding, unit, and pseudonaturality cells in their natural (positive) directions. There is a collection of restrictions related to trivial crossings, where one of the 'branches' of the crossing is empty. In detail: for the braiding cells $B_{\vec{X}(\vec{A},\vec{B}|\vec{C})\vec{Y}}$ and $B_{\vec{X}(\vec{A}|\vec{B},\vec{C})\vec{Y}}$, we require all three sequences $\vec{A}$, $\vec{B}$ and $\vec{C}$ to be non-empty. Similarly, for the pseudo-naturality cells $N_{\vec{X}(f,\vec{B})\vec{Y}}$ we require $\vec{B}$ to be non-empty, and for $N_{\vec{X}(\vec{A},f)\vec{Y}}$ we require $\vec{A}$ to be non-empty. In both types of pseudonaturality cell, if 'f' denotes a crossing $\beta_{\vec{A},\vec{B}}$, we require $\vec{A}$ and $\vec{B}$ to be non-empty. These restrictions should give the reader pause, since we must ensure that every structural 2-cell corresponds to some zig-zag of rewrites. The reason they are admissible is that the excluded braiding and pseudonaturality cells (where some sequence is empty) are all equal to some unit cell or composite of unit cells, by the unit axioms and Propositions 3.12 and 3.13.

Lemma 4.4. *This rewriting system is strongly normalising.*

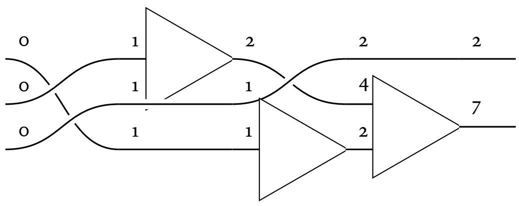
Proof. As in the non-braided case, we shall define a well-founded partially ordered set of 'weightings', and associate a weighting with each (positive) 1-cell; then we shall show that for each rewrite rule the target has a strictly smaller weight than the source.

In the non-braided case, the weightings were simply natural numbers. Here, the set of weightings is taken to be $\mathbb{N}^4$ with its lexicographic ordering. Thus to each positive 1-cell we associate four numbers, the **total overbraid width**, the **total crossing weight**, the **total prefix weight** and the **number of trivial crossings**. To define these, we need two auxiliary notions, the *width* and the *weight* of a wire or sequence of wires in a 1-cell. Both these are defined in an iterative way, proceeding (in terms of the diagrams) from left to right; the width (weight) of a sequence of wires is simply the sum of the individual widths (weights).

The width of each wire is initially 1, i.e. in the identity 1-cell (represented by an empty sequence of basic 1-cells) each wire has a width of 1. For each multiarrow cell, the width of the output wire is defined to be 1+ the width of the sequence of input wires. For each crossing cell, the width of each output wire is simply equal to the width of the corresponding input wire. For example, the following diagram shows a 1-cell with every wire annotated with its width.



The weight of a wire is defined in a related way. Weights are initially zero; for each multiarrow cell the weight of the output wire is 1+ the weight of the sequence of input wires; for each crossing cell, the weight of each output wire is the sum of the weight and the width of the corresponding input wire. By way of illustration, here is the same 1-cell as above, now annotated with weights:



We denote the width function wd(), and the weight function wt().

Our weighting functions are defined as follows:

- The *overbraid width* of a crossing $\beta_{\vec{A},\vec{B}}$ is $2\mathrm{wd}(\vec{X}) - 1$, and the total overbraid width of a 1-cell is the sum of the overbraid widths of its crossings.

- The *crossing weight* of a crossing $\beta_{\vec{A},\vec{B}}$ is $\mathrm{wd}(\vec{A},\vec{B}) \times \mathrm{wt}(\vec{A},\vec{B})$. The total crossing weight of a 1-cell is the sum of the crossing weights of its crossings

- The *prefix weight* of a multiarrow or crossing cell is the weight of its prefix. The total prefix weight of a 1-cell is the sum of the prefix weights of all the basic 1-cells.

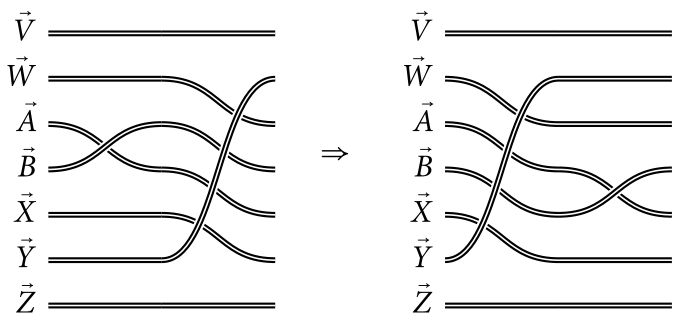
- A crossing $\beta_{\vec{A},\vec{B}}$ is *trivial* if either $\vec{A}$ or $\vec{B}$ is empty. The *number of trivial crossings* is the number of trivial crossings.

We must verify that every rewrite rule reduces the aggregate weighting.

- The overbraiding rule $B_{\vec{A},\vec{B}|\vec{C}}$ changes the overbraid width of the cells it acts on from $4\mathrm{wd}(\vec{C})-2$ to $2\mathrm{wd}(\vec{C})-1$. The overbraid widths of other crossings are unchanged. Since we are taking $\vec{C}$ to be non-empty, we know that $2\mathrm{wd}(\vec{C}) - 1 > 0$, hence the total overbraid width is reduced.

- The underbraiding rule $B_{\vec{A}|\vec{B},\vec{C}}$ changes the overbraid width of the cells it acts on from $2\mathrm{wd}(\vec{B},\vec{C}) - 1$ to $2\mathrm{wd}(\vec{B}) + 2\mathrm{wd}(\vec{C}) - 2 = 2\mathrm{wd}(\vec{B},\vec{C}) - 2$, reducing it by 1. The overbraid width of other crossings is unaffected.

- The pseudonaturality cells do not increase the overbraid width of any crossing. (If a multiarrow cell is being moved across the overbraid part of a crossing, the overbraid width of that crossing is reduced by 1. Otherwise overbraid widths are unaffected.) In the case of a pseudonaturality cell $N_{f,\vec{B}}$ or $N_{\vec{A},f}$ where 'f' represents a multiarrow cell, the crossing weight of the affected crossing is clearly reduced (and other crossings are unaffected). The more subtle case is that where 'f' is another crossing cell. For example, the general case of $N_{\vec{V}(\beta_{\vec{W}(\vec{A},\vec{B})\vec{X}},\vec{Y})\vec{Z}}$ looks like this:



On the left-hand side, the total crossing weight is

$$\mathrm{wd}(\vec{A},\vec{B})\times\mathrm{wt}(\vec{A},\vec{B})+\mathrm{wd}(\vec{W},\vec{A},\vec{B},\vec{X},\vec{Y})\times\big(\mathrm{wt}(\vec{W},\vec{A},\vec{B},\vec{X},\vec{Y})+\mathrm{wd}(\vec{A},\vec{B})\big),$$

and on the right-hand side the total crossing weight is

$$\mathrm{wd}(\vec{W},\vec{A},\vec{B},\vec{X},\vec{Y})\times\mathrm{wt}(\vec{W},\vec{A},\vec{B},\vec{X},\vec{Y})+\mathrm{wd}(\vec{A},\vec{B})\times\big(\mathrm{wt}(\vec{A},\vec{B})+\mathrm{wd}(\vec{A},\vec{B})\big).$$

So the difference (left−right) is

$$\mathrm{wd}(\vec{W},\vec{A},\vec{B},\vec{X},\vec{Y}) \times \mathrm{wd}(\vec{A},\vec{B}) - \mathrm{wd}(\vec{A},\vec{B})^2,$$

which is equal to $\mathrm{wd}(\vec{W},\vec{X},\vec{Y}) \times \mathrm{wd}(\vec{A},\vec{B})$. Since we are taking $\vec{Y}$, $\vec{A}$ and $\vec{B}$ to be non-empty, this difference is > 0. The other type of pseudonaturality cell is handled in a similar way.

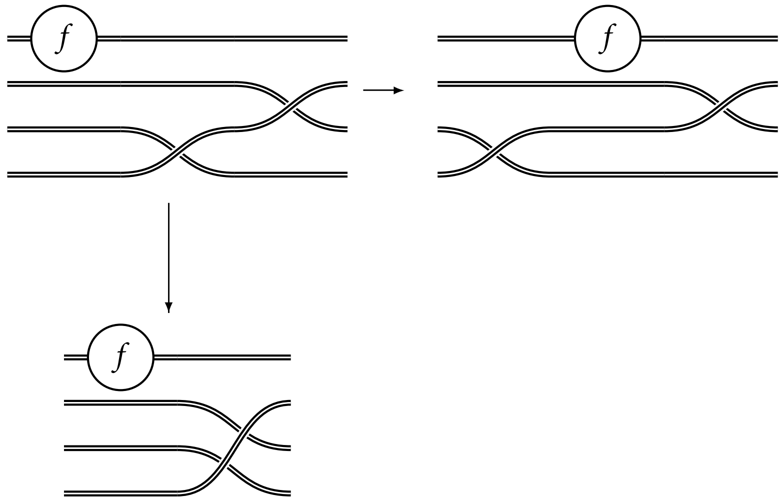
- The interchange cells do not change the overbraid width nor the crossing weight of any crossing, and (as in the non-braided case) they reduce the total prefix weight.
- The unit cells do not affect any of the other measures, but reduce the number of trivial crossings.

□

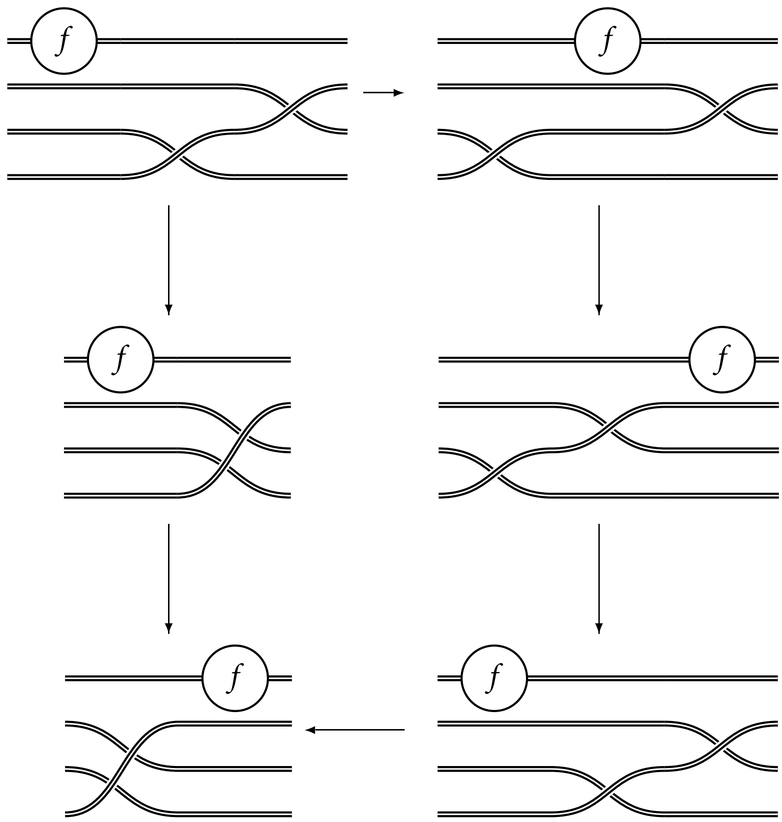
Proposition 4.5. *The system is coherently confluent.*

Proof. We consider the possible ways that two rules could be applied to the same 1-cell. In the case where two rules apply to non-overlapping portions of a 1-cell, the rules can be applied in either order, and the resulting 2-cells are equal. Thus we need to consider conflicts, i.e. situations where two rules can apply to overlapping portions.

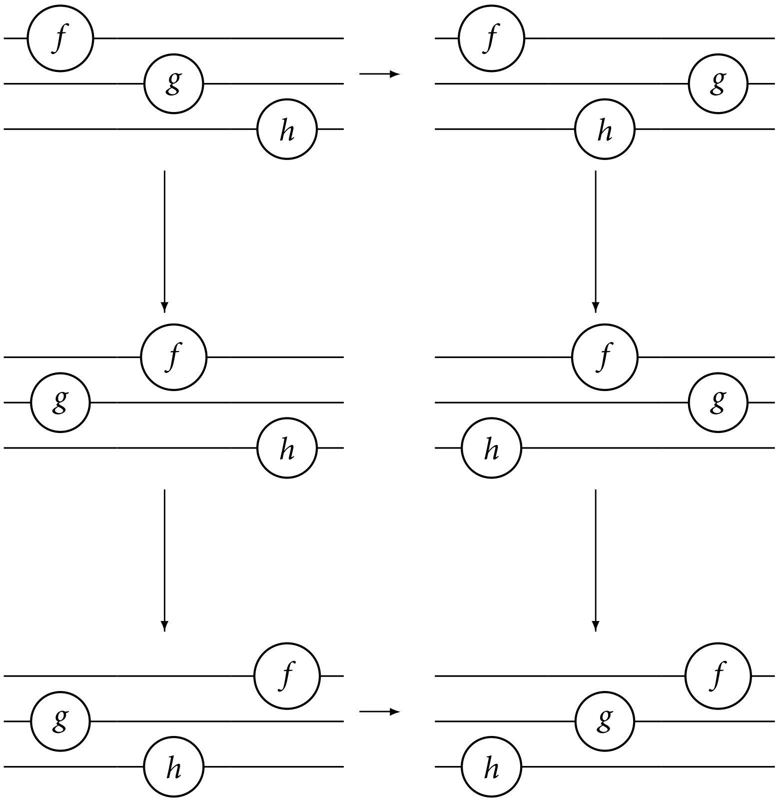
First, consider the interchange rule. There is a class of conflicts where some basic 1-cell is moved past a point where some other 2-cell could apply. For example, there is a conflict between interchange and underbraiding:



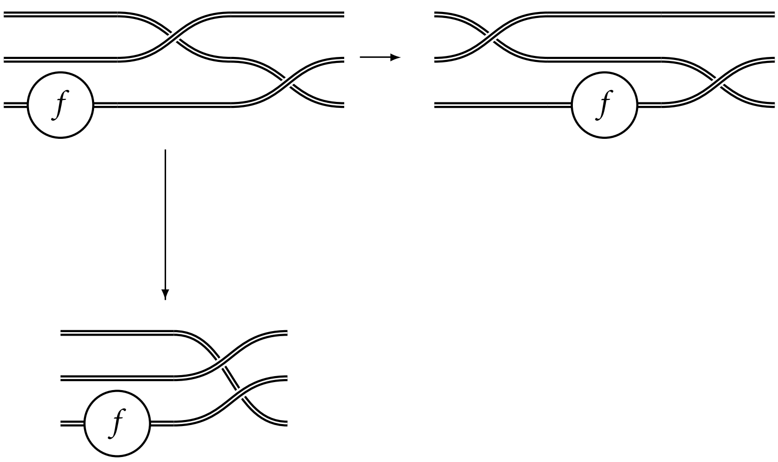
where f is either a multiarrow or a crossing. This diagram illustrates the style in which we shall show the conflicts. We omit extraneous prefix and suffix wires, and leave the arrows unlabelled (since it is always clear which rule applies by looking at the source and target). This case can of course be completed as



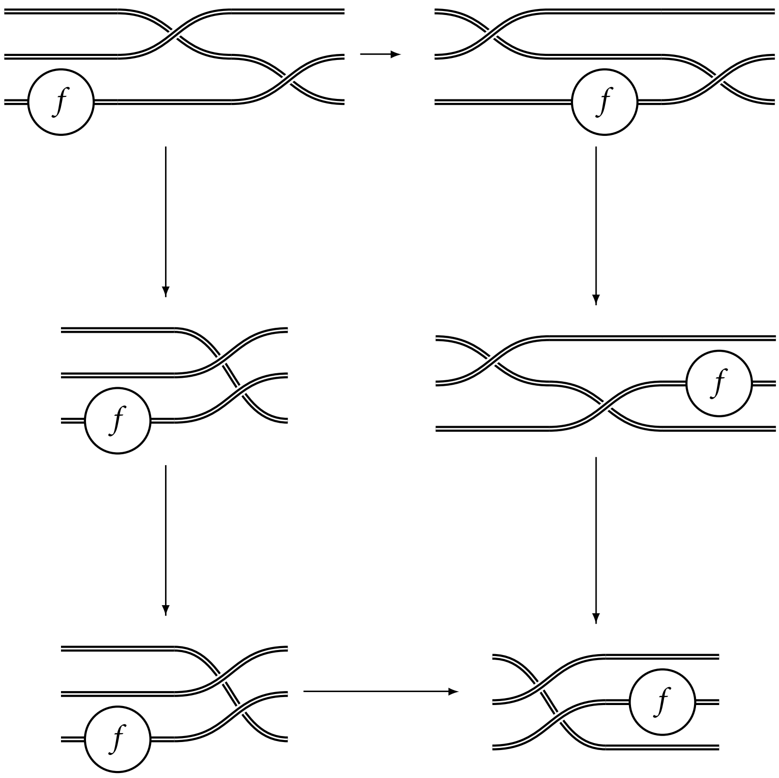
and it is clear that a similar completion is possible for all conflicts of this sort. Where, as in this case, there is an obvious reason below for some diagram of 2-cells to commute, no comment will be made below. Where the commutativity follows from the axioms in the definition of braiding for a monoidal bicategory, we indicate which axiom(s) are used. Note that interchange can conflict with itself, as in



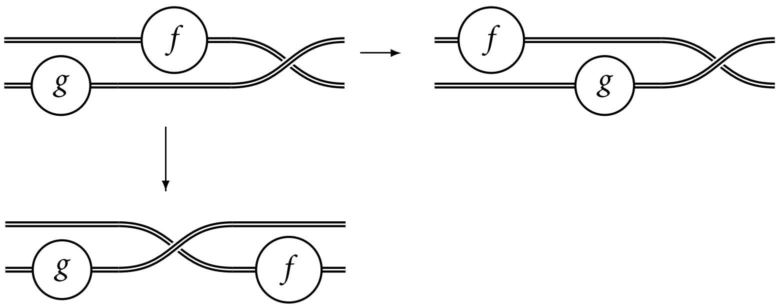
This is just another instance of the general situation described above, and does not require a separate treatment. This does not quite exhaust the possible ways that interchange can participate in a conflict. One remaining possibility occurs with the underbraiding rule, as follows.



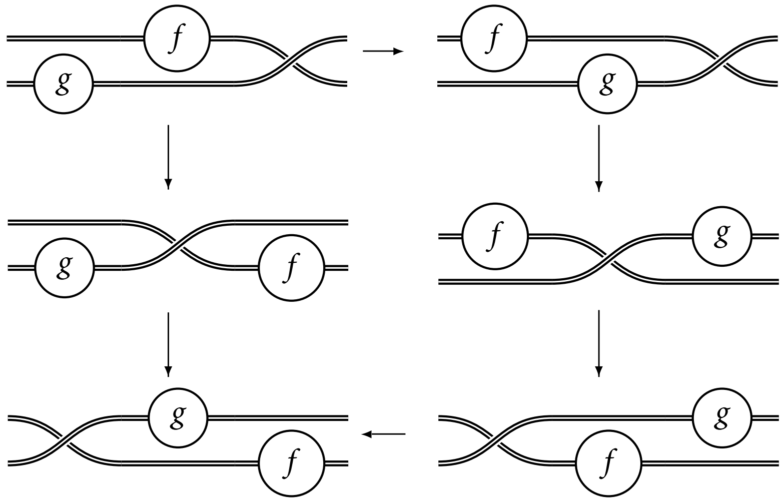
which can be completed as



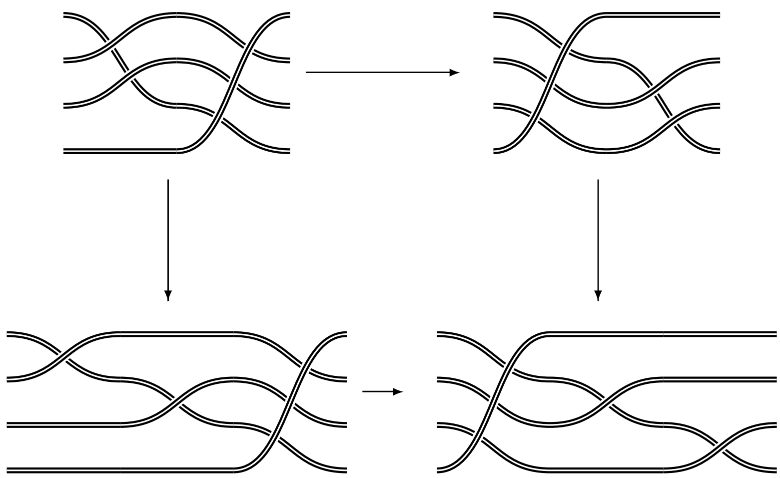
The other remaining case involving the interchange rule is where we have



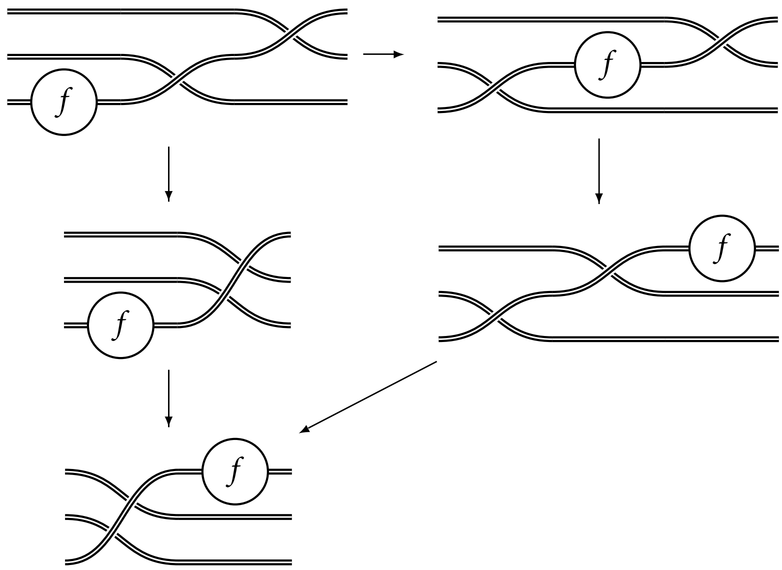
which can be completed as



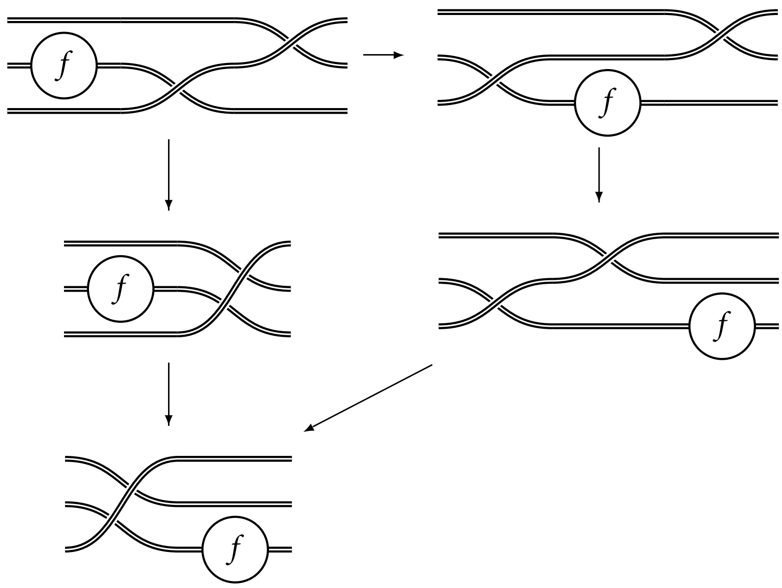
The pseudonaturality rules permit another general class of conflicts, where a sequence of basic 1-cells that could have some rule applied to it could alternatively be moved over or under another bunch of strands using the pseudonaturality rule. These conflicts can clearly be coherently completed in a natural way, as seen in the following illustrative example, which uses the underbraiding rule:



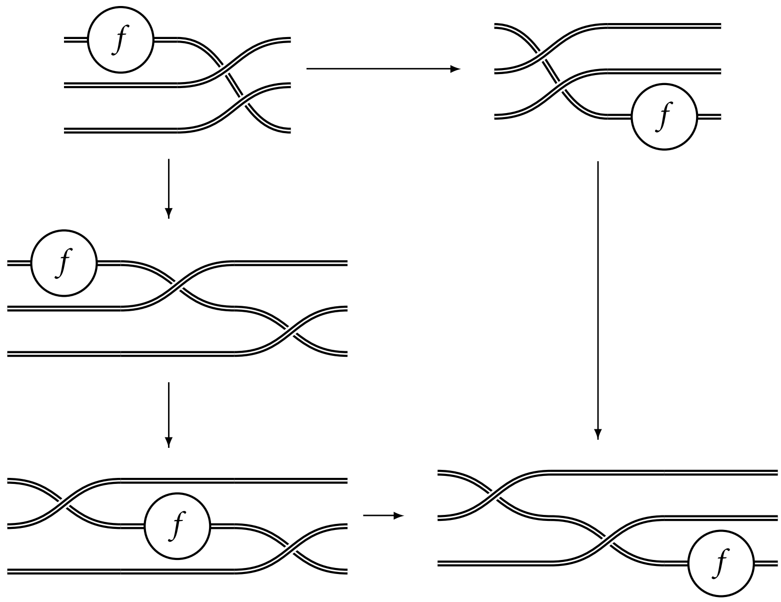
The pseudonaturality rules can also conflict with the over- and underbraiding rules in an only slightly more interesting way. With the overbraiding rule, there are two such basic possibilities, shown with their completions:



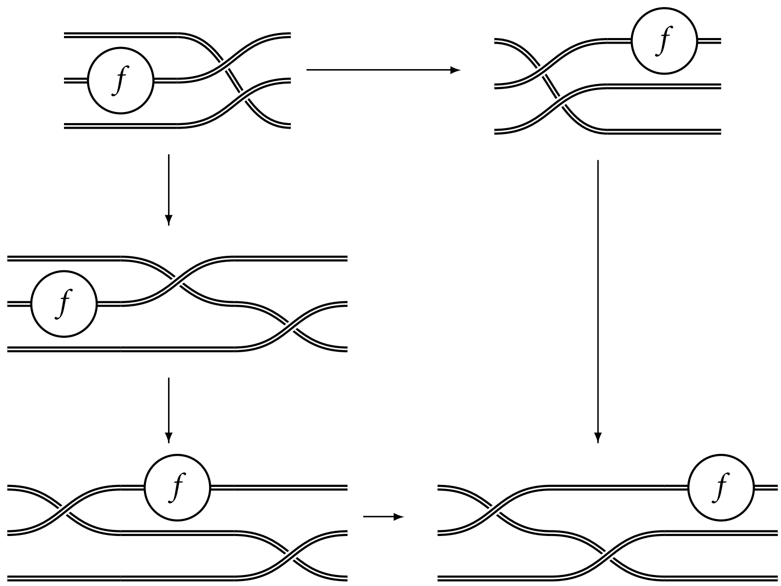
and



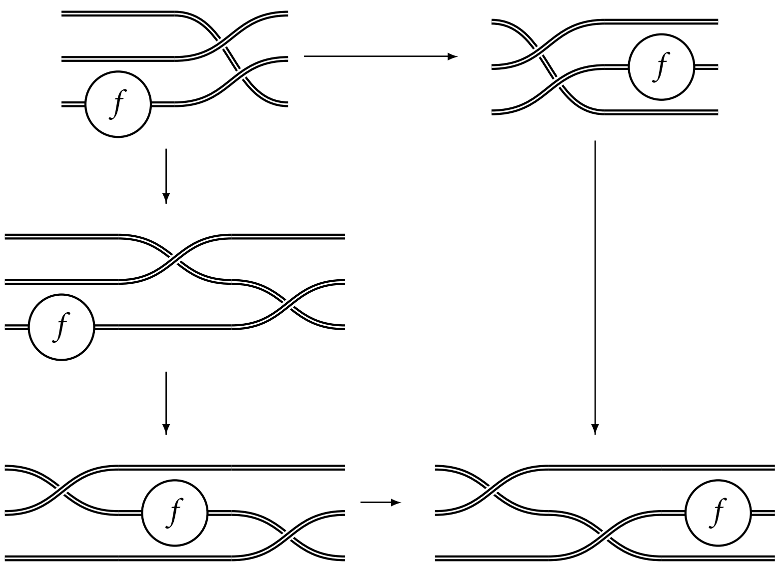
With the underbraiding rule, there are three, which all follow fundamentally the same pattern:



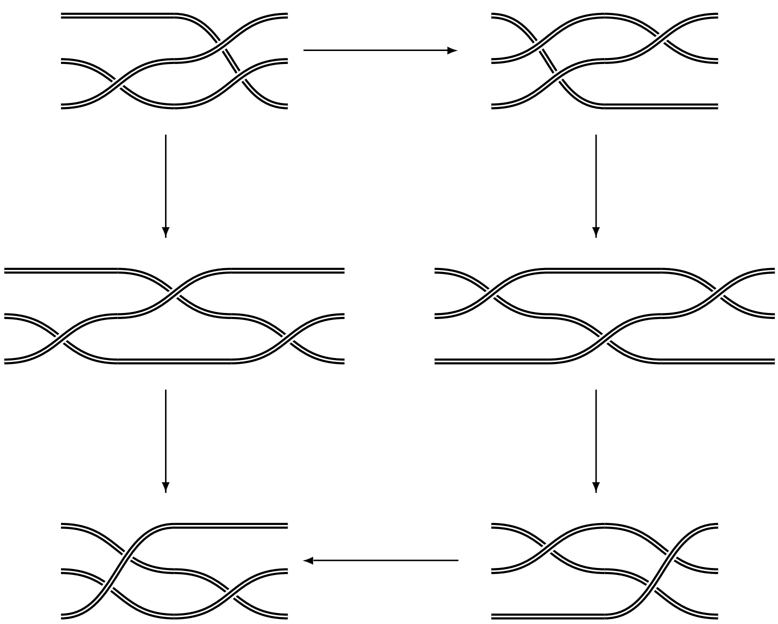
and the second:



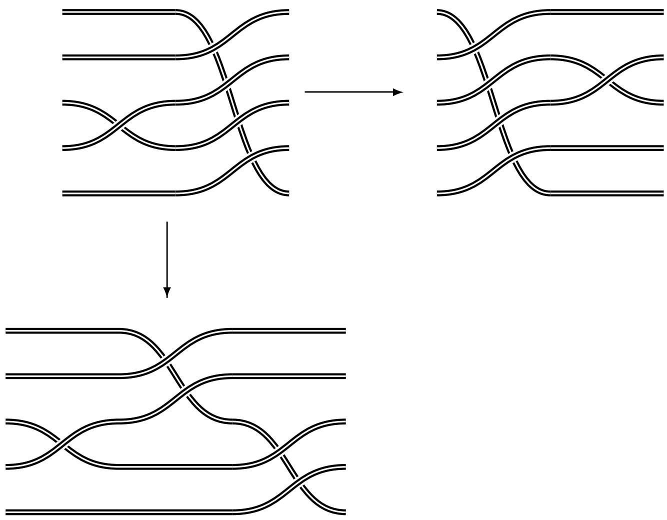
and the third:



There is also a more interesting kind of conflict between the underbraiding and pseudonaturality rules, as follows:

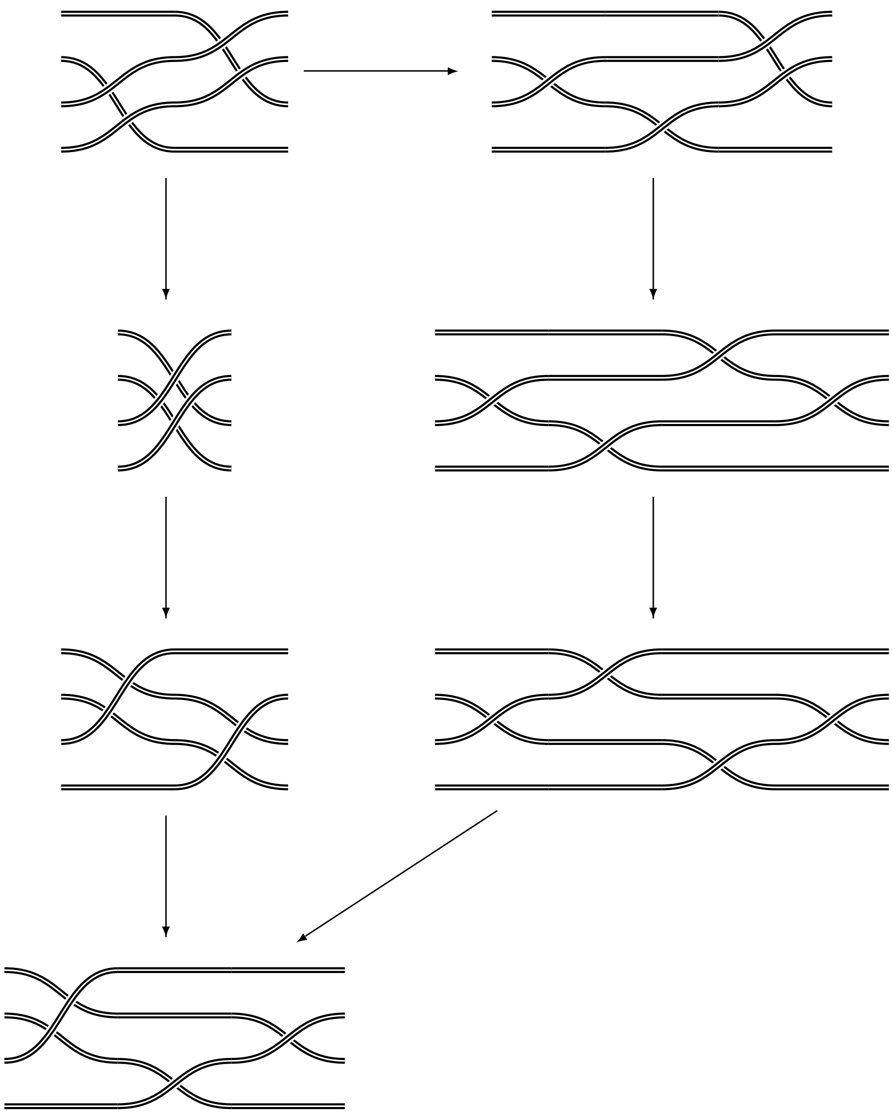


This diagram of 2-cells commutes by the Yang-Baxter axiom, i.e. the fourth axiom in our definition of braiding. In fact the conflict shown above is not the most general case of this situation; the general case looks like this:



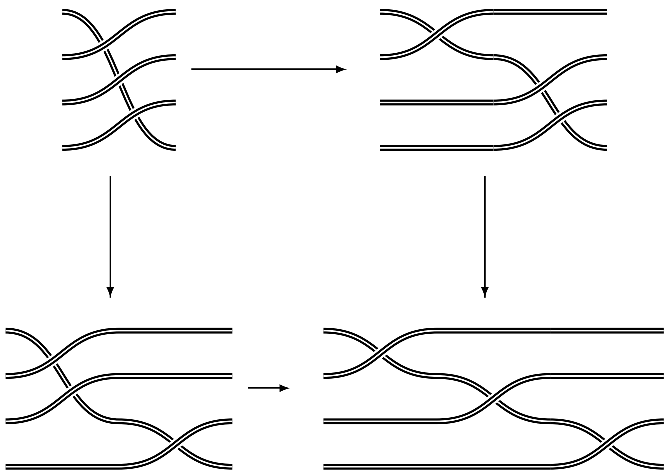
and can be completed in a similar way, once the underbraiding rule has been applied twice to both the resulting 1-cells. It is an easy exercise to show that the resulting diagram of 2-cells commutes.

There is also a non-trivial way in which the overbraiding and underbraiding rules can conflict with each other:



This corresponds to the mysterious third axiom in the definition of braiding. Of course the first two steps of the right-hand path could have been applied in the other order.

The final case to consider is conflict between the overbraiding rule and itself, or similarly between the underbraiding rule and itself. In the case of the overbraiding rule, the situation is this:



which corresponds to the second axiom. The obvious analogue for underbraiding corresponds to the first axiom.

Notice that we have set things up in such a way that there are no non-trivial conflicts with the unit rules. (The unit axioms were used to justify the decision to exclude empty sequences from the domain of application of the other rules.) Systematic consideration of the rules shows that we have exhausted the possible conflicts, so the proof is complete. □

Chapter 5

A Calculus of Components

5.1 The problem

Even with the help of the coherence theorem, it can be extremely tedious to prove even simple facts about pseudomonoids. The problem is essentially notational rather than mathematical: in an ordinary category, we can use commutative diagrams to establish equations between arrows. It is almost always easier to understand a diagram than an explicit sequence of equalities between expressions, because

- The types of the arrows are clear from the diagram.
- The diagram abstracts away from certain details of the proof: if two equations could be applied in either order, the diagram does not have to choose an order arbitrarily.
- The global structure of the whole argument can be seen at a glance.

In a bicategory, to prove an equation between 2-cells by similar means, we should need three-dimensional diagrams. This poses practical problems – paper is two-dimensional – and also requires the reader to visualise a three-dimensional structure, something that most humans do not find easy. This technique has sometimes been attempted nonetheless: by Lack (1995, Section 3.4), for example. The more usual alternative is to use a sequence of equations between string or pasting diagrams. As well as using a lot of paper, such proofs are often difficult to follow.

Now, there is certainly one monoidal bicategory in which this problem does not arise. Working in the monoidal bicategory Cat, one typically reasons about natural transformations via their components. For example, instead of the pasting equation

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] & \mathbb{D} \end{tikzcd} \quad \text{with } \Downarrow s \text{ and } \Downarrow t \quad = \quad \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow u"] & \mathbb{D} \end{tikzcd}$$

one would typically draw the commutative triangle

$$\begin{tikzcd} F(C) \arrow[rr, "u_C"] \arrow[dr, "s_C"'] & & H(C) \\ & G(C) \arrow[ur, "t_C"'] & \end{tikzcd}$$

for an arbitrary object $C \in \mathbb{C}$, considering the components of the natural transformations rather than the natural transformations as a whole. It turns out that this 'calculus of components' is in fact a valid mode of reasoning in *any* monoidal bicategory. We prove this by formalising the component-based reasoning, and showing how the resulting formal language may be interpreted in a monoidal bicategory. As a corollary, we show that certain types of theorem hold in any monoidal bicategory just when they hold in Cat. For example, the famous coherence theorem for monoidal categories implies a coherence theorem for general pseudomonoids.

The calculus of components may be used in situations of the following form: suppose we have a collection of 1-cells of the form

$$A_1 \otimes \cdots \otimes A_n \to B.$$

(In a general monoidal bicategory, some particular bracketing needs to be chosen for the tensor.) These can be combined by tensor and composition to create composite 1-cells of the same shape (i.e. where the target is a single object rather than a tensor of objects), as in a multicategory. For example, if we have 1-cells $F : A \otimes B \to C$ and $G : X \otimes Y \to B$, there is a natural composite $A \otimes X \otimes Y \to C$. In Cat, we could say that this composite takes the value $F(a, G(x, y))$ on objects $a \in A$, $x \in X$ and $y \in Y$. Such an expression can be used to describe the composite in any monoidal bicategory, where we regard the elements a, x and y in a purely formal way. Now suppose that we have another 1-cell $H : X \otimes Y \to B$, and a 2-cell $t : G \Rightarrow H$. There is a composite with components $F(a, H(x, y))$ (with a, x and y again being formal elements of A, X and Y), and there is clearly a 2-cell from '$F(a, G(x, y))$' to '$F(a, H(x, y))$', built from t and F. If we were in Cat, this 2-cell would have components $t_{a,x,y} : F(a, G(x, y)) \to F(a, H(x, y))$; and again, this description is adequate to describe the equivalent 2-cell in any monoidal bicategory, provided that we treat the 'elements' and 'components' in a purely formal fashion. Furthermore, as we shall see, one can perform equational reasoning (or equivalently and more importantly diagrammatic reasoning) with these components precisely as if they *were* components of ordinary natural transformations.

So the purpose of this chapter is to show how the notational difficulties associated with monoidal bicategories may, in some cases, be overcome by using a formal language to specify 1-cells, 2-cells, and equations between 2-cells in a monoidal bicategory. The syntax of the language is designed in such a way that a proof using the language closely resembles – typically, is formally identical to – a proof using categories, functors and natural transformations in the usual way. In other words, provided that one uses only admissible techniques, a proof in the 2-monoidal 2-category Cat may be reinterpreted as a general proof that applies to any monoidal bicategory. In particular, we shall be able to prove various general facts about pseudomonoids simply by reusing the usual proof of the corresponding fact for monoidal categories. The language could be regarded as a higher-dimensional analogue of the formal language developed Jay (1989) for monoidal categories.

This formal language is not completely general. The fundamental restriction (mentioned above) is that it may only be used to talk about theories whose 1-cells are of the form

$$A_1 \otimes \cdots \otimes A_n \to B.$$

In particular, the braiding on a braided Gray monoid is not of this form, and we will need an extension of the language to prove facts about structures that interact with the braiding, such as the braiding of a braided pseudomonoid. This extended language is developed in the next chapter.

Most of the formal development is fairly routine. The interpretation of a theory is defined first for Gray monoids, and then extended to arbitrary monoidal bicategories using coherence. The only potentially difficult part is to account for the non-strictness of the interchange law in a Gray monoid. In an earlier version of this chapter, some rather notationally-intimidating proofs were needed to account for that; however the coherence results of Chapter 4 now save us the trouble.[1]

5.2 The language

The language is used to prove equations between 2-cells that hold in every model of a given theory. We require the theory to be presented as a collection of objects,

[1] In particular, they save *you, the reader* the trouble, the author being unable to spare himself retroactively.

(multi-)1-cells, 2-cells and equations between 2-cells.

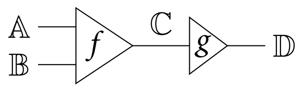
A *theory* $\mathcal{T}$ consists of:

- A set $\mathcal{T}_0$ of *objects*,
- For every non-empty sequence $\mathbb{A}_1, \ldots, \mathbb{A}_n, \mathbb{B}$ of elements of T_0, a set $\mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$ of *1-cells* $(\mathbb{A}_1, \ldots, \mathbb{A}_n) \to \mathbb{B}$. We suppose that the different sets of 1-cells are pairwise disjoint.
- Sets of 2-cells and equations, as described below.

Before we attempt a formal description of the 2-cells and equations, we shall introduce the part of the language that describes 1-cells.

5.2.1 1-cells

A 1-cell will be described by a *1-cell sequent*: suppose we are considering a theory with objects $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and 1-cells $f : (\mathbb{A}, \mathbb{B}) \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$. Then the arrow



might be represented by the sequent

$$A \in \mathbb{A}, B \in \mathbb{B} \vdash g(f(A, B)) \in \mathbb{D}.$$

The names in the context – to the left of the turnstile – should be regarded as bound, so for example the sequent

$$X \in \mathbb{A}, Y \in \mathbb{B} \vdash g(f(X, Y)) \in \mathbb{D}$$

is equivalent to the one above, modulo the renaming of bound variables, and represents the same 1-cell. (We assume throughout that we have some infinite set of names on which to draw. Names here will be represented by upper-case italic letters.) Formally, the derivation rules for 1-cell sequents are as follows:

- The axiom rule: for every $\mathbb{C} \in \mathcal{T}_0$,

$$\frac{}{A \in \mathbb{C} \vdash A \in \mathbb{C}}\ (\mathbb{C})$$

- The application rule: for every $f \in \mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$,

$$\frac{\Gamma_1 \vdash \alpha_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \alpha_n \in \mathbb{A}_n}{\Gamma_1, \ldots, \Gamma_n \vdash f(\alpha_1, \ldots, \alpha_n) \in \mathbb{B}}\ f(\bullet)$$

where the sets of names in the Γ_i are assumed pairwise disjoint.

(Whenever several contexts are mentioned together, we shall *always* assume that their sets of names are pairwise disjoint, usually without explicitly mentioning this assumption.) So a 1-cell sequent is simply a formal composite of formal 1-cells. It is clear (and easily proved by induction) that every derivable 1-cell sequent has a unique derivation. Also:

Definition 5.1. Let $A_1 \in \mathbb{A}_1, \cdots, A_n \in \mathbb{A}_n \vdash \beta \in \mathbb{B}$ be a derivable 1-cell sequent; and for every $1 \leq i \leq n$, let $\Gamma_i \vdash \alpha_i \in \mathbb{A}_i$ be a derivable 1-cell sequent. Then we write $[A_i := \alpha_i]_{i=1}^n$ to denote the multiple substitution

$$[A_1 := \alpha_1, \cdots, A_n := \alpha_n].$$

On occasion this is abbreviated to $[A_i := \alpha_i]_i$, where the value of n is apparent from the context.

Later, we sometimes need to use nested sequences: if

$$(A^1_1, \cdots, A^1_{n_i}), \ldots, (A^n_1, \cdots, A^n_{n_n})$$

is a nested sequence of objects, with α^i_j being a similarly-numbered nested sequence of 1-cell expressions, then

$$[[A^i_j := \alpha^i_j]^{n_i}_{j=1}]^n_{i=1}$$

denotes the substitution

$$[A^1_1 := \alpha^1_1, \cdots, A^1_{n_1} := \alpha^1_{n_1}, \quad \cdots, \quad A^n_1 := \alpha^n_1, \cdots, A^n_{n_n} := \alpha^n_{n_n}].$$

Lemma 5.2 (1-cell substitution). *If $A_1 \in \mathbb{A}_1, \cdots, A_n \in \mathbb{A}_n \vdash \beta \in \mathbb{B}$ is a derivable 1-cell sequent, and for every $1 \leq i \leq n$, $\Gamma_i \vdash \alpha_i \in \mathbb{A}_i$ is a derivable 1-cell sequent, with the names in the Γ_i pairwise disjoint, then*

$$\Gamma_i, \cdots, \Gamma_n \vdash \beta[A_i := \alpha_i]^n_{i=1}$$

is a derivable 1-cell sequent,

Proof. Consider a derivation of β, and make the substitution throughout the derivation, as follows. For each sequent $\Gamma \vdash \gamma \in \mathbb{C}$ in the derivation, replace γ by $\gamma[A_i := \alpha_i]_i$, and replace each name $A_i \in \mathbb{A}_i$ of Γ by the contents of Γ_i. Each application rule is thus transformed into another instance of that application rule, and each axiom $A_i \in \mathbb{A}_i \vdash A_i \in \mathbb{A}_i$ is transformed into $\Gamma_i \vdash \alpha_i \in \mathbb{A}_i$, which is derivable by assumption. □

5.2.2 2-cells

Now we can define the 2-cells of $\mathcal{T}$. In addition to the objects and 1-cells, we have: for every pair $\Gamma \vdash \alpha \in \mathbb{B}$, $\Gamma \vdash \beta \in \mathbb{B}$ of derivable 1-cell sequents, a set $\mathcal{T}_{\mathbb{B}}^{\Gamma}[\alpha, \beta]$ of 2-cells. Of course, we take these to be invariant under renaming, so if σ is a permutation of the set of names then we require

$$\mathcal{T}_{\mathbb{B}}^{\Gamma^\sigma}[\alpha^\sigma, \beta^\sigma] = \mathcal{T}_{\mathbb{B}}^{\Gamma}[\alpha, \beta].$$

Any two *different* sets of 2-cells (i.e. sets that are not presumed equal by the above) are taken to be disjoint. A 2-cell sequent is of the form $\Gamma \vdash \phi : \alpha \to \beta \in \mathbb{B}$. The derivation rules for 2-cell sequents are:

- The identity rule:

$$\frac{\Gamma \vdash \alpha \in \mathbb{A}}{\Gamma \vdash 1_\alpha : \alpha \to \alpha \in \mathbb{A}}\ 1$$

- The axiom rule: for every $t \in \mathcal{T}_{\mathbb{B}}^{\Gamma}[\alpha, \beta]$, with $\Gamma = (A_1 \in \mathbb{A}_1, \ldots, A_n \in \mathbb{A}_n)$,

$$\frac{\Gamma_1 \vdash \gamma_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \gamma_n \in \mathbb{A}_n}{\Gamma_1 \cdots \Gamma_n \vdash t_{\gamma_1, \ldots, \gamma_n} : \alpha[A_i := \gamma_i]_i \to \beta[A_i := \gamma_i]_i \in \mathbb{B}}\ t_\bullet$$

  where the sets of names in the Γ_i are pairwise disjoint.

- The composition rule:

$$\frac{\Gamma \vdash \phi : \beta \to \gamma \in \mathbb{A} \quad \Gamma \vdash \psi : \alpha \to \beta \in \mathbb{A}}{\Gamma \vdash \phi \cdot \psi : \alpha \to \gamma \in \mathbb{A}}\ \text{comp}$$

- The 1-cell application rule: for every $f \in \mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$,

$$\frac{\Gamma_1 \vdash \phi_1 : \alpha_1 \to \beta_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \phi_n : \alpha_n \to \beta_n \in \mathbb{A}_n}{\Gamma_1, \cdots, \Gamma_n \vdash f(\phi_1, \ldots, \phi_n) : f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n) \in \mathbb{B}}\ f(\to)$$

  The names in the Γ_i are again required to be pairwise disjoint.

Since the conclusion of each rule has a distinct syntactic form, each derivable 2-cell sequent has a unique derivation. Notice that, if $\Gamma \vdash \phi : \alpha \to \beta \in \mathbb{B}$ is a derivable 2-cell sequent, then $\Gamma \vdash \alpha \in \mathbb{B}$ and $\Gamma \vdash \beta \in \mathbb{B}$ are derivable 1-cell sequents. We also observe that substituting a 1-cell expression in a 2-cell expression, or vice versa, results in a derivable 2-cell sequent:

Lemma 5.3 (1-in-2 substitution). *Let $\Delta \vdash \phi : \alpha \to \beta \in \mathbb{C}$ be a derivable 2-cell sequent, with*

$$\Delta = B_1 \in \mathbb{B}_1, \cdots, B_n \in \mathbb{B}_n.$$

For each $1 \leq i \leq n$*, let*

$$\Gamma_i \vdash \gamma_i \in \mathbb{B}_i$$

be a derivable 1-cell sequent. Then the 2-cell sequent

$$\Gamma_1, \cdots, \Gamma_n \vdash \phi[B_i := \gamma_i]_i : \alpha[B_i := \gamma_i]_i \to \beta[B_i := \gamma_i]_i \in \mathbb{C},$$

is also derivable, assuming that the sets of names in the Γ_i *and* Δ *are pairwise disjoint.*

Proof. Consider a derivation of $\Delta \vdash \phi : \alpha \to \beta \in \mathbb{C}$, and make the substitution throughout the derivation. In detail: for every sequent in the derivation, in the context replace each $B_i \in \mathbb{B}_i$ by Γ_i and in the expression replace each B_i by γ_i. For each derivation rule, this substitution results in another instance of the same rule, and the 1-cell substitution lemma ensures that every 1-cell sequent in the derivation remains derivable. The conclusion of this new derivation is

$$\Gamma_1, \cdots, \Gamma_n \vdash \phi[B_i := \gamma_i]_i : \alpha[B_i := \gamma_i]_i \to \beta[B_i := \gamma_i]_i \in \mathbb{C},$$

which is therefore derivable. □

Lemma 5.4 (2-in-1 substitution). *Let* $\Gamma \vdash \gamma \in \mathbb{C}$ *be a derivable 1-cell sequent, for some*

$$\Gamma = (A_1 \in \mathbb{A}_1, \ldots, A_n \in \mathbb{A}_n).$$

If the 2-cell sequents

$$\Delta_1 \vdash \phi_1 : \alpha_1 \to \beta_1 \in \mathbb{A}_1 \quad \cdots \quad \Delta_n \vdash \phi_n : \alpha_n \to \beta_n \in \mathbb{A}_n$$

are all derivable, then so is

$$\Delta_1, \cdots, \Delta_n \vdash \gamma[A_i := \phi_i]_i : \gamma[A_i := \alpha_i]_i \to \gamma[A_i := \beta_i]_i \in \mathbb{B}.$$

The names in the Δ_i *must be pairwise disjoint, of course.*

Proof. This is simply a matter of repeatedly applying the 1-cell application rule. Formally, the proof is by induction on γ. If γ is a constant, i.e. Γ is the singleton $C \in \mathbb{C}$ and γ is just C, then the conclusion is equal to the hypothesis and the claim is trivial. Suppose, then, that $\gamma = f(\gamma_1, \ldots, \gamma_m)$ for some $m \in \mathbb{N}$, 1-cell symbol $f \in \mathcal{T}(\mathbb{B}_1, \ldots, \mathbb{B}_m; \mathbb{C})$ and 1-cell expressions γ_i.

Clearly the context Γ must be equal to $(\Gamma_1, \cdots, \Gamma_m)$, where each Γ_i contains the names used in γ_i. We want to reindex the names so that it is clear to which γ_i each belongs, for which we introduce a simple technical device that is also used in several other proofs below. For each $1 \le i \le m$, let m_i be the number of names in Γ_i. Then, given $1 \le i \le m$ and $1 \le j \le m_i$, define $\langle i, j\rangle$ to be $j + \sum_{r<i} m_r$. The effect of this definition is that

$$\Gamma_i = \left(A_{\langle i,1\rangle} \in \mathbb{A}_{\langle i,1\rangle}, \cdots, A_{\langle i,m_i\rangle} \in \mathbb{A}_{\langle i,m_i\rangle}\right)$$

or, in abbreviated notation, that

$$\Gamma_i = \left(A_{\langle i,j\rangle} \in \mathbb{A}_{\langle i,j\rangle}\right)_{j=1}^{m_i}$$

The inductive hypothesis implies that, for each $1 \le i \le m$, the 2-cell sequent

$$\Gamma_i \vdash \gamma_i[A_{\langle i,j\rangle} := \phi_{\langle i,j\rangle}]_{j=1}^{m_i} : \gamma_i[A_{\langle i,j\rangle} := \alpha_{\langle i,j\rangle}]_{j=1}^{m_i} \to \gamma_j[A_{\langle i,j\rangle} := \beta_{\langle i,j\rangle}]_{j=1}^{m_i} \in \mathbb{B}_i.$$

is derivable. Now apply the 1-cell application rule to these sequents and the 1-cell symbol f, and the conclusion follows. □

5.2.3 Equations

The final ingredient that we need to complete our description of $\mathcal{T}$ is a collection of equations between 2-cells. Formally, we declare that for each pair $\Gamma \vdash \alpha \in \mathbb{A}$ and $\Gamma \vdash \beta \in \mathbb{A}$ of 1-cell sequents, there is a set that we denote $\mathcal{T}_=(\Gamma \vdash \alpha \to \beta \in \mathbb{A})$. This set consists of pairs (ϕ, ψ), where $\Gamma \vdash \phi : \alpha \to \beta \in \mathbb{A}$ and $\Gamma \vdash \psi : \alpha \to \beta \in \mathbb{A}$ are derivable 2-cell sequents. We also require the collection of equations to be closed under renaming, so if σ is a permutation of the set of names then

$$(\phi, \psi) \in \mathcal{T}_=(\Gamma \vdash \alpha \to \beta \in \mathbb{A}) \Leftrightarrow (\phi^\sigma, \psi^\sigma) \in \mathcal{T}_=(\Gamma^\sigma \vdash \alpha^\sigma \to \beta^\sigma \in \mathbb{A}).$$

In fact this restriction is not strictly necessary, since the axiom rule for equations (below) permits any 1-in-2 substitution instance of an equation to be used, but we retain it for the sake of consistency.

The derivation rules for equations are as follows. There are three rules expressing the fact that any equality worth the name should be reflexive, symmetric, and transitive:

- For every derivable 2-cell sequent $\Gamma \vdash \phi : \alpha \to \beta \in \mathbb{A}$, we have reflexivity:

$$\frac{}{\Gamma \vdash \phi = \phi : \alpha \to \beta \in \mathbb{A}} =_r$$

- symmetry:

$$\frac{\Gamma \vdash \phi = \psi : \alpha \to \beta \in \mathbb{A}}{\Gamma \vdash \psi = \phi : \alpha \to \beta \in \mathbb{A}} =_s$$

- and transitivity:

$$\frac{\Gamma \vdash \psi = \chi : \alpha \to \beta \in \mathbb{A} \quad \Gamma \vdash \phi = \psi : \alpha \to \beta \in \mathbb{A}}{\Gamma \vdash \phi = \chi : \alpha \to \beta \in \mathbb{A}} =_t$$

There is an axiom rule for equations, which allows us to use any 1-in-2 substitution instance of an equation axiom:

- For every $(\Delta, \mathbb{A}, \alpha, \beta, \phi, \psi) \in \mathcal{T}_=$, with $\Delta = (B_1 \in \mathbb{B}_1, \cdots, B_n \in \mathbb{B}_n)$,

$$\frac{\Gamma_1 \vdash \gamma_1 \in \mathbb{B}_1 \quad \cdots \quad \Gamma_n \vdash \gamma_n \in \mathbb{B}_n}{\Gamma_1 \cdots \Gamma_n \vdash \phi[B_i := \gamma_i]_i = \psi[B_i := \gamma_i]_i : \alpha[B_i := \gamma_i]_i \to \beta[B_i := \gamma_i]_i \in \mathbb{A}} \ (\phi = \psi)$$

The next two rules express the idea that the 2-cell construction rules should preserve equality:

- The composition rule preserves equality:

$$\frac{\Gamma \vdash \phi = \phi' : \beta \to \gamma \in \mathbb{A} \quad \Gamma \vdash \psi = \psi' : \alpha \to \beta \in \mathbb{A}}{\Gamma \vdash \phi \cdot \psi = \phi' \cdot \psi' : \alpha \to \gamma \in \mathbb{A}} \ \text{comp}$$

- The 1-cell application rule preserves equality: for every $f \in \mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$,

$$\frac{\Gamma_1 \vdash \phi_1 = \phi_1' : \alpha_1 \to \beta_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \phi_n = \phi_n' : \alpha_n \to \beta_n \in \mathbb{A}_n}{\Gamma_1, \cdots, \Gamma_n \vdash f(\phi_1, \ldots, \phi_n) = f(\phi_1', \ldots, \phi_n') : f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n) \in \mathbb{B}} \ f(\to)$$

Next, there are equations expressing the fact that 1-cell application preserves identities and composition:

- 1-cell application preserves identities: for every $f \in \mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$,

$$\frac{\Gamma_1 \vdash \alpha_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \alpha_n \in \mathbb{A}_n}{\Gamma_1, \cdots, \Gamma_n \vdash f(1_{\alpha_1}, \ldots, 1_{\alpha_n}) = 1_{f(\alpha_1, \ldots, \alpha_n)} \in \mathbb{B}} \ f(1)$$

- 1-cell application preserves composition: for every $f \in \mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$,

$$\frac{\Gamma_1 \vdash \phi_1 : \beta_1 \to \gamma_1 \in \mathbb{A}_1 \ \ \Gamma_1 \vdash \psi_1 : \alpha_1 \to \beta_1 \in \mathbb{A}_n \quad \ldots \quad \Gamma_n \vdash \phi_n : \beta_n \to \gamma_n \in \mathbb{A}_n \ \ \Gamma_n \vdash \psi_n : \alpha_n \to \beta_n \in \mathbb{A}_n}{\Gamma_1, \cdots, \Gamma_n \vdash f(\phi_1, \ldots, \phi_n) \cdot f(\psi_1, \ldots, \psi_n) = f(\phi_1 \cdot \psi_1, \ldots, \phi_n \cdot \psi_n) : f(\alpha_1, \ldots, \alpha_n) \to f(\gamma_1, \ldots, \gamma_n) \in \mathbb{B}} \ f(\cdot)$$

The identity 2-cells should be units for composition:

- Left identity:

$$\frac{\Gamma \vdash \phi : \alpha \to \beta \in \mathbb{B}}{\Gamma \vdash 1_\beta \cdot \phi = \phi : \alpha \to \beta \in \mathbb{B}} \ l_\beta$$

- Right identity:

$$\frac{\Gamma \vdash \phi : \alpha \to \beta \in \mathbb{B}}{\Gamma \vdash \phi \cdot 1_\alpha = \phi : \alpha \to \beta \in \mathbb{B}}\, r_\beta$$

Finally, we have:

- The naturality rule: for every $t \in \mathcal{T}^\Gamma_{\mathbb{B}}[\alpha, \beta]$, with $\Gamma = (A_1 \in \mathbb{A}_1, \ldots, A_n \in \mathbb{A}_n)$,

$$\frac{\Gamma_1 \vdash \phi_1 : \gamma_1 \to \delta_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \phi_n : \gamma_n \to \delta_n \in \mathbb{A}_n}{\Gamma_1 \cdots \Gamma_n \vdash \beta[A_i := \phi_i]_i \cdot t_{\gamma_1,\ldots,\gamma_n} = t_{\delta_1,\ldots,\delta_n} \cdot \alpha[A_i := \phi_i]_i : \alpha[A_i := \gamma_i]_i \to \beta[A_i := \delta_i]_i \in \mathbb{B}}\, t_\natural$$

  where the sets of names in the Γ_i are pairwise disjoint.

The following proposition is just a sanity check of the equation rules.

Proposition 5.5. *If*

$$\Gamma \vdash \phi = \psi : \alpha \to \beta \in \mathbb{B}$$

is a derivable equation sequent, then $\Gamma \vdash \phi : \alpha \to \beta \in \mathbb{B}$ *and* $\Gamma \vdash \psi : \alpha \to \beta \in \mathbb{B}$ *are derivable 2-cell sequents.*

Proof. An easy induction over the derivation, using Lemma 5.3 for the axiom rule and Lemmas 5.3 and 5.4 for the naturality rule. □

5.3 Interpretation in a Gray monoid

To interpret these sequents in a target monoidal bicategory $\mathcal{B}$, we need an interpretation, or model, of $\mathcal{T}$. This section defines the interpretation in a Gray monoid, so suppose for now that $\mathcal{B}$ is a Gray monoid. A model $v : \mathcal{T} \to \mathcal{B}$ consists of:

- for every $\mathbb{A} \in \mathcal{T}_0$, an object $v(\mathbb{A})$ in $\mathcal{B}$;
- for every $f \in \mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$, a 1-cell

$$v(f) : v(\mathbb{A}_1) \otimes \cdots \otimes v(\mathbb{A}_n) \to v(\mathbb{B});$$

  in $\mathcal{B}$.

- for every $t \in \mathcal{T}^\Gamma_{\mathbb{B}}[\alpha, \beta]$, a 2-cell

$$v(t) : [\![\Gamma \vdash \alpha \in \mathbb{B}]\!]_v \Rightarrow [\![\Gamma \vdash \beta \in \mathbb{B}]\!]_v$$

  where the function $[\![-]\!]_v$ is as defined for 1-cell sequents below,

- such that for every $(\Gamma, \alpha, \beta, \phi, \psi) \in \mathcal{T}_=$ we have

$$[\![\Gamma \vdash \phi : \alpha \to \beta]\!]_v = [\![\Gamma \vdash \psi : \alpha \to \beta]\!]_v$$

where the function $[\![-]\!]_v$ is as defined for 2-cell sequents below.

If $\Gamma = (A_1 \in \mathbb{A}_1, \cdots, A_n \in \mathbb{A}_n)$ is a context, we write $v(\Gamma)$ as an abbreviation for $v(\mathbb{A}_1) \otimes \cdots \otimes v(\mathbb{A}_n)$.

5.3.1 Interpretation of 1-cells

The semantic interpretation $[\![-]\!]_v$ of a 1-cell derivation is defined by induction:

- $\left[\!\!\left[\dfrac{}{A \in \mathbb{C} \vdash A \in \mathbb{C}}\,(\mathbb{C})\right]\!\!\right]_v = 1_{v(\mathbb{C})}$,
- $\left[\!\!\left[\dfrac{\dfrac{[\pi_1]}{\Gamma_1 \vdash \alpha_1 \in \mathbb{A}_1} \quad \cdots \quad \dfrac{[\pi_n]}{\Gamma_n \vdash \alpha_n \in \mathbb{A}_n}}{\Gamma_1, \ldots, \Gamma_n \vdash f(\alpha_1, \ldots, \alpha_n) \in \mathbb{B}}\, f(\bullet)\right]\!\!\right]_v$

$$= v(f) \circ \left(\left[\!\!\left[\frac{[\pi_1]}{\Gamma_1 \vdash \alpha_1 \in \mathbb{A}_1}\right]\!\!\right]_v \otimes \cdots \otimes \left[\!\!\left[\frac{[\pi_n]}{\Gamma_n \vdash \alpha_n \in \mathbb{A}_n}\right]\!\!\right]_v \right).$$

Since a derivable 1-cell sequent has a unique derivation, we may regard the input to the interpretation function as a sequent rather than a derivation. Clearly $[\![\Gamma \vdash \alpha \in \mathbb{B}]\!]_v$ is always a 1-cell $v(\Gamma) \to v(\mathbb{B})$. Where Γ and $\mathbb{B}$ are evident from the context, we shall abbreviate $[\![\Gamma \vdash \alpha \in \mathbb{B}]\!]_v$ to $[\![\gamma]\!]_v$.

5.3.2 Semantics of substitution

In order to define the interpretation of 2-cells (and, later, to show soundness) we shall need a careful analysis of the semantics of 1-cell substitution. The reason this is non-trivial is that the interchange law of a Gray monoid does not hold on the nose: there is some work to be done to account for this.

Definition 5.6. Given arrows $f_1, \ldots, f_n$ of a Gray monoid, we write $\otimes_{i=1}^n f_i$ to mean

$$f_1 \otimes \cdots \otimes f_n.$$

Definition 5.7. Given arrows $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ of a Gray monoid, let $\mathrm{I}_{i=1}^n(f_i, g_i)$ denote the interchange isomorphism

$$\bigotimes_{i=1}^n f_i \circ \bigotimes_{i=1}^n g_i \to \bigotimes_{i=1}^n (f_i \circ g_i).$$

If $((f^i_j)^{n_i}_{j=1})^n_{i=1}$ and $((g^i_j)^{n_i}_{j=1})^n_{i=1}$ are nested sequences of 1-cells, then the interchange isomorphism

$$\bigotimes_{i=1}^{n}\bigotimes_{j=1}^{n^i_j} f^i_j \circ \bigotimes_{i=1}^{n}\bigotimes_{j=1}^{n^i_j} g^i_j \to \bigotimes_{i=1}^{n}\bigotimes_{j=1}^{n^i_j}(f^i_j \circ g^i_j)$$

is denoted $(\mathrm{I}^{n_i}_{j=1})^n_{i=1}(f^i_j, g^i_j)$.

Definition 5.8. Now let $\Gamma \vdash \beta \in \mathbb{C}$ be a derivable 1-cell sequent, where

$$\Gamma = (B_i \in \mathbb{B}_i)^n_{i=1},$$

and for each $1 \leq i \leq n$ let $\Delta_i \vdash \alpha_i \in \mathbb{B}_i$ be a derivable 1-cell sequent. We shall explicitly define a 'normalisation' isomorphism

$$\mathrm{norm}^{\Gamma}_{\mathbb{C}}((\alpha_i)^n_{i=1}, \beta) : [\![\beta]\!]_\nu \circ \bigotimes_{i=i}^{n}[\![\alpha_i]\!]_\nu \xrightarrow{\cong} [\![\beta[B_i := \alpha_i]^n_{i=1}]\!]_\nu$$

The definition is by recursion over β. If $\beta = B_1$, then we use the identity. So suppose that $\beta = f(\beta_1, \ldots, \beta_m)$ for some natural number m and 1-cell symbol $f \in \mathcal{T}(\mathbb{C}_1, \cdots, \mathbb{C}_m; \mathbb{D})$. For each $1 \leq i \leq m$, let m_i denote the number of names that occur in β_i. Let B^i_j denote $B_{j+\sum_{r<i} m_r}$ and α^i_j denote $\alpha_{j+\sum_{r<i} m_r}$. Thus $(B_i)^n_{i=1}$ is divided into the nested sequence $((B^i_j)^{m_i}_{j=1})^m_{i=1}$, where the inner sequence $(B^i_j)^{m_i}_{j=1}$ contains just the names that occur in β_i. In a similar way, divide the context Γ into $(\Gamma_1, \cdots, \Gamma_m)$ so that each Γ_i contains the names that occur in β_i, i.e.

$$\Gamma_i = (B^i_j \in \mathcal{B}^{i_j})^{m_i}_{j=1},$$

where the nested sequence $(\mathcal{B}^i_j)$ is defined in the obvious way.

Now, define $\mathrm{norm}^{\Gamma}_{\mathbb{D}}((\alpha_i)^n_{i=1}, \beta)$ to be the composite

$$\begin{array}{c}
\nu(f) \circ \bigotimes_{i=1}^{m}[\![\beta_i]\!]_\nu \circ \bigotimes_{i=1}^{m}\bigotimes_{j=1}^{m_i}[\![\alpha^i_j]\!]_\nu \\
\Big\downarrow \nu(f) \circ \mathrm{I}^m_{i=1}([\![\beta_i]\!]_\nu, \otimes^{m_j}_{j=1}[\![\alpha^i_j]\!]_\nu) \\
\nu(f) \circ \bigotimes_{i=1}^{m}([\![\beta_i]\!]_\nu \circ \bigotimes_{j=1}^{m_i}[\![\alpha^i_j]\!]_\nu) \\
\Big\downarrow \nu(f) \circ \otimes^m_{i=i} \mathrm{norm}^{\Gamma_i}_{\mathbb{C}_i}((\alpha^i_j)^{m_i}_{j=1}, \beta_i) \\
\nu(f) \circ \bigotimes_{i=1}^{m}[\![\beta_i[B^i_j := \alpha^i_j]^{m_i}_{j=1}]\!]_\nu
\end{array}$$

This completes the recursive definition.

We shall need the fact that the norm operation has a kind of associativity property, described in the following Proposition. This is absolutely fundamental, since it is used to show that Gray monoids have enough coherence for our language to be soundly interpretable. Fortunately it follows immediately from the coherence theorem of Chapter 4.

Proposition 5.9 (Double norm). *Let $\Gamma \vdash \gamma \in \mathbb{D}$ be a derivable 1-cell sequent, where $\Gamma = (C_i \in \mathbb{C}_i)_{i=1}^n$. For each $1 \le i \le n$, let $\Delta_i \vdash \beta_i \in \mathbb{C}_i$ be a derivable 1-cell sequent, where $\Delta_i = (B_j^i \in \mathbb{B}_j^i)_{j=1}^{n_i}$. For each $1 \le i \le n$ and $1 \le j \le n_i$, let $\Xi_j^i \vdash \alpha_j^i \in \mathbb{B}_j^i$ be a derivable 1-cell sequent. Then the diagram*

$$\begin{tikzcd}
\llbracket\gamma\rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket\beta_i\rrbracket_v \circ \bigotimes_{i=1}^{n} \bigotimes_{j=1}^{n_i} \llbracket\alpha_j^i\rrbracket_v \arrow[r, "\llbracket\gamma\rrbracket_v \circ \mathrm{I}_{i=1}^n(\llbracket\beta_i\rrbracket_v, \otimes_{j=1}^{n_i} \llbracket\alpha_j^i\rrbracket_v)"] \arrow[d, "\mathrm{norm}_{\mathbb{D}}^{\Gamma}\left((\beta_i)_{i=1}^n, \gamma\right) \circ \otimes_{i=1}^n \otimes_{j=1}^{n_i} \llbracket\alpha_j^i\rrbracket_v"] & \llbracket\gamma\rrbracket_v \circ \bigotimes_{i=1}^{n} (\llbracket\beta_i\rrbracket_v \circ \bigotimes_{j=1}^{n_i} \llbracket\alpha_j^i\rrbracket_v) \arrow[d, "\llbracket\gamma\rrbracket_v \circ \otimes_{i=1}^n \mathrm{norm}_{\mathbb{C}_i}^{\Delta_i}\left((\alpha_j^i)_{j=1}^{n_i}, \beta_i\right)"] \\
\llbracket\gamma[C_i := \beta_i]_{i=1}^n\rrbracket_v \circ \bigotimes_{i=1}^{n} \bigotimes_{j=1}^{n_i} \llbracket\alpha_j^i\rrbracket_v \arrow[d, "\mathrm{norm}_{\mathbb{D}}^{(\Delta_i)_{i=1}^n}\left(((\alpha_j^i)_{j=1}^{n_i})_{i=1}^n, \gamma[C_i := B_i]_{i=1}^n\right)"] & \llbracket\gamma\rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket\beta_i[B_j^i := \alpha_j^i]_{j=1}^{n_i}\rrbracket_v \arrow[d, "\mathrm{norm}_{\mathbb{D}}^{\Gamma}\left((\beta_i[B_j^i := \alpha_j^i]_{j=1}^{n_i})_{i=1}^n, \gamma\right)"] \\
\llbracket\gamma[C_i := \beta_i]_{i=1}^n[[B_j^i := \alpha_j^i]_{j=1}^{n_i}]_{i=1}^n\rrbracket_v \arrow[r, "="'] & \llbracket\gamma[C_i := \beta_i[B_j^i := \alpha_j^i]_{j=1}^{n_i}]_{i=1}^n\rrbracket_v
\end{tikzcd}$$

commutes.

Proof. Immediate from 4.1. ☐

5.3.3 Interpretation of 2-cells

The semantic interpretation of a 2-cell derivation is also defined by induction. As for 1-cells, we define the interpretation on derivations, but since each derivable 2-cell sequent has a unique derivation the interpretation may be applied directly to derivable sequents. Also, we sometimes omit the type information when it is obvious from the context.

- $\left\llbracket \dfrac{\Gamma \vdash \alpha \in \mathbb{A}}{\Gamma \vdash 1_\alpha : \alpha \to \alpha \in \mathbb{A}}\, 1 \right\rrbracket_v = 1_{\llbracket\Gamma\vdash\alpha\in\mathbb{A}\rrbracket_v}$,

- $\left\llbracket \dfrac{\dfrac{[\pi_1]}{\Gamma \vdash \phi : \beta \to \gamma \in \mathbb{A}} \qquad \dfrac{[\pi_2]}{\Gamma \vdash \psi : \alpha \to \beta \in \mathbb{A}}}{\Gamma \vdash \phi \cdot \psi : \alpha \to \gamma \in \mathbb{A}}\, \mathrm{comp} \right\rrbracket_v = \llbracket\phi\rrbracket_v \cdot \llbracket\psi\rrbracket_v$.

- $\left\llbracket \dfrac{\Gamma_1 \vdash \gamma_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \gamma_n \in \mathbb{A}_n}{\Gamma_1 \cdots \Gamma_n \vdash t_{\gamma_1,\dots,\gamma_n} : \alpha[A_i := \gamma_i]_i \to \beta[A_i := \gamma_i]_i \in \mathbb{B}}\, t_\bullet \right\rrbracket_v$

is defined to be the composite

$$\begin{tikzcd}
\llbracket \alpha[A_i := \gamma_i]_i \rrbracket_v \arrow[d, "\mathrm{norm}^{-1}"'] & \llbracket \beta[A_i := \gamma_i]_i \rrbracket_v . \\
\llbracket \alpha \rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_v \arrow[r, "v(t) \circ \otimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_v"] & \llbracket \beta \rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_v \arrow[u, "\mathrm{norm}"']
\end{tikzcd}$$

- $$\left\llbracket \frac{\dfrac{[\pi_1]}{\Gamma_1 \vdash \phi_1 : \alpha_1 \to \beta_1 \in \mathbb{A}_1} \quad \cdots \quad \dfrac{[\pi_n]}{\Gamma_n \vdash \phi_n : \alpha_n \to \beta_n \in \mathbb{A}_n}}{\Gamma_1, \cdots, \Gamma_n \vdash f(\phi_1, \ldots, \phi_n) : f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n) \in \mathbb{B}} f(\to) \right\rrbracket_v$$

$$= v(f) \circ \otimes_{i=1}^{n} \llbracket \phi_i \rrbracket_v .$$

Proposition 5.10 (Semantics of 1-in-2 substitution). *Let*

$$\Gamma_1 \vdash \gamma_1 \in \mathbb{B}_1 \quad \cdots \quad \Gamma_n \vdash \gamma_n \in \mathbb{B}_n$$

be derivable 1-cell sequents, and let

$$\Delta \vdash \phi : \alpha \to \beta \in \mathbb{C}$$

be a derivable 2-cell sequent with $\Delta = (B_1 \in \mathbb{B}_1, \cdots, B_n \in \mathbb{B}_n)$. *Then*

$$\llbracket \Gamma_1 \cdots \Gamma_n \vdash \phi[B_i := \gamma_i]_i : \alpha[B_i := \gamma_i]_i \to \beta[B_i := \gamma_i]_i \in \mathbb{C} \rrbracket_v$$

is equal to the composite

$$\begin{tikzcd}
\llbracket \alpha[B_i := \gamma_i]_i \rrbracket_v \arrow[d, "\mathrm{norm}^{-1}"'] & \llbracket \beta[B_i := \gamma_i]_i \rrbracket_v . \\
\llbracket \alpha \rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_v \arrow[r, "\llbracket \phi \rrbracket_v \circ \otimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_v"] & \llbracket \beta \rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_v \arrow[u, "\mathrm{norm}"']
\end{tikzcd}$$

Proof. The proof is by induction over the derivation of ϕ: the non-trivial cases are 1-cell application and the axiom rule. For 1-cell application, let $\phi = f(\phi_1, \ldots, \phi_m)$. As before, let m_i be the number of names occurring in ϕ_i, and let $\langle i, j \rangle = j + \sum_{r<i} m_i$.

Now consider the diagram

$$\begin{tikzcd}
\nu(f)\circ\bigotimes_{i=1}^{m}\llbracket\alpha_i[B_{\langle i,j\rangle}:=\gamma_{\langle i,j\rangle}]_{j=1}^{m_i}\rrbracket_\nu \arrow[r, "\nu(f)\circ\bigotimes_{i=1}^{m}\llbracket\phi_i[B_{\langle i,j\rangle}:=\gamma_{\langle i,j\rangle}]_{j=1}^{m_i}\rrbracket_\nu"] \arrow[d, "\nu(f)\circ\bigotimes_{i=1}^{m}\mathrm{norm}^{-1}((\gamma_{\langle i,j\rangle})_j,\alpha_i)"'] & \nu(f)\circ\bigotimes_{i=1}^{m}\llbracket\beta_i[B_{\langle i,j\rangle}:=\gamma_{\langle i,j\rangle}]_{j=1}^{m_i}\rrbracket_\nu \\
\nu(f)\circ\bigotimes_{i=1}^{m}\Big(\llbracket\alpha_i\rrbracket_\nu\circ\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu\Big) \arrow[r, "\nu(f)\circ\bigotimes_{i=1}^{m}\big(\llbracket\phi_i\rrbracket_\nu\circ\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu\big)"] \arrow[d, "\nu(f)\circ\mathrm{I}_{i=1}^{m}(\llbracket\alpha_i\rrbracket_\nu,\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu)^{-1}"'] & \nu(f)\circ\bigotimes_{i=1}^{m}\Big(\llbracket\beta_i\rrbracket_\nu\circ\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu\Big) \arrow[u, "\nu(f)\circ\bigotimes_{i=1}^{m}\mathrm{norm}((\gamma_{\langle i,j\rangle})_j,\beta_i)"'] \\
\nu(f)\circ\bigotimes_{i=1}^{m}\llbracket\alpha_i\rrbracket_\nu\circ\bigotimes_{i=1}^{m}\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu \arrow[r, "\nu(f)\circ\bigotimes_{i=1}^{m}\llbracket\phi_i\rrbracket_\nu\circ\bigotimes_{i=1}^{m}\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu"'] & \nu(f)\circ\bigotimes_{i=1}^{m}\llbracket\beta_i\rrbracket_\nu\circ\bigotimes_{i=1}^{m}\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu \arrow[u, "\nu(f)\circ\mathrm{I}_{i=1}^{m}(\llbracket\beta_i\rrbracket_\nu,\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu)"']
\end{tikzcd}$$

where the upper square commutes by definition, and the lower square commutes by naturality of I. Thus the outside commutes: the left-hand vertical edge is equal, by definition, to $\mathrm{norm}((\gamma_i)_{i=1}^n,\alpha)^{-1}$, and similarly the right-hand edge is equal to $\mathrm{norm}((\gamma_i)_{i=1}^n,\beta)$. The top edge is equal, again by definition, to $\llbracket\phi[B_i := \gamma_i]_{i=1}^n\rrbracket_\nu$, and the lower edge to $\llbracket\phi\rrbracket_\nu\circ\bigotimes_{i=1}^n\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu$.

For the axiom rule, let $\phi = t_{\delta_1,\dots,\delta_m}$, where $t\in\mathcal{T}_{\mathbb{C}}^{(A_i\in\mathbb{A}_i)_{i=1}^m}[\lambda,\mu]$. Note that

$$\alpha = \lambda[A_i := \delta_i]_{i=1}^m \quad\text{and}\quad \beta = \lambda[A_i := \delta_i]_{i=1}^l.$$

Let m_i be the number of names occurring in δ_i, and let $\langle i,j\rangle = j+\sum_{r<i}m_i$. Note in particular that

$$\bigotimes_{i=1}^{n}\llbracket\gamma_i\rrbracket_\nu = \bigotimes_{i=1}^{m}\bigotimes_{j=1}^{m_i}\llbracket\gamma_{\langle i,j\rangle}\rrbracket_\nu.$$

Now consider the diagram

$$\begin{tikzcd}
\llbracket \lambda[A_i := \delta_i[B_{\langle i,j\rangle} := \gamma_{\langle i,j\rangle}]_{j=1}^{m_i}]_{i=1}^{m} \rrbracket_\nu \arrow[r, "\llbracket \phi[B_i := \gamma_i]_{i=1}^{n} \rrbracket_\nu"] \arrow[d, "\mathrm{norm}^{-1}"'] & \llbracket \mu[A_i := \delta_i[B_{\langle i,j\rangle} := \gamma_{\langle i,j\rangle}]_{j=1}^{m_i}]_{i=1}^{m} \rrbracket_\nu \\
\llbracket \lambda \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \llbracket \delta_i[B_{\langle i,j\rangle} := \gamma_{\langle i,j\rangle}]_{j=1}^{m_i} \rrbracket_\nu \arrow[r, "\nu(t) \circ \otimes_{i=1}^{m}(\cdots)"] \arrow[d, "\llbracket \lambda \rrbracket_\nu \circ \otimes_i \mathrm{norm}^{-1}"'] & \llbracket \mu \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \llbracket \delta_i[B_{\langle i,j\rangle} := \gamma_{\langle i,j\rangle}]_{j=1}^{m_i} \rrbracket_\nu \arrow[u, "\mathrm{norm}"'] \\
\llbracket \lambda \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \left(\llbracket \delta_i \rrbracket_\nu \circ \bigotimes_{j=1}^{m_i} \llbracket \gamma_{\langle i,j\rangle} \rrbracket_\nu \right) \arrow[r, "\nu(t) \circ \otimes_{i=1}^{m}(\cdots)"] \arrow[d, "\llbracket \lambda \rrbracket_\nu \circ \mathrm{I}^{-1}"'] & \llbracket \mu \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \left(\llbracket \delta_i \rrbracket_\nu \circ \bigotimes_{j=1}^{m_i} \llbracket \gamma_{\langle i,j\rangle} \rrbracket_\nu \right) \arrow[u, "\llbracket \mu \rrbracket_\nu \circ \otimes_{i=1}^{m} \mathrm{norm}"'] \\
\llbracket \lambda \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \llbracket \delta_i \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \bigotimes_{j=1}^{m_i} \llbracket \gamma_{\langle i,j\rangle} \rrbracket_\nu \arrow[r, "\nu(t) \circ (\cdots)"'] & \llbracket \mu \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \llbracket \delta_i \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \bigotimes_{j=1}^{m_i} \llbracket \gamma_{\langle i,j\rangle} \rrbracket_\nu \arrow[u, "\llbracket \mu \rrbracket_\nu \circ \mathrm{I}"']
\end{tikzcd}$$

The upper square commutes by definition of $\llbracket \phi[B_i := \gamma_i]_{i=1}^{n} \rrbracket_\nu$, and the other two squares commute by the 2-categorical interchange law, hence the outside commutes. Now, applying the Double norm result (Proposition 5.9) to the vertical sides gives that the outside of the following diagram commutes:

$$\begin{tikzcd}
\llbracket \alpha[B_i := \gamma_i]_{i=1}^{n} \rrbracket_\nu \arrow[r, "\llbracket \phi[B_i := \gamma_i]_{i=1}^{n} \rrbracket_\nu"] \arrow[d, "\mathrm{norm}^{-1}"'] & \llbracket \beta[B_i := \gamma_i]_{i=1}^{n} \rrbracket_\nu \\
\llbracket \alpha \rrbracket_\nu \circ \bigotimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_\nu \arrow[r, "\llbracket \phi \rrbracket_\nu \circ \otimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_\nu"] \arrow[d, "\mathrm{norm}^{-1} \circ \otimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_\nu"'] & \llbracket \beta \rrbracket_\nu \circ \bigotimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_\nu \arrow[u, "\mathrm{norm}"'] \\
\llbracket \lambda \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \llbracket \delta_i \rrbracket_\nu \circ \bigotimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_\nu \arrow[r, "\nu(t) \circ \cdots"'] & \llbracket \mu \rrbracket_\nu \circ \bigotimes_{i=1}^{m} \llbracket \delta_i \rrbracket_\nu \circ \bigotimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_\nu \arrow[u, "\mathrm{norm} \circ \otimes_{i=1}^{n} \llbracket \gamma_i \rrbracket_\nu"']
\end{tikzcd}$$

The lower square also commutes, by definition of $\llbracket \phi \rrbracket_\nu$, hence the upper square commutes as required. □

Proposition 5.11 (Semantics of 2-in-1 substitution). *Let* $\Gamma \vdash \gamma \in \mathbb{C}$ *be a derivable 1-cell sequent, for some*

$$\Gamma = (A_1 \in \mathbb{A}_1, \ldots, A_n \in \mathbb{A}_n).$$

Let

$$\Delta_1 \vdash \phi_1 : \alpha_1 \to \beta_1 \in \mathbb{A}_1 \quad \cdots \quad \Delta_n \vdash \phi_n : \alpha_n \to \beta_n \in \mathbb{A}_n$$

be derivable 2-cell sequents. Then

$$[\![\Delta_1, \cdots, \Delta_n \vdash \gamma[A_i := \phi_i]_i : \gamma[A_i := \alpha_i]_i \to \gamma[A_i := \beta_i]_i \in \mathbb{B}.]\!]_v$$

is equal to the composite

$$\begin{tikzcd}
{[\![\gamma[A_i := \alpha_i]_i]\!]_v} \arrow[d, "\mathrm{norm}^{-1}"'] & {[\![\gamma[A_i := \beta_i]_i]\!]_v.} \\
{[\![\gamma]\!]_v \circ \bigotimes_{i=1}^{n} [\![\alpha_i]\!]_v} \arrow[r, "{[\![\gamma]\!]_v \circ \otimes_{i=1}^{n} [\![\phi_i]\!]_v}"] & {[\![\gamma]\!]_v \circ \bigotimes_{i=1}^{n} [\![\beta_i]\!]_v} \arrow[u, "\mathrm{norm}"']
\end{tikzcd}$$

Proof. Compared with the previous Proposition, this one is easy to prove! The proof is by induction over γ. If γ is a constant then it is trivial, so let $\gamma = f(\gamma_1, \ldots, \gamma_m)$ for some $f \in \mathcal{T}(\mathbb{B}_1, \cdots, \mathbb{B}_m; \mathbb{C})$. For each $1 \le i \le m$, let m_i be the number of names in γ_i, and let $\langle i, j \rangle = j + \sum_{r<i} m_r$. We have to show that the outside of the diagram

$$\begin{tikzcd}
{v(f) \circ \bigotimes_{i=1}^{m} [\![\gamma_i[A_{\langle i,j\rangle} := \alpha_{\langle i,j\rangle}]_{j=1}^{m_i}]\!]_v} \arrow[r, "{v(f) \circ \otimes_{i=1}^{m} [\![\gamma_i[A_{\langle i,j\rangle} := \phi_{\langle i,j\rangle}]_{j=1}^{m_i}]\!]_v}"] \arrow[d, "{v(f) \circ \otimes_{i=1}^{m} \mathrm{norm}^{-1}}"'] & {v(f) \circ \bigotimes_{i=1}^{m} [\![\gamma_i[A_{\langle i,j\rangle} := \beta_{\langle i,j\rangle}]_{j=1}^{m_i}]\!]_v} \\
{v(f) \circ \bigotimes_{i=1}^{m} \big([\![\gamma_i]\!]_v \circ \bigotimes_{j=1}^{m_i} [\![\alpha_{\langle i,j\rangle}]\!]_v\big)} \arrow[r, "{v(f) \circ \otimes_{i=1}^{m} \big([\![\gamma_i]\!]_v \circ \otimes_{j=1}^{m_i} [\![\phi_{\langle i,j\rangle}]\!]_v\big)}"] \arrow[d, "{v(f) \circ \mathrm{I}^{-1}}"'] & {v(f) \circ \bigotimes_{i=1}^{m} \big([\![\gamma_i]\!]_v \circ \bigotimes_{j=1}^{m_i} [\![\beta_{\langle i,j\rangle}]\!]_v\big)} \arrow[u, "{v(f) \circ \otimes_{i=1}^{m} \mathrm{norm}}"'] \\
{v(f) \circ \bigotimes_{i=1}^{m} [\![\gamma_i]\!]_v \circ \bigotimes_{i=1}^{m} \bigotimes_{j=1}^{m_i} [\![\alpha_{\langle i,j\rangle}]\!]_v} \arrow[r, "{v(f) \circ \otimes_{i=1}^{m} [\![\gamma_i]\!]_v \circ \otimes_{i=1}^{m} \otimes_{j=1}^{m_i} [\![\phi_{\langle i,j\rangle}]\!]_v}"'] & {v(f) \circ \bigotimes_{i=1}^{m} [\![\gamma_i]\!]_v \circ \bigotimes_{i=1}^{m} \bigotimes_{j=1}^{m_i} [\![\beta_{\langle i,j\rangle}]\!]_v} \arrow[u, "{v(f) \circ \mathrm{I}}"']
\end{tikzcd}$$

commutes. The upper square commutes by the inductive hypothesis, and the lower square by naturality of I, hence the outside commutes as required. □

5.4 Soundness

To make use of the equation rules, we need to show that the derivable equations hold in every model.

Proposition 5.12 (Soundness). *If*

$$\Gamma \vdash \phi = \psi : \alpha \to \beta \in \mathbb{B}$$

is a derivable equation sequent and v *is an interpretation then*

$$[\![\Gamma \vdash \phi : \alpha \to \beta \in \mathbb{B}]\!]_v = [\![\Gamma \vdash \psi : \alpha \to \beta \in \mathbb{B}]\!]_v.$$

Proof. We consider in turn each of the derivation rules for equations.

- The reflexivity, symmetry and transitivity rules need no comment.
- For the axiom rule, let $(\phi, \psi) \in \mathcal{T}_=(\Delta \vdash \alpha \to \beta \in \mathbb{A})$ and let

  $$\Gamma_1 \vdash \gamma_1 \in \mathbb{B}_1 \quad \cdots \quad \Gamma_n \vdash \gamma_n \in \mathbb{B}_n$$

  be derivable 1-cell sequents. We must show that

  $$\begin{aligned} &[\![\Gamma_1 \cdots \Gamma_n \vdash \phi[B_i := \gamma_i]_i : \alpha[B_i := \gamma_i]_i \to \beta[B_i := \gamma_i]_i]\!]_v \\ &\quad = [\![\Gamma_1 \cdots \Gamma_n \vdash \psi[B_i := \gamma_i]_i : \alpha[B_i := \gamma_i]_i \to \beta[B_i := \gamma_i]_i]\!]_v. \end{aligned}$$

  By definition of model, we already know that $[\![\phi]\!]_v = [\![\psi]\!]_v$, hence

  $$[\![\phi]\!]_v \circ \bigotimes_{i=1}^{n} [\![\gamma_i]\!]_v = [\![\psi]\!]_v \circ \bigotimes_{i=1}^{n} [\![\gamma_i]\!]_v$$

  which, by Proposition 5.10, implies that

  $$[\![\phi[B_i := \gamma_i]_{i=1}^{n}]\!]_v = [\![\psi[B_i := \gamma_i]_{i=1}^{n}]\!]_v$$

  as required.
- Since both horizontal and vertical composition preserve equality in a 2-category (or indeed in a bicategory), the rules expressing that composition and 1-cell application preserve equality are sound.
- Let $f \in \mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$ and let

  $$\Gamma_1 \vdash \alpha_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \alpha_n \in \mathbb{A}_n$$

  be derivable 1-cell sequents. Then

  $$[\![f(1_{\alpha_1}, \ldots, 1_{\alpha_n})]\!]_v$$

$$= \nu(f) \circ \left([\![1_{\alpha_1}]\!]_\nu \otimes \cdots \otimes [\![1_{\alpha_n}]\!]_\nu \right)$$

$$= \nu(f) \circ \left(1_{[\![\alpha_1]\!]_\nu} \otimes \cdots \otimes 1_{[\![\alpha_n]\!]_\nu} \right)$$

$$= \nu(f) \circ 1_{[\![\alpha_1]\!]_\nu \otimes \cdots \otimes [\![\alpha_n]\!]_\nu}$$

$$= 1_{\nu(f) \circ ([\![\alpha_1]\!]_\nu \otimes \cdots \otimes [\![\alpha_n]\!]_\nu)}$$

$$= [\![1_{f(\alpha_1, \ldots, \alpha_n)}]\!]_\nu$$

showing that 1-cell application preserves identities.

- Let $f \in \mathcal{T}(\mathbb{A}_1, \ldots, \mathbb{A}_n; \mathbb{B})$ and let

  $$\Gamma_1 \vdash \phi_1 : \beta_1 \to \gamma_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \phi_n : \beta_n \to \gamma_n \in \mathbb{A}_n$$

  and

  $$\Gamma_1 \vdash \psi_1 : \alpha_1 \to \beta_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \psi_n : \alpha_n \to \beta_n \in \mathbb{A}_n$$

  be derivable 2-cell sequents. Now, we have

  $$[\![\Gamma_1, \cdots, \Gamma_n \vdash f(\phi_1, \ldots, \phi_n) \cdot f(\psi_1, \ldots, \psi_n) \in \mathbb{B}]\!]_\nu$$

  $$= [\![\Gamma_1, \cdots, \Gamma_n \vdash f(\phi_1, \ldots, \phi_n) \in \mathbb{B}]\!]_\nu \cdot [\![\Gamma_1, \cdots, \Gamma_n \vdash f(\psi_1, \ldots, \psi_n) \in \mathbb{B}]\!]_\nu$$

  $$= \left(\nu(f) \circ (\bigotimes_{i=1}^{n} [\![\phi_i]\!]_\nu)\right) \cdot \left(\nu(f) \circ (\bigotimes_{i=1}^{n} [\![\psi_i]\!]_\nu)\right)$$

  $$= \nu(f) \circ \left(\bigotimes_{i=1}^{n} [\![\phi_i]\!]_\nu \cdot \bigotimes_{i=1}^{n} [\![\psi_i]\!]_\nu\right)$$

  $$= \nu(f) \circ \bigotimes_{i=1}^{n} \left([\![\phi_i]\!]_\nu \cdot [\![\psi_i]\!]_\nu\right)$$

  $$= \nu(f) \circ \bigotimes_{i=1}^{n} \left([\![\phi_i \cdot \psi_i]\!]_\nu\right)$$

  $$= [\![f(\phi_1 \cdot \psi_1, \ldots, \phi_n \cdot \psi_n)]\!]_\nu$$

  showing that 1-cell application does indeed preserve composition.

- The left and right identity rules are obviously sound, since $[\![1_\alpha]\!]_\nu = 1_{[\![\alpha]\!]_\nu}$ is a strict unit for vertical composition in a Gray monoid.

- For the naturality rule, let $t \in \mathcal{T}_{\mathbb{B}}^{\Gamma}[\alpha, \beta]$ where

  $$\Gamma = (A_1 \in \mathbb{A}_1, \cdots, A_n \in \mathbb{A}_n),$$

and let

$$\Gamma_1 \vdash \phi_1 : \gamma_1 \to \delta_1 \in \mathbb{A}_1 \quad \cdots \quad \Gamma_n \vdash \phi_n : \gamma_n \to \delta_n \in \mathbb{A}_n$$

be derivable 2-cell sequents. We wish to show that

$$[\![\beta[A_i := \phi_i]_i \cdot t_{\gamma_1,\ldots,\gamma_n}]\!]_v = [\![t_{\delta_1,\ldots,\delta_n} \cdot \alpha[A_i := \phi_i]_i]\!]_v,$$

i.e. that the diagram

$$\begin{tikzcd}
[\![\alpha[A_i := \gamma_i]_i]\!]_v \arrow[r, "{[\![t_{\gamma_1,\ldots,\gamma_n}]\!]_v}"] \arrow[d, "{[\![\alpha[A_i := \phi_i]_i]\!]_v}"'] & [\![\beta[A_i := \gamma_i]_i]\!]_v \arrow[d, "{[\![\beta[A_i := \phi_i]_i]\!]_v}"] \\
[\![\alpha[A_i := \delta_i]_i]\!]_v \arrow[r, "{[\![t_{\delta_1,\ldots,\delta_n}]\!]_v}"'] & [\![\beta[A_i := \delta_i]_i]\!]_v
\end{tikzcd}$$

commutes. Expanding the definitions of these four arrows gives the diagram

$$\begin{tikzcd}
[\![\alpha]\!]_v \circ \bigotimes_{i=1}^{n} [\![\gamma_i]\!]_v \arrow[r, "{v(t) \circ \otimes_{i=1}^{n} [\![\gamma_i]\!]_v}"] & [\![\beta]\!]_v \circ \bigotimes_{i=1}^{n} [\![\gamma_i]\!]_v \arrow[d, "\mathrm{norm}"] \\
[\![\alpha[A_i := \gamma_i]_i]\!]_v \arrow[u, "\mathrm{norm}^{-1}"] \arrow[d, "\mathrm{norm}^{-1}"'] & [\![\beta[A_i := \gamma_i]_i]\!]_v \arrow[d, "\mathrm{norm}^{-1}"] \\
[\![\alpha]\!]_v \circ \bigotimes_{i=1}^{n} [\![\gamma_i]\!]_v \arrow[d, "{[\![\alpha]\!]_v \circ \otimes_{i=1}^{n} [\![\phi_i]\!]_v}"'] & [\![\beta]\!]_v \circ \bigotimes_{i=1}^{n} [\![\gamma_i]\!]_v \arrow[d, "{[\![\beta]\!]_v \circ \otimes_{i=1}^{n} [\![\phi_i]\!]_v}"] \\
[\![\alpha]\!]_v \circ \bigotimes_{i=1}^{n} [\![\delta_i]\!]_v \arrow[d, "\mathrm{norm}"'] & [\![\beta]\!]_v \circ \bigotimes_{i=1}^{n} [\![\delta_i]\!]_v \arrow[d, "\mathrm{norm}"] \\
[\![\alpha[A_i := \delta_i]_i]\!]_v \arrow[d, "\mathrm{norm}^{-1}"'] & [\![\beta[A_i := \delta_i]_i]\!]_v \\
[\![\alpha]\!]_v \circ \bigotimes_{i=1}^{n} [\![\gamma_i]\!]_v \arrow[r, "{v(t) \circ \otimes_{i=1}^{n} [\![\delta_i]\!]_v}"'] & [\![\beta]\!]_v \circ \bigotimes_{i=1}^{n} [\![\delta_i]\!]_v \arrow[u, "\mathrm{norm}"']
\end{tikzcd}$$

Cancelling the norm arrows gives

$$\begin{tikzcd}
\llbracket\alpha\rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket\gamma_i\rrbracket_v \arrow[r, "\nu(t) \circ \bigotimes_{i=1}^{n}\llbracket\gamma_i\rrbracket_v"] \arrow[d, "\llbracket\alpha\rrbracket_v \circ \bigotimes_{i=1}^{n}\llbracket\phi_i\rrbracket_v"'] & \llbracket\beta\rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket\gamma_i\rrbracket_v \arrow[d, "\llbracket\beta\rrbracket_v \circ \bigotimes_{i=1}^{n}\llbracket\phi_i\rrbracket_v"] \\
\llbracket\alpha\rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket\gamma_i\rrbracket_v \arrow[r, "\nu(t) \circ \bigotimes_{i=1}^{n}\llbracket\delta_i\rrbracket_v"'] & \llbracket\beta\rrbracket_v \circ \bigotimes_{i=1}^{n} \llbracket\delta_i\rrbracket_v
\end{tikzcd}$$

which clearly commutes, by the interchange law for composition in a 2-category. Hence the naturality rule is sound.

□

5.5 Interpretation in a general monoidal bicategory

We have shown how the language may be soundly interpreted in a Gray monoid. In this section, we use coherence to interpret the language soundly in any monoidal bicategory.

Let $\mathcal{B}$ be a monoidal bicategory, and let $\mathrm{Gr}(\mathcal{B})$ be the corresponding Gray monoid, as defined by Gurski (2006, Chapter 10), with monoidal biequivalences

$$\mathcal{B} \underset{\mathbf{e}}{\overset{\mathbf{f}}{\rightleftarrows}} \mathrm{Gr}(\mathcal{B}).$$

We briefly recall the definitions of $\mathrm{Gr}(\mathcal{B})$ and **e**:

- an object of $\mathrm{Gr}(\mathcal{B})$ is a finite sequence of objects of $\mathcal{B}$, which we regard as a formal tensor of those objects. So the tensor product of two objects is just their concatenation as lists.
- a 'basic 1-cell' of $\mathrm{Gr}(\mathcal{B})$

$$\langle f, x, y, \alpha, \beta\rangle : (X_i)_{i=1}^{x}(A_i)_{i=1}^{m}(Y_i)_{i=1}^{y} \to (X_i)_{i=1}^{x}(B_i)_{i=1}^{n}(Y_i)_{i=1}^{y}$$

consists of:

  - a bracketing[2] α of the subsequence $(A_i)_{i=1}^{m}$,
  - a bracketing β of the subsequence $(B_i)_{i=1}^{n}$,

[2] By *bracketing* we mean what Gurski calls a 'choice of association' (his Definition 10.3.1). This can include copies of the unit object: for example, $((\mathbb{I} \otimes A) \otimes B) \otimes \mathbb{I}$ is a bracketing of (A, B).

– a 1-cell $f : \bigotimes_\alpha A_i \to \bigotimes_\beta B_i$ in $\mathcal{B}$, where $\bigotimes_\alpha$ denotes the iterated tensor according to the bracketing α.

- Let us define the *canonical tensor* of a sequence of objects or 1-cells to associate to the right, for example $\bigotimes(A, B, C, D) = A \otimes (B \otimes (C \otimes D))$. On basic 1-cells, $\mathbf{e}(\langle h, x, y, \alpha, \beta\rangle)$ is defined to be a composite of three 1-cells:

$$\begin{array}{c} \bigotimes((X_i)_{i=1}^{x}(A_i)_{i=1}^{m}(Y_i)_{i=1}^{y}) \\ \Big\downarrow \cong \\ (\bigotimes_{i=1}^{x} X_i) \otimes \big((\bigotimes_{\alpha} A_i) \otimes (\bigotimes_{i=1}^{y} Y_i)\big) \\ \Big\downarrow (\bigotimes_{i=1}^{x} X_i) \otimes h \otimes (\bigotimes_{i=1}^{y} Y_i) \\ (\bigotimes_{i=1}^{x} X_i) \otimes \big((\bigotimes_{\beta} B_i) \otimes (\bigotimes_{i=1}^{y} Y_i)\big) \\ \Big\downarrow \cong \\ \bigotimes((X_i)_{i=1}^{x}(B_i)_{i=1}^{n}(Y_i)_{i=1}^{y}) \end{array}$$

where the first and third arrows are associator 1-cells.(Gurski assumes that some associator has been chosen for each pair of bracketings. We shall further assume that each chosen associator has minimal complexity, where the complexity of an associator is the number of basic structural 1-cells it is built from. In addition we assume that, if α and β are bracketings, the chosen associator $\beta \to \alpha$ is the inverse of the chosen associator $\alpha \to \beta$. This is clearly consistent with the previous requirement.)

- a 1-cell of $\mathrm{Gr}(\mathcal{B})$ is a composable finite sequence of basic 1-cells.

- On 1-cells, $\mathbf{e}(f_1, \ldots, f_n)$ is defined to be

$$e(f_n) \circ \cdots \circ e(f_1).$$

- a 2-cell $f \Rightarrow g$ is a 2-cell $\mathbf{e}(f) \Rightarrow \mathbf{e}(g)$ in $\mathcal{B}$.

- $\mathbf{e}$ is the identity on 2-cells.

- Horizontal composition of 2-cells in $\mathrm{Gr}(\mathcal{B})$ is non-trivial, and inserts the structural isomorphisms necessary to make the types match.

Our primary interest in $\mathbf{e}$ is that it serves as a machine for inserting structural 1-cells and 2-cells where appropriate. Unfortunately for us, the very simplicity of its definition means that it will often insert structural 1-cells unnecessarily. This is no bar to defining an interpretation in a general monoidal bicategory, but it would make the resulting interpretation more complicated than necessary. Therefore it is convenient to introduce a new operation $\mathbf{e}'$ on the 1-cells of $\mathrm{Gr}(\mathcal{B})$, which produces a 1-cell in $\mathcal{B}$ that is isomorphic to, but generally simpler than, the one produced by $\mathbf{e}$. We will also explicitly define an invertible 2-cell $\epsilon(h) : \mathbf{e}(h) \Rightarrow \mathbf{e}'(h)$, for every 1-cell h of $\mathrm{Gr}(\mathcal{B})$.

The first difference between $\mathbf{e}'$ and $\mathbf{e}$ is that we redefine the value of $\mathbf{e}'$ on a basic 1-cell $\langle h, x, y, \alpha, \beta \rangle$. We suppress the first and/or third (associator) arrow when its source and target are equal. In this case our minimal-complexity criterion ensures that these associators are identities, hence there is an obvious invertible 2-cell

$$\epsilon(\langle h, x, y, \alpha, \beta \rangle) : \mathbf{e}(\langle h, x, y, \alpha, \beta \rangle) \Rightarrow \mathbf{e}'(\langle h, x, y, \alpha, \beta \rangle)$$

built from the unit 2-cells of the bicategory $\mathcal{B}$. For example, let A and B be objects of $\mathcal{B}$, and consider the one-element sequences (A) and (B), which are objects of $\mathrm{Gr}(\mathcal{B})$. Given $h : A \to B$ in $\mathcal{B}$, there is a basic 1-cell $\langle h, 0, 0, (-), (-) \rangle : (A) \to (B)$ in $\mathrm{Gr}(B)$; in fact this is precisely $\mathbf{f}(h)$, according to Gurski's triequivalence $\mathbf{f}$. Now, applying $\mathbf{e}$ to this basic 1-cell gives the composite

$$A \xrightarrow{1} A \xrightarrow{h} B \xrightarrow{1} B,$$

whereas applying $\mathbf{e}'$ just gives $h : A \to B$.

The second case in which $\mathbf{e}$ may introduce unnecessary 1-cells comes when two basic 1-cells are juxtaposed within a 1-cell of $\mathrm{Gr}(\mathcal{B})$. Suppose, for example, that we have a 1-cell containing $\langle h, x, y, \alpha, \beta \rangle$ followed by $\langle h', x', y', \alpha', \beta' \rangle$. Applying $\mathbf{e}$ to this pair gives a composite of six 1-cells, the third and fourth of which are:

$$
\begin{array}{c}
\left(\bigotimes_{i=1}^{x} X_i\right) \otimes \left(\left(\bigotimes_{\beta} B_i\right) \otimes \left(\bigotimes_{i=1}^{n} B_i\right)\right) \\
\Big\downarrow \cong \\
\bigotimes\left((X_i)_{i=1}^{x}(B_i)_{i=1}^{n}(Y_i)_{i=1}^{y}\right) \\
= \bigotimes\left((X'_i)_{i=1}^{x'}(B'_i)_{i=1}^{n'}(Y'_i)_{i=1}^{y'}\right) \\
\Big\downarrow \cong \\
\left(\bigotimes_{i=1}^{x'} X'_i\right) \otimes \left(\left(\bigotimes_{\alpha'} B'_i\right) \otimes \left(\bigotimes_{i=1}^{y'} Y'_i\right)\right)
\end{array}
$$

We define $\mathbf{e}'$ so that:

- If these associators are mutually inverse, both are suppressed.
- Otherwise, this composite is replaced by the relevant chosen associator of minimal complexity.

And we define ϵ in the obvious way.

Finally, we define $\mathbf{e}'$ on the 2-cells of $\mathrm{Gr}(\mathcal{B})$. If $\xi : (h_i) \to (k_i)$ is a 2-cell in $\mathrm{Gr}(\mathcal{B})$, let $\mathbf{e}'(\xi)$ be the composite

$$\mathbf{e}'((h_i)) \xrightarrow{\epsilon^{-1}((h_i))} \mathbf{e}((h_i)) \xrightarrow{\mathbf{e}(\xi)} \mathbf{e}((k_i)) \xrightarrow{\epsilon((k_i))} \mathbf{e}'((k_i)).$$

We shall say in stages what it means to give an interpretation $\nu : \mathcal{T} \to \mathcal{B}$, and use ν to simultaneously define:

- an interpretation $\nu_{\mathrm{Gr}} : \mathcal{T} \to \mathrm{Gr}(\mathcal{B})$, and
- the semantic function $[\![-]\!]_\nu$.

The interpretation ν is given as follows:

- To give the object part of ν, one gives, for every object $\mathbb{A}$ of $\mathcal{T}$, an object $\nu(\mathbb{A})$ of $\mathcal{B}$. Using this, we define the object part of ν_{Gr} by letting $\nu_{\mathrm{Gr}}(\mathbb{A}) = \mathbf{f}(\nu(\mathbb{A}))$.
- To give the 1-cell part of ν, one gives, for every $h \in \mathcal{T}(\mathbb{A}_1, \dots, \mathbb{A}_n; \mathbb{B})$, a 1-cell $\nu(h) : \mathbf{e}(\otimes_{i=1}^{n} \nu_{\mathrm{Gr}}(\mathbb{A}_i)) \to \nu(\mathbb{B})$. Using this, we define $\nu_{\mathrm{Gr}}(h)$ to be the basic 1-cell
$$(\nu_{\mathrm{Gr}}(\mathbb{A}_i))_{i=1}^{n} \to \nu_{\mathrm{Gr}}(\mathbb{B})$$
for which $\mathbf{e}'(\nu_{\mathrm{Gr}}(h)) = \nu(h)$.

- The interpretation $[\![\gamma]\!]_v$ of a 1-cell is defined to be $\mathbf{e}'([\![\gamma]\!]_{v_{\mathrm{Gr}}})$.
- To give the 2-cell part of v, one gives, for every $t \in \mathcal{T}^{\mathrm{T}}_{\mathbb{B}}(\alpha, \beta)$, a 2-cell

$$v(t) : [\![\alpha]\!]_v \to [\![\beta]\!]_v$$

in $\mathcal{B}$. Since $\mathbf{e}$ is a bijection on 2-cells, so is $\mathbf{e}'$, hence there is a unique 2-cell

$$v_{\mathrm{Gr}}(t) : [\![\alpha]\!]_{v_{\mathrm{Gr}}} \to [\![\beta]\!]_{v_{\mathrm{Gr}}}$$

for which $\mathbf{e}'(v_{\mathrm{Gr}}(t)) = v(t)$.

- The interpretation $[\![\phi]\!]_v$ of a 2-cell is defined to be $\mathbf{e}([\![\phi]\!]_{v_{\mathrm{Gr}}})$.
- Additionally we demand that, for every equation $(\phi, \psi) \in \mathcal{T}_{=}(\Gamma \vdash \alpha \to \beta \in \mathbb{B})$, $[\![\phi]\!]_v = [\![\psi]\!]_v$. Note that this is so if and only if $[\![\phi]\!]_{v_{\mathrm{Gr}}} = [\![\psi]\!]_{v_{\mathrm{Gr}}}$.
- Since the interpretation $[\![-]\!]_{v_{\mathrm{Gr}}}$ is known to be sound, the interpretation $[\![-]\!]_v$ is also sound.

5.6 Towards a braided extension

Thanks to Theorem 4.2, there is no serious obstacle to defining a version of the language applicable to braided monoidal bicategories. With each context one could associate a positive braid on that context, and arrange matters so that the associated expression (on the right of the turnstile) uses the variables in the order that they 'emerge' from the braid. This could then be used to treat braided structures, such as braided pseudomonoids, via components.

Only lack of time has prevented us from developing the braided extension in full detail here.

Chapter 6

Pseudomonoids

A monoid, in a monoidal category, consists of an object A equipped with a unit $u : I \to A$ and a multiplication $m : A \otimes A \to A$, satisfying the obvious unit and associativity axioms. In a monoidal *bicategory*, the corresponding notion is that of a pseudomonoid, where the unit and associativity laws hold, not on the nose, but up to coherent isomorphism. The primeval example is of course that of monoidal categories, which are pseudomonoids in the monoidal bicategory Cat; though for present purposes we are particularly interested in promonoidal categories, which are pseudomonoids in Prof.

Little has been published about pseudomonoids per se, though the definition is given by Day and Street (1997). However, the equivalent notion of *pseudomonad* (in a tricategory or Gray-category) has received more attention: from Marmolejo (1997, 1999, 2004); Lack (2000); Tanaka (2005), among others.

Definition 6.1. A pseudomonoid $\mathbb{C}$ in a monoidal bicategory $\mathcal{B}$ is a normal pseudofunctor $1 \to \mathcal{B}$. More concretely, it consists of an object $\mathbb{C}$, 1-cells

$$\begin{aligned} J &: \mathbb{I} \to \mathbb{C}, \\ P &: \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}, \end{aligned}$$

and invertible 2-cells

$$\begin{tikzcd}
\mathbb{C}\otimes(\mathbb{C}\otimes\mathbb{C}) \arrow[rr, "a_{\mathbb{C},\mathbb{C},\mathbb{C}}"] \arrow[d, "\mathbb{C}\otimes P"'] & & (\mathbb{C}\otimes\mathbb{C})\otimes\mathbb{C} \arrow[d, "P\otimes\mathbb{C}"] \\
\mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C} \arrow[u, phantom, "\overset{\Rightarrow}{\mathfrak{a}}"] & \mathbb{C}\otimes\mathbb{C} \arrow[l, "P"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{I}\otimes\mathbb{C} \arrow[rr, "J\otimes\mathbb{C}"] \arrow[dr, "l_{\mathbb{C}}"'] & \arrow[d, phantom, "\overset{\Rightarrow}{\mathfrak{l}}"] & \mathbb{C}\otimes\mathbb{C} \arrow[dl, "P"] \\
& \mathbb{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}\otimes\mathbb{I} \arrow[rr, "\mathbb{C}\otimes J"] \arrow[dr, "r_{\mathbb{C}}"'] & \arrow[d, phantom, "\overset{\Rightarrow}{\mathfrak{r}}"] & \mathbb{C}\otimes\mathbb{C} \arrow[dl, "P"] \\
& \mathbb{C} &
\end{tikzcd}$$

subject to the two equations below (stated in the Gray monoid setting). Since the 2-cells are assumed to be invertible, we shall permit ourselves to omit the arrow. Also, here and elsewhere, we write $\mathbb{C}^2$ for $\mathbb{C}\otimes\mathbb{C}$.

$$\begin{tikzcd}
\mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[dr, "\mathbb{C}\otimes J\otimes\mathbb{C}"] \arrow[dd, "1"'] & \\
\arrow[r, phantom, "\mathbb{C}\otimes\mathfrak{l}"] & \mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C} \arrow[dl, "\mathbb{C}\otimes P"] \arrow[dd, "P\otimes\mathbb{C}"] \\
\mathbb{C}\otimes\mathbb{C} \arrow[dd, "P"'] \arrow[r, phantom, "\mathfrak{a}"] & \\
& \mathbb{C}\otimes\mathbb{C} \arrow[dl, "P"] \\
\mathbb{C} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[dr, "\mathbb{C}\otimes J\otimes\mathbb{C}"] \arrow[dd, "1"'] & \\
\arrow[r, phantom, "\mathfrak{r}\otimes\mathbb{C}"] & \mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C} \arrow[dl, "P\otimes\mathbb{C}"] \\
\mathbb{C}\otimes\mathbb{C} \arrow[dd, "P"'] & \\
& \\
\mathbb{C} &
\end{tikzcd}
\tag{6.0.1}$$

and

$$\begin{tikzcd}
& \mathbb{C}^3 \arrow[d, "P\otimes\mathbb{C}"'] \arrow[dr, "\mathbb{C}\otimes P"] & \\
\mathbb{C}^4 \arrow[ur, "\mathbb{C}^2\otimes P"] \arrow[d, "P\otimes\mathbb{C}^2"'] \arrow[r, phantom, "\sim"] & \mathbb{C}^2 \arrow[dr, "P"] \arrow[r, phantom, "\underset{\mathfrak{a}}{\Leftarrow}"] & \mathbb{C}^2 \arrow[d, "P"] \\
\mathbb{C}^3 \arrow[ur, "\mathbb{C}\otimes P"] \arrow[dr, "P\otimes\mathbb{C}"'] \arrow[rr, phantom, "\Downarrow\mathfrak{a}"] & & \mathbb{C} \\
& \mathbb{C}^2 \arrow[ur, "P"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathbb{C}^3 \arrow[dr, "\mathbb{C}\otimes P"] \arrow[d, phantom, "\Downarrow\mathbb{C}\otimes\mathfrak{a}"] & \\
\mathbb{C}^4 \arrow[ur, "\mathbb{C}^2\otimes P"] \arrow[d, "P\otimes\mathbb{C}^2"'] \arrow[dr, "\mathbb{C}\otimes P\otimes\mathbb{C}"] & {} & \mathbb{C}^2 \arrow[d, "P"] \\
\mathbb{C}^3 \arrow[dr, "P\otimes\mathbb{C}"'] \arrow[r, phantom, "\underset{\mathfrak{a}\otimes\mathbb{C}}{\Leftarrow}"] & \mathbb{C}^3 \arrow[ur, "\mathbb{C}\otimes P"] \arrow[d, "P\otimes\mathbb{C}"] \arrow[r, phantom, "\swarrow\mathfrak{a}"] & \mathbb{C} \\
& \mathbb{C}^2 \arrow[ur, "P"'] &
\end{tikzcd}
\tag{6.0.2}$$

The first of these equations corresponds to the triangle axiom relating α, λ and ρ, and the second to Mac Lane's pentagon axiom.

6.1 Some facts about pseudomonoids

If we express the results of Kelly (1964) in the language of general pseudomonoids, we obtain the three equations below. In the following sections, we shall prove that they hold in general, by showing how Kelly's argument can be applied, via the calculus of components, to any pseudomonoid $\mathbb{C}$.

$$
\begin{tikzcd}
\mathbb{I}\otimes\mathbb{C}^2 \arrow[r, "J\otimes\mathbb{C}^2"] \arrow[dr, "\mathfrak{l}\otimes\mathbb{C}", "1"'] & \mathbb{C}^3 \arrow[r, "\mathbb{C}\otimes P"] \arrow[d, "P\otimes\mathbb{C}"] \arrow[dr, phantom, "\mathfrak{a}"] & \mathbb{C}^2 \arrow[d, "P"] \\
 & \mathbb{C}^2 \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{I}\otimes\mathbb{C}^2 \arrow[r, "J\otimes\mathbb{C}^2"] \arrow[d, "\mathbb{I}\otimes P"'] \arrow[dr, phantom, "\sim"] & \mathbb{C}^3 \arrow[d, "\mathbb{C}\otimes P"] \\
\mathbb{I}\otimes\mathbb{C} \arrow[r, "J\otimes\mathbb{C}"] \arrow[dr, "\mathfrak{l}", "1"'] & \mathbb{C}^2 \arrow[d, "P"] \\
 & \mathbb{C}
\end{tikzcd}
\tag{6.1.1}
$$

$$
\begin{tikzcd}
\mathbb{C}^2\otimes\mathbb{I} \arrow[r, "\mathbb{C}^2\otimes J"] \arrow[dr, "\mathbb{C}\otimes\mathfrak{r}", "1"'] & \mathbb{C}^3 \arrow[r, "P\otimes\mathbb{C}"] \arrow[d, "\mathbb{C}\otimes P"] \arrow[dr, phantom, "\mathfrak{a}"] & \mathbb{C}^2 \arrow[d, "P"] \\
 & \mathbb{C}^2 \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{C}^2\otimes\mathbb{I} \arrow[r, "\mathbb{C}^2\otimes J"] \arrow[d, "P\otimes\mathbb{I}"'] \arrow[dr, phantom, "\sim"] & \mathbb{C}^3 \arrow[d, "P\otimes\mathbb{C}"] \\
\mathbb{C}\otimes\mathbb{I} \arrow[r, "\mathbb{C}\otimes J"] \arrow[dr, "\mathfrak{r}", "1"'] & \mathbb{C}^2 \arrow[d, "P"] \\
 & \mathbb{C}
\end{tikzcd}
\tag{6.1.2}
$$

$$
\begin{tikzcd}
\mathbb{I}\otimes\mathbb{I} \arrow[dr, "\mathbb{I}\otimes J"] \arrow[dd, "J\otimes\mathbb{I}"'] & \\
 & \mathbb{I}\otimes\mathbb{C} \arrow[dd, "J\otimes\mathbb{C}"] \\
\mathbb{C}\otimes\mathbb{I} \arrow[dr, "\mathbb{C}\otimes J"'] \arrow[dd, "1"', "\mathfrak{r}"] \arrow[ur, phantom, "\sim"] & \\
 & \mathbb{C}\otimes\mathbb{C} \arrow[dl, "P"] \\
\mathbb{C} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{I}\otimes\mathbb{I} \arrow[dr, "\mathbb{I}\otimes J"] \arrow[dd, "J\otimes\mathbb{I}"'] & \\
 & \mathbb{I}\otimes\mathbb{C} \arrow[dd, "J\otimes\mathbb{C}"] \arrow[dddl, "1"', "\mathfrak{l}"] \\
\mathbb{C}\otimes\mathbb{I} \arrow[dd, "1"'] & \\
 & \mathbb{C}\otimes\mathbb{C} \arrow[dl, "P"] \\
\mathbb{C} &
\end{tikzcd}
\tag{6.1.3}
$$

Lack (1995, section 3.4) describes an interesting geometrical way to prove these equations, using certain four-dimensional diagrams.[1] Marmolejo (1997, Proposition 8.1) gives a more down-to-earth version of the argument, using pasting diagrams. Instead we will show how they follow from Kelly's proof for ordinary monoidal categories, by the calculus of components.

6.2 Pseudomonoids via the calculus of components

Next we see how the language of components may be used to reason about pseudomonoids. We will define the theory $\mathcal{M}$ of pseudomonoids, and show that reasoning in the formal language corresponds precisely to the usual modes of reasoning about monoidal categories, and that a model of $\mathcal{M}$ is precisely a pseudomonoid. The theory concerns a single object $\mathbb{C}$, so $\mathcal{M}_0 = \{\mathbb{C}\}$. There are two basic 1-cells, $J : \mathbb{I} \to \mathbb{C}$ and $P : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, so formally we have $\mathcal{M}(;\mathbb{C}) = \{J\}$ and $\mathcal{M}(\mathbb{C},\mathbb{C};\mathbb{C}) := \{P\}$. There are six basic 2-cells, corresponding to $\mathfrak{a}$, $\mathfrak{l}$, $\mathfrak{r}$ and their inverses. The 2-cell $\mathfrak{a}$ goes from

$$A \in \mathbb{C}, B \in \mathbb{C}, C \in \mathbb{C} \vdash P(A, P(B, C)) \in \mathbb{C}$$

to

$$A \in \mathbb{C}, B \in \mathbb{C}, C \in \mathbb{C} \vdash P(P(A, B), C) \in \mathbb{C};$$

formally, for every three distinct names A, B and C we have

$$\mathcal{M}_{\mathbb{C}}^{(A\in\mathbb{C},B\in\mathbb{C},C\in\mathbb{C})}[P(A, P(B, C)), P(P(A, B), C)] = \{\mathfrak{a}\}.$$

To make the notation appear more familiar, we shall write $A * B$ to mean $P(A, B)$, and I to mean $J()$. Thus $\mathfrak{a}$ is a 2-cell with components

$$\mathfrak{a}_{A,B,C} : A * (B * C) \to (A * B) * C$$

for A, B, $C \in \mathbb{C}$. In fact we want $\mathfrak{a}$ to be an invertible 2-cell, which formally means that there is another 2-cell $\mathfrak{a}^{-1}$ with components

$$\mathfrak{a}^{-1}_{A,B,C} : (A * B) * C \to A * (B * C)$$

such that $\mathfrak{a}_{A,B,C} \cdot \mathfrak{a}^{-1}_{A,B,C}$ and $\mathfrak{a}^{-1}_{A,B,C} \cdot \mathfrak{a}_{A,B,C}$ are identities. In terms of the formal definition of the theory $\mathcal{M}$, this means

$$\mathcal{M}_{\mathbb{C}}^{\Gamma}[P(P(A, B), C), P(A, P(B, C))] = \{\mathfrak{a}^{-1}\},$$

[1] Lack is working in the slightly more general context of enriched bicategories: a pseudomonoid is an enriched bicategory with one object.

$$(\Gamma, \mathbb{C}, P(P(A,B),C), P(P(A,B),C), \mathfrak{a}^{-1}_{A,B,C} \cdot \mathfrak{a}_{A,B,C}, 1_{P(P(A,B),C)}) \in \mathcal{M}_{=},$$

$$(\Gamma, \mathbb{C}, P(A,P(B,C)), P(A,P(B,C)), \mathfrak{a}_{A,B,C} \cdot \mathfrak{a}^{-1}_{A,B,C}, 1_{P(A,P(B,C))}) \in \mathcal{M}_{=},$$

where $\Gamma = (A \in \mathbb{C}, B \in \mathbb{C}, C \in \mathbb{C})$.

In the same way, we want $\mathfrak{l}$ to be an invertible 2-cell with components

$$\mathfrak{l}_A : I * A \to A$$

for $A \in \mathbb{C}$, and $\mathfrak{r}$ to be an invertible 2-cell with components

$$\mathfrak{r}_A : A * I \to A$$

for $A \in \mathbb{C}$. (Formally, the theory $\mathcal{M}$ contains 2-cells $\mathfrak{l}$, $\mathfrak{l}^{-1}$, $\mathfrak{r}$ and $\mathfrak{r}^{-1}$, and four equations expressing that $\mathfrak{l}^{-1}$ is inverse to $\mathfrak{l}$ and $\mathfrak{r}^{-1}$ is inverse to $\mathfrak{r}$.)

Finally, we have the pentagon and triangle axioms. Consider the pentagon:

$$\begin{tikzcd}
A*(B*(C*D)) \arrow[r, "\mathfrak{a}_{A,B,C*D}"] \arrow[d, "A*\mathfrak{a}_{B,C,D}"'] & (A*B)*(C*D) \arrow[r, "\mathfrak{a}_{A*B,C,D}"] & (((A*B)*C)*D) \\
A*((B*C)*D) \arrow[rr, "\mathfrak{a}_{A,B*C,D}"'] & & (A*(B*C))*D \arrow[u, "\mathfrak{a}_{A,B,C}*D"']
\end{tikzcd}$$

Of course, this diagram is just a convenient way of writing the equation

$$\mathfrak{a}_{A*B,C,D} \cdot \mathfrak{a}_{A,B*C,D} = P(\mathfrak{a}_{A,B,C}, D) \cdot \mathfrak{a}_{A,B*C,D} \cdot (A * \mathfrak{a}_{B,C,D})$$

which we take as a formal equation of $\mathcal{M}$. Thus any model of $\mathcal{M}$ must satisfy

$$[\![\mathfrak{a}_{A*B,C,D} \cdot \mathfrak{a}_{A,B*C,D}]\!]_v = [\![P(\mathfrak{a}_{A,B,C}, D) \cdot \mathfrak{a}_{A,B*C,D} \cdot (A * \mathfrak{a}_{B,C,D})]\!]_v.$$

Similarly the triangle:

$$\begin{tikzcd}
A*(I*C) \arrow[rr, "\mathfrak{a}_{A,I,C}"] \arrow[dr, "A*\mathfrak{l}_C"'] & & (A*I)*C \arrow[dl, "\mathfrak{r}_A*C"] \\
& A*C &
\end{tikzcd}$$

represents the equation

$$(\mathfrak{r}_A * C) \cdot \mathfrak{a}_{A,I,C} = A * \mathfrak{l}_C,$$

hence a model of $\mathcal{M}$ must also satisfy

$$[\![(\mathfrak{r}_A * C) \cdot \mathfrak{a}_{A,I,C}]\!]_v = [\![A * \mathfrak{l}_C]\!]_v.$$

Now, let us consider an interpretation of $\mathcal{M}$ in a Gray monoid $\mathcal{B}$. We shall omit $v()$, writing just $\mathbb{C}$ instead of $v(\mathbb{C})$, and P instead of $v(P)$, etc. So an interpretation of $\mathcal{M}$ consists of an object $\mathbb{C}$, 1-cells $J : \mathbb{I} \to \mathbb{C}$ and $P : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, and invertible 2-cells

$$\begin{tikzcd} & \mathbb{C}^2 \arrow[dr, "P"] & \\ \mathbb{C}^3 \arrow[ur, "\mathbb{C}\otimes P"] \arrow[dr, "P\otimes\mathbb{C}"'] \arrow[rr, phantom, "\Downarrow\mathfrak{a}"] & & \mathbb{C} \\ & \mathbb{C}^2 \arrow[ur, "P"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C}\otimes\mathbb{C} \arrow[dr, "P"'] & & \mathbb{I}\otimes\mathbb{C} \arrow[ll, "J\otimes\mathbb{C}"'] \arrow[dl, "1"] \\ & \mathbb{C} \arrow[u, phantom, "\overset{\Rightarrow}{\mathfrak{l}}"] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}\otimes\mathbb{C} \arrow[dr, "P"'] & & \mathbb{C}\otimes\mathbb{I} \arrow[ll, "\mathbb{C}\otimes J"'] \arrow[dl, "1"] \\ & \mathbb{C} \arrow[u, phantom, "\overset{\Rightarrow}{\mathfrak{r}}"] & \end{tikzcd}$$

where we have written $\mathbb{C}^2$ as an abbreviation for $\mathbb{C} \otimes \mathbb{C}$, etc., subject to equations corresponding to the pentagon and triangle conditions. Let us first consider the pentagon equation

$$[\![\mathfrak{a}_{A*B,C,D} \cdot \mathfrak{a}_{A,B,C*D}]\!]_v = [\![(\mathfrak{a}_{A,B,C} * D) \cdot \mathfrak{a}_{A,B*C,D} \cdot (A * \mathfrak{a}_{B,C,D})]\!]_v.$$

equivalently

$$[\![\mathfrak{a}_{A*B,C,D}]\!]_v \cdot [\![\mathfrak{a}_{A,B,C*D}]\!]_v = [\![(\mathfrak{a}_{A,B,C} * D)]\!]_v \cdot [\![\mathfrak{a}_{A,P(B,C),D}]\!]_v \cdot [\![P(A, \mathfrak{a}_{B,C,D})]\!]_v.$$

By definition, $[\![\mathfrak{a}_{A,B,C*D}]\!]_v : [\![A * (B * (C * D))]\!]_v \Rightarrow [\![(A * B) * (C * D)]\!]_v$ is

$$\begin{tikzcd} {[\![A * (B * (C * D))]\!]_v} \arrow[d, "\mathrm{norm}^{-1}"'] & {[\![(A * B) * (C * D)]\!]_v} \\ {[\![A * (B * X)]\!]_v \circ (\mathbb{C} \otimes \mathbb{C} \otimes P)} \arrow[r, "\mathfrak{a} \circ (\mathbb{C} \otimes \mathbb{C} \otimes P)"'] & {[\![(A * B) * X]\!]_v \circ (\mathbb{C} \otimes \mathbb{C} \otimes P)} \arrow[u, "\mathrm{norm}"'] \end{tikzcd}$$

On the left we have

$$
\begin{aligned}
&[\![A * (B * (C * D))]\!]_v \\
&\quad = P \circ (\mathbb{C} \otimes [\![B * (C * D)]\!]_v) \\
&\quad = P \circ (\mathbb{C} \otimes (P \circ (\mathbb{C} \otimes [\![C * D]\!]_v))) \\
&\quad = P \circ (\mathbb{C} \otimes (P \circ (\mathbb{C} \otimes P))) \\
&\quad = P \circ (\mathbb{C} \otimes P) \circ (\mathbb{C} \otimes \mathbb{C} \otimes P)
\end{aligned}
$$

and the norm^{-1} map is just the identity. The reason is essentially that every occurrence of P has one argument equal to a constant. Thus the right-hand side is more interesting: we have

$$[\![(A * B) * (C * D)]\!]_v = P \circ (P \otimes P)$$

and the norm map is the isomorphism

$$P \circ (P \otimes \mathbb{C}) \circ (\mathbb{C} \otimes \mathbb{C} \otimes P) \to P \circ (P \otimes P).$$

So this 2-cell is

$$
\begin{tikzcd}
 & & \mathbb{C}^2 \arrow[dr, "P"] & \\
\mathbb{C}^4 \arrow[r, "\mathbb{C} \otimes \mathbb{C} \otimes P"] \arrow[drr, bend right, "P \otimes P"'] \arrow[dr, phantom, "\sim"] & \mathbb{C}^3 \arrow[ur, "\mathbb{C} \otimes P"] \arrow[dr, "P \oplus \mathbb{C}"] \arrow[rr, phantom, "\Downarrow \mathfrak{a}"] & & \mathbb{C} \\
 & & \mathbb{C}^2 \arrow[ur, "P"'] &
\end{tikzcd}
$$

In a similar way, one may calculate that

$$[\![\mathfrak{a}_{P(A,B),C,D}]\!]_v$$

is equal to

$$
\begin{tikzcd}
 & & \mathbb{C}^2 \arrow[dr, "P"] & \\
\mathbb{C}^4 \arrow[r, "P \otimes \mathbb{C} \otimes \mathbb{C}"'] \arrow[urr, bend left, "P \otimes P"] \arrow[ur, phantom, "\sim"] & \mathbb{C}^3 \arrow[ur, "\mathbb{C} \oplus P"] \arrow[dr, "P \otimes \mathbb{C}"'] \arrow[rr, phantom, "\Downarrow \mathfrak{a}"] & & \mathbb{C} \\
 & & \mathbb{C}^2 \arrow[ur, "P"'] &
\end{tikzcd}
$$

Note that, according to the definition of Gray monoid, in fact

$$P \otimes P = (\mathbb{C} \otimes P) \circ (P \otimes \mathbb{C} \otimes \mathbb{C}),$$

and this interchange cell is the identity. Thus the composite

$$[\![\mathfrak{a}_{A*B,C,D} \cdot \mathfrak{a}_{A,B,C*D}]\!]_v = [\![\mathfrak{a}_{A*B,C,D}]\!]_v \cdot [\![\mathfrak{a}_{A,B,C*D}]\!]_v$$

is

$$\begin{tikzcd}
& \mathbb{C}^3 \arrow[d, "P\otimes\mathbb{C}"'] \arrow[dr, "\mathbb{C}\otimes P"] & \\
\mathbb{C}^4 \arrow[ur, "\mathbb{C}^2\otimes P"] \arrow[r, phantom, "\sim"] \arrow[d, "P\otimes\mathbb{C}^2"'] & \mathbb{C}^2 \arrow[r, phantom, "\underset{\mathfrak{a}}{\Leftarrow}"] \arrow[dr, "P"'] & \mathbb{C}^2 \arrow[d, "P"] \\
\mathbb{C}^3 \arrow[ur, "\mathbb{C}\oplus P"] \arrow[r, phantom, "\Downarrow\mathfrak{a}"] \arrow[dr, "P\otimes\mathbb{C}"'] & & \mathbb{C} \\
& \mathbb{C}^2 \arrow[ur, "P"'] &
\end{tikzcd}$$

Calculating

$$[\![(\mathfrak{a}_{A,B,C} * D) \cdot \mathfrak{a}_{A,B*C,D} \cdot (A * \mathfrak{a}_{B,C,D})]\!]_v.$$

is less interesting, since all the norm maps are identities, so it is just

$$\begin{tikzcd}
& \mathbb{C}^3 \arrow[dr, "\mathbb{C}\otimes P"] & \\
\mathbb{C}^4 \arrow[ur, "\mathbb{C}^2\otimes P"] \arrow[rr, phantom, "\Downarrow\mathbb{C}\otimes\mathfrak{a}"] \arrow[d, "P\otimes\mathbb{C}^2"'] \arrow[dr, "\mathbb{C}\oplus P\oplus\mathbb{C}"] & & \mathbb{C}^2 \arrow[d, "P"] \\
\mathbb{C}^3 \arrow[r, phantom, "\underset{\mathfrak{a}\otimes\mathbb{C}}{\Leftarrow}"] \arrow[dr, "P\otimes\mathbb{C}"'] & \mathbb{C}^3 \arrow[ur, "\mathbb{C}\oplus P"] \arrow[r, phantom, "\Swarrow\mathfrak{a}"] \arrow[d, "P\otimes\mathbb{C}"] & \mathbb{C} \\
& \mathbb{C}^2 \arrow[ur, "P"'] &
\end{tikzcd}$$

Thus the pentagon axiom, in a Gray monoid, is precisely equation (6.0.2). In a similar way, the triangle axiom corresponds to equation (6.0.1). The interpretation in a general monoidal bicategory works in a similar way; the monoidal biequivalence $\mathbf{e}'$ causes structural 1-cells and 2-cells to be inserted where necessary. Let us consider the 2-cell $\mathfrak{a}$: its interpretation needs to be a 2-cell

$$[\![A * (B * C)]\!]_v \to [\![(A * B) * C]\!]_v,$$

and $[\![A * (B * C)]\!]_v$ is simply

$$\mathbb{C} \otimes (\mathbb{C} \otimes \mathbb{C}) \xrightarrow{\mathbb{C}\otimes P} \mathbb{C} \otimes \mathbb{C} \xrightarrow{P} \mathbb{C}$$

whereas $[\![(A * B) * C]\!]_v$ is

$$\mathbb{C} \otimes (\mathbb{C} \otimes \mathbb{C}) \xrightarrow{a_{\mathbb{C},\mathbb{C},\mathbb{C}}} (\mathbb{C} \otimes \mathbb{C}) \otimes \mathbb{C} \xrightarrow{P\otimes\mathbb{C}} \mathbb{C} \otimes \mathbb{C} \xrightarrow{P} \mathbb{C},$$

hence $\mathfrak{a}$ should be a 2-cell

$$\begin{tikzcd}
\mathbb{C} \otimes (\mathbb{C} \otimes \mathbb{C}) \arrow[r, "\mathbb{C} \otimes P"] \arrow[d, "a_{\mathbb{C},\mathbb{C},\mathbb{C}}"'] \arrow[dr, phantom, "\swarrow \mathfrak{a}"] & \mathbb{C} \otimes \mathbb{C} \arrow[dr, "P"] & \\
(\mathbb{C} \otimes \mathbb{C}) \otimes \mathbb{C} \arrow[r, "P \otimes \mathbb{C}"'] & \mathbb{C} \otimes \mathbb{C} \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}$$

The axioms also sport structural 2-cells. For example, the pentagon equation becomes the requirement that

$$\begin{tikzcd}
\mathbb{C} \otimes (\mathbb{C} \otimes (\mathbb{C} \otimes \mathbb{C})) \arrow[r, "\mathbb{C} \otimes (\mathbb{C} \otimes P)"] \arrow[dd, "a_{\mathbb{C},\mathbb{C},\mathbb{C}\otimes\mathbb{C}}"'] \arrow[dr, phantom, "\swarrow a_{\mathbb{C},\mathbb{C},P}"] & \mathbb{C} \otimes (\mathbb{C} \otimes \mathbb{C}) \arrow[r, "\mathbb{C} \otimes P"] \arrow[d, "a_{\mathbb{C},\mathbb{C},\mathbb{C}}"'] \arrow[dr, phantom, "\swarrow \mathfrak{a}"] & \mathbb{C} \otimes \mathbb{C} \arrow[ddr, "P"] & \\
& (\mathbb{C} \otimes \mathbb{C}) \otimes \mathbb{C} \arrow[dr, "P \otimes \mathbb{C}"] & & \\
(\mathbb{C} \otimes \mathbb{C}) \otimes (\mathbb{C} \otimes \mathbb{C}) \arrow[ur, "(\mathbb{C} \otimes \mathbb{C}) \otimes P"] \arrow[dr, "P \otimes (\mathbb{C} \otimes \mathbb{C})"'] \arrow[dd, "a_{\mathbb{C}\otimes\mathbb{C},\mathbb{C},\mathbb{C}}"'] \arrow[ddr, phantom, "\swarrow a_{P,\mathbb{C},\mathbb{C}}"] & \cong & \mathbb{C} \otimes \mathbb{C} \arrow[r, "P"'] & \mathbb{C} \\
& \mathbb{C} \otimes (\mathbb{C} \otimes \mathbb{C}) \arrow[ur, "\mathbb{C} \otimes P"'] \arrow[d, "a_{\mathbb{C},\mathbb{C},\mathbb{C}}"'] \arrow[dr, phantom, "\swarrow \mathfrak{a}"] & & \\
((\mathbb{C} \otimes \mathbb{C}) \otimes \mathbb{C}) \otimes \mathbb{C} \arrow[r, "(P \otimes \mathbb{C}) \otimes \mathbb{C}"'] & (\mathbb{C} \otimes \mathbb{C}) \otimes \mathbb{C} \arrow[r, "P \otimes \mathbb{C}"'] & \mathbb{C} \otimes \mathbb{C} \arrow[uur, "P"'] &
\end{tikzcd}$$

must be equal to

$$
\begin{tikzcd}
 & \mathbb{C}\otimes(\mathbb{C}\otimes(\mathbb{C}\otimes\mathbb{C})) \arrow[r, "\mathbb{C}\otimes(\mathbb{C}\otimes P)"] \arrow[d, "\mathbb{C}\otimes a_{\mathbb{C},\mathbb{C},\mathbb{C}}"'] \arrow[ddl, bend right, "a_{\mathbb{C},\mathbb{C},\mathbb{C}\otimes\mathbb{C}}"'] \arrow[dr, phantom, "\swarrow \mathbb{C}\otimes\mathfrak{a}"] & \mathbb{C}\otimes(\mathbb{C}\otimes\mathbb{C}) \arrow[dr, "\mathbb{C}\otimes P"] & & \\
 & \mathbb{C}\otimes((\mathbb{C}\otimes\mathbb{C})\otimes\mathbb{C}) \arrow[r, "\mathbb{C}\otimes(P\otimes\mathbb{C})"'] \arrow[dd, "a_{\mathbb{C},\mathbb{C}\otimes\mathbb{C},\mathbb{C}}"'] \arrow[ddr, phantom, "\swarrow a_{\mathbb{C},P,\mathbb{C}}"] & \mathbb{C}\otimes(\mathbb{C}\otimes\mathbb{C}) \arrow[r, "\mathbb{C}\otimes P"'] \arrow[dd, "a_{\mathbb{C},\mathbb{C},\mathbb{C}}"'] \arrow[ddr, phantom, "\swarrow \mathfrak{a}"] & \mathbb{C}\otimes\mathbb{C} \arrow[ddr, "P"] & \\
(\mathbb{C}\otimes\mathbb{C})\otimes(\mathbb{C}\otimes\mathbb{C}) \arrow[r, phantom, "\Leftarrow_{\pi^{-1}_{A,B,C,D}}"] \arrow[ddr, bend right, "a_{\mathbb{C}\otimes\mathbb{C},\mathbb{C},\mathbb{C}}"'] & & & & \\
 & (\mathbb{C}\otimes(\mathbb{C}\otimes\mathbb{C}))\otimes\mathbb{C} \arrow[r, "(\mathbb{C}\otimes P)\otimes\mathbb{C}"'] \arrow[d, "a_{\mathbb{C},\mathbb{C},\mathbb{C}}\otimes\mathbb{C}"'] \arrow[dr, phantom, "\Downarrow \mathfrak{a}\otimes\mathbb{C}"] & (\mathbb{C}\otimes\mathbb{C})\otimes\mathbb{C} \arrow[r, "P\otimes\mathbb{C}"'] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C} \\
 & ((\mathbb{C}\otimes\mathbb{C})\otimes\mathbb{C})\otimes\mathbb{C} \arrow[r, "(P\otimes\mathbb{C})\otimes\mathbb{C}"'] & (\mathbb{C}\otimes\mathbb{C})\otimes\mathbb{C} \arrow[ur, "P\otimes\mathbb{C}"'] & &
\end{tikzcd}
$$

and the triangle equation becomes the requirement that

$$
\begin{tikzcd}
\mathbb{C}\otimes(\mathbb{I}\otimes\mathbb{C}) \arrow[r, "a_{\mathbb{C},\mathbb{I},\mathbb{C}}"] \arrow[d, "\mathbb{C}\otimes(J\otimes\mathbb{C})"'] \arrow[dr, phantom, "\nearrow a_{\mathbb{C},J,\mathbb{C}}"] & (\mathbb{C}\otimes\mathbb{I})\otimes\mathbb{C} \arrow[d, "(\mathbb{C}\otimes J)\otimes\mathbb{C}"] \arrow[ddr, bend left, "r_{\mathbb{C}}\otimes\mathbb{C}"] & \\
\mathbb{C}\otimes(\mathbb{C}\otimes\mathbb{C}) \arrow[r, "a_{\mathbb{C},\mathbb{C},\mathbb{C}}"'] \arrow[d, "\mathbb{C}\otimes P"'] \arrow[dr, phantom, "\nearrow \mathfrak{a}"] & (\mathbb{C}\otimes\mathbb{C})\otimes\mathbb{C} \arrow[r, phantom, "\Rightarrow_{\mathfrak{r}\otimes\mathbb{C}}"] \arrow[dr, "P\otimes\mathbb{C}"'] & \\
\mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C} & \mathbb{C}\otimes\mathbb{C} \arrow[l, "P"]
\end{tikzcd}
$$

be equal to

$$\begin{tikzcd}
 & \mathbb{C}\otimes(\mathbb{I}\otimes\mathbb{C}) \arrow[dl, "\mathbb{C}\otimes(J\otimes\mathbb{C})"'] \arrow[dd, "\mathbb{C}\otimes l_{\mathbb{C}}"] \arrow[dr, "a_{\mathbb{C},\mathbb{I},\mathbb{C}}"] & \\
\mathbb{C}^3 \arrow[dr, "\mathbb{C}\otimes P"'] \arrow[r, phantom, "\underset{\mathfrak{r}\otimes\mathbb{C}}{\Rightarrow}"] & \arrow[r, phantom, "\underset{\mu_{\mathbb{C},\mathbb{C}}}{\Rightarrow}"] & (\mathbb{C}\otimes\mathbb{I})\otimes\mathbb{C} \arrow[dl, "r_{\mathbb{C}}\otimes\mathbb{C}"] \\
 & \mathbb{C}\otimes\mathbb{C} \arrow[d, "P"'] & \\
 & \mathbb{C} &
\end{tikzcd}$$

(In fact the 'raw' version of the equation, as it emerges from the interpretation, has μ^{-1} on the left-hand side, rather than μ on the right as we have written.)

6.3 Calculating in the theory of pseudomonoids

Now we may use the language to prove various facts about pseudomonoids, essentially using the formal interpretation of the language as a translation tool that allows us to transfer proofs from the familiar setting of monoidal categories. As a first example, consider the simple fact that

$$\mathfrak{l}_{I*A} = I * \mathfrak{l}_A : I * (I * A) \to I * A.$$

The usual proof runs as follows: since $\mathfrak{l}$ is a natural transformation, we have a naturality square

$$\begin{tikzcd}
I * (I * A) \arrow[r, "I * \mathfrak{l}_A"] \arrow[d, "\mathfrak{l}_{I*A}"'] \arrow[dr, phantom, "\natural"] & I * A \arrow[d, "\mathfrak{l}_A"] \\
I * A \arrow[r, "\mathfrak{l}_A"'] & A
\end{tikzcd}$$

which, since $\mathfrak{l}_A$ is invertible, implies the claim. The formal proof in our language is precisely the same: the naturality square is an instance of the naturality axiom; we then compose with $\mathfrak{l}_A^{-1}$ (using the axiom that composition preserves equality), then use our axiom relating $\mathfrak{l}$ and $\mathfrak{l}^{-1}$ to derive

$$1_A \cdot \mathfrak{l}_{I*A} = 1_A \cdot I * \mathfrak{l}_A,$$

and finally use the identity axiom (and symmetry and transitivity) to conclude that

$$\mathfrak{l}_{I*A} = I * \mathfrak{l}_A : I * (I * A) \to I * A$$

as required. In a Gray monoid, this shows that

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "J\otimes\mathbb{C}"'] \arrow[rrr, bend left=30, "J\otimes(P\circ(J\otimes\mathbb{C}))"] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] \arrow[r, phantom, bend left=30, "\sim"] & \mathbb{C} \arrow[r, "J\otimes\mathbb{C}"'] \arrow[rr, bend right=50, "1"'] \arrow[rr, phantom, bend right=25, "\Downarrow\mathfrak{l}"] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "J\otimes\mathbb{C}"'] \arrow[rrr, bend left=30, "J\otimes(P\circ(J\otimes\mathbb{C}))"] \arrow[rr, bend right=50, "1"'] \arrow[rr, phantom, bend right=25, "\Downarrow\mathfrak{l}"] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] \arrow[r, phantom, bend left=30, "\sim"] & \mathbb{C} \arrow[r, "J\otimes\mathbb{C}"'] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}$$

Since we may cancel the common norm cell, this can equivalently be stated as:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "J\otimes\mathbb{C}"'] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C} \arrow[r, "J\otimes\mathbb{C}"'] \arrow[rr, bend right=50, "1"'] \arrow[rr, phantom, bend right=25, "\Downarrow\mathfrak{l}"] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "J\otimes\mathbb{C}"'] \arrow[rr, bend right=50, "1"'] \arrow[rr, phantom, bend right=25, "\Downarrow\mathfrak{l}"] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C} \arrow[r, "J\otimes\mathbb{C}"'] & \mathbb{C}\otimes\mathbb{C} \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}$$

For a more substantial application, consider the triangle (6.1.1):

$$\begin{tikzcd}
I * (A * B) \arrow[rr, "\mathfrak{a}_{I,A,B}"] \arrow[dr, "\mathfrak{l}_{A*B}"'] & & (I * A) * B \arrow[dl, "\mathfrak{l}_A * B"] \\
& A * B &
\end{tikzcd}$$

The proof of this, as given by Kelly (1964) for monoidal categories, runs as follows. Consider the diagram

$$
\begin{tikzcd}
\big(A,(I,(C,D))\big) \arrow[rr, "\mathfrak{a}_{A,I,(C*D)}"] \arrow[dd, "A*\mathfrak{a}_{I,C,D}"'] \arrow[dr, "{(A,\mathfrak{l}_{(C,D)})}"] \arrow[d, phantom, "?"] & & (A*I)*(C*D) \arrow[rr, "\mathfrak{a}_{(A*I),C,D}"] \arrow[dl, "\mathfrak{r}_A*(C,D)"'] \arrow[dr, phantom, "\natural"] & & ((A*I)*C)*D \arrow[dl, "(\mathfrak{r}_A*C)*D"'] \\
 & A*(C*D) \arrow[rr, "\mathfrak{a}_{A,C,D}"'] \arrow[dr, phantom, "\natural"] & & (A*C)*D & \\
A*((I*C)*D) \arrow[ur, "A*(\mathfrak{l}_C*D)"] \arrow[rrrr, "\mathfrak{a}_{A,I*C,D}"'] & & & & (A*(I*C))*D \arrow[ul, "(A*\mathfrak{l}_C)*D"'] \arrow[uu, "\mathfrak{a}_{A,I,C}*D"']
\end{tikzcd}
$$

The outside commutes by the pentagon axiom. The quadrilaterals commute by naturality, and the unmarked triangles by the triangle axiom. Since $\mathfrak{a}_{A,C,D}$ is invertible, it follows that the triangle marked '?' commutes. Now set $A = I$, and use the naturality and invertibility of $\mathfrak{l}$ to conclude that the required triangle commutes. It's easy to see that all this reasoning is formalisable in the language, hence

$$\llbracket \mathfrak{a}_{I,A,B} \cdot \mathfrak{l}_{A*B} \rrbracket_v = \llbracket \mathfrak{l}_A * B \rrbracket_v$$

in any model. By a dual argument,

$$\llbracket \mathfrak{r}_{A*B} \cdot \mathfrak{a}_{A,B,I} \rrbracket_v = \llbracket A * \mathfrak{r}_B \rrbracket_v$$

as well. Also, we can show that $\llbracket \mathfrak{l}_I \rrbracket_v = \llbracket \mathfrak{r}_I \rrbracket_v$, as follows:

$$(\mathfrak{l}_I * A) \cdot \mathfrak{a}_{I,I,A} = \mathfrak{l}_{I*A}$$

by the triangle we proved above, which is equal to $I * \mathfrak{l}_A$ by the naturality argument at the start of this section, which in turn is equal to $(\mathfrak{r}_I * A) \cdot \mathfrak{a}_{I,I,A}$ by the triangle axiom. Since $\mathfrak{a}_{I,I,A}$ is invertible, we have that $\mathfrak{l}_{J()} * A = \mathfrak{r}_{J()} * A$. Then use the invertibility and naturality of $\mathfrak{r}$ to conclude $\mathfrak{l}_I = \mathfrak{r}_I$, as required.

Again, this reasoning may easily be formalised in the language, and as promised the language allows us to transfer proofs from the setting of ordinary monoidal categories to that of arbitrary pseudomonoids.

6.4 Braided pseudomonoids

In the context of a braided monoidal bicategory $\mathcal{B}$, we can define what it means to have a braiding on a pseudomonoid in $\mathcal{B}$. Observe that it is not possible to define *symmetric* pseudomonoids in this setting: there is simply no way to express the desired equation. To define symmetric pseudomonoids in general, one needs some additional structure on the braiding of the monoidal bicategory: this additional structure is called a *syllepsis*, and consists of an invertible modification between the identity transformation on the tensor pseudofunctor and the transformation with components

$$A \otimes B \xrightarrow{s_{A,B}} B \otimes A \xrightarrow{s_{B,A}} A \otimes B,$$

subject to coherence conditions. However, all the general facts that we need concerning symmetric promonoidal categories are true more generally in the braided case, hence we have no need to consider symmetry explicitly in the abstract setting.

Definition 6.2. Let $\mathbb{C}$ be a pseudomonoid in the braided monoidal bicategory $\mathcal{B}$. A *braiding* for $\mathbb{C}$ is a 2-cell $\mathfrak{s}$:

$$\begin{tikzcd}
\mathbb{C} \otimes \mathbb{C} \arrow[rr, "s_{\mathbb{C},\mathbb{C}}"] \arrow[dr, "P"'] & \arrow[d, phantom, "\Rightarrow \mathfrak{s}"] & \mathbb{C} \otimes \mathbb{C} \arrow[dl, "P"] \\
& \mathbb{C} &
\end{tikzcd}$$

subject to two equations, which (in a Gray monoid) are as follows:

$$
\begin{tikzcd}
 & \mathbb{C}^3 \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[d, "P\otimes\mathbb{C}"'] \arrow[drr, phantom, "s_{\mathbb{C},P}"] & & \mathbb{C}^3 \arrow[ll, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[d, "\mathbb{C}\otimes P"] \arrow[ddr, bend left, "P\otimes\mathbb{C}"] & \\
 & \mathbb{C}^2 \arrow[dr, "P"'] \arrow[dl, phantom, "\mathfrak{a}"] \arrow[rr, phantom, "\mathfrak{s}"', shift right=4] & & \mathbb{C}^2 \arrow[ll, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[dl, "P"] \arrow[dr, phantom, "\mathfrak{a}"] & \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} & & \mathbb{C}^2 \arrow[ll, "P"]
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & \mathbb{C}^3 \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[dr, phantom, "\mathbb{C}\otimes\mathfrak{s}"', shift right=6] & & \mathbb{C}^3 \arrow[ll, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{C}}\otimes\mathbb{C}"] \arrow[ddr, bend left, "P\otimes\mathbb{C}"] \arrow[dl, phantom, "\mathfrak{s}\otimes\mathbb{C}", shift left=6] & \\
 & & \mathbb{C}^3 \arrow[ul, "\mathbb{C}\otimes s_{\mathbb{C},\mathbb{C}}"] \arrow[u, phantom, "S_{\mathbb{C}|\mathbb{C},\mathbb{C}}"] \arrow[dll, "\mathbb{C}\otimes P"] \arrow[drr, "P\otimes\mathbb{C}"'] \arrow[d, phantom, "\mathfrak{a}"] & & \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} & & \mathbb{C}^2 \arrow[ll, "P"]
\end{tikzcd}
\tag{6.4.1}
$$

and

$$
\begin{tikzcd}
& \mathbb{C}^3 \arrow[rr, "s_{\mathbb{C}^2,\mathbb{C}}"] \arrow[d, "P\otimes\mathbb{C}"'] \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[drr, phantom, "s_{P,\mathbb{C}}"] & & \mathbb{C}^3 \arrow[d, "\mathbb{C}\otimes P"] \arrow[ddr, bend left, "P\otimes\mathbb{C}"] & \\
& \mathbb{C}^2 \arrow[rr, "s_{\mathbb{C},\mathbb{C}}"] \arrow[dr, "P"'] \arrow[dl, phantom, "\mathfrak{a}"] \arrow[r, phantom, "\mathfrak{s}"'] & & \mathbb{C}^2 \arrow[dl, "P"] \arrow[dr, phantom, "\mathfrak{a}"] & \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} & & \mathbb{C}^2 \arrow[ll, "P"]
\end{tikzcd}
=
\begin{tikzcd}
& \mathbb{C}^3 \arrow[rr, "s_{\mathbb{C}^2,\mathbb{C}}"] \arrow[dr, "\mathbb{C}\otimes s_{\mathbb{C},\mathbb{C}}"'] \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[drr, phantom, "S_{\mathbb{C},\mathbb{C}|\mathbb{C}}"] & & \mathbb{C}^3 \arrow[ddr, bend left, "P\otimes\mathbb{C}"] & \\
& & \mathbb{C}^3 \arrow[ur, "s_{\mathbb{C},\mathbb{C}}\otimes\mathbb{C}"'] \arrow[dll, "\mathbb{C}\otimes P"] \arrow[drr, "P\otimes\mathbb{C}"'] \arrow[l, phantom, "\mathbb{C}\otimes\mathfrak{s}"] \arrow[r, phantom, "\mathfrak{s}\otimes\mathbb{C}"] \arrow[d, phantom, "\mathfrak{a}"] & & \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} & & \mathbb{C}^2 \arrow[ll, "P"]
\end{tikzcd}
\tag{6.4.2}
$$

Definition 6.3. A *braided pseudomonoid* is a pseudomonoid equipped with a braiding.

Observe that, if $\mathfrak{s}$ is a braiding for $\mathbb{C}$ with respect to the monoidal bicategory braiding s, then the inverse of the right mate of $\mathfrak{s}$, with respect to $s_{\mathbb{C},\mathbb{C}}$ and $1_{\mathbb{C}}$, is a braiding with respect to s^*. We shall denote this dual braiding as $\mathfrak{s}^*$. (Note that, by Lemma 2.41, $\mathfrak{s}^*$ is also the right mate of the inverse of $\mathfrak{s}$.)

6.5 Another approach to braided pseudomonoids

This section concerns the equation

$$
\begin{tikzcd}
& \mathbb{I}\otimes\mathbb{C} \arrow[dl, "J\otimes\mathbb{C}"'] \arrow[dd, "1"] & \\
\mathbb{C}^2 \arrow[dr, "P"'] \arrow[r, phantom, "\mathfrak{l}"] & {} & \mathbb{C}\otimes\mathbb{I} \arrow[ul, "s_{\mathbb{C},\mathbb{I}}"'] \arrow[dl, "1"] \arrow[l, phantom, "U_{\mathbb{C}|\mathbb{I}}"] \\
& \mathbb{C} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{C}^2 \arrow[ddr, bend right, "P"'] & \mathbb{I}\otimes\mathbb{C} \arrow[l, "J\otimes\mathbb{C}"'] \arrow[d, phantom, "s_{\mathbb{C},J}"] & \mathbb{C}\otimes\mathbb{I} \arrow[l, "s_{\mathbb{C},\mathbb{I}}"'] \arrow[dl, "\mathbb{C}\otimes J"] \arrow[ddl, bend left, "1"] \\
{} \arrow[r, phantom, "\mathfrak{s}"] & \mathbb{C}^2 \arrow[ul, "s_{\mathbb{C},\mathbb{C}}"] \arrow[d, "P"] \arrow[r, phantom, "\mathfrak{r}"] & {} \\
& \mathbb{C} &
\end{tikzcd}
\qquad (6.5.1)
$$

We'll first show that this equation holds in every braided pseudomonoid, and then we'll show that, in the presence of axioms (6.0.2), (6.4.1) and (6.4.2), the equations (6.1.1) and (6.5.1) together imply (6.0.1). This gives a useful alternative axiomatisation of braided pseudomonoids.

In the case of ordinary braided monoidal categories, this equation corresponds to the triangle:

$$
\begin{tikzcd}
I\otimes A \arrow[rr, "\sigma_{I,A}"] \arrow[dr, "\lambda_A"'] & & A\otimes I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}
$$

and the alternative axiomatisation consists of the ordinary pentagon and hexagon equations together with this triangle and the triangle

$$
\begin{tikzcd}
I\otimes(A\otimes B) \arrow[rr, "\alpha"] \arrow[dr, "\lambda_{A\otimes B}"'] & & (I\otimes A)\otimes B \arrow[dl, "\lambda_A\otimes B"] \\
& A\otimes B &
\end{tikzcd}
$$

The advantage of this axiomatisation, in the general case just as in the case of ordinary monoidal categories, is that ρ appears just once; hence it may be eliminated from the data and defined in terms of σ and λ.

In fact this can be proved quite easily using the braided extension of the calculus of components that was mentioned briefly in Section 5.6. Sadly time constraints have made it impossible to incorporate this improvement in adequate detail, so instead we give a direct proof using pasting diagrams. This may serve at least to indicate what a dramatic simplification is made possible by component-based reasoning.

Remark 6.4. Notice that, if we can prove that this equation holds of every braided pseudomonoid, then in particular it holds of the dual braiding $\mathfrak{s}^*$, so we have

$$
\begin{tikzcd}
 & \mathbb{I}\otimes\mathbb{C} \arrow[dl, "J\otimes\mathbb{C}"'] \arrow[dd, "1"] & \\
\mathbb{C}^2 \arrow[dr, "P"'] \arrow[r, phantom, "\mathfrak{l}"] & {} \arrow[r, phantom, "U^*_{\mathbb{C}|\mathbb{I}}"] & \mathbb{C}\otimes\mathbb{I} \arrow[ul, "s^*_{\mathbb{C},\mathbb{I}}"'] \arrow[dl, "1"] \\
 & \mathbb{C} &
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C}^2 \arrow[dd, bend right=40, "P"'] & \mathbb{I}\otimes\mathbb{C} \arrow[l, "J\otimes\mathbb{C}"'] \arrow[d, phantom, "s^*_{\mathbb{C},J}"] & \mathbb{C}\otimes\mathbb{I} \arrow[l, "s^*_{\mathbb{C},\mathbb{I}}"'] \arrow[dl, "\mathbb{C}\otimes J"] \arrow[dd, bend left=40, "1"] \\
 {} \arrow[r, phantom, "\mathfrak{s}^*"] & \mathbb{C}^2 \arrow[ul, "s^*_{\mathbb{C},\mathbb{C}}"'] \arrow[d, "P"] \arrow[r, phantom, "\mathfrak{r}"] & {} \\
 & \mathbb{C} &
\end{tikzcd}
$$

Taking mates then gives

$$
\begin{tikzcd}
 & \mathbb{I}\otimes\mathbb{C} \arrow[dl, "J\otimes\mathbb{C}"'] \arrow[dd, "1"] \arrow[dr, "s_{\mathbb{I},\mathbb{C}}"] & \\
\mathbb{C}^2 \arrow[dr, "P"'] \arrow[r, phantom, "\mathfrak{l}"] & {} \arrow[r, phantom, "U_{\mathbb{I}|\mathbb{C}}"] & \mathbb{C}\otimes\mathbb{I} \arrow[dl, "1"] \\
 & \mathbb{C} &
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C}^2 \arrow[dd, bend right=40, "P"'] \arrow[dr, "s_{\mathbb{C},\mathbb{C}}"'] & \mathbb{I}\otimes\mathbb{C} \arrow[l, "J\otimes\mathbb{C}"'] \arrow[r, "s_{\mathbb{I},\mathbb{C}}"] \arrow[d, phantom, "s_{J,\mathbb{C}}"] & \mathbb{C}\otimes\mathbb{I} \arrow[dl, "\mathbb{C}\otimes J"] \arrow[dd, bend left=40, "1"] \\
 {} \arrow[r, phantom, "\mathfrak{s}"] & \mathbb{C}^2 \arrow[d, "P"] \arrow[r, phantom, "\mathfrak{r}"] & {} \\
 & \mathbb{C} &
\end{tikzcd}
\tag{6.5.2}
$$

So a proof of (6.5.1) will also establish (6.5.2). This is an example of how the duality principle can be used to establish facts about the braiding $\mathfrak{s}$, not only about the dual braiding $\mathfrak{s}^*$.

These equations generalise the one of Joyal and Street (1993, Prop. 2.1, part 1), and indeed the essence of the lengthy argument here is contained in the two-line sketch proof therein. The proof is originally due to Kelly (1964) – that he considered a symmetry, rather than a braiding, does not affect the proof.

At the end of the proof, we shall need to appeal to the following lemma. It corresponds to the fact that, in a monoidal category, the functors $I \otimes -$ and $- \otimes I$ are faithful.

Lemma 6.5. *Let A be some object of $\mathcal{B}$, let $f, g : A \to \mathbb{C}$ and let $\gamma, \delta : f \Rightarrow g$. If*

$$
\begin{tikzcd}
\mathbb{C}\times A \arrow[r, bend left, "\mathbb{C}\times f"] \arrow[r, bend right, "\mathbb{C}\times g"'] \arrow[r, phantom, "\Downarrow\mathbb{C}\times\gamma"] & \mathbb{C}\times\mathbb{C} \arrow[r, "P"] & \mathbb{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C}\times A \arrow[r, bend left, "\mathbb{C}\times f"] \arrow[r, bend right, "\mathbb{C}\times g"'] \arrow[r, phantom, "\Downarrow\mathbb{C}\times\delta"] & \mathbb{C}\times\mathbb{C} \arrow[r, "P"] & \mathbb{C}
\end{tikzcd}
$$

then $\gamma = \delta$. Dually, if

$$\begin{tikzcd}
A \times \mathbb{C} \arrow[r, bend left, "f \times \mathbb{C}"] \arrow[r, bend right, "g \times \mathbb{C}"'] \arrow[r, phantom, "\Downarrow \gamma\times\mathbb{C}"] & \mathbb{C} \times \mathbb{C} \arrow[r, "P"] & \mathbb{C}
\end{tikzcd}
=
\begin{tikzcd}
A \times \mathbb{C} \arrow[r, bend left, "f \times \mathbb{C}"] \arrow[r, bend right, "g \times \mathbb{C}"'] \arrow[r, phantom, "\Downarrow \delta\times\mathbb{C}"] & \mathbb{C} \times \mathbb{C} \arrow[r, "P"] & \mathbb{C}
\end{tikzcd}$$

then $\gamma = \delta$.

Proof. Easy, via the calculus of components. □

Proposition 6.6. *Equation* (6.5.1) *holds in any braided pseudomonoid, hence (by the discussion above) so does* (6.5.2).

Proof. Consider the 2-cell

$$\begin{tikzcd}
 & \mathbb{C} \otimes \mathbb{I} \otimes \mathbb{C} \arrow[dl, "\mathbb{C} \otimes J \otimes \mathbb{C}"'] \arrow[dd, "1"] & & \mathbb{C}^2 \otimes \mathbb{I} \arrow[ll, "s_{\mathbb{C},\mathbb{C}\otimes\mathbb{I}}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{C}} \oplus \mathbb{I}"] \arrow[dd, "1"] \\
\mathbb{C}^3 \arrow[r, phantom, "\mathbb{C} \otimes \iota"] \arrow[dr, "\mathbb{C} \otimes P"'] & \arrow[r, phantom, "\mathbb{C} \otimes U_{\mathbb{C}|\mathbb{I}}"] & \mathbb{C}^2 \otimes \mathbb{I} \arrow[ul, "\mathbb{C} \oplus s_{\mathbb{C},\mathbb{I}}"] \arrow[dl, "1"] \arrow[u, phantom, "S_{\mathbb{C}|\mathbb{C},\mathbb{I}}"] & \\
 & \mathbb{C}^2 \arrow[dr, "P"'] & \arrow[d, phantom, "\mathfrak{s}"] & \mathbb{C}^2 \arrow[ll, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[dl, "P"] \\
 & & \mathbb{C} &
\end{tikzcd}$$

By Proposition 3.12, this is equal to

$$\begin{tikzcd}
 & \mathbb{C} \otimes \mathbb{I} \otimes \mathbb{C} \arrow[dl, "\mathbb{C} \otimes J \otimes \mathbb{C}"'] \arrow[dd, "1"] & & \mathbb{C}^2 \otimes \mathbb{I} \arrow[ll, "s_{\mathbb{C},\mathbb{C}\otimes\mathbb{I}}"'] \arrow[dd, "1"] \\
\mathbb{C}^3 \arrow[r, phantom, "\mathbb{C} \otimes \iota"] \arrow[dr, "\mathbb{C} \otimes P"'] & & & \\
 & \mathbb{C}^2 \arrow[dr, "P"'] & \arrow[d, phantom, "\mathfrak{s}"] & \mathbb{C}^2 \arrow[ll, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[dl, "P"] \\
 & & \mathbb{C} &
\end{tikzcd}$$

which is equal, by equation (6.0.1), to

$$
\begin{tikzcd}
 & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[dll, "\mathbb{C}\otimes J\otimes\mathbb{C}"'] \arrow[dd, "1"] & & \mathbb{C}^2\otimes\mathbb{I} \arrow[ll, "s_{\mathbb{C},\mathbb{C}\otimes\mathbb{I}}"'] \arrow[dd, "1"] \\
\mathbb{C}^3 \arrow[r, phantom, "\mathfrak{r}\otimes\mathbb{C}"] \arrow[drr, "P\oplus\mathbb{C}"] \arrow[dd, "\mathbb{C}\otimes P"'] & & & & \\
 & \arrow[r, phantom, "\mathfrak{a}"] & \mathbb{C}^2 \arrow[dr, "P"'] & \arrow[r, phantom, "\mathfrak{s}"] & \mathbb{C}^2 \arrow[ll, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[dl, "P"] \\
\mathbb{C}^2 \arrow[rrr, "P"'] & & & \mathbb{C} &
\end{tikzcd}
$$

Since s is pseudo-natural, this is equal to

$$
\begin{tikzcd}
 & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[dl, "\mathbb{C}\oplus J\oplus\mathbb{C}"'] \arrow[d, phantom, "s_{\mathbb{C},\mathbb{C}\otimes J}"] & & \mathbb{C}^2\otimes\mathbb{I} \arrow[ll, "s_{\mathbb{C},\mathbb{C}\otimes\mathbb{I}}"'] \arrow[dll, "\mathbb{C}^2\oplus J"'] \arrow[dd, "1"] \\
\mathbb{C}^3 \arrow[dr, "P\oplus\mathbb{C}"] \arrow[dd, "\mathbb{C}\otimes P"'] & \mathbb{C}^3 \arrow[l, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[drr, "\mathbb{C}\oplus P"] \arrow[d, phantom, "s_{\mathbb{C},P}"] & \arrow[r, phantom, "\mathbb{C}\otimes\mathfrak{r}"] & \\
\arrow[r, phantom, "\mathfrak{a}"] & \mathbb{C}^2 \arrow[dr, "P"'] & \arrow[d, phantom, "\mathfrak{s}"] & \mathbb{C}^2 \arrow[ll, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[dl, "P"] \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} &
\end{tikzcd}
$$

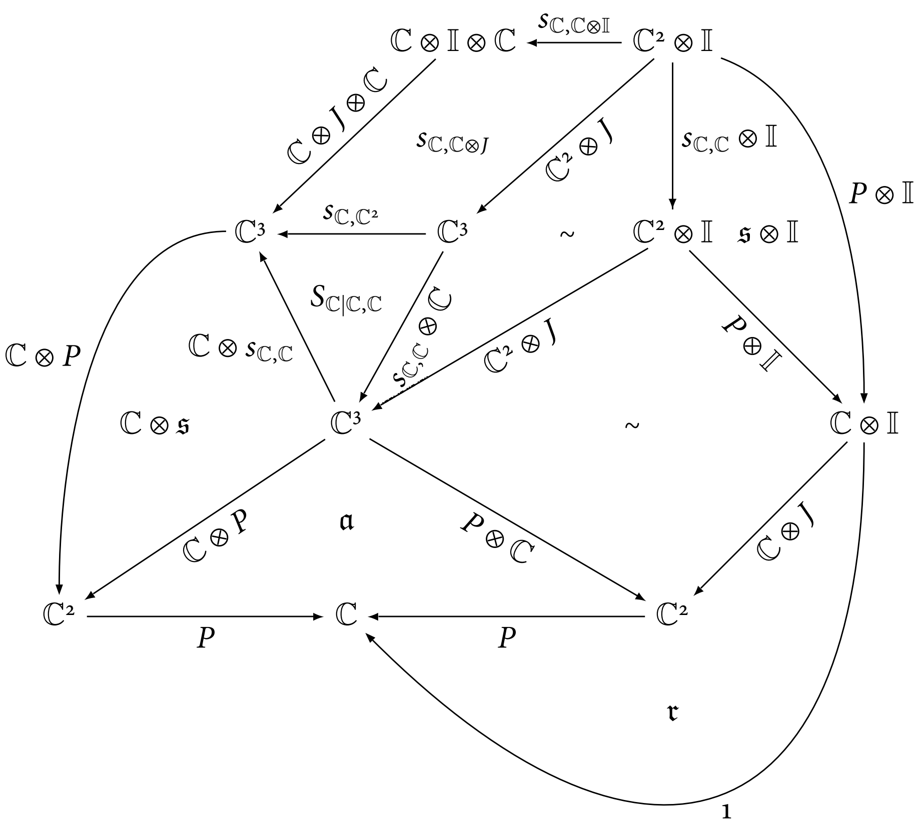
which, by equation (6.1.2), equals

$$
\begin{tikzcd}
 & & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[dll, "\mathbb{C}\oplus J\oplus\mathbb{C}"'] & \mathbb{C}^2\otimes\mathbb{I} \arrow[l, "s_{\mathbb{C},\mathbb{C}\otimes\mathbb{I}}"'] \arrow[dl, "\mathbb{C}^2\oplus J"] \arrow[ddr, bend left, "P\otimes\mathbb{I}"] \arrow[dll, phantom, "s_{\mathbb{C},\mathbb{C}\otimes J}"] \arrow[d, phantom, "\sim"] & \\
 & \mathbb{C}^3 \arrow[d, "P\otimes\mathbb{C}"'] \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[dr, phantom, "s_{\mathbb{C},P}"] & & \mathbb{C}^3 \arrow[ll, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[d, "\mathbb{C}\otimes P"] \arrow[ddr, bend left, "P\otimes\mathbb{C}"] & & \\
 & \mathbb{C}^2 \arrow[dr, "P"] \arrow[dl, phantom, "\mathfrak{a}"] & \mathfrak{s} & \mathbb{C}^2 \arrow[ll, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[dl, "P"'] \arrow[dr, phantom, "\mathfrak{a}"] & & \mathbb{C}\otimes\mathbb{I} \arrow[dl, "\mathbb{C}\oplus J"] \arrow[dlll, bend left=60, "1"] \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} & & \mathbb{C}^2 \arrow[ll, "P"] \arrow[d, phantom, "\mathfrak{r}"] & \\
 & & & & &
\end{tikzcd}
$$

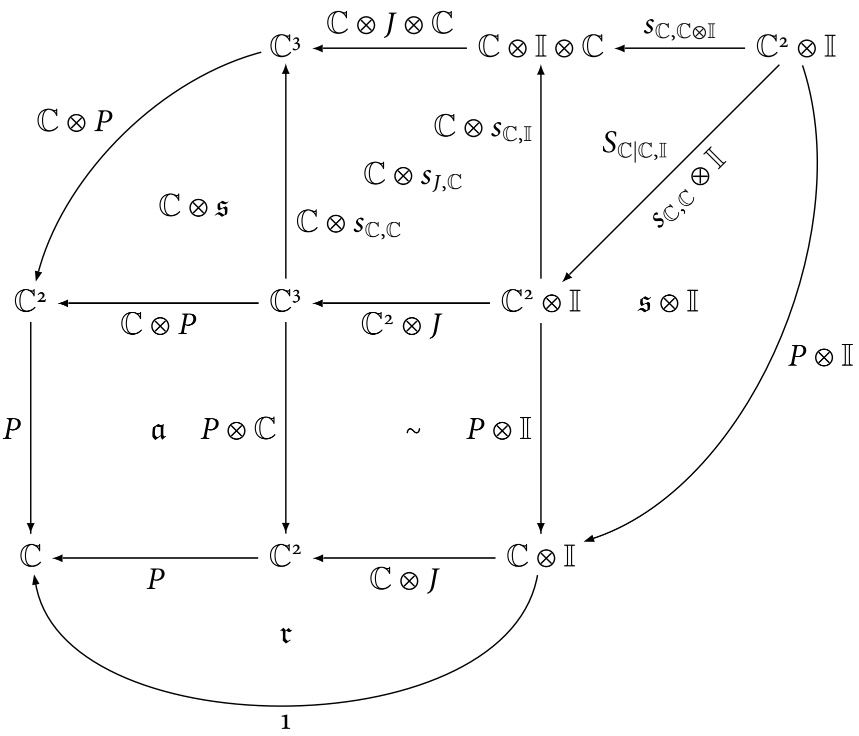
By (6.4.1), this is equal to

$$
\begin{tikzcd}
 & & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[dll, "\mathbb{C}\oplus J\oplus\mathbb{C}"'] & \mathbb{C}^2\otimes\mathbb{I} \arrow[l, "s_{\mathbb{C},\mathbb{C}\otimes\mathbb{I}}"'] \arrow[dl, "\mathbb{C}^2\oplus J"] \arrow[ddr, bend left, "P\otimes\mathbb{I}"] \arrow[dll, phantom, "s_{\mathbb{C},\mathbb{C}\otimes J}"] \arrow[d, phantom, "\sim"] & \\
 & \mathbb{C}^3 \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[dr, phantom, "S_{\mathbb{C}|\mathbb{C},\mathbb{C}}"] & & \mathbb{C}^3 \arrow[ll, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{C}}\otimes\mathbb{C}"] \arrow[ddr, bend left, "P\otimes\mathbb{C}"] & & \\
 & \mathbb{C}\otimes\mathfrak{s} & \mathbb{C}^3 \arrow[ul, "\mathbb{C}\otimes s_{\mathbb{C},\mathbb{C}}"] \arrow[dll, "\mathbb{C}\oplus P"'] \arrow[drr, "P\oplus\mathbb{C}"] \arrow[d, phantom, "\mathfrak{a}"] & \mathfrak{s}\otimes\mathbb{C} & & \mathbb{C}\otimes\mathbb{I} \arrow[dl, "\mathbb{C}\oplus J"] \arrow[dlll, bend left=60, "1"] \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} & & \mathbb{C}^2 \arrow[ll, "P"] \arrow[d, phantom, "\mathfrak{r}"] & \\
 & & & & &
\end{tikzcd}
$$

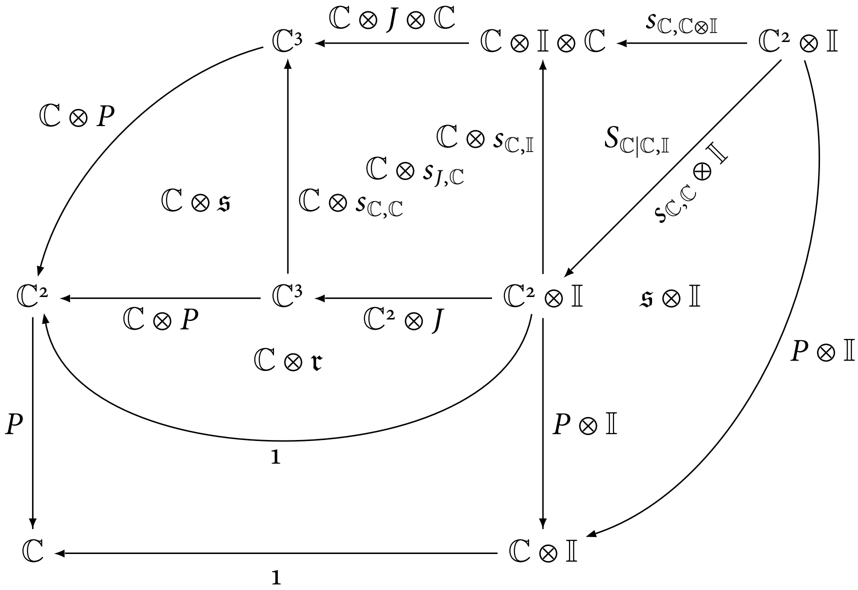
which, since ⊗ is a pseudo-functor, is equal to



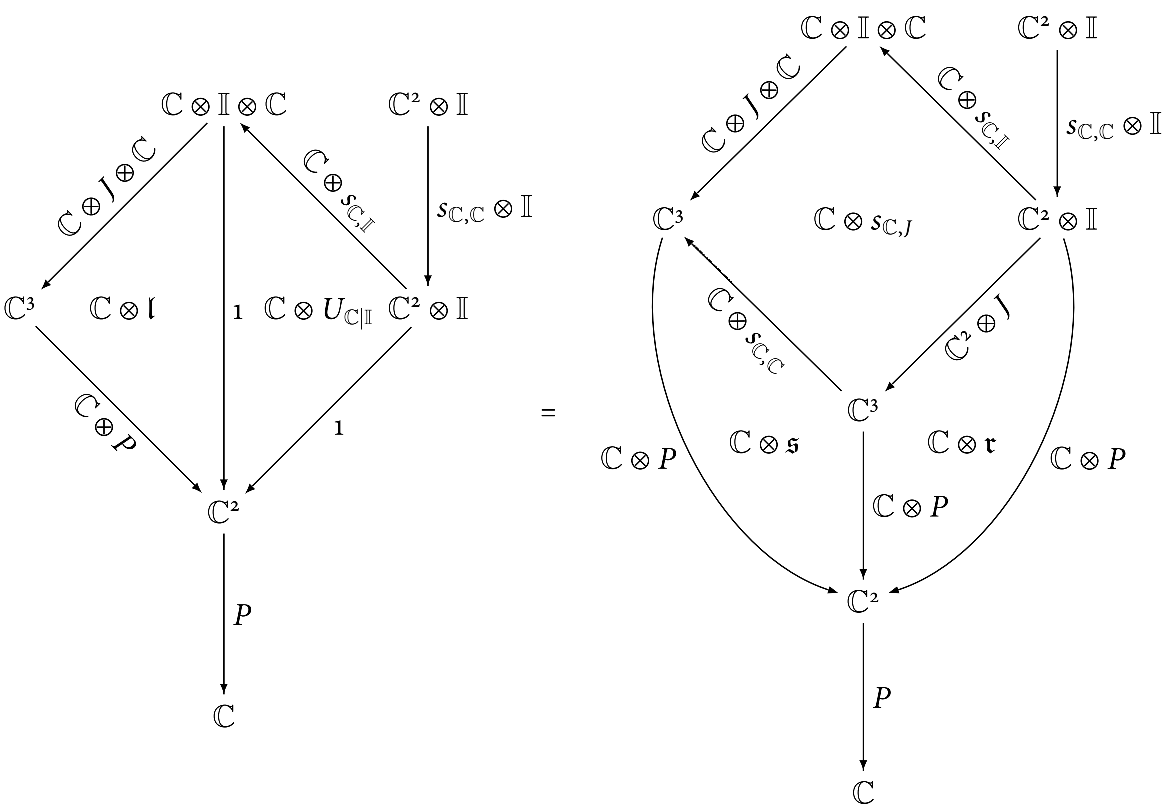
Since S is a modification, this in turn is equal to



which by (6.1.2) is equal to



If we compare this with the diagram we began with, and cancel the invertible 2-cells $S_{\mathbb{C}|\mathbb{C},\mathbb{I}}$ and $\mathfrak{s} = \mathfrak{s} \otimes \mathbb{I}$, we have that



and since $s_{\mathbb{C},\mathbb{C}}$ is an equivalence, it follows that

$$
\begin{tikzcd}
 & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[dl, "\mathbb{C}\oplus J\oplus\mathbb{C}"'] \arrow[dd, "1"] & \\
\mathbb{C}^3 \arrow[dr, "\mathbb{C}\oplus P"'] \arrow[r, phantom, "\mathbb{C}\otimes\mathfrak{l}"] & \arrow[r, phantom, "\mathbb{C}\otimes U_{\mathbb{C}|\mathbb{I}}"] & \mathbb{C}^2\otimes\mathbb{I} \arrow[ul, "\mathbb{C}\oplus s_{\mathbb{C},\mathbb{I}}"'] \arrow[dl, "1"] \\
 & \mathbb{C}^2 \arrow[d, "P"] & \\
 & \mathbb{C} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{C}^3 \arrow[ddrr, bend right, "\mathbb{C}\otimes P"'] & & \mathbb{C}\otimes s_{\mathbb{C},J} \arrow[ll, "\mathbb{C}\otimes J\otimes\mathbb{C}"'] \arrow[d, phantom, "\mathbb{C}\otimes s_{\mathbb{C},J}"] & & \mathbb{C}^2\otimes\mathbb{I} \arrow[ll, "\mathbb{C}\otimes s_{\mathbb{C},\mathbb{I}}"'] \arrow[dll, "\mathbb{C}^2\oplus J"] \arrow[ddll, bend left, "\mathbb{C}\otimes P"] \\
 & \arrow[d, phantom, "\mathbb{C}\otimes\mathfrak{s}"] & \mathbb{C}^3 \arrow[ull, "\mathbb{C}\oplus s_{\mathbb{C},\mathbb{C}}"] \arrow[d, "\mathbb{C}\otimes P"] & \arrow[d, phantom, "\mathbb{C}\otimes\mathfrak{r}"] & \\
 & \, & \mathbb{C}^2 \arrow[d, "P"] & \, & \\
 & & \mathbb{C} & &
\end{tikzcd}
$$

Now Lemma 6.5 yields the claim. □

Proposition 6.7. *Let the object $\mathbb{C}$, the 1-cells P and J, and the 2-cells $\mathfrak{a}$, $\mathfrak{l}$, $\mathfrak{r}$ and $\mathfrak{s}$ be given, as in the definition of braided pseudomonoid. Suppose that equations (6.0.2), (6.4.1), (6.4.2) and (6.1.1) are satisfied. If (6.5.1) or (6.5.2) is satisfied then the structure is indeed a braided pseudomonoid.*

It follows that a braided pseudomonoid may be defined using just the 2-cells $\mathfrak{a}$, $\mathfrak{l}$, and $\mathfrak{s}$, subject to equations (6.0.2), (6.4.1), (6.4.2) and (6.1.1). The 2-cell $\mathfrak{r}$ can be defined, if necessary, using equation (6.5.1) or (6.5.2).

Proof. We shall assume equations (6.4.1), (6.1.1) and (6.5.1), and derive (6.0.1). Consider the 2-cell

$$
\begin{tikzcd}
 & \mathbb{I}\otimes\mathbb{C}^2 \arrow[d, "J\otimes\mathbb{C}^2"'] \arrow[rrd, phantom, "s_{\mathbb{C},J\otimes\mathbb{C}}"] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[d, "\mathbb{C}\otimes J\otimes\mathbb{C}"] \arrow[dddr, bend left, "1"] \arrow[dr, phantom, "\mathfrak{r}\otimes\mathbb{C}"] & \\
 & \mathbb{C}^3 \arrow[d, "P\otimes\mathbb{C}"'] \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[rrd, phantom, "s_{\mathbb{C},P}"] & & \mathbb{C}^3 \arrow[ll, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[d, "\mathbb{C}\otimes P"] \arrow[ddr, bend left, "P\otimes\mathbb{C}"] & \\
 & \mathbb{C}^2 \arrow[dr, "P"] \arrow[dl, phantom, "\mathfrak{a}"] \arrow[rr, phantom, "\mathfrak{s}"'] & & \mathbb{C}^2 \arrow[ll, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[dl, "P"'] \arrow[dr, phantom, "\mathfrak{a}"] & \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} & & \mathbb{C}^2 \arrow[ll, "P"]
\end{tikzcd}
$$

By equation (6.4.1), this is equal to

$$
\begin{tikzcd}
 & \mathbb{I}\otimes\mathbb{C}^2 \arrow[d, "J\otimes\mathbb{C}^2"'] \arrow[rrd, phantom, "s_{\mathbb{C},J\otimes\mathbb{C}}"] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[d, "\mathbb{C}\otimes J\otimes\mathbb{C}"] \arrow[dddr, bend left, "1"] \arrow[dr, phantom, "\mathfrak{r}\otimes\mathbb{C}"] & \\
 & \mathbb{C}^3 \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[rrd, phantom, "s_{\mathbb{C}|\mathbb{C},\mathbb{C}}"] \arrow[dd, phantom, "\mathbb{C}\otimes\mathfrak{s}"'] & & \mathbb{C}^3 \arrow[ll, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{C}}\otimes\mathbb{C}"] \arrow[ddr, bend left, "P\otimes\mathbb{C}"] \arrow[dd, phantom, "\mathfrak{s}\otimes\mathbb{C}"] & \\
 & & \mathbb{C}^3 \arrow[ul, "\mathbb{C}\otimes s_{\mathbb{C},\mathbb{C}}"] \arrow[dll, "\mathbb{C}\otimes P"] \arrow[drr, "P\otimes\mathbb{C}"'] \arrow[d, phantom, "\mathfrak{a}"] & & \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} & & \mathbb{C}^2 \arrow[ll, "P"]
\end{tikzcd}
$$

which, since S is a modification, is equal to

$$
\begin{tikzcd}
\mathbb{I}\otimes\mathbb{C}^2 \arrow[d, "J\otimes\mathbb{C}^2"'] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{I}}\oplus\mathbb{C}"'] \arrow[d, "\mathbb{C}\otimes J\otimes\mathbb{C}"] \arrow[dddd, bend left=60, "1"] \arrow[dl, phantom, bend right, "S_{\mathbb{C}|\mathbb{I},\mathbb{C}}"] \\
\mathbb{C}^3 \arrow[ddd, "\mathbb{C}\otimes P"'] \arrow[r, phantom, "\sim"] & \mathbb{I}\otimes\mathbb{C}^2 \arrow[ul, "\mathbb{I}+s_{\mathbb{C},\mathbb{C}}"] \arrow[dd, "J\otimes\mathbb{C}^2"] \arrow[r, phantom, "s_{\mathbb{C},J}\otimes\mathbb{C}"] & \mathbb{C}^2 \arrow[ddl, "s_{\mathbb{C},\mathbb{C}}\oplus\mathbb{C}"'] \arrow[ddd, "P\otimes\mathbb{C}"] \arrow[dd, phantom, "\mathfrak{r}\otimes\mathbb{C}"] \\
& & \\
\arrow[r, phantom, "\mathbb{C}\otimes\mathfrak{s}"] & \mathbb{C}^3 \arrow[uul, "\mathbb{C}\oplus s_{\mathbb{C},\mathbb{C}}"] \arrow[dl, "\mathbb{C}\oplus P"] \arrow[dr, "P\oplus\mathbb{C}"'] \arrow[d, phantom, "\mathfrak{a}"] \arrow[r, phantom, "\mathfrak{s}\otimes\mathbb{C}"] & \\
\mathbb{C}^2 \arrow[r, "P"'] & \mathbb{C} & \mathbb{C}^2 \arrow[l, "P"]
\end{tikzcd}
$$

By equation (6.5.1), this is

$$
\begin{tikzcd}
\mathbb{I}\otimes\mathbb{C}^2 \arrow[dd, "J\otimes\mathbb{C}^2"'] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{I}}\otimes\mathbb{C}"'] \arrow[dd, "1"] \arrow[dl, phantom, bend right, "S_{\mathbb{C}|\mathbb{I},\mathbb{C}}"] \\
& \mathbb{I}\otimes\mathbb{C}^2 \arrow[ul, "\mathbb{I}\otimes s_{\mathbb{C},\mathbb{C}}"] \arrow[d, "J\otimes\mathbb{C}^2"'] \arrow[dr, "1"] \arrow[r, phantom, "U_{\mathbb{C}|\mathbb{I}}\otimes\mathbb{C}"] \arrow[dl, phantom, "\sim"] \arrow[dr, phantom, bend left, "\mathfrak{l}\otimes\mathbb{C}"] & \\
\mathbb{C}^3 \arrow[dr, "\mathbb{C}\otimes P"'] \arrow[dr, phantom, bend left, "\mathbb{C}\otimes\mathfrak{s}"] & \mathbb{C}^3 \arrow[l, "\mathbb{C}\otimes s_{\mathbb{C},\mathbb{C}}"'] \arrow[r, "P\otimes\mathbb{C}"'] \arrow[d, "\mathbb{C}\otimes P"] \arrow[dr, phantom, "\mathfrak{a}"] & \mathbb{C}^2 \arrow[d, "P"] \\
& \mathbb{C}^2 \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}
$$

which, by (6.1.1), is equal to

$$
\begin{tikzcd}
& \mathbb{I}\otimes\mathbb{C}^2 \arrow[ddl, "J\oplus\mathbb{C}^2"'] \arrow[dr, phantom, "S_{\mathbb{C}|\mathbb{I},\mathbb{C}}"] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{I}}\otimes\mathbb{C}"] \arrow[dd, "1"] \\
& & \mathbb{I}\otimes\mathbb{C}^2 \arrow[ul, "\mathbb{I}\otimes s_{\mathbb{C},\mathbb{C}}"] \arrow[dl, "J\oplus\mathbb{C}^2"'] \arrow[d, "\mathbb{I}\otimes P"'] \arrow[dr, "1"] \arrow[ur, phantom, "U_{\mathbb{C}|\mathbb{I}}\otimes\mathbb{C}"'] \arrow[ll, phantom, "\sim"] & \\
\mathbb{C}^3 \arrow[dr, "\mathbb{C}\otimes P"'] \arrow[r, phantom, "\mathbb{C}\otimes\mathfrak{s}"'] & \mathbb{C}^3 \arrow[l, "\mathbb{C}\otimes s_{\mathbb{C},\mathbb{C}}"'] \arrow[d, "\mathbb{C}\otimes P"] \arrow[r, phantom, "\sim"] & \mathbb{I}\otimes\mathbb{C} \arrow[dl, "J\oplus\mathbb{C}"'] \arrow[dr, "1"] \arrow[d, phantom, "\wr"] & \mathbb{C}^2 \arrow[d, "P"] \\
& \mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C}
\end{tikzcd}
$$

which, since the ~ cells are natural, is equal to

$$
\begin{tikzcd}
& \mathbb{I}\otimes\mathbb{C}^2 \arrow[ddl, "J\oplus\mathbb{C}^2"'] \arrow[ddrr, bend right, "\mathbb{I}\otimes P"'] \arrow[dr, phantom, "S_{\mathbb{C}|\mathbb{I},\mathbb{C}}"] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{I}}\otimes\mathbb{C}"] \arrow[dd, "1"] \\
& \arrow[r, phantom, "\mathbb{I}\otimes\mathfrak{s}"] & \mathbb{I}\otimes\mathbb{C}^2 \arrow[ul, "\mathbb{I}\otimes s_{\mathbb{C},\mathbb{C}}"] \arrow[d, "\mathbb{I}\otimes P"'] \arrow[dr, "1"] \arrow[ur, phantom, "U_{\mathbb{C}|\mathbb{I}}\otimes\mathbb{C}"'] & \\
\mathbb{C}^3 \arrow[dr, "\mathbb{C}\otimes P"'] \arrow[r, phantom, "\sim"] & & \mathbb{I}\otimes\mathbb{C} \arrow[dl, "J\oplus\mathbb{C}"'] \arrow[dr, "1"] \arrow[d, phantom, "\wr"] & \mathbb{C}^2 \arrow[d, "P"] \\
& \mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C}
\end{tikzcd}
$$

By (6.1.1), this is equal to

$$\begin{tikzcd}
 & \mathbb{I}\otimes\mathbb{C}^2 \arrow[dddl, bend right, "J\otimes\mathbb{C}^2"'] \arrow[ddd, "1"'] \arrow[ddr, bend right, "\mathbb{I}\otimes P"'] \arrow[dr, phantom, "\mathbb{I}\otimes\mathfrak{s}"] \arrow[ddl, phantom, "\mathfrak{l}\otimes\mathbb{C}"] \arrow[rr, phantom, "S_{\mathbb{C}|\mathbb{I},\mathbb{C}}"'] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{I}}\otimes\mathbb{C}"] \arrow[dd, "1"] \\
 & & \mathbb{I}\otimes\mathbb{C}^2 \arrow[ul, "\mathbb{I}\otimes s_{\mathbb{C},\mathbb{C}}"] \arrow[d, "\mathbb{I}\otimes P"] \arrow[dr, "1"] \arrow[r, phantom, "U_{\mathbb{C}|\mathbb{I}}\otimes\mathbb{C}"] & \\
 & & \mathbb{I}\otimes\mathbb{C} \arrow[dr, "1"] & \mathbb{C}^2 \arrow[d, "P"] \\
\mathbb{C}^3 \arrow[r, "P\otimes\mathbb{C}"] \arrow[drr, "\mathbb{C}\otimes P"'] & \mathbb{C}^2 \arrow[rr, "P"'] \arrow[dr, phantom, "\mathfrak{a}"] & & \mathbb{C} \\
 & & \mathbb{C}^2 \arrow[ur, "P"'] &
\end{tikzcd}$$

which equals

$$\begin{tikzcd}
 & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[dll, bend right, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[dl, "s_{\mathbb{C},\mathbb{I}}\otimes\mathbb{C}"] \arrow[dd, "1"] \arrow[dl, phantom, bend right, "S_{\mathbb{C}|\mathbb{I},\mathbb{C}}"] \\
\mathbb{I}\otimes\mathbb{C}^2 \arrow[d, "J\otimes\mathbb{C}^2"'] \arrow[dr, "1"] \arrow[dr, phantom, "\mathfrak{l}\otimes\mathbb{C}"'] & \mathbb{I}\otimes\mathbb{C}^2 \arrow[l, "\mathbb{I}\otimes s_{\mathbb{C},\mathbb{C}}"'] \arrow[dr, "1"'] \arrow[dr, phantom, "U_{\mathbb{C}|\mathbb{I}}\otimes\mathbb{C}"] & \\
\mathbb{C}^3 \arrow[r, "P\otimes\mathbb{C}"'] \arrow[d, "\mathbb{C}\otimes P"'] \arrow[dr, phantom, "\mathfrak{a}"] & \mathbb{C}^2 \arrow[dr, "P"'] & \mathbb{C}^2 \arrow[l, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[d, "P"] \arrow[dl, phantom, "\mathfrak{s}"'] \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C}
\end{tikzcd}$$

By one of the unit axioms in the definition of braiding for a monoidal bicategory, this is

$$\begin{tikzcd}
\mathbb{I}\otimes\mathbb{C}^2 \arrow[d, "J\otimes\mathbb{C}^2"'] \arrow[dr, "1"] \arrow[d, phantom, shift left=6ex, "\mathfrak{l}\otimes\mathbb{C}"] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[d, "1"] \\
\mathbb{C}^3 \arrow[r, "P\otimes\mathbb{C}"'] \arrow[d, "\mathbb{C}\otimes P"'] \arrow[dr, phantom, "\mathfrak{a}"] & \mathbb{C}^2 \arrow[dr, "P"'] & \mathbb{C}^2 \arrow[l, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[d, "P"] \arrow[dl, phantom, "\mathfrak{s}"] \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C}
\end{tikzcd}$$

which, since s is pseudo-natural, is equal to

$$\begin{tikzcd}
\mathbb{I}\otimes\mathbb{C}^2 \arrow[dd, "J\otimes\mathbb{C}^2"'] \arrow[dr, phantom, "s_{\mathbb{C},J\otimes\mathbb{C}}"] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[dl, "\mathbb{C}\oplus J\oplus\mathbb{C}"'] \arrow[dd, "1"] \\
& \mathbb{C}^3 \arrow[dl, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[dr, "\mathbb{C}\oplus P"] \arrow[r, phantom, "\mathbb{C}\otimes\mathfrak{l}"] \arrow[d, phantom, "s_{\mathbb{C},P}"] & {} \\
\mathbb{C}^3 \arrow[r, "P\otimes\mathbb{C}"'] \arrow[d, "\mathbb{C}\otimes P"'] \arrow[dr, phantom, "\mathfrak{a}"] & \mathbb{C}^2 \arrow[dr, "P"'] & \mathbb{C}^2 \arrow[l, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[d, "P"] \arrow[dl, phantom, "\mathfrak{s}"] \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C}
\end{tikzcd}$$

If we now compare this with the 2-cell we began with, and cancel the common invertible 2-cell

$$
\begin{tikzcd}
& \mathbb{I}\otimes\mathbb{C}^2 \arrow[d, "J\otimes\mathbb{C}^2"'] \arrow[rr, phantom, shift right=8ex, "s_{\mathbb{C},J\otimes\mathbb{C}}"] & & \mathbb{C}\otimes\mathbb{I}\otimes\mathbb{C} \arrow[ll, "s_{\mathbb{C},\mathbb{I}\otimes\mathbb{C}}"'] \arrow[d, "\mathbb{C}\otimes J\otimes\mathbb{C}"] \\
& \mathbb{C}^3 \arrow[d, "P\otimes\mathbb{C}"'] \arrow[ddl, bend right, "\mathbb{C}\otimes P"'] \arrow[rr, phantom, shift right=8ex, "s_{\mathbb{C},P}"] & & \mathbb{C}^3 \arrow[ll, "s_{\mathbb{C},\mathbb{C}^2}"'] \arrow[d, "\mathbb{C}\otimes P"] \\
& \mathbb{C}^2 \arrow[dr, "P"'] \arrow[dl, phantom, "\mathfrak{a}"] \arrow[rr, phantom, shift right=5ex, "\mathfrak{s}"] & & \mathbb{C}^2 \arrow[ll, "s_{\mathbb{C},\mathbb{C}}"'] \arrow[dl, "P"] \\
\mathbb{C}^2 \arrow[rr, "P"'] & & \mathbb{C} &
\end{tikzcd}
$$

we obtain equation (6.0.1), as claimed. □

Chapter 7

Cayley's Theorem for Pseudomonoids

In its traditional form, Cayley's theorem says that every finite group is (isomorphic to) a subgroup of one of the symmetric groups. What the proof actually shows is that every monoid M is a submonoid of the monoid of endofunctions $|M| \to |M|$, via the function that maps the element $x \in M$ to the function $x \circ -$.

A 'Cayley Theorem for monoidal categories' has been used by Joyal and Street (1993, Proposition 1.3) to show that every monoidal category is monoidally equivalent to a *strict* monoidal category. It turns out that we need the corresponding theorem for promonoidal categories; and rather than just prove another special case, it seems wise to generalise. So the purpose of this chapter is to state and prove a Cayley Theorem for pseudomonoids.

As motivation, we begin by reviewing the one-dimensional case, of monoids in a monoidal category.

7.1 Cayley's theorem for monoids

I'm not aware of any literature that specifically addresses the question of giving an analogue of Cayley's theorem for monoids in a monoidal category, but it is a special case of a standard construction. A monoid in the monoidal category $\mathcal{V}$ can be regarded as a one-object $\mathcal{V}$-enriched category.[1] Then we can apply the enriched Yoneda Lemmas of Kelly (1982, sections 1.9 and 2.4) to this $\mathcal{V}$-category. Since it requires some effort to extract the specifics of the one-object case from Kelly's constructions, we give an overview here.

Let $\mathcal{V}$ be a monoidal category, and let M be a monoid in $\mathcal{V}$, with unit $e : I \to M$ and multiplication $m : M \otimes M \to M$.

The Cayley-Yoneda theorems are best stated in terms of modules:

[1] This operation – reimagining a monoid as a one-object category – is often called *suspension*.

Definition 7.1. A *right M-module* consists of an object $X \in \mathcal{V}$ and a map

$$x : X \otimes M \to X$$

such that the diagrams

$$\begin{tikzcd} X \otimes M \otimes M \arrow[r, "x \otimes M"] \arrow[d, "X \otimes m"'] & X \otimes M \arrow[d, "x"] \\ X \otimes M \arrow[r, "x"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, "1"'] & X \otimes I \arrow[d, "X \otimes e"] \\ X & X \otimes M \arrow[l, "x"] \end{tikzcd}$$

commute.

Remark 7.2. Note that (M, m) is a right M-module.

Definition 7.3. Let (X, x) and (Y, y) be right M-modules. A *morphism of modules* from X to Y is a map $f : X \to Y$ for which the diagram

$$\begin{tikzcd} X \otimes M \arrow[r, "x"] \arrow[d, "f \otimes M"'] & X \arrow[d, "f"] \\ Y \otimes M \arrow[r, "y"'] & Y \end{tikzcd}$$

commutes. The set of such maps will be denoted $\mathrm{Mod}_M(X, Y)$.

Kelly proves two versions of his Yoneda Lemma, which he refers to as 'weak' and 'strong'. We shall call our corresponding theorems the 'external' and 'internal' Cayley Theorem, respectively. The internal theorem requires some additional properties of $\mathcal{V}$, but the external one is perfectly general.

7.1.1 The external Cayley Theorem

Theorem 7.4 (External Cayley). *For every right M-module X, the natural transformation*

$$\phi_X : \mathrm{Mod}_M(M, X) \to \mathcal{V}(I, X),$$

defined as $\phi_X(f) = f \cdot e$, is invertible with $\phi_X^{-1}(z)$ equal to the composite

$$M \xrightarrow{\cong} I \otimes M \xrightarrow{z \otimes M} X \otimes M \xrightarrow{x} X.$$

Proof. Let $f : M \to X$ be a map of modules. We'll first show that $\phi_X^{-1}(\phi_X(f)) = f$: consider the diagram

$$\begin{tikzcd}
M \arrow[r, "\cong"] & I\otimes M \arrow[r, "e\otimes M"] \arrow[dr, "\cong"'] & M\otimes M \arrow[r, "f\otimes M"] \arrow[d, "m"] & X\otimes M \arrow[d, "x"] \\
 & & M \arrow[r, "f"'] & X
\end{tikzcd}$$

The triangle commutes by the left-unit law for the monoid M, and the square because f is a map of modules. The upper edge is $\phi_X^{-1}(\phi_X(f))$ by definition, and the lower edge is equal to f.

Now let $z : I \to X$, and consider the diagram

$$\begin{tikzcd}
 & X \arrow[dr, "\cong"] & & \\
I \arrow[ur, "z"] \arrow[r, "\cong"'] \arrow[d, "e"'] & I\otimes I \arrow[r, "z\otimes I"'] \arrow[d, "I\otimes e"'] & X\otimes I \arrow[d, "X\otimes e"] & \\
M \arrow[r, "\cong"'] & I\otimes M \arrow[r, "z\otimes M"'] & X\otimes M \arrow[r, "x"'] & X
\end{tikzcd}$$

The cells commute by naturality and functoriality of tensor, and by the unit condition in the definition of module the upper edge is equal to z. The lower edge is $\phi_X(\phi_X^{-1}(z))$ by definition. Thus ϕ_X and ϕ_X^{-1} are indeed mutually inverse.

We must also show that $\phi_X^{-1}(z)$ is a map of modules, so let $z : I \to X$ and consider the diagram

$$\begin{tikzcd}
M\otimes M \arrow[r, "\cong"] \arrow[d, "m"'] & I\otimes M\otimes M \arrow[r, "z\otimes M\otimes M"] \arrow[d, "I\otimes m"] & X\otimes M\otimes M \arrow[r, "x\otimes M"] \arrow[d, "X\otimes m"] & X\otimes M \arrow[d, "m"] \\
M \arrow[r, "\cong"'] & I\otimes M \arrow[r, "z\otimes M"'] & X\otimes M \arrow[r, "x"'] & X
\end{tikzcd}$$

The squares commute, from left to right, by the naturality of the left-unit isomorphism, the functoriality of tensor, and the fact that X is a module. Thus the outer edge commutes, showing that $\phi_X^{-1}(z)$ is a map of modules. $\square$

Remark 7.5. More can be said about ϕ_X, as follows. Each of the sets $\mathrm{Mod}_M(M, X)$ and $\mathcal{V}(I, X)$ is a right $\mathcal{V}(I, M)$-module in a natural way, using the M-module structure of X. Then ϕ_X is a map of modules with respect to these module structures. We omit[2] the details of this, but they are quite routine.

Remark 7.6. The set $\mathcal{V}(I, M)$ inherits the monoid structure from M: its unit is $u : I \to M$, and the product of $a, b : I \to M$ is the composite

$$I \xrightarrow{\cong} I \otimes I \xrightarrow{a \otimes b} M \otimes M \xrightarrow{m} M.$$

This is the *underlying ordinary monoid* of the abstract monoid M.

To relate Theorem 7.4 to the ordinary Cayley Theorem, take the module X to be M itself. Then we have that $\mathcal{V}(I, M)$, the underlying ordinary monoid of M, is isomorphic to $\mathrm{Mod}_M(M, M)$, which is a submonoid of $\mathcal{V}(M, M)$. We should like to be able to say that this ϕ_M is an isomorphism of monoids, i.e. that it preserves the monoid structure.

Proposition 7.7. *The isomorphism*

$$\phi_M : \mathrm{Mod}_M(M, M) \to \mathcal{V}(I, M)$$

is a map of monoids.

Proof. For the unit, $\phi_M(1_M) = 1_M \cdot e = e$. For the multiplication, let $f, g : M \to M$ be right module morphisms: we must show that $\phi_M(f \cdot g)$:

$$I \xrightarrow{e} M \xrightarrow{g} M \xrightarrow{f} M$$

is equal to $\phi_M(f) \otimes \phi_M(g)$:

$$I \xrightarrow{\cong} I \otimes I \xrightarrow{(f \cdot e) \otimes (g \cdot e)} M \otimes M \xrightarrow{m} M.$$

Consider the diagram

$$\begin{tikzcd}
I \otimes I \arrow[r, "I \otimes (g \cdot e)"] & I \otimes M \arrow[r, "e \otimes M"] \arrow[dr, "\cong"] & M \otimes M \arrow[r, "f \otimes M"] \arrow[d, "m"] & M \otimes M \arrow[d, "m"] \\
I \arrow[u, "\cong"] \arrow[r, "g \cdot e"'] & M \arrow[u, "\cong"] \arrow[r, "1"'] & M \arrow[r, "f"'] & M
\end{tikzcd}$$

[2]An earlier version of this chapter included these details: I removed them, as an unneeded distraction.

in which the right-hand square commutes because f is a map of right M-modules, and the upper triangle by the right-unit law for M. By the functoriality of tensor, the top edge is equal to $(f \cdot e) \otimes (g \cdot e)$, so the upper outer edge is $\phi_M(f) \otimes \phi_M(g)$ and the lower edge is $\phi_M(f \cdot g)$, which are therefore equal as required. □

7.1.2 The internal Cayley Theorem

The theorem above is external in the sense that it essentially concerns the set $\mathcal{V}(I, X)$ rather than the module X itself. If the monoidal category $\mathcal{V}$ is closed, then we can state an internal version, purely in terms of arrows in $\mathcal{V}$ itself. So let $\mathcal{V}$ be biclosed, i.e. suppose we are given functors

$$\multimap : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$$
$$\multimapinv : \mathcal{V} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{V},$$

and natural isomorphisms with components

$$\mathrm{curry}_{A,B,C} : \mathcal{V}(A \otimes B, C) \cong \mathcal{V}(A, B \multimap C),$$
$$\mathrm{curry}'_{A,B,C} : \mathcal{V}(A \otimes B, C) \cong \mathcal{V}(B, C \multimapinv A).$$

Then the theorem is:

Theorem 7.8 (Internal Cayley). *Let (X, x) be a right M-module. Then the diagram*

$$X \xrightarrow{\mathrm{curry}_{X,M,X}(x)} M \multimap X \overset{h}{\underset{k}{\rightrightarrows}} (M \otimes M) \multimap X$$

is a coequaliser diagram, where the maps h and k are defined as follows. The map h is obtained by currying

$$(M \multimap X) \otimes M \otimes M \xrightarrow{\mathrm{ev}_X^M \otimes M} X \otimes M \xrightarrow{x} X,$$

and k is obtained by currying

$$(M \multimap X) \otimes M \otimes M \xrightarrow{(M \multimap X) \otimes m} (M \multimap X) \otimes M \xrightarrow{\mathrm{ev}_X^M} X,$$

where the map ev_X^M is the result of uncurrying the identity on $M \multimap X$.

Although the statement of the theorem uses only the right-closed structure $\multimap$, the left-closed structure $\multimapinv$ plays a crucial role in the proof. We will not give a detailed proof here[3], since this is something of a digression, but explain the outline.

[3] Again, I did type out a detailed proof. So this is not just an excuse for laziness!

The preliminary observations are that:

1. For every object $X \in \mathbb{C}$, the object $X \multimapinv X$ is a monoid in a canonical way, via the definable 'internal unit' and 'internal composition' maps:
$$\begin{aligned} u &: I \to X \multimapinv X, \\ m &: (X \multimapinv X) \otimes (X \multimapinv X) \to X \multimapinv X. \end{aligned}$$

2. A right M-module is precisely a map of monoids $M \to X \multimapinv X$ for some X.

3. It follows from this that a right M-module structure on X induces a canonical right M-module structure on $A \otimes X$ for each object $A \in \mathcal{V}$, because there is an 'internal tensor' map
$$X \multimapinv X \longrightarrow (A \otimes X) \multimapinv (A \otimes X),$$
which is compatible with the internal unit and compositions.

4. In a similar way, a right M-module structure on X induces a canonical right M-module structure on $X \multimapinv A$, for each object A.

Our second observation is trivial to prove: simply curry the diagrams in Definition 7.1. The others are proved in Kelly (1982, Section 1.6). Now for the proof itself: suppose we are given an object $A \in \mathcal{V}$ and a map $g : A \to M \multimap X$ such that $hg = kg$. We can curry g to get a map
$$g' : A \otimes M \to X,$$
and it is not hard to check that, since $hg = kg$, this g' is a map of modules from $A \otimes M$ to X; where the module structure on $A \otimes M$ is obtained from that of M by point (3) above. Then we curry g' with respect to the left-closure, to obtain a map
$$g'' : M \to X \multimapinv A,$$
which is a map of modules with respect to the module structure on $X \multimapinv A$ obtained from (4). By the external Cayley Theorem, g'' is thus equal to
$$M \xrightarrow{\cong} I \otimes M \xrightarrow{f \otimes M} (X \multimapinv A) \otimes M \longrightarrow X \multimapinv A$$
for some unique $f : I \to X \multimapinv A$, where the unlabelled arrow above comes from the M-module structure on $X \multimapinv A$. From this it can be deduced that g' is equal to
$$A \otimes M \xrightarrow{f' \otimes M} X \otimes M \xrightarrow{x} X$$

for some unique $f' : A \to X$ (obtained by uncurrying f). Thus g is equal to

$$A \xrightarrow{f'} X \xrightarrow{\text{curry}_{X,M,X}(x)} M \multimap X$$

for this unique f', which is precisely the universal property of the equaliser diagram.

Remark 7.9. Above we have deduced the internal theorem from the external one. Conversely, the external theorem can be obtained as a corollary of the internal one, under an additional assumption on $\mathcal{V}$. The additional assumption is that the functor $\mathcal{V}(I,-) : \mathcal{V} \to \text{Set}$ should preserve equalisers. In particular, this is always so if $\mathcal{V}$ has small coproducts, because in that case $\mathcal{V}$ is *tensored* (in the sense of Kelly, 1982, Section 2.7) as a Set-category, and so the functor $\mathcal{V}(I,-)$ has a left adjoint. Then we can obtain the external theorem simply by applying the functor $\mathcal{V}(I,-)$ to the equaliser diagram of the internal theorem.

7.2 Cayley's Theorem for Pseudomonoids

Both these theorems, internal and external, admit generalisation to the higher-dimensional setting. But since it is sufficient for our applications, we shall confine ourselves to the external version.

7.2.1 Modules for pseudomonoids

First we must give a suitable definition of $\mathbb{C}$-module, where $\mathbb{C}$ is a pseudomonoid. In fact, we want to define a bicategory $\text{Mod}_{\mathbb{C}}$ of right $\mathbb{C}$-modules.

Definition 7.10. A *right* $\mathbb{C}$*-module* (X, x) for the pseudomonoid $\mathbb{C}$ consists of a 1-cell $x : X \otimes \mathbb{C} \to X$ together with invertible 2-cells

$$\begin{tikzcd}
 & X \otimes \mathbb{C} \arrow[dd, "x"] \\
X \otimes \mathbb{I} \arrow[ur, "X \otimes J"] \arrow[dr, "r_X"'] \arrow[r, phantom, "\nearrow \chi^J_X"] & \\
 & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \otimes (\mathbb{C} \otimes \mathbb{C}) \arrow[r, "a_{X,\mathbb{C},\mathbb{C}}"] \arrow[d, "X \otimes P"'] & (X \otimes \mathbb{C}) \otimes \mathbb{C} \arrow[r, "x \otimes \mathbb{C}"] \arrow[d, phantom, "\Nearrow \chi^P_X"] & X \otimes \mathbb{C} \arrow[d, "x"] \\
X \otimes \mathbb{C} \arrow[rr, "x"'] & & X
\end{tikzcd}$$

such that the following two equations hold:

$$\begin{tikzcd}[column sep=small]
 & X \otimes \mathbb{C}^3 \arrow[rr, "x\otimes\mathbb{C}^2"] \arrow[dl, "X\otimes\mathbb{C}\otimes P"'] \arrow[dr, "X\otimes P\otimes\mathbb{C}"] & & X \otimes \mathbb{C}^2 \arrow[dr, "x\otimes\mathbb{C}"] \arrow[d, phantom, "\Nearrow \chi^P_X\otimes\mathbb{C}"] & \\
X \otimes \mathbb{C}^2 \arrow[dr, "X\otimes P"'] \arrow[r, phantom, "\overset{\Rightarrow}{X\otimes\mathfrak{a}}"] & {} & X \otimes \mathbb{C}^2 \arrow[rr, "x\otimes\mathbb{C}"] \arrow[dl, "X\otimes P"] & {} \arrow[d, phantom, "\Nearrow \chi^P_X"] & X \otimes \mathbb{C} \arrow[dl, "x"] \\
 & X \otimes \mathbb{C} \arrow[rr, "x"'] & & X &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & X \otimes \mathbb{C}^3 \arrow[rr, "x\otimes\mathbb{C}^2"] \arrow[dl, "X\otimes\mathbb{C}\otimes P"'] \arrow[d, phantom, "\sim"] & & X \otimes \mathbb{C}^2 \arrow[dl, "X\otimes P"'] \arrow[dr, "x\otimes\mathbb{C}"] & \\
X \otimes \mathbb{C}^2 \arrow[rr, "x\otimes\mathbb{C}"] \arrow[dr, "X\otimes P"'] & {} \arrow[d, phantom, "\Nearrow \chi^P_X"] & X \otimes \mathbb{C} \arrow[dr, "x"] \arrow[rr, phantom, "\overset{\Rightarrow}{\chi^P_X}"] & & X \otimes \mathbb{C} \arrow[dl, "x"] \\
 & X \otimes \mathbb{C} \arrow[rr, "x"'] & & X &
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \otimes \mathbb{I} \otimes \mathbb{C} \arrow[r, "X\otimes J\otimes\mathbb{C}"] \arrow[dr, "1"'] & X \otimes \mathbb{C}^2 \arrow[r, "x\otimes\mathbb{C}"] \arrow[d, "X\otimes P"] \arrow[dl, phantom, "\Nearrow X\otimes\mathfrak{l}"] \arrow[dr, phantom, "\Nearrow \chi^P_X"] & X \otimes \mathbb{C} \arrow[d, "x"] \\
 & X \otimes \mathbb{C} \arrow[r, "x"'] & X
\end{tikzcd}
=
\begin{tikzcd}
X \otimes \mathbb{I} \otimes \mathbb{C} \arrow[r, "X\otimes J\otimes\mathbb{C}"] \arrow[dr, "1"'] & X \otimes \mathbb{C}^2 \arrow[d, "x\otimes\mathbb{C}"] \arrow[dl, phantom, "\Nearrow \chi^J_X\otimes\mathbb{C}"] \\
 & X \otimes \mathbb{C} \arrow[d, "x"] \\
 & X
\end{tikzcd}$$

Definition 7.11. Given right $\mathbb{C}$-modules (X, x) and (Y, y), a *map of* $\mathbb{C}$*-modules* from X to Y consists of a 1-cell $f : X \to Y$ together with an invertible 2-cell

$$\begin{tikzcd}
X \otimes \mathbb{C} \arrow[r, "f \otimes \mathbb{C}"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi^f"] & Y \otimes \mathbb{C} \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

such that

$$\begin{tikzcd}[column sep=small]
& X \otimes \mathbb{C}^2 \arrow[rr, "f\otimes\mathbb{C}^2"] \arrow[dl, "X\otimes P"'] \arrow[dr, "X\otimes\mathbb{C}"] & & Y \otimes \mathbb{C}^2 \arrow[dr, "y\otimes\mathbb{C}"] \arrow[d, phantom, "\Nearrow \phi^f\otimes\mathbb{C}"] & \\
X \otimes \mathbb{C} \arrow[rr, phantom, "\underset{\chi^P_X}{\Rightarrow}"] \arrow[dr, "x"'] & & X \otimes \mathbb{C} \arrow[rr, "f\otimes\mathbb{C}"] \arrow[dl, "x"] \arrow[dr, phantom, "\Nearrow \phi^f"] & & Y \otimes \mathbb{C} \arrow[dl, "y"] \\
& X \arrow[rr, "f"'] & & X &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& X \otimes \mathbb{C}^2 \arrow[rr, "f\otimes\mathbb{C}^2"] \arrow[dl, "X\otimes P"'] \arrow[dr, phantom, "\sim"] & & Y \otimes \mathbb{C}^2 \arrow[dl, "Y\otimes P"'] \arrow[dr, "y\otimes\mathbb{C}"] & \\
X \otimes \mathbb{C} \arrow[rr, "f\otimes\mathbb{C}"] \arrow[dr, "x"'] \arrow[drr, phantom, "\Nearrow \phi^f"] & & Y \otimes \mathbb{C} \arrow[rr, phantom, "\underset{\chi^P_Y}{\Rightarrow}"] \arrow[dr, "y"] & & X \otimes \mathbb{C} \arrow[dl, "y"] \\
& X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \otimes \mathbb{I} \arrow[r, "f \otimes \mathbb{I}"] \arrow[d, "X \otimes J"'] \arrow[dr, phantom, "\sim"] \arrow[dd, bend right=80, "1"'] & Y \otimes \mathbb{I} \arrow[d, "Y \otimes J"] \\
X \otimes \mathbb{C} \arrow[r, "f \otimes \mathbb{C}"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi^f"] \arrow[l, phantom, "\underset{\chi^J_X}{\Rightarrow}"] & Y \otimes \mathbb{C} \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
X \otimes \mathbb{I} \arrow[r, "f \otimes \mathbb{I}"] \arrow[dd, "1"] & Y \otimes \mathbb{I} \arrow[dd, "1"'] \arrow[dr, "Y \otimes J"] & \\
& \arrow[r, phantom, "\underset{\chi^J_Y}{\Rightarrow}"] & Y \otimes \mathbb{C} \arrow[dl, "y"] \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

Definition 7.12. The bicategory $\mathrm{Mod}_{\mathbb{C}}$ of right $\mathbb{C}$-modules is defined as follows. An object is a right $\mathbb{C}$-module, a 1-cell is a map of right $\mathbb{C}$-modules, and a 2-cell γ :

$f \Rightarrow g : X \to Y$ is a 2-cell $f \Rightarrow g$ in $\mathcal{B}$ such that

$$
\begin{tikzcd}
X \otimes \mathbb{C} \arrow[r, "f \otimes \mathbb{C}"'] \arrow[r, bend left=50, "g \otimes \mathbb{C}"] \arrow[r, phantom, bend left=25, "\Uparrow \gamma \otimes \mathbb{C}"] \arrow[d, "x"'] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow \phi^f"] & Y \otimes \mathbb{C} \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
X \otimes \mathbb{C} \arrow[r, "g \otimes \mathbb{C}"] \arrow[d, "x"'] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow \phi^g"] & Y \otimes \mathbb{C} \arrow[d, "y"] \\
X \arrow[r, "g"] \arrow[r, bend right=50, "f \otimes \mathbb{C}"'] \arrow[r, phantom, bend right=25, "\Uparrow \gamma"] & Y
\end{tikzcd}
$$

Composition is defined as in $\mathcal{B}$, and given $f : X \to Y$ and $g : Y \to Z$, the 2-cell ϕ^{gf} is the pasting

$$
\begin{tikzcd}
X \otimes \mathbb{C} \arrow[r, "f \otimes \mathbb{C}"] \arrow[d, "x"'] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow \phi^f"] & Y \otimes \mathbb{C} \arrow[r, "g \otimes \mathbb{C}"] \arrow[d, "y"] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow \phi^g"] & Z \otimes \mathbb{C} \arrow[d, "z"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

which is easily verified to satisfy the necessary equations.

7.2.2 Modules via Components

The calculus of components is very useful here. We shall consider what the definitions mean in terms of components, which in practice is essentially the same thing as considering what they imply in the case $\mathcal{B} = \mathrm{Cat}$.

A right $\mathbb{C}$-module consists of an object $\mathbb{X}$ together with a functor

$$x : \mathbb{X} \otimes \mathbb{C} \to \mathbb{X};$$

we shall write this functor in infix form as $\bullet$. There are also invertible 2-cells with components

$$\alpha^{\bullet}_{X,A,B} : X \bullet (A \otimes B) \to (X \bullet A) \bullet B$$

and

$$\rho^{\bullet}_{X} : X \bullet I \to X$$

where we take X to be a (generic) element of $\mathbb{X}$, and A, B to be elements of $\mathbb{C}$. These are just the 2-cells that were denoted χ^P and χ^J above. The coherence conditions say that the diagrams

$$\begin{tikzcd}
X \bullet (A \otimes (B \otimes C)) \arrow[r, "\alpha^\bullet"] \arrow[d, "X \bullet \alpha"'] & (X \bullet A) \bullet (B \otimes C) \arrow[r, "\alpha^\bullet"] & ((X \bullet A) \bullet B) \bullet C \\
X \bullet ((A \otimes B) \otimes C) \arrow[rr, "\alpha^\bullet"'] & & (X \bullet (A \otimes B)) \bullet C \arrow[u, "\alpha^\bullet \bullet C"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \bullet (I \otimes A) \arrow[rr, "\alpha^\bullet"] \arrow[dr, "X \bullet \lambda_A"'] & & (X \bullet I) \bullet A \arrow[dl, "\rho^\bullet_X \bullet A"] \\
& X \bullet A &
\end{tikzcd}$$

must commute, for all $X \in \mathbb{X}$ and $A, B, C \in \mathbb{C}$. It is clear that, in particular, $\mathbb{C}$ itself is a right $\mathbb{C}$-module, when equipped in the obvious way.

Now, if we have two right $\mathbb{C}$-modules $\mathbb{X}$ and $\mathbb{Y}$, a map of modules $F : \mathbb{X} \to \mathbb{Y}$ is a 1-cell F equipped with an invertible 2-cell with components

$$F^\bullet_{X,A} : F(X \bullet A) \to FX \bullet A$$

such that the diagrams

$$\begin{tikzcd}
F(X \bullet (A \otimes B)) \arrow[rr, "F^\bullet_{X,A\otimes B}"] \arrow[d, "F(\alpha^\bullet_{X,A,B})"'] & & FX \bullet (A \otimes B) \arrow[d, "\alpha^\bullet_{FX,A,B}"] \\
F((X \bullet A) \bullet B) \arrow[r, "F^\bullet_{X \bullet A,B}"'] & F(X \bullet A) \bullet B \arrow[r, "F^\bullet_{X,A} \bullet B"'] & (FX \bullet A) \bullet B
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(X \bullet I) \arrow[rr, "F^\bullet_{X,I}"] \arrow[dr, "F(\rho^\bullet_X)"'] & & FX \bullet I \arrow[dl, "\rho^\bullet_{FX}"] \\
& FX &
\end{tikzcd}$$

commute.

Finally, given maps of modules F and $G : \mathbb{X} \to \mathbb{Y}$, a module 2-cell is given by a

2-cell $\gamma : F \to G$ such that the diagram of components

$$\begin{tikzcd} F(X \bullet A) \arrow[r, "F^\bullet_{X,A}"] \arrow[d, "\gamma_{X\bullet A}"'] & FX \bullet A \arrow[d, "\gamma_X \bullet A"] \\ G(X \bullet A) \arrow[r, "G^\bullet_{X,A}"'] & GX \bullet A \end{tikzcd}$$

commutes for all $X \in X$ and $A \in \mathbb{C}$.

7.2.3 External Cayley for Pseudomonoids

We shall use the component presentation to prove the External Cayley theorem.

Proposition 7.13 (External Cayley for Pseudomonoids). *For every right $\mathbb{C}$-module $\mathbb{X}$, the functor*

$$\phi_{\mathbb{X}} : \mathrm{Mod}_{\mathbb{C}}(\mathbb{C}, \mathbb{X}) \to \mathcal{B}(\mathbb{I}, \mathbb{X})$$

defined by $\phi_{\mathbb{X}}(F) = F(I)$ (and in the obvious way on 2-cells) is an equivalence of categories, with equivalence-inverse $\phi'_{\mathbb{X}} : \mathcal{B}(\mathbb{I}, \mathbb{X}) \to \mathrm{Mod}_{\mathbb{C}}(\mathbb{C}, \mathbb{X})$ defined as

$$\phi'_{\mathbb{X}}(X()) = X() \bullet -$$

and made into a map of modules by the invertible 2-cell $\alpha^\bullet_{X(),-,-}$.

Proof. Let $F : \mathbb{C} \to \mathbb{X}$ be a map of $\mathbb{C}$-modules; then $\phi'_{\mathbb{X}}(\phi_{\mathbb{X}}(F)) = F(I) \bullet -$ which, by the natural isomorphism $F^\bullet_{I,-}$, is naturally isomorphic to $F(I \otimes -)$ which, by the right-unit isomorphism of $\mathbb{C}$, is naturally isomorphic to F.

Let $X()$ (in component notation) be a 1-cell $\mathbb{I} \to \mathbb{X}$. Then

$$\phi_{\mathbb{X}}(\phi'_{\mathbb{X}}(X())) = X() \bullet I,$$

which is naturally isomorphic to $X()$ by the natural isomorphism $\rho^\bullet_X$. □

Proposition 7.14. *The functor $\phi_{\mathbb{C}} : \mathrm{Mod}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \to \mathcal{B}(\mathbb{I}, \mathbb{C})$ can be equipped with the structure of a strong monoidal functor, with respect to the natural monoidal structures on the categories $\mathrm{Mod}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ and $\mathcal{B}(\mathbb{I}, \mathbb{C})$.*

Proof. The unit for the monoidal structure of $\mathrm{Mod}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is of course the identity map, and $\phi_{\mathbb{C}}(1) = 1(I) \cong I$. For the 'tensor' part of the monoidal structure of $\phi_{\mathbb{C}}$, we take the composite

$$G(F(I)) \xrightarrow[\cong]{G(\lambda^{-1}_{FI})} G(I \otimes FI) \xrightarrow[\cong]{G^\bullet_{I,FI}} GI \otimes FI.$$

The unit condition holds by the triangle in the definition of map of modules, so it remains to show the associativity condition. Let F, G, and H be module maps $\mathbb{C} \to \mathbb{C}$, and consider the diagram

$$
\begin{tikzcd}
HGFI \arrow[r, "H\lambda_{GFI}"] \arrow[d, "HG\lambda_{FI}"'] & H(I \otimes GFI) \arrow[r, "H^{\bullet}_{I,GFI}"] \arrow[d, "H(I \otimes G\lambda_{FI})"] & HI \otimes GFI \arrow[d, "HI \otimes G\lambda_{FI}"] \\
HG(I \otimes FI) \arrow[r, "H\lambda_{G(I\otimes FI)}"'] \arrow[d, "H(G^{\bullet}_{I,FI})"'] & H(I \otimes G(I \otimes FI)) \arrow[r, "H^{\bullet}_{I,G(I\otimes FI)}"'] \arrow[d, "H(I \otimes G^{\bullet}_{I,FI})"] & HI \otimes G(I \otimes FI) \arrow[d, "HI \otimes G^{\bullet}_{I,FI}"] \\
H(GI \otimes FI) \arrow[r, "H\lambda_{GI\otimes FI}"] \arrow[d, "H^{\bullet}_{GI,FI}"'] \arrow[dr, "H(\lambda_{GI} \otimes FI)"'] & H(I \otimes (GI \otimes FI)) \arrow[r, "H^{\bullet}_{I,GI\otimes FI}"'] \arrow[d, "H(\alpha_{I,GI,FI})"] & HI \otimes (GI \otimes FI) \arrow[dd, "\alpha_{HI,GI,FI}"] \\
HGI \otimes FI \arrow[dr, "H(\lambda_{GI}) \otimes FI"'] & H((I \otimes GI) \otimes FI) \arrow[d, "H^{\bullet}_{I\otimes GI,FI}"] & \\
& H(I \otimes GI) \otimes FI \arrow[r, "H^{\bullet}_{I,GI} \otimes FI"'] & (HI \otimes GI) \otimes FI
\end{tikzcd}
$$

where the quadrilaterals commute by naturality, the triangle by definition of pseudomonoid, and the pentagonal region by the definition of morphism of modules.

□

So we have shown that, for any pseudomonoid $\mathbb{C} \in \mathcal{B}$, the monoidal category $\mathcal{B}(\mathbb{I}, \mathbb{C})$ is monoidally equivalent to $\mathrm{Mod}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$. In the case $\mathcal{B} = \mathrm{Cat}$, this shows that every monoidal category is monoidally equivalent to a *strict* monoidal category. However we are more interested in the case $\mathcal{B} = \mathrm{Prof}$, and some of the implications in this case are explored in the next chapter.

Chapter 8

Linear Logic without Units

In this, the title chapter, we at last address the question that we set out to answer at the outset. We consider the notion of model described in the introduction, and bring to bear all the machinery of earlier chapters to find a simple formulation of it.

8.1 Promonoidal categories

As we have foreshadowed, we are mainly interested in the monoidal bicategory Prof of categories and profunctors. Its objects are categories, and a 1-cell $\mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{D}$ is a profunctor from $\mathbb{C}$ to $\mathbb{D}$; which is to say, a functor $\mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathrm{Set}$. (Note that some authors use the converse direction, so what we call a profunctor from $\mathbb{C}$ to $\mathbb{D}$ they would call a profunctor from $\mathbb{D}$ to $\mathbb{C}$. There are good arguments in favour of both choices, and we have simply chosen the one that is easiest for our purposes.)

Profunctors are composed by a convolution operation, where the composite

$$\mathbb{B} \xrightarrow{F} \mathbb{C} \xrightarrow{G} \mathbb{D}$$

is defined by the coend formula

$$GF(B,D) = \int^{C} F(B,C) \times G(C,D).$$

The 2-cells of Prof are just ordinary natural transformations. It is easy enough to check that Prof is a bicategory. The monoidal structure is more subtle, and although it has long been assumed that Prof is a symmetric monoidal bicategory, the first rigorous published proof was given very recently by Carboni et al. (2007). The monoidal structure is given by the ordinary cartesian product of categories.

A *promonoidal category* is a pseudomonoid in Prof. More concretely, a promonoidal category consists of a category $\mathbb{C}$ equipped with profunctors

$$P : \mathbb{C} \times \mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{C}$$

and

$$J : 1 \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{C}$$

together with natural isomorphisms with components

$$\mathfrak{a}_{A,B,C,D} : \int^X P(A,X;D) \times P(B,C;X) \to \int^X P(A,B;X) \times P(X,C;D),$$

$$\mathfrak{l}_{A,B} : \int^X J(X) \times P(X,A;B) \to \mathbb{C}(A,B),$$

and

$$\mathfrak{r}_{A,B} : \int^X J(X) \times P(A,X;B) \to \mathbb{C}(A,B),$$

satisfying the pseudomonoid axioms.

A functor $F : \mathbb{C} \to \mathbb{D}$ induces profunctors $\mathbb{C}(-, F=) : \mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{D}$ and $\mathbb{C}(F-, =) : \mathbb{D} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{C}$. These extend in an obvious way to bijective-on-objects embeddings

$$\mathrm{Cat} \to \mathrm{Prof}$$

and

$$\mathrm{Cat}^{\mathrm{op}} \to \mathrm{Prof},$$

which (by Yoneda) are locally full and faithful. In particular, the covariant embedding gives every monoidal category the structure of a promonoidal category in a natural way. Concretely, one takes $P(A,B;C) = \mathbb{C}(A \otimes B, C)$ and $J(A) = \mathbb{C}(I, A)$.

8.1.1 Braiding and symmetry

Clearly a braided promonoidal category must be a braided pseudomonoid in Prof: that is, a promonoidal category $\mathbb{C}$ additionally equipped with a natural isomorphism with components

$$\sigma_{A,B,C} : P(A,B;C) \to P(B,A;C)$$

satisfying the braiding axioms.

We have not given a general definition of symmetric pseudomonoid, because to do so would require venturing into the deep waters of *sylleptic* monoidal bicategories. However we can easily say what it means for a braided promonoidal category to be symmetric; we simply demand that

$$\sigma_{A,B,C} = \sigma^{-1}_{B,A,C}.$$

It is clear that, in the symmetric case, each of the two braiding axioms implies the other.

8.2 Modelling linear logic without units

As explained in the introduction, the appropriate structures for modelling the unitless fragment of multiplicative intuitionistic linear logic are promonoidal categories where the multiplication P is represented by a functor, but the unit J generally is not.

Definition 8.1. A semi symmetric monoidal category (semi SMC) consists of a category $\mathbb{C}$ equipped with functors

$$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

and

$$J : \mathbb{C} \to \mathrm{Set}$$

and natural isomorphisms with components

$$\alpha_{A,B,C} : A \otimes (B \otimes C),$$

$$\sigma_{A,B} : A \otimes B \to B \otimes A,$$

and

$$\lambda_{A,B} : \int^X J(X) \times \mathbb{C}(X \otimes A, B) \to \mathbb{C}(A, B)$$

satisfying the axioms for a symmetric promonoidal category. The associativity (pentagon) and symmetry (hexagon) axioms do not involve the unit, so are the same as for an ordinary symmetric monoidal category. In detail, the diagrams

$$\begin{tikzcd}
A \otimes \big(B \otimes (C \otimes D)\big) \arrow[r, "\alpha"] \arrow[d, "A \otimes \alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & \big((A \otimes B) \otimes C\big) \otimes D \\
A \otimes \big((B \otimes C) \otimes D\big) \arrow[rr, "\alpha"'] & & \big(A \otimes (B \otimes C)\big) \otimes D \arrow[u, "\alpha \otimes D"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha"] \arrow[d, "A \otimes \sigma"'] & (A \otimes B) \otimes C \arrow[r, "\sigma"] & C \otimes (A \otimes B) \arrow[d, "\alpha"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha"'] & (A \otimes C) \otimes B \arrow[r, "\sigma \otimes B"'] & (C \otimes A) \otimes B
\end{tikzcd}$$

must commute. The unit axiom says that the diagram

$$\begin{tikzcd}
\int^X J(X) \times \mathbb{C}((X \otimes B) \otimes C, Z) \arrow[r, "\int^X J(X) \times \mathbb{C}(\alpha_{X,B,C}, Z)"] \arrow[d, "\cong"'] & \int^X J(X) \times \mathbb{C}(X \otimes (B \otimes C), Z) \arrow[dd, "\lambda_{B \otimes C, Z}"] \\
\int^{X,Y} J(X) \times \mathbb{C}(X \otimes B, Y) \times \mathbb{C}(Y \otimes C, Z) \arrow[d, "\int^Y \lambda_{B,Y} \times \mathbb{C}(Y \otimes C, Z)"'] & \\
\int^Y \mathbb{C}(B, Y) \times \mathbb{C}(Y \otimes C, Z) \arrow[r, "\cong"'] & \mathbb{C}(B \otimes C, Z)
\end{tikzcd}$$

must commute for all $B,C,Z \in \mathbb{C}$.

Definition 8.2. A semi symmetric monoidal closed category (semi SMCC) is a semi SMC $\mathbb{C}$ together with a functor

$$\multimap : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$$

and a natural isomorphism with components

$$\mathrm{curry}_{A,B,C} : \mathbb{C}(A \otimes B, C) \to \mathbb{C}(A, B \multimap C).$$

The purpose of this chapter is to find a simpler formulation of the notion of semi SMCC. We should like to formulate the notion in a way that does not involve coends, for one thing. In fact we give two such formulations, which are superficially very different. Before delving into the details, we give a summary of our two presentations.

8.2.1 The presentation via linear elements

There is an obvious notion of *unitless SMCC*, essentially obtained taking the ordinary definition of SMCC and erasing those parts that mention the unit object. In the introduction, we saw that unitless SMCCs are not adequate to model the unitless fragment of IMLL, because of the need to represent sequents whose left-hand side is empty. However, it turns out that the gap between unitless SMCCs and semi SMCCs is smaller than it seems: surprisingly, no additional structure is actually required! In fact a semi SMCC can be described as a unitless SMCC satisfying a certain *property*. Here we briefly describe the property in question.

Given a unitless SMCC $\mathbb{C}$, and an object $A \in \mathbb{C}$, let us define a *linear element* of A to be a natural transformation with components

$$\gamma_X : X \to A \otimes X$$

such that, for all X and $Y \in \mathbb{C}$, the diagram

$$\begin{tikzcd} & X \otimes Y \arrow[dl, "\gamma_{X\otimes Y}"'] \arrow[dr, "\gamma_X \otimes Y"] & \\ A \otimes (X \otimes Y) \arrow[rr, "\alpha_{A,X,Y}"'] & & (A \otimes X) \otimes Y \end{tikzcd}$$

commutes. It's easy to see that, in an ordinary monoidal category, the linear elements of A are in bijective correspondence with morphisms $I \to A$. Now, there is an obvious functor

$$\mathrm{Lin} : \mathbb{C} \to \mathrm{Set}$$

that takes each object $A \in \mathbb{C}$ to the set of linear elements on $\mathbb{C}$. (In an ordinary monoidal category, it is isomorphic to the hom functor $\mathbb{C}(I, -)$.) Furthermore, there is a canonical natural transformation with components

$$\ell_{A,B} : \mathrm{Lin}(A \multimap B) \to \mathbb{C}(A, B);$$

if γ is a linear element of $A \multimap B$, then $\ell_{A,B}(\gamma)$ is just the composite

$$A \xrightarrow{\gamma_A} (A \multimap B) \otimes A \xrightarrow{\mathrm{ev}^A_B} B$$

where ev^A is the counit of the adjunction $- \otimes A \dashv A \multimap -$.

Our additional condition, then, is as follows: a unitless SMCC $\mathbb{C}$ can be regarded as a semi SMCC just when the natural transformation ℓ is invertible, so that every arrow $A \to B$ comes from a unique linear element of $A \multimap B$. It is easy to see that this condition is satisfied when $\mathbb{C}$ has a unit object, and in Section 8.3 we shall prove that in fact it is satisfied if and only if $\mathbb{C}$ has a promonoidal unit.

8.2.2 The 'ψ' presentation

Our second presentation is given in terms of structure rather than properties. We show that to give a semi SMCC is to give a category $\mathbb{C}$ equipped with an associative, symmetric[1] functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, a functor $\multimap : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ through which hom factors up to isomorphism,

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[r, "\multimap"] \arrow[dr, "\mathrm{hom}"'] & \mathbb{C} \arrow[d, dotted, "\exists J"] \arrow[dl, phantom, "\cong"] \\
 & \mathrm{Set}
\end{tikzcd}$$

and a natural isomorphism

$$\psi_{A,B,C} \quad : \quad (A \otimes B) \multimap C \quad \to \quad A \multimap (B \multimap C)$$

such that the diagram

$$\begin{tikzcd}
\big(A \otimes (B \otimes C)\big) \multimap D \arrow[r, "\alpha_{A,B,C} \multimap D"] \arrow[dd, "\psi_{A,B\otimes C,D}"'] & \big((A \otimes B) \otimes C\big) \multimap D \arrow[d, "\psi_{A\otimes B,C,D}"] \\
 & (A \otimes B) \multimap (C \multimap D) \arrow[d, "\psi_{A,B,C\multimap D}"] \\
A \multimap \big((B \otimes C) \multimap D\big) \arrow[r, "A \multimap \psi_{B,C,D}"'] & A \multimap \big(B \multimap (C \multimap D)\big)
\end{tikzcd}$$

commutes.

8.3 The promonoidal unit

Although we are primarily interested in semi SMCCs, the basic argument here works in any braided promonoidal category, so we shall work at that level of generality and specialise to our intended application at the end.

The essential tool here is the Cayley Theorem for pseudomonoids: in the case $\mathcal{B} = \mathbf{Cat}$, it shows that every monoidal category is monoidally equivalent to a *strict*

[1] *I.e.*, equipped with the standard associativity and symmetry isomorphisms and coherence laws of a symmetric monoidal category.

monoidal category. In the case $\mathcal{B} = \mathbf{Prof}$, which is the case we're interested in here, it allows us to construct a canonical representation for the promonoidal unit. (This turns out to be particularly powerful in the braided, or symmetric, case.)

In summary, the story goes as follows. It is a familiar fact that, in a semigroup, all units must be equal: if i and j are both units, then $i = ij = j$. The existence or otherwise of a unit is really a *property* of a semigroup; there is no choice of how to define the unit. By a similar argument, in a pseudomonoid (or 'pseudosemigroup', a pseudomonoid without the unit structure) all units are isomorphic. Being good category theorists, we don't care about the difference between isomorphic structures; so we have no real choice in how to define the unit of a pseudomonoid: given the tensor structure (P and $\mathfrak{a}$), either it is possible to define a unit, or it isn't. We might say that the unit is 'essentially property-like' (cf. Kelly and Lack, 1997). This point may seem rather irrelevant, since the easiest way to demonstrate the existence of a unit is often to exhibit one. However, it turns out that in the case of promonoidal categories that is not necessarily true. There is a canonical representative for the unit – the functor here denoted Lin – defined solely in terms of the tensor structure (P and α) that is isomorphic to the unit whenever there is a unit to be isomorphic to. Furthermore, in the case of braided or symmetric promonoidal categories, we can define a simple test of whether or not there is a unit. So we may *define* a braided promonoidal category purely in terms of the tensor, associator and braiding, subject to a condition that determines the existence of a unit. Should an actual unit be required, we are free to use the canonical unit Lin.

Now for the details. Let $\mathbb{C}$ be a promonoidal category. The monoidal category $\mathbf{Prof}(1, \mathbb{C})$ is $[\mathbb{C}, \mathrm{Set}]$, equipped with Day's convolution tensor. The monoidal category $\mathrm{Mod}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ does not really have a simpler description than the general one given above: an object is a profunctor $\mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{C}$ together with a natural transformation

$$\begin{tikzcd}
\mathbb{C} \times \mathbb{C} \arrow[r, "P", "\shortmid" marking] \arrow[d, "F \times \mathbb{C}"', "\shortmid" marking] \arrow[dr, phantom, "\swarrow \phi^F"] & \mathbb{C} \arrow[d, "F"] \\
\mathbb{C} \times \mathbb{C} \arrow[r, "P"', "\shortmid" marking] & \mathbb{C}
\end{tikzcd}$$

satisfying the conditions of Definition 7.3. So ϕ^F is a natural isomorphism with components

$$\phi^F_{A,B,C} : \int^X P(A, B; X) \times F(X; C) \to \int^X F(A; X) \times P(X, B; C).$$

Now let us write $\mathbb{E}$ as an abbreviation for the monoidal category $\mathrm{Mod}_{\mathbb{C}}(\mathbb{C},\mathbb{C})$, and let $A \in \mathbb{C}$. We have a sequence of natural isomorphisms

$$\begin{array}{rcll} JA & \cong & [\mathbb{C},\mathrm{Set}](\mathbb{C}(A,-),J) & \text{by Yoneda} \\ & \cong & \mathbb{E}(\phi'_{\mathbb{C}}(\mathbb{C}(A,-)),\phi'_{\mathbb{C}}(J)) & \text{since } i_{\mathbb{C}} \text{ is full and faithful} \\ & \cong & \mathbb{E}(\phi'_{\mathbb{C}}(\mathbb{C}(A,-)),I) & \text{applying } m_I \end{array}$$

where $\phi'_{\mathbb{C}}$ is the monoidal equivalence $[\mathbb{C},\mathrm{Set}] \to \mathbb{E}$ described by Proposition 7.13. If we define the functor $\mathrm{Lin} : \mathbb{C} \to \mathrm{Set}$ as $\mathrm{Lin}(A) := \mathbb{E}(\phi'_{\mathbb{C}}(\mathbb{C}(A,-)),I)$, then we have exhibited a natural isomorphism $\theta : J \Rightarrow \mathrm{Lin}$. An element of the set $\mathrm{Lin}(A)$ is a natural transformation γ with components

$$\gamma_{X,Y} : P(A,X;Y) \to \mathbb{C}(X,Y)$$

such that the diagram

$$\begin{tikzcd} \int^X P(L,M;X) \times P(A,X;N) \arrow[rr, "\int^X P(L,M;X) \times \gamma_{X,N}"] \arrow[dd, "\alpha_{A,L,M}"'] & & \int^X P(L,M;X) \times \mathbb{C}(X,N) \arrow[d, "\cong"] \\ & & P(L,M;N) \\ \int^X P(A,L;X) \times P(X,M;N) \arrow[rr, "\int^X \gamma_{L,X} \times P(X,M;N)"'] & & \int^X \mathbb{C}(L,X) \times P(X,M;N) \arrow[u, "\cong"'] \end{tikzcd} \tag{8.3.1}$$

commutes for all L, M and N in $\mathbb{C}$. We shall refer to the elements of this set as the *linear elements* of A.

For the purposes of this thesis, of course, we're particularly interested in the case where the profunctor P is represented by a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. In this case, a linear element is a natural transformation with components

$$\gamma_{X,Y} : \mathbb{C}(A \otimes X, Y) \to \mathbb{C}(X,Y)$$

which, by Yoneda, can be represented by a natural transformation with components

$$\gamma_X : X \to A \otimes X.$$

With this representation, the condition boils down to the requirement that the diagram

$$\begin{tikzcd} & X \otimes Y \arrow[dl, "\gamma_{X\otimes Y}"'] \arrow[dr, "\gamma_X \otimes Y"] & \\ A \otimes (X \otimes Y) \arrow[rr, "\alpha_{A,X,Y}"'] & & (A \otimes X) \otimes Y \end{tikzcd}$$

should commute, for all X, $Y \in \mathbb{C}$.

Since an element of $\mathrm{Lin}(A)$ is a natural transformation

$$\gamma_{X,Y} : P(A, X; Y) \to \mathbb{C}(X, Y),$$

for every A there is an obvious natural transformation λ'^A with components

$$\lambda'^A_{X,Y} : \mathrm{Lin}(A) \times P(A, X; Y) \to \mathbb{C}(X, Y),$$

dinatural in A.

Proposition 8.3. *The diagram*

$$\begin{tikzcd} JA \times P(A, X; Y) \arrow[dd, "\theta_A \times P(A,X;Y)"'] \arrow[dr, "\lambda^A_{X,Y}"] & \\ & \mathbb{C}(X, Y) \\ \mathrm{Lin}A \times P(A, X; Y) \arrow[ur, "\lambda'^A_{X,Y}"'] & \end{tikzcd}$$

commutes, for all A, X and $Y \in \mathbb{C}$.

Proof. This is a direct consequence of the definition of θ: any apparent complexity here is notational rather than mathematical. First we shall calculate the effect of $\theta_A : JA \to \mathrm{Lin}A$ on an element $e \in JA$. We'll consider in sequence the three isomorphisms that define θ. The first takes e to the natural transformation with X-component

$$(f : A \to X) \mapsto J(f)(e);$$

this must then be mapped by the second to a natural transformation

$$P(A, X; Y) \to \int^Z JZ \times P(Z, X; Y)$$

natural in X and Y. The elements of $\int^Z JZ \times P(Z, X; Y)$ are equivalence classes of pairs $\langle j, p\rangle$, with $j \in JZ$ and $p \in P(Z, X; Y)$ for some $Z \in \mathbb{C}$. Our element is mapped to the A-indexed natural family of functions

$$(p \in P(A, X; Y)) \mapsto [\langle e, p\rangle].$$

where $[\langle e, p\rangle]$ denotes the equivalence class containing $\langle e, p\rangle$. The final natural isomorphism takes this element to the A-indexed natural family of functions

$$(p \in P(A, X; Y)) \mapsto \lambda_{X,Y}([\langle e, p\rangle]).$$

Now it's easy to show that the diagram commutes: let $\langle e, p\rangle$ be an element of $JA \times P(A, X; Y)$. The vertical arrow maps it to the pair $\langle f, p\rangle$, where f is the linear element displayed above. λ' then maps this to $\lambda^A_{X,Y}(e, p)$, as required. □

There is an apparent asymmetry here: although it was easy to define λ', there is no obvious way to define a corresponding ρ' – unless of course our promonoidal category is braided, of which more below. This asymmetry derives from the fact that $\mathbb{E}$ is defined using *right* $\mathbb{C}$-modules, and of course it would be possible to define a dual version using left modules, which would also be monoidally isomorphic to $\mathrm{Prof}(1, \mathbb{C})$, hence to $\mathbb{E}$. Using this, we could define a 'co-linear elements' functor $\mathrm{Lin}' : \mathbb{C} \to \mathrm{Set}$, with a canonical natural isomorphism

$$\rho'^A_{X,Y} : \mathrm{Lin}'A \times P(X, A; Y) \to \mathbb{C}(X, Y).$$

However, in general there is no canonical natural isomorphism between Lin and Lin′. In the braided case, there is. So in that case we may simply take Lin (or equivalently Lin′) to be the unit, which role it is able to fulfil if and only if λ' (equivalently ρ') is invertible. More formally we have:

Proposition 8.4. *Let $\mathbb{C}$ be a category, and $P : \mathbb{C} \times \mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{C}$ a profunctor. Let α be an associator satisfying the pentagon condition, and let σ be a braiding satisfying the hexagon conditions. There exists a unit $J : 1 \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{C}$ (with coherent unit isomorphisms λ and ρ) if and only if the natural transformation*

$$\lambda' : \int^A \mathrm{Lin}A \times P(A, X; Y) \to \mathbb{C}(X, Y),$$

defined above, is invertible.

Proof. We have already established the 'only if' direction, so let $\mathbb{C}$, P, α and σ be given, and define $\mathrm{Lin} : \mathbb{C} \to \mathrm{Set}$ and λ'. Suppose that λ' is invertible.

By the presentation of braided pseudomonoids given in Section 6.5, it suffices to show that

$$
\begin{tikzcd}
1\times\mathbb{C}^2 \arrow[r, "\mathrm{Lin}\times\mathbb{C}^2", "\shortmid"{marking}] \arrow[d, "1\times P"', "\shortmid"{marking}] \arrow[dr, phantom, "\sim"] & \mathbb{C}^3 \arrow[d, "\mathbb{C}\times P", "\shortmid"{marking}] \\
1\times\mathbb{C} \arrow[r, "\mathrm{Lin}\times\mathbb{C}", "\shortmid"{marking}] \arrow[dr, bend right, "1"'] \arrow[dr, phantom, "\lambda'"] & \mathbb{C}^2 \arrow[d, "P", "\shortmid"{marking}] \\
& \mathbb{C}
\end{tikzcd}
=
\begin{tikzcd}
1\times\mathbb{C}^2 \arrow[r, "\mathrm{Lin}\times\mathbb{C}^2", "\shortmid"{marking}] \arrow[dr, bend right, "1"'] \arrow[dr, phantom, "\lambda'\times\mathbb{C}"] & \mathbb{C}^3 \arrow[r, "\mathbb{C}\times P", "\shortmid"{marking}] \arrow[d, "P\times\mathbb{C}", "\shortmid"{marking}] \arrow[dr, phantom, "\alpha"] & \mathbb{C}^2 \arrow[d, "P", "\shortmid"{marking}] \\
& \mathbb{C}^2 \arrow[r, "P"', "\shortmid"{marking}] & \mathbb{C}
\end{tikzcd}
$$

Concretely, this amounts to showing that the diagram

$$
\begin{tikzcd}
\int^{A,X} P(A,X;N)\times \mathrm{Lin}A\times P(L,M;X) \arrow[r, "\int^X \lambda'_{X,N}\times P(L,M;X)"] \arrow[dd, "\int^A \mathrm{Lin}A\times\alpha_{A,L,M,N}"'] & \int^X \mathbb{C}(X,N)\times P(L,M;X) \arrow[d, "\cong"] \\
& P(L,M;N) \\
\int^{A,X} \mathrm{Lin}A\times P(A,L;X)\times P(X,M;N) \arrow[r, "\int^X \lambda'_{L,X}\times P(X,M;N)"'] & \int^X \mathbb{C}(L,X)\times P(X,M;N) \arrow[u, "\cong"']
\end{tikzcd}
$$

commutes, which is an immediate consequence of (8.3.1). □

8.3.1 Application to semi SMCCs

In the case where $\mathbb{C}$ is a semi SMCC, we have the isomorphism

$$\int^A \mathrm{Lin}A\times P(A,X;Y)\cong\int^A \mathrm{Lin}(A)\times\mathbb{C}(A,X\multimap Y)\cong \mathrm{Lin}(X\multimap Y),$$

and the natural transformation $\lambda'_{X,Y} : \mathrm{Lin}(X\multimap Y)\to\mathbb{C}(X,Y)$ is precisely the natural transformation that we called ℓ in Section 8.2.1. Therefore Proposition 8.4 does indeed justify the presentation described in Section 8.2.1.

8.4 The ψ presentation

In the closed case we have a functor $\multimap$ such that

$$P(A,B;C)\cong\mathbb{C}(A,B\multimap C),$$

The left-unit isomorphism of a promonoidal category has components

$$\lambda_{A,B} : \int^{X} J(X) \times P(X, A; B) \to \mathbb{C}(A, B),$$

the left-hand side of which is isomorphic to

$$\int^{X} J(X) \times \mathbb{C}(X, A \multimap B)$$

which in turn is isomorphic to $J(A \multimap B)$. So we can take λ to be an isomorphism

$$\lambda_{A,B} : J(A \multimap B) \to \mathbb{C}(A, B),$$

thereby eliminating the need to mention coends. The problem (if we want a fully coherent presentation) is to reconcile the fact that the associativity and symmetry are defined using $\otimes$, whereas the unit is defined using $\multimap$. Abstractly, we may consider that we have two isomorphic multiplication profunctors, say P and Q where

$$P(A, B; C) = \mathbb{C}(A \otimes B, C)$$

and

$$Q(A, B; C) = \mathbb{C}(A, B \multimap C),$$

with the associativity and symmetry isomorphisms defined on P, and the unit isomorphism defined on Q. Of course a unit isomorphism may be defined on P by using the isomorphism with Q, and the coherence condition for the unit expressed in terms of this composite. Here it simplifies matters to back up and address the question at the level of structure internal to a general monoidal bicategory. (The symmetry does not play an essential role in this part of the argument, so there is no need to assume a braiding here.) We shall use the language of components, and to make the notation more familiar we shall write $A \otimes B$ to mean $P(A, B)$ and $A \odot B$ to mean $Q(A, B)$. Also we'll write I to mean $J()$. So (symmetry aside) we have invertible 2-cells with components

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$

$$\lambda_A : I \odot A \to A,$$

and

$$\chi_{A,B} : A \otimes B \to A \odot B,$$

this last corresponding to the currying isomorphism. We assume that α satisfies the pentagon condition, and that the diagram of components

$$\begin{tikzcd}
I \otimes (A \otimes B) \arrow[rr, "\alpha_{I,A,B}"] \arrow[d, "\chi_{I,A\otimes B}"'] & & (I \otimes A) \otimes B \arrow[d, "\chi_{I,A} \otimes B"] \\
I \odot (A \otimes B) \arrow[r, "\lambda_{A\otimes B}"'] & A \otimes B \arrow[u, phantom, "(\boldsymbol{\lambda\alpha\chi})"] & (I \odot A) \otimes B \arrow[l, "\lambda_A \otimes B"]
\end{tikzcd}$$

commutes. Now, define ψ to be the unique invertible 2-cell with components

$$\psi_{A,B,C} : A \odot (B \otimes C) \to (A \odot B) \odot C,$$

such that the diagram

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[rr, "\alpha_{A,B,C}"] \arrow[d, "\chi_{A,B\otimes C}"'] & & (A \otimes B) \otimes C \arrow[d, "\chi_{A,B} \otimes C"] \\
A \odot (B \otimes C) \arrow[r, "\psi_{A,B,C}"'] & (A \odot B) \odot C \arrow[u, phantom, "(\boldsymbol{\alpha\chi\psi})"] & (A \odot B) \otimes C \arrow[l, "\chi_{A\odot B,C}"]
\end{tikzcd}$$

commutes. In the abstract this seems a rather odd thing to construct, but in our concrete example it corresponds (via Yoneda) to a natural isomorphism $(X \otimes Y) \multimap Z \cong X \multimap (Y \multimap Z)$, an internal analogue of currying. We shall consider the relationship between χ and ψ, with the aim of replacing the former by the latter.

Lemma 8.5. *If diagram* $(\boldsymbol{\alpha\chi\psi})$ *commutes, then diagram* $(\boldsymbol{\lambda\alpha\chi})$ *commutes if and only if the following does.*

$$\begin{tikzcd}
I \odot (A \otimes B) \arrow[r, "\psi_{I,A,B}"] \arrow[d, "\lambda_{A\otimes B}"'] \arrow[dr, phantom, "(\boldsymbol{\lambda\chi\psi})"] & (I \odot A) \odot B \arrow[d, "\lambda_A \odot B"] \\
A \otimes B \arrow[r, "\chi_{A,B}"'] & A \odot B
\end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd}
I \otimes (A \otimes B) \arrow[rr, "\alpha_{I,A,B}"] \arrow[dd, "\chi_{I,A\otimes B}"'] & & (I \otimes A) \otimes B \arrow[dd, "\chi_{I,A} \otimes B"] \\
& (I \odot A) \odot B \arrow[d, "\lambda_A \odot B"] & \\
I \odot (A \otimes B) \arrow[ur, "\psi_{I,A,B}"] \arrow[dr, "\lambda_{A\otimes B}"'] & A \odot B & (I \odot A) \otimes B \arrow[ul, "\chi_{I\odot A,B}"'] \arrow[dl, "\lambda_A \otimes B"] \\
& A \otimes B \arrow[u, "\chi_{A,B}"'] &
\end{tikzcd}$$

The upper region is an instance of ($\boldsymbol{\alpha\chi\psi}$), and the quadrilateral at lower-right commutes by naturality. Since all the components are invertible, it follows that the outside ($\boldsymbol{\lambda\alpha\chi}$) commutes if and only if the left-hand quadrilateral ($\boldsymbol{\lambda\chi\psi}$) does. □

Lemma 8.6. *If diagram* ($\boldsymbol{\alpha\chi\psi}$) *commutes, then so does*

$$\begin{tikzcd}
A \odot \big(B \otimes (C \otimes D)\big) \arrow[r, "\psi"] \arrow[d, "A \otimes \alpha"'] & (A \odot B) \odot (C \otimes D) \arrow[r, "\psi"] \arrow[d, phantom, "(\boldsymbol{\alpha\psi})"] & \big((A \odot B) \odot C\big) \odot D \\
A \odot \big((B \otimes C) \otimes D\big) \arrow[rr, "\psi"'] & & \big(A \odot (B \otimes C)\big) \odot D \arrow[u, "\psi \odot D"']
\end{tikzcd}$$

Proof. We can use ($\boldsymbol{\alpha\chi\psi}$) to show that ($\boldsymbol{\alpha\psi}$) is equivalent to the pentagon condition. Consider the diagram shown in Figure 8.1. The unlabelled arrows are constructed using repeated instances of χ: for example the vertical arrow

$$(A \otimes B) \otimes (C \otimes D) \to (A \odot B) \odot (C \otimes D)$$

is equal to the diagonal of the commutative square

$$\begin{tikzcd}
(A \otimes B) \otimes (C \otimes D) \arrow[r, "\chi_{A,B} \otimes (C \otimes D)"] \arrow[d, "\chi_{A\otimes B, C\otimes D}"'] & (A \odot B) \otimes (C \otimes D) \arrow[d, "\chi_{A\odot B, C\otimes D}"] \\
(A \otimes B) \odot (C \otimes D) \arrow[r, "\chi_{A,B} \odot (C \otimes D)"'] & (A \odot B) \odot (C \otimes D)
\end{tikzcd}$$

The cells that involve these arrows thus commute by a combination of naturality and condition ($\boldsymbol{\alpha\chi\psi}$). So the central pentagon commutes if and only if the outside does. □

$$
\begin{tikzcd}
A \odot \big(B \otimes (C \otimes D)\big) \arrow[rr, "\psi"] \arrow[ddd, "A \odot \alpha"'] & & (A \odot B) \odot (C \otimes D) \arrow[rr, "\psi"] & & \big((A \odot B) \odot C\big) \odot D \\
& A \otimes \big(B \otimes (C \otimes D)\big) \arrow[ul] \arrow[r, "\alpha"] \arrow[d, "A \otimes \alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[u] \arrow[r, "\alpha"] & \big((A \otimes B) \otimes C\big) \otimes D \arrow[ur] & \\
& A \otimes \big((B \otimes C) \otimes D\big) \arrow[rr, "\alpha"'] \arrow[dl] & & \big(A \otimes (B \otimes C)\big) \otimes D \arrow[u, "\alpha \otimes D"'] \arrow[dr] & \\
A \odot \big((B \otimes C) \otimes D\big) \arrow[rrrr, "\psi"'] & & & & \big(A \odot (B \otimes C)\big) \odot D \arrow[uuuu, "\psi \odot D"']
\end{tikzcd}
$$

Figure 8.1: Diagram used in the proof of Lemma 8.6

Lemma 8.7. *If* $(\boldsymbol{\lambda\chi\psi})$ *and* $(\boldsymbol{\alpha\psi})$ *commute, then so does* $(\boldsymbol{\alpha\chi\psi})$.

Proof. Consider the diagram shown in Figure 8.2. The regions marked with the symbol ♮ commute by naturality, the other three quadrilaterals commute by $(\boldsymbol{\lambda\chi\psi})$, and the outside is an instance of $(\boldsymbol{\alpha\psi})$. Thus the inner pentagonal region commutes, which is precisely $(\boldsymbol{\alpha\chi\psi})$. □

So χ and ψ are interdefinable: given χ, we can define ψ using diagram $(\boldsymbol{\alpha\chi\psi})$, and alternatively given ψ we can define χ using diagram $(\boldsymbol{\lambda\chi\psi})$. If we take ψ rather than χ to be part of our defining data, then it suffices to take $(\boldsymbol{\alpha\psi})$ as an additional axiom (in addition to the pentagon condition). Diagram $(\boldsymbol{\lambda\chi\psi})$ commutes by construction so, by Lemma 8.7, condition $(\boldsymbol{\alpha\chi\psi})$ holds; then by Lemma 8.5 condition $(\boldsymbol{\lambda\alpha\chi})$ holds too.

In the concrete case we're considering, this justifies the presentation of Section 8.2.2.

8.5 The star-autonomous Case

Finally, we consider full (non-intuitionistic) multiplicative linear logic. The appropriate notion of model (for the unitless fragment) is easy to find:

Definition 8.8. A *semi star-autonomous category* is a semi SMC $\mathbb{C}$ equipped with an equivalence $-^* : \mathbb{C} \to \mathbb{C}^{\mathrm{op}}$ and a natural isomorphism with components $\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(A, (B \otimes \mathbb{C}^*)^*)$.

We shall write $B \multimap C$ as an abbreviation for $(B \otimes C^*)^*$. The ψ presentation becomes remarkably simple in the star-autonomous case:

Proposition 8.9. *To give a semi star-autonomous category is to give a category* $\mathbb{C}$ *equipped with an associative, symmetric functor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, *an equivalence* $-^* : \mathbb{C} \to \mathbb{C}^{\mathrm{op}}$, *and a functor* $J : \mathbb{C} \to \mathrm{Set}$ *with a natural isomorphism*

$$J((A \otimes B^*)^*) \cong \mathbb{C}(A, B).$$

Proof. Define $\psi_{A,B,C}$ to be the composite

$$\big((A \otimes B) \otimes C^*\big)^* \xrightarrow{(\alpha_{A,B,C^*})^*} \big(A \otimes (B \otimes C^*)\big)^* \xrightarrow{(A \otimes n_{B \otimes C^*})^*} \Big(A \otimes \big((B \otimes C^*)^*\big)^*\Big)^*.$$

With this definition, the diagram $(\boldsymbol{\alpha\psi})$ commutes as a consequence of the pentagon condition. □

$$\begin{tikzcd}
I \odot \big(B \otimes (C \otimes D)\big) \arrow[rr, "\psi_{I,B,C\otimes D}"] \arrow[dr, "\lambda_{B\otimes(C\otimes D)}"] \arrow[ddd, "I \odot \alpha_{B,C,D}"'] & & (I \odot B) \odot (C \otimes D) \arrow[rr, "\psi_{I\odot B,C,D}"] \arrow[d, "\lambda_B \odot (C \otimes D)"'] \arrow[dr, phantom, "\natural_\psi"] & & \big((I \odot B) \odot C\big) \odot D \arrow[dl, "(\lambda_B \odot C) \odot D"'] \\
\arrow[dr, phantom, "\natural_\lambda"] & B \otimes (C \otimes D) \arrow[r, "\chi_{B,C\otimes D}"] \arrow[d, "\alpha"] & B \odot (C \otimes D) \arrow[r, "\psi_{B,C,D}"] & (B \odot C) \odot D & \\
& (B \otimes C) \otimes D \arrow[rr, "\chi_{B\otimes C,D}"'] & & (B \otimes C) \odot D \arrow[u, "\chi_{B,C} \odot D"'] & \\
I \odot \big((B \otimes C) \otimes D\big) \arrow[ur, "\lambda_{(B\otimes C)\otimes D}"] \arrow[rrrr, "\psi_{I,B\otimes C,D}"'] & & & & \big(I \odot (B \otimes C)\big) \odot D \arrow[ul, "\lambda_{B\otimes C} \odot D"'] \arrow[uuu, "\psi \odot D"']
\end{tikzcd}$$

Figure 8.2: Diagram used in the proof of Lemma 8.7

Chapter 9

Compact Closed Categories without Units

It is, of course, a routine matter to specialise this definition of semi star-autonomous category to the compact closed case. This chapter gives an elementary axiomatisation of semi compact closure, and shows its equivalence to the 'abstract' notion. The present definition is completely algebraic, and is perhaps easier to understand and use. The definition itself is not really new: Hines (1999, §3.5) has a similar-looking definition, which seems to be strictly weaker than the present one, and Došen and Petrić (2005) give a more general version.[1]

Definition 9.1. A *category with tensor* $\mathbb{C}$ is a category equipped with a tensor product

$$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C},$$

together with natural isomorphisms having components

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$
$$\sigma_{A,B} : A \otimes B \to B \otimes A$$

such that $\sigma^{-1}_{B,A} = \sigma_{A,B}$, and subject to the pentagon and hexagon conditions found in the usual definition of symmetric monoidal category.

Although the development is stated in terms of a symmetric tensor, it is perfectly possible – with only a little more work – to carry it through when the tensor is merely braided. The string diagrams, in particular, should make it clear which direction of braiding is required in any particular definition. Note that an additional

[1] The Došen-Petrić axioms are intended to define a semi star-autonomous category, therefore assume two tensors $\otimes$ and $\invamp$, related by a linear distributivity. If one takes the two tensors to be equal, and the linear distributivity to be the ordinary associativity, then one recovers the present definition with some redundancy.

$$\begin{tikzcd}
X \otimes Y \arrow[r, "X \otimes \eta^A_Y"] \arrow[dr, "\eta^A_{X \otimes Y}"'] & X \otimes (Y \otimes (A^* \otimes A)) \arrow[d, "\alpha_{X,Y,A^* \otimes A}"] \\
& (X \otimes Y) \otimes (A^* \otimes A)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes A^*) \otimes (X \otimes Y) \arrow[d, "\alpha_{A^* \otimes A,X,Y}"'] \arrow[dr, "\varepsilon^A_{X \otimes Y}"] & \\
((A^* \otimes A) \otimes X) \otimes Y \arrow[r, "\varepsilon^A_X \otimes Y"'] & X \otimes Y
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "\eta^A_A"'] & A \\
A \otimes (A^* \otimes A) \arrow[r, "\alpha_{A,A^*,A}"'] & (A \otimes A^*) \otimes A \arrow[u, "\varepsilon^A_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A^* \arrow[r, "1"] \arrow[d, "\eta^A_{A^*}"'] & A^* \\
A^* \otimes (A^* \otimes A) \arrow[r, "\theta_{A^*,A^*,A}"'] & (A \otimes A^*) \otimes A^* \arrow[u, "\varepsilon^A_{A^*}"']
\end{tikzcd}$$

Figure 9.1: Coherence conditions for a semi compact closed category

axiom is needed in the braided case, specifically the braid dual of the second cancellation condition, and braid-dual versions of the lemmas need to be proved. We also introduce an abbreviation that will be useful in the next definition: let θ denote the unique canonical natural isomorphism with components

$$\theta_{A,B,C} : A \otimes (B \otimes C) \to (C \otimes A) \otimes B.$$

(This may be defined as either $\alpha_{C,A,B}.\sigma_{A \otimes B,C}.\alpha_{A,B,C}$ or $(\sigma_{A,C} \otimes B)\alpha_{A,C,B}(A \otimes \sigma_{B,C})$; the hexagon condition says precisely that these must be equal.)

Definition 9.2. A *semi compact closed category* is a category $\mathbb{C}$ with tensor, equipped with: for every object $A \in \mathbb{C}$, a *dual object* A^*, and natural transformations η^A and ε^A with components

$$\begin{aligned} \eta^A_X &: X \to X \otimes (A^* \otimes A) \\ \varepsilon^A_X &: (A \otimes A^*) \otimes X \to A \end{aligned}$$

These natural transformations are called the *unit* and *counit* of A, and are required to satisfy the four axioms shown in Fig. 9.1.

The plan for the rest of this chapter is as follows. In §9.1 we develop the theory of semi compact closed categories directly from the axioms, since it is instructive to see how readily this may be done, and how similar it is to the ordinary theory of compact closure. (But see later for an alternative, indirect, approach.) §9.2 then shows that every semi compact closed category is (degenerately) semi star-

Figure 9.2: Diagrammatic notation for η and ε

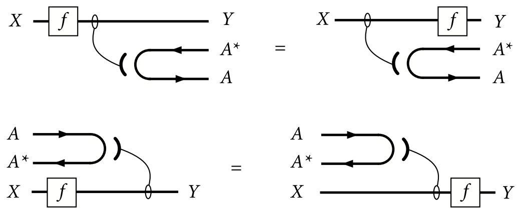


Figure 9.3: Diagrammatic form of the naturality conditions

autonomous in the sense of Chapter 8.

§9.3 is independent of the previous sections, and shows how an arbitrary semi compact closed category may be fully and faithfully embedded in an ordinary compact closed category (which has one additional object playing the role of the unit). This embedding preserves the tensor and duality on the nose, which makes it possible to transfer most of our knowledge about compact closed categories to the unitless situation, and in particular to deduce the main results of §9.1.

9.1 Direct development

We shall use string diagrams (Joyal and Street, 1991), to make the calculations easier to follow. Our diagrams are to be read from left to right, and we notate η and ε as in Fig. 9.2. Diagrammatic forms of the axioms are shown in Figs. 9.3–9.5. The first task is to show how the duality operation can be extended to a contravariant functor, in such a way that η and ε are both dinatural in A. Given an arrow $f : A \to B$, we define $f^* : B^* \to A^*$ as shown in Fig. 9.6. Note that, directly from the second cancellation axiom, we have $1_A^* = 1_{A^*}$ for all $A \in \mathbb{C}$, thus our putative functor preserves identities (which is a good start). It is surprisingly complicated to prove directly that it also preserves composition, but it will be easy once we have the right lemmas.

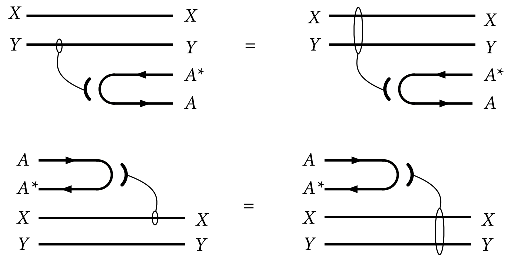


Figure 9.4: Diagrammatic form of the associativity conditions

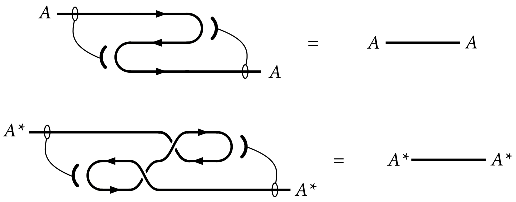


Figure 9.5: Diagrammatic form of the cancellation conditions

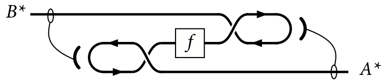


Figure 9.6: Given $f : A \to B$, we define $f^* : B^* \to A^*$ using this diagram

Lemma 9.3. *For all X, A and $Y \in \mathbb{C}$, the following diagrams commute.*

$$\begin{tikzcd}
(X \otimes (A \otimes A^*)) \otimes Y \arrow[r, "\alpha^{-1}_{X,A\otimes A^*,Y}"] \arrow[d, "\sigma_{A\otimes A^*,X} \otimes Y"'] & X \otimes ((A \otimes A^*) \otimes Y) \arrow[d, "X \otimes \varepsilon^A_Y"] \\
((A \otimes A^*) \otimes X) \otimes Y \arrow[r, "\varepsilon^A_X \otimes Y"'] & X \otimes Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "X \otimes \eta^A_Y"] \arrow[d, "\eta^A_X \otimes Y"'] & X \otimes (Y \otimes (A^* \otimes A)) \arrow[d, "X \otimes \sigma_{A^*\otimes A,Y}"] \\
(X \otimes (A^* \otimes A)) \otimes Y \arrow[r, "\alpha^{-1}_{X,A^*\otimes A,Y}"'] & X \otimes ((A^* \otimes A) \otimes Y)
\end{tikzcd}$$

Proof. The proof of the first diagram, by string diagram manipulation, is shown in Fig. 9.7. (Perhaps the least obvious step is the penultimate one, which uses the naturality of σ.) The second is proved by a symmetrical argument: Fig. 9.8 shows the diagrammatic form of its statement. $\square$

Lemma 9.4. *For any objects X, A, B, Y, and arrow $f : A \to B$, the following diagram commutes. (The associativities have been suppressed to make it more comprehensible.)*

$$\begin{tikzcd}
X \otimes B^* \otimes Y \arrow[r, "X \otimes f^* \otimes Y"] \arrow[d, "\eta^A_X \otimes B^* \otimes Y"'] & X \otimes A^* \otimes Y \\
X \otimes A^* \otimes A \otimes B^* \otimes Y \arrow[r, "X \otimes A^* \otimes f \otimes B^* \otimes Y"'] & X \otimes A^* \otimes B \otimes B^* \otimes Y \arrow[u, "X \otimes A^* \otimes \varepsilon^B_Y"']
\end{tikzcd}$$

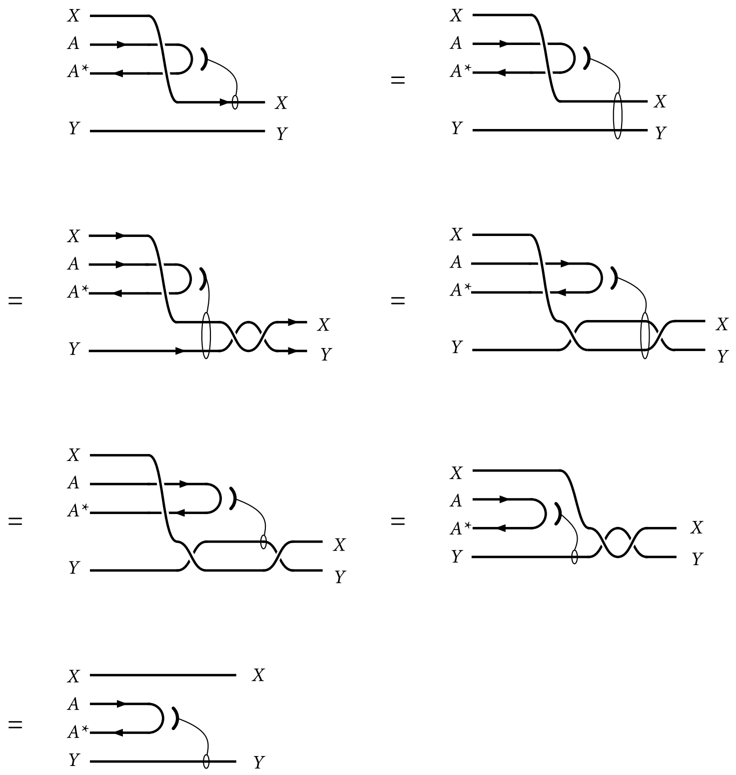


Figure 9.7: A diagrammatic proof of Lemma 9.3

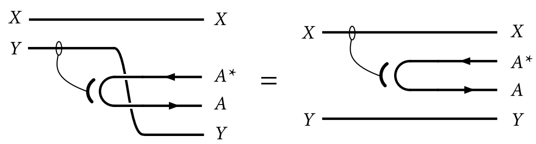


Figure 9.8: The second part of Lemma 9.3

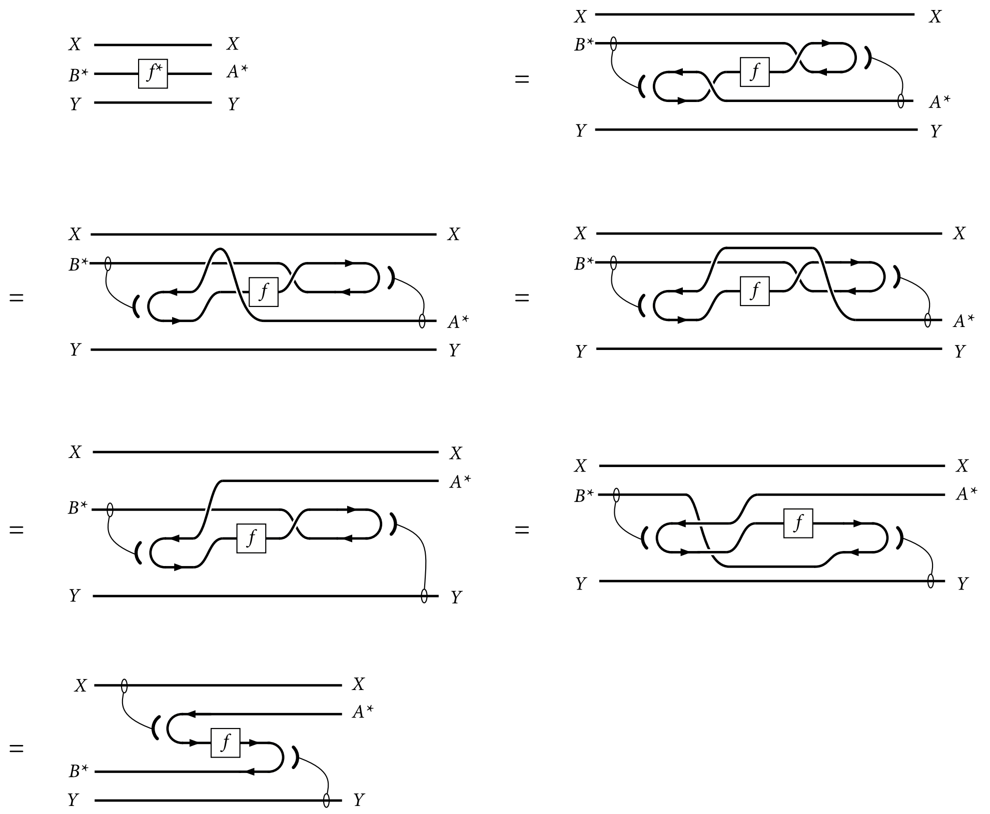
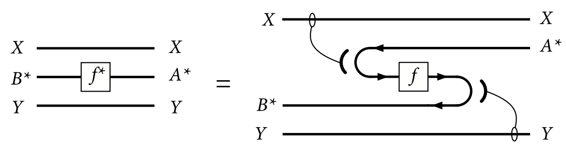


Figure 9.9: A diagrammatic proof of Lemma 9.4

In string diagram terms, this says



Proof. The proof is again by string diagram manipulation, shown in Fig. 9.9. Both parts of Lemma 9.3 are used. □

Lemma 9.5. *For all X, A, $Y \in \mathbb{C}$, the following diagram commutes.*

$$\begin{tikzcd}
X \otimes A \otimes Y \arrow[r, "1"] \arrow[d, "\eta^A_X \otimes A \otimes Y"'] & X \otimes A \otimes Y \\
X \otimes A^* \otimes A \otimes A \otimes Y \arrow[r, "X \otimes \sigma_{A^*, A\otimes A, Y}"'] & X \otimes A \otimes A \otimes A^* \otimes Y \arrow[u, "X \otimes A \otimes \varepsilon^A_Y"']
\end{tikzcd}$$

(The associativities have again been suppressed.)

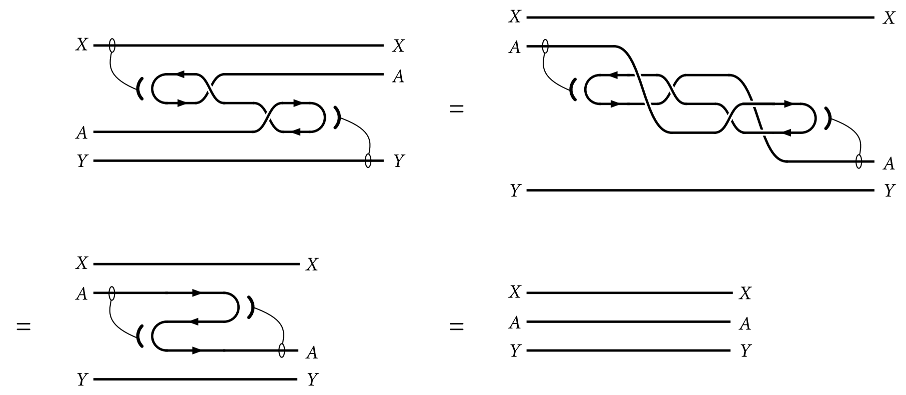


Figure 9.10: Proof of Lemma 9.5

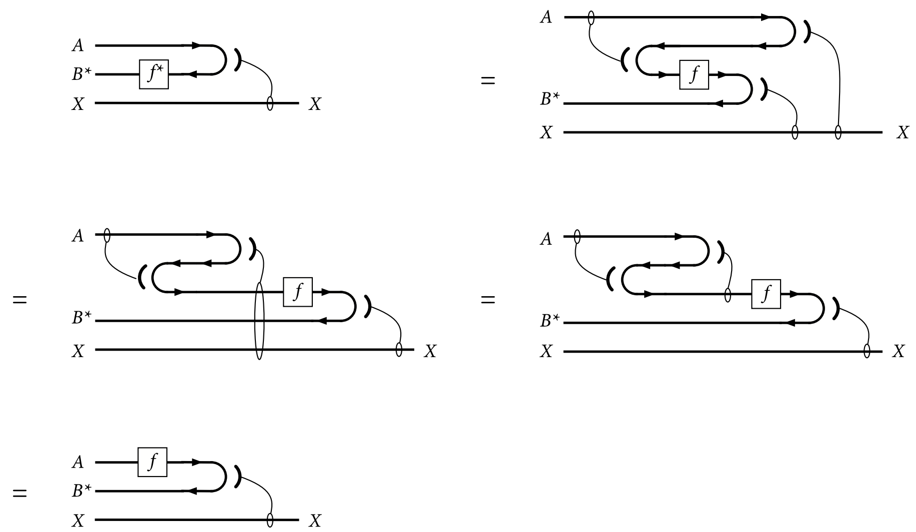


Figure 9.11: Proof that ε is dinatural (Prop. 9.6)

Proof. s See Fig. 9.10. The first step uses both parts of Lemma 9.3.[2] □

All the hard work was in the lemmas: everything else is comparatively straightforward.

Proposition 9.6. *The natural transformations η and ε are also dinatural in the superscript variable.*

Proof. See Fig. 9.11 for a proof that ε is dinatural. The proof for η may be obtained by turning the string diagrams upside down. □

Proposition 9.7. *The duality preserves composition, i.e. given $f : A \to B$ and $g : B \to C$, we have $(gf)^* = f^* g^*$.*

[2] In the braided case, it uses the braid-dual analogue of that lemma.

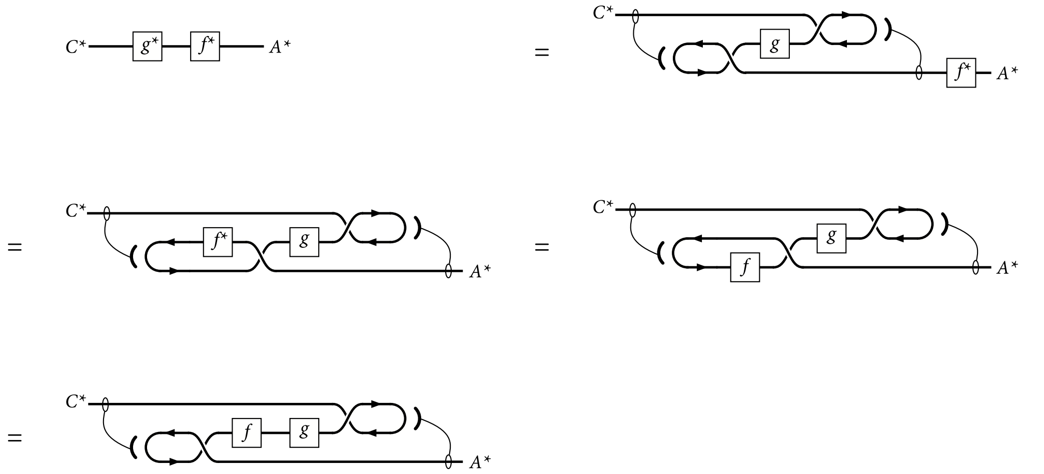


Figure 9.12: Proof of Prop. 9.7

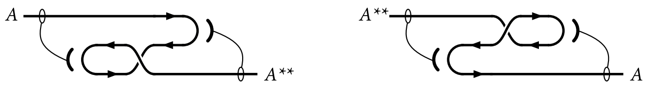


Figure 9.13: How to construct a natural isomorphism $A \cong A^{**}$

Proof. See Fig. 9.12. The third equality uses the dinaturality of η. □

Proposition 9.8. *There is a natural isomorphism $A \cong A^{**}$.*

Proof. Fig. 9.13 shows how to construct a natural isomorphism $A \cong A^{**}$. Naturality is immediate from the naturality and dinaturality of η and ε, and the naturality of σ. Figs. 9.14 and 9.15 show that these maps are indeed mutually inverse, hence determine an isomorphism. Notice that the second step in Fig. 9.14 uses Lemma 9.5. □

Proposition 9.9. *For each object A, there is an adjunction $A \otimes - \dashv A^* \otimes -$, which determines a natural isomorphism*

$$\mathbb{C}(B \otimes A, C) \cong \mathbb{C}(A, B^* \otimes C)$$

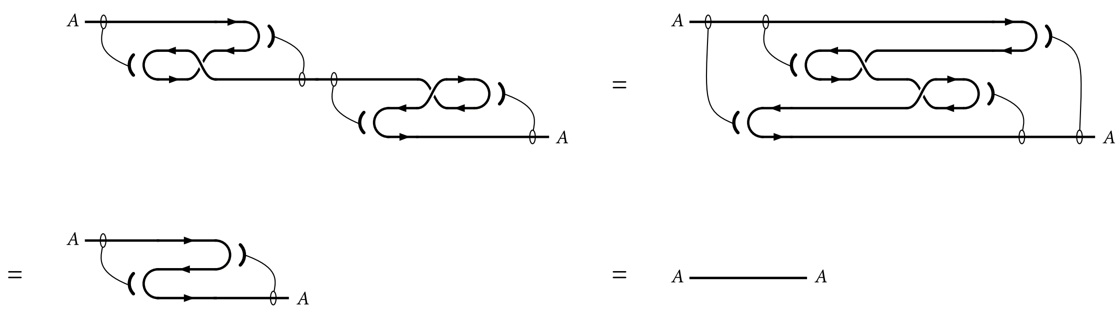


Figure 9.14: The maps from Fig. 9.13 compose to give the identity on A

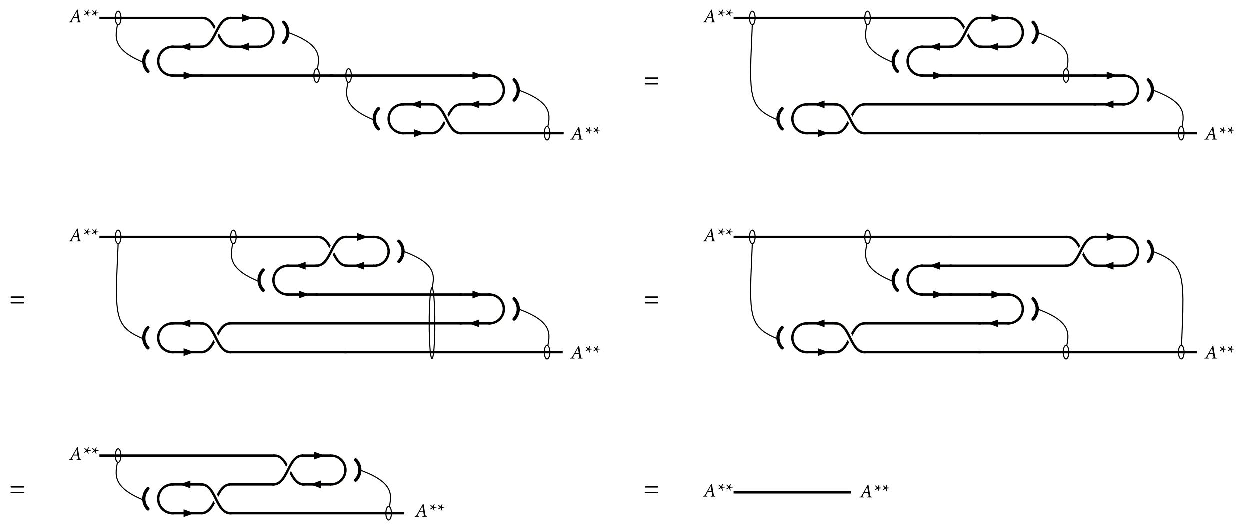


Figure 9.15: The maps from Fig. 9.13 compose to give the identity on A^{**}

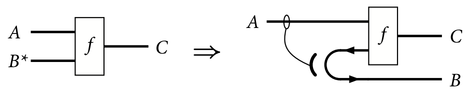


Figure 9.16: The natural transformation γ

Proof. There are obvious natural transformations

$$\gamma_{A,B,C} : \mathbb{C}(A \otimes B^*, C) \to \mathbb{C}(A, C \otimes B),$$
$$\delta_{A,B,C} : \mathbb{C}(A, B^* \otimes C) \to \mathbb{C}(B \otimes A, C)$$

illustrated in Figs. 9.16–9.17. We need to show that one of these natural transformations is invertible, which we shall do by showing that they are in some sense mutually inverse. We begin by showing that the composite

$$\mathbb{C}(A \otimes B^*, C) \xrightarrow{\gamma} \mathbb{C}(A, C \otimes B) \xrightarrow{\cong} \mathbb{C}(A, C \otimes B^{**})$$

$$\xrightarrow{\cong} \mathbb{C}(A, B^{**} \otimes C) \xrightarrow{\delta} \mathbb{C}(B^* \otimes A, C) \xrightarrow{\cong} \mathbb{C}(A \otimes B^*, C)$$

is the identity (where the unlabelled isomorphisms are symmetry or involution maps). Consider some $f : A \otimes B^* \to C$: the result of applying this composite to f is shown in Fig. 9.18. Fig. 9.19 shows that this is equal to f. (The first step combines several uses of naturality and associativity conditions.) Therefore γ has a post-inverse and δ a pre-inverse.

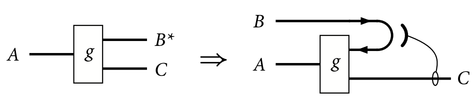


Figure 9.17: The natural transformation δ

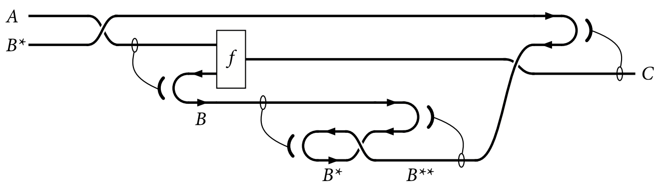


Figure 9.18: The result of applying γ and then δ to some $f : A \otimes B^* \to C$

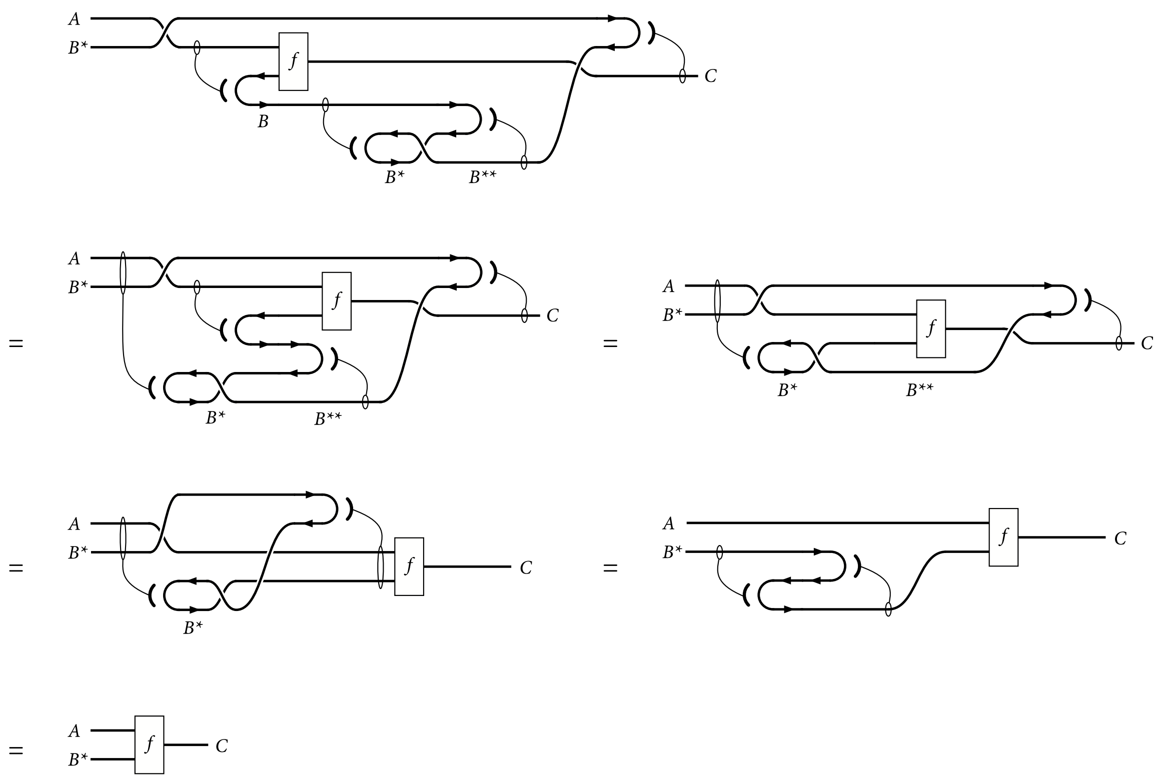


Figure 9.19: The map shown in Fig. 9.18 is equal to f

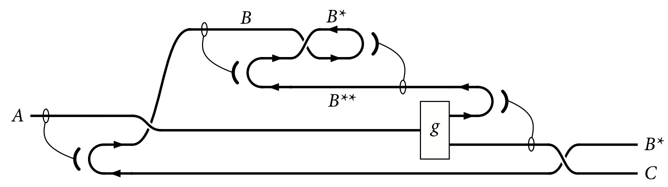


Figure 9.20: The result of applying δ and then γ to some $g : A \to B^* \otimes C$,

Similarly one may take a map $g : A \to B^* \otimes C$, and apply to it the composite

$$\mathbb{C}(A, B^* \otimes C) \xrightarrow{\delta} \mathbb{C}(B \otimes A, C) \xrightarrow{\cong} \mathbb{C}(B^{**} \otimes A, C)$$

$$\xrightarrow{\cong} \mathbb{C}(A \otimes B^{**}, C) \xrightarrow{\gamma} \mathbb{C}(A, C \otimes B^*) \xrightarrow{\cong} \mathbb{C}(A, B^* \otimes C)$$

as shown in Fig. 9.20. This is equal to g – the proof is obtained by turning all the diagrams in Fig. 9.19 upside down – hence γ also has a pre-inverse and δ a post-inverse. Therefore both are invertible, as claimed. □

9.2 The promonoidal structure

This section shows that a semi compact closed category is semi star-autonomous in the sense of Chapter 8. The proof relies on the characterisation of semi SMC categories via linear elements, as given in Section 8.2.1. With this machinery available it is easy to prove the main result of this section:

Proposition 9.10. *A semi compact closed category is semi star-autonomous.*

Proof. Let $\mathbb{C}$ be a semi compact closed category. By assumption it is equipped with a symmetric tensor, and if we define

$$A \multimap B := A^* \otimes B$$

then Prop.9.9 shows that we have a natural isomorphism

$$\delta_{A,B,C} : \mathbb{C}(B \otimes A, C) \xleftarrow{\cong} \mathbb{C}(A, B \multimap C).$$

It remains only to construct an inverse to the function $\ell_{A,B} : \mathrm{Lin}(A^* \otimes B) \to \mathbb{C}(A, B)$.

If we represent a linear element $x \in \mathrm{Lin}(A^* \otimes B)$ as shown in Fig. 9.21, note that, by the definition of δ, the arrow $\ell_{A,B}(x)$ is as shown in Fig. 9.22.

Given a map $f : A \to B$, define $\ell^{-1}_{A,B}(f)$ to be the natural transformation whose

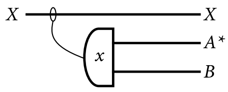


Figure 9.21: The diagrammatic representation of a linear element $x \in \mathrm{Lin}(A^* \otimes B)$

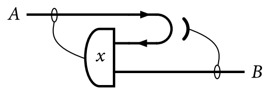


Figure 9.22: The diagrammatic representation of $\ell_{A,B}(x)$

component at X is

$$X \xrightarrow{\eta^A_X} X \otimes (A^* \otimes A) \xrightarrow{X \otimes (A^* \otimes f)} X \otimes (A^* \otimes B).$$

For any $f : A \to B$, the arrow $\ell_{A,B}(l^{-1}_{A,B}(f))$ is the composite

$$A \xrightarrow{\eta^A_A} A \otimes (A^* \otimes A) \xrightarrow{A \otimes (A^* \otimes f)} A \otimes (A^* \otimes B) \xrightarrow{\alpha_{A,A^*,B}} (A \otimes A^*) \otimes B \xrightarrow{\varepsilon^A_B} B.$$

By the naturality of α and ϵ, and the first cancellation condition, this is indeed equal to f. Conversely suppose we have a linear element $x \in \mathrm{Lin}(A^* \otimes B)$. Since it will be convenient to use a string diagram calculation here, we introduce a diagrammatic notation for this linear element, shown in Fig. 9.21. Now Fig. 9.23 shows a proof that $\ell^{-1}_{A,B}(l_{A,B}(x))$ is equal to x. □

9.3 Embedding theorem

If we wanted to add a unit object I to a semi star-autonomous category $\mathbb{C}$, we would also have to add an infinite family of other objects such as I^*, $I^* \otimes A$ for $A \in \mathbb{C}$, and so on. In the compact closed case, there is no such obstacle, since I^* is always isomorphic to I, and we may take $I^* = I$ without essential loss of generality. This

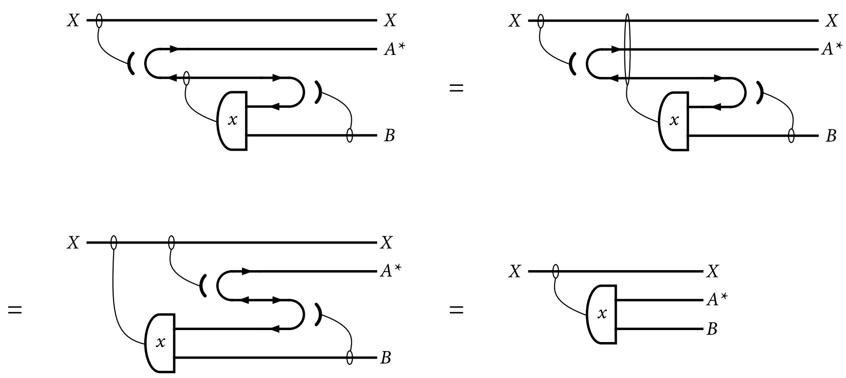


Figure 9.23: Proof that $\ell^{-1}_{A,B}(\ell_{A,B}(x)) = x$. (See Prop. 9.10)

raises the hope that it may always be possible to fully embed any semi compact closed category $\mathbb{C}$ into a compact closed category $\mathbb{C}'$, in such a way the objects of $\mathbb{C}'$ are essentially just the objects of $\mathbb{C}$ plus a unit object. It turns out that such an embedding is possible, as this section shows.

Recall the '**e**' construction of Joyal and Street (1993), there used to prove the case $\mathcal{B}$ = Cat of the Cayley Theorem (our Proposition 7.13). Given a monoidal category $\mathbb{C}$, the category $\mathbf{e}(\mathbb{C})$ is defined as follows. An object of $\mathbf{e}(\mathbb{C})$ is a pair (F, γ^F) of a functor $F : \mathbb{C} \to \mathbb{C}$ and a natural isomorphism with components

$$\gamma^F_{A,B} : F(A \otimes B) \to F(A) \otimes B.$$

A morphism $\delta : (F, \gamma^F) \to (G, \gamma^G)$ is a natural transformation $F \Rightarrow G$ such that the diagram

$$\begin{tikzcd} F(A \otimes B) \arrow[r, "\gamma^F_{A,B}"] \arrow[d, "\delta_{A\otimes B}"'] & F(A) \otimes B \arrow[d, "\delta_A \otimes B"] \\ G(A \otimes B) \arrow[r, "\gamma^G_{A,B}"] & G(A) \otimes B \end{tikzcd}$$

commutes for all A and $B \in \mathbb{C}$.

The tensor product $(F, \gamma^F) \otimes (G, \gamma^G)$ is defined to be (FG, γ^{FG}), where $\gamma^{FG}_{A,B}$ is the composite

$$FG(A \otimes B) \xrightarrow{F\gamma^G_{A,B}} F(GA \otimes B) \xrightarrow{\gamma^F_{GA,B}} FGA \otimes B.$$

The tensor product of two arrows is their horizontal composite as natural transformations. The tensor unit I is simply the identity functor, with the identity natural transformation.

There is an functor $L : \mathbb{C} \to \mathbf{e}(\mathbb{C})$, where $L(A) := (A \otimes -, \alpha_{A,-,-})$ for objects $A \in \mathbb{C}$, and $L(f) := f \otimes -$ for arrows f. Note that the unit object of $\mathbb{C}$ plays no part in the construction of the category $\mathbf{e}(\mathbb{C})$ or the functor L, so that everything so far makes sense for a semi compact closed category. Furthermore:

Proposition 9.11. *When $\mathbb{C}$ is a semi compact closed category, thr functor L is full and faithful.*

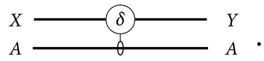
Proof. Joyal and Street's proof of this claim (for $\mathbb{C}$ a monoidal category) uses the tensor unit in an essential way, so we need to find a new proof that uses semi compact closure instead. The functor L induces, for every X and $Y \in \mathbb{C}$, a function $\mathbb{C}(X, Y) \to \mathbf{e}(\mathbb{C})(LX, LY)$. We'll describe an inverse to this function, showing that it is invertible and hence that L is full and faithful.

Let δ be a natural transformation $LX \Rightarrow LY$. Thus δ consists of components $\delta_A : X \otimes A \to Y \otimes A$, natural in A and such that the diagram

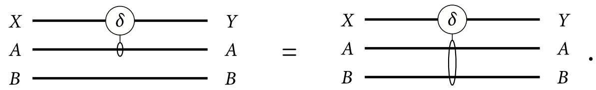
$$\begin{tikzcd}
X \otimes (A \otimes B) \arrow[r, "\alpha_{X,A,B}"] \arrow[d, "\delta_{A\otimes B}"'] & (X \otimes A) \otimes B \arrow[d, "\delta_A \otimes B"] \\
Y \otimes (A \otimes B) \arrow[r, "\alpha_{Y,A,B}"] & (Y \otimes A) \otimes B
\end{tikzcd}$$

commutes for all $A, B \in \mathbb{C}$.

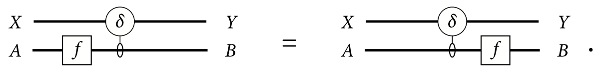
It will be convenient to use string diagrams in the proof: we'll picture δ_A as



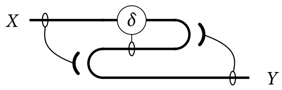
In string diagram terms, the commutative square above is a rewiring condition of the sort we have seen above:



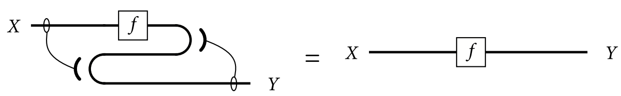
The naturality of δ means that functions can pass through the loop:



Now we can define our inverse to the action of L, to take δ to the following arrow $f : X \to Y$:



We must show that $f \otimes A = \delta_A$, for any $A \in \mathbb{C}$. The proof is a routine string diagram manipulation, shown in Fig. 9.24. This shows that the passage $\mathbf{e}(\mathbb{C})(LX, LY) \to \mathbb{C}(X, Y) \to \mathbf{e}(\mathbb{C})(LX, LY)$ is the identity. For the other direction, we need to show that



which is immediate. □

Now define $\mathbb{E}$ to be the full subcategory of $\mathbf{e}(\mathbb{C})$ determined by the objects that have adjoints. This is clearly a compact closed category.

Proposition 9.12. *The image of L is contained in $\mathbb{E}$. Specifically, for every $X \in \mathbb{C}$, the*

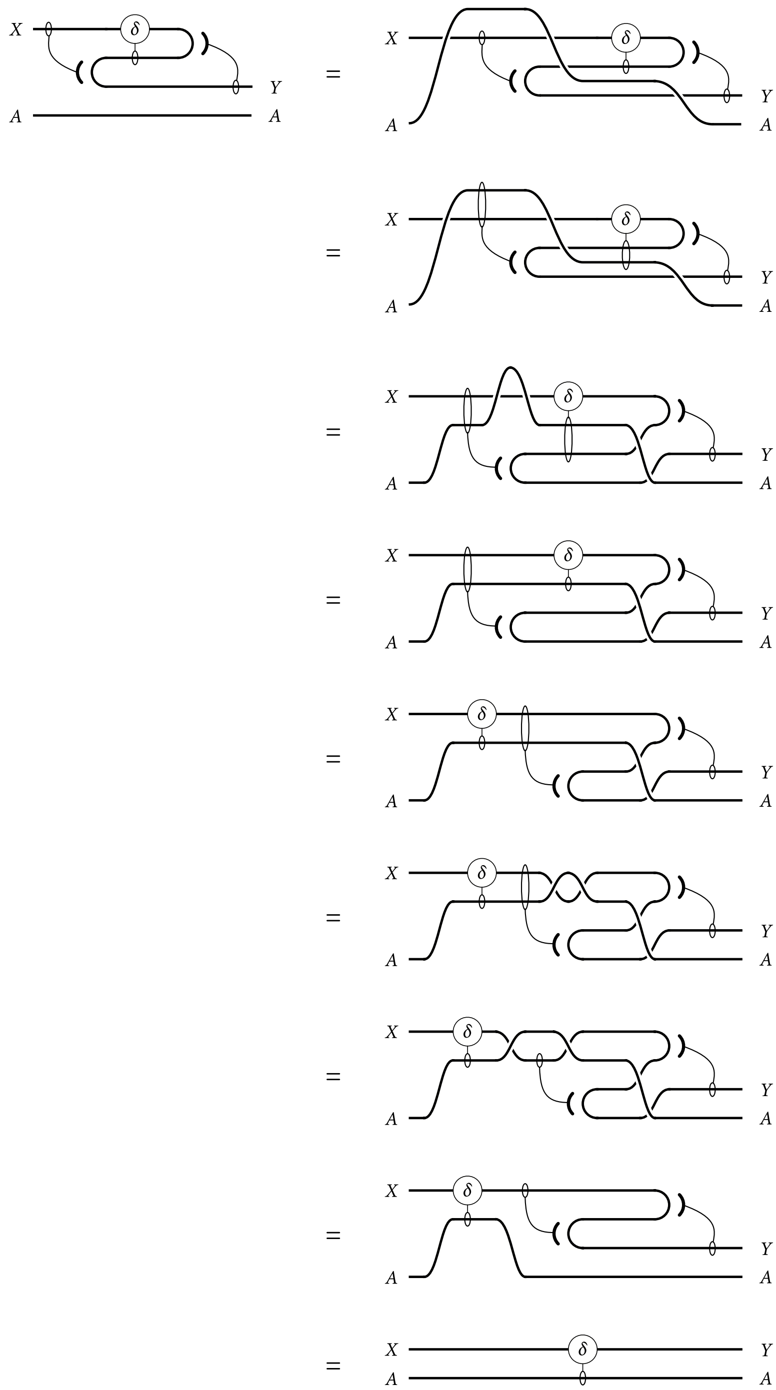
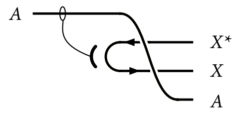


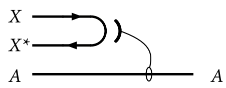
Figure 9.24: Proof that $f \otimes A = \delta_A$, used in Prop. 9.11.

object $L(X)$ is adjoint to $L(X^*)$ in $\mathbf{e}(\mathbb{C})$.

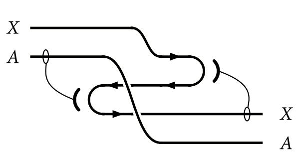
Proof. Define $\eta^{LX} : I \to L(X^*)L(X)$ to have components



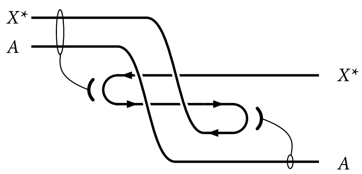
and $\varepsilon^{LX} : L(X)L(X^*) \to I$ to have components



These are clearly natural in A, and it's easy to verify that they satisfy the condition making them maps of $\mathbf{e}(\mathbb{C})$. To show that they really do define an adjunction between $L(X)$ and $L(X^*)$, we need to show that



and



are both equal to the identity. This is an easy exercise in manipulations of the sort that are by now routine. □

Finally, let $\mathbb{C}'$ be the full subcategory of $\mathbb{E}$ determined by those objects that are either isomorphic to $L(X)$ for some X, or isomorphic to I. This subcategory is closed under the tensor and adjoint operations, so it's compact closed. The image of the (full and faithful) functor L is contained in $\mathbb{C}'$ by definition. Thus $\mathbb{C}$ is embedded, in a structure-preserving fashion, in a compact closed category that has essentially only one extra object, the unit object. (If in fact $\mathbb{C}$ had a unit object all along, this functor will be an equivalence.)

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